

WEIGHTED BMO AND HANKEL OPERATORS BETWEEN BERGMAN SPACES

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ABSTRACT. We introduce a family of weighted BMO spaces in the Bergman metric on the unit ball of \mathbb{C}^n and use them to characterize complex functions f such that the big Hankel operators H_f and $H_{\bar{f}}$ are both bounded or compact from a weighted Bergman space into a weighted Lebesgue space with possibly different exponents and different weights. As a consequence, when the symbol function f is holomorphic, we characterize bounded and compact Hankel operators $H_{\bar{f}}$ between weighted Bergman spaces. In particular, this resolves two questions left open in [7, 12].

1. INTRODUCTION

Let \mathbb{B}_n be the open unit ball in \mathbb{C}^n and dv the usual Lebesgue volume measure on \mathbb{B}_n , normalized so that the volume of \mathbb{B}_n is one. Given a parameter $\alpha > -1$ we write

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z),$$

where c_α is a positive constant such that $v_\alpha(\mathbb{B}_n) = 1$.

Denote by $H(\mathbb{B}_n)$ the space of holomorphic functions on \mathbb{B}_n . For $0 < p < \infty$ the weighted Bergman space $A_\alpha^p := A_\alpha^p(\mathbb{B}_n)$ consists of functions $f \in H(\mathbb{B}_n)$ that are in the Lebesgue space $L_\alpha^p := L^p(\mathbb{B}_n, dv_\alpha)$. The corresponding norm is given by

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right)^{1/p}.$$

When $p = 2$, the space A_α^2 is a reproducing kernel Hilbert space: for each $z \in \mathbb{B}_n$ there is a function $K_z^\alpha \in A_\alpha^2$ such that $f(z) = \langle f, K_z^\alpha \rangle_\alpha$

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whenever $f \in A_\alpha^2$. Here

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}_n} f \bar{g} dv_\alpha$$

is the natural inner product in L_α^2 . K_z^α is called the reproducing kernel of the Bergman space A_α^2 . It is explicitly given by the formula

$$K_z^\alpha(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n.$$

We also let k_z^α denote the normalized reproducing kernel at z . Thus

$$k_z^\alpha(w) = K_z^\alpha(w) / \sqrt{K_z^\alpha(z)} = \frac{(1 - |z|^2)^{(n+1+\alpha)/2}}{(1 - \langle w, z \rangle)^{n+1+\alpha}}.$$

The orthogonal projection $P_\alpha : L_\alpha^2 \rightarrow A_\alpha^2$ is an integral operator given by

$$P_\alpha f(z) = \int_{\mathbb{B}_n} \frac{f(w) dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad f \in L^2(\mathbb{B}_n, dv_\alpha).$$

The (big) Hankel operator H_f^β with symbol f is defined by

$$H_f^\beta g = (I - P_\beta)(fg).$$

We are interested in the mapping properties of H_f^β between different Lebesgue spaces.

Hankel operators are closely related to Toeplitz operators and have been extensively studied by many authors in recent decades. For analytic f , Axler [3] first characterized the boundedness and compactness of $H_{\bar{f}}$ on the unweighted Bergman space of the unit disk. Later on, Axler's result was generalized in [1, 2] to weighted Bergman spaces of the unit ball in \mathbb{C}^n . For general symbol functions, Zhu [15] first established the connection between size estimates of Hankel operators and the mean oscillation of the symbols in the Bergman metric. This idea was further investigated in a series of papers [5], [6], and [4] in the context of bounded symmetric domains, and in [8, 9] in the context of strongly pseudo convex domains.

The main purpose of this paper is to characterize real-valued functions $f \in L_\beta^q$ such that H_f^β is bounded or compact from A_α^p to L_β^q with $1 < p \leq q < \infty$. This is equivalent to characterizing complex-valued functions $f \in L_\beta^q$ such that both H_f^β and $H_{\bar{f}}^\beta$ are bounded or compact between the above spaces. As a consequence, we will characterize holomorphic symbols $f \in A_\beta^1$ such that H_f^β is bounded or compact from A_α^p to L_β^q with $1 < p \leq q < \infty$. Our characterizations are based on a family of weighted BMO spaces in the Bergman metric.

Most previous results of this type are for bounded and compact Hankel operators from A_α^p to L_α^p . When f is holomorphic, Janson [7] and Wallstén [12] characterized bounded and compact Hankel operators between weighted Bergman spaces (in the Hilbert space case) with different weights on the unit disk and the unit ball, respectively. Our results generalize theirs and solve two cases left open by them.

In the following, the notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$, and the notation $A \asymp B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. PRELIMINARIES AND AUXILIARY RESULTS

In this section we collect some preliminary results that are needed for the proof of the main theorems. We begin with notation for the rest of the paper. For any two points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in \mathbb{C}^n , we write

$$\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n,$$

and

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}.$$

For any $a \in \mathbb{B}_n$ with $a \neq 0$ we denote by $\varphi_a(z)$ the Möbius transformation on \mathbb{B}_n that interchanges the points 0 and a . It is known that

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B}_n,$$

where $s_a = 1 - |a|^2$, P_a is the orthogonal projection from \mathbb{C}^n onto the one dimensional subspace $[a]$ generated by a , and Q_a is the orthogonal projection from \mathbb{C}^n onto the orthogonal complement of $[a]$. When $a = 0$, $\varphi_a(z) = -z$. It is known that φ_a satisfies the following properties:

$$\varphi_a \circ \varphi_a(z) = z, \quad 1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}. \quad (2.1)$$

For $z, w \in \mathbb{B}_n$, the distance between z and w induced by the Bergman metric is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For $z \in \mathbb{B}_n$ and $r > 0$, the Bergman metric ball at z is given by

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

We refer to [17] for more information about automorphisms and the Bergman metric on \mathbb{B}_n .

A sequence $\{a_k\}$ of points in \mathbb{B}_n is called a *separated sequence* (in the Bergman metric) if there exists a positive constant $\delta > 0$ such that

$\beta(a_i, a_j) > \delta$ for any $i \neq j$. The following result is Theorem 2.23 in [17].

Lemma 2.1. *There exists a positive integer N such that for any $0 < r < 1$ we can find a sequence $\{a_k\}$ in \mathbb{B}_n with the following properties:*

- (i) $\mathbb{B}_n = \cup_k D(a_k, r)$.
- (ii) *The sets $D(a_k, r/4)$ are mutually disjoint.*
- (iii) *Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.*

Any sequence $\{a_k\}$ satisfying the conditions of the above lemma will be called an r -lattice in the Bergman metric. Obviously any r -lattice is separated. The following integral estimate is well known and can be found in [17, Theorem 1.12] for example.

Lemma 2.2. *Let $t > -1$ and $s > 0$. There is a positive constant C such that*

$$\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \leq C (1 - |z|^2)^{-s}$$

for all $z \in \mathbb{B}_n$.

We also need a well-known variant of the previous lemma.

Lemma 2.3. *Let $\{z_k\}$ be a separated sequence in \mathbb{B}_n and let $n < t < s$. Then*

$$\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^t}{|1 - \langle z, z_k \rangle|^s} \leq C (1 - |z|^2)^{t-s}, \quad z \in \mathbb{B}_n.$$

Lemma 2.3 above can be deduced from Lemma 2.2 after noticing that, if a sequence $\{z_k\}$ is separated, then there is a constant $r > 0$ such that the Bergman metric balls $D(z_k, r)$ are pairwise disjoint. The following result is from [13].

Lemma 2.4. *Given real numbers b and c , consider the integral operator on \mathbb{B}_n defined by*

$$S_{b,c}f(z) = \int_{\mathbb{B}_n} \frac{f(w)(1 - |w|^2)^b dv(w)}{|1 - \langle z, w \rangle|^c}.$$

Let $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, and

$$\lambda = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

Then the operator $S_{b,c}$ is bounded from L^p_α to L^q_β if and only if

$$\alpha + 1 < p(b+1), \quad c \leq n+1+b+\lambda.$$

We show that, under the same conditions, with an extra (unbounded) factor $\beta(z, w)$ in the integrand, the modified operator is still bounded from L_α^p to L_β^q . Thus we consider the operator

$$T_{b,c}f(z) = \int_{\mathbb{B}_n} \frac{f(w) \beta(z, w)}{|1 - \langle z, w \rangle|^c} (1 - |w|^2)^b dv(w).$$

Proposition 2.5. *Let b and c be real numbers. Let $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, and*

$$\lambda = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

If $\alpha+1 < p(b+1)$ and $c \leq n+1+b+\lambda$, then $T_{b,c}$ is bounded from L_α^p to L_β^q .

Proof. Pick $\varepsilon > 0$ so that $\alpha+1 < p(b+1-\varepsilon)$ and $\beta - q\varepsilon > -1$. Since $\beta(z, w)$ grows logarithmically, we have

$$\beta(z, w) = \beta(0, \varphi_z(w)) \leq C(1 - |\varphi_z(w)|^2)^{-\varepsilon}.$$

It follows from (2.1) that

$$|T_{b,c}f(z)| \leq C(1 - |z|^2)^{-\varepsilon} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{b-\varepsilon}}{|1 - \langle z, w \rangle|^{c-2\varepsilon}} |f(w)| dv(w).$$

Thus $T_{b,c}$ is bounded from L_α^p to L_β^q if the operator $S_{b-\varepsilon, c-2\varepsilon}$ is bounded from L_α^p to $L_{\beta-\varepsilon q}^q$. The desired result then follows from the previous lemma. \square

In a similar manner, the following version of Lemma 2.2 can be obtained. The proof is left to the interested reader.

Lemma 2.6. *Let $t > -1$, $s > 0$, and $d > 0$. There is a positive constant C such that*

$$\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t \beta(z, w)^d dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+s}} \leq C(1 - |z|^2)^{-s}$$

for all $z \in \mathbb{B}_n$.

The rest of this section is devoted to the proof of Theorem 2.8, which can be interpreted as some sort of tangential maximum principle. We begin with the following elementary fact.

Lemma 2.7. *Suppose F and G are holomorphic functions on \mathbb{B}_2 . If*

$$F(z_1, z_2) = G(z_1, z_2), \quad z_k = (u/\sqrt{2})e^{i\theta_k}, \quad (2.2)$$

where θ_k are arbitrary real numbers and u is arbitrary from the unit disk \mathbb{D} , then $F = G$ on \mathbb{B}_2 .

Proof. Suppose

$$F(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl} z_1^k z_2^l, \quad G(z_1, z_2) = \sum_{k,l=0}^{\infty} b_{kl} z_1^k z_2^l.$$

Then we have

$$\sum_{k,l=0}^{\infty} a_{kl} \left(\frac{u}{\sqrt{2}} \right)^{k+l} e^{ik\theta_1} e^{il\theta_2} = \sum_{k,l=0}^{\infty} b_{kl} \left(\frac{u}{\sqrt{2}} \right)^{k+l} e^{ik\theta_1} e^{il\theta_2}$$

for all $u \in \mathbb{D}$ and all real θ_1 and θ_2 . By the uniqueness of Fourier coefficients on the torus, we must have $a_{kl} = b_{kl}$ for all k and l . This shows that $F = G$ on \mathbb{B}_2 . \square

For $n > 1$, $f \in H(\mathbb{B}_n)$, and $z \in \mathbb{B}_n - \{0\}$ we will write

$$|\nabla_t f(z)| = \sup \left\{ \left| \frac{\partial f}{\partial u}(z) \right| : \|u\| = 1, u \in [z]^\perp \right\}$$

and call it the complex tangential gradient of f at z .

Theorem 2.8. *Let $n > 1$ and $f \in H(\mathbb{B}_n)$. If $|\nabla_t f(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$, then f is constant.*

Proof. We will first prove the case $n = 2$. In this case, the condition

$$|\nabla_t f(z)| \rightarrow 0, \quad |z| \rightarrow 1^-, \quad (2.3)$$

is equivalent to

$$\lim_{|z| \rightarrow 1^-} \left(\bar{z}_2 \frac{\partial f}{\partial z_1}(z) - \bar{z}_1 \frac{\partial f}{\partial z_2}(z) \right) = 0.$$

Let

$$z_1 = \frac{u}{\sqrt{2}} e^{i\theta_1}, \quad z_2 = \frac{u}{\sqrt{2}} e^{i\theta_2},$$

where $u \in \mathbb{D}$ and θ_k are arbitrary real numbers. Since

$$|z_1|^2 + |z_2|^2 = |u|^2,$$

we see that $|z| \rightarrow 1^-$ if and only if $|u| \rightarrow 1^-$. Thus condition (2.3) implies that

$$\lim_{|u| \rightarrow 1^-} \left[e^{-i\theta_2} \frac{\partial f}{\partial z_1} \left(\frac{u}{\sqrt{2}} e^{i\theta_1}, \frac{u}{\sqrt{2}} e^{i\theta_2} \right) - e^{-i\theta_1} \frac{\partial f}{\partial z_2} \left(\frac{u}{\sqrt{2}} e^{i\theta_1}, \frac{u}{\sqrt{2}} e^{i\theta_2} \right) \right] = 0.$$

For any fixed θ_k the expression inside the brackets above is an analytic function $F(u)$ on the unit disk. By the classical maximum principle for analytic functions on the unit disk, we conclude that the condition

in (2.3) implies that $F(u)$ is identically zero on \mathbb{D} . Therefore, the condition in (2.3) implies that

$$e^{i\theta_1} \frac{\partial f}{\partial z_1} \left(\frac{u}{\sqrt{2}} e^{i\theta_1}, \frac{u}{\sqrt{2}} e^{i\theta_2} \right) = e^{i\theta_2} \frac{\partial f}{\partial z_2} \left(\frac{u}{\sqrt{2}} e^{i\theta_1}, \frac{u}{\sqrt{2}} e^{i\theta_2} \right)$$

for all $u \in \mathbb{D}$. Multiply the above equation by $u/\sqrt{2}$, we conclude that

$$z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) = z_2 \frac{\partial f}{\partial z_2}(z_1, z_2) \quad (2.4)$$

whenever

$$z_k = \frac{u}{\sqrt{2}} e^{i\theta_k}, \quad k = 1, 2.$$

By Lemma 2.7, we see that the condition in (2.3) implies that the identity in (2.4) must hold for all $z = (z_1, z_2)$ in the unit ball.

Write

$$f(z) = \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j$$

and assume that the identity in (2.4) holds for all $z = (z_1, z_2) \in \mathbb{B}_2$. Then

$$\sum_{i,j=0}^{\infty} i a_{ij} z_1^i z_2^j = \sum_{i,j=0}^{\infty} j a_{ij} z_1^i z_2^j.$$

This gives $i a_{ij} = j a_{ij}$ for all i and j , which implies that $a_{ij} = 0$ whenever $i \neq j$. Writing $a_j = a_{jj}$, we obtain

$$f(z) = \sum_{j=0}^{\infty} a_j (z_1 z_2)^j.$$

In this case, we have

$$\bar{z}_2 \frac{\partial f}{\partial z_1}(z) - \bar{z}_1 \frac{\partial f}{\partial z_2}(z) = (|z_2|^2 - |z_1|^2) \sum_{j=1}^{\infty} j a_j (z_1 z_2)^{j-1}.$$

Consider the case in which

$$z_1 = u \sin \theta, \quad z_2 = u \cos \theta,$$

where $u \in \mathbb{D}$ is arbitrary and $0 < \theta < \pi/4$ is fixed. Then

$$\bar{z}_2 \frac{\partial f}{\partial z_1}(z) - \bar{z}_1 \frac{\partial f}{\partial z_2}(z) = |u|^2 \cos(2\theta) \sum_{j=1}^{\infty} j a_j (u^2 \sin \theta \cos \theta)^{j-1}.$$

Let $|u| \rightarrow 1^-$ and apply the classical maximum principle on the unit disk, we must have $a_j = 0$ for all $j \geq 1$, namely, f is constant. This completes the proof of the theorem in the case $n = 2$.

Next let us assume that $n \geq 3$ and $|\nabla_t f(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$ for some $f \in H(\mathbb{B}_n)$. We want to show that f is constant. For any $z = (z_1, z_2, z_3, \dots, z_n) \in \mathbb{B}_n$, the vector $(z_2, -z_1, 0, \dots, 0)$ is perpendicular to z . Therefore,

$$\lim_{|z| \rightarrow 1^-} \left(\bar{z}_2 \frac{\partial f}{\partial z_1}(z) - \bar{z}_1 \frac{\partial f}{\partial z_2}(z) \right) = 0.$$

We proceed to show that this implies that f is independent of the first two variables.

Fix $z = (z_1, z_2, z_3, \dots, z_n) \in \mathbb{B}_n$ (we specifically mention that it is OK if some of the z_k are 0) and write

$$1 - r^2 = |z_3|^2 + \dots + |z_n|^2,$$

where $0 < r \leq 1$. Consider the function

$$g(w_1, w_2) = f(rw_1, rw_2, z_3, \dots, z_n),$$

where $w = (w_1, w_2) \in \mathbb{B}_2$. It is clear that $|w| \rightarrow 1^-$ if and only if

$$|rw_1|^2 + |rw_2|^2 + |z_3|^2 + \dots + |z_n|^2 \rightarrow 1^-.$$

So we also have

$$\lim_{|w| \rightarrow 1^-} \left[\bar{w}_2 \frac{\partial g}{\partial w_1}(w) - \bar{w}_1 \frac{\partial g}{\partial w_2}(w) \right] = 0.$$

By the $n = 2$ case that we have already proved, g must be constant. If we choose $w \in \mathbb{B}_2$ such that $rw_k = z_k$ for $k = 1, 2$. Then

$$f(z_1, z_2, z_3, \dots, z_n) = g(w_1, w_2) = g(0, 0) = f(0, 0, z_3, \dots, z_n).$$

Repeat the argument for the first and k th variable, where $k \geq 3$, and let k run from 3 to n . The result is

$$f(z_1, z_2, z_3, \dots, z_n) = f(0, 0, 0, \dots, 0).$$

Since $z = (z_1, \dots, z_n) \in \mathbb{B}_n$ is arbitrary, we have shown that f is constant. \square

For $f \in H(\mathbb{B}_n)$ we write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right), \quad z \in \mathbb{B}_n$$

and call $|\nabla f(z)|$ the complex gradient of f at z . As a consequence of Theorem 2.8, we obtain the following maximum principle in terms of the invariant gradient $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$.

Corollary 2.9. *Let $n > 1$ and $f \in H(\mathbb{B}_n)$. If $(1 - |z|^2)^{-1/2} |\tilde{\nabla} f(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$, then f is constant.*

Proof. This follows from [17, Theorem 7.22] and Theorem 2.8. \square

3. A FAMILY OF WEIGHTED BMO SPACES

Let $\gamma \in \mathbb{R}$. For any positive radius r and every exponent p with $1 \leq p < \infty$, the space $BMO_{r,\gamma}^p$ consists of those functions $f \in L_{loc}^p(\mathbb{B}_n)$ (the space of locally L^p integrable functions on \mathbb{B}_n) such that

$$\|f\|_{BMO_{r,\gamma}^p} = \sup \left\{ (1 - |z|^2)^\gamma MO_{p,r}(f)(z) : z \in \mathbb{B}_n \right\} < \infty,$$

where

$$MO_{p,r}(f)(z) = \left[\frac{1}{v_\sigma(D(z,r))} \int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^p dv_\sigma(w) \right]^{1/p}$$

is the p -mean oscillation of f at z in the Bergman metric. Here

$$\widehat{f}_r(z) = \frac{1}{v_\sigma(D(z,r))} \int_{D(z,r)} f(w) dv_\sigma(w)$$

is the averaging function of f and $dv_\sigma(z) = (1 - |z|^2)^\sigma dv(z)$. At first glance, the function $MO_{p,r}(f)$ seems to depend on the real parameter σ , but the weight factor $(1 - |z|^2)^\sigma$ in dv_σ is essentially canceled out by the extra factor $(1 - |z|^2)^\sigma$ in $v_\sigma(D(z,r)) \asymp (1 - |z|^2)^{n+1+\sigma}$. As a consequence, the space $BMO_{r,\gamma}^p$ is actually independent of the weight parameter σ . In particular, this independence on σ is a consequence of the following lemma.

Lemma 3.1. *Let $1 \leq p < \infty$, $f \in L_{loc}^p(\mathbb{B}_n)$, and $r > 0$. Then $f \in BMO_{r,\gamma}^p$ if and only if there exists some constant $C > 0$ such that for any $z \in \mathbb{B}_n$, there is a constant λ_z satisfying*

$$\frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^p dv_\sigma(w) \leq C. \quad (3.1)$$

Proof. If $f \in BMO_{r,\gamma}^p$, then (3.1) holds with $C = \|f\|_{BMO_{r,\gamma}^p}^p$ and $\lambda_z = \widehat{f}_r(z)$. Conversely, if (3.1) is satisfied, then by the triangle inequality for the L^p -norm, $MO_{p,r}(f)(z)$ is less than or equal to

$$\left[\frac{1}{v_\sigma(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^p dv_\sigma(w) \right]^{\frac{1}{p}} + |\widehat{f}_r(z) - \lambda_z|.$$

By Hölder's inequality,

$$\begin{aligned} |\widehat{f}_r(z) - \lambda_z| &= \left| \frac{1}{v_\sigma(D(z,r))} \int_{D(z,r)} (f(w) - \lambda_z) dv_\sigma(w) \right| \\ &\leq \left[\frac{1}{v_\sigma(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^p dv_\sigma(w) \right]^{\frac{1}{p}}. \end{aligned}$$

It follows that

$$(1 - |z|^2)^\gamma MO_{p,r}(f)(z) \leq 2C^{1/p}$$

for all $z \in \mathbb{B}_n$, so that $f \in BMO_{r,\gamma}^p$. \square

For a continuous function f on \mathbb{B}_n let

$$\omega_r(f)(z) = \sup \{|f(z) - f(w)| : w \in D(z, r)\}.$$

The function $\omega_r(f)(z)$ is called the oscillation of f at the point z in the Bergman metric. For any $r > 0$ and $\gamma \in \mathbb{R}$, let $BO_{r,\gamma}$ denote the space of continuous functions f on \mathbb{B}_n such that

$$\|f\|_{BO_{r,\gamma}} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \omega_r(f)(z) < \infty.$$

Lemma 3.2. *Let $r > 0$, $\gamma \in \mathbb{R}$, and f be a continuous function on \mathbb{B}_n . If $\gamma \geq 0$, then $f \in BO_{r,\gamma}$ if and only if there is a constant $C > 0$ such that*

$$|f(z) - f(w)| \leq C \frac{\beta(z, w) + 1}{\min(1 - |z|, 1 - |w|)^\gamma}$$

for all z and w in \mathbb{B}_n . If $\gamma < 0$, then $f \in BO_{r,\gamma}$ if and only if there is a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C \frac{\beta(z, w) + 1}{[\max(1 - |z|, 1 - |w|)]^{-\gamma}} |1 - \langle z, w \rangle|^{-2\gamma}$$

for all z and w in \mathbb{B}_n .

Proof. Assume that $f \in BO_{r,\gamma}$. If $\beta(z, w) \leq r$, the result is clear, because then

$$|1 - \langle z, w \rangle| \asymp 1 - |z| \asymp 1 - |w|.$$

Fix any $z, w \in \mathbb{B}_n$ with $\beta(z, w) > r$. Let $\lambda(t)$, $0 \leq t \leq 1$, be the geodesic in the Bergman metric from z to w . Let $N = [\beta(z, w)/r] + 1$ and $t_i = i/N$, $0 \leq i \leq N$, where $[x]$ denotes the largest integer less than or equal to x . Set $z_i = \lambda(t_i)$, $0 \leq i \leq N$. Then

$$\beta(z_i, z_{i+1}) = \frac{\beta(z, w)}{N} \leq r.$$

Therefore,

$$\begin{aligned} |f(z) - f(w)| &\leq \sum_{i=1}^N |f(z_{i-1}) - f(z_i)| \leq \sum_{i=1}^N \omega_r(f)(z_i) \\ &\leq \|f\|_{BO_{r,\gamma}} \sum_{i=1}^N (1 - |z_i|)^{-\gamma}. \end{aligned}$$

If $\gamma \geq 0$, it follows from the obvious inequality

$$(1 - |z_i|) \geq \min(1 - |z|, 1 - |w|)$$

that

$$\begin{aligned} |f(z) - f(w)| &\leq \|f\|_{BO_{r,\gamma}} \frac{N}{\min(1 - |z|, 1 - |w|)^\gamma} \\ &\leq \|f\|_{BO_{r,\gamma}} \frac{\beta(z, w)/r + 1}{\min(1 - |z|, 1 - |w|)^\gamma} \\ &\leq \max(1, 1/r) \|f\|_{BO_{r,\gamma}} \frac{\beta(z, w) + 1}{\min(1 - |z|, 1 - |w|)^\gamma}. \end{aligned}$$

If $\gamma < 0$, the result is proved in the same way once the inequality

$$1 - |z_i| \leq \frac{2|1 - \langle z, w \rangle|^2}{\max(1 - |z|^2, 1 - |w|^2)} \quad (3.2)$$

is established. To prove this, simply note that the Möbius transformation φ_z sends the geodesic joining z and w to the geodesic joining 0 and $\varphi_z(w)$. This gives

$$1 - |\varphi_z(w)|^2 \leq 1 - |\varphi_z(z_i)|^2.$$

Developing this inequality using (2.1), we get

$$\frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} \leq \frac{1 - |z_i|^2}{|1 - \langle z, z_i \rangle|^2} \leq \frac{2}{|1 - \langle z, z_i \rangle|},$$

which gives

$$|1 - \langle z, z_i \rangle| \leq \frac{2|1 - \langle z, w \rangle|^2}{1 - |w|^2}.$$

Interchanging the roles of z and w , we get (3.2).

The converse implication is obvious. \square

A consequence of the above lemma is that the space $BO_{r,\gamma}$ is independent of the choice of r . So we will simply write $BO_\gamma = BO_{1,\gamma}$ and

$$\|f\|_{BO_\gamma} = \|f\|_{BO_{1,\gamma}} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \omega_1(f)(z).$$

Let $0 < p < \infty$, $\gamma \in \mathbb{R}$, and $r > 0$. We say that $f \in BA_{r,\gamma}^p$ if $f \in L_{loc}^p(\mathbb{B}_n)$ and

$$\|f\|_{BA_{r,\gamma}^p} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \left[\widehat{|f|_r^p}(z) \right]^{1/p} < \infty.$$

We proceed to show that the space $BA_{r,\gamma}^p$ is also independent of r .

For $\sigma > -1$ and $c > 0$ the generalized Berezin transform $B_{c,\sigma}(\varphi)$ of a function $\varphi \in L^1(\mathbb{B}_n, dv_\sigma)$ is defined as

$$B_{c,\sigma}(\varphi)(z) = (1 - |z|^2)^c \int_{\mathbb{B}_n} \frac{\varphi(w)}{|1 - \langle w, z \rangle|^{n+1+c+\sigma}} dv_\sigma(w).$$

In the case when $c = n + 1 + \sigma$, this coincides with the ordinary Berezin transform $B_\sigma \varphi(z) = \langle \varphi k_z^\sigma, k_z^\sigma \rangle_\sigma$.

Lemma 3.3. *Let $0 < p \leq q < \infty$, $\alpha, \beta > -1$, and $f \in L_{loc}^q(\mathbb{B}_n)$. Set*

$$\gamma = (n + 1 + \beta)/q - (n + 1 + \alpha)/p, \quad d\mu_{f,\beta} = |f|^q dv_\beta.$$

The following conditions are equivalent:

- (i) *The embedding $i : A_\alpha^p \rightarrow L^q(\mathbb{B}_n, d\mu_{f,\beta})$ is bounded.*
- (ii) *$f \in BA_{r,\gamma}^q$ for some (or all) $r > 0$.*
- (iii) *$(1 - |z|^2)^{\gamma q} B_{c,\sigma}(|f|^q) \in L^\infty(\mathbb{B}_n)$ for all $\sigma > -1 + \gamma q$ and all $c > \max(0, -\gamma q)$.*

Proof. By [14, Theorem 50], condition (i) is equivalent to

$$\mu_{f,\beta}(D(z, r)) \leq C(1 - |z|^2)^{(n+1+\alpha)q/p}$$

for some (or all) $r > 0$. Since

$$\widehat{|f|_r^q}(z) \asymp \frac{\mu_{f,\beta}(D(z, r))}{(1 - |z|^2)^{n+1+\beta}},$$

it follows that (i) and (ii) are equivalent.

Since $|1 - \langle z, w \rangle| \asymp (1 - |z|^2)$ for $w \in D(z, r)$, we have

$$\begin{aligned} B_{c,\sigma}(|f|^q)(z) &= (1 - |z|^2)^c \int_{\mathbb{B}_n} \frac{|f(w)|^q dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} \\ &\geq (1 - |z|^2)^c \int_{D(z,r)} \frac{|f(w)|^q dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} \\ &\asymp \widehat{|f|_r^q}(z). \end{aligned}$$

This proves that (iii) implies (ii).

To finish the proof, let $\{a_j\}$ be an r -lattice. Then

$$\begin{aligned} B_{c,\sigma}(|f|^q)(z) &= (1 - |z|^2)^c \int_{\mathbb{B}_n} \frac{|f(w)|^q dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} \\ &\leq (1 - |z|^2)^c \sum_{j=1}^{\infty} \int_{D(a_j,r)} \frac{|f(w)|^q dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}}. \end{aligned}$$

By the estimate in (2.20) on page 63 of [17], we have

$$|1 - \langle z, w \rangle| \asymp |1 - \langle z, a_j \rangle|, \quad w \in D(a_j, r).$$

Thus

$$B_{c,\sigma}(|f|^q)(z) \lesssim (1 - |z|^2)^c \sum_{j=1}^{\infty} \frac{(1 - |a_j|^2)^{\sigma-\beta}}{|1 - \langle z, a_j \rangle|^{n+1+c+\sigma}} \mu_{f,\beta}(D(a_j, r)).$$

So condition (ii) implies that

$$B_{c,\sigma}(|f|^q)(z) \lesssim (1 - |z|^2)^c \sum_{j=1}^{\infty} \frac{(1 - |a_j|^2)^{n+1+\sigma-\gamma q}}{|1 - \langle z, a_j \rangle|^{n+1+c+\sigma}}.$$

Since $\sigma > -1 + \gamma q$, or $n + 1 + \sigma - \gamma q > n$, an application of Lemma 2.3 shows that condition (ii) implies (iii). \square

As a consequence of the previous result, we see that the space $BA_{r,\gamma}^p$ is independent of the choice of r . Thus we will simply call it BA_γ^p .

Lemma 3.4. *Let $f \in L_{loc}^1(\mathbb{B}_n)$ and $r > 0$. Then \widehat{f}_r is continuous.*

Proof. The proof is elementary and we omit the details here. \square

Theorem 3.5. *Suppose $r > 0$, $\gamma \in \mathbb{R}$, $1 \leq p < \infty$, and $f \in L_{loc}^p(\mathbb{B}_n)$. The following conditions are equivalent:*

- (a) $f \in BMO_{r,\gamma}^p$.
- (b) $f = f_1 + f_2$ with $f_1 \in BO_\gamma$ and $f_2 \in BA_\gamma^p$.
- (c) For some (or all) $\sigma > \max(-1, -1 + \gamma p)$ and for each $c > \max(0, -2\gamma p)$ we have

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(w) - \widehat{f}_r(z)|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) < \infty.$$

- (d) For some (or all) $\sigma > \max(-1, -1 + \gamma p)$ and for each $c > \max(0, -2\gamma p)$ there is a function λ_z such that

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(w) - \lambda_z|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) < \infty.$$

Proof. That (c) implies (d) is obvious, and the implication (d) \Rightarrow (a) is a consequence of Lemma 3.1 and the inequality

$$\begin{aligned} & \frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - \lambda_z|^p dv_\sigma(w) \\ & \lesssim \int_{\mathbb{B}_n} |f(w) - \lambda_z|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w), \end{aligned}$$

which follows from the well-known facts that

$$|1 - \langle z, w \rangle| \asymp (1 - |z|^2), \quad v_\sigma(D(z, r)) \asymp (1 - |z|^2)^{n+1+\sigma},$$

for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$.

The proof of (a) \Rightarrow (b) can be done as in [16, Theorem 5]. Indeed, since r is arbitrary, it suffices to show that

$$BMO_{2r,\gamma}^p \subset BO_\gamma + BA_\gamma^p.$$

Given $f \in BMO_{2r,\gamma}^p$ and points $z, w \in \mathbb{B}_n$ with $\beta(z, w) \leq r$, we have

$$\begin{aligned} |\widehat{f}_r(z) - \widehat{f}_r(w)| &\leq |\widehat{f}_r(z) - \widehat{f}_{2r}(z)| + |\widehat{f}_{2r}(z) - \widehat{f}_r(w)| \\ &\leq \frac{1}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(u) - \widehat{f}_{2r}(z)| dv_\sigma(u) \\ &\quad + \frac{1}{v_\sigma(D(w, r))} \int_{D(w, r)} |f(u) - \widehat{f}_{2r}(z)| dv_\sigma(u). \end{aligned}$$

Since $v_\sigma(D(w, r)) \asymp v_\sigma(D(z, r))$ for $w \in D(z, r)$, and since $D(z, r)$ and $D(w, r)$ are both contained in $D(z, 2r)$, it follows from Hölder's inequality that the two integral summands above are both bounded by constant times $(1 - |z|^2)^{-\gamma} \|f\|_{BMO_{2r,\gamma}^p}$. This together with Lemma 3.4 proves that \widehat{f}_r belongs to $BO_{r,\gamma} = BO_\gamma$. On the other hand, we can prove that the function $g = f - \widehat{f}_r$ is in BA_γ^p whenever $f \in BMO_{2r,\gamma}^p$. In fact, it is rather easy to see that $f \in BMO_{2r,\gamma}^p$ implies that $f \in BMO_{r,\gamma}^p$. By the triangle inequality for L^p integrals,

$$[\widehat{|g|^p}_r(z)]^{1/p} \leq \left[\frac{1}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(u) - \widehat{f}_r(z)|^p dv_\sigma(u) \right]^{1/p} + \omega_r(\widehat{f}_r)(z).$$

Since $\widehat{f}_r \in BO_{r,\gamma}$ and $f \in BMO_{r,\gamma}^p$, we deduce that g belongs to BA_γ^p .

To show that (b) implies (c), first observe that it follows from Lemma 2.2 that the integral appearing in part (c) is dominated by

$$(1 - |z|^2)^{\gamma p} \left(B_{c,\sigma}(|f|^p)(z) + |\widehat{f}_r(z)|^p \right),$$

and by Hölder's inequality, we have $|\widehat{f}_r(z)|^p \leq \widehat{|f|^p}_r(z)$. Thus Lemma 3.3 shows that $f \in BA_\gamma^p$ implies condition (c). On the other hand, if $f \in BO_\gamma$, we write

$$f(w) - \widehat{f}_r(z) = \frac{1}{v_\sigma(D(z, r))} \int_{D(z, r)} (f(w) - f(\zeta)) dv_\sigma(\zeta)$$

and use Lemma 3.2 and the triangle inequality to obtain

$$|f(w) - \widehat{f}_r(z)| \leq C \|f\|_{BO_\gamma} \frac{\beta(z, w) + 1}{\min(1 - |z|, 1 - |w|)^\gamma}, \quad \gamma \geq 0,$$

and

$$|f(w) - \widehat{f}_r(z)| \leq C \|f\|_{BO_\gamma} \frac{(\beta(z, w) + 1) |1 - \langle z, w \rangle|^{-2\gamma}}{(1 - |z|)^{-\gamma}}, \quad \gamma < 0.$$

In both cases, the integral estimates in Lemmas 2.2 and 2.6 show that (c) holds if $f \in BO_\gamma$. This shows that condition (b) implies (c) and completes the proof of the theorem. \square

One of the consequences of the above result is that the space $BMO_{r,\gamma}^p$ is also independent of r . So from now on it will simply be denoted by BMO_γ^p .

Next we are going to identify the space of all holomorphic functions in BMO_γ^p with certain Bloch-type spaces. Recall that for $\alpha \geq 0$, the Bloch type space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B}_n)$ consists of holomorphic functions f in \mathbb{B}_n for which

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\alpha |\nabla f(z)| < \infty.$$

Note that the complex gradient $\nabla f(z)$ can be replaced by the radial derivative $Rf(z)$. We will simply obtain equivalent norms. When $\alpha > 1/2$, a description can also be obtained using the invariant gradient $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$. That is, for $\alpha > 1/2$, a holomorphic function f belongs to \mathcal{B}^α if and only if

$$|f(0)| + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{\alpha-1} |\tilde{\nabla} f(z)| < \infty,$$

and this quantity defines an equivalent norm in \mathcal{B}^α . Note that when $n = 1$, this is true for all $\alpha \geq 0$, because in this case we actually have $\tilde{\nabla} f(z) = (1 - |z|^2)f'(z)$.

When $0 < \alpha < 1$, the Bloch type space \mathcal{B}^α coincides (with equivalent norms) with the holomorphic Lipschitz space $\Lambda_{1-\alpha} = \Lambda_{1-\alpha}(\mathbb{B}_n)$ consisting of all holomorphic functions f in \mathbb{B}_n such that

$$\|f\|_{\Lambda_{1-\alpha}} = |f(0)| + \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^{1-\alpha}} : z, w \in \mathbb{B}_n, z \neq w \right\} < \infty.$$

Note that when $\alpha = 0$, the space \mathcal{B}^α consists of holomorphic functions f with bounded partial derivatives. Equivalently, \mathcal{B}^0 consists of all holomorphic functions f such that

$$\sup \left\{ \frac{|f(z) - f(w)|}{|z - w|} : z \neq w \right\} < \infty.$$

This space is not what is usually called the Lipschitz space Λ_1 . We refer to [17, Chapter 7] for all these properties of Bloch and Lipschitz type spaces.

Proposition 3.6. *Let $\gamma \in \mathbb{R}$, $1 \leq p < \infty$, and $f \in H(\mathbb{B}_n)$. Then $f \in BMO_\gamma^p$ if and only if*

$$\|f\|_{\gamma,*} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma |\tilde{\nabla} f(z)| < \infty.$$

Proof. We will show that condition (d) of Theorem 3.5 is satisfied with $\lambda_z = f(z)$ if $\|f\|_{\gamma,*} < \infty$. To this end, let $\sigma > \max(-1, -1 + p\gamma)$ and $c > \max(0, -\gamma p)$. By [11, Lemma 7] and Lemma 2.2, we have

$$\begin{aligned} & \int_{\mathbb{B}_n} |f(w) - f(z)|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) \\ & \leq C \int_{\mathbb{B}_n} |\tilde{\nabla} f(w)|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) \\ & \leq C \|f\|_{\gamma,*}^p (1 - |z|^2)^{c+\gamma p} \int_{\mathbb{B}_n} \frac{dv_{\sigma-\gamma p}(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} \\ & \leq C. \end{aligned}$$

Thus $\|f\|_{\gamma,*} < \infty$ implies that $f \in BMO_\gamma^p$.

To prove the other implication, we use the inequality

$$|\tilde{\nabla} f(z)|^p \lesssim \frac{1}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - f(z)|^p dv_\sigma(w),$$

which appears on page 182 of [17]. By the triangle inequality for L^p spaces,

$$|\tilde{\nabla} f(z)| \leq MO_{p,r}(f)(z) + |f(z) - \hat{f}_r(z)|.$$

Applying Lemma 2.24 of [17] to the function $g(w) = f(w) - \hat{f}_r(z)$ and the point z , we find a constant C such that

$$|f(z) - \hat{f}_r(z)| \leq C MO_{p,r}(f)(z)$$

for all $z \in \mathbb{B}_n$. Thus $f \in BMO_\gamma^p$ implies $\|f\|_{\gamma,*} < \infty$. \square

Corollary 3.7. *Let $\gamma \in \mathbb{R}$ and $1 \leq p < \infty$. Then*

(a) *If $n = 1$, we have*

$$H(\mathbb{B}_n) \cap BMO_\gamma^p = \begin{cases} \mathcal{B}^{1+\gamma}, & \gamma \geq -1, \\ \mathbb{C}, & \gamma < -1. \end{cases}$$

(b) *If $n > 1$, we have*

$$H(\mathbb{B}_n) \cap BMO_\gamma^p = \begin{cases} \mathcal{B}^{1+\gamma}, & \gamma > -1/2, \\ \mathbb{C}, & \gamma < -1/2. \end{cases}$$

(c) *If $n > 1$ and $\gamma = -\frac{1}{2}$, the space $H(\mathbb{B}_n) \cap BMO_\gamma^p$ consists of those holomorphic functions f on \mathbb{B}_n with*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{-1/2} |\tilde{\nabla} f(z)| < \infty.$$

Proof. Parts (a) and (b) are consequence of Proposition 3.6, the remarks preceding it, and Corollary 2.9. Part (c) is just a restatement of Proposition 3.6. \square

According to [17, Theorem 7.2], the space $BMO_{-\frac{1}{2}}^p$ contains non-constant functions. In fact, any function in \mathcal{B}^α with $0 < \alpha < 1/2$ is in $BMO_{-\frac{1}{2}}^p$. However, this space differs from $\mathcal{B}^{1/2}$. See Corollary 2.9.

In the next theorem we record some characterizations obtained for Bloch type spaces, which are of some independent interest.

Theorem 3.8. *Suppose $r > 0$, $1 \leq p < \infty$, $\sigma > -1$, and $f \in H(\mathbb{B}_n)$. Let $\gamma \geq -1$ if $n = 1$; and $\gamma > -1/2$ if $n > 1$. Then the following statements are equivalent.*

- (i) $f \in \mathcal{B}^{1+\gamma}$.
- (ii) There is a constant $C > 0$ such that

$$\frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - f(z)|^p dv_\sigma(w) \leq C$$

for all $z \in \mathbb{B}_n$.

- (iii) There is a constant $C > 0$ such that

$$\frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - \widehat{f}_r(z)|^p dv_\sigma(w) \leq C$$

for all $z \in \mathbb{B}_n$.

- (iv) For each $z \in \mathbb{B}_n$ there exists a complex number λ_z such that

$$\sup_{z \in \mathbb{B}_n} \frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - \lambda_z|^p dv_\sigma(w) < \infty.$$

- (v) For some (or all) $\eta > \max(-1, -1 + \gamma p)$ and for each $c > \max(0, -2\gamma p)$ we have

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} |f(w) - f(z)|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\eta}} dv_\eta(w) < \infty.$$

Proof. In the proof of Proposition 3.6, we proved the implications (i) \Rightarrow (v) \Rightarrow (ii) \Rightarrow (i). The other equivalences are obtained from Theorem 3.5 and Lemma 3.1. \square

When $\gamma = 0$, the equivalences of (i)-(iv) in Theorem 3.8 is just [17, Theorem 5.22], and the equivalence with (v) appears in [10].

4. BOUNDED HANKEL OPERATORS

The main result of this section is the following result, which characterizes bounded Hankel operators induced by real-valued symbols between weighted Bergman spaces.

Theorem 4.1. *Let $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, $f \in L^q_\beta$, and*

$$\gamma = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

Then $H_f^\beta, H_{\bar{f}}^\beta : A_\alpha^p \rightarrow L^q_\beta$ are both bounded if and only if $f \in BMO_\gamma^q$.

We are going to prove Theorem 4.1 with a series of lemmas and propositions. For $t \geq 0$ let

$$K_z^{\beta,t}(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1+\beta+t}}.$$

Also, for $f \in L^q_\beta$ and $z \in \mathbb{B}_n$, we consider the function $MO_{\beta,q,t}f$ defined by

$$MO_{\beta,q,t}f(z) = \|fh_z^t - \overline{g_z(z)}h_z^t\|_{q,\beta},$$

where, for $z \in \mathbb{B}_n$, the function g_z (that depends on f and t) is given by

$$g_z(w) = \frac{P_\beta(\bar{f}h_z^t)(w)}{h_z^t(w)}, \quad w \in \mathbb{B}_n,$$

with the function h_z^t defined by

$$h_z^t(w) = h_z^{\alpha,\beta,t}(w) = \frac{K_z^{\beta,t}(w)}{\|K_z^{\beta,t}\|_{p,\alpha}}, \quad w \in \mathbb{B}_n.$$

Clearly, $\|h_z^t\|_{p,\alpha} = 1$. When $t = 0$, it is easily seen that $\overline{g_z(z)} = B_\beta f(z)$ is the Berezin transform of f at the point z . Also, since $h_z^t(w)$ never vanishes on \mathbb{B}_n , the function g_z is holomorphic on \mathbb{B}_n .

Lemma 4.2. *Let $1 < p \leq q < \infty$ and $\alpha, \beta > -1$. Let γ be as in Theorem 4.1 and $t \geq 0$ such that*

$$n+1+\beta+t > (n+1+\alpha)/p.$$

If $MO_{\beta,q,t}f \in L^\infty(\mathbb{B}_n)$, then $f \in BMO_\gamma^q$.

Proof. Since $n+1+\beta+t > (n+1+\alpha)/p$, Lemma 2.2 gives us the estimate

$$\|K_z^{\beta,t}\|_{p,\alpha} \asymp (1 - |z|^2)^{(n+1+\alpha)/p - (n+1+\beta+t)}.$$

It follows that $[MO_{\beta,q,t}f(z)]^q$ is comparable to

$$(1 - |z|^2)^{\gamma q + (n+1+\beta)(q-1) + tq} \int_{\mathbb{B}_n} \frac{|f(w) - \overline{g_z(z)}|^q}{|1 - \langle z, w \rangle|^{(n+1+\beta+t)q}} dv_{\beta}(w),$$

which dominates

$$\frac{(1 - |z|^2)^{\gamma q}}{|D(z, r)|} \int_{D(z, r)} |f(w) - \overline{g_z(z)}|^q dv(w).$$

This shows that, if $MO_{\beta,q,t}f \in L^{\infty}(\mathbb{B}_n)$, the condition in Lemma 3.1 is satisfied with $\lambda_z = \overline{g_z(z)}$, so $f \in BMO_{\gamma}^q$. \square

The following result gives the necessity in Theorem 4.1. It generalizes, in several directions, Proposition 8.19 in [18], where the method of proof is based on Hilbert space techniques. A different method was used in [16] to deal with the case $\alpha = \beta$ and $p = q$. Our method here is more flexible and allows us to obtain the result in much more generality.

Proposition 4.3. *Let $1 < p, q < \infty$, $\alpha, \beta > -1$, and $t \geq 0$. Then for any $f \in L_{\beta}^q$ we have*

$$MO_{\beta,q,t}f(z) \lesssim \|H_f^{\beta} h_z^t\|_{q,\beta} + \|H_{\bar{f}}^{\beta} h_z^t\|_{q,\beta}.$$

Proof. By the triangle inequality and the definition of Hankel operators, we have

$$\begin{aligned} MO_{\beta,q,t}f(z) &= \|fh_z^t - \overline{g_z(z)} h_z^t\|_{q,\beta} \\ &\leq \|fh_z^t - P_{\beta}(fh_z^t)\|_{q,\beta} + \|P_{\beta}(fh_z^t) - \overline{g_z(z)} h_z^t\|_{q,\beta} \\ &= \|H_f^{\beta} h_z^t\|_{q,\beta} + \|P_{\beta}(fh_z^t) - \overline{g_z(z)} h_z^t\|_{q,\beta}. \end{aligned}$$

For any $g \in A_{\beta}^1(\mathbb{B}_n)$ it is easy to check that

$$\overline{g(z)} h_z^t = P_{\beta+t}(\overline{g} h_z^t). \quad (4.1)$$

This together with the boundedness of $P_{\beta+t}$ on L_{β}^q yields

$$\begin{aligned} \|P_{\beta}(fh_z^t) - \overline{g_z(z)} h_z^t\|_{q,\beta} &= \|P_{\beta}(fh_z^t) - P_{\beta+t}(\overline{g_z} h_z^t)\|_{q,\beta} \\ &= \|P_{\beta+t}(P_{\beta}(fh_z^t) - \overline{g_z} h_z^t)\|_{q,\beta} \\ &\leq \|P_{\beta+t}\|_{L_{\beta}^q} \cdot \|P_{\beta}(fh_z^t) - \overline{g_z} h_z^t\|_{q,\beta}. \end{aligned}$$

Finally,

$$\begin{aligned}
\|P_\beta(fh_z^t) - \bar{g}_z h_z^t\|_{q,\beta} &\leq \|fh_z^t - P_\beta(fh_z^t)\|_{q,\beta} + \|fh_z^t - \bar{g}_z h_z^t\|_{q,\beta} \\
&= \|H_f^\beta h_z^t\|_{q,\beta} + \|\bar{f} h_z^t - g_z h_z^t\|_{q,\beta} \\
&= \|H_f^\beta h_z^t\|_{q,\beta} + \|\bar{f} h_z^t - P_\beta(\bar{f} h_z^t)\|_{q,\beta} \\
&= \|H_f^\beta h_z^t\|_{q,\beta} + \|H_{\bar{f}}^\beta h_z^t\|_{q,\beta}.
\end{aligned}$$

This proves the result with constant $C = (1 + \|P_{\beta+t}\|_{L_\beta^q})$. \square

Note that the proposition above does not require $p \leq q$. The next two propositions, which require the condition $p \leq q$, will establish the sufficiency of Theorem 4.1.

Proposition 4.4. *Let $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, and*

$$\gamma = (n + 1 + \beta)/q - (n + 1 + \alpha)/p.$$

If $f \in BA_\gamma^q$, then $H_f^\beta : A_\alpha^p \rightarrow L_\beta^q$ is bounded.

Proof. Since $q > 1$, the Bergman projection P_β is bounded on L_β^q . Thus

$$\|H_f^\beta g\|_{q,\beta} \leq \|fg\|_{q,\beta} + \|P_\beta(fg)\|_{q,\beta} \lesssim \|fg\|_{q,\beta} = \|g\|_{L^q(d\mu_{f,\beta})}.$$

The result then follows from Lemma 3.3. \square

We note that the proof of the previous proposition also works for $1 = p < q < \infty$. In order to show that H_f^β is bounded if $f \in BO_\gamma$ with $\gamma < 0$, we need the following result.

Lemma 4.5. *Let $s \geq \beta > -1$, $1 < q < \infty$, $f \in L_\beta^q$, and $g \in H^\infty$. Then*

$$\|H_f^\beta g\|_{q,\beta} \leq C \|H_f^s g\|_{q,\beta}.$$

Proof. Since $g \in H^\infty$, we have $gf \in L_\beta^q$. Also,

$$\begin{aligned}
\|H_f^\beta g\|_{q,\beta} &= \|(I - P_\beta)(gf)\|_{q,\beta} \\
&\leq \|(I - P_s)(gf)\|_{q,\beta} + \|(P_s - P_\beta)(gf)\|_{q,\beta} \\
&= \|H_f^s g\|_{q,\beta} + \|(P_\beta - P_s)(gf)\|_{q,\beta}.
\end{aligned}$$

Since P_s is bounded on L_β^q , the reproducing formula yields $P_\beta P_s(gf) = P_s(gf)$. Thus

$$(P_\beta - P_s)(gf) = (P_\beta - P_\beta P_s)(gf) = P_\beta(I - P_s)(gf) = P_\beta(H_f^s g).$$

This gives

$$\|(P_\beta - P_s)(gf)\|_{q,\beta} \leq \|P_\beta\| \cdot \|H_f^s g\|_{q,\beta},$$

so we obtain the desired inequality with $C = 1 + \|P_\beta\|$. \square

Proposition 4.6. *Let $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, and*

$$\gamma = (n + 1 + \beta)/q - (n + 1 + \alpha)/p.$$

If $f \in BO_\gamma$, then $H_f^\beta : A_\alpha^p \rightarrow L_\beta^q$ is bounded.

Proof. We first consider the case $\gamma \geq 0$. For $g \in H^\infty$, which is dense in A_α^p , we have

$$\begin{aligned} \|H_f^\beta g\|_{q,\beta}^q &= \int_{\mathbb{B}_n} |H_f^\beta g(z)|^q dv_\beta(z) \\ &= \int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} \frac{(f(z) - f(w))g(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} dv_\beta(w) \right|^q dv_\beta(z) \\ &\leq \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|f(z) - f(w)||g(w)|}{|1 - \langle z, w \rangle|^{n+1+\beta}} dv_\beta(w) \right)^q dv_\beta(z). \end{aligned}$$

By Lemma 3.2,

$$\|H_f^\beta g\|_{q,\beta}^q \lesssim \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(1 + \beta(z, w))|g(w)| dv_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta} \min(1 - |z|, 1 - |w|)^\gamma} \right)^q dv_\beta(z).$$

Write

$$I_1(g) = \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|g(w)| dv_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta} \min(1 - |z|, 1 - |w|)^\gamma} \right)^q dv_\beta(z),$$

and split the inner integral in two parts, $I_{1,1}(g)$ over $|w| \leq |z|$ and $I_{1,2}(g)$ over $|w| > |z|$. Since

$$\min(1 - |z|, 1 - |w|)^\gamma = (1 - |z|)^\gamma$$

for $|w| \leq |z|$, we have

$$\begin{aligned} I_{1,1}(g) &\lesssim \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|g(w)| dv_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta}} \right)^q dv_{\beta-q\gamma}(z) \\ &= \int_{\mathbb{B}_n} |S_{b,c}(|g|)(z)|^q dv_{\beta-q\gamma}(z), \end{aligned}$$

where $S_{b,c}$ is the integral operator appearing in Theorem 2.4 with $b = \beta$ and $c = n + 1 + \beta$. Notice that $\beta - q\gamma > -1$ is equivalent to

$$n \left(\frac{1}{q} - \frac{1}{p} \right) < \frac{1 + \alpha}{p},$$

which is automatically satisfied since $p \leq q$. Applying Theorem 2.4, we obtain $I_{1,1}(g) \lesssim \|g\|_{p,\alpha}^q$, provided $1 + \alpha < p(1 + b)$ and $c \leq n + 1 + b + \lambda$. Since $b = \beta$ and

$$\lambda = \frac{n + 1 + (\beta - q\gamma)}{q} - \frac{n + 1 + \alpha}{p} = 0,$$

the condition $c \leq n + 1 + b + \lambda$ is satisfied with equality. It remains to check that the condition

$$1 + \alpha < p(1 + b) = p(1 + \beta) \quad (4.2)$$

is satisfied. Since $\gamma \geq 0$ and $q \geq p$, we have

$$0 \leq \gamma = (n + 1 + \beta)/q - (n + 1 + \alpha)/p \leq (\beta - \alpha)/p.$$

This gives $\alpha \leq \beta$, so (4.2) holds since $p > 1$.

Similarly, we have

$$I_{1,2}(g) \lesssim \int_{\mathbb{B}_n} |S_{b,c}(|g|)(z)|^q dv_\beta(z)$$

with $b = \beta - \gamma$, $c = n + 1 + \beta$, and $\lambda = \gamma$. We want to apply Theorem 2.4 to estimate $I_{1,2}(g)$. In this case, the condition $c \leq n + 1 + b + \lambda$ in Theorem 2.4 holds with equality. The other condition in Theorem 2.4 is $\alpha + 1 < p(1 + \beta - \gamma)$, which is equivalent to

$$\frac{p(n + 1 + \beta)}{q} < p\beta + p + n.$$

If q' is the conjugate exponent of q , the above condition is equivalent to

$$n \left(\frac{1}{q} - \frac{1}{p} \right) < \frac{1 + \beta}{q'},$$

which is automatically satisfied since $p \leq q$. Hence, by Theorem 2.4, we have $I_{1,2}(g) \lesssim \|g\|_{p,\alpha}^q$. This together with the previous estimate yields $I_1(g) \lesssim \|g\|_{p,\alpha}^q$. The remaining estimate $I_2(g) \lesssim \|g\|_{p,\alpha}^q$ with

$$I_2(g) := \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|g(w)|^\beta(z, w) dv_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta} \min(1 - |z|, 1 - |w|)^\gamma} \right)^q dv_\beta(z),$$

can be proved in a similar manner, using Proposition 2.5 instead of Theorem 2.4. The proof of the case $\gamma \geq 0$ is now complete.

If $\gamma < 0$ and $g \in H^\infty$, we use Lemma 4.5, with $s \geq \beta$ big enough so that $p(s + \gamma + 1) > \alpha + 1$, to obtain

$$\|H_f^\beta g\|_{q,\beta}^q \lesssim \|H_f^s g\|_{q,\beta}^q \leq \int_{\mathbb{B}_n} \left[\int_{\mathbb{B}_n} \frac{|f(z) - f(w)| |g(w)|}{|1 - \langle z, w \rangle|^{n+1+s}} dv_s(w) \right]^q dv_\beta(z).$$

By Lemma 3.2,

$$|f(z) - f(w)| \leq C \frac{\beta(z, w) + 1}{(1 - |w|)^{-\gamma}} |1 - \langle z, w \rangle|^{-2\gamma}.$$

Therefore,

$$\|H_f^\beta g\|_{q,\beta}^q \lesssim \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(\beta(z, w) + 1) |g(w)|}{|1 - \langle z, w \rangle|^{n+1+s+2\gamma}} dv_{s+\gamma}(w) \right)^q dv_\beta(z),$$

and the boundedness of $H_f^\beta : A_\alpha^p \rightarrow L_\beta^q$ follows from Theorem 2.4 and Proposition 2.5 again. \square

The proof of Theorem 4.1 is now complete: the necessity of the condition $f \in BMO_\gamma^q$ follows from Lemma 4.2 and Proposition 4.3. Since $f \in BMO_\gamma^q$ if and only if $\bar{f} \in BMO_\gamma^q$, the sufficiency is a consequence of Theorem 3.5, Proposition 4.4, and Proposition 4.6.

As an immediate consequence of Theorem 4.1 and Proposition 3.6, we obtain the following result that characterizes the boundedness of Hankel operators with conjugate holomorphic symbols.

Corollary 4.7. *Let $f \in A_\beta^1$, $1 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, and*

$$\gamma = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

- (1) *For $n = 1$ we have*
 - (a) *If $\gamma \geq -1$, then $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded if and only if $f \in \mathcal{B}^{1+\gamma}$.*
 - (b) *If $\gamma < -1$, then $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded if and only if f is constant.*
- (2) *For $n > 1$ we have*
 - (a) *If $\gamma > -1/2$, then $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded if and only if $f \in \mathcal{B}^{1+\gamma}$.*
 - (b) *If $\gamma < -1/2$, then $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded if and only if f is constant.*
 - (c) *If $\gamma = -1/2$, then $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded if and only if*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{-1/2} |\widetilde{\nabla} f(z)| < \infty.$$

Proof. Since $f \in A_\beta^1$, the Hankel operator $H_{\bar{f}}$ is densely defined. If $H_{\bar{f}} : A_\alpha^p \rightarrow L_\beta^q$ is bounded, by testing the boundedness on the function 1 we see that $f \in A_\beta^q$. Since $\mathcal{B}^{1+\gamma} \subset A_\beta^q$, the result follows from Theorem 4.1 and Proposition 3.6. \square

In the case $q = p = 2$, this recovers the results of Janson and Wallstén [7, 12], where the case $\gamma = -1$ for $n = 1$ and the case $\gamma = -1/2$ for $n > 1$ were left open. Thus we have resolved these open cases.

5. WEIGHTED VMO SPACES

Let $\gamma \in \mathbb{R}$. For any positive radius r and every exponent p with $1 \leq p < \infty$, the space $VMO_{r,\gamma}^p$ consists of those functions f in $BMO_{r,\gamma}^p$ such that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\gamma MO_{p,r}(f)(z) = 0.$$

Again, the space $VMO_{r,\gamma}^p$ is actually independent of the weight parameter σ . Similarly as before, for $r > 0$, we define $VO_{r,\gamma}$ as the space of functions f in $BO_{r,\gamma}$ satisfying

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\gamma \omega_r(f)(z) = 0,$$

and $VA_{r,\gamma}^p$ as the space of functions f in $BA_{r,\gamma}^p$ satisfying

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\gamma \left[\widehat{|f|_r^p}(z) \right]^{1/p} = 0.$$

The following result shows that $VA_{r,\gamma}^p$ does not depend on r .

Lemma 5.1. *Let $0 < p \leq q < \infty$, $\alpha > -1$, $\beta > -1$, $f \in L_{loc}^q(\mathbb{B}_n)$, and*

$$\gamma = (n + 1 + \beta)/q - (n + 1 + \alpha)/p, \quad d\mu_{f,\beta} = |f|^q dv_\beta.$$

The following are equivalent:

- (i) *If $\{f_k\}$ is a bounded sequence in A_α^p and $f_k \rightarrow 0$ uniformly on every compact subset of \mathbb{B}_n , then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{B}_n} |f_k(z)|^q d\mu_{f,\beta}(z) = 0.$$

- (ii) *$f \in VA_{r,\gamma}^q$ for some (or all) $r > 0$.*
 (iii) *The condition*

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma q} B_{c,\sigma}(|f|^q)(z) = 0$$

holds for all $\sigma > \max(-1, -1 + \gamma q)$ and all $c > \max(0, -\gamma q)$.

Proof. By the corresponding little-oh result of Theorem 50 in [14], we know that (i) is equivalent to

$$\lim_{|z| \rightarrow 1} \frac{\mu_{f,\beta}(D(z,r))}{(1 - |z|^2)^{(n+1+\alpha)q/p}} = 0$$

for some (or all) $r > 0$. The equivalence of (i) and (ii) is a consequence of this result and the fact that

$$\widehat{|f|_r^q}(z) \asymp \frac{\mu_{f,\beta}(D(z,r))}{(1 - |z|^2)^{n+1+\beta}}.$$

That (iii) implies (ii) follows from the fact that

$$\widehat{|f|_r^q}(z) \lesssim B_{c,\sigma}(|f|^q)(z),$$

which has been shown in the proof of Lemma 3.3.

It remains to prove that (ii) implies (iii). Let $f \in VA_{r,\gamma}^q$. By definition, we have

$$(1 - |z|^2)^{\gamma q} B_{c,\sigma}(\widehat{|f|_r^q})(z) = (1 - |z|^2)^{c+\gamma q} \int_{\mathbb{B}_n} \frac{\widehat{|f|_r^q}(w) dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}}.$$

For $0 < s < 1$ let

$$I_1(s) = (1 - |z|^2)^{c+\gamma q} \int_{|w| \leq s} \frac{\widehat{|f|_r^q}(w) dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}},$$

and

$$I_2(s) = (1 - |z|^2)^{c+\gamma q} \int_{s < |w| < 1} \frac{\widehat{|f|_r^q}(w) dv_\sigma(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}}.$$

Let ε be an arbitrary positive number. By (ii), and then by Lemma 2.2, there exists an $s > 0$ such that

$$I_2(s) \lesssim \varepsilon (1 - |z|^2)^{c+\gamma q} \int_{s < |w| < 1} \frac{(1 - |w|^2)^{\sigma-\gamma q} dv(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} \lesssim \varepsilon.$$

Since $f \in VA_{r,\gamma}^q \subset BA_{r,\gamma}^q$, we know that

$$\widehat{|f|_r^q}(w) \lesssim (1 - |w|^2)^{-q\gamma}.$$

Since $|1 - \langle z, w \rangle| \gtrsim (1 - |w|^2)$, we obtain

$$\begin{aligned} I_1(s) &\leq (1 - |z|^2)^{c+\gamma q} \int_{|w| \leq s} \frac{dv_\sigma(w)}{(1 - |w|^2)^{n+1+c+\sigma+\gamma q}} \\ &\lesssim \frac{(1 - |z|^2)^{c+\gamma q}}{(1 - s^2)^{n+1+c+\gamma q}}. \end{aligned}$$

Hence, we can find a $\delta \in (0, 1)$ such that $I_1(s) \lesssim \varepsilon$ whenever $1 - \delta < |z| < 1$. Combining the above two inequalities for $I_1(s)$ and $I_2(s)$ we deduce for $1 - \delta < |z| < 1$ that

$$(1 - |z|^2)^{\gamma q} B_{c,\sigma}(\widehat{|f|_r^q})(z) \lesssim \varepsilon.$$

Therefore,

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma q} B_{c,\sigma}(\widehat{|f|_r^q})(z) = 0.$$

Let $d\mu_{f,\sigma} = |f|^q dv_\sigma$. Since

$$\widehat{|f|_r^q}(z) \asymp \widehat{\mu_{f,\sigma}}(z) := \frac{\mu_{f,\sigma}(D(z, r))}{(1 - |z|^2)^{n+1+\sigma}}, \quad (5.1)$$

the above equation is equivalent to

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma q} B_{c,\sigma}(\widehat{\mu_{f,\sigma}})(z) = 0.$$

By [14, Lemma 52], we have

$$B_{c,\sigma}(\mu_{f,\sigma})(z) \lesssim B_{c,\sigma}(\widehat{\mu_{f,\sigma}})(z),$$

where

$$B_{c,\sigma}(\mu_{f,\sigma})(z) = (1 - |z|^2)^c \int_{\mathbb{B}_n} \frac{d\mu_{f,\sigma}(w)}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}}.$$

Thus we obtain

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma q} B_{c,\sigma}(\mu_{f,\sigma})(z) = 0,$$

which is the same as

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma q} B_{c,\sigma}(|f|^q)(z) = 0.$$

This proves (ii) implies (iii) and completes the proof of the lemma. \square

The next result shows that $VO_{r,\gamma}$ does not depend on r .

Lemma 5.2. *Let $\gamma \in \mathbb{R}$ and $r_1, r_2 > 0$. If $f \in VO_{r_1,\gamma}$, then $f \in VO_{r_2,\gamma}$.*

Proof. If $r_1 > r_2$, the result is obvious. So we assume that $r_1 < r_2$ and fix $z \in \mathbb{B}_n$. It follows from the continuity of f on \mathbb{B}_n that

$$\omega_{r_2}(f)(z) = \sup\{|f(z) - f(\zeta)|, \zeta \in \overline{D(z, r_2)}\},$$

and we can find $w \in \overline{D(z, r_2)}$ such that

$$|f(z) - f(w)| = \omega_{r_2}(f)(z).$$

Let $\lambda = \lambda(t)$, $0 \leq t \leq 1$, be the geodesic in the Bergman metric from z to w . Then λ lies entirely in $\overline{D(z, r_2)}$. As in the proof of Lemma 3.2, we let $N = [r_2/r_1] + 2$ and $t_i = i/N$, $0 \leq i \leq N$, where $[x]$ denotes the largest integer less than or equal to x . Set $z_i = \lambda(t_i)$, $0 \leq i \leq N$. Since $N \geq r_2/r_1 + 1 > r_2/r_1$, we have

$$\beta(z_{i-1}, z_i) = \frac{\beta(z, w)}{N} \leq \frac{r_2}{N} < r_1.$$

Because z_i is in the closure of $D(z, r_2)$, there exists a constant $K > 0$, independent of i , such that

$$\frac{1}{K}(1 - |z|^2)^\gamma \leq (1 - |z_i|^2)^\gamma \leq K(1 - |z|^2)^\gamma.$$

Since $f \in VO_{r_1,\gamma}$, we know that

$$\lim_{|z_i| \rightarrow 1} (1 - |z_i|^2)^\gamma \omega_{r_1}(f)(z_i) = 0.$$

But $|z_i| \rightarrow 1$ as $|z| \rightarrow 1$. So for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1 - |z_i|^2)^\gamma \omega_{r_1}(f)(z_i) < \frac{\varepsilon}{NK}$$

whenever $1 - |z| < \delta$. Thus

$$\begin{aligned} |f(z) - f(w)| &\leq \sum_{i=1}^N |f(z_{i-1}) - f(z_i)| \leq \sum_{i=1}^N \omega_{r_1}(f)(z_i) \\ &\leq \frac{\varepsilon}{K(1 - |z_i|^2)^\gamma} \leq \frac{\varepsilon}{K} \cdot \frac{K}{(1 - |z|^2)^\gamma} \\ &= \frac{\varepsilon}{(1 - |z|^2)^\gamma}. \end{aligned}$$

Therefore,

$$\omega_{r_2}(f)(z) = |f(z) - f(w)| \leq \frac{\varepsilon}{(1 - |z|^2)^\gamma}$$

for $1 - \delta < |z| < 1$, which shows that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\gamma \omega_{r_2}(f)(z) = 0,$$

or $f \in VO_{r_2, \gamma}$. □

Because of Lemmas 5.1 and 5.2, we can denote $VA_{r, \gamma}^p$ and $VO_{r, \gamma}$ by VA_γ^p and VO_γ , respectively. Just as in the big-oh case, we have the following result for $VMO_{r, \gamma}^p$.

Theorem 5.3. *Suppose $r > 0$, $\gamma \in \mathbb{R}$, $1 \leq p < \infty$, and $f \in BMO_\gamma^p$. The following conditions are equivalent:*

- (a) $f \in VMO_{r, \gamma}^p$.
- (b) $f = f_1 + f_2$ with $f_1 \in VO_\gamma$ and $f_2 \in VA_\gamma^p$.
- (c) For some (or all) $\sigma > \max(-1, -1 + \gamma p)$ and for each $c \geq \max(n + 1 + \sigma, n + 1 + \sigma - 2\gamma)$, we have

$$\lim_{|z| \rightarrow 1} \int_{\mathbb{B}_n} |f(w) - \widehat{f}_r(z)|^p \frac{(1 - |z|^2)^{c + \gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) = 0.$$

- (d) For some (or all) $\sigma > \max(-1, -1 + \gamma p)$ and for each $c \geq \max(n + 1 + \sigma, n + 1 + \sigma - 2\gamma)$, there is a function λ_z such that

$$\lim_{|z| \rightarrow 1} \int_{\mathbb{B}_n} |f(w) - \lambda_z|^p \frac{(1 - |z|^2)^{c + \gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w) = 0.$$

- (e) For some (or all) $\sigma > -1$ there is a function λ_z such that

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - \lambda_z|^p dv_\sigma(w) = 0.$$

Proof. That (c) implies (d) is obvious. That (d) implies (e) follows from the simple inequality

$$\begin{aligned} & \frac{(1 - |z|^2)^{\gamma p}}{v_\sigma(D(z, r))} \int_{D(z, r)} |f(w) - \lambda_z|^p dv_\sigma(w) \\ & \lesssim \int_{\mathbb{B}_n} |f(w) - \lambda_z|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w). \end{aligned}$$

An easy modification of the proof of Lemma 3.1 shows that (e) implies (a). That (a) implies (b) follows easily from the proof of (a) implying (b) in Theorem 3.5.

Thus we only need to prove that (b) implies (c). Suppose that (b) holds. As in the proof of Theorem 3.5, from Lemma 5.1 it is not difficult to see that (c) is satisfied for $f \in VA_\gamma^p$. Now, for $f \in VO_\gamma$, it is obvious that $f \in BO_\gamma$. Set

$$I(z) = \int_{\mathbb{B}_n} |f(w) - \widehat{f}_r(z)|^p \frac{(1 - |z|^2)^{c+\gamma p}}{|1 - \langle z, w \rangle|^{n+1+c+\sigma}} dv_\sigma(w).$$

Making the change of variables $w = \varphi_z(\zeta)$, we obtain

$$I(z) = \int_{\mathbb{B}_n} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p \frac{(1 - |z|^2)^{\gamma p}}{|1 - \langle z, \zeta \rangle|^{n+1+\sigma-c}} dv_\sigma(\zeta). \quad (5.2)$$

In the case $\gamma \geq 0$, it follows from the proof of Theorem 3.5 and the invariance of the Bergman metric that

$$\begin{aligned} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)| & \lesssim \|f\|_{BO_\gamma} \frac{\beta(\varphi_z(\zeta), z) + 1}{\min(1 - |z|, 1 - |\varphi_z(\zeta)|)^\gamma} \\ & \lesssim \|f\|_{BO_\gamma} \frac{\beta(\zeta, 0) + 1}{\min(1 - |z|, 1 - |\varphi_z(\zeta)|)^\gamma}. \end{aligned}$$

Let

$$E = \{\zeta \in \mathbb{B}_n : |\varphi_z(\zeta)| \leq |z|\}.$$

For $\zeta \in E$ we have $1 - |\varphi_z(\zeta)|^2 \geq 1 - |z|^2$ and

$$|f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} \lesssim (\beta(\zeta, 0) + 1)^p. \quad (5.3)$$

For $\zeta \in \mathbb{B}_n \setminus E$ we have $1 - |\varphi_z(\zeta)|^2 \leq 1 - |z|^2$ and

$$\begin{aligned} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} & \lesssim \frac{(\beta(\zeta, 0) + 1)^p (1 - |z|^2)^{\gamma p}}{(1 - |\varphi_z(\zeta)|^2)^{\gamma p}} \\ & \lesssim (\beta(\zeta, 0) + 1)^p (1 - |\zeta|^2)^{-\gamma p}. \end{aligned}$$

Since $\sigma > -1 + \gamma p \geq -1$, we have

$$\int_{\mathbb{B}_n} (\beta(\zeta, 0) + 1)^p dv_\sigma(\zeta) < \infty,$$

and

$$\int_{\mathbb{B}_n} (\beta(\zeta, 0) + 1)^p (1 - |\zeta|^2)^{-\gamma p} dv_\sigma(\zeta) < \infty.$$

Let

$$H(\zeta) = \begin{cases} (\beta(\zeta, 0) + 1)^p, & \zeta \in E \\ (\beta(\zeta, 0) + 1)^p (1 - |\zeta|^2)^{-\gamma p}, & \zeta \in \mathbb{B}_n \setminus E \end{cases}$$

The above argument shows that $H(\zeta)$ is in $L^1(\mathbb{B}_n, dv_\sigma)$ and, since $c \geq n + 1 + \sigma$, we have

$$|f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} |1 - \langle z, \zeta \rangle|^{c-(n+1+\sigma)} \lesssim H(\zeta) \quad (5.4)$$

for $\gamma \geq 0$ and all $z \in \mathbb{B}_n$.

If $\gamma < 0$, it follows from the proof of Theorem 3.5 again that

$$\begin{aligned} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)| &\lesssim \|f\|_{BO_\gamma} \frac{(\beta(\varphi_z(\zeta), z) + 1)}{(1 - |z|^2)^{-\gamma}} |1 - \langle z, \varphi_z(\zeta) \rangle|^{-2\gamma} \\ &= \|f\|_{BO_\gamma} \frac{(\beta(\zeta, 0) + 1)(1 - |\zeta|^2)^\gamma}{(1 - |\varphi_z(\zeta)|^2)^\gamma} \\ &= \|f\|_{BO_\gamma} \frac{(\beta(\zeta, 0) + 1) |1 - \langle z, \zeta \rangle|^{2\gamma}}{(1 - |z|^2)^\gamma}. \end{aligned}$$

Since $c \geq n + 1 + \sigma - 2\gamma p$, we also have

$$|f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} |1 - \langle z, \zeta \rangle|^{c-(n+1+\sigma)} \lesssim G(\zeta) \quad (5.5)$$

for all $z \in \mathbb{B}_n$, where $G(\zeta) = (\beta(\zeta, 0) + 1)^p$ is in $L^1(\mathbb{B}_n, dv_\alpha)$.

Fix any $\zeta \in \mathbb{B}_n$ and let $t = \beta(\zeta, 0)$. Since $\beta(\varphi_z(\zeta), z) = \beta(\zeta, 0) = t$ and $f \in VO_\gamma$, we get

$$\lim_{|z| \rightarrow 1} |f \circ \varphi_z(\zeta) - f(z)|^p (1 - |z|^2)^{\gamma p} \leq \lim_{|z| \rightarrow 1} (1 - |z|^2)^{\gamma p} \omega_t(f)(z)^p = 0. \quad (5.6)$$

On the other hand, we have

$$\begin{aligned} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p &\lesssim |f \circ \varphi_z(\zeta) - f(z)|^p + |f(z) - \widehat{f}_r(z)|^p \\ &\lesssim |f \circ \varphi_z(\zeta) - f(z)|^p + \\ &\quad + \frac{1}{v_\sigma(D(z, r))^p} \int_{D(z, r)} |f(z) - f(t)|^p dv_\sigma(t) \\ &\lesssim |f \circ \varphi_z(\zeta) - f(z)|^p + \omega_r(f)(z)^p. \end{aligned}$$

Therefore,

$$\begin{aligned} &|f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} \\ &\lesssim |f \circ \varphi_z(\zeta) - f(z)|^p (1 - |z|^2)^{\gamma p} + \omega_r(f)(z)^p (1 - |z|^2)^{\gamma p}, \end{aligned}$$

which tends to zero as $|z| \rightarrow 1$, because $f \in VO_\gamma$ and (5.6). Since $c \geq n + 1 + \sigma$, we also have

$$\lim_{|z| \rightarrow 1^-} |f \circ \varphi_z(\zeta) - \widehat{f}_r(z)|^p (1 - |z|^2)^{\gamma p} |1 - \langle z, \zeta \rangle|^{c - (n+1+\sigma)} = 0.$$

Thus in all cases, due to (5.4) and (5.5), we can apply Lebesgue's Dominated Convergence Theorem (bearing in mind the expression for $I(z)$ given in (5.2)) to obtain $I(z) \rightarrow 0$ as $|z| \rightarrow 1^-$, which is (c). This completes the proof of the theorem. \square

Since condition (b) in the theorem above is independent of r , we see that the space $VMO_{\gamma,r}^p$ is actually independent of r . Thus we will simply use the notation VMO_γ^p .

Notice that in (c) and (d) of Theorem 5.3 we require a somewhat stronger condition $c \geq n + 1 + \sigma$ than $c > 0$ in Theorem 3.5. It would be nice to know whether it is possible to replace condition $c \geq n + 1 + \sigma$ by $c > 0$ with $c > -\gamma p$ in (c) and (d) here. Anyway, for our main purpose here (to characterize compactness of Hankel operators) condition $c \geq n + 1 + \sigma$ is enough.

For $\alpha > 0$ let $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(\mathbb{B}_n)$ denote the closure of the set of polynomials in \mathcal{B}^α . The space \mathcal{B}_0^α consists exactly of those holomorphic functions f such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |\nabla f(z)| = 0.$$

As before, the complex gradient can be replaced by the radial derivative Rf . Furthermore, for $\alpha > 1/2$, a function f is in \mathcal{B}_0^α if and only if the function $(1 - |z|^2)^{\alpha-1} |\widetilde{\nabla} f(z)|$ belongs to $\mathbb{C}_0(\mathbb{B}_n)$. Again, we refer to [17, Chapter 7] for all these facts. With minor modifications in the proof of Proposition 3.6 together with Corollary 2.9 we obtain the following result.

Proposition 5.4. *Let $\gamma \in \mathbb{R}$ and $1 \leq p < \infty$. Then $H(\mathbb{B}_n) \cap VMO_\gamma^p = \mathcal{B}_0^{1+\gamma}$ for $n = 1$ and $\gamma > -1$, or for $n > 1$ and $\gamma > -1/2$. In all other cases, the space $H(\mathbb{B}_n) \cap VMO_\gamma^p$ consists of only constants.*

Proof. The details are left to the interested reader. \square

6. COMPACT HANKEL OPERATORS

In this section we prove the following characterization of compact Hankel operators between weighted Bergman spaces.

Theorem 6.1. *Let $1 < p \leq q < \infty$, $\alpha, \beta > -1$, $f \in L_\beta^q$, and*

$$\gamma = \frac{n + 1 + \beta}{q} - \frac{n + 1 + \alpha}{p}.$$

Then both $H_f^\beta, H_{\bar{f}}^\beta : A_\alpha^p \rightarrow L_\beta^q$ are compact if and only if $f \in VMO_\gamma^q$.

Again, we are going to prove Theorem 6.1 with several lemmas. We begin with the necessity.

Lemma 6.2. *Let p, q, α, β and γ be as in Theorem 6.1. If both $H_f^\beta, H_{\bar{f}}^\beta : A_\alpha^p \rightarrow L_\beta^q$ are compact, then $f \in VMO_\gamma^q$.*

Proof. Fix a nonnegative t such that $n+1+\beta+t > (n+1+\alpha)/p$. It is easy to see that h_z^t converges to zero uniformly on compact subsets of \mathbb{B}_n as $|z| \rightarrow 1^-$. Since each h_z^t is a unit vector in A_α^p , we conclude that $h_z^t \rightarrow 0$ weakly in A_α^p as $|z| \rightarrow 1^-$. It follows from the compactness of H_f^β that

$$\lim_{|z| \rightarrow 1^-} \|H_f^\beta h_z^t\|_{q,\beta} = 0.$$

The same is true if f is replaced by \bar{f} . By Proposition 4.3 we have

$$\lim_{|z| \rightarrow 1^-} MO_{q,\beta,t}(f)(z) = 0.$$

In other words, we have

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{c+\gamma q} \int_{\mathbb{B}_n} |f(w) - \overline{g_z(z)}|^q \frac{dv_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+c+\beta}} = 0,$$

where $c = (q-1)(n+1+\beta) + tq$. This implies condition (e) in Theorem 5.3 with $\lambda_z = \overline{g_z(z)}$, so $f \in VMO_\gamma^q$. \square

The sufficiency will follow from the next two results.

Lemma 6.3. *Let p, q, α, β and γ be as in Theorem 6.1. If $f \in VA_\gamma^q$, then $H_f^\beta : A_\alpha^p \rightarrow L_\beta^q$ is compact.*

Proof. Let $\{g_n\}$ be a bounded sequence in A_α^p converging to zero uniformly on compact subsets of \mathbb{B}_n . We must prove that $\|H_f^\beta g_n\|_{q,\beta} \rightarrow 0$. Following the proof of Proposition 4.4, we know that

$$\|H_f^\beta g_n\|_{q,\beta} \lesssim \|g_n\|_{L^q(d\mu_{f,\beta})}$$

with $d\mu_{f,\beta} = |f|^q dv_\beta$. The desired result then follows from Lemma 5.1. \square

Lemma 6.4. *Let p, q, α, β and γ be as in Theorem 6.1. If $f \in VO_\gamma$, then $H_f^\beta : A_\alpha^p \rightarrow L_\beta^q$ is compact.*

Proof. Let $\{g_n\}$ be a bounded sequence in A_α^p converging to zero uniformly on compact subsets of \mathbb{B}_n . By Lemma 4.5 and the density of H^∞ in A_α^p , for any $\delta \geq \beta$ we have

$$\|H_f^\beta g\|_{q,\beta} \leq C \|H_f^\delta g\|_{q,\beta}, \quad g \in A_\alpha^p.$$

We will be done if we can prove that

$$\lim_{n \rightarrow \infty} \|H_f^\delta g_n\|_{q,\beta} = 0$$

for some $\delta \geq \beta$.

Since $\beta > -1$, we can find some $\eta > 0$ satisfying $\beta - \eta \max(q, q') > -1$. We then choose some $\delta \geq \beta$ large enough so that $c = \eta q + \delta - \beta$ satisfies

$$c \geq n + 1 + \sigma + \max(0, -2\gamma q),$$

with $\sigma = \beta - \eta q$. In fact, this is the same as

$$\delta \geq n + 1 + 2\beta + \max(0, -2\gamma q) - 2\eta q.$$

So the choice $\delta = n + 1 + 2\beta + \max(0, -2\gamma q)$ works. Let $\varepsilon > 0$ be arbitrary. Since $VO_\gamma \subset VMO_\gamma^q$, by part (c) of Theorem 5.3 and with the above c and σ , we may choose t_1 sufficiently close to 1 so that

$$(1 - |w|^2)^{(\eta q + \delta - \beta) + \gamma q} \int_{\mathbb{B}_n} \frac{|f(z) - \widehat{f}_r(w)|^q}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_{\beta - \eta q}(z) < \varepsilon \quad (6.1)$$

for all $t_1 < |w| < 1$.

Fix $r > 0$. By the definition of VO_γ , there exists t_2 , $0 < t_2 < 1$, such that

$$\omega_r(f)(w) < \varepsilon(1 - |w|^2)^{-\gamma}, \quad |w| > t_2.$$

We have

$$\|H_f^\delta g_n\|_{q,\beta}^q \leq \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|f(z) - f(w)| |g_n(w)|}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_\delta(w) \right)^q dv_\beta(z).$$

Let $t = \max(t_1, t_2)$ and split the inner integral above in two parts: one for $|w| \leq t$ and the other for $|w| > t$. The integral on $|w| \leq t$ can be made as small as we want because of the uniform convergence to zero on compact subsets of g_n . For the other, we will use our assumption $f \in VO_\gamma$.

Since

$$|f(z) - f(w)| \leq |f(z) - \widehat{f}_r(w)| + |f(w) - \widehat{f}_r(w)|,$$

we get two integrals. The first one involves the function

$$I_1(z) = \int_{|w| > t} \frac{|f(w) - \widehat{f}_r(w)| |g_n(w)|}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_\delta(w).$$

Since

$$f(w) - \widehat{f}_r(w) = \frac{1}{v_\delta(D(w, r))} \int_{D(w, r)} (f(w) - f(\zeta)) dv_\delta(\zeta),$$

we obtain for $|w| > t$ that

$$|f(w) - \widehat{f}_r(w)| \leq \omega_r(f)(w) < \varepsilon (1 - |w|^2)^{-\gamma}.$$

Therefore,

$$\begin{aligned} I_1 &:= \int_{\mathbb{B}_n} I_1(z)^q dv_\beta(z) \lesssim \varepsilon^q \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{|g_n(w)| dv_{\delta-\gamma}(w)}{|1 - \langle z, w \rangle|^{n+1+\delta}} \right)^q dv_\beta(z) \\ &= \varepsilon^q \int_{\mathbb{B}_n} S_{b,d}(|g_n|)(z)^q dv_\beta(z), \end{aligned}$$

with $d = n + 1 + \delta$ and $b = \delta - \gamma$. Now we want to apply Theorem 2.4 to show that $S_{b,d} : L_\alpha^p \rightarrow L_\beta^q$ is bounded. In the notation of Theorem 2.4 we have $\lambda = \gamma$ and

$$n + 1 + b + \lambda = n + 1 + \delta = d.$$

It remains to check the condition

$$\alpha + 1 < p(b + 1) = p(1 + \delta - \gamma),$$

which is easily seen to be equivalent to

$$n \left(\frac{1}{q} - \frac{1}{p} \right) < (1 + \delta) - \frac{1 + \beta}{q}, \quad (6.2)$$

By the proof of Proposition 4.6, we have

$$n \left(\frac{1}{q} - \frac{1}{p} \right) < \frac{1 + \beta}{q'} = (1 + \beta) - \frac{1 + \beta}{q},$$

where q' is the conjugate exponent of q . Since $\beta \leq \delta$, we see that (6.2) is indeed true. Therefore, by Theorem 2.4, we have

$$I_1 \lesssim \varepsilon^q \|g_n\|_{p,\alpha}^q \leq C \varepsilon^q.$$

It remains to deal with

$$I_2 := \int_{\mathbb{B}_n} I_2(z)^q dv_\beta(z),$$

where

$$I_2(z) = \int_{|w|>t} \frac{|f(z) - \widehat{f}_r(w)| |g_n(w)|}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_\delta(w).$$

By Hölder's inequality and Lemma 2.2,

$$\begin{aligned} I_2(z)^q &\lesssim \left[\int_{|w|>t} \frac{|f(z) - \widehat{f}_r(w)|^q |g_n(w)|^q}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_{\delta+\eta q}(w) \right] \\ &\quad \cdot \left[\int_{\mathbb{B}_n} \frac{dv_{\delta-\eta q'}(w)}{|1 - \langle z, w \rangle|^{n+1+\delta}} \right]^{q/q'} \\ &\lesssim (1 - |z|^2)^{-\eta q} \left(\int_{|w|>t} \frac{|f(z) - \widehat{f}_r(w)|^q |g_n(w)|^q}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_{\delta+\eta q}(w) \right). \end{aligned}$$

Thus I_2 is dominated by

$$\int_{|w|>t} |g_n(w)|^q \left[(1 - |w|^2)^{\eta q} \int_{\mathbb{B}_n} \frac{|f(z) - \widehat{f}_r(w)|^q}{|1 - \langle z, w \rangle|^{n+1+\delta}} dv_{\beta-\eta q}(z) \right] dv_{\delta}(w).$$

By (6.1), we get

$$I_2 \lesssim \varepsilon \int_{|w|>t} |g_n(w)|^q dv_{\beta-\gamma q}(w) \lesssim \varepsilon \|g_n\|_{A_{\alpha}^p}^q.$$

For the last inequality we used the fact that $A_{\alpha}^p \subseteq A_{\beta-\gamma q}^q$, which can be obtained from Theorem 69 in [14]. Putting everything together we conclude that $\|H_f^{\delta} g_n\|_{q,\beta} \rightarrow 0$ as $n \rightarrow \infty$. This finishes the proof. \square

To summarize, the necessity of Theorem 6.1 is proved by Lemma 6.2. Since $f \in VMO_{\gamma}^q$ if and only if $\bar{f} \in VMO_{\gamma}^q$, the sufficiency is a consequence of Theorem 5.3, Lemma 6.3, and Lemma 6.4.

As a direct consequence of Theorem 6.1 and Proposition 5.4, we obtain the following characterization of compactness of Hankel operators with conjugate holomorphic symbols.

Corollary 6.5. *Let $f \in A_{\beta}^1$, $1 < p \leq q < \infty$, $\alpha, \beta > -1$, and*

$$\gamma = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}.$$

If $n = 1$ and $\gamma > -1$, or if $n > 1$ and $\gamma > -1/2$, then $H_{\bar{f}}^{\beta} : A_{\alpha}^p \rightarrow L_{\beta}^q$ is compact if and only if $f \in \mathcal{B}_0^{1+\gamma}$. In all other cases, $H_{\bar{f}}^{\beta} : A_{\alpha}^p \rightarrow L_{\beta}^q$ is compact if and only if f is constant.

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