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Norming Sets

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Abstract

The main objective of this report is the study of the Logvinenko-Sereda sets for different function spaces. It consists in characterizing the subsets $G \subset \Omega$ such that there is a constant $C > 0$ where

$$\|f\|^2 \leq C \int_G |f|^2 \, dm.$$

Following to the proof that appears in the book of V. Havin and B. Jöricke we have obtained the Logvinenko-Sereda theorem for the Paley-Wiener space. Moreover, for the same function space we have found another argument based on the proof of Daniel H. Luecking for the Bergman space in the ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$. In this case, we have taken the same structure of the proof with the translations group and euclidean balls instead of the automorphism group and hyperbolic balls. Next, considering the same idea as for the Paley-Wiener space we have achieved the Logvinenko-Sereda theorem for the Classic Fock space. Finally, we have finished with the analogous result for the space of polynomials in the torus.

Keywords: *Bergman space, Fock space, Functional analysis, Harmonic analysis, Harmonic functions, Logvinenko-Sereda sets, Paley-Wiener space.*

Contents

Chapter 1

Motivation

First of all, we consider a function $f \in L^2(\mathbb{T})$. We have that

$$\|f\|^2 = \int_{\mathbb{T}} |f|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

However, if we do not have information of the function on the all set $\mathbb{T}$ or $\mathbb{Z}$, we need to find sets A and B such that

$$\|f\|^2 \simeq \int_{\mathbb{T}\setminus A} |f(z)|^2 \; dm(z) + \sum_{n\in\mathbb{Z}\setminus B} |\hat{f}(n)|^2,$$

to determine the function f. So, our aim is to characterize these sets. We will say that (A, B) is a strong a-pair.

Now, let us consider the particular case where $B = [-N, N]$, $N \in \mathbb{N}$, and $\mathbb{T} \setminus A$ is relatively dense, that is, there is a constant $\delta > 0$ and $R > 0$ such that

$$m(I(x, R) \cap (\mathbb{T} \setminus A)) \geq \delta m(I(x, R))$$

for all $x \in \mathbb{T}$.

We can decompose the functions $f \in L^2(\mathbb{T})$ as the sum of orthogonal function f_1 and f_2 such that

$$supp \hat{f}_1 \subset \mathbb{Z} \setminus [-N, N] \;\text{ and }\; supp \hat{f}_2 \subset [-N, N].$$

Hence, let us see that $(A, [-N, N])$ is a strong a-pair. We only need to prove that

$$\|f\|^2 \lesssim \int_{\mathbb{T}\setminus A} |f(z)|^2 \; dm(z) + \sum_{n\in\mathbb{Z}\setminus B} |\hat{f}(n)|^2,$$

since the other inequality holds clearly. For this, we will use that $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$.

(1)
$$\|f_1\|^2 = \|\hat{f}_1\|^2 = \sum_{n\in\mathbb{Z}\setminus[-N,N]} |\hat{f}_1(n)|^2 = \sum_{n\in\mathbb{Z}\setminus[-N,N]} |\hat{f}(n)|^2$$

since $supp \hat{f}_2 \subset [-N, N]$.

(2) First of all, as $supp \hat{f}_2 \subset [-N, N]$ we have that f_2 is a polynomial on $\mathbb{T}$. Moreover, as we will see in the Chapter 5, since $\mathbb{T} \setminus A$ is relatively dense we can use the Logvinenko-Sereda Theorem and we obtain

$$\begin{aligned}\|f_2\|^2 &\lesssim \int_{\mathbb{T}\setminus A} |f_2|^2\, dm \lesssim \int_{\mathbb{T}\setminus A} |f|^2\, dm + \int_{\mathbb{T}\setminus A} |f_1|^2\, dm\\ &\lesssim \int_{\mathbb{T}\setminus A} |f|^2\, dm + \int_{\mathbb{T}} |f_1|^2\, dm \lesssim \int_{\mathbb{T}\setminus A} |f|^2\, dm + \sum_{n\in\mathbb{Z}\setminus[-N,N]} |\hat{f}(n)|,\end{aligned}$$

where $|f_2|^2 = |f - f_1|^2 \leq |f|^2 + |f_1|^2$.

Therefore, we have

$$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 \lesssim \int_{\mathbb{T}\setminus A} |f|^2\, dm + \sum_{n\in\mathbb{Z}\setminus[-N,N]} |\hat{f}(n)|,$$

that is, $(A, [-N, N])$ is a strong a-pair, where $\mathbb{T} \setminus A$ is relatively dense and $N \in \mathbb{N}$.

Analogously, we consider a function $f \in L^2(\mathbb{R}^n)$. Moreover, it verifies that

$$\|f\|^2 = \int_{\mathbb{R}^n} |f|^2\, dm = \int_{\mathbb{R}^n} |\hat{f}|^2\, dm.$$

As above, the aim is to find the sets A and B such that

$$\|f\|^2 \simeq \int_{\mathbb{R}^n\setminus A} |f|^2\, dm + \int_{\mathbb{R}^n\setminus B} |\hat{f}|^2\, dm.$$

In such case, we will say also that (A, B) is a strong a-pair.

In this case, we consider a relatively dense set $\mathbb{R}^n \setminus A$ and a bounded set B.
Now, we decompose the function $f \in L^2(\mathbb{R}^n)$ as the sum of orthogonal functions f_1 and f_2 such that

$$supp \hat{f}_1 \subset \mathbb{R}^n \setminus B \;\text{ and }\; supp \hat{f}_2 \subset B.$$

Let us see that (A, B) is a strong a-pair. Only we need to prove that

$$\|f\|^2 \lesssim \int_{\mathbb{R}^n\setminus A} |f(z)|^2\, dm(z) + \int_{\mathbb{R}^n\setminus B} |\hat{f}(z)|^2\, dm(z),$$

since the other inequality holds clearly. For it, we will use that $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$.

(1)

$$\|f_1\|^1 = \|\hat{f}_1\|^2 = \int_{\mathbb{R}^n \setminus B} |\hat{f}_1(z)|^2 \, dm(z) = \int_{\mathbb{R}^n \setminus B} |\hat{f}(z)|^2 \, dm(z)$$

since $supp\hat{f}_2 \subset B$.

(2) As $supp\hat{f}_2 \subset B$ we have that f_2 is a Paley-Wiener function. On the other hand, as we see will see in the Chapter 4, since $\mathbb{R}^n \setminus A$ is relatively dense we can use the Logvinenko-Sereda Theorem and we obtain

$$\begin{aligned}\|f_2\|^2 &\lesssim \int_{\mathbb{R}^n \setminus A} |f_2|^2 \, dm \lesssim \int_{\mathbb{R}^n \setminus A} |f|^2 \, dm + \int_{\mathbb{R}^n \setminus A} |f_1|^2 \, dm \\ &\lesssim \int_{\mathbb{R}^n \setminus A} |f|^2 \, dm + \int_{\mathbb{R}^n} |f_1|^2 \, dm \lesssim \int_{\mathbb{R}^n \setminus A} |f|^2 \, dm + \int_{\mathbb{R}^n \setminus B} |\hat{f}|^2 \, dm,\end{aligned}$$

where $|f_2|^2 = |f - f_1|^2 \leq |f|^2 + |f_1|^2$.

Hence , we have

$$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 \lesssim \int_{\mathbb{R}^n \setminus A} |f|^2 \, dm + \int_{\mathbb{R}^n \setminus B} |\hat{f}|^2 \, dm,$$

that is, (A, B) is a strong a-pair, where $\mathbb{R}^n \setminus A$ is a relatively dense set and B is a bounded set .

Notice that if we assume the pair (A, B), where $B = supp\hat{f}$ is bounded, is a strong a-pair, we have

$$\|f\| \lesssim \int_{\mathbb{T} \setminus A} |f|^2 \, dm$$

in the first case or

$$\|f\| \lesssim \int_{\mathbb{R}^n \setminus A} |f|^2 \, dm$$

in the second case. And as we will see in this work, this implies that $\mathbb{T} \setminus A$ and $\mathbb{R}^n \setminus A$ must be relatively dense.

Chapter 2

Classic Proof of Logvinenko-Sereda Theorem

In this chapter we will give a complete description of the pair (A, B) forming a strong a-pair with any bounded spectrum B. For that, we have based on the classical proof of the Logvinenko-Sereda Theorem that appears in [2, p. 112]. Notice that we have adapted the given proof of the sufficiency in one dimension to several variables.

First of all, we need define the *Poisson measure* on $\mathbb{R}^n$ as

$$\Pi := \prod_{i=1}^{n} (\pi(1 + x_i^2))^{-1} m,$$

m being the Lebesgue measure on $\mathbb{R}^n$.

Remark 2.0.1. In this chapter, we will use the following notation:

$$\mathbb{R}_+^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_j \geq 0, \forall j \in \{1, \ldots, n\}\}.$$

Now, we show the type of functions that we will use to prove the Theorem on Two Constants, which is necessary in the proof of the Logvinenko-Sereda Theorem.

Definition 2.0.2. We call a distribution $f \in S'(\mathbb{R}^n)$ a *plus-function* if *supp*$\hat{f} \subset \mathbb{R}_+^n$. A distribution $f \in S'(\mathbb{T}^n)$ a *plus-function* if *supp*$\hat{f} \subset \mathbb{R}_+^n \cap \mathbb{Z}^n$.

Definition 2.0.3. We denote by $H^p(\mathbb{R}^n)$ ($H^p(\mathbb{T}^n)$ respectively), $1 \leq p < \infty$, the set of the functions $f \in L^p(\mathbb{R}^n)$ ($f \in L^p(\mathbb{T}^n)$ respectively). It is called the Hardy class.

Now let us prove a the Jensen's Inequality on $\mathbb{T}^n$ and $\mathbb{R}^n$ that we will use in the following results.

Proposition 2.0.4. *If $f \in H^1(\mathbb{T}^n)$ then*

$$\log |\hat{f}(0)| \leq \int_{\mathbb{T}^n} \log |f| \; dm. \tag{2.0.1}$$

Proof. According with the classical inequality for geometric and arithmetic means,

$$\exp \int \log |f| \; d\mu \leq \int |f| \; d\mu$$

for any probability measure μ and $f \in L^1(\mu)$. We may apply it to $\mu = m$ and $f \in L^1(\mathbb{T}^n)$. Hence, for $f \in H^1(\mathbb{T}^n)$ we have

$$\log |\hat{f}(0)| \leq \int_{\mathbb{T}^n} \log |f| \; dm.$$

□

Proposition 2.0.5. *If $f \in L^1(\Pi)$ with supp$\hat{f} \subset \mathbb{R}^n_+$, then*

$$\int_{\mathbb{R}^n} \log |f| \; d\Pi \geq \log \left| \int_{\mathbb{R}^n} f \; d\Pi \right|. \tag{2.0.2}$$

Proof. Before to prove the Jensen's inequality, we need the following lemma:

Lemma 2.0.6. *For $f \in H^1(\mathbb{R}^n)$ we have the following:*

(i) $f \cdot \prod_{i=1}^n (x_i + ia_i)^{-1} \in H^1(\mathbb{R}^n)$ *for all* $a_j > 0$.

(ii) If $w = \prod_{j=1}^n \frac{x_j - i}{x_j + i}$, *then* $f \cdot w \in H^1$.

(iii) $\int_{\mathbb{R}^n} f(x) \; dx = 0$.

(iv) $\int_{\mathbb{R}^n} f \cdot w \; d\Pi = 0$.

(v) $\int_{\mathbb{R}^n} f \cdot w^n \; d\Pi = 0$, $n = 1, 2, \ldots$

(vi) If $f \in L^1(\Pi)$ with supp$\hat{f} \subset \mathbb{R}^n_+$, then for any $\varepsilon > 0$ we have

$$f_\varepsilon := f \cdot \prod_{j=1}^n (\varepsilon_j x_j + i)^{-2} \in H^1.$$

(vii) The equalities (v) are valid for any function $f \in L^1(\Pi)$ with supp$\hat{f} \subset \mathbb{R}^n_+$.

Proof. Now we will prove all the points.

(i)

$$\begin{aligned}\int_{\mathbb{R}^n} f(t) \cdot e^{-i\xi t} \cdot \prod_{j=1}^n (t_j + ia_j)^{-1} \; dt &= \int_{\mathbb{R}^n} f(t) \prod_{j=1}^n \left(\frac{1}{i} \int_0^\infty e^{-\eta_j a_j} e^{-it_j \eta_j} \; d\eta_j \right) e^{-\xi t} \; dt \\ &= \left(\frac{2\pi}{i} \right)^n \int_{\mathbb{R}^n_+} e^{-a\eta} \hat{f}(\xi - \eta) \; d\eta = 0\end{aligned}$$

if $\xi_j < 0$ for all $j \in \{1, \ldots, n\}$, since $\xi_j - \eta_j < 0$.

(ii) First of all, let us see that $f \cdot \frac{x_1-i}{x_1+i} \in H^1$. And as we have

$$\frac{x_1 - i}{x_1 + i} = f - 2if \cdot \frac{1}{x_1 + i},$$

we only need to prove that $f \cdot \frac{1}{x_1+i} \in H^1$.

$$\begin{aligned} \int_{\mathbb{R}^n} f(t) \cdot e^{-i\xi t} \cdot (t_1 + i)^{-1}\ dt &= \int_{\mathbb{R}^n} f(t) \left(\frac{1}{i} \int_0^\infty e^{-\eta} e^{-it_1\eta}\ d\eta \right) e^{-\xi t}\ dt \\ &= \left(\frac{2\pi}{i} \right) \int_0^\infty e^{-\eta} \hat{f}(\xi_1 - \eta_1, \xi_2, \ldots, \xi_n)\ d\eta = 0 \end{aligned}$$

if $\xi_j < 0$ for all $j \in \{1, \ldots, n\}$, since $\xi_1 - \eta_1 < 0$.
Therefore, iterating this for all $\frac{x_1-i}{x_1+i}$ we obtain the result.

(iii) Since $\hat{f}$ is continuous, it vanishes in $\mathbb{R}^n \setminus \mathbb{R}^n_+$ and

$$\hat{f}(0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x)\ dx = 0.$$

(iv)

$$fw \prod_{j=1}^n \frac{1}{1 + x_j^2} = f \prod_{j=1}^n \frac{x_j - i}{x_j + i} \frac{1}{1 + x_j^2} = f \prod_{j=1}^n \frac{1}{(x_j + i)^2} \in H^1$$

since it follows from (i) and (iii).

(v) It follows by induction from (ii) and (iv).

(vi) Clearly, $f \in L^1(m)$. We only have to check that *supp*$\hat{f}_\varepsilon \subset \mathbb{R}^n_+$. If $h \in S(\mathbb{R}^n)$ is a plus-function, then

$$\int_{\mathbb{R}^n} f_\varepsilon(x) h(x)\ dx = f \left[h \prod_{j=1}^n (\varepsilon_j x_j + i)^{-2} \right] = 0$$

because $h \prod_{j=1}^n (\varepsilon_j x_j + i)^{-2}$ is a plus-function and it is in $S(\mathbb{R}^n)$ by (i).

(vii) By Lebesgue theorem on dominated convergence,

$$\int_{\mathbb{R}^n} f \cdot w^n\ d\Pi = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f_\varepsilon \cdot w^n\ d\Pi = 0$$

according to (v) and (vi), since $\left| \prod_{j=1}^n (\varepsilon_j t_j + i)^{-2} \right| \leq 1$ for $t \in \mathbb{R}^n$.

$\square$

Now we assume that $f \in L^1(\Pi)$ is a plus-function. We put

$$F(e^{i\theta_1}, \dots, e^{i\theta_n}) := f(-\cot\frac{\theta_1}{2}, \dots, -\cot\frac{\theta_n}{2}), \quad \theta_j \in (0, 2\pi).$$

The variables θ_j and $x_j = -\cot\frac{\theta_j}{2}$ are connected by the equalities:

$$x_j = -i\frac{1+e^{i\theta_j}}{e^{i\theta_j}-1},$$
$$d\theta_j = \frac{2dx_j}{1+x_j^2}.$$

Hence,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} |F(e^{i\theta_1}, \dots, e^{i\theta_n})| \, d\theta_1 \dots d\theta_n = 2^n \int_{\mathbb{R}^n} |f(t)| \prod_{j=1}^n \frac{1}{1+t_j^2} \, dt < +\infty,$$

and

$$\hat{F}(-k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(e^{i\theta}) e^{ik\cdot\theta} \, d\theta = \int_{\mathbb{R}^n} f \cdot w^k \, d\Pi = 0, \quad k \in \mathbb{N}^n,$$

by (vii) in the previous lemma. Therefore, $F \in H^1(\mathbb{T}^n)$ and F verifies (2.0.1), which is equivalent to (2.0.5).

□

As we have proved the Jensen inequalities, we proceed now with the proof of the Theorem on Two Constants.

Theorem 2.0.7 (Theorem on Two Constants)**.** *Let $p \in [1, +\infty)$, $\gamma > 0$. Suppose $S \subset \mathbb{R}^n$ is Lebesgue measurable and $\Pi_x(S) \geq \gamma$, $x \in \mathbb{R}^n$. If $f \in H^p(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |P(f)(x)|^p \, dm \leq 2 \left(\int_S |f|^p \, dm \right)^\gamma \|f\|_p^{p(1-\gamma)}. \tag{2.0.3}$$

Proof. First, we put $\Pi_x(A) := \Pi(A - x)$, $x \in \mathbb{R}^n$, $A \subset \mathbb{R}^n$. That is,

$$\Pi_x(A) = \frac{1}{\pi^n} \int_A \prod_{j=1}^n \frac{1}{1+(x_j-t_j)^2} \, dt, \quad x \in \mathbb{R}^n.$$

Notice that Π_x is a probability measure on $\mathbb{R}^n$.

Before proving this theorem, we need the following two points.

- Let $f \in L^1(\Pi)$. We put

  $$P(f)(x) = \int_{\mathbb{R}^n} f \, d\Pi_x = \int_{\mathbb{R}^n} f(x+t) \, d\Pi(t), \quad x \in \mathbb{R}^n.$$

  If $f \in L^1(\Pi)$ is a plus-function, then the same is true for the function $t \mapsto f(x+t)$. Hence,

  $$\log|P(f)(x)| \leq P(\log|f|)(x). \tag{2.0.4}$$

- If $f \geq 0$ then

$$\int_{\mathbb{R}^n} P(f)(x)\ dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x+t)\ dx\ d\Pi(t) = \int_{\mathbb{R}_n} f(x)\ dx.$$

In particular, for any measurable set $S \subset \mathbb{R}^n$

$$\int_{\mathbb{R}^n}\int_S f\ d\Pi_x\ dx = \int_{\mathbb{R}^n} P(\chi_S f)(x)\ dx = \int_S f(x)\ dx. \tag{2.0.5}$$

Now, we fix $x \in \mathbb{R}^n$ and put $k := \Pi_x(S)$, $k' = \Pi_x(S^c)$, $\lambda(A) := k^{-1}\Pi_x(A \cap S)$ and $\lambda'(A) := (k')^{-1}\Pi_x(A \cap S^c)$, where $A \subset \mathbb{R}^n$. Notice that $k' > 0$ since otherwise there is nothing to prove. So we have two probability measures λ and λ'. Using (2.0.4) and the inequality for geometric and arithmetic means we obtain that

$$\begin{aligned}
p\log|P(f)(x)| &\leq k\int_S (\log|f|^p)\ d\lambda + k'(\log|f|^p)\ d\lambda' \\
&\leq k\log\left(\int_S |f|^p\ d\lambda\right) + k'\left(\int_{S^c}|f|^p\ d\lambda'\right) \\
&= k\log\frac{1}{k} + k'\log\frac{1}{k'} + k\log\int_S |f|^p\ d\Pi_x + k'\log\int_{S^c}|f|^p\ d\Pi_x \qquad (2.0.6)\\
&\leq \log 2 + \gamma\log\int_S |f|^p\ d\Pi_x + (k-\gamma)\log\int_S |f|^p\ d\Pi_x + k'\log\int_{S^c}|f|^p\ d\Pi_x.
\end{aligned}$$

We use the inequality $k\log\frac{1}{k} + (1-k)\log\frac{1}{1-k} \leq \log 2$. The sum of the last two numbers in (2.0.6) is not greater than $(k-\gamma+k')\log\int_{\mathbb{R}^n}|f|^p\ d\Pi_x = (1-\gamma)\log\int_{\mathbb{R}^n}|f|^p\ d\Pi_x$. It follows from (2.0.6) that

$$|P(f)(x)|^p \leq 2\left(\int_S |f|^p\ d\Pi_x\right)^{\gamma}\left(\int_{\mathbb{R}^n}|f|^p\ d\Pi_x\right)^{1-\gamma}, \quad x \in \mathbb{R}^n.$$

Integrating this estimate over $\mathbb{R}^n$ with respect the measure m, applying the Hölder inequality and (2.0.5), we obtain the expression (2.0.3).

□

As we have the background necessary, we continue with the main result of this chapter.

Theorem 2.0.8 (The Logvinenko-Sereda Theorem)**.** *For a measurable subset $G \subset \mathbb{R}^n$ the following are equivalent:*

(1) There a constant $C > 0$ such that

$$\int_{\mathbb{R}^n}|f|^2\ dm \leq C\int_G |f|^2\ dm$$

for every function $f \in L^2(\mathbb{R}^n)$ with bounded spectrum. We will say that G is a norming set.

(2) *There is a cube $K \subset \mathbb{R}^n$ and a constant $\gamma > 0$ such that*

$$|(K + x) \cap G| \geq \gamma$$

for all $x \in \mathbb{R}^n$.

Proof. **Necessity:**

As any subspace of functions of $L^2(\mathbb{R}^n)$ with bounded spectrum is shift-invariant, we only have to prove the following: if $\mathcal{E}$ is a shift-invariant non-trivial subspace of $L^1(\mathbb{R}^n)$, then every set that satisfies (1) for every $f \in \mathcal{E}$ also verifies (2).

Assume $f \in \mathcal{E}$ with $\|f\|_2 = 1$. We put $\omega_f(\delta) = \sup\left\{\int_e |f|^2\, dm \,:\, m(e) \leq \delta\right\}$. Moreover, $\omega_f(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Now, we consider $f_h(x) := f(x - h)$, $h \in \mathbb{R}^n$. It is easy to see that

$$\int_e |f_h|^2\, dm \leq \omega_f(m(e))$$

for any Lebesgue measurable set $e \subset \mathbb{R}^n$.

Suppose G satisfies (1) for every $f \in \mathbb{E}$. We can find a arge cube K satisfying

$$\int_{K^c} |f|^2\, dm \leq 1/2C,$$

being C the constant of (1). Then, we have

$$\int_{(K+h)^c} |f_h|^2\, dm \leq 1/2C$$

for any $h \in \mathbb{R}^n$. By (1) and by the shift invariance of $\mathcal{E}$ we obtain

$$\begin{aligned}\frac{1}{C} = \frac{1}{C}\int_{\mathbb{R}^n} |f_h|^2\, dm \leq \int_G |f_h|^2\, dm &= \int_{G\cap(K+h)} |f_h|^2\, dm + \int_{G\cap(K+h)^c} |f_h|^2\, dm \\ &\leq \omega_f(m(G \cap (K + h))) + \frac{1}{2C}\end{aligned}$$

for $h \in \mathbb{R}^n$.

Therefore, $\omega_f(m(G \cap (K + h))) \geq \frac{1}{2C}$ for any $h \in \mathbb{R}^n$ and $(m(G \cap (K + h))$ is bounded off zero by a constant depending on f and C, but not on h.

Sufficiency:

Here we use the information on the Poisson transform P and the measures Π_x.

Lemma 2.0.9. *The following assertions are equivalent:*

(a) *$G \subset \mathbb{R}^n$ satisfies (2).*

(b) $\inf\{\Pi_x(S) : x \in \mathbb{R}^n\} > 0$.

Proof. Assume (a) is true and take K and γ in (2), where $K = (-L, L)^n$. So, for $x \in \mathbb{R}^n$

$$\begin{aligned}\Pi_x(G) &\geq \int_{(K+x)\cap G} d\Pi_x = \frac{1}{\pi^n}\int_{x_n-L}^{x_n+L}\cdots\int_{x_1-L}^{x_1+L}\chi_G(t_1,\dots,t_n)\prod_{i=1}^n \frac{1}{1+(x_i-t_i)^2}\ dt_1\dots dt_n \\ &\geq \pi^{-n}\frac{1}{(1+L^2)^n}m(G\cap(K+x)) \geq \pi^{-n}\frac{1}{(1+L^2)^n}\gamma.\end{aligned}$$

Now we suppose that $\Pi_x(S) \geq \sigma$, $x \in \mathbb{R}$, where σ is positive and not depending on x. We put $K := (-L, L)^n$, $K_j := (-2^jL, 2^jL)^n$, $a(x) := m(G \cap (K + x))$. Then, we obtain

$$\begin{aligned}\pi^n\sigma &\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\chi_G(t_1,\dots,t_n)\prod_{i=1}^n\frac{1}{1+(x_i-t_i)^2}\ dt_1\dots dt_n \\ &= \int_{K+x}\chi_G(t)\prod_{i=1}^n\frac{1}{1+(x_i-t_i)^2}\ dt + \sum_{j=0}^{\infty}\int_{K_{j+1}\setminus K_j}\chi_G(t)\prod_{i=1}^n\frac{1}{1+(x_i-t_i)^2}\ dt \\ &\leq a(x) + \sum_{j=0}^{\infty}2^{-2nj}L^{-2n}m(G\cap(K_{j+1}+x)) \leq a(x) + \frac{2^{2n}}{L^n}.\end{aligned}$$

If L is large enough, then we have $a(x) \geq \frac{\pi\sigma}{2}$.

□

Now, we will show some properties of the operator

$$P(\varphi)(x) = \frac{1}{\pi^n}\int_{\mathbb{R}^n}\varphi(t)\prod_{i=1}^n\frac{1}{1+(x_i-t_i)^2}\ dt, \quad x \in \mathbb{R}^n. \tag{2.0.7}$$

(A) If $\varphi \in L^2(\mathbb{R}^n)$, then $P(\varphi)$ is the convolution $f * k$, where

$$k = \frac{1}{\pi^n}\prod_{i=1}^n\frac{1}{1+x_i^2}.$$

As follows from $\hat{k}(\xi) = \frac{1}{(2\pi)^n}\exp\left(-\sum_{i=1}^n|\xi_i|\right)$, $\xi \in \mathbb{R}^n$ for $\varphi \in L^2(\mathbb{R}^n)$ and for m-almost all $\xi \in \mathbb{R}^n$

$$\widehat{P(\varphi)} = \frac{\hat{\varphi}(\xi)}{(2\pi)^n}\exp\left(-\sum_{i=1}^n|\xi_i|\right).$$

(B) If $p \in [1, +\infty]$, $\varphi \in L^p(\mathbb{R}^n, m)$, then $P(\varphi) \in L^p(\mathbb{R}^n, m)$ and $\|P(\varphi)\|_p \leq \|\varphi\|_p$.

Proof. Suppose $1 < p < \infty$, $q := p/(p-1)$. Then

$$\begin{aligned}|P(\varphi)(x)|^p &\leq \int (k(x-t))^{1/p}|\varphi(t)|(k(x-t))^{1/q}\ dt \\ &\leq \left(\int k(x-t)|\varphi(t)|^p\ dt\right)\left(\int k(x-t)\ dt\right)^{p/q} = P(|\varphi|)(x), \quad x \in \mathbb{R}^n.\end{aligned}$$

Integrating this estimate with respect to x and using the equality $\int P(|\varphi|^p)\ dm = \int |\varphi|^p\ dm$, we obtain the result. The proof is even simpler if $p = 1$ or $+\infty$.

$\square$

(C) From above $\|P(\varphi)\|_2$ can be estimated by $\|\varphi\|_2$. The inverse estimate is in general false, but it becomes true for plus-functions with bounded spectrum. If $\varphi \in H^2$ and $\sup \hat{f} \subset (0, l)^n$, then

$$\|\varphi\|_2 \leq (2\pi)^n e^{nl} \|P(\varphi)\|_2. \tag{2.0.8}$$

Proof. If $\varphi \in H^2$, then (2.0.7) can be written as

$$\widehat{P(\varphi)} = \frac{\hat{\varphi}(\xi)}{(2\pi)^n} \exp\left(-\sum_{i=1}^n |\xi_i|\right).$$

Hence, we obtain that

$$|\hat{\varphi}(\xi)| = (2\pi)^n \exp\left(\sum_{i=1}^n |\xi_i|\right) |\widehat{P(\varphi)}(\xi)| \leq (2\pi)^n e^{2l} |\widehat{P(\varphi)}(\xi)| \tag{2.0.9}$$

for $\xi \in \mathbb{R}^n$. Now we obtain (2.0.8) by the inequality (2.0.9) and the Plancherel theorem.

$\square$

Assume that $f \in L^2(\mathbb{R}^n)$ with $supp \hat{f} \subset (a, b)^n$, $b - a = l$. Now we consider the following function

$$\varphi := f \cdot \exp\left(-\sum_{i=1}^n iax_i\right).$$

As we can see $\varphi \subset H^2$ and $|\varphi| \equiv |f|$. By (C),

$$\int_{\mathbb{R}^n} |f|^2\ dm = \int_{\mathbb{R}^n} |\varphi|^2\ dm \leq (2\pi)^{2n} e^{2nl} \int_{\mathbb{R}^n} |P(\varphi)|^2\ dm. \tag{2.0.10}$$

Since G is relatively dense, we have $\Pi_x(G) \geq \sigma$ for all $x \in \mathbb{R}^n$. Now, applying the Theorem on Two Constants (Theorem 2.0.7), we obtain

$$\|P(\varphi)\|_2^2 \leq 2 \left(\int_G |\varphi|^2\right)^\sigma \|\varphi\|_2^{2(1-\sigma)} = 2 \left(\int_G |f|^2\right)^\sigma \|f\|_2^{2(1-\sigma)}.$$

This estimate combined with (2.0.10) gives

$$\|f\|_2^2 \leq (2(2\pi)^{2n} e^{2l})^{1/\sigma} \int_G |f|^2\ dm.$$

$\square$

Chapter 3

Bergman spaces

In this chapter we will give a description of the norming set G for the Bergman space. For that, we have used the proofs which appear in the articles [3] and [4] de Daniel H. Luecking.

Hence, we start by defining the functions of the Bergman space.

Definition 3.0.1. Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk. If $p > 0$, A^p denotes the Bergman space of functions f which are analytic in $\mathbb{D}$ and $|f|^p$ is integrable on $\mathbb{D}$.

In the following results we will need pass from the balls

$$D(a, R) = \{z \in \mathbb{D} \,:\, |z - a| < r(1 - |a|)\}, \quad r \in (0, 1),$$

to balls in terms of the pseudohyperbolic metric. So, we have to define the following:

$$\Delta(a, r) = \left\{ z \in \mathbb{D} \,|\, \frac{|z - a|}{|1 - \overline{a}z|} < r \right\}.$$

Now, we show some properties of these pseudohyperbolic disks.

Proposition 3.0.2. *(i) If $z \in D(a, R)$ and $2R/(1 + R^2) \leq r < 1$ then*

$$D(a, R) \subset \Delta(z, r).$$

(ii) There exist constants $C(r)$ depending only on r such that

$$\frac{(1 - |a|)^2}{C(r)} \leq m(\Delta(a, r)) \leq C(r)(1 - |a|)^2.$$

Proof. (i) To prove (i) simply estimate

$$\frac{|z - z'|}{|1 - \overline{z}z'|} \leq \frac{2R}{1 + R^2}$$

for $z, z' \in D(a, R)$. This estimate is simplified by the fact that the maximum occurs at $z = a + R(1 - a)$ and $z' = a - R(1 - a)$ when $0 < a < 1$.

(ii) The proof of (ii) is an estimate of the diameter of $\Delta(a, r)$.

□

The following lemma will be useful in th proofs.

Lemma 3.0.3. *Given $\varepsilon > 0$, there are constants $C_1, C_2 > 0$ and a radius $R > 0$ depending on ε such that for all pesudohyperbolic disk $\Delta(\xi, R)$ with $\xi \in \mathbb{D}$, there is a A^p-function $f = f_{\Delta(\xi,R)}$ such that*

- $\int_{\mathbb{D}} |f|^p \, dm = C_1$,
- $\int_{\mathbb{D}\setminus\Delta(\xi,R)} |f|^p \, dm < \varepsilon$,
- $\sup_{y\in\Delta(\xi,R)} |f(y)|^p \leq C_2/m(\Delta(\xi, R))$.

Proof. Given $\varepsilon > 0$. We consider the function

$$f_w(z) := \frac{(1 - |w|^2)^{2/p}}{(1 - z\overline{w})^{4/p}} \in A^p, \quad z \in \mathbb{D}$$

for some $w \in \mathbb{D}$ fixed.

Let us see that the function satisfies all the properties.

$$\int_{\mathbb{D}} |f_w(z)|^p \, dz = \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} \, dz = \frac{(1 - |w|^2)^2}{(1 - |w|^2)^2} = \pi$$

since we use the property of the Bergman kernel:

$$\int_{\mathbb{D}} \left| \frac{1}{(1 - z\overline{w})^2} \right|^2 \, dz = \frac{1}{(1 - |w|^2)^2}.$$

Moreover, if we take a disk $D(a, R)$ and as $\frac{(1-|w|^2)^2}{|1-z\overline{w}|^4} \leq \frac{4}{(1-|w|)^2}$, we obtain

$$\int_{\mathbb{D}\setminus\Delta(w,R)} \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} \, dz \leq \int_{\mathbb{D}\setminus\Delta(w,R)} \frac{4}{(1 - |w|)^2} \, dz \leq 4\frac{m(\mathbb{D} \setminus \Delta(w, R))}{(1 - |w|)^2} \to 0, \quad R \to 1.$$

So, for that ε there is R such that

$$\int_{\mathbb{D}\setminus D(w,R)} |f_w(z)|^p \, dz < \varepsilon.$$

The last property is verified because

$$|f_w|^p = \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} \leq \frac{4}{(1 - |w|)^2} \leq \frac{4C(R)}{m(\Delta(a, R))}$$

by Proposition 3.0.2. □

We continue with the main result of this section.

Theorem 3.0.4 (The Logvinenko-Sereda Theorem)**.** *For a measurable subset $G \subset \mathbb{D}$ the following are equivalent:*

(1) There is a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f|^p \, dm \leq C \int_G |f|^p \, dm$$

for every $f \in A^p$. We will say that G is a norming set.

(2) There is a constant $\delta > 0$ and a radius $R \in (0,1)$ such that

$$m(G \cap D(a,R)) > \delta m(D(a,R))$$

for all $a \in \mathbb{D}$.

Remark 3.0.5. Using estimates like those we can show that (2) is equivalent to:
(2') There exist $\delta_0 > 0$ and $0 < R_0 < 1$ such that

$$m(G \cap \Delta(a,R_0)) > \delta_0 m(\Delta(a,R_0)), \quad a \in \mathbb{D}.$$

Proof of the Theorem 3.0.4. The proof that (1) implies (2) is relatively simple. We will prove that (1) implies (2'). So, given $\varepsilon \leq \frac{\pi}{2C}$ and applying Lemma 3.0.3 there is a radius $R > 0$ such that for all pseudohyperbolic disks $\Delta(\xi, R)$ there is a function $f_{\Delta(\xi,R)}$ verifying the properties of the lemma. Hence, we obtain that

$$\begin{aligned}
\frac{m(G \cap \Delta(\xi,R))}{m(\Delta(\xi,R))} &\geq \frac{1}{4C(R)} \int_{G \cap \Delta(\xi,R)} |f_{B(\xi,R)}(x)|^2 \, dm(x) \\
&\geq \frac{1}{4C(R)} \left(\int_G |f_{\Delta(\xi,R)}(x)|^2 \, dm(x) - \int_{\mathbb{D} \setminus \Delta(\xi,R)} |f_{\Delta(\xi,R)}(x)|^2 \, dm(x) \right) \\
&\geq \frac{1}{4C(R)} \left(\frac{\pi}{C} - \varepsilon \right) \geq \frac{\pi}{8CC(R)}
\end{aligned}$$

where $\delta = \frac{\pi}{8CC(R)}$.

The proof that (2) implies (1) is the difficult one to prove. It requires the following three lemmas. We assume that δ and R are given by (2) and fixed. The constants used only depends on R and p unless explicitly stated otherwise. In particular they do not depend on the function f. As R is fixed we abbreviate $D(a,R)$ by $D(a)$. If the analytic function f is given and $\lambda \in (0,1)$ we define the set

$$Ef_\lambda(a) = \{z \in D(a) \, : \, |f(z)| > \lambda |f(a)|\}$$

and the operator

$$B_\lambda f(a) = \frac{1}{m(Ef_\lambda(a))} \int_{Ef_\lambda(a)} |f|^p \, dm.$$

Now, we will assume $p = 1$, the proof of the general case can be obtained with only minor modifications on replacing $|f|$ by $|f|^p$.

Lemma 3.0.6. *Let f be analytic in $\mathbb{D}$ and $a \in \mathbb{D}$. Then*

$$\frac{m(Ef_\lambda(a))}{m(D(a))} \geq \frac{\log \frac{1}{\lambda}}{\log \frac{B_\lambda f(a)}{|f(a)|} + \log \frac{1}{\lambda}}.$$

Proof. We consider $w \in \mathbb{D}$. Applying Jensen's inequality and elementary estimates we have

$$\begin{aligned}
\log|f(w)| &\leq \frac{1}{m(D(w))} \int_{D(w)} \log|f|\; dm \\
&= \frac{1}{m(D(w))} \int_{D(w)\setminus Ef_\lambda(w)} \log|f|\; dm + \frac{1}{m(D(w))} \int_{Ef_\lambda(w)} \log|f|\; dm \\
&\leq \frac{m(D(w)) - m(Ef_\lambda(w))}{m(D(w))} \log \lambda|f(w)| + \frac{m(Ef_\lambda(w))}{m(D(w))} \frac{1}{m(Ef_\lambda(w))} \int_{Ef_\lambda(w)} \log|f|\; dm \\
&\leq \frac{m(D(w)) - m(Ef_\lambda(w))}{m(D(w))} \log \lambda|f(w)| + \frac{m(Ef_\lambda(w))}{m(D(w))} \log B_\lambda f(w).
\end{aligned}$$

In the last inequality, we use the concavity of the log. Now, we subtract $\log|f(w)|$ from the both sides.

$$0 \leq \frac{m(D(w)) - m(Ef_\lambda(w))}{m(D(w))} \log \lambda + \frac{m(Ef_\lambda(w))}{m(D(w))} \log \frac{B_\lambda f(w)}{|f(w)|}.$$

As $\log \lambda < 0$ and $\log \frac{B_\lambda f(w)}{|f(w)|} > 0$, we have that

$$\frac{m(Ef_\lambda(w))}{m(D(w))} \geq \frac{\log \frac{1}{\lambda}}{\log \frac{B_\lambda f(w)}{|f(w)|} + \log \frac{1}{\lambda}}.$$

□

The aim of this lemma is to show eventually that $Ef_\lambda(a)$ takes a large enough fraction of $D(a)$ to include some of $G \cap D(a)$. This will no be true for all $a \in \mathbb{D}$ because $\frac{B_\lambda f(a)}{|f(a)|}$ may be very larger. Hence, we will use the following two lemmas to show that the set where $\frac{B_\lambda f(a)}{|f(a)|}$ is not very larger is sufficient.

Lemma 3.0.7. *Let $\varepsilon > 0$ and $f \in A^1$. Define the set*

$$\mathcal{A} = \{a \in \mathbb{D} \,:\, |f(a)| < \frac{\varepsilon}{m(D(a))} \int_{D(a)} |f|\; dm\}.$$

There is a constant C depending only on R such that

$$\int_{\mathcal{A}} |f|\; dm \leq C\varepsilon \int_U |f|\; dm.$$

Proof. For $a \in \mathcal{A}$ we have

$$|f(a)| \leq \varepsilon \int_{\mathbb{D}} |f(z)| \frac{1}{m(D(a))} \chi_{D(a)}(z)\ dm(z).$$

Integrating over $\mathcal{A}$ and using Fubini's Theorem, we obtain

$$\int_{\mathcal{A}} |f(a)|\ dm(a) \leq \varepsilon \int_{\mathbb{D}} |f(z)| \int_{\mathcal{A}} \frac{1}{m(D(a))} \chi_{D(a)}(z)\ dm(a)\ dm(z).$$

Using the Proposition 3.0.2 (i) with $r = 2R/(1+R^2)$ we can write

$$\chi_{D(a)}(z) \leq \chi_{\Delta(a,r)}(z) = \chi_{\Delta(z,r)}(a).$$

Therefore, we have that

$$\int_{\mathcal{A}} \frac{1}{m(D(a))} \chi_{D(a)}(z)\ dm(a) \leq \int_{\Delta(z,r)} \frac{1}{m(D(a))}\ dm(a).$$

Moreover, if $a \in \Delta(z,r)$, there is a constant C^* such that

$$m(D(a)) \geq \frac{(1-|z|)^2}{C^*}.$$

Combining this with Proposition 3.0.2 (ii) we have

$$\int_{\mathcal{A}} |f(a)|\ dm(a) \leq C(r) C^* \varepsilon \int_{\mathbb{D}} |f(z)|\ dm(z).$$

□

The only use made of Lemma 3.0.7 is in the proof of the following. If $p \neq 1$, we need change $|f|$ to $|f|^p$ and ε^3 to $\varepsilon^{1+2/p}$ in the following lemma. We assume from now on than $\lambda < 1/2$.

Lemma 3.0.8. *Let $\varepsilon \in (0,1)$ and $f \in A^1$. Define the set*

$$\mathcal{B} = \{s \in \mathbb{D} : |f(a)| < \varepsilon^3 B_\lambda f(a)\}.$$

Then there is a constant C depending on R (and p) such that

$$\int_{\mathcal{B}} |f|\ dm \leq C\varepsilon \int_{\mathbb{D}} |f|\ dm.$$

Proof. We write

$$\int_{\mathcal{B}} |f|\ dm = \int_{\mathcal{B}\cap\mathcal{A}} |f|\ dm + \int_{\mathcal{B}\setminus\mathcal{A}} |f|\ dm.$$

The first integral can be estimated by Lemma 3.0.7. For the second integral we use the Fubini's Theorem as in the previous lemma.

$$\int_{\mathcal{B}\setminus\mathcal{A}} |f|\ dm \leq \varepsilon^3 \int_{\mathbb{D}} |f(z)| \int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a))} \chi_{Ef_\lambda(a)}(z)\ dm(a)\ dm(z).$$

Now, we need show that the inner integral is bounded. Since $\chi_{Ef_\lambda(a)} \leq \chi_{D(a)}(z)$, we use the argument in Lemma 3.0.7 and we can show

$$\frac{1}{m(Ef_\lambda(a))} \leq \frac{C}{\varepsilon^2 m(D(a))}$$

whenever $a \notin \mathcal{A}$. We will do showing that any disk D centered at a contains a concentric disk D' of area $(1/C)\varepsilon^2 m(D)$ with the following property. Whenever f is analytic and

$$|f(a)| \geq \varepsilon \frac{1}{m(D)} \int_D |f|\ dm.$$

Then $|f(z)| > \frac{1}{2}|f(a)| > \lambda |f(a)|$ on D'.

Without of generality we take $a = 0$, $D = \mathbb{D}$, and

$$\frac{1}{\pi} \int_{\mathbb{D}} |f|\ dm = 1.$$

Our hypothesis then is $|f(0)| \geq \varepsilon$. There is a constant $\Gamma_0 > 1$ (depending ony on p) such that $|f(z)| < \Gamma_0$ on the set $|z| = \frac{1}{2}$. Assuming $|z| < \frac{1}{4}$ we obtain

$$2\pi |f(z) - f(0)| \leq \left| \int_{|t|=1/2} f(t) \left(\frac{1}{t-z} - \frac{1}{t} \right)\ dt \right| \leq \Gamma_0 \cdot 8\pi |z|.$$

Choosing $|z| < \varepsilon / 8\Gamma_0$ we have that

$$|f(z)| > |f(0)| - \frac{\varepsilon}{2} > \frac{1}{2}|f(0)|$$

on a ball about zero of volume $\pi(\varepsilon/8\Gamma_0)^2$. Translating this to the ball $D(a)$ we obtain that $Ef_\lambda(a)$ contains a ball of area $(\varepsilon^2/C)m(D(a))$ whenever $a \notin \mathcal{A}$. Therefore, we have the following.

$$\int_{\mathcal{B}\setminus\mathcal{A}} |f|\ dm \leq C\varepsilon \int_{\mathbb{D}} |f(z)|\ dm(z).$$

Hence, we have proved the lemma.

□

Let $\mathcal{F} = \mathbb{D} \setminus \mathcal{B} = \{a \in \mathbb{D} : |f(a)| \geq \varepsilon^3 B_\lambda f(a)\}$. If we choose ε such that $\varepsilon C < 1/2$, we have

$$\int_{\mathbb{D}} |f|\ dm < 2 \int_{\mathcal{F}} |f|\ dm. \tag{3.0.1}$$

For $a \in \mathcal{F}$ we have $\frac{B_\lambda f(a)}{|f(a)|} \leq \frac{1}{\varepsilon^3}$. Hence, if we choose $\lambda \leq \varepsilon^{6/\delta}$, we obtain

$$\frac{m(Ef_{\lambda(a)})}{m(D(a))} > \frac{(2/\delta)\log(1/\varepsilon^3)}{\log(1/\varepsilon^3) + (2/\delta)\log(1/\varepsilon^3)} > 1 - \frac{\delta}{2}.$$

Consequently, (2) implies, for $a \in \mathcal{F}$,

$$m(G \cap Ef_\lambda(a)) > \frac{1}{2}\delta m(D(a)) \tag{3.0.2}$$

where the choice of λ depends only on R, δ and p. As (3.0.2), we have

$$\frac{1}{m(D(a))}\int_G \chi_{B(a)}(z)|f(a)|\ dm \geq \frac{1}{2}\delta\lambda|f(a)|, \quad a \in \mathcal{F}.$$

Integrating over $\mathcal{F}$ and using Fubini's Theorem, we obtain

$$\begin{aligned}\frac{1}{2}\delta\lambda\int_{\mathcal{F}}|f|\ dm &\leq \int_G |f(z)|\int_{\mathcal{F}}\frac{1}{m(B(a))}\chi_{B(a)}(z)\ dm(a)\ dm(z)\\ &\leq C\int_G |f(z)|\ dm(z).\end{aligned}$$

Hence, using the inequality (3.0.1) we have

$$C\int_G |f(z)|\ dm(z) \geq \frac{1}{4}\delta\lambda\int_{\mathbb{D}}|f|dm.$$

So, this complete the proof. □

Chapter 4

Paley-Wiener space

In this chapter, we will give a complete description of the pair (A, B) forming a strong a-pair with any bound spectrum B as in Chapter 2. However, here we will show an original proof using the same structure as for Bergman spaces.

Now, we give a definition of the functions of Paley-Wiener space.

Definition 4.0.1. We say that $f \in PW_K$ if $f \in L^2(\mathbb{R}^n)$ and $suppf \subset K$. That is,

$$PW_K = \left\{ f \in L^2(\mathbb{R}^n) \mid supp\hat{f} \subset K \right\}.$$

Moreover, using the Paley-Wiener theorem we obtain that

$$PW_K = \left\{ f \in \mathcal{H}(\mathbb{C}^n) : f \in L^2(\mathbb{R}^n) \text{ and } \exists A, C \in \mathbb{R}^+ \text{ such that } |f(z)| \leq Ce^{A|z|} \right\}.$$

We will need the following property, which appears in [7, pp. 95-96], to prove the main result of this chapter.

Proposition 4.0.2. *If $f \in PW_K$ then*

$$\int_{\mathbb{R}^n} |f(x+iy)|^2 \, dm(x) \leq e^{2A|y|} \int_{\mathbb{R}^n} |f(x)|^2 \, dm(x).$$

Remark 4.0.3. If we integrate respect to the imaginary part y, we obtain that

$$\begin{aligned} \int_{|\Im z|<R} |f(z)|^2 \, d\sigma(z) &\leq \int_{|y|<R} e^{2A|y|} \, dm(y) \cdot \int_{\mathbb{R}^n} |f(x)|^2 \, dm(x) \\ &\leq e^{2AR} \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)} \int_{\mathbb{R}^n} |f(x)|^2 \, dm(x). \end{aligned}$$

Notice that we use the measure m on $\mathbb{R}^n$ and the measure σ on $\mathbb{C}^n$.

The following lemmma will be useful to prove the necessity of Logvinenko-Sereda Theorem.

Lemma 4.0.4. *Given $\varepsilon > 0$, there is a constant $C' > 0$ and a radius $R > 0$ depending on ε such that for all ball $B(\xi, R)$ with $\xi \in \mathbb{R}^n$, there is a PW-function $f = f_{B(\xi,R)}$ on $\mathbb{R}^n$ such that*

- $\int_{\mathbb{R}^n} |f|^2 \, dm = 1$,
- $\int_{\mathbb{R}^n \setminus B(\xi,R)} |f|^2 \, dm < \varepsilon$,
- $\sup_{y \in B(\xi,R)} |f(y)|^2 \leq C'$.

Proof. Given $\varepsilon > 0$. We consider the function $f(x) = \frac{1}{\sqrt{m(K)}} \hat{\chi_K}(x - \xi) \in PW_K$. Let us see that f verify the conditions.

Applying the Plancherel theorem, we obtain the first property

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dm(x) = \frac{1}{m(K)} \int_{R^n} |\hat{\chi_K}(x - \xi)|^2 \, dm(x) = \frac{1}{m(K)} \int_{R^n} |\chi_K(x) e^{i\xi x}|^2 \, dm(x) = 1.$$

Moreover, as we can see

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(\xi,R)} |f(x)|^2 \, dm(x) &= \frac{1}{m(K)} \int_{R^n \setminus B(\xi,R)} |\hat{\chi_K}(x - \xi)|^2 \, dm(x) \\ &= \frac{1}{m(K)} \int_{R^n \setminus B(0,R)} |\hat{\chi_K}(y)|^2 \, dm(y) \to 0 \text{ when } R \to \infty. \end{aligned}$$

So, for that ε there is $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(\xi,R)} |f(x)|^2 \, dx = \frac{1}{m(K)} \int_{R^n \setminus B(\xi,R)} |\hat{\chi_K}(x - \xi)|^2 \, dx < \varepsilon.$$

The last property is verified because

$$||f||_\infty \leq \frac{1}{(2\pi)^n \sqrt{m(K)}} \int_{\mathbb{R}^n} |\chi_K(x - \xi)| \, dm(x) = \frac{\sqrt{m(K)}}{(2\pi)^n}.$$

□

Now we will prove the Logvinenko-Sereda Theorem of the Paley-Wiener functions with the balls of $\mathbb{C}^n$.

Theorem 4.0.5 (The Logvinenko-Sereda Theorem)**.** *For a measurable set $G \subset \mathbb{R}^n$ the following are equivalent:*

(1) There is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |f|^2 \, dm \leq C \int_G |f|^2 \, dm$$

for every $f \in PW_K$. We will say that G is a norming set.

(2) There is a constant $\delta > 0$ and a radius $R > 0$ such that

$$m(G \cap B(x, R)) \geq \delta m(B(x, R))$$

for all $x \in \mathbb{R}^n$.

Proof. The proof that (1) implies (2) is the easiest. In this proof, we consider a function with a certain properties.

So, given $\varepsilon \leq \frac{1}{2C}$ and applying Lemma 4.0.4 there is a radius $R > 0$ such that for all balls $B(\xi, R)$ there is a function $f_{B(\xi,R)}$ verifying the properties of the lemma. Hence, we obtain that

$$\begin{aligned}
m(G \cap B(\xi, R)) &\geq \frac{1}{C'} \int_{G \cap B(\xi,R)} |f_{B(\xi,R)}(x)|^2 \, dm(x) \\
&\geq \frac{1}{C'} \left(\int_G |f_{B(\xi,R)}(x)|^2 \, dm(x) - \int_{\mathbb{R}^n \setminus B(\xi,R)} |f_{B(\xi,R)}(x)|^2 \, dm(x) \right) \\
&\geq \frac{1}{C'} \left(\frac{1}{C} - \varepsilon \right) \geq \frac{1}{2CC'} = \delta m(B(\xi, R))
\end{aligned}$$

where $\delta = \frac{1}{2CC' m(B(\xi,R))}$.

The proof that (2) implies (1) is the difficult one. We will use the followings lemmas to facilitate the proof. We fix R as in (2). If $f \in PW_K$ and $\lambda \in (0, 1)$ we define the set

$$Ef_\lambda(a, R) = \{z \in B(a, R) \subset \mathbb{C}^n : |f(z)| > \lambda |f(a)|\}$$

and the operator

$$B_\lambda f(a, R) = \frac{1}{m(Ef_\lambda(a, R))} \int_{Ef_\lambda(a,R)} |f|^2 \, d\sigma.$$

Lemma 4.0.6. *If $f \in PW_K$ and $a \in \mathbb{R}^n$, then*

$$\frac{m(Ef_\lambda(a, R))}{m(B(a, R))} \geq \frac{\log\left(\frac{1}{\lambda^2}\right)}{\log \frac{B_\lambda f(a,R)}{|f(a)|^2} + \log\left(\frac{1}{\lambda^2}\right)}.$$

Proof. As we will see the placement and the size of the ball do not matter in the proof of the lemma. So, we consider $a = 0$ and $m(B(a, R')) = 1$. Applying the mean value inequality we have

$$\begin{aligned}
\log |f(0)|^2 &\leq \int_{B(0,R')} \log |f|^2 \, d\sigma = \int_{B(0,R') \setminus Ef_\lambda(0,R')} \log |f|^2 \, d\sigma + \int_{Ef_\lambda(0,R')} \log |f|^2 \, d\sigma \\
&\leq [1 - m(Ef_\lambda(0, R'))] \log \lambda^2 |f(0)|^2 + m(Ef_\lambda(0, R')) \frac{1}{m(Ef_\lambda(0, R'))} \int_{Ef_\lambda(0,R')} \log |f|^2 \, d\sigma.
\end{aligned}$$

By the concavity of log

$$\log|f(0)|^2 \leq [1-m(Ef_\lambda(0,R'))]\log\lambda^2|f(0)|^2 + m(Ef_\lambda(0,R'))\log B_\lambda f(0,R').$$

Subtracting $\log|f(0)|^2$ from both sides

$$0 \leq [1-m(Ef_\lambda(0,R'))]\log\lambda^2 + m(Ef_\lambda(0,R'))\log\frac{B_\lambda f(0,R')}{|f(0)|^2}.$$

As $\log\lambda < 0$ and $\log\frac{B_\lambda f(0,R')}{|f(0)|^2} > 0$ then

$$m(Ef_\lambda(0,R')) \geq \frac{\log\left(\frac{1}{\lambda^2}\right)}{\log\frac{B_\lambda f(0,R')}{|f(0)|^2} + \log\left(\frac{1}{\lambda^2}\right)}. \tag{4.0.1}$$

Now, let us see that the lemma it is verifying for all $a \in \mathbb{R}^n$ and for all radius $R > 0$. We know that

$$m(B(a,R)) = \left(\frac{R}{R'}\right)^{2n} m(B(0,R')) = \left(\frac{R}{R'}\right)^{2n}, \quad \forall R > 0.$$

Moreover, knowing that we can deduce that

$$\begin{aligned} m(Ef^*_\lambda(a,R)) &= m\left(\{z \in B(a,R) \ : \ |f^*(z)| > \lambda|f^*(a)|\}\right) \\ &= \left(\frac{R}{R'}\right)^{2n} m\left(\left\{z \in B(0,R') \ : \ |f(z)| > \lambda|f(0)|\right\}\right) \\ &= \left(\frac{R}{R'}\right)^{2n} m(Ef_\lambda(0,R')), \quad \forall R > 0, \end{aligned}$$

and

$$\begin{aligned} Bf^*_\lambda(a,R) &= \frac{1}{m(Ef^*_\lambda(a,R))}\int_{Ef^*_\lambda(a,R)} |f^*(z)|^2 \, d\sigma(z) \\ &= \left(\frac{R'}{R}\right)^{2n} \frac{1}{m(Ef_\lambda(0,R'))}\int_{Ef_\lambda(0,R')} |f(\zeta)|^2 \, \left(\frac{R}{R'}\right)^{2n} d\sigma(\zeta) \\ &= \frac{1}{m(Ef_\lambda(0,R'))}\int_{Ef_\lambda(0,R')} |f(\zeta)|^2 \, d\sigma(\zeta) = Bf_\lambda(0,R') \end{aligned}$$

where $f^*(z) = f\left((z-a)\frac{R'}{R}\right)$.

So, using the equation (4.0.1) we obtain that

$$\frac{m(Ef^*_\lambda(a,R))}{m(B(a,R))} \geq \frac{\log\left(\frac{1}{\lambda^2}\right)}{\log\frac{B_\lambda f^*(a)}{|f^*(a)|^2} + \log\left(\frac{1}{\lambda^2}\right)}.$$

□

The goal of this lemma is to show that $Ef_\lambda(a,R)$ take a large enough fraction of $B(a,R)$ to include some of $G \cap B(a,R)$. This will not be true for all $a \in \mathbb{R}^n$ since $\frac{B_\lambda f(a,R)}{|f(a)|^2}$ may be larger. Therefore, we will use the following two lemmas to show that the set where $\frac{B_\lambda f(a,R)}{|f(a)|^2}$ is not very larger is sufficient.

Lemma 4.0.7. *Let $\varepsilon > 0$ and $f \in PW_K$. Define the set*

$$\mathcal{A} = \left\{ a \in \mathbb{R}^n : |f(a)|^2 < \frac{\varepsilon}{m(B(a,R))} \int_{B(a,R)} |f|^2 \, d\sigma \right\}.$$

Then there is a constant C depending on R such that

$$\int_{\mathcal{A}} |f|^2 \, dm < C\varepsilon \int_{\mathbb{R}^n} |f|^2 \, dm.$$

Proof. For $a \in \mathcal{A}$ we have

$$|f(a)|^2 < \frac{\varepsilon}{m(B(a,R))} \int_{B(a,R)} |f|^2 \, d\sigma = \varepsilon \int_{\mathbb{C}^n} |f(z)|^2 \frac{1}{m(B(a,R))} \chi_{B(a,R)}(z) \, d\sigma(z).$$

Integrating respect to a and applying the Fubini's theorem

$$\begin{aligned} \int_{\mathcal{A}} |f(a)|^2 \, dm(a) &< \varepsilon \int_{\mathbb{C}^n} |f(z)|^2 \left(\int_{\mathcal{A}} \frac{1}{m(B(a,R))} \chi_{B(a,R)}(z) \, dm(a) \right) d\sigma(z) \\ &= \varepsilon \int_{|\operatorname{Im}(z)|<R} |f(z)|^2 \left(\int_{\mathcal{A}} \frac{1}{m(B(a,R))} \chi_{B(a,R)}(z) \, dm(a) \right) d\sigma(z) \\ &\leq \varepsilon \int_{|\operatorname{Im}(z)|<R} |f(z)|^2 \, d\sigma(z). \end{aligned}$$

Then, applying Remark (4.0.3), we obtain

$$\int_{\mathcal{A}} |f(a)|^2 \, dm(a) < \varepsilon C(R) \int_{\mathbb{R}^n} |f(x)|^2 \, dm(x).$$

□

The only use made of Lemma 4.0.7 is in the proof of the following.

Lemma 4.0.8. *If $\lambda < 1/2$, $\varepsilon \in (0,1)$ and $f \in PW_K$. Define the set*

$$\mathcal{B} = \left\{ a \in \mathbb{R}^n : |f(a)|^2 < \varepsilon^{n+1} B_\lambda f(a,R) \right\}.$$

Then there is a constant C depending on R such that

$$\int_{\mathcal{B}} |f|^2 \, dm < C\varepsilon \int_{\mathbb{R}^n} |f|^2 \, dm.$$

Proof. We write

$$\int_{\mathcal{B}} |f|^2 \, dm = \int_{\mathcal{B}\cap\mathcal{A}} |f|^2 \, dm + \int_{\mathcal{B}\setminus\mathcal{A}} |f|^2 \, dm.$$

We estimate the first integral by the Lemma 4.0.7. For the second we use the Fubini's Theorem as in the Lemma 4.0.7.

$$\begin{aligned} \int_{\mathcal{B}\setminus\mathcal{A}} |f|^2 \, dm &< \varepsilon^{n+1} \int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a))} \left(\int_{Ef_\lambda(a)} |f(z)|^2 \, d\sigma(z) \right) \, dm(a) \\ &= \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a))} \chi_{Ef_\lambda(a)}(z) \, dm(a) \right) \, d\sigma(z). \end{aligned}$$

Now we will show that any ball $B(a,R)$ at a contains a concentric ball B' of volume $C_{2n}(\sqrt{\varepsilon}/4\sqrt{n}\Gamma_0)^{2n}$, where $C_{2n} = \pi^n/\Gamma(n+1)$. If $f \in PW_K$ and

$$|f(a)|^2 \geq \varepsilon \frac{1}{m(B(a,R))} \int_{B(a,R)} |f|^2 \, d\sigma$$

then $|f(z)| > \frac{1}{2}|f(a)| > \lambda |f(a)|$ on B'.

Without loss of generality we can assume that $a = 0$ and

$$\frac{1}{m(B(0,R))} \int_{B(0,R)} |f|^2 \, d\sigma = 1.$$

Then $|f(0)| \geq \sqrt{\varepsilon}$. Moreover, there is a constant Γ_0 such that $|f(z)| < \Gamma_0$ on the set $|z| = \frac{1}{2}$.

Assuming $|z| < 1/4$ we have that

$$|f(z) - f(0)| < |z| \cdot \sup_{|z|<1/4} |\nabla f(z)|$$

since

$$f(z) - f(0) = \int_0^1 \frac{d}{dt}(f(zt)) \, dt.$$

Moreover, as $f \in PW_K$ we have that f holomorphic in $\mathbb{C}^n$. Then applying the Cauchy's formula to each component, we obtain that

$$\left| \frac{\partial f}{\partial z_k} \right| \leq \frac{\Gamma_0}{2\pi} \left| \int_{|z_k|=1/4} \frac{1}{z_k^2} \, dz_k \right| \leq 2\Gamma_0.$$

Thus, $|\nabla f| \leq 2\sqrt{n}\Gamma_0$ in $|z| < 1/4$ and so

$$|f(0)| - |f(z)| \leq |f(z) - f(0)| < 2\sqrt{n}\Gamma_0 |z| \quad \text{ for } |z| < 1/4.$$

Hence, if we consider $|z| < \sqrt{\varepsilon}/4\sqrt{n}\Gamma_0$ we obtain that

$$|f(z)| \geq |f(0)| - \sqrt{\varepsilon}/2 > |f(0)|/2$$

on a ball of volume $C_{2n}(\sqrt{\varepsilon}/4\sqrt{n}\Gamma_0)^{2n}$, where $C_{2n} = \pi^n/\Gamma(n+1)$. Translating this to the ball $B(a,R)$ we obtain other ball of volume $C_{2n}(\sqrt{\varepsilon}/4\sqrt{n}\Gamma_0)^{2n}$ contained in $Ef_\lambda(a,R)$.

Therefore, we have that

$$\begin{aligned}
\int_{\mathcal{B}\setminus\mathcal{A}} |f|^2 \, dm &< \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a,R))} \chi_{Ef_\lambda(a,R)}(z) \, dm(a) \right) \, d\sigma(z) \\
&\leq \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{(4\sqrt{n}\Gamma_0 R)^{2n}}{\varepsilon^n m(B(a,R))} \chi_{B(a,R)}(z) \, dm(a) \right) \, d\sigma(z) \\
&\leq \varepsilon^{n+1} \int_{|\operatorname{Im}(z)|<R} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{(4\sqrt{n}\Gamma_0 R)^{2n}}{\varepsilon^n m(B(a,R))} \chi_{B(a,R)}(z) \, dm(a) \right) \, d\sigma(z) \\
&< (4\sqrt{n}\Gamma_0 R)^{2n} \varepsilon \int_{|\operatorname{Im}(z)|<R} |f(z)|^2 \, d\sigma(z).
\end{aligned}$$

Applying the Remark 4.0.3, we obtain that

$$\int_{\mathcal{B}\setminus\mathcal{A}} |f|^2 \, dm < C\varepsilon \int_{\mathbb{R}^n} |f|^2 \, dm.$$

$\square$

Hence, we have proved the lemma.

Let $\mathcal{F} = \mathbb{R}^n \setminus \mathcal{B} = \{a \in \mathbb{R}^n : |f(a)|^2 \geq \varepsilon^{n+1} B_\lambda f(a)\}$.

We choose ε such that $\varepsilon C < 1/2$ we have

$$\int_{\mathbb{R}^n} |f|^2 \, dm < 2 \int_{\mathcal{F}} |f|^2 \, dm$$

since

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^2 \, dm = \int_{\mathcal{F}} |f|^2 \, dm + \int_{\mathcal{B}} |f|^2 \, dm &< \int_{\mathcal{F}} |f|^2 \, dm + C\varepsilon \int_{\mathbb{R}^n} |f|^2 \, dm \\
&< \int_{\mathcal{F}} |f|^2 \, dm + \frac{1}{2} \int_{\mathbb{R}^n} |f|^2 \, dm
\end{aligned}$$

by lemma 4.0.8.

For $a \in \mathcal{F}$ we have $\frac{B_\lambda f(a)}{|f(a)|^2} \leq \frac{1}{\varepsilon^{n+1}}$. So if we choose $\lambda < \varepsilon^{(n+1)/\delta_0}$ we get

$$\frac{m(Ef_\lambda(a,R))}{m(B(a,R))} > \frac{(2/\delta_0)\log(1/\varepsilon^{n+1})}{\log(1/\varepsilon^{n+1}) + (2/\delta_0)\log(1/\varepsilon^{n+1})} > 1 - \delta_0/2.$$

Consequently, (2) implies for $a \in \mathcal{F}$

$$m(G \cap Ef_\lambda(a,R)) > \frac{\delta_0}{2} m(B(a,R))$$

where λ depend of R and δ_0. So,

$$\frac{1}{m(B(a,R))}\int_G \chi_{B(a,R)}(x)|f(x)|^2\ dm(x) > \frac{1}{2}\delta_0\lambda^2|f(a)|^2, \quad a \in \mathcal{F}$$

because

$$\frac{1}{m(B(a,R))}\int_G \chi_{B(a,R)}(x)|f(x)|^2\ dm(x) > \frac{\lambda^2|f(a)|^2}{m(B(a,R))}\int_G \chi_{Ef_\lambda(a,R)}(x)\ dm(x)$$
$$> \frac{\lambda^2|f(a)|^2 m(Ef_\lambda(a,R)\cap G)}{m(B(a,R))} > \frac{\delta_0\lambda^2|f(a)|^2}{2}.$$

Integrating over $\mathcal{F}$

$$\int_G |f(x)|^2\left(\int_{\mathcal{F}}\frac{1}{m(B(a,R))}\chi_{B(a,R)}(x)\ dm(a)\right)\ dm(x) \geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}}|f(a)|^2\ dm(a).$$

Analogous to previous lemmas

$$\int_G |f(x)|^2\ dm(x) \geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}}|f(a)|^2\ dm(a).$$

As we know

$$\int_{\mathbb{R}^n}|f|^2\ dm < 2\int_{\mathcal{F}}|f|^2\ dm.$$

Finally,

$$\int_G |f|^2\ dm \geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}}|f|^2\ dm > \frac{1}{4}\delta_0\lambda^2\int_{\mathbb{R}^n}|f|^2\ dm.$$

$\square$

Chapter 5

Classical Fock space

In this chapter we will give a description of the norming set G for the Classical Fock space. For that, we will show an original proof with the same structure of the previous chapter. However, here we use the article [6] to understand the Classical Fock space.

Now, we define this functional space.

Definition 5.0.1. We say that $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ if $f \in \mathcal{H}(\mathbb{C}^n)$ and

$$\|f\|^2 := \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z) < \infty.$$

That is, we define the Fock space as

$$\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n) = \{f \in \mathcal{H}(\mathbb{C}^n) : \|f\| < \infty\}.$$

Remark 5.0.2. The Fock space $\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ is a Hilbert space with reproductive kernel

$$K(z,w) := \sum_{\alpha\in\mathbb{N}_0^n} \varphi_\alpha(z)\varphi_\alpha(\overline{w}) = \sum_{\alpha\in\mathbb{N}_0^n} \frac{(z\overline{w})^\alpha}{\alpha!} = e^{z\cdot\overline{w}}$$

where a ortonormal basis of this space is $\{\varphi_\alpha\}_{\alpha\in\mathbb{N}_0^n} = \left\{\dfrac{z^\alpha}{\sqrt{\alpha!}}\right\}_{\alpha\in\mathbb{N}_0^n}$. Notice that we are using the multi-index notation.

We will need the following two results to prove the main theorem of this chapter.

Proposition 5.0.3. *Let $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and $w \in \mathbb{C}$. The translations*

$$\mathcal{T}_w(f) = e^{2\overline{w}\cdot z-|w|^2} f(z-w)$$

act isometrically in $\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$.

Proof. We consider a function $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and we fix $w \in \mathbb{C}^n$. So, we have the function

$$g_w(z) = e^{2\overline{w}\cdot z - |w|^2} f(z-w).$$

Now, let us compute $\|g_w\|^2$.

$$\begin{aligned}
\|g_w\|^2 &= \int_{\mathbb{C}^n} |g_w(z)|^2 e^{-2|z|^2}\, dm(z) = e^{-2|w|^2} \int_{\mathbb{C}^n} |e^{2\overline{w}\cdot z} f(z-w)|^2 e^{-2|z|^2}\, dm(z) \\
&= \int_{\mathbb{C}^n} e^{-2|w|^2 + 4\Re(\overline{w}\cdot z) - 2|z|^2} |f(z-w)|^2\, dm(z) = \int_{\mathbb{C}^n} |f(z-w)|^2 e^{-2|z-w|^2}\, dm(z) \\
&= \int_{\mathbb{C}^n} |f(\eta)|^2 e^{-2|\eta|^2}\, dm(\eta) = \|f\|^2.
\end{aligned}$$

□

Lemma 5.0.4. *Let $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and a ball $B(a,r) \subset \mathbb{C}^n$, then we have that there is a constant $\Gamma_0 > 0$ such that $|f(z)|^2 e^{-2|z|^2} < \Gamma_0$ for all $z \in \partial B(a,r)$.*

Proof. Let us consider the holomorphic function $F(z) = e^{|w|^2 - 2z\cdot\overline{w}} f(z)$ where $w \in \partial B(a,r)$. So, applying the mean value inequality we obtain that

$$\begin{aligned}
|f(w)|^2 e^{-2|w|^2} = |F(w)|^2 &\le \frac{1}{m(B(w,2r))} \int_{B(w,2r)} |F(z)|^2\, dm(z) \\
&= \frac{1}{m(B(w,2r))} \int_{B(w,2r)} |f(z)|^2 e^{-2|z|^2} e^{2|z-w|^2}\, dm(z) \\
&\le \frac{1}{m(B(w,2r))} \int_{B(w,2r)} |f(z)|^2 e^{-2|z|^2}\, dm(z) < \infty
\end{aligned}$$

for $w \in \partial B(a,r)$. □

The following lemma will be useful to prove the necessity of Logvinenko-Sereda Theorem.

Lemma 5.0.5. *Given $\varepsilon > 0$, there is a constant $C' > 0$ and a radius $R > 0$ depending on ε such that for all ball $B(\xi, R)$ with $\xi \in \mathbb{C}^n$, there is a $\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$-function $f = f_{B(\xi,R)}$ such that*

- $\int_{\mathbb{C}^n} |f|^2 e^{-2|z|^2}\, dm(z) = 1,$
- $\int_{\mathbb{C}^n \setminus B(\xi,R)} |f|^2 e^{-2|z|^2}\, dm(z) < \varepsilon,$
- $\sup_{y \in B(\xi,R)} |f(y)|^2 e^{-2|y|^2} \le C'.$

Proof. Given $\varepsilon > 0$ and fixed $\xi \in \mathbb{C}^n$. We consider the function

$$g_\xi(z) := \sqrt{(2\pi^{-1})^n} e^{2\overline{\xi}\cdot z - |\xi|^2} K(z-\xi, 0) = \sqrt{(2\pi^{-1})^n} e^{2\overline{\xi}\cdot z - |\xi|^2}$$

where K is the reproductive kernel of the Fock space $\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and $\xi \in \mathbb{C}^n$.

Now, let us compute $\|g_\xi\|^2$

$$\begin{aligned}\|g_\xi\|^2 &= \int_{\mathbb{C}^n} |g_\xi(z)|^2 e^{-|z|^2}\, dm(z) = (2\pi^{-1})^n \int_{\mathbb{C}^n} |e^{2\bar{\xi}\cdot z - |\xi|^2}|^2 e^{-2|z|^2}\, dm(z)\\ &= (2\pi^{-1})^n \int_{\mathbb{C}^n} e^{-2|z|^2-2|\xi|^2+4\Re(\overline{w}\cdot\xi)}\, dm(z) = (2\pi^{-1})^n \int_{\mathbb{C}^n} e^{-2|z-\xi|^2}\, dm(z)\\ &= (2\pi^{-1})^n \int_{\mathbb{C}^n} e^{-2|\eta|^2}\, dm(\eta) = (2/\pi)^n \left(\int_{\mathbb{C}^n} e^{-2|\eta|^2}\, dm(\eta)\right)^n = (2/\pi)^n(\pi/2)^n = 1\end{aligned}$$

where we use the change of variable $\eta = z - \xi$. Hence, we have that $\|g_\xi\|^2 = 1$.

Moreover, we can see that

$$\begin{aligned}\int_{\mathbb{C}^n\setminus B(\xi,R)} |g_\xi(z)|^2 e^{-|z|^2}\, dm(z) &= \left(\frac{2}{\pi}\right)^n \int_{\mathbb{C}^n\setminus B(\xi,R)} e^{-2|z-\xi|^2}\, dm(z)\\ &= \left(\frac{2}{\pi}\right)^n \int_{\mathbb{C}^n\setminus B(0,R)} e^{-2|\eta|^2}\, dm(\eta)\end{aligned}$$

where we use the change of variable $\eta = z - \xi$.

So, if $R \to \infty$ we have that

$$\int_{\mathbb{C}^n\setminus B(0,R)} e^{-2|\eta|^2}\, dm(\eta) = \left(\frac{\pi}{2}\right)^n - \int_{B(0,R)} e^{-2|\eta|^2}\, dm(\eta) \to 0.$$

Hence, for that $\varepsilon > 0$ there is $R > 0$ such that

$$\int_{\mathbb{C}^n\setminus B(\xi,R)} |g_\xi(z)|^2 e^{-|z|^2}\, dm(z) = 1 - \left(\frac{2}{\pi}\right)^n \int_{B(0,R)} e^{-2|\eta|^2}\, dm(\eta) < \varepsilon.$$

The last property is verified because

$$|g_\xi(y)|^2 e^{-2|y|^2} = \left(\frac{2}{\pi}\right)^n e^{-2|y-\xi|^2} \le \left(\frac{2}{\pi}\right)^n$$

for $y \in B(\xi, R)$. $\square$

Next, we will prove the Logvinenko-Sereda Theorem for the Fock space in several variables.

Theorem 5.0.6 (The Logvinenko-Sereda Theorem)**.** *For a measurable set $G \subset \mathbb{C}^n$ the following are equivalent:*

(1) There is a constant $C > 0$ such that

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z) \le C \int_G |f(z)|^2 e^{-2|z|^2}\, dm(z)$$

for every $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$. We will say that G is a norming set.

(2) There is a constant $\delta > 0$ and a radius $R > 0$ such that

$$m(G \cap B(z,R)) \geq \delta m(B(z,R))$$

for all $z \in \mathbb{C}^n$.

Proof. First, we will start with (1) implies (2). So, given $\varepsilon \leq \frac{1}{2C}$ and applying Lemma 5.0.5 there is a radius $R > 0$ such that for all balls $B(\xi, R)$ there is a function $f_{B(\xi,R)}$ verifying the properties of the lemma. Hence, we obtain that

$$\begin{aligned} m(G \cap B(\xi,R)) &\geq \frac{1}{C'} \int_{G\cap B(\xi,R)} |f_{B(\xi,R)}(z)|^2 e^{-2|z|^2} \, dm(z) \\ &\geq \frac{1}{C'} \left(\int_G |f_{B(\xi,R)}(z)|^2 e^{-2|z|^2} \, dm(z) - \int_{\mathbb{C}^n \setminus B(\xi,R)} |f_{B(\xi,R)}(z)|^2 e^{-2|z|^2} \, dm(z) \right) \\ &\geq \frac{1}{C'} \left(\frac{1}{C'} - \varepsilon \right) \geq \frac{1}{2CC'} = \delta m(B(\xi,R)) \end{aligned}$$

where $\delta = \frac{1}{2CC'm(B(\xi,R))}$.

Now, let us prove that (2) implies (1). We will use the following lemma to prove this implication. We fix R as in (2). If $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and $\lambda \in (0,1)$ we define the set

$$Ef_\lambda(a,R) = \{z \in B(a,R) \subset \mathbb{C}^n \; : \; |f(z)|e^{-|z|^2} > \lambda |f(a)| e^{-|a|^2}\}$$

and the operator

$$B_\lambda f(a,R) = \frac{1}{m(Ef_\lambda(a,R))} \int_{Ef_\lambda(a,R)} |f(z)|^2 e^{-2|z|^2} \, dm(z).$$

Lemma 5.0.7. *If $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$, $a \in \mathbb{C}^n$ and $\lambda < e^{-R^2}$, then*

$$\frac{m(Ef_\lambda(a,R))}{m(B(a,R))} \geq \frac{\log\left(\frac{1}{\lambda^2 e^{2R^2}}\right)}{\log\left(\frac{e^{2R^2} B_\lambda f(a,R)}{|f(a)|^2 e^{-2|a|^2}}\right) + \log\left(\frac{1}{\lambda^2 e^{2R^2}}\right)}.$$

Proof. Let us consider the holomorphic function $F(z) = e^{|w|^2 - 2z\cdot\overline{w}} f(z)$. So, applying the mean value inequality we have

$$\begin{aligned} \log|F(w)|^2 &\leq \frac{1}{m(B(w,R))} \int_{B(w,R)} \log(|F(z)|^2) \, dm(z) \\ &= \frac{1}{m(B(w,R))} \int_{B(w,R)\setminus Ef_\lambda(w,R)} \log(|F(z)|^2) \, dm(z) \\ &+ \frac{1}{m(B(w,R))} \int_{Ef_\lambda(w,R)} \log(|F(z)|^2) \, dm(z) \\ &\leq \frac{m(B(w,R)) - m(Ef_\lambda(w,R))}{m(B(w,R))} \log \lambda^2 |F(w)|^2 e^{2R^2} \\ &+ \frac{m(Ef_\lambda(w,R))}{m(B(w,R))} \frac{1}{m(Ef_\lambda(w,R))} \int_{Ef_\lambda(w,R)} \log(|F(z)|^2) \, dm(z). \end{aligned}$$

By the concavity of log

$$\log|F(w)|^2 \leq \frac{m(B(w,R)) - m(Ef_\lambda(w,R))}{m(B(w,R))}\log\lambda^2|F(w)|^2e^{2R^2} + \frac{m(Ef_\lambda(w,R))}{m(B(w,R))}\log(e^{2R^2}B_\lambda f(w,R)).$$

Subtracting $\log|F(w)|^2$ from both sides

$$0 \leq [m(B(w,R)) - m(Ef_\lambda(w,R))]\log\lambda^2e^{2R^2} + m(Ef_\lambda(w,R))\log\frac{e^{2R^2}B_\lambda f(w,R)}{|F(w)|^2}.$$

As $\log(\lambda e^{R^2}) < 0$ and $\log\frac{e^{2R^2}B_\lambda f(w,R)}{|F(w)|^2} > 0$ then

$$\frac{m(Ef_\lambda(w,R))}{m(B(w,R))} \geq \frac{\log\left(\frac{1}{\lambda^2e^{2R^2}}\right)}{\log\frac{e^{2R^2}B_\lambda f(w,R')}{|f(w)|^2e^{-2|w|^2}} + \log\left(\frac{1}{\lambda^2e^{2R^2}}\right)}. \tag{5.0.1}$$

□

The aim of this lemma is to show that $Ef_\lambda(a,R)$ takes a large enough fraction of $B(a,R)$ to include some of $G\cap B(a,R)$. This will be true for all $a\in\mathbb{C}^n$ because $\frac{e^{2R^2}B_\lambda f(a)}{|f(a)|^2e^{-2|a|^2}}$ may be very larger. Hence, we will us the following two lemmas to show that the set where $\frac{e^{2R^2}B_\lambda f(a)}{|f(a)|^2e^{-2|a|^2}}$ is not very larger is sufficient.

Lemma 5.0.8. *Let $\varepsilon > 0$ and $f\in\mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$. Define the set*

$$\mathcal{A} = \left\{a\in\mathbb{C}^n : |f(a)|^2e^{-2|a|^2} < \frac{\varepsilon}{m(B(a,R))}\int_{B(a,R)}|f(z)|^2e^{-2|z|^2}\,dm(z)\right\}.$$

Then there is a constant depending on R such that

$$\int_{\mathcal{A}}|f(z)|^2e^{-2|z|^2}\,dm(z) < C\varepsilon\int_{\mathbb{C}^n}|f(z)|^2e^{-2|z|^2}\,dm(z).$$

Proof. For $a\in\mathcal{A}$ we have

$$\begin{aligned}|f(a)|^2e^{-2|z|^2} &< \frac{\varepsilon}{m(B(a,R))}\int_{B(a,R)}|f(z)|^2e^{-2|z|^2}\,dm(z)\\ &= \frac{\varepsilon}{m(B(a,R))}\int_{\mathbb{C}^n}|f(z)|^2e^{-2|z|^2}\chi_{B(a,R)(z)}\,dm(z).\end{aligned}$$

Integrating respect to a and applying the Fubini's theorem

$$\begin{aligned}\int_{\mathcal{A}}|f(a)|^2e^{-2|z|^2}\,dm(a) &< \varepsilon\int_{\mathbb{C}^n}|f(z)|^2e^{-2|z|^2}\left(\int_{\mathcal{A}}\frac{1}{m(B(a,R))}\chi_{B(a,R)}(z)\,dm(a)\right)\,dm(z)\\ &\leq \varepsilon\int_{\mathbb{C}^n}|f(z)|^2e^{-2|z|^2}\,dm(z).\end{aligned}$$

□

The only use made of Lemma 5.0.8 is in the proof of the following.

Lemma 5.0.9. *If $\lambda < 1/2$, $\varepsilon \in (0,1)$ and $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$. Define the set*

$$\mathcal{B} = \{a \in \mathbb{C} : |f(a)|^2 e^{-2|z|^2} < \varepsilon^{n+1} B_\lambda f(a,R)\}.$$

Then there is a constant C depending on R such that

$$\int_{\mathcal{B}} |f(z)|^2 e^{-2|z|^2}\, dm(z) < C\varepsilon \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z).$$

Proof. We write

$$\int_{\mathcal{B}} |f(z)|^2 e^{-2|z|^2}\, dm(z) = \int_{\mathcal{B}\cap\mathcal{A}} |f(z)|^2 e^{-2|z|^2}\, dm(z) + \int_{\mathcal{B}\setminus\mathcal{A}} |f(z)|^2 e^{-2|z|^2}\, dm(z).$$

We estimate the first integral by the Lemma (5.0.8). Analogous to its proof we have that

$$\begin{aligned}\int_{\mathcal{B}\setminus\mathcal{A}} |f(z)|^2 e^{-2|z|^2}\, dm(z) &< \varepsilon^{n+1} \int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a,R))} \left(\int_{Ef_\lambda(a,R)} |f(z)|^2 e^{-2|z|^2}\, dm(z) \right) dm(a) \\ &= \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2} \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a,R))} \chi_{Ef_\lambda(a,R)}(z)\, dm(a) \right) dm(z).\end{aligned}$$

Now we will show that any ball $B(a,R)$ at a contains a concentric ball B' of volume $C_{2n}(\sqrt{\varepsilon}/8\Gamma_0)^{2n}$, where $C_{2n} = \pi^n/\Gamma(n+1)$. If $f \in \mathcal{F}^2_{2|z|^2}(\mathbb{C}^n)$ and

$$|f(z)|^2 e^{-2|z|^2} \geq \varepsilon \frac{1}{m(B(a,R))} \int_{B(a,R)} |f(z)|^2 e^{-2|z|^2}\, dm(z)$$

then $|f(z)| e^{-|z|^2} > \frac{1}{2} |f(a)| e^{-|a|^2} > \lambda |f(a)| e^{-|a|^2}$.

Without loss of generality we can assume that $a = 0$ and

$$\frac{1}{m(B(0,R))} \int_{B(a,R)} |f(z)|^2 e^{-2|z|^2}\, dm(z) = 1.$$

Then $|f(0)| \geq \sqrt{\varepsilon}$. Moreover, applying the same idea of the Lemma 5.0.4 there is a constant Γ_0 such that $|F(z)| < \Gamma_0$ on the set $|z| = \frac{1}{2}$.

Fixing $|w| < 1/4$ and assuming $|z| < 1/4$ we have that

$$|f(0)| - |e^{|w|^2 - 2z\overline{w}} f(z)| \leq |F(0) - F(z)| \leq 4\Gamma_0 |z|.$$

So, we obtain

$$|f(0)| - |e^{-|w|^2} f(w)| \leq 4\Gamma_0 |w|$$

for $|w| < 1/4$. Hence, if we consider $|w| < \sqrt{\varepsilon}/8\Gamma_0$ we obtain that

$$|e^{-|w|^2} f(w)| \geq |f(0)| - \sqrt{\varepsilon}/2 > |f(0)|/2$$

on a ball of volume $C_{2n}(\sqrt{\varepsilon}/8\Gamma_0)^{2n}$, where $C_{2n} = \pi^n/\Gamma(n+1)$. Translating this to the ball $B(a,R)$ we obtain other ball of volume $C_{2n}(\sqrt{\varepsilon}/8\Gamma_0)^{2n}$ contained in $Ef_\lambda(a,R)$.

Therefore, we have that

$$\begin{aligned}
\int_{\mathcal{B}\setminus\mathcal{A}} |f(z)|^2 e^{-2|z|^2}\, dm(z) &< \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2} \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a,R))} \chi_{Ef_\lambda(a,R)}(z)\, dm(a) \right) dm(z) \\
&< \varepsilon^{n+1} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2} \left(\int_{\mathcal{B}\setminus\mathcal{A}} C_{2n}^{-1} \left(\frac{8\Gamma_0}{\sqrt{\varepsilon}} \right)^{2n} \chi_{B(a,R)}(z)\, dm(a) \right) dm(z) \\
&< \varepsilon (8\Gamma_0 R)^{2n} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z).
\end{aligned}$$

□

Let $\mathcal{F} = \mathbb{C}^n \setminus \mathcal{B} = \{a \in \mathbb{C}^n : |f(a)|^2 e^{-2|z|^2} \geq \varepsilon^{n+1} B\lambda f(a,R)\}$.
We choose ε such that $\varepsilon C < 1/2$ we have

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z) < 2 \int_{\mathcal{F}} |f(z)|^2 e^{-2|z|^2}\, dm(z)$$

since

$$\begin{aligned}
\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z) &= \int_{\mathcal{F}} |f(z)|^2 e^{-2|z|^2}\, dm(z) + \int_{\mathcal{B}} |f(z)|^2 e^{-2|z|^2}\, dm(z) \\
&< \int_{\mathcal{B}} |f(z)|^2 e^{-2|z|^2}\, dm(z) + C\varepsilon \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z) \\
&< \int_{\mathcal{B}} |f(z)|^2 e^{-2|z|^2}\, dm(z) + \frac{1}{2} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\, dm(z).
\end{aligned}$$

For $a \in \mathcal{F}$ we have $\frac{B_\lambda f(a,R)}{|f(z)|^2 e^{-2|z|^2}} \leq \frac{1}{\varepsilon^{n+1}}$. So, if we choice $\lambda < e^{-R^2} \left(\frac{\varepsilon^{n+1}}{e^{2R^2}} \right)^{1/\delta_0}$ we get

$$\frac{m(Ef_\lambda(a,R))}{m(B(a,R))} > \frac{(2/\delta_0)\log(e^{2R^2}/\varepsilon^{(n+1)})}{\log(e^{2R^2}/\varepsilon^{(n+1)}) + (2/\delta_0)\log(e^{2R^2}/\varepsilon^{(n+1)})} > 1 - \delta_0/2.$$

Consequently, (2) implies for $a \in \mathcal{F}$

$$m(G \cap Ef_\lambda(a,R)) > \frac{\delta_0}{2} m(B(a,R))$$

where λ depend of R and δ_0. So,

$$\frac{1}{m(B(a,R))} \int_G \chi_{B(a,R)}(z) |f(z)|^2 e^{-2|z|^2}\, dm(z) > \frac{1}{2}\delta_0 \lambda^2 |f(a)|^2 e^{-2|a|^2}, \quad a \in \mathcal{F}$$

because

$$\frac{1}{m(B(a,R))}\int_G \chi_{B(a,R)}(z)|f(z)|^2 e^{-2|z|^2}\ dm(z) > \frac{\lambda^2|f(a)|^2 e^{-2|a|^2}}{m(B(a,R))}\int_G \chi_{Ef_\lambda(a,R)}(z)\ dm(z)$$
$$> \frac{\lambda^2|f(a)|^2 e^{-2|a|^2} m(Ef_\lambda(a,R)\cap G)}{m(B(a,R))} > \frac{\delta_0\lambda^2|f(a)|^2 e^{-2|a|^2}}{2}.$$

Integrating over $\mathcal{F}$

$$\int_G |f(z)|^2 e^{-2|z|^2}\left(\int_{\mathcal{F}} \frac{1}{m(B(a,R))}\chi_{B(a,R)}(z)\ dm(a)\right)\ dm(z) \geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}} |f(a)|^2 e^{-2|a|^2}\ dm(a).$$

Analogous to previous lemmas

$$\int_G |f(z)|^2 e^{-2|z|^2}\ dm(z) \geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}} |f(a)|^2 e^{-2|a|^2}\ dm(a).$$

As we know

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\ dm < 2\int_{\mathcal{F}} |f(z)|^2 e^{-2|z|^2}\ dm.$$

Finally,

$$\begin{aligned}\int_G |f(z)|^2 e^{-2|z|^2}\ dm(z) &\geq \frac{1}{2}\delta_0\lambda^2\int_{\mathcal{F}} |f(z)|^2 e^{-2|z|^2}\ dm(z)\\ &> \frac{1}{4}\delta_0\lambda^2\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2}\ dm(z).\end{aligned}$$

□

Chapter 6

Space of polynomials

In this chapter we will give a description of the norming set G for the space of polynomials. For that, we will use the same structure of the last chapters to give an original proof. Here we use the article [5] to understand the space of polynomials.

So, we define this functional space.

Definition 6.0.1. We say that $f \in \mathcal{P}_n(\mathbb{C})$ if $f \in \mathbb{C}[x]$ and $\deg f \leq n$. That is,

$$\mathcal{P}_n(\mathbb{C}) = \{f \in \mathbb{C}[x] \quad | \quad \deg f \leq n\}.$$

Remark 6.0.2. $f \in \mathcal{P}_n(\mathbb{C})$ is a Hilbert space with the norm

$$\|f\| = \int_{\mathbb{T}} |f(z)|^2 \, dm(z)$$

and a orthonormal basis $\left\{\frac{1}{\sqrt{2\pi}}, \frac{z}{\sqrt{2\pi}}, \ldots, \frac{z^n}{\sqrt{2\pi}}\right\}$.

The following two results will be useful in the proof the main theorem of this chapter.

Proposition 6.0.3. *If $f \in \mathcal{P}_n(\mathbb{C})$, then there is a constant $\Gamma_0 > 0$ such that $|f| < \Gamma_0$ for $|z - a| = r$.*

Proof. We consider the polynomial

$$f = \sum_{k=0}^{n} a_k z^k, \quad a_j \in \mathbb{C}.$$

As f is holomorphic in $\mathbb{C}$, we can take the power series of f in a

$$f = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(z - a)^k.$$

So, we obtain that

$$|f| \leq \sum_{k=0}^{n} \left| \frac{f^{(k)}(a)}{k!} \right| r^k = \Gamma_0 < \infty.$$

□

Notice that we will use the measure m on $\mathbb{T}$ and the measure σ on $\mathbb{C}$.

Proposition 6.0.4. *If $f \in \mathcal{P}_n(\mathbb{C})$, then there is a constant C depending on n and R such that*

$$\int_{C_n} |f|^2 \, d\sigma(z) < C(R,n) \int_{\mathbb{T}} |f|^2 \, dm.$$

Proof. We consider the polynomial

$$f = \sum_{k=0}^{n} a_k z^k, \quad a_j \in \mathbb{C}.$$

So, we obtain that

$$\begin{aligned}\int_{C_n} |f(z)|^2 \, d\sigma(z) &= \int_{1-R/n}^{1+R/n} \int_0^{2\pi} |f(re^{i\theta})|^2 \, dm(\theta) \, dm(r) = 2\pi \sum_{j=0}^{n} |a_j|^2 \int_{1-R/n}^{1+R/n} r^j \, dm(r) \\ &+ \sum_{j\neq k} a_j \overline{a}_k \left(\int_{1-R/n}^{1+R/n} r^{j-k} \, dm(r) \right) \left(\int_0^{2\pi} e^{i(j-k)\theta} \, dm(\theta) \right) \\ &\leq 2\pi \left(1 + \frac{R}{n}\right)^n \frac{2R}{n} \sum_{j=0}^{n} |a_j|^2 = \left(1 + \frac{R}{n}\right)^n \frac{2R}{n} \int_{\mathbb{T}} |f|^2 \, dm(z).\end{aligned}$$

$\square$

We will need the following lemma to prove the necessity of Logvinenko-Sereda Theorem.

Lemma 6.0.5. *Given $\varepsilon > 0$, there is a constant $C' > 0$, a radius $R > 0$ and $n_0 \in \mathbb{N}$ such that there is for all arc $I(\xi, R/n)$ with $\xi \in \mathbb{T}$ where $n > n_0$, there is polynomial $f_{I(\xi,R/n)}$ on $\mathbb{T}$ such that*

- $\int_{\mathbb{T}} |f|^2 \, dm = 1$,
- $\int_{\mathbb{T}\setminus I(\xi,R/n)} |f|^2 \, dm < \varepsilon$,
- $\sup_{y \in I(\xi,R/n)} |f(y)|^2 \leq C' \cdot (n+1)$

Proof. Given $\varepsilon > 0$. We consider the reproducing kernel of the Hilbert space $\mathcal{P}_n(\mathbb{C})$

$$K(z,w) = \sum_{j=0}^{n} \frac{z^j \overline{w}^j}{2\pi} = \frac{1}{2\pi} \frac{(z\overline{w})^{n+1} - 1}{z\overline{w} - 1}$$

for $w \in \mathbb{T}$ fixed.

Let us see the first property.

$$\int_{\mathbb{T}} |K(z,w)|^2 \, dm(z) = \frac{1}{2\pi} \sum_{j=0}^{n} |w|^2 = \frac{n+1}{2\pi}$$

by orthogonality of elements of the basis.

Now, we consider the new function

$$f_w(z) = \sqrt{\frac{2\pi}{n+1}} K(z,w).$$

This function f_w verify the last property too, since

$$|f_w|^2 \leq \frac{2\pi}{n+1}\left(\frac{n+1}{2\pi}\right)^2 = \frac{n+1}{2\pi}.$$

Now, let us see the second property.

As we know that

$$f_w(z) = \frac{1}{\sqrt{2\pi(n+1)}}\frac{1}{w^n}\frac{z^{n+1}-w^{n+1}}{z-w}.$$

So, we obtain that

$$|f_w(z)|^2 = \frac{1}{2\pi(n+1)}\left|\frac{z^{n+1}-w^{n+1}}{z-w}\right|^2.$$

Without loss of generality, we assume that $w = 1$ since for other values of w we can take rotations. So, we obtain

$$|f_1(z)|^2 = \frac{1}{2\pi(n+1)}\left|\frac{z^{n+1}-1}{z-1}\right|^2.$$

Hence, we have

$$\int_{\mathbb{T}\setminus I(\xi,R/n)} |f_1(z)|^2 \, dm(z) = \frac{1}{2\pi(n+1)}\int_{\theta_0}^{2\pi-\theta_0}\left|\frac{e^{(n+1)\theta i}-1}{e^{\theta i}-1}\right|^2 \, dm(\theta)$$

where $\theta_0 = 2\arcsin\left(\frac{R}{2n}\right)$. So,

$$\begin{aligned}
\int_{\mathbb{T}\setminus I(\xi,R/n)} |f_1(z)|^2 \, dm(z) &\leq \frac{4}{2\pi(n+1)}\int_{\theta_0}^{2\pi-\theta_0}\frac{1}{|e^{\theta i}-1|^2} \, dm(\theta) \\
&= \frac{1}{\pi(n+1)}\int_{\theta_0}^{2\pi-\theta_0}\frac{1}{1-\cos(\theta)} \, dm(\theta) \\
&= \frac{1}{\pi(n+1)}\left[\cot\left(\frac{\theta_0}{2}\right) - \cot\left(\frac{2\pi-\theta_0}{2}\right)\right] \\
&= \frac{4n\sqrt{1-\frac{R^2}{4n^2}}}{\pi(n+1)R} \leq \frac{4n}{\pi(n+1)R} < \frac{4}{\pi R} = \varepsilon
\end{aligned}$$

where $R = \frac{4}{\pi\varepsilon}$. Notice that there is $n_0 \geq \frac{R}{2}$ such that the last inequalities holds for $n > n_0$.

$\square$

Now, we will prove the Logvinenko-Sereda Theorem of the space of polynomials.

Theorem 6.0.6 (The Logvinenko-Sereda Theorem)**.** *For a measurable set $E_n \subset \mathbb{T}$ the following are equivalent*

(1) There is a constant $C > 0$ such that

$$\int_{\mathbb{T}} |f_n|^2 \, dm \leq C \int_{E_n} |f|^2 \, dm$$

for every $f \in \mathcal{P}_n(\mathbb{C})$. We will say that E_n is a norming set.

(2) There is a constant $\delta > 0$ and radius $R > 0$ such that

$$m(I(x, R/n) \cap E_n) \geq \delta m(I(x, R/n))$$

for all $x \in \mathbb{T}$.

Proof. The proof that (1) implies (2) is the easiest. In this proof, we consider a function with a certain properties.

So, given $\varepsilon \leq 1/2C$ and applying Lemma 6.0.5 there is a radius $R > 0$ and $n_0 \in \mathbb{N}$ such that for all arcs $I(\xi, R/n)$ with $n > n_0$ there is a function $f_{I(\xi,R/n)}$ verifying the properties of the lemma. Hence, we obtain that

$$\begin{aligned}
m(E_n \cap I(\xi, R/n)) &\geq \frac{1}{C'} \int_{E_n \cap I(\xi,R/n)} |f_{I(\xi,R/n)}(z)|^2 \, dm(z) \\
&\geq \frac{1}{C'} \left(\int_G |f_{I(\xi,R/n)}(z)|^2 \, dm(z) - \int_{\mathbb{T} \setminus I(\xi,R/n)} |f_{I(\xi,R/n)}(z)|^2 \, dm(z) \right) \\
&\geq \frac{1}{C'} \left(\frac{1}{C} - \varepsilon \right) \geq \frac{1}{2CC'(n+1)} = \frac{1}{2CC'} \frac{1}{n+1} \frac{\arcsin(R/2n)}{\arcsin(R/2n)} \\
&\geq \frac{\arcsin(R/2n)}{2CC'\pi} \left(\frac{4}{2+R} \right) = \frac{1}{2CC'\pi(R+2)} m(I(\xi, R/n))
\end{aligned}$$

where $\delta = \frac{1}{2CC'\pi(R+2)}$.

Notice that (2) holds for $n \leq n_0$.

Now, let us prove that (2) implies (1). We will use the following lemmas to facilitate the proof. We fix R as in (2). If $f \in \mathcal{P}_n(\mathbb{C})$ and $\lambda \in (0, 1)$ we define the set

$$Ef_\lambda(a, R/n) = \{z \in D(a, R/n) \subset C_n \quad | \quad |f(z)| > \lambda |f(a)|\}$$

where C_n is the annulus $C_n = \{z \in \mathbb{C} \mid 1 - R/n < |z| < 1 + R/n\}$ and the operator

$$B_\lambda f(a, R/n) = \frac{1}{m(Ef_\lambda(a, R/n))} \int_{Ef_\lambda(a,R/n)} |f|^2 \, d\sigma.$$

Lemma 6.0.7. *If $f \in \mathcal{P}_n(\mathbb{C})$ and $a \in \mathbb{T}$, then*

$$\frac{m(Ef_\lambda(a,R/n))}{m(I(a,R/n))} \geq \frac{\log\left(\frac{1}{\lambda^2}\right)}{\log \frac{B_\lambda f(a,R/n)}{|f(a)|^2} \log\left(\frac{1}{\lambda^2}\right)}.$$

Proof. Applying the mean value we have

$$\begin{aligned}\log|f(w)|^2 &\leq \frac{1}{m(D(w,R/n))}\int_{D(w,R/n)} \log|f|^2 \, d\sigma \\ &= \frac{1}{m(D(w,R/n))}\int_{D(w,R/n)\setminus Ef_\lambda(w,R/n)} \log|f|^2 \, d\sigma \\ &+ \frac{1}{m(D(w,R/n))}\int_{Ef_\lambda(w,R/n)} \log|f|^2 \, d\sigma.\end{aligned}$$

By the concavity of the log

$$\begin{aligned}\log|f(w)|^2 \leq& \frac{m(D(w,R/n)) - m(Ef_\lambda(w,R/n))}{m(D(w,R/n))}\log(\lambda^2|f(w)|^2) \\ &+ \frac{m(Ef_\lambda(w,R/n))}{m(D(w,R/n))}\log Bf_\lambda(w,R/n).\end{aligned}$$

Subtracting $\log|f(w)|^2$ from both sides

$$0 \leq (m(D(w,R/n)) - m(Ef_\lambda(w,R/n)))\log\lambda^2 + m(Ef_\lambda(w,R/n))\log\left(\frac{Bf_\lambda(w,R/n)}{|f(w)|^2}\right).$$

As $\log\lambda < 0$ and $\log\left(\frac{Bf_\lambda(w,R/n)}{|f(w)|^2}\right) > 0$ then

$$\frac{m(Ef_\lambda(w,R/n))}{m(I(w,R/n))} \geq \frac{\log\left(\frac{1}{\lambda^2}\right)}{\log \frac{B_\lambda f(w,R/n)}{|f(w)|^2} \log\left(\frac{1}{\lambda^2}\right)}.$$

□

The goal of this lemma is to show that $Ef_\lambda(a,R/n)$ takes a large enough fraction of $I(a,R/n)$ to include some of $G \cap I(a,R/n)$. This will be true for all $a \in \mathbb{T}$ because $\frac{B_\lambda f(a,R/n)}{|f(a)|^2}$ may be very larger. Hence, we will us the following two lemmas to show that the set where $\frac{B_\lambda f(a,R/n)}{|f(a)|^2}$ is not very larger is sufficient.

Lemma 6.0.8. *Let $\varepsilon > 0$ and $f \in \mathbb{P}_n(\mathbb{C})$. Define the set*

$$\mathcal{A} = \left\{a \in \mathbb{T} \mid |f(a)|^2 < \frac{\varepsilon}{m(D(a,R/n))}\int_{D(a,R/n)} |f|^2 \, d\sigma\right\}.$$

Then there is a constant C depending on R and n such that

$$\int_{\mathcal{A}} |f|^2 \, dm < C\varepsilon \int_{\mathbb{T}} |f|^2 \, dm.$$

Proof. For $a \in \mathcal{A}$ we have

$$|f(a)|^2 < \frac{\varepsilon}{m(D(a,R/n))} \int_{D(a,R/n)} |f|^2 \, d\sigma = \varepsilon \int_{C_n} |f(z)|^2 \frac{1}{m(D(a,R/n))} \chi_{D(a,R/n)}(z) \, d\sigma(z).$$

Integrating respect to a and applying the Fubini's theorem

$$\begin{aligned} \int_{\mathcal{A}} |f(a)|^2 \, dm(a) &< \varepsilon \int_{C_n} |f(z)|^2 \left(\int_{\mathcal{A}} \frac{1}{m(D(a,R/n))} \chi_{D(a,R/n)}(z) \, dm(a) \right) \, d\sigma(z) \\ &\leq 2\varepsilon \cdot \frac{\arcsin(R/n)}{m(D(1,R/n))} \int_{C_n} |f(z)|^2 \, d\sigma(z). \end{aligned}$$

Then, applying the Proposition 6.0.4, we obtain

$$\int_{\mathcal{A}} |f(a)|^2 \, dm(a) < \varepsilon C(R,n) \int_{\mathbb{T}} |f(z)|^2 \, dm(z).$$

□

The only use made of Lemma 6.0.8 is in the proof of the following.

Lemma 6.0.9. *If $\lambda < 1/2$, $\varepsilon \in (0,1)$ and $f \in \mathcal{P}_n(\mathbb{C})$. Define the set*

$$\mathcal{B} = \left\{ a \in \mathbb{T} \mid |f(a)|^2 < \varepsilon^2 B_\lambda f(a, R/n) \right\}.$$

Then there is a constant C depending on R and n such that

$$\int_{\mathcal{B}} |f(z)|^2 \, dm < C\varepsilon \int_{\mathbb{T}} |f|^2 \, dm.$$

Proof. We write

$$\int_{\mathcal{B}} |f|^2 \, dm = \int_{\mathcal{B} \cap \mathcal{A}} |f|^2 \, dm + \int_{\mathcal{B} \setminus \mathcal{A}} |f|^2 \, dm.$$

We estimate the first integral by the Lemma 6.0.8. Analogous to its proof we have that

$$\begin{aligned} \int_{\mathcal{B} \setminus \mathcal{A}} |f|^2 \, dm &< \varepsilon^2 \int_{\mathcal{B} \setminus \mathcal{A}} \frac{1}{m(Ef_\lambda(a,R/n))} \left(\int_{Ef_\lambda(a,R/n)} |f(z)|^2 \, dm(z) \right) \, dm(a) \\ &= \varepsilon^2 \int_{C_n} |f(z)|^2 \left(\int_{\mathcal{B} \setminus \mathcal{A}} \frac{1}{m(Ef_\lambda(a,R/n))} \chi_{Ef_\lambda(a,R/n)}(z) \, dm(a) \right) \, dm(z). \end{aligned}$$

Now we will show that any disk $D(a,R/n)$ at a contains a concentric disk D' of area $\pi \frac{R^2}{16n^2\Gamma_0^2} \varepsilon$. If $f \in \mathcal{P}_n(\mathbb{C})$ and

$$|f(a)|^2 \geq \varepsilon \frac{1}{m(D(a,R/n))} \int_{D(a,R/n)} |f|^2 \, d\sigma$$

then $|f(z)| > \frac{1}{2}|f(a)| > \lambda|f(a)|$ on D'.

Without loss of generality we can assume that $a = 1$ and

$$\frac{1}{m(D(1,R/n))}\int_{D(1,R/n)} |f|^2\, d\sigma = 1.$$

Then $|f(1)| \geq \sqrt{\varepsilon}$. Moreover, applying the Proposition 6.0.3 there is a constant $\Gamma_0 > 0$ such that $|f(z)| < \Gamma_0$ on the set $|z-1| < R/2n$.

Assuming $|z-1| < R/4n$ we have that

$$\begin{aligned}
2\pi|f(z) - f(1)| &\leq \Gamma_0 \int_{|w-1|=R/2n} \left|\frac{1}{w-z} - \frac{1}{w-1}\right| dm(w) \\
&= 2n\Gamma_0 \frac{|z-1|}{R}\int_{|w-1|=R/2n} \frac{1}{|w-z|}\, dm(w) \\
&\leq \Gamma_0 \frac{4\pi n}{R}|z-1|.
\end{aligned}$$

Hence, if we consider $|z-1| < \frac{R}{4n\Gamma_0}\sqrt{\varepsilon}$ we obtain

$$|f(z)| \geq |f(1)| - \sqrt{\varepsilon}/2 > |f(1)|/2$$

on a disk of area $\pi\frac{R^2}{16n^2\Gamma_0^2}\varepsilon$. Translating this to the disk $D(a, R/n)$ we obtain other disk of area $\pi\frac{R^2}{16n^2\Gamma_0^2}\varepsilon$ contained in $Ef_\lambda(a, R/n)$.

Therefore, we have that

$$\begin{aligned}
\int_{\mathcal{B}\setminus\mathcal{A}} |f(z)|^2\, dm(z) &< \varepsilon^2 \int_{C_n} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{1}{m(Ef_\lambda(a,R/n))}\chi_{Ef_\lambda(a,R/n)}(z)\, dm(a)\right) d\sigma(z) \\
&\leq \varepsilon \int_{C_n} |f(z)|^2 \left(\int_{\mathcal{B}\setminus\mathcal{A}} \frac{16^2\Gamma_0^2}{m(D(a,R/n))}\chi_{D(a,R/n)}(z)\, dm(a)\right) d\sigma(z) \\
&\leq \varepsilon\ 512\ \Gamma_0^2 \frac{\arcsin(R/n)}{m(D(1,R/n))}\int_{C_n} |f(z)|^2\, d\sigma(z).
\end{aligned}$$

Applying the Proposition 6.0.4, we obtain that

$$\int_{\mathcal{B}\setminus\mathcal{A}} |f|^2\, dm \leq C\varepsilon \int_{\mathbb{T}} |f|^2\, dm.$$

$\square$

Let $\mathcal{F} = \mathbb{T} \setminus \mathcal{B} = \{a \in \mathbb{T} : |f(a)|^2 \geq \varepsilon^2 B_\lambda f(a, R/n)$.

We choose ε such that $\varepsilon C < 1/2$ we have

$$\int_{\mathbb{T}} |f|^2\, dm < 2\int_{\mathcal{F}} |f|^2\, dm$$

since

$$\int_{\mathbb{T}} |f|^2 \, dm = \int_{\mathcal{F}} |f|^2 \, dm + \int_{\mathcal{B}} |f|^2 \, dm < \int_{\mathcal{F}} |f|^2 \, dm + C\varepsilon \int_{\mathbb{T}} |f|^2 \, dm$$
$$< \int_{\mathcal{F}} |f|^2 \, dm + \frac{1}{2} \int_{\mathbb{T}} |f|^2 \, dm$$

by Lemma 6.0.9.

For $a \in \mathcal{F}$ we have $\frac{B_\lambda f(a,R/n)}{|f(a)|^2} \leq \frac{1}{\varepsilon^2}$. So if we choose $\lambda < \varepsilon^{2/\delta_0}$ we get

$$\frac{m(Ef_\lambda(a,R/n))}{m(D(a,R/n))} > \frac{(2/\delta_0)\log(1/\varepsilon^2)}{\log(1/\varepsilon^2) + (2/\delta_0)\log(1/\varepsilon^2)} > 1 - \delta_0/2.$$

Consequently, (2) implies for $a \in \mathcal{F}$

$$m(G \cap Ef_\lambda(a,R/n)) > \frac{\delta_0}{2} m(D(a,R/n))$$

where λ depend of R, δ_0 and n. So,

$$\frac{1}{m(D(a,R/n))} \int_{E_n} \chi_{D(a,R/n)}(z)|f(z)|^2 \, dm(z) > \frac{1}{2}\delta_0 \lambda^2 |f(a)|^2, \quad a \in \mathcal{F}$$

because

$$\frac{1}{m(D(a,R/n))} \int_{E_n} \chi_{D(a,R/n)}(z)|f(z)|^2 \, dm(z) > \frac{\lambda^2 |f(a)|^2}{m(D(a,R/n))} \int_{E_n} \chi_{Ef_\lambda(a,R/n)}(z) \, dm(z)$$
$$> \frac{\lambda^2 |f(a)|^2 m(Ef_\lambda(a,R/n) \cap E_n)}{m(D(a,R/n))} > \frac{\delta_0 \lambda^2 |f(a)|^2}{2}.$$

Integrating over $\mathcal{F}$

$$\int_{E_n} |f(z)|^2 \left(\int_{\mathcal{F}} \frac{1}{m(D(a,R/n))} \chi_{D(a,R/n)}(z) \, dm(a) \right) \, dm(z) \geq \frac{1}{2}\delta_0 \lambda^2 \int_{\mathcal{F}} |f(a)|^2 \, dm(a).$$

Analogous to previous lemmas

$$C \int_{E_n} |f(z)|^2 \, dm(z) \geq \frac{1}{2}\delta_0 \lambda^2 \int_{\mathcal{F}} |f(a)|^2 \, dm(a)$$

As we know

$$\int_{\mathbb{T}} |f|^2 \, dm < 2 \int_{\mathcal{F}} |f|^2 \, dm.$$

Finally,

$$C \int_{E_n} |f(z)|^2 \, dm(z) \geq \frac{1}{2}\delta_0 \lambda^2 \int_{\mathcal{F}} |f(z)|^2 \, dm(z) > \frac{1}{4}\delta_0 \lambda^2 \int_{\mathbb{T}} |f(z)|^2 \, dm(z).$$

□

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