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Edge excitations and topological order in a rotating Bose gas

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The edge excitations and related topological orders of correlated states of a fast rotating Bose gas are studied. Using exact diagonalization of small systems, we compute the energies and number of edge excitations, as well as the boson occupancy near the edge for various states. The chiral Luttinger-liquid theory of Wen is found to be a good description of the edges of the bosonic Laughlin and other states identified as members of the principal Jain sequence for bosons. However, we find that in a harmonic trap the edge of the state identified as the Moore-Read (Pfaffian) state shows a number of anomalies. An experimental way of detecting these correlated states is also discussed.

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It has been argued[1–3] that quantum fluctuations can destroy a Bose-Einstein condensate (BEC) if it rotates very fast. Since large amounts of angular momentum can be imparted to a cold atomic gas, experimentalists have been able to create systems with a large number of vortices, N_v.[3–5] Thus, the question of what happens when N_v eventually becomes comparable to the number of particles, N, has been raised.[1,2] In Ref. 2, it was shown that for $\nu = N/N_v \sim 10$, the BEC and the Abrikosov vortex lattice are destroyed and replaced by a series of "vortex liquid" states, some of which are *incompressible* and exhibit good overlap with bosonic versions of wave functions known from fractional quantum Hall effect (FQHE).

Good overlap is usually a strong indication that some of the correlations of a state obtained from exact diagonalization are captured by a model wave function. However, Wen[6] has emphasized that FQHE states possess a unique type of *quantum* order, which he dubbed "topological order," that provides a better way to classify them. In a deep sense, topological order can be regarded as a measure of the quantum entanglement existing between the particles in a correlated quantum Hall (QH) state.[7]

In this paper we study the topological order of the vortex liquids as reflected in their edge properties.[6] *In the rotating frame*, edge excitations are the low-lying excitations of the vortex liquid[8] and, contrary to the ground states,[1,2,9,10] so far they have received little attention. Based on the strong similarities with electron FQHE physics, chiral Luttinger liquids[11] and similar edge excitations[6] are expected, but an explicit demonstration is lacking for a harmonically confined gas of bosons under rotation. This is provided here by numerically diagonalizing the Hamiltonian of small systems.

The topological order can be studied at the edge of fairly small droplets of QH liquid.[6,11,12] In what follows, we shall focus on three types of states that have been identified in previous works on small droplets.[1,9] In the high angular momentum end, we study the edge properties of the Laughlin state (corresponding to a bulk filling fraction $\nu = \frac{1}{2}$). By decreasing the total angular momentum, we come across the compact composite-fermion states discussed in Ref. 9. Amongst these states, we find evidence for the topological order (and related edge structures) that correspond to bulk states with filling fractions $\nu = \frac{2}{3}$ and $\nu = \frac{3}{4}$ of the principal Jain sequence. At even lower angular momentum, we study the edge properties of the state identified[1] as the finite-sized Moore-Read (or Pfaffian) state[13] ($\nu = 1$). We observe a number of anomalies that persist up to the largest sizes studied here ($N = 13$). The bulk properties of the states at these filling fractions have also been studied in exact diagonalizations in edgeless geometries[2,10] which have established their interpretation in terms of the Laughlin $\left(\nu = \frac{1}{2}\right)$, composite fermion $\left(\nu = \frac{2}{3}, \frac{3}{4}\right)$, and Moore-Read ($\nu = 1$) states. For the latter, calculations on the sphere[10] and on the torus[2] found the correct shift and ground-state degeneracy (which are also a consequence of the topological order[6]).

An ultracold gas that rotates rapidly in a cylindrically symmetric harmonic trap acquires a pancake shape and eventually becomes quasi-two-dimensional (quasi-2D) when the chemical potential $\mu < \hbar\omega_\parallel$, where $\omega_\parallel$ is the axial trapping frequency.[15] Furthermore, the Coriolis force acts as an effective Lorentz force, which in a quasi-2D system leads to Landau levels (LL's) separated by an energy $2\hbar\omega_\perp$.[15] For $\mu < 2\hbar\omega_\perp$, all atoms lie in the lowest Landau level (LLL), and the total single-particle energy is proportional (up to a constant) to the angular momentum. Here we shall be interested in this limit, which has been already achieved in the experiments.[4] In the rotating frame, the Hamiltonian (relative to the zero-point energy) is

$$H_{\mathrm{LLL}} = T + U_2 = \hbar(\omega_\perp - \Omega)L + g\sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j), \qquad (1)$$

with $\hbar L$ being the total axial angular momentum and $g = \sqrt{8\pi}\hbar\omega_\perp(a_s/\ell_\parallel)\ell^2$ the effective coupling for a gas harmonically confined to two dimensions ($a_s < \ell_\parallel$ being the scattering length that characterizes the atom-atom interaction in three dimensions, $\ell_\parallel = \sqrt{\hbar/M\omega_\parallel}$ the axial oscillator length, and $\ell = \sqrt{\hbar/M\omega_\perp}$, M being the atom mass).

Laughlin state. For $L = L_0 = N(N-1)$, the ground state of H_{LLL} is[1,8,9]

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TABLE I. Number of edge states (NoS) vs excitation angular momentum m. In the data for the Moore-Read (Pfaffian) state $\lambda=0$ for a *pure* three-body interaction and $\lambda=1$ for a *pure* two-body interaction.

$m=L-L_0$	0	1	2	3	4	5	6
Laughlin ($N=5$, $L=20$)	1	1	2	3	5	7	10
Laughlin ($N=6$, $L=30$)	1	1	2	3	5	7	11
Laughlin ($N\to\infty$)	1	1	2	3	5	7	11
{4, 2} CF ($N=6$, $L=20$)	1	2	5	8			
{5, 2} CF ($N=7$, $L=30$)	1	2	5	9	15		
Jain $\nu=\frac{2}{3}(N\to\infty)$	1	2	5	10	20	36	65
Moore-Read ($\lambda=0$, $N=8$, $L=24$)	1	1	3	5	10	15	
Moore-Read ($\lambda=0$, even $N\to\infty$)	1	1	3	5	10	16	28
Moore-Read ($\lambda=1$, $N=12$, $L=60$)	1	4	10	21			
Moore-Read ($\lambda=0$, $N=7$, $L=18$)	1	2	4	7	12		
Moore-Read ($\lambda=0$, odd $N\to\infty$)	1	2	4	7	13	21	35
Moore-Read ($\lambda=1$, $N=13$, $L=72$)	1	6	14	29			

$$\Phi_0(z_1,\ldots,z_N)=\left\{\prod_{i<j}(z_i-z_j)^2\right\}e^{-\sum_{i=1}^{N}|z_i|^2/2}, \tag{2}$$

where $z_j\equiv(x_j+i\,y_j)/\ell$. This wave function vanishes when any two particles coincide and therefore $U_2|\Phi_0\rangle=0$. This property is maintained if it is multiplied by an arbitrary symmetric polynomial of the z_i. Edge excitations of the above state are generated[8,11] by elementary symmetric polynomials of the form $s_m=\Sigma_{i_1<i_2<\cdots<i_m}z_{i_1}z_{i_2}\cdots z_{i_m}$. Thus, L is increased by m units and, according to (1), the excitation energy is $\hbar(\omega_\perp-\Omega)m$. In the rotating frame, the edge excitations are the lowest energy excitations of a rotating Bose gas in the Laughlin state.[8] Their degeneracy for $L=L_0+m$ is given by the number of distinct ways m can be written as a sum of smaller non-negative integers [i.e., partitions of $m,p(m)$]. For example, for $m=4$ there are five degenerate states: $[s_4,s_3s_1,(s_2)^2,s_2(s_1)^2,(s_1)^4]\times\Phi_0$. The properties of these wave functions are captured by an effective-field theory,[8,11] which treats these excitations as a noninteracting phonon system with Hamiltonian $H_{\rm edge}=\hbar(\omega_\perp-\Omega)L$, where $L=L_0+\Sigma_{m>0}mb_m^\dagger b_m$, and $[b_m,b_n^\dagger]=\delta_{mn}$, commuting otherwise. Hence, the number of edge states (NoS) for a given m can be obtained from a generating (or partition) function: $Z(q)=\mathrm{Tr}\,q^{L-L_0}=\Pi_{m>0}(1-q^m)^{-1}=\Sigma_{m>0}p(m)q^m$. In Table I, we compare the theoretical NoS ($N\to\infty$) with the numerical results, finding excellent agreement for $m\leqslant N$. The deviations are due to finite-size effects and can be accounted by a "truncated" function $Z^N(q)$, where the product is restricted to $0<m\leqslant N$.

Another prediction of the theory[11,8] is the form of the ground-state boson occupancy, $n(l)$, just below the highest occupied orbital, $l_{\max}=2(N-1)$. The following ratios of $n(l)/n(l_{\max})$ are predicted: 1:2:3:4 for $l=l_{\max},\ldots,l_{\max}-3$. For $N=7$ bosons in the Laughlin state, we find 1.0:2.0:2.9:3.5, which is in good agreement with the theory given that its validity is limited to $l_{\max}-l\lesssim\sqrt{N}\sim3$ (Ref. 11) Similar agreement was found for the fermion Laughlin state in Ref. 11.

Principal Jain sequence. The incompressibility of the Laughlin droplet can be regarded as a consequence of a statistical transmutation: at large N_v, a boson binds one vortex and becomes a composite object (called composite fermion, CF)[23] that behaves as a *spinless* fermion. In a Laughlin droplet, the CF's fill up the N lowest angular momentum orbitals in the LLL. Compressing the droplet and therefore decreasing L requires promoting CF's to higher LL's and costs a finite amount of energy. Thus, states with lower angular momentum will contain CF's in higher CF LL's, and this leads to the ansatz[9,23]

$$\Phi(z_1,\ldots,z_N)=\mathcal{P}\left[\prod_{i<j}(z_i-z_j)\Phi_{\rm CF}^{[\{N_i\}]}(\mathbf{r}_1,\ldots,\mathbf{r}_N)\right], \tag{3}$$

where $\mathcal{P}$ projects onto the LLL (Ref. 9) and $\Phi_{\rm CF}^{[\{N_i\}]}$ is a Slater determinant with $\{N_i\}_{i=1}^p$ CF's filling the lowest angular momentum orbitals of $p\geqslant1$ CF LL's, and $L=L_0=N(N-1)/2+\Sigma_{i=0}^{p-1}N_i[N_i-(2i+1)]/2$ ($N_0=N$ and $N_i=0$ for $i>0$ in the Laughlin state). In what follows, we focus on the CF states {4, 2} and {5, 2}, which have been shown[9] to have good overlap with the exact states at $L_0=20(N=6)$ and $L_0=30(N=7)$, respectively. For a relatively large range of $L-L_0>0$ (3 for {4, 2} and 4 for {5, 2}) the lowest energy state is a center-of-mass excitation of the state at $L=L_0$, and the interaction energy is unchanged. Edge excitations are those states with energy lower than the bulk gap (Δ, cf. Fig. 1).[12,14] According to Refs. 8,11, these states exhibit two branches of edge phonons described by.

$$H_{\rm edge}=E_0+\sum_{m>0}\hbar[\omega_m^{(b)}b_m^\dagger b_m+\omega_m^{(d)}d_m^\dagger d_m], \tag{4}$$

and $L=L_0+\Sigma_{m>0}m[b_m^\dagger b_m+d_m^\dagger d_m]$, with $[d_m,d_n^\dagger]=[b_m,b_n^\dagger]=\delta_{nm}$, commuting otherwise. Note that the edge excitations are not degenerate in energy. However, one can still compute the NoS from $z(q)=\mathrm{Tr}\,q^{L-L_0}=\Pi_{m>0}(1-q^m)^{-2}$. In Table I we compare the numerical results to the theoretical predictions for the NoS, finding perfect agreement for $m=1,2$. For higher m the results are affected by finite-size effects. However, in the state {5, 2} the NoS for $m=3$ is quite close to its

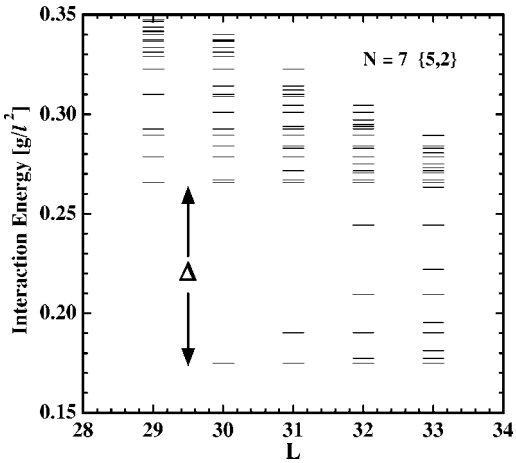

FIG. 1. Spectrum of the state $L=30$ with $N=7$. The gap $\Delta=0.091g/\ell^2$. Below the gap, we observe two phonon branches along with their multiphonon excitations.

$N\to\infty$ value, and therefore we concentrate on this state for further analysis. We next try to reproduce the energies of the 16 edge states of {5, 2} from $m=1$ to 3 using $\omega_{m=2}^{(b)}$ and $\omega_{m=1,2}^{(d)}$ as the only fitting parameters [$\omega_1^{(b)}$ has only a kinetic-energy contribution by Kohn's theorem[8]]. The interaction part of $\hbar\omega_1^{(d)}$ can be extracted from the $m=1$ data: $\hbar\omega_1^{(d)}=0.015g/\ell^2$. For $m=2$, there are the five following states: $2^{-1/2}(b_1^\dagger)^2|0\rangle, b_2^\dagger|0\rangle, d_1^\dagger b_1^\dagger|0\rangle, 2^{-1/2}(d_1^\dagger)^2|0\rangle, d_2^\dagger|0\rangle$, whose energies can be obtained from Eq. (4). For $m=3$ the states and energies can be written down in a similar fashion. The comparison with numerics for $m=2,3$ is given in Table II. Thus, these states are good representatives of the topological order of the filling $\nu=\frac{2}{3}$ from the principal Jain sequence for bosons.[6,8]

Finally, we also find evidence (to be reported elsewhere[16]) for the edge structures (three phonon branches) corresponding to $\nu=\frac{3}{4}$ However, the NoS for $m>1$ is affected by finite-size effects for $N=6,7$.

Moore-Read (MR) or Pfaffian state. Ground states of CF's with lower angular momentum are obtained by placing CF's in higher effective LL's. Eventually, when the number of occupied levels $p\to+\infty$, the CF's would not feel any effective Coriolis force and the resulting state should be compressible. However, in such a state the CF's can pair and condense into a BCS state, which would render the state incompressible again.[17] Since CF's are spinless, the pairing takes place in p wave and the BCS wave function is a Pfaffian,[13,17] which must be multiplied by $\Pi_{i<j}(z_i-z_j)$ to yield a bosonic wave function,

$$\Phi_{\rm MR}=\prod_{i<j}(z_i-z_j)\mathcal{A}\left[\frac{1}{z_1-z_2}\cdots\frac{1}{z_{N-1}-z_N}\right], \qquad (5)$$

where $\mathcal{A}$ stands for antisymmetrization (see, e.g., Refs. 1,18) of the bracketed product, which is the Moore-Read (MR) wave function.[13] Soon after the MR state was introduced, it was pointed out[19] that it is a zero interaction-energy eigenstate of a three-body potential, $U_3=g_3\Sigma_{i<j<k}\delta(\mathbf{r}_i-\mathbf{r}_j)\delta(\mathbf{r}_i-\mathbf{r}_k)$. Thus, it is convenient to work with a modified Hamiltonian, $H'_{\rm LLL}=T+\lambda U_2+(1-\lambda)U_3$, so that for $\lambda=0$ the ground state at angular momentum $L_0^{\rm MR}=N(N-2)/2$ (even N) or $L_0^{\rm MR}=(N-1)^2/2$ (odd N) is *exactly* the MR state. Indeed, this modification is not entirely artificial as the MR state can describe the exact ground state close to a Feshbach resonance.[20]

For the *exact* MR state (i.e., $\lambda=0$) besides the polynomials s_m introduced above, Wen,[6] and Milovanovic and Read[18] found a branch of fermionic edge excitations which are generated by replacing the Pfaffian in $\Phi_{\rm MR}$ by $\mathcal{A}[z_1^{n_1}\cdots z_F^{n_F}(z_{F+1}-z_{F+2})^{-1}\cdots(z_{N-1}-z_N)^{-1}]$, where $n_1,\ldots,n_F$ are non-negative integers. Thus the angular momentum is increased by $L-L_0^{\rm MR}=\Sigma_{k=1}^F(n_k+\frac{1}{2})$. This spectrum, together with the phonon branch related to s_m is described[6] by $L=L_0^{\rm MR}+\Sigma_{m>0}[m\, b_m^\dagger b_m+(m-\frac{1}{2})c_{m-1/2}^\dagger c_{m-1/2}]$, where $b_m, b_m^\dagger$ are the phonon operators, and the fermions $\{c_{m-1/2},c_{n-1/2}^\dagger\}=\delta_{mn}$, anticommuting otherwise. However, due to the paired nature of the state, even and odd N are different. For instance, to compute the NoS, one must define[6] $Z^{\rm even}(q)=\frac{1}{2}{\rm Tr}[1+(-1)^F]q^{L-L_0}$ and $Z^{\rm odd}(q)=\frac{1}{2}{\rm Tr}[1-(-1)^F]q^{L-L_0}$, since the parity $(-1)^F$, with $F=\Sigma_{m>0}c_{m-1/2}^\dagger c_{m-1/2}$, is a good quantum number.[6] The numerical results are compared with the NoS for $N\to\infty$ in Table I. Perfect agreement is found for $m\leqslant[N/2]$. Higher values of m are shown to illustrate the effects of finite size; the observed differences from the $N\to\infty$ values can be also accounted for by the theory.[21] Furthermore, using the effective-field theory,[16] we have also obtained the behavior of $n(l)$ near the edge. For N even the predicted ratios of $n(l)/n(l_{\rm max})$, where $l_{\rm max}=N-2$, are 1:2:3:… . For $N=8$, we numerically find 1.0:2.2:3.4. However, for N odd the ratios of $n(l\leqslant l_{\rm max})/n(l_{\rm max})(l_{\rm max}=N-1)$ behave differently (Ref. 16): 1:1:2:3. For $N=9$ we numerically find 1.0:1.0:2.5:4.1, and the scaling observed from smaller systems shows a trend of convergence to the predicted ratios. Although $n(l)$ had been analyzed in Ref. 6 for a fermion MR state, the different behavior of $n(l)$ for odd N had not been described.

As soon as the two-body interaction is turned on, i.e., already for small $\lambda>0$, we observe that the *plateau* at $L>L_0^{\rm MR}$ is lost for small $L-L_0^{\rm MR}$. For pure two-body interaction ($\lambda=1$), the states at $L_0^{\rm MR}+1$ for $N=5,9,11,13$ and at $L_0^{\rm MR}+2$ for $N=6,7,8,10,12$ have lower interaction energy than the ground state at $L_0^{\rm MR}$. Thus, U_2 strongly perturbs the plateau of the exact MR state by favoring states with angular

TABLE II. Interaction energies (in units of g/ℓ^2) of the edge excitations of the {5, 2} state ($N=7$, $L=30$, see Fig. 1). The predictions for multiphonon states are given in brackets. Deviations are due to nonlinear terms not included in Eq. (4).

State No.	$m=2$ (th.)	$m=3$ (th.)	State No.	$m=3$ (th.)
1	0 (0)	0 (0)	6	0.034 (0.030)
2	0.002 [$\hbar\omega_2^{(b)}$]	0.002 (0.002)	7	0.047 (0.045)
3	0.015 (0.015)	0.006 [$\hbar\omega_3^{(b)}$]	8	0.069 (0.069)
4	0.034 (0.030)	0.015 (0.015)	9	0.088 (0.084)
5	0.069 [$\hbar\omega_2^{(d)}$]	0.020 (0.017)	10	[$\hbar\omega_3^{(d)}\geqslant\Delta$]

momenta of compact CF states. If, regardless of this fact, one counts the number of states at $L > L_0^{\mathrm{MR}}$ with interaction energies less than the first excited state at L_0^{MR}, the results do not seem to converge and disagree with the NoS expected for the MR state (see Table I for the observed NoS at $\lambda = 1$ for $N = 12$, 13). We also observe a rapid deterioration of the overlap of the state at L_0^{MR} with the exact MR state: from 0.91 ($N=5$) and 0.90 ($N=6$) to 0.68 ($N=8$) and 0.74 ($N=9$). These discrepancies might be due to a very slow convergence as N grows towards a well-defined bulk MR state. However, we note that for very large N, local-density arguments[22] show that the edge of the MR state will reconstruct. The anomalies in the edge of the small systems observed here are striking in view of the good behavior exhibited by the Jain states, which are also approximate wave functions.

Experimental consequences. It is possible to excite the surface modes by inducing a small time-dependent deformation of the harmonic trap. Within linear response, the energy injected by an m-polar deformation is proportional to the oscillator strength $f_m = \Sigma_\alpha |\langle L_0 + m, \alpha | O_m | L_0 \rangle|^2$, ($O_m = \Sigma_{i=1}^N z_i^m$), where α runs over all edge excitations. The theory predicts (Ref. 8) $f_m = m\nu R^{2m}$, where $R = \sqrt{N/\nu}$ is the droplet radius. For the dipole, $f_1 = N$, as required by Kohn's theorem,[8] and confirmed by our numerics. For the quadrupole ($m=2$) $f_2 = 2N^2/\nu$. We have numerically tested the accuracy of this formula for the $m=2$ modes of the states in Table I (except the MR state at $\lambda = 1$), finding that already for $N=5$, 6 the deviations are no larger than 7%. This result suggests an experimental way of estimating the filling fraction, provided f_2 and N (or f_1) can be measured: $\nu_{\mathrm{exp}}^{-1} = f_2/2N^2$. More details will be provided elsewhere.[16]

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