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Interpolation and Sampling Arrays in Spaces of Polynomials

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To my family.

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Abstract

We study the discretisation procedure of homogeneous polynomials in the unit sphere $\mathbb{S} \cong \mathbb{CP}^1$. This can be seen as a basic model of a more general problem of discretisation of sections of holomorphic line bundles over compact complex manifolds.

Our aim is to obtain geometric necessary and sufficient conditions describing the discretising sequences. An important model for such sequences are the so-called Fekete arrays, which can be seen as nets adapted to the geometry of the sphere.

The tools used in such description go back to the signal processing theory pioneered by Beurling and Landau.

Key words: Interpolation Sequences, Sampling Sequences, Fekete Points, Bergman Kernel, Beurling-Landau density.

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Prologue

The Shannon-Whittaker Theorem [6] states that a band limit signal f with band-width τ, i.e., $f \in L^2(\mathbb{R})$ with spectrum in $(-\tau, \tau)$, can be recovered from its samples at the Nyquist rate $f\left(\frac{k}{2\tau}\right)$, $k \in \mathbb{Z}$, through the cardinal series

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2\tau}\right) \operatorname{sinc}\left[2\tau\left(t - \frac{k}{2\tau}\right)\right],$$

where $\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$. Moreover,

$$2\tau \int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{2\tau}\right)\right|^2.$$

This theorem was well-known in the engineering community as in the Shannon's theory of information in the 40's. The use of the previous result is discerning when a function in the preceding space PW_τ can be recovered from its samples. Landau in [11] gives an answer of the interpolation and sampling problems in this setting in terms of density.

In this work we are focused in the interpolation and sampling problems in the space of polynomials $\mathcal{P}_k$ over $\mathbb{C}$. One aim is obtain necessary and sufficient geometric conditions describing the appropiate discretising sequences.

The memoir is divided in five sections. The first one is an introduction to the interpolation and sampling problems, the space where we set these questions and the Bergman Kernel, which is essential for our purpose. The Fekete arrays are defined in the second chapter. These can be seen as nets with a limit density, so that can be perturbed to obtain sampling or interpolating arrays.

In the third one, a Landau's technique is used to obtain necessary conditions in terms of the Beurling-Landau densities. Besides, we get upper and lower bounds for the Kantorovich-Wasserstein distance between the Fekete measure and its limit measure.

Next, we study the connection between sampling and interpolation arrays in our setting and the respective sequences in the Bargmann-Fock space. As a consequence, we find that there are no arrays fot $\mathcal{P}_k$ which are simultaneously interpolating and sampling. Finally, in the last chapter we find sufficient density conditions. The report ends with the bibliography and an index with the main concepts.

1

Introduction

1.1 Preliminaries

In this work we will study the sampling and interpolation problems for the space $\mathcal{P}_k$ of holomorphic polynomials of degree k on the sphere $\mathbb{S}^2 \simeq \mathbb{PC}^1$. The metric we consider is the usual Fubini-Study metric induced by $\phi(z) = \log(1 + |z|^2)$. In particular, the volume form is

$$d\mathcal{V}(z) = \frac{i}{2\pi}\partial\bar{\partial}\phi(z) = \frac{dm(z)}{(1+|z|^2)^2}, \qquad z \in \mathbb{C} \tag{1.1}$$

where $dm(z)$ denotes the Lebesgue measure normalised in $\mathbb{C}$ so that $m(\mathbb{D}) = 1$. As in [1, p. 18], the unit sphere with equation $x_1^2 + x_2^2 + x_3^2 = 1$ can be seen as the complex projective plane by taking

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

for every point except $(0, 0, 1)$, the point at infinity, and this correspondence is one to one. Indeed,

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}$$
$$x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}.$$

This is the chart we will use through the work. The distance we consider is the *chordal distance*, whose expression in the plane after the stereographic projection is

$$d(z, w) = \frac{2|z - w|}{(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}}.$$

The group of isometries of the Riemann sphere with the Fubini-Study metric is the projective special group $\mathbf{PSU(2)}$. If we denote the center of a group G by $\mathcal{Z}(G)$, then

$$\mathbf{PSU(2)} = \mathbf{SU(2)}/\mathcal{Z}(\mathbf{SU(2)}) \cong \mathbf{SO(3)} = \{A \in \mathbf{O(3)} : \det(A) = +1\}$$

which can be seen as rotations of the sphere. Notice that for some calculations we can use one of this rotations to move a point of the plane to 0, making the computation easier. Given an 'invariant' function, i.e., which depends on the chordal distance, the equation

$$\int_{\mathbb{C}} f(d(z,w))d\mathcal{V}(w) = \int_{\mathbb{C}} f(d(0,w))d\mathcal{V}(w)$$

holds, since the Fubini-Study metric inherits the invariance.

The space $(\mathcal{P}_k, ||\cdot||_\phi)$ is a Hilbert space, with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\bar{g}(z)\frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \qquad f, g \in \mathcal{P}.$$

So the norm is

$$||p_k||^2 = \int_{\mathbb{C}} \frac{|p_k(z)|^2}{(1+|z|^2)^k} d\mathcal{V}(z), \qquad p_k \in \mathcal{P}_k.$$

We will denote the set of holomorphic functions in Ω by $\mathcal{O}(\Omega)$, and the set of harmonic function on Ω by $\mathcal{H}(\Omega)$.

Note. An important result for many estimates is the measure of an annulus

$$\mathcal{V}(a \le d(z,w) \le b) = \int_{\frac{a^2}{4-a^2}}^{\frac{b^2}{4-b^2}} \frac{dt}{(1+t)^2}. \tag{1.2}$$

In particular, for b not close to 2,

$$\mathcal{V}(a \le d(z,w) \le b) \approx \left[-\frac{1}{1+t}\right]_{\frac{a^2}{4-a^2}}^{\frac{b^2}{4-b^2}} \simeq b^2 - a^2 = (b-a)(a+b).$$

And in case $b = a + \delta$, with $\delta \ll a$, then

$$\mathcal{V}(a \le d(z,w) \le a+\delta) \simeq a\delta.$$

Proof. We can assume that $w = 0$, by invariance, so that

$$\mathcal{V}(a \le d(z,w) \le b) = \int_{a \le d(z,0) \le b} \frac{dm(z)}{(1+|z|^2)^2}.$$

Since $d^2(z,0) = \frac{4|z|^2}{1+|z|^2}$, we have

$$a \leq d(z,w) \leq b \Leftrightarrow \frac{a}{\sqrt{4-a^2}} \leq |z| \leq \frac{b}{\sqrt{4-b^2}},$$

and therefore

$$\mathcal{V}(a \leq d(z,w) \leq b) = \int_{\frac{a}{\sqrt{4-a^2}} \leq r \leq \frac{b}{\sqrt{4-b^2}}} \frac{2r dr}{(1+r^2)^2} = \int_{\frac{a^2}{4-a^2}}^{\frac{b^2}{4-b^2}} \frac{dt}{(1+t)^2}.$$

Notice that we can estimate of the measure of a disc of radius r as r^2.

1.2 The Bergman Kernel

Many of the properties of the space $\mathcal{P}_k$ are coded in the reproducing kernel. We may apply the Riesz representation theorem to see that there is an element $K(z,\cdot) \in \mathcal{P}_k$ such that the linear functional $\phi_z : \mathcal{P}_k \to \mathbb{C}$, $\phi_z(f) = f(z)$ is represented by inner product with K.

Definition 1.1. *The Bergman Kernel is the function $K(z,w)$ such that*

$$p(z) = \int_{\mathbb{C}} K(z,w)p(w)\frac{d\mathcal{V}(w)}{(1+|w|^2)^k},$$

for all $p \in \mathcal{P}_k(\mathbb{C})$.

If $\{e_j\}_{j=0}^k$ an orthonormal basis of $\mathcal{P}_k$ then

$$K(z,w) = \sum_{j=0}^{k} e_j(z)\overline{e_j(w)}, \quad z,w \in \mathbb{C}.$$

This kernel cannot be expressed explicitly in most occasions, nevertheless in this case it is possible. We take the normalization of the monomials $\left\{\frac{z^j}{\|z^j\|}\right\}$ as orthonormal basis. Here

$$\|z^j\| = \int_{\mathbb{C}} \frac{|z|^{2j}}{(1+|z|^2)^k} d\mathcal{V}(z) = \int_0^\infty r^{2j} \frac{2r dr}{(1+r^2)^{k+2}}$$

$$= \int_0^\infty r^j \frac{dr}{(1+r)^{k+2}} = \frac{\Gamma(j+1)\Gamma(k-j+1)}{\Gamma(k+2)}$$

$$= \beta(j+1, k-j+1).$$

Hence, the kernel is

$$K(z,w)=\sum_{j=0}^{k}\frac{(z\bar{w})^j}{\beta(j+1,k-j+1)}=(k+1)!\sum_{j=0}^{k}\frac{(z\bar{w})^j}{j!\Gamma(k-j+1)}$$
$$=(k+1)(1+z\bar{w})^k. \tag{1.3}$$

The Bergman Kernel is conjugate symmetric and, by the definition before,

$$K(z,z)=\int_{\mathbb{C}}|K(z,w)|^2\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}.$$

A direct computation shows that

$$\frac{|K(z,w)|}{\sqrt{K(z,z)}\sqrt{K(w,w)}}$$
$$=\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}}=\left[1-\left(\frac{d(z,w)}{2}\right)^2\right]^{k/2}. \tag{1.4}$$

The proof of the main theorems uses the good behaviour of the Bergman Kernel that we can observe in the following lemma.

Lemma 1.2. *It is satisfied*

1.

$$\sup_k\sup_{z\in\mathbb{C}}k\int_{\mathbb{C}}\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}}d\mathcal{V}(w)<\infty,$$

and

$$\sup_k\sup_{z\in\mathbb{C}}k^{3/2}\int_{\mathbb{C}}d(z,w)\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}}d\mathcal{V}(w)<\infty.$$

2. If $\Omega=D(z',r/\sqrt{k})$, *then*

$$k^2\iint_{\Omega\times\Omega^c}\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}}d\mathcal{V}(z)d\mathcal{V}(w)\lesssim r,$$

and

$$k^2\iint_{\Omega\times\Omega^c}\frac{|1+z\bar{w}|^{2k}}{(1+|z|^2)^{k}(1+|w|^2)^{k}}d\mathcal{V}(z)d\mathcal{V}(w)\lesssim r.$$

where $\lesssim$ *denotes to be bounded by a constant which does not depend on* k *and* r.

Proof. 1. By the invariance before, we can consider $z = 0$, then

$$\int_{\mathbb{C}} \frac{|1+z\bar{w}|^k}{(1+|w|^2)^{k/2}} \frac{dm(w)}{(1+|w|^2)^2} = \int_{\mathbb{C}} \frac{dm(w)}{(1+|w|^2)^{2+k/2}}$$
$$= \int_0^\infty \frac{2\rho d\rho}{(1+\rho^2)^{k/2+2}} = \int_0^\infty \frac{dt}{(1+t)^{k/2+2}} = \frac{1}{1+\frac{k}{2}},$$

and the result follows. The second integral is straightfoward when the distance is less or equal than $1/\sqrt{k}$. For the other case, denote $R = \{w \in \mathbb{C} : d(z,w) \geq 1/\sqrt{k}\}$. Then (by (1.4))

$$\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}} = e^{-\frac{k}{2}\log\left(\frac{1}{1-d^2(z,w)}\right)} \leq e^{-\frac{k}{2}C},$$

where $C = -\log(1-1/k)$. So

$$\int_R d(z,w)\frac{|1+z\bar{w}|^k}{(1+|z|^2)^{k/2}(1+|w|^2)^{k/2}} d\mathcal{V}(w) \leq e^{-\frac{k}{2}C}\int_R d(z,w)d\mathcal{V}(w)$$
$$\lesssim e^{-\frac{k}{2}C}\int_R d\mathcal{V}(w) \lesssim e^{-\frac{k}{2}C}.$$

2. If $d(z,w) \leq 1/2$ since $e^t \simeq 1-t$ when $t \approx 0$ then

$$\left[1-\left(\frac{d(z,w)}{2}\right)^2\right]^{k/2} \simeq e^{-\frac{k}{8}d(z,w)^2}, \tag{1.5}$$

where d is the chordal distance.
For the third integral, assume $\Omega = D(0, r\sqrt{k})$. We can take a partition of Ω^c into 'dyadic' shells defined by

$$\Omega_j^c = \left\{w \in \mathbb{C} : 2^{j-1}\frac{r}{\sqrt{k}} \leq d(z,w) \leq 2^j\frac{r}{\sqrt{k}}\right\} \quad (j = 1, \dots, J),$$

where $J = E[1 + \log_2 \frac{\sqrt{k}}{r}]$, where $E[x]$ is the integer part of x. Hence,

$$\int_{\Omega^c}\left[1-\left(\frac{d(z,w)}{2}\right)^2\right]^{\frac{k}{2}} d\mathcal{V}(w) \leq \int_{\Omega^c} e^{-\frac{d(z,w)^2}{8}k}d\mathcal{V}(w)$$
$$\leq \sum_{j=1}^J \int_{\Omega_j^c} e^{-4^{j-2}\frac{r^2}{2}}d\mathcal{V}(w) = \sum_{j=1}^J e^{-4^{j-2}\frac{r^2}{2}}\mathcal{V}(\Omega_j^c)$$
$$\lesssim \sum_{j=1}^J e^{-4^{j-2}\frac{r^2}{2}}4^j\frac{r^2}{k} \simeq \frac{r^2}{k}\sum_{j=1}^J e^{-4^{j-2}\frac{r^2}{2}}4^j.$$

Therefore, the estimate is

$$\int_{\Omega} \frac{r^2}{k} d\mathcal{V}(z) = \frac{r^2}{k}\frac{r^2}{k} = \frac{r^4}{k^2},$$

as we take a small r, the result follows. The second estimate is similar, with the change

$$\begin{aligned}\int_{\Omega^c} \left[1 - \left(\frac{d(z,w)}{2}\right)^2\right]^k d\mathcal{V}(w) &\leq \int_{\Omega^c} e^{-\frac{d(z,w)^2}{4}k} d\mathcal{V}(w) \\ &\leq \sum_{j=1}^{J} \int_{\Omega_j^c} e^{-4^{j-2}r^2} d\mathcal{V}(w) = \sum_{j=1}^{J} e^{-4^{j-2}r^2} \mathcal{V}(\Omega_j^c) \\ &\lesssim \sum_{j=1}^{J} e^{-4^{j-2}r^2} 4^j \frac{r^2}{k} \simeq \frac{r^2}{k} \sum_{j=1}^{J} e^{-4^{j-2}r^2} 4^j.\end{aligned}$$

1.3 Interpolation and Sampling Problems

There exist general notions of sampling and interpolation for Hilbert spaces of holomorphic functions with reproducing kernels.

On the one hand, we say that Λ is an interpolation set if $\{\tilde{K}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz sequence, or equivalently, if for all $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2$ there exists $f \in \mathcal{H}$ such that

$$\langle f, \tilde{K}_\lambda \rangle = c_\lambda, \quad \lambda \in \Lambda.$$

On the other hand, we say that Λ is a sampling set if $\{\tilde{K}_\lambda\}_{\lambda\in\Lambda}$ is a frame, that is, if there exists $C > 0$ such that

$$\frac{1}{C} \sum_{\lambda\in\Lambda} \left|\langle f, \tilde{K}_\lambda \rangle\right|^2 \leq \|f\|^2 \leq C \sum_{\lambda\in\Lambda} \left|\langle f, \tilde{K}_\lambda \rangle\right|^2, \quad f \in \mathcal{H}.$$

Notice that

$$\langle f, \tilde{K}_\lambda \rangle = \frac{1}{\sqrt{k+1}} f(\lambda) e^{-\frac{k}{2}\phi(\lambda)} = \frac{1}{\sqrt{k+1}} \frac{f(\lambda)}{(1+|\lambda|^2)^{k/2}}$$

and therefore, the precise definitions in our setting are

1.

$$\Lambda \in \mathrm{Int}(\mathcal{P}_k) \Leftrightarrow \forall \{c_\lambda\} \in l^2,\ \exists f \in \mathcal{P}_k : f(\lambda) = \sqrt{k+1}(1+|\lambda|^2)^{k/2} c_\lambda,\ \lambda \in \Lambda.$$

2.

$$\Lambda \in \mathrm{Samp}(\mathcal{P}_k) \Leftrightarrow \exists C > 0 :$$
$$\frac{1}{C}\sum_{\lambda\in\Lambda}\frac{1}{k+1}\frac{f(\lambda)^2}{(1+|\lambda|^2)^k} \leq \|f\|^2 \leq C\sum_{\lambda\in\Lambda}\frac{1}{k+1}\frac{f(\lambda)^2}{(1+|\lambda|^2)^k}, \quad f\in\mathcal{H}.$$

If instead of $\{c_\lambda\} \subset l^2$, we consider $v_\lambda = \sqrt{k+1}(1+|\lambda|^2)^{k/2}c_\lambda$, we get an equivalent formulation of the interpolation

$$\forall \{v_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda), \exists f \in \mathcal{P}_k : f(\lambda) = v_\lambda, \quad \lambda \in \Lambda.$$

Here

$$l^2(\Lambda_k) = \left\{ \{v_\lambda\}_{\lambda\in\Lambda_k} : \|v_\lambda\|_{l^2(\Lambda_k)} = \frac{1}{k}\sum_{\lambda\in\Lambda_k}\frac{|v_\lambda|^2}{(1+|\lambda|^2)^k} < \infty \right\}.$$

A very basic intuition is that when the sequences are 'sparse', they are interpolating; and when they are 'dense', they are sampling. Notice that the interpolation and sampling problems are trivial for a fixed level k. If we have a set Λ_k with less than $k+1$ points we can find an interpolating polynomial. Analogously, a sampling set is characterized by more than $k+1$ points. The point in our work is to perform the sampling and the interpolation uniformly in k.

Definition 1.3. *Let $\Lambda = \{\Lambda_k\}_k$ be a sequence of finite sets in $\mathbb{C}$. We call Λ a sampling array if there are k_0 and positive constants A, B not depending on k, such that Λ_k is a sampling set at each level $k \geq k_0$ with the same sampling constants.*

Analogously, Λ an interpolation array if there are k_0 and a positive constant C not depending on k, such that Λ_k is a interpolation set at each level $k \geq k_0$ with the same interpolation constant.

Our aim is to obtain conditions for an array Λ to be sampling or interpolating.

Now, we want to state a pointwise estimate for $p \in \mathcal{P}_k$, so we need a local control on ϕ.

Lemma 1.4. *Given $r > 0$ and $w \in \mathbb{C}$, for $z \in D(w, r/\sqrt{k})$ it holds that*

$$|\phi(z) - \phi(w) - h_w(z)| \leq \frac{2r^2}{k}(\log 2 + 2)$$

where $h_w(z) \in \mathcal{H}(D(w, r/\sqrt{k}))$ and $h_w(w) = 0$.

Proof. From the first Green's identity [8, p. 31], the Green-Riesz representation of ϕ is

$$\phi(z) = \int_{\partial D(w,r/\sqrt{k})} P(z,\zeta)\phi(\zeta)d\sigma(\zeta) + \int_{D(w,r/\sqrt{k})} G(z,\eta)d\mathcal{V}(\eta), \quad z \in D(w,r/\sqrt{k}),$$

where G and P are the Green function and the Poisson kernel respectively. Then,

$$\phi(z) - \phi(w) = h_w(z) + \int_{D(w,r/\sqrt{k})} [G(z,\eta) - G(w,\eta)]d\mathcal{V}(\eta)$$

where $h_w(z) \in \mathcal{H}(D(w,r/\sqrt{k}))$ and $h_w(w) = 0$. Hence,

$$\begin{aligned} |\phi(z) - \phi(w) - h_w(z)| &\le \int_{D(w,r/\sqrt{k})} [G(z,\eta) - G(w,\eta)]d\mathcal{V}(\eta) \\ &\le 2\int_{D(w,r/\sqrt{k})} G(z,\eta)dm(\eta) \le \frac{2r^2}{k}(\log 2 + 2). \end{aligned}$$

by the following estimates for the Green's function in a disc.

$$\begin{aligned} \int_{D(w,r)} G(z,\eta)dm(z) &= \int_{D(w,r)} \log\left|\frac{r^2 - \overline{(\eta - z)}(z-w)}{r(\eta - z)}\right| dm(z) \\ &\le r^2 \log 2 + \int_{D(\eta,2r)} \log\left(\frac{r}{|\eta - z|}\right) dm(z) \\ &= r^2\log 2 + 4r^2 \int_{\mathbb{D}} \log\frac{1}{|\zeta|}dm(\zeta) = r^2\log 2 - 2r^2 \\ &\le r^2(\log 2 + 2). \end{aligned}$$

□

Lemma 1.5. *Given $r > 0$, there exist constants A, $B > 0$ such that for all $f \in \mathcal{O}(\mathbb{C})$, $w \in \mathbb{C}$*

1.
$$\frac{|f(w)|^p}{(1+|w|^2)^{p\frac{k}{2}}} \lesssim kC(r)\int_{D(w,r/\sqrt{k})} |f(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^{p\frac{k}{2}}},$$

with $C(r) = \frac{1}{r^2}e^{-pr^2(2+\log 2)}\left(1+r^2\right)^2$.

2.
$$|\nabla(|f|^2e^{-k\phi})(w)|^2 \lesssim kD(r)\int_{D(w,r/\sqrt{k})} |f(z)|^2\frac{d\mathcal{V}(z)}{(1+|z|^2)^k},$$

with $D(r) = \frac{1}{r^2}(1+r^2)^2 e^{-2r^(2+\log 2)}$.

Proof. Take $H_w \in \mathcal{O}(D(w, r/\sqrt{k}))$ such that $\Re H_w = h_w$ as in the previous Lemma 1.4 and define

$$g(z) = f(z) e^{\frac{k}{2} H_w(z)},$$

notice that $|g(w)| = |f(w)|$. Then,

$$\begin{aligned}
\frac{|f(w)|^p}{(1+|w|^2)^{p\frac{k}{2}}} &= \frac{|g(w)|^p}{(1+|w|^2)^{p\frac{k}{2}}} \lesssim \frac{k}{r^2} \int_{D(w,r/\sqrt{k})} \frac{|g(\zeta)|^p}{(1+|w|^2)^{p\frac{k}{2}}} dm(\zeta) \\
&= \frac{k}{r^2} \int_{D(w,r/\sqrt{k})} |f(\zeta)|^p e^{-p\frac{k}{2}[\phi(w) - h_w(\zeta)]} dm(\zeta) \\
&\leq \frac{k}{r^2} \int_{D(w,r/\sqrt{k})} |f(z)|^p e^{-p\frac{k}{2}\left[\frac{2r^2}{k}(2+\log 2) + \phi(z)\right]} dm(z) \\
&\leq \frac{k}{r^2} e^{-pr^2(2+\log 2)} (1+r^2) \int_{D(w,r/\sqrt{k})} |f(z)|^p e^{-p\frac{k}{2}\phi(z)} d\mathcal{V}(z).
\end{aligned}$$

For the second part, notice that

$$|\nabla(|f|e^{-k\phi})(w)|^2 = |f'(w) - \phi'(w) f(w)|^2 e^{-k\phi(w)}$$

and if $g(z) = f(z) e^{\frac{k}{2} H_w(z)}$ this is equal to

$$\begin{aligned}
|g'(w)|^2 e^{-k\phi(w)} &\lesssim \frac{k}{r^2} \int_{D(w,r/\sqrt{k})} |g(z)|^2 e^{-k\phi(w)} dm(z) \\
&\leq \frac{k}{r^2} (1+r^2)^2 e^{-2r^2(2+\log 2)} \int_{D(w,r/\sqrt{k})} |f(z)|^2 e^{-k\phi(z)} d\mathcal{V}(z).
\end{aligned}$$

□

With the previous estimates, we can find a *Plancherel-Polya inequality*.

Definition 1.6. *A family Λ_k is δ-separated at level k if*

$$d(z, w) \geq \frac{\delta}{\sqrt{k}}, \qquad z, w \in \Lambda_k, \quad z \neq w.$$

Lemma 1.7. (Plancherel-Polya inequality) *Let $\{\lambda_j\}_j$ be points in $\mathbb{C}$ such that are δ-separated at level k, where $0 < \delta < 1$. Then*

$$\frac{1}{k} \sum_j \frac{|p_k(\lambda_j)|^p}{(1+|\lambda_j|^2)^{p\frac{k}{2}}} \lesssim \delta^{-2} \int_{\mathbb{C}} |p_k(z)|^p \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}p}} \qquad (1 \leq p < \infty)$$

for any $p_k \in \mathcal{P}_k$, where the constant does not depend on k.

Proof. It is enough to notice that for each λ_j we have

$$\frac{|p_k(\lambda_j)|^p}{(1+|\lambda_j|^2)^{p\frac{k}{2}}} \lesssim \int_{D(\lambda_j, \delta/\sqrt{k})} |p_k(z)|^p \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}p}}.$$

The separation implies the lemma.

2

The Fekete Points and Lagrange's Polynomials

The Fekete arrays are models, or at least examples, of well-distributed sequences. These arrays can be seen as nets, which, after rescaling, have a *limit* density.

2.1 Definition and properties

Definition 2.1. *A configuration $\mathcal{F}_k$ of $k+1$ points $\{\lambda_j^k\}_{j=0}^k$ in $\mathbb{C}$ is called a Fekete configuration if it maximizes the pointwise norm of the Vandermonde-type determinant*

$$\begin{vmatrix} e_0(\lambda_0^k) & \cdots & e_0(\lambda_k^k) \\ \vdots & \ddots & \vdots \\ e_k(\lambda_0^k) & \cdots & e_k(\lambda_k^k) \end{vmatrix} e^{-k\phi(\lambda_0^k)} \cdots e^{-k\phi(\lambda_k^k)}$$

where $\{e_j(z)\}_{j=0}^k$ is a basis for $\mathcal{P}_k$.

These points are related with the following polynomials, which have interest for themselves.

Definition 2.2. *The Lagrange polynomials $\{l_j\}_{j=0}^k$ associated to $\{\lambda_j^k\}$ are defined as*

$$l_j(z) = \frac{\begin{vmatrix} e_0(\lambda_0^k) & \cdots & e_0(z) & \cdots & e_0(\lambda_k^k) \\ \vdots & & \vdots & & \vdots \\ e_k(\lambda_0^k) & \cdots & e_k(z) & \cdots & e_k(\lambda_k^k) \end{vmatrix}}{\begin{vmatrix} e_0(\lambda_0^k) & \cdots & e_0(\lambda_j^k) & \cdots & e_0(\lambda_k^k) \\ \vdots & & \vdots & & \vdots \\ e_k(\lambda_0^k) & \cdots & e_k(\lambda_j^k) & \cdots & e_k(\lambda_k^k) \end{vmatrix}}$$

for a given basis $\{e_j\}_{j=0}^k$ of $\mathcal{P}_k$.

They satisfy the following properties

a) For every level set k,
$$\frac{l_j(\lambda_i^k)}{(1+|\lambda_i^k|^2)^{\frac{k}{2}}} = \delta_{ij}.$$

b) The norm satisfies $\sup_{z\in\mathbb{C}} \dfrac{|l_j(z)|}{(1+|z|^2)^{\frac{k}{2}}} = 1.$

The first property implies that this set $\{l_j\}_j$ is linerly independent. By definition $\sharp\{l_j\}_j = k+1 = \dim\mathcal{P}_k$, then it spans the whole $\mathcal{P}_k$.

Let us see that every Fekete configuration is a separated family.

Lemma 2.3. *Let $\mathcal{F}_k$ be a Fekete configuration for $\mathcal{P}_k$. Then, there exists $\delta > 0$ such that*
$$d(z,w) \geq \frac{\delta}{\sqrt{k}}, \qquad z, w \in \mathcal{F}_{k+1} \quad (z \neq w).$$

Proof. We proceed by contradiction. If that is not the case, there are points $z_k, w_k \in \mathcal{F}_k$ such that $\sqrt{k}d(z_k,w_k) \to 0$ and $z_k \neq w_k$. Now take Lagrange polynomials l_k, so that
$$\frac{l_k(z)}{(1+|z|^2)^{\frac{k}{2}}} = \begin{cases} 0, & z = z_k, \\ 1, & z = w_k. \end{cases}$$

As $l_k \in \mathcal{P}_k$, then
$$1 = \left| \frac{l_k(z_k)}{(1+|z_k|^2)^{\frac{k}{2}}} - \frac{l_k(w_k)}{(1+|w_k|^2)^{\frac{k}{2}}} \right| \leq \sup_{z\in\mathbb{C}} |\nabla(l_k e^{-\frac{k}{2}\phi})| d(z_k,w_k) \to 0,$$

by Lemma 1.5. □

2.2 Fekete Arrays, Sampling and Interpolation

At this moment, we can prove that every set of Fekete points can be slightly perturbed to get interpolation or sampling arrays, with control on the constants. The use of $\mathcal{F}_{(1\pm\epsilon)k}$ as if $(1\pm\epsilon)k$ are integer is due to a simplification, we are referring that we should replace $(1\pm\epsilon)k$ by its integer part. This notation is maintened throughout the text.

Lemma 2.4. *Let k be a positive integer, and let ϵ be a number satisfying $1/k \lesssim \epsilon \lesssim 1$. The set*
$$\Lambda_k = \mathcal{F}_{(1+\epsilon)k}$$

is a sampling set at level k with sampling constants A, B such that $1 \lesssim A < B \lesssim 1/\epsilon^2$. On the other hand, the set

$$\Lambda_k = \mathcal{F}_{(1-\epsilon)k}$$

is an interpolation set at level k with interpolation constant C satisfying $C \lesssim 1/\epsilon^2$.

Proof. We take the notation $e^{-k\phi(x)} = \frac{1}{(1+|x|^2)^2}$ for ease. Let us prove the interpolation part, so let l_j be Lagrange functions for $\Lambda_k = \mathcal{F}_{k(1-\epsilon)} = \{x_j\}_j$, so that

a) $\frac{l_j(x_i)}{(1+|x_i|^2)^{(1-\epsilon)\frac{k}{2}}} = \delta_{ij}$,
b) $\sup_{x\in\mathbb{C}} |l_j(x)|^2 e^{-(1-\epsilon)k\phi(x)} = 1$.

Let

$$Q_j(x) = l_j(x)\left(\frac{K_{\frac{\epsilon}{2}k}(x,x_j)}{K_{\frac{\epsilon}{2}k}(x_j,x_j)}\right)^2,$$

where the second factor helps in the decay of Q_j away form x_j, and the notation K_α means the Bergman kernel for $\mathcal{P}_\alpha$. Given $\{v_j\} \subset l^2$, the interpolating polynomial is

$$p(x) = \sqrt{k+1}\sum_j v_j\overline{Q_j(x_j)}e^{-\frac{1}{2}k\phi(x_j)}Q_j(x)e^{\epsilon k\phi(x_j)},$$

since $p(x_i) = \sqrt{k+1}e^{-\frac{k}{2}\phi(x_i)}v_i$. Notice that $p \in \mathcal{P}_k$ since $\Lambda_k = \mathcal{F}_{(1-\epsilon)k}$ and $K_{\frac{\epsilon}{2}k}$ is a polynomial of degree less or equal than $\frac{\epsilon}{2}k$, so we need to show that

$$\|p\|^2 = \int_{\mathbb{C}} \frac{|p(z)|^2}{(1+|z|^2)^k}d\mathcal{V}(z) \le C\sum_j |v_j|^2,$$

with C independent of k. Notice that,

$$\begin{aligned}
|p(x)| &\le \sqrt{k+1}\sum_j |v_j||l_j(x_j)|e^{-(\frac{1}{2}-\epsilon)k\phi(x_j)}|Q_j(x)| \\
&\lesssim \sqrt{k}\sum_j |v_j|e^{\frac{1}{2}(1-\epsilon)k\phi(x_j)}e^{-(\frac{1}{2}-\epsilon)k\phi(x_j)}|Q_j(x)| \\
&= \sqrt{k}\sum_j |v_j|e^{\frac{\epsilon}{2}k\phi(x_j)}|Q_j(x)|.
\end{aligned}$$

Then, by Cauchy-Schwarz

$$|p(x)|^2 \lesssim k\left[\sum_j |v_j|^2 e^{\frac{\epsilon}{2}k\phi(x_j)}|Q_j(x)|\right]\left[\sum_j e^{\frac{\epsilon}{2}k\phi(x_j)}|Q_j(x)|\right]. \tag{2.1}$$

By a pointwise estimate on $|Q_j(x)|$, we can estimate the second part as follows

$$\sum_j e^{\frac{\epsilon}{2}k\phi(x_j)}|Q_j(x)| \lesssim \sum_j \frac{1}{(\epsilon k)^2} e^{-\epsilon k\phi(x_j)} e^{(1-\epsilon)\frac{k}{2}\phi(x)} |K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{\frac{\epsilon}{2}k\phi(x_j)}$$
$$\lesssim \frac{1}{(\epsilon k)^2} e^{(1-\epsilon)\frac{k}{2}\phi(x)} \sum_j |K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)},$$

where the first estimation is due to the identity $K_{\frac{\epsilon}{2}k}(x_j,x_j) = \frac{\epsilon}{2}ke^{\frac{\epsilon}{2}k\phi(x_j)}$ and the estimate 2.2. By the Plancherel-Polya inequality 1.7, and since $\{x_j\}$ is separated $\delta/(\sqrt{\epsilon}k)$, the Lemma 2.3

$$\sum_j e^{\frac{\epsilon}{2}k\phi(x_j)}|Q_j(x)| \lesssim \frac{1}{k^2} e^{(1-\epsilon)\frac{k}{2}\phi(x)} k^2 e^{\epsilon\frac{k}{2}\phi(x)} \simeq e^{\frac{k}{2}\phi(x)}.$$

With this

$$|p(x)|^2 e^{-k\phi(x)} \lesssim k\sum_j |v_j|^2 e^{\frac{\epsilon}{2}k\phi(x_j)} |l_j(x)| \frac{|K_{\frac{\epsilon}{2}k}(x,x_j)|^2}{(k\epsilon)^2 e^{k\epsilon\phi(x_j)}} e^{-\frac{k}{2}\phi(x)}$$
$$\lesssim \frac{1}{k}\sum_j |v_j|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)} |K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\frac{k\epsilon}{2}\phi(x)}.$$

Integrating

$$\int_{\mathbb{C}} |p(x)|^2 e^{-k\phi(x)} d\mathcal{V}(x) \lesssim \frac{1}{k}\sum_j |v_j|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)} \int_{\mathbb{C}} |K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\epsilon\frac{k}{2}\phi(x)} d\mathcal{V}(x)$$
$$= \frac{1}{k}\sum_j |v_j|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)} K_{\frac{\epsilon}{2}k}(x_j,x_j)$$
$$\simeq \frac{1}{k}\sum_j |v_j|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)} \frac{\epsilon}{2}ke^{\frac{\epsilon}{2}k\phi(x_j)}$$
$$\simeq \sum_j |v_j|^2.$$

For the sampling part, the set is $\Lambda_k = \mathcal{F}_{(1+\epsilon)k} = \{x_j\}_j$.The inequality

$$\|p\|^2 \gtrsim \frac{1}{k}\sum_j |p(x_j)|^2 e^{-k\phi(x_j)}$$

follows from Plancherel-Polya (Lemma 1.7), since

$$d(x_i,x_j) \gtrsim \frac{1}{\sqrt{(1+\epsilon)k}} \gtrsim \frac{1}{\sqrt{k}} \quad (i \neq j).$$

To see the inequality

$$\|p\|^2 \lesssim \frac{1}{k}\sum_j |p(x_j)|^2 e^{-k\phi(x_j)},$$

with constants independent of k let, given $x \in \mathbb{C}$ fix,

$$p_x(y) = p(y)\left(\frac{K_{\frac{\epsilon}{2}k}(x,y)}{K_{\frac{\epsilon}{2}k}(x,x)}\right)^2.$$

Notice that $p_x(x) = p(x)$ and that $p_x \in \mathcal{P}_{(1+\epsilon)k}$. Let l_j the Lagrange polynomials associated to $\{x_j\}$. Since they form a basis of the space $\mathcal{P}_{(1+\epsilon)k}$, there exist c_j such that

$$p_x = \sum_j c_j l_j.$$

Evaluating on x_i, we obtain $p_x(x_i) = c_i l_i(x_i)$, thus

$$c_i = \frac{p_x(x_i)}{l_i(x_i)} = p_x(x_i)e^{-(1+\epsilon)\frac{k}{2}\phi(x_i)},$$

by the property 2.2. Hence

$$p_x = \sum_j p_x(x_j)e^{-(1+\epsilon)\frac{k}{2}\phi(x_j)}l_j,$$

and

$$\begin{aligned}
|p(x)| = |p_x(x)| &\leq \sum_j |p_x(x_j)|e^{-(1+\epsilon)\frac{k}{2}\phi(x_j)}e^{(1+\epsilon)\frac{k}{2}\phi(x)} \\
&\simeq \sum_j |p(x_j)|\frac{|K_{\frac{\epsilon}{2}k}(x,x_j)|^2}{(k\epsilon)^2 e^{\epsilon k\phi(x)}}e^{-(1+\epsilon)\frac{k}{2}\phi(x_j)}e^{(1+\epsilon)\frac{k}{2}\phi(x)} \\
&\approx \frac{1}{(k\epsilon)^2}e^{(1-\epsilon)\frac{k}{2}\phi(x)}\sum_j |p(x_j)|e^{-\frac{k}{2}\phi(x_j)}|K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)}.
\end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned}
|p(x)|^2 e^{-k\phi(x)} \lesssim \frac{1}{k^4}e^{-\epsilon k\phi(x)} &\left\{\sum_j |p(x_j)|^2 e^{-k\phi(x_j)}|K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)}\right\} \\
&\cdot \sum_j |K_{\frac{\epsilon}{2}k}(x,x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)}.
\end{aligned}$$

By the Plancherel-Polya inequality (Lemma 1.7), the second term is bounded by $k^2 e^{\frac{\epsilon}{2}k\phi(x_j)}$, and therefore

$$|p(x)|^2 e^{-k\phi(x)} \lesssim \frac{1}{k^2} e^{-\frac{\epsilon}{2}k\phi(x)} \sum_j |p(x_j)|^2 e^{-k\phi(x_j)} |K_{\frac{\epsilon}{2}k}(x, x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)}.$$

Integrating,

$$\int_{\mathbb{C}} |p(x)|^2 e^{-k\phi(x)} d\mathcal{V}(x) \lesssim \frac{1}{k^2} \sum_j |p(x_j)|^2 e^{-k\phi(x_j)} e^{-\frac{\epsilon}{2}k\phi(x_j)} \int_{\mathbb{C}} |K_{\frac{\epsilon}{2}k}(x, x_j)|^2 e^{-\frac{\epsilon}{2}k\phi(x_j)} d\mathcal{V}$$

$$= \frac{1}{k^2} \sum_j |p(x_j)|^2 e^{-k\phi(x_j)} e^{-\frac{\epsilon}{2}k\phi(x_j)} K_{\frac{\epsilon}{2}k}(x_j, x_j) \approx \frac{1}{k} \sum_j |p(x_j)|^2 e^{-k\phi(x_j)},$$

as desired. □

3

A Landau's Classical Technique, Density and Equidistribution

In [11] Landau introduced a method to obtain necessary density conditions for sampling and interpolation. We shall see now how this method is adapted to our setting.

We shall also see how the method is used to estimate from below and above the number of Fekete points in a disc. Besides, we get an upper and lower bounds for the Kantorovich-Wasserstein distance between the Fekete measure and its limit measure.

3.1 Landau's Technique

Let Ω be a measurable set of $\mathbb{C}$. We will denote the restriction operator by T_Ω, a linear operator on $\mathcal{P}_k$ defined by

$$(T_\Omega p)(z) = \int_{\mathbb{C}} K(z,w)p(w)\chi_\Omega(w)\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}, \quad p \in \mathcal{P}_k,$$

where χ_Ω is the indicator function of Ω.

It is known that the previous operator is self-adjoint and non-negative, that is, for all $p \in \mathcal{P}_k$,

$$\langle T_\Omega p, p\rangle = \langle p, T_\Omega p\rangle,$$

and

$$\langle T_\Omega p, p\rangle = 0 \Leftrightarrow p = 0.$$

Moreover, by definition

$$\langle T_\Omega p, p\rangle = \langle \chi_\Omega p, p\rangle = \int_\Omega |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k},$$

where the previous scalar product is defined on a 'bigger' Hilbert space, i. e., f belongs to the previous space if

$$\int_{\mathbb{C}} |f(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} < \infty$$

but it is not necessary to be holomorphic.

Let us see that $\|T_\Omega\| \leq 1$. By duality, first

$$\langle T_\Omega p, f\rangle = \int_{\mathbb{C}} \left\{ \int_\Omega K(z,w)p(w)\frac{d\mathcal{V}(w)}{(1+|w|^2)^k} \right\} \overline{f(z)} \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$
$$\int_\Omega p(w) \left\{ \int_{\mathbb{C}} K(z,w)\overline{f(z)}\frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \right\} \frac{d\mathcal{V}(w)}{(1+|w|^2)^k}$$
$$= \int_\Omega p(w)\overline{f(w)}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}.$$

Hence,

$$\|T_\Omega p\| \leq \sup_{\|f\|\leq 1} \left(\int_\Omega |f(w)|^2 \frac{d\mathcal{V}(w)}{(1+|w|^2)^k} \right)^{1/2} \left(\int_\Omega |p(w)|^2 \frac{d\mathcal{V}(w)}{(1+|w|^2)^k} \right)^{1/2}$$
$$\leq \sup_{\|f\|\leq 1} \|f\| \cdot \|p\| \leq \|p\|.$$

Finally, this yields

$$\|T_\Omega\| = \sup_{\|p\|\leq 1} \|T_\Omega p\| \leq 1.$$

The spectral theorem allows to find an orthonormal basis $\{p_j\}_{j=0}^k$ of $\mathcal{P}_k$ consisting of eigenfunctions,

$$T_\Omega(p_j) = \lambda_j(\Omega)p_j, \quad (j = 0, \ldots, k).$$

The eigenvalues $\lambda_j(\Omega)$ lie between 0 and 1, and we can order them in a non-increasing way,

$$\lambda_0(\Omega) \geq \lambda_1(\Omega) \geq \lambda_2(\Omega) \geq \ldots \geq \lambda_k(\Omega) \geq 0$$

Using that

$$K(z,z) = \sum_{j=0}^k |p_j(z)|^2,$$

we can compute the trace of the operator:

$$\sum_{j=0}^k \lambda_j(\Omega) = \sum_{j=0}^k \langle T_\Omega p_j, p_j\rangle = \sum_{j=0}^k \int_\Omega |p_j(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} = \int_\Omega K(z,z)\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}.$$

By the same reason, the Hilbert-Schmidt norm of the operator can be expressed as

$$\sum_{j\geq 1}\lambda_j^2(\Omega) = \iint_{\Omega\times\Omega} |K(z,w)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}.$$

This follows from the identities

$$\int_{\mathbb{C}} (T_\Omega p_j)(z)\overline{(T_\Omega p_j)(z)}\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$

$$= \int_{\mathbb{C}} \left(\int_\Omega K(z,\zeta)p_j(\zeta)\frac{d\mathcal{V}(\zeta)}{(1+|\zeta|^2)^k}\right)\left(\int_\Omega \overline{K(z,\eta)p_j(\eta)}\frac{d\mathcal{V}(\eta)}{(1+|\eta|^2)^k}\right)\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$

$$\iint_{\Omega\times\Omega} p_j(\zeta)\overline{p_j(\eta)}\left\{\int_{\mathbb{C}} K(z,\eta)\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\right\}\frac{d\mathcal{V}(\zeta)}{(1+|\zeta|^2)^k}\frac{d\mathcal{V}(\eta)}{(1+|\eta|^2)^k}$$

$$\iint_{\Omega\times\Omega} p_j(\zeta)\overline{p_j(\eta)}K(\eta,\zeta)\frac{d\mathcal{V}(\zeta)}{(1+|\zeta|^2)^k}\frac{d\mathcal{V}(\eta)}{(1+|\eta|^2)^k}.$$

Using that $\sum_j p_j(\zeta)\overline{p_j(\eta)} = K(\zeta,\eta)$ and $\lambda_j^2(\Omega) = \langle T_\Omega p_j, T_\Omega p_j\rangle$, we get the result.

Lemma 3.1. *Let $0<\gamma<1$ and denote by $n(\Omega,\gamma)$ the number of eigenvalues $\lambda_j(\Omega)$ which are strictly bigger than γ. Then we have the lower bound*

$$n(\Omega,\gamma) \geq \int_\Omega |K(z,z)|\frac{d\mathcal{V}(z)}{(1+|z|^2)^k} - \frac{1}{1-\gamma}\iint_{\Omega\times\Omega^c} |K(z,w)|^2\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k},$$

and the upper bound

$$n(\Omega,\gamma) \leq \int_\Omega |K(z,z)|\frac{d\mathcal{V}(z)}{(1+|z|^2)^k} + \frac{1}{\gamma}\iint_{\Omega\times\Omega^c} |K(z,w)|^2\frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}.$$

Proof. We have

$$\chi_{(\gamma,1]}(z) \geq z - \frac{z(1-z)}{1-\gamma} \quad (0\leq z\leq 1),$$

hence

$$n(\Omega,\gamma) = \sum_j \chi_{(\gamma,1]}(\lambda_j(\Omega)) \geq \sum_j \lambda_j(\Omega) - \frac{1}{1-\gamma}\sum_j\left(\lambda_j(\Omega)-\lambda_j^2(\Omega)\right),$$

and by the previous calculations of the traces of T_Ω and T_Ω^2, this is

$$
\begin{aligned}
&\int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} - \frac{1}{1-\gamma}\Big[\int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \\
&- \iint_{\Omega\times\Omega} |K(z,w)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}\Big] \\
&\geq \int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} - \frac{1}{1-\gamma}\Big[\iint_{\Omega\times\mathbb{C}} |K(z,w)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k} \\
&- \iint_{\Omega\times\Omega} |K(z,w)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k}\Big]
\end{aligned}
$$

and this proves the first part.

The proof of the second statement is similar using the inequality

$$\chi_{(\gamma,1]}(z) \leq z + \frac{z(1-z)}{\gamma} \quad (0 \leq z \leq 1).$$

□

Lemma 3.2. *Let Λ_k be a δ-separated sampling set at level k with sampling constants A, B. Then for any $z \in \mathbb{C}$ and $r > 0$,*

$$\#(\Lambda_k \cap D(z, \frac{r+\delta}{\sqrt{k}})) \geq n(D(z, \frac{r}{\sqrt{k}}), \gamma)$$

where λ is some constant lying between 0 and 1, such that $1/(1-\gamma)$ is bounded by the sampling constant B times a constant which only may depend on δ.

Proof. Let $\{p_j\}_j$ be the orthonormal basis of eigenvectors associated to the eigenvalues $\lambda_j^{(k)}(\Omega)$, where $\Omega = D(z, \frac{r}{\sqrt{k}})$. Let $N = \#(\Lambda_k \cap D(z, \frac{r+\delta/2}{\sqrt{k}}))$. We study the case when $N < k+1$, so take

$$p = \sum_{j=1}^{N+1} c_j p_j,$$

of the first eigenfunctions such that not all c_j are 0 and

$$p(\lambda) = 0, \quad \lambda \in \Lambda_k \cap D(z, \frac{r+\delta/2}{\sqrt{k}}).$$

Since Λ_k is a sampling set, it holds that

$$
\begin{aligned}
\|p\|^2 &\leq \frac{B}{k} \sum_{\lambda \in \Lambda \setminus D(z, \frac{r+\delta/2}{\sqrt{k}})} \frac{|p(\lambda)|^2}{(1+|\lambda|^2)^k} \\
&\leq C(\delta) B \int_{D(\lambda, \frac{\delta/2}{\sqrt{k}})} |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \\
&\leq C(\delta) B \int_{\mathbb{C}\setminus\Omega} |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}.
\end{aligned}
$$

Hence, by the pythagorean and the spectral theorems

$$\lambda_{N+1}(\Omega)\|p\|^2 = \langle T_\Omega p, p\rangle = \int_\Omega |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \leq \left(1 - \frac{1}{C(\delta)B}\right)\|p\|^2.$$

Therefore,

$$\lambda_{N+1}(\Omega) \leq \gamma'$$

and

$$n_k(\Omega, \gamma') \leq N.$$

□

Lemma 3.3. *Let Λ_k be a δ-separated interpolation set at level k with interpolation constant C, then for any $z \in \mathbb{C}$ and $r > 0$,*

$$\#(\Lambda_k \cap D(z, \frac{r-\delta}{\sqrt{k}})) \leq n(D(z, \frac{r}{\sqrt{k}}), \gamma)$$

where λ is some constant lying between 0 and 1, such that $1/\gamma$ is bounded by the interpolation constant C times a constant which only may depend on δ.

Proof. Let W denote the orthonormal complement in $\mathcal{P}_k$ of the subspace of polynomials vanishing in Λ_k. Since Λ_k is an interpolation set at level k, for any set of values $\{v_\lambda\}_{\lambda\in\Lambda_k} \subset \mathbb{C}$, there exist a polynomial p with $p(\lambda) = \lambda, \quad \lambda \in \Lambda_k$ and

$$\|p\|^2 \leq \frac{C}{k} \sum_{\lambda\in\Lambda_k} \frac{|p(\lambda)|^2}{(1+|\lambda|^2)^k}.$$

If we take the projection of p onto W, this is another solution to the interpolation problem with the same control in the norm, since the values of the projection are null. Moreover, if $p \in W$ then it is the unique interpolant in W, so every polynomial in W satisfies the control in the norm.

Let $\{z_k\}_{j=1}^N$ the elements in $\Lambda_k \cap D(z, \frac{r-\delta}{\sqrt{k}})$. For each one we can find a polynomial such that

$$\frac{p_i(z_j)}{(1+|z_j|^2)^{\frac{k}{2}}} = \delta_{ij},$$

so they are linearly independent. Denote by F the linear subspace spanned by the previous polynomials, and take a polynomial $p \in F$, then

$$\|p\|^2 \leq \frac{C}{k} \sum_{\lambda\in\Lambda_k\cap D(z,\frac{r-\delta}{\sqrt{k}})} \frac{|p(\lambda)|^2}{(1+|\lambda|^2)^k} < D(\delta)C\int_\Omega |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$

where $\Omega = D(z, \frac{r}{\sqrt{k}})$. The last inequality comes from Plancherel-Polya (Lemma 1.7). Therefore,

$$\frac{\langle T_\Omega p, p\rangle}{\|p\|^2} = \frac{1}{\|p\|^2}\int_\Omega |p(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} > \frac{1}{D(\delta)C} = \gamma,$$

for any polynomial of F. Finally, by the min-max theorem

$$\lambda_N = \min_{p_j\,(1\le j\le N-1)} \max \{\langle T_\Omega^{(k)} s, s\rangle : s \in \langle p_j\rangle \text{ and } \|s\| = 1\} > \lambda,$$

it holds that $n_k(\Omega, \gamma)$. □

Finally, we have an estimate of the number of points of a sampling or interpolation set in a disc in terms of the Bergman kernel. The proofs are straightfoward and follow from the previous statements, so we omit them.

Lemma 3.4. *Let Λ_k be a δ-separated sampling set at level k with sampling constants A, B. Then for any $z \in \mathbb{C}$ and $r > 0$,*

$$\#(\Lambda_k \cap D(z, \frac{r+\delta}{\sqrt{k}})) \ge \int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} - M \iint_{\Omega\times\Omega^c} |K(z,w)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k},$$

where $\Omega = D(z, r/\sqrt{k})$ and the constant M is bounded by the sampling constant B times a constant which only may depend on δ.

Lemma 3.5. *Similarly, if Λ_k is a δ-separated interpolation set at level k with interpolation constant C, then for any $z \in \mathbb{C}$ and $r > 0$,*

$$\#(\Lambda_k \cap D(z, \frac{r-\delta}{\sqrt{k}})) \le \int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} + M \iint_{\Omega\times\Omega^c} |K(z,w)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}\frac{d\mathcal{V}(w)}{(1+|w|^2)^k},$$

where $\Omega = D(z, r/\sqrt{k})$ and the constant M is bounded by the interpolation constant C times a constant which only may depend on δ.

3.2 Density Conditions and Equidistribution

We can characterize the sampling and interpolating arrays in terms of the number of points of the array in a disc normalised by the measure of this disc. The separation of the Fekete points can be interpreted in terms of density as follows.

Lemma 3.6. *If $\mathcal{F}_{k+1}$ is a Fekete configuration, there is $R > 0$ so that for any $z \in \mathbb{C}$, $D(z, R/\sqrt{k})$ contains at least one point of the Fekete family.*

The geometric conditions can be found in the following lemma.

Lemma 3.7. *If Λ_k is a δ-separated sampling set at level k with sampling constants A, B, then for any $z \in \mathbb{C}$ and $r > 0$,*

$$\frac{\#(\Lambda_k \cap \Omega)}{k \int_\Omega d\mathcal{V}(z)} > 1 - \frac{M}{r},$$

where $\Omega = D(z, r/\sqrt{k})$ and the constant M is bounded by the sampling constant B times a constant which only may depend on δ.

Similarly, if Λ_k is a δ-separated interpolation set at level k with interpolation constant C, then for any $z \in \mathbb{C}$ and $r > 0$,

$$\frac{\#(\Lambda_k \cap \Omega)}{k \int_\Omega d\mathcal{V}(z)} < 1 + \frac{M}{r},$$

where $\Omega = D(z, r/\sqrt{k})$ and the constant M is bounded by the sampling constant C times a constant which only may depend on δ.

Proof. Let us prove the sampling part, the other case is similar using the corresponding lemma. So assume that Λ_k is a δ-separated sampling set at level k, and let $\Omega = D(z, \frac{r}{\sqrt{k}})$. We can estimate the measure of annuli as in (**??**), so the separation condition with the estimate of the volume of annuli imply the bound

$$\#(\Lambda_k \cap (D(z, \frac{r+\delta}{\sqrt{k}}) \backslash D(z, \frac{r}{\sqrt{k}}))) \leq r\delta,$$

Hence by the estimate of the Bergman Kernel (Lemma 1.2) and the Lemma 3.4 for the estimate of the number of points in a disc, we have

$$\#(\Lambda_k \cap \Omega) \geq (k+1) \int_\Omega d\mathcal{V}(z) - \delta_2 r,$$

as the integral is of order r^2, since, by the invariance

$$\int_{D(z,r/\sqrt{k})} d\mathcal{V}(z) = \int_{D(0,r/\sqrt{k})} \frac{dm(z)}{(1+|z|^2)^2} = \int_0^{r^2/k} \frac{dt}{(1+t)^2} \approx r^2/k$$

the result follows. □

In the case of the Fekete points we can say about the density of these in any disc the following.

Theorem 3.8. *For every $r > 0$,*

$$\frac{\#(\mathcal{F}_k \cap D(z,r))}{\#\mathcal{F}_k} = \left(1 + O\left(\left(r\sqrt{k}\right)^{-1}\right)\right) \int_{D(z,r)} d\mathcal{V}(w)$$

uniformly in $z \in \mathbb{C}$.

Proof. Denote $\Omega = D(z, \frac{R}{\sqrt{k}})$, and let l_λ be the Lagrange polynomial associated to $\lambda \in \mathcal{F}_k$, we are allowed to write

$$\#(\mathcal{F}_k \cap \Omega) - \int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} = \sum_{\lambda \in \mathcal{F}_k \cap \Omega} \int_{\mathbb{C}} l_\lambda(z) \frac{K(\lambda,z)}{(1+|\lambda|^2)^{\frac{k}{2}}} \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$
$$-\int_\Omega \sum_{\lambda \in \mathcal{F}_k} l_\lambda(z) \frac{K(\lambda,z)}{(1+|\lambda|^2)^{\frac{k}{2}}} \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} = \int_{\mathbb{C}\setminus\Omega} \sum_{\lambda \in \mathcal{F}_k \cap \Omega} l_\lambda(z) \frac{K(\lambda,z)}{(1+|\lambda|^2)^{\frac{k}{2}}} \frac{d\mathcal{V}(z)}{(1+|z|^2)^k}$$
$$- \int_\Omega \sum_{\lambda \in \mathcal{F}_k \cap (\mathbb{C}\setminus\Omega)} l_\lambda(z) \frac{K(\lambda,z)}{(1+|\lambda|^2)^{\frac{k}{2}}} \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} = A_1 - A_2.$$

Let us estimate A_1, then

$$|A_1| \le \int_{\Omega^c} \sum_{\lambda \in \mathcal{F}_k \cap \Omega} |K(z,\lambda)| e^{-k/2\phi(z)} e^{-k/2\phi(\lambda)} d\mathcal{V}(z).$$

By the sub-mean property (Lemma 1.5)

$$\frac{|K(\lambda,z)|}{(1+|\lambda|^2)^{\frac{k}{2}}} \lesssim \left(\frac{\delta}{\sqrt{k}}\right)^{-2} \int_{D(\lambda,\delta/\sqrt{k})} |K(z,w)| \frac{d\mathcal{V}(w)}{(1+|w|^2)^{\frac{k}{2}}},$$

taking δ the constant of separation of $\mathcal{F}_k$, we get

$$\sum_{\lambda \in \mathcal{F}_k \cap D(z, \frac{R-\delta}{\sqrt{k}})} \frac{|K(\lambda,z)|}{(1+|\lambda|^2)^{\frac{k}{2}}} \le k \iint_{\Omega^c \times \Omega} |K(z,w)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}}} \frac{d\mathcal{V}(w)}{(1+|w|^2)^{\frac{k}{2}}} \lesssim R.$$

On the other hand, the estimate for annuli together with the separation condition imply

$$\#\left(\mathcal{F}_k \cap \left(D(z, \frac{R}{\sqrt{k}}) \setminus D(z, \frac{R-\delta}{\sqrt{k}})\right)\right) \lesssim R,$$

hence by the estimates of the Bergman Kernel (Lemma 1.2)

$$\int_{\Omega^c} \sum_{\mathcal{F}_k \cap \left(D(z,\frac{R}{\sqrt{k}}) \setminus D(z,\frac{R-\delta}{\sqrt{k}})\right)} \frac{|K(\lambda,z)|}{(1+|\lambda|^2)^{\frac{k}{2}}} \lesssim R \sup_\lambda \int_{\mathbb{C}} |K(z,\lambda)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}}} \lesssim R.$$

Then, $|A_1| \lesssim R$ and $|A_2| \lesssim R$, which can be estimated in the same way. By the form of the Bergman Kernel and the volume form we get

$$\#(\mathcal{F}_k \cap \Omega) = \int_\Omega |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}}} + O(R) = (1 + O(R^{-1}))k \int_\Omega d\mathcal{V}(z).$$

Moreover,

$$\#\mathcal{F}_k = \int_{\mathbb{C}} |K(z,z)| \frac{d\mathcal{V}(z)}{(1+|z|^2)^{\frac{k}{2}}} = (1 + O(k^{-1}))k \int_\Omega d\mathcal{V}(z),$$

these statements complete the proof. □

Definition 3.9. *Let $\nu_\Lambda^-(R)$, respectively $\nu_\Lambda^+(R)$, denote the infimum, respectively supremum, of the ratio*

$$\frac{\#(\Lambda_k \cap \Omega)}{k \int_\Omega d\mathcal{V}(z)}$$

where $\Omega = D(z,r)$, over all $z \in \mathbb{C}$, and all k, r such that $R/\sqrt{k} \leq r \leq 2$. The lower and upper Beurling-Landau densities are defined by

$$D^-(\Lambda) = \liminf_{R\to\infty} \nu_\Lambda^-(R) \qquad D^+(\Lambda) = \limsup_{R\to\infty} \nu_\Lambda^+(R).$$

Corollary 3.10. *For any $\epsilon > 0$, there exist*

1. *a sampling array Λ with $D^+(\Lambda) < 1 + \epsilon$,*
2. *an interpolation array Λ with $D^-(\Lambda) > 1 + \epsilon$.*

We have two probability measures, the Fekete measure $\mu_k = \frac{1}{k+1}\sum_{j=0}^k \delta_{\lambda_j^k}$ and its weak limit measure $\mathcal{V}$, which is called the equilibrium measure. This last fact is due to the Theorem of Berman, Boucksom and Witt [7].

To metrize the weak convergence we use the following definition.

Definition 3.11. *Given two probability measures μ and ν on a metric space X, one defines the Kantorovich-Wasserstein distance W is defined as*

$$W(\mu,\nu) = \inf\left\{\iint_{X\times X} d(x,y)d\rho(x,y)\right\}$$

where the infimum is taken over all Borel probability measures ρ on $X \times X$ with marginals $\rho(\cdot, X) = \mu$ and $\rho(X, \cdot) = \nu$. Equivalently,

$$W(\mu,\nu) = \sup_{f\in Lip_{1,1}(X)} \left\{\left|\int_X f d(\mu - \nu)\right|\right\},$$

where $Lip_{1,1}(X)$ is the collection of all functions f on X satisfying $|f(x)-f(y)| \leq d(x,y)$.

We say that the measure μ converges weakly to v, $\mu \overset{w}{\to} v$, if

$$\int_X f(x)d\mu(x) \to \int_X f(x)dv(x)$$

for every $f \in L^\infty(X) \cap \mathcal{C}(X)$.

The next theorem gives us the testing of the Fekete points are in a sense optimally distributed.

Theorem 3.12. *Let μ_k the empirical measure associated to a Fekete sequence at level k and W the Kantorovich-Wasserstein distance. Then, there exists $C > 0$ such that*

$$\frac{1}{C}\frac{1}{\sqrt{k}} \leq W(\mu_k, \mathcal{V}) \leq C\frac{1}{\sqrt{k}}$$

as $k \to \infty$.

Proof. To prove the lower bound we consider the function

$$f_k(x) = \text{dist}(x, \mathcal{F}_k),$$

which belongs to $\text{Lip}_{1,1}(\mathbb{C})$ and vanishes on the Fekete points. This last fact and the use of the second definition imply

$$W(\mu_k, \mathcal{V}) \geq \left| \int_{\mathbb{C}} f_k(d\mu_k - d\mathcal{V}) \right| = \int_{\mathbb{C}} f_k d\mathcal{V}.$$

On the other hand, by Lemma 2.3

$$\int_{\mathbb{C}} f_k d\mathcal{V} \geq \frac{\delta}{\sqrt{k}} \cdot \mathcal{V}(\mathbb{C} \setminus \bigcup_{\lambda \in \mathcal{F}_k} D(\lambda, \delta/\sqrt{k})) \geq \frac{\delta}{\sqrt{k}}(1 - \frac{C\delta^2 \# \mathcal{F}_k}{k}),$$

since f_k is bounded below by the radius of the disc $D(\lambda, \delta)$, $\lambda \in \mathcal{F}_k$. Being $\#\mathcal{F}_k = k+1$, we can choose δ such that $C\delta^2/k\#\mathcal{F}_k = 1/2$, it holds that

$$W(\mu_k, \mathcal{V}) \gtrsim \frac{1}{\sqrt{k}}.$$

For the upper estimate, we will use an alternative definition of the Kantorovich-Wasserstein distance. If S is the set of complex measures on $\mathbb{C}\times\mathbb{C}$ with marginals μ and $\mathcal{V}$ respectively, then

$$W(\mu, \mathcal{V}) = \inf_{\rho \in S} \iint_{\mathbb{C}\times\mathbb{C}} d(x,y)|d\rho(x,y)|.$$

This alternative definition coincides with the previous one since

$$\inf_{\rho\in S}\iint_{\mathbb{C}\times\mathbb{C}} d(x,y)|d\rho(x,y)| \le W(\mu,\mathcal{V}),$$

and for $\rho \in S$,

$$\left|\int_{\mathbb{C}} f d(\mu-\mathcal{V})\right| = \left|\iint_{\mathbb{C}\times\mathbb{C}} (f(x)-f(y))d\rho\right| \le \iint_{\mathbb{C}\times\mathbb{C}} d(x,y)|d\rho(x,y)|.$$

Hence

$$W(\mu,\mathcal{V}) \le \inf_{\rho\in S}\iint_{\mathbb{C}\times\mathbb{C}} d(x,y)|d\rho(x,y)|.$$

In order to get the upper bound for $W(\mu_k,\mathcal{V})$ consider the complex measure

$$d\rho(x,y) = \frac{1}{k+1}\sum_{\lambda\in\mathcal{F}_k} \delta_\lambda(x) l_\lambda(y)\frac{K(\lambda,y)}{(1+|\lambda|^2)^{\frac{k}{2}}} dV(y),$$

where l_λ are the Lagrange polynomials. Thus

$$\begin{aligned} W(\mu_k,\mathcal{V}) \le \iint_{\mathbb{C}\times\mathbb{C}} d(x,y)|d\rho(x,y)| &= \frac{1}{k+1}\sum_{\lambda\in\mathcal{F}_k}\int_{\mathbb{C}} d(\lambda,y) l_\lambda(y)\frac{K(\lambda,y)}{(1+|\lambda|^2)^{\frac{k}{2}}} dV(y) \\ &\le \sum_{\lambda\in\mathcal{F}_k}\int_{\mathbb{C}} d(x,y)|1+z\bar{\lambda}|^k\frac{d\mathcal{V}(y)}{(1+|y|^2)^{\frac{k}{2}}} \lesssim 1/\sqrt{k} \end{aligned}$$

by the estimates of the Bergman kernel (Lemma 1.2). □

4

Simultaneously Sampling and Interpolation Arrays

In this section, we will prove that there are no arrays which are interpolating and sampling at the same time. To see this we need to study a related result in the Bargmann-Fock space, which can be seen as a limit space of $\mathcal{P}_k$ as $k \to \infty$.

4.1 The Bargmann-Fock Space $\mathcal{BF}^p$

Definition 4.1. *Given $p \in [1, \infty)$, the Bargmann-Fock space $\mathcal{BF}^p$ consists of entire functions such that*

$$\|f\|_p^p = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2/2} dm(z) < \infty.$$

When $p = \infty$, the norm is

$$\|f\|_\infty = \sup_{\mathbb{C}} |f| e^{-|z|^2/2}.$$

We will relate sampling and interpolating arrays for $\mathcal{P}_k$ to sampling and interpolating sequences for $\mathcal{BF}^2$. Notice that the Bergman kernel for this space is

$$K(z, \sigma) = e^{\bar{\sigma} z},$$

and the normalised kernel is

$$\frac{K(z,\sigma)}{\sqrt{K(\sigma,\sigma)}} = e^{\bar{\sigma} z - \frac{1}{2}|\sigma|^2}.$$

The general definitions of sampling and interpolation sequences for a Hilbert space given in Section 1 take, for the Hilbert space $\mathcal{BF}^2$, the following form.

Definition 4.2. *A sequence Σ is sampling for $\mathcal{BF}^2$ if and only if there are A, $B>0$ such that*

$$A\|f\|_2^2 \le \sum_{\sigma\in\Sigma} |f(\sigma)|^2 e^{-|\sigma|^2} \le B\|f\|_2^2,$$

and it is interpolation for $\mathcal{BF}^2$ if given any values $\{\nu_\sigma\}_{\sigma\in\Sigma}$ such that

$$\sum_{\sigma\in\Sigma} |v_\sigma|^2 e^{-|\sigma|^2} < +\infty,$$

there is a function $f \in \mathcal{BF}^2$ such that $f(\sigma) = \nu_\sigma$. In particular, there is a constant $C>0$ so that

$$\|f\|^2 \le C \sum_{\sigma\in\Sigma} |f(\sigma)|^2 e^{-|\sigma|^2}.$$

We shall see that the rescaled interpolation or sampling arrays

$$\Sigma_k = \{\sqrt{k}\lambda : \lambda \in \Lambda_k\}$$

tend to interpolating or sampling sequences of $\mathcal{BF}^2$.

Definition 4.3. *Let Σ_k be a collection of separated sequences, with a uniform separation constant for all k, and let Σ be another separated sequence. We say that Σ_k converges weakly to Σ if the corresponding measures $\mu_k = \sum_{\sigma\in\Sigma_k}\delta_\sigma$ converge weakly to $\sum_{\sigma\in\Sigma}\delta_\sigma$. If this is the case, we will write $\Sigma \in W(\Sigma_k)$.*

Let $\bar\partial$ denote the operator

$$\bar\partial : \mathcal{C}^\infty(\mathbb{C}, \Lambda^{p,q}) \to \mathcal{C}^\infty(\mathbb{C}, \Lambda^{p,q+1}),$$

where $\mathcal{C}^\infty(\mathbb{C}, \Lambda^{p,q})$ is the space of smooth forms of bi-degree (p,q).

The following result by Hörmander will be used to solve the $\bar\partial$-equation in our setting (see [4, p. 7, Theorem 1.3]). Let $\Omega \subset \mathbb{C}$ and $\psi \in \mathcal{C}^2(\Omega)$ with $\Delta\psi \ge 0$. For any $f \in L^2_{\text{loc}}(\Omega)$, there is a solution u of $\frac{\partial u}{\partial \bar z} = f$, satisfying

$$\int |u(z)|^2 e^{-\psi(z)} dm(z) \le \int \frac{|f(z)|^2}{\Delta\psi(z)} e^{-\psi(z)} dm(z). \tag{4.1}$$

This will allow to verify that any weak limit of a partial subsequence of a reescaled interpolating, or sampling, array on $\mathcal{P}_k$ is an interpolation, or sampling, array on $\mathcal{BF}^2$.

Theorem 4.4. *Let $f \in C^1_c(\bar{\mathbb{C}})$ and $z \in \mathbb{C}$ then*

$$f(z) = m\int_{\mathbb{C}} \left(\frac{1+\bar\zeta z}{1+|\zeta|^2}\right)^{m-1} f(\zeta) d\mathcal{V}(\zeta) - \int_{\mathbb{C}} \left(\frac{1+\bar\zeta z}{1+|\zeta|^2}\right)^{m} \frac{\frac{\partial f(\zeta)}{\partial\bar\zeta}}{\zeta - z} dm(\zeta),$$

for all $m \in \mathbb{N}$.

Proof. Fix $z \in \Omega$ where Ω is a big disc such the support of f is contanied there and take $\epsilon > 0$ such that the disc $D = D(z, \epsilon) \subset \Omega$. If we apply the Stokes theorem [9, p. 2] to the form

$$\left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^m \frac{f(\zeta)}{\zeta - z} d\zeta$$

on the domain $\Omega \setminus D$ with the boundary positively oriented, we get

$$\int_{\Omega\setminus D} \left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^m \frac{\frac{\partial f(\zeta)}{\partial \bar{\zeta}}}{\zeta - z} dm(\zeta) - m \int_{\Omega\setminus D} \left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^{m-1} f(\zeta) d\mathcal{V}(\zeta)$$
$$= \int_{\Omega\setminus D} \bar{\partial} \left[\left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^m \frac{f(\zeta)}{\zeta - z}\right] d\zeta$$
$$= \frac{i}{2} \int_{\partial\Omega} \left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^m \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{i}{2} \int_{\partial D} \left(\frac{1+\bar{\zeta}z}{1+|\zeta|^2}\right)^m \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Observe that $1/(\zeta - z)$ is holomorphic in the new domain. The result follows from the fact that $1/(\zeta - z)$ is locally integrable, taking $\epsilon \to 0$ and extending Ω to the whole plain, so the first integral is zero and the second is $-\pi f(z)$, where the sign is due to the orientation of the small disk D. □

Theorem 4.5. *Given $f \in \mathcal{P}_k$, let*

$$u(z) = -\int_{\mathbb{C}} \left(\frac{1+z\bar{\zeta}}{1+|\zeta|^2}\right)^k \frac{f(\zeta)}{z - \zeta} dm(\zeta).$$

Then u satisfies $\frac{\partial u}{\partial \bar{z}} = f$, and

$$\int_{\mathbb{C}} |u(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \leq \frac{1}{2k+8} \int_{\mathbb{C}} |f(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^{k+2}}. \tag{4.2}$$

This is a refinement of Hörmander's theorem due to M. Andersson and B. Berndtsson. The proof of the theorem is a particular case of [3, Theorem 8] and the previous theorem.

Proof. From the previous Theorem, we have the form of the solution.The estimate in terms of the data function is due to the Hörmander estimates (4.1). Take $\psi(z) = (k+2)\log(1+|z|^2)$, hence $\Delta\psi(z) = 4\,(k+2)\,\frac{1}{(1+|z|^2)^2}$ and

$$\int_{\mathbb{C}} |u(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \leq \frac{1}{2k+8} \int_{\mathbb{C}} |f(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^{k+2}}.$$

Let us see now in what sense the space $\mathcal{BF}^2$ is the limit of the spaces $\mathcal{P}_k$.

Lemma 4.6. *Given any function $f \in \mathcal{BF}^2$, and any big $M > 0$, there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, there are polynomials $p_k \in \mathcal{P}_k$ such that*

$$\int_{|z|<M/\sqrt{k}} |f(\sqrt{k}z) - p_k(z)|^2 e^{-k|z|^2} dm(z) \lesssim \frac{1}{k}\|f\|^2$$

and

$$\int_{|z|>M/\sqrt{k}} |p_k(z)|^2 e^{-k\phi(z)} d\mathcal{V}(z) \lesssim \frac{1}{k}\|f\|^2,$$

where the previous norm $\|\cdot\|$ in L^2 coincides with the norm in $\mathcal{P}_k$. In particular $||p_k||^2 \backsimeq ||f||^2/k$ for all $k \geq k_0$.

Proof. Let χ be a cutoff function supported in a disc of radius M centered at the origin and equal to 1 in $D(0, M/2)$, with M such that $|\nabla\chi| \leq 4/M$. We put $\chi_k(z) = \chi(z\sqrt{k})$ and define $g_k(z) = f(\sqrt{k}z)\chi_k(z)$, which is not holomorphic, but grows like a polynomial of degree k in the disk.

We are looking for a modification of g_k of the form $p_k = g_k - u_k$. On the one hand, for p_k to be holomorphic we need $\bar{\partial}g_k = \bar{\partial}u_k$. On the other hand, we also want that the modification does not affect greatly the value of g_k in $D(0, M/2)$, i.e., p_k grows like a polynomial of degree k. This is due to Hörmander's estimates (4.2).

Finally, let us see that p_k is a holomorphic polynomial. The Cauchy formula yields, for R sufficiently big

$$\frac{p_k^{(j)}(0)}{j!} = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{p_k(\zeta)}{\zeta^{j+1}} d\zeta = -\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{u_k(\zeta)}{\zeta^{j+1}} d\zeta.$$

Taking modulus, if supp $\frac{\partial g_k}{\partial \bar{z}} = T$, T compact,

$$\frac{1}{2\pi}\int_{|\zeta|=R} \left| \int_{\mathbb{C}} \left(\frac{1+\zeta\bar{\eta}}{1+|\eta|^2}\right)^k \frac{\frac{\partial g_k(\eta)}{\partial \bar{\eta}}}{\zeta - \eta} dm(\eta) \right| \frac{|d\zeta|}{|\zeta|^{j+1}}$$
$$\lesssim \left(\max_T \left|\frac{\partial g_k}{\partial \bar{z}}\right|\right) \frac{1}{R^{j-k}} \left\{ \int_{\mathbb{C}} \frac{|\eta|^k dm(\eta)}{(1+|\eta|^2)^k |\zeta - \eta|} \right\},$$

and this goes to 0 when $j > k$ and $R \to \infty$. Thus $p_k^{(j)(0)=0}$, for all $j > k$, and $p_k \in \mathcal{P}_k$.

We also want that

$$\int_{|z|<M/\sqrt{k}} |f(\sqrt{k}z) - p_k(z)|^2 e^{-k|z|^2} dm(z)$$

$$\begin{aligned}
&\simeq \int_{|z|<M/\sqrt{k}} |f(\sqrt{k}z) - p_k(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \\
&\leq \int_{M/\sqrt{4k}<|z|<M/\sqrt{k}} |f(\sqrt{k}z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} + \int_{|z|<M/\sqrt{k}} |u_k(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \\
&\qquad\qquad \lesssim \frac{\|f\|^2}{k} + \int_{M/\sqrt{4k}<|z|<M/\sqrt{k}} |\bar{\partial} g_k(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \\
&\qquad \lesssim \frac{\|f\|^2}{k} + \int_{M/\sqrt{4k}<|z|<M/\sqrt{k}} |\bar{\partial}\chi(z\sqrt{k}) f(\sqrt{k}z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \lesssim \frac{\|f\|^2}{k},
\end{aligned}$$

where last inequalities come from the estimates $|\nabla\chi| \leq 4/M$ and the estimates of Hörmander.

Finally, notice that

$$\int_{|z|>M/\sqrt{k}} |p_k(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} = \int_{|z|>M/\sqrt{k}} |u_k(z)|^2 \frac{d\mathcal{V}(z)}{(1+|z|^2)^k} \lesssim \frac{\|f\|^2}{k}.$$

□

4.2 Interpolation and Sampling arrays and Interpolation and Sampling Sequences for $\mathcal{BF}^2$

Let us see now the relation between sampling and interpolating arrays for $\mathcal{P}_k$ the sampling and interpolating arrays for $\mathcal{BF}^2$.

Theorem 4.7. *Let Λ_k be a separated array for $\mathcal{P}_k$ and let Σ be any weak limit of a partial subsequence of Σ_k.*

1. *If Λ_k is a sampling array, then Σ is a sampling sequence for $\mathcal{BF}^2$.*
2. *If Λ_k is an interpolation array, then Σ is an interpolation sequence for $\mathcal{BF}^2$.*

Proof. Let us start with the interpolation part. Assume that Σ is the weak limit of a partial subsequence of Σ_k still denoted by Σ_k. Take a sequence of values $\{v_\sigma\}_{\sigma\in\Sigma} \subset \mathbb{C}$ with $\sum_{\sigma\in\Sigma} |v_\sigma|^2 e^{-|\sigma|^2} < \infty$. We will construct a sequence of functions $f_k \in \mathcal{O}(D(0, M_k))$, with $M_k \to \infty$, such that

$$\sup_k \int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} dm(z) < \infty$$

and $\lim_k f_k(\sigma) = v_\sigma$ for all $\sigma \in \Sigma$. Thus we will conclude by a normal family argument that there is an interpolating function $f \in \mathcal{BF}^2$ with $f(\sigma) = v_\sigma$. We may assume without loss of generality that, except for a finite number of points, $v_\sigma = 0$, since

$$\limsup_{k\to\infty} \int_{|z|<M_k} |f_k(z)|^2 e^{-|z|^2} dm(z) \le C \sum_\sigma |v_\sigma|^2 e^{-|\sigma|^2},$$

where C is a constant independent of the number of non-zero terms. We can find an increasing sequence M_k such that $M_k \to \infty$ and $M_k/\sqrt{k} \to 0$. Take some given values v_σ and denote by $\Sigma' \subset \Sigma$ the finite set of points $\sigma \in \Sigma$ such that they do not vanish. For k big enough $|\sigma/\sqrt{k}| < M_k$ for all $\sigma \in \Sigma'$, and there is an associated $\lambda_\sigma^k \in \Lambda_k$ such that $\sqrt{k}\lambda_\sigma^k \to \sigma$ since $\Sigma_k \to \Sigma$ weakly.

If we consider the interpolation problem with data v_σ at the points λ_σ^k, $\sigma \in \Sigma'$, by hypothesis there is a polynomial $p_k \in \mathcal{P}_k$ so that $p_k(\lambda_\sigma^k) = v_\sigma$ and

$$\|p_k\|^2 \le \frac{C}{k} \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-k|\lambda_\sigma^k|^2}, \tag{4.3}$$

and thus

$$\int_{|z|\le M_k/\sqrt{k}} |p_k(z)|^2 e^{-k|z|^2} dm(z) \lesssim \|p_k\|^2 \le \frac{C}{k} \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-k|\lambda_\sigma^k|^2}.$$

The functions $f_k(z) = p_k(\sqrt{k}z)$ are holomorphic in $D(0, M_k)$ and satisfy, by (4.3),

$$\int_{|z|\le M_k} |f_k(z)|^2 e^{-k|z|^2} dm(z) \le C \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-k|\lambda_\sigma^k|^2},$$

and if we let $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \int_{|z|\le M_k} |f_k(z)|^2 e^{-k|z|^2} dm(z) \lesssim \sum_{\sigma\in\Sigma'} |v_\sigma|^2 e^{-|\sigma|^2}.$$

Finally, as $f_k \in \mathcal{O}(D(0, M_k))$ and we have seen that the family is uniformly bounded, then by the Montel's Theorem the family $\{f_k\}$ is normal, hence there exists a subsequence $\{f_{k_j}\}$ which converges uniformly to the desired function.

Let us prove now the sampling part. Given any function f in the Fock space, take a large $M > 0$ so that

$$\int_{|z|>M} |f(z)|^2 e^{-|z|^2} dm(z) \le 0.1\|f\|^2.$$

We can construct a sequence of polynomials p_k such that the conclusions of the previous Lemma 4.6 hold. For such p_k the sampling property of Λ_k yields

$$\|p_k\|^2 \lesssim \frac{1}{k} \sum_{\lambda\in\Lambda_k} \frac{|p_k(\lambda)|^2}{(1+|\lambda|^2)^k}.$$

Since all p_k have small L^2 norm outside $D(0, M/\sqrt{k})$, the mean value property (Lemma 1.5) implies

$$\|p_k\|^2 \lesssim \frac{1}{k} \sum_{\lambda \in \Lambda_k \cap D(0,M/\sqrt{k})} \frac{|p_k(\lambda)|^2}{(1+|\lambda|^2)^k},$$

and as $k\|p_k\|^2 \simeq \|f\|^2$ (Lemma 4.6), taking weak limits of Σ_k, it holds that

$$\|f\|^2 \lesssim \sum_{|\sigma| \leq M} |f(\sigma)|^2 e^{-|\sigma|^2}.$$

□

Theorem 4.8. *There are no arrays for $\mathcal{P}_k$ which are simultaneously interpolating and sampling.*

Proof. By reductio ad absurdum, we can suppose that there exists such array. Then, by the previous theorem, there is an associated sequence Σ which is both sampling and interpolating for $\mathcal{BF}^2$, and no such thing exists [12, p. 27]. □

5

Sufficient Density Conditions

Finally, we find necessary and sufficient conditions for an array to be sampling or interpolating. To get these conditions we will study such arrays with conditions of type L^∞ or L^1. The characterizations we will use are only possible in the one dimensional case, in fact, only the necessary conditions we have found work on higher dimensions.

5.1 Conditions for Sampling

Definition 5.1. *A separated array $\Lambda = \{\Lambda_k\}_k$ is an L^∞-sampling array if there is k_0 and a constant $0 < C < \infty$ such that, for each $k \geq k_0$ and any polynomial $p \in \mathcal{P}_k$ we have*

$$\sup_{z\in\mathbb{C}} \frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}} \leq C \sup_{\lambda\in\Lambda_k} \frac{|p(\lambda)|}{(1+|\lambda|^2)^{\frac{k}{2}}}.$$

Proposition 5.2. *If $\Lambda = \{\Lambda_k\}_k$ is an L^∞-sampling array, then $\{\Lambda_{(1+\epsilon)k}\}_k$ is L^2-sampling.*

Proof. We know by hypothesis that for any $p \in \mathcal{P}_k$,

$$\sup_{z\in\mathbb{C}} \frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}} \leq C \sup_{\Lambda_k} \frac{|p(\lambda)|}{(1+|\lambda|^2)^{\frac{k}{2}}},$$

so define for any $y \in \mathbb{C}$ and $p \in \mathcal{P}_k$ the polynomial in $\mathcal{P}_{k(1+\epsilon)}$

$$p_y(x) = p(x)\left[\frac{K_{\frac{\epsilon}{2}k}(x,y)}{K_{\frac{\epsilon}{2}k}(y,y)}\right]^2.$$

Let us take $y \in \mathbb{C}$ a point where $\frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}}$ attains its maximum. Then

$$\sup_{z\in\mathbb{C}} \frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}} = \frac{|p(y)|}{(1+|y|^2)^{\frac{k}{2}}} = \frac{|p_y(y)|}{(1+|y|^2)^{\frac{k}{2}}}$$
$$\leq C \sup_{\Lambda_{k(1+\epsilon)}} \frac{|p_y(\lambda)|}{(1+|\lambda|^2)^{\frac{k}{2}}} \leq C \sup_{\Lambda_{k(1+\epsilon)}} \frac{|p(\lambda)|}{(1+|\lambda|^2)^{\frac{k}{2}}}. \quad (5.1)$$

Since $\Lambda_{k(1+\epsilon)}$ is sampling, for any $z \in \mathbb{C}$,

$$\frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}} = \frac{|p_z(z)|}{(1+|z|^2)^{\frac{k}{2}}} \lesssim \sup_{\Lambda_{k(1+\epsilon)}} \frac{|p(\lambda)|}{(1+|z|^2)^{\frac{k}{2}}} \left| \frac{K_{\frac{\epsilon}{2}k}(\lambda,z)}{K_{\frac{\epsilon}{2}k}(z,z)} \right|^2$$
$$\leq \sum_{\Lambda_{k(1+\epsilon)}} \frac{|p(\lambda)|}{(1+|z|^2)^{\frac{k}{2}}} \left| \frac{K_{\frac{\epsilon}{2}k}(\lambda,z)}{K_{\frac{\epsilon}{2}k}(z,z)} \right|^2 .$$

Thus, integrating both sides we get

$$\int_{\mathbb{C}} \frac{|p(z)|}{(1+|z|^2)^{\frac{k}{2}}} d\mathcal{V}(z) \lesssim \frac{1}{\epsilon k} \sum_{\Lambda_{k(1+\epsilon)}} \frac{|p(\lambda)|}{(1+|\lambda|^2)^{\frac{k}{2}}}.$$

Hence,

$$\int_{\mathbb{C}} \frac{|p(z)|^2}{(1+|z|^2)^{k}} d\mathcal{V}(z) \lesssim \frac{1}{\epsilon k} \sum_{\Lambda_{k(1+\epsilon)}} \frac{|p(\lambda)|^2}{(1+|\lambda|^2)^{k}}.$$

□

Theorem 5.3. *Let Λ be a separated array. Then Λ is a sampling array for $\mathcal{P}_k$ if and only if there are $\epsilon > 0$, $r > 0$ and $k_0 > 0$ such that for all $k \geq k_0$,*

$$\frac{\sharp(\Lambda_k \cap \Omega)}{k \int_\Omega d\mathcal{V}(z)} > 1 + \epsilon$$

where $\Omega = D(z, r/\sqrt{k})$ and for all $z \in \mathbb{C}$.

Proof. Consider arrays $\Lambda_k \subset \mathbb{C}$ and $\Sigma_k \subset D(0, M_k)$ the corresponding dilated sequences in the plane. Since Σ_k are separated there is a subsequence converging weakly to Σ that for simplicity we keep denoting by Σ_k. The hypothesis implies that

$$\frac{\sharp(\Sigma \cap D(y, r_0))}{r_0^2} \geq (1+\epsilon)\,.$$

By a theorem of Seip and Wallsten [18, Theorem 1.1], Σ is sampling for the space of functions $\mathcal{BF}^\infty$. Those are entire functions f such that

$$\sup_{z\in\mathbb{C}} |f| e^{-\frac{1}{2}|z|^2} < \infty.$$

On the other hand, we may extract a converging subsequence of polynomials f_k to $f \in \mathcal{BF}^\infty$ such that $|f(0)| = 1$ and $f_{|\Sigma} = 0$, and this is a contradiction with the fact that Σ is sampling for $\mathcal{BF}^\infty$. We can conclude that actually $\{\Lambda_{(1-\epsilon)k}\}$ is L^∞-sampling and therefore Λ_k is L^2-sampling by Proposition 5.2.

We turn now to the necessity of the density condition, so we assume that Λ_k is a sampling array. Thus, we know that the density of Λ is bigger or equal than a critical level (Lemma 3.7). We will prove that there is $\epsilon > 0$ so that $\{\Lambda_{(1-\epsilon)k}\}$ is still an L^2-sampling array, so the inequality is strict.

We know that any weak limit $\Sigma \in W(\Lambda_k)$ is a sampling sequence in $\mathcal{BF}^2(\mathbb{C})$ (Theorem 4.7), so by a characterization of Seip and Wallsten the lower Beurling density $D^-(\Sigma)$ in [18], it is bounded below by 1. Let us show that then there is an $\epsilon > 0$ so that $\{\Lambda_{(1-2\epsilon)k}\}$ is L^∞-sampling.

By contradiction, suppose for any n and k very big, there are polynomials $p_k \in \mathcal{P}_k$ such that $\|p_k\|_\infty = 1$ and $\|p_{k|\Lambda_{(1-1/n)k}}\|_\infty = o(1)$. We fix n and we construct functions $f_n \in \mathcal{BF}^\infty(\mathbb{C})$ of norm one such that $f_n(0) = 1$ and $f_{n|\Sigma_n} \equiv 0$, where Σ_k is a weak limit of a subsequence of $\Lambda_{(1-1/n)k}$ as $k \to \infty$.

We take another subsequence of the functions f_n of the separated sequences Σ_n in such a way that Σ_n converge weakly to Σ, $f_n \to f$ with $f \in \mathcal{BF}^\infty(\mathbb{C})$, norm one and $f(0) = 1$ and $f_{|\Sigma} \equiv 0$. And this is a contradiction with $D^-(\Sigma) > 1$ (Theorem 5.1). Finally, the Proposition 5.2 implies the result, since there should exist an L^2 sampling array. □

5.2 Conditions for Interpolation

Definition 5.4. *A separated array $\Lambda = \{\Lambda_k\}$ is an L^1-interpolation array if there is k_0 and a constant $0 < C < \infty$ such that, for each $k \geq k_0$ and any set of vectors $\{\nu_\lambda\}_{\lambda\in\Lambda_k}$, there is a polynomial $p \in \mathcal{P}_k$ so that*

$$p(\lambda) = \nu_\lambda, \quad \lambda \in \Lambda_k,$$

and

$$\int_{\mathbb{C}} |p(z)|e^{-k/2\phi(z)} d\mathcal{V}(z) \leq \frac{C}{k} \sum_{\lambda\in\Lambda_k} |\nu_\lambda| e^{-k/2\phi(z)}.$$

Proposition 5.5. *The constant of L^1 interpolation at level k is comparable to the smallest constant A_k such that*

$$\sup_{z\in\mathbb{C}} \frac{1}{k} \left| \sum_{\Lambda_k} a_\lambda K(z,\lambda) \right| e^{\frac{k}{2}} \leq A_k \sup_{\Lambda_k} |a_\lambda| e^{-\frac{k}{2}\phi(\lambda)},$$

where $\{a_\lambda\}_{\lambda\in\Lambda_k} \subset \mathbb{C}$.

Proof. The result will follow from a duality argument. The main fact is that a linear operator and its dual have the same norm [16, Theorem 4.10, p. 93]. Let us consider the restriction operator

$$\begin{aligned} R :& \mathcal{P}_k^1 \to l^1(\Lambda_k) \\ & f \mapsto \{f(\lambda)e^{-\frac{k}{2}\phi(\lambda)}\}_{\lambda\in\Lambda_k} \end{aligned}$$

which is injective over the space $\mathcal{P}_k/N$, where $N = \{f \in \mathcal{P}_k^1 : f_{|\Lambda_k} \equiv 0\}$. Then the range of the adjoint R^* is the whole $(\mathcal{P}_k^1/N)^* \simeq P_k^\infty$ with same values on Λ_k, this is

$$R^* : l^\infty(\lambda_k) \to \mathcal{P}_k^\infty/N$$

is injective. For $f \in \mathcal{P}_k^1$ and $a \in l^\infty$ we have

$$\langle Rf, a\rangle = \langle f, R^*a\rangle.$$

Let us check that $R^*a = \sum_\lambda a_\lambda \tilde{K}_\lambda$, where $\tilde{K}_\lambda$ are the normalised reproductive kernels:

1. $\langle Rf, v\rangle = \sum_{\lambda\in\Lambda_k} f(\lambda)e^{-\frac{k}{2}\phi(\lambda)}\bar{a}_\lambda$.
2. $\langle f, R^*a\rangle = \sum_{\lambda\in\Lambda_k} \bar{a}_\lambda\langle f, \tilde{K}_\lambda\rangle = \sum_{\lambda\in\Lambda_k} \bar{a}_\lambda f(\lambda)e^{-\frac{k}{2}\phi(\lambda)}$.

Thus the condition $\|R^*a\|_{\mathcal{P}_k^\infty} \le c\|a\|_\infty$ is

$$\left\|\sum_{\lambda\in\Lambda_k} a_\lambda\tilde{K}_\lambda\right\|_{\mathcal{P}_k^\infty} \le c\|a\|_\infty.$$

□

Proposition 5.6. *If $\Lambda = \{\Lambda_k\}$ is an L^1-interpolation array, then $\{\Lambda_{(1-\epsilon)k}\}$ is an L^2-interpolation array.*

Proof. If Λ is an L^1 interpolation array then for each $\lambda \in \Lambda_{(1-2\epsilon)k}$ we can build a Lagrange polinomial $l_\lambda \in \mathcal{P}_{(1-2\epsilon)k}$ so that $l_\lambda(\lambda')e^{-\frac{k}{2}\phi(\lambda')} = \delta_{\lambda\lambda'}$ and $\|l_\lambda\|_{L^1} \le C/k$. Then by the sub-mean property (Lemma 1.5) we obtain $\sup_{\mathbb{C}} |l_\lambda(z)|e^{-\frac{k}{2}\phi(z)} \le Ck\|l_\lambda\|_{L^1} \le C^2$. Thus we can use the same argument as in Theorem 4.7 and prove that $\{\Lambda_{(1-\epsilon)k}\}$ is an L^2-interpolation array. □

Proposition 5.7. *Let Λ be an L^2-interpolation array, then there is $\epsilon > 0$ such that $\{\Lambda_{(1+\epsilon)k}\}$ is L^2-interpolating.*

Proof. We know that any weak limit is an interpolating sequence in $\mathcal{BF}^2(\mathbb{C})$ (Theorem 4.7), and by a characterization of Seip [17], the upper Beurling density $D^+(\Sigma) < 1$. Let us show that then there is an $\epsilon > 0$ such that $\{\Lambda_{(1+2\epsilon)k}\}$ is L^1-interpolating.

By contradiction, suppose that for any n the interpolation constants at level k, C_k for $\Lambda_{(1+1/n)k}$ blow up. Thus, by the dual description of C_k (Proposition 5.5) we can find sequences of vectors $\{a_\lambda\}_{\lambda\in\Lambda_{(1+1/n)k}}$ so that $\sup_{\Lambda_{(1+1/n)k}} |a_\lambda| = 1$ and

$$\sup_{x\in\mathbb{C}} \frac{1}{k}\left|\sum_{\Lambda_{(1+1/n)k}} a_k \tilde{K}(x,\lambda)\right| = o(1), \quad \text{as } k\to\infty.$$

If we fix n and by passing to a subsequence in around the points λ_k^* where $|a_k|$ attains its maximum value, we can extract a subsequence of $\Lambda_{(1+1/n)k}$ as $k\to\infty$ that scaled appropriately converges weakly to the separated sequence $\Sigma_n \subset \mathbb{C}$. Morevover, after taking a subsequence again, there are subsequences $a_\lambda^k \to a_\sigma^n$ for all $\sigma\in\Sigma_n$. We are going to prove in this case

$$f_n(z) = \sum_{\sigma\in\Sigma_n} a_\sigma^n e^{\bar{\sigma}z - 1/2|\sigma|^2} \equiv 0,$$

with $|a_0| = 1$, and $\sup_\sigma |a_\sigma^n| \le 1$. To see this we will prove that for any $\epsilon > 0$,

$$\sup_{|z|<1} |f_n(z)|e^{-|z|^2} \le \epsilon.$$

Since Σ_n is separated and $|a_\sigma^n| \le 1$, the decay of the Bargmann-Fock kernel away from the diagonal implies that for any $\epsilon > 0$ it is possible to find $R > 0$ such that

$$\sup_{|z|<1}\left|\sum_{\sigma\in\Sigma_n,|\sigma|>R} a_\sigma^n e^{\bar{\sigma}z-1/2|\sigma|^2}\right| e^{-1/2|z|^2} \le \epsilon.$$

Due to the Bargmann-Fock kernel is the limit of the Bergman kernel, both decay away from the diagonal, this is

$$\frac{1}{k+1}K(z/\sqrt{k}, w/\sqrt{k}) = \left(1+\frac{z\bar{w}}{k}\right)^k \to e^{z\bar{w}}.$$

So we only need to care about the points $\sigma\in\Sigma_n\cap D(0,R)$. It holds that

$$\frac{1}{k}\sum_{\Lambda_{(1+1/n)k}\cap D(\lambda_k^*,R/\sqrt{k})} a_k K(x,\lambda) \mapsto \frac{1}{\pi}\sum_{\sigma\in\Sigma_n,|\sigma|<R} a_\sigma^n e^{\bar{\sigma}z-1/2|\sigma|^2-1/2|z|^2}$$

uniformly in $|z| < 1$. Thus,

$$\frac{1}{k}\left|\sum_{\Lambda_{(1+1/n)k}\cap D(\lambda_k^*,R/\sqrt{k})} a_k, K(x,\lambda)\right| \le \epsilon$$

if k is big enough, because the global sum for all $\lambda \in \Lambda_{(1+1/n)k}$ converges to zero and the terms λ outside the disc $D(\lambda_k^*, R/\sqrt{k})$ are small when R is big since $\Lambda_{(1+1/n)k}$ is separated and there is a fast decay of the normalized reproducing kernel away from the diagonal.

Finally, we have to prove that $f_n \equiv 0$ and $\{a_\sigma^n\}$ is uniformly bounded sequence with $a_0 = 1$. We can take a subsequence as $n \to \infty$ and we find $\Sigma_n \to \Sigma$ weakly and there is a bounded sequence $\{a_\sigma\}$ such that $f(z) = \sum a_\sigma e^{\bar{\sigma} z - |\sigma|^2/2} \equiv 0$ and $|a_\sigma^0| = 1$. This is clear not possible since Σ is a weak limit and thus it has $D^+(\Lambda) < 1$, thus Λ is interpolating for the L^1 Bargmann-Fock space and this means that by duality

$$\sup_\sigma |a_\sigma| \leq C \sup_{z\in\mathbb{C}} \left| \sum a_\sigma e^{\bar{\sigma} z - |\sigma|^2/2} \right| e^{-|z|^2}.$$

Hence we have proved that $\{\Lambda_{(1+2\epsilon)k}\}$ is L^1-interpolation. We finish the proof by observing that by a previous proposition (Porposition 5.6) this implies that $\{\Lambda_{(1+\epsilon)k}\}$ is L^2-interpolation. □

A simple sufficient condition for interpolation is the following

Lemma 5.8. *If*

$$\sup_k \sup_{\lambda\in\Lambda_k} \frac{1}{k+1} \sum_{\lambda'\neq\lambda} |K(\lambda,\lambda')| e^{-k\phi(\lambda')} < 1,$$

then $\{\Lambda_k\}_k$ *is an interpolation array.*

Proof. Consider the restriction operator at level k

$$\begin{aligned} \mathcal{R}_k : \mathcal{P}_k &\longrightarrow l^2(\Lambda_k) \\ p &\mapsto \{p(\lambda)\}_{\lambda\in\Lambda_k}. \end{aligned}$$

In order to see that $\mathcal{R}_k$ is surjective, consider the approximate extension at level k:

$$\begin{aligned} \mathcal{E}_k :\ & l^2(\Lambda_k) \longrightarrow \mathcal{P}_k \\ & \{v_\lambda\}_{\lambda\in\Lambda_k} \mapsto \mathcal{E}_k(v)(z) = \frac{1}{k+1} \sum_{\lambda\in\Lambda_k} v_\lambda e^{-k\phi(\lambda)} K(\lambda, z). \end{aligned}$$

We shall see that $\|I - \mathcal{R}_k\mathcal{E}_k\| < 1$, so that $(\mathcal{R}_k\mathcal{E}_k)^{-1}$ is well-defined. Then the operator $\mathcal{E}_k(\mathcal{R}_k\mathcal{E}_k)^{-1}$ solves the interpolation problem. We shall see also that these estimates are uniform in k.

Let us see first that the norm of $\mathcal{E}_k$ is uniform in k. We have, by duality

$$\|\mathcal{E}_k\| = \sup_{p\in\mathcal{P}_k \|p\|\le 1} |\langle \mathcal{E}_k, p\rangle| = \sup_{p\in\mathcal{P}_k \|p\|\le 1} \frac{1}{k+1}\left|\sum_{\lambda\in\Lambda} v_\lambda e^{-k\phi(\lambda)}\langle K_\lambda, p\rangle\right|$$
$$= \sup_{p\in\mathcal{P}_k \|p\|\le 1} \frac{1}{k+1}\left|\sum_{\lambda\in\Lambda} v_\lambda e^{-k\phi(\lambda)}\bar{p}(\lambda)\right|.$$

By Cauchy-Schwarz

$$\|\mathcal{E}_k\| = \sup_{p\in\mathcal{P}_k \|p\|\le 1}\left(\frac{1}{k+1}\sum_{\lambda\in\Lambda}|v_\lambda|^2 e^{-k\phi(\lambda)}\right)^{\frac{1}{2}}\left(\frac{1}{k+1}\sum_{\lambda\in\Lambda}|p(\lambda)|^2 e^{-k\phi(\lambda)}\right)^{\frac{1}{2}}$$
$$\lesssim \|v\|_{l^2(\Lambda_k)} \sup_{p\in\mathcal{P}_k \|p\|\le 1}\left(\frac{1}{k+1}\sum_{\lambda\in\Lambda}|p(\lambda)|^2 e^{-k\phi(\lambda)}\right)^{\frac{1}{2}}.$$

Since Λ_k is separated, Plancherel-Polya guarantees the the second factor is uniformly bounded in k.

Let us estimates now $\|I - \mathcal{R}_k\mathcal{E}_k\|$. Given $v \in l^2(\Lambda_k)$, we have

$$\|\{\mathcal{E}_k(v)(\lambda) - v_\lambda\}_{\lambda\in\Lambda_k}\|^2_{l^2(\Lambda_k)} = \frac{1}{k}\sum_{\lambda\in\Lambda_k}|\mathcal{E}_k(v)(\lambda) - v_\lambda|^2 e^{-k\phi(\lambda)},$$

where

$$\mathcal{E}_k(v)(\lambda) - v_\lambda = \frac{1}{k+1}\sum_{\lambda'\ne\lambda} v_{\lambda'} e^{-k\phi(\lambda')}K(\lambda,\lambda')$$

Let

$$\sigma = \sup_k \sup_{\lambda\in\Lambda_k}\frac{1}{k+1}\sum_{\lambda'\ne\lambda}|K(\lambda,\lambda')|e^{-k\phi(\lambda')} < 1.$$

By Cauchy-Schwarz

$$\|(\mathcal{R}_k\mathcal{E}_k)(v) - v\|^2_{l^2(\Lambda_k)} \le \frac{1}{k}\sum_{\lambda\in\Lambda_k}\frac{1}{(k+1)^2}\left\{\sum_{\lambda'\ne\lambda}|v_{\lambda'}|^2 e^{-k\phi(\lambda')}|K(\lambda,\lambda')|\right\}$$
$$\left\{\sum_{\lambda'\ne\lambda} e^{-k\phi(\lambda')}|K(\lambda,\lambda')|\right\} e^{-k\phi(\lambda)}$$

$$\leq \sigma \frac{1}{k} \frac{1}{k+1} \sum_{\lambda \in \Lambda_k} \sum_{\lambda' \neq \lambda} |v_{\lambda'}|^2 e^{-k\phi(\lambda')} |K(\lambda, \lambda')| e^{-k\phi(\lambda)}$$

$$= \frac{\sigma}{k} \sum_{\lambda' \neq \lambda} |v_{\lambda'}|^2 e^{-k\phi(\lambda')} \left(\frac{1}{k+1} \sum_{\lambda \in \Lambda_k} |K(\lambda, \lambda')| e^{-k\phi(\lambda)} \right)$$

$$= \frac{\sigma^2}{k} \sum_{\lambda' \neq \lambda} |v_{\lambda'}|^2 e^{-k\phi(\lambda')} = \sigma^2 \|v\|^2_{l^2(\Lambda_k)}.$$

Thus

$$\|\mathcal{R}_k \mathcal{E}_k - I\| \leq \sigma < 1$$

as desired.

Theorem 5.9. *Let Λ be a separated array, then Λ is an interpolation array for $\mathcal{P}_k$ if and only if there are $\epsilon > 0$, $r > 0$ and $k_0 > 0$ such that for all $k \geq k_0$,*

$$\frac{\sharp(\Lambda_k \cap \Omega)}{k \int_\Omega d\mathcal{V}(z)} < 1 - \epsilon$$

where $\Omega = D(z, r/\sqrt{k})$ and for all $z \in \mathbb{C}$.

Proof. The neccesity part is a straightfoward chain of propositions. If we have that Λ_k is an L^1 interpolation array, then by Proposition 5.6 $\Lambda_{(1-\epsilon)k}$ is an L^2 interpolation array. Then Proposition 5.7 implies that Λ_k is an L^2 interpolation array. Finally, Corollary 3.10 gives the result.

Let us indicate how to prove the sufficiency part. Since by hypothesis the density is small the corresponding sequence Σ_k is an interpolation sequence for the $\mathcal{BF}^2$ space in $\mathbb{C}$. Actually since the separation constant is uniform and the density is uniform then by a theorem of Seip and Wallstén, the constants of interpolation for all the sequences Σ_k will be uniformly bounded, for $k \geq k_0$. Thus we can construct functions f^k_λ such that $|f^k_\lambda(0)| = 1$, $\|f^k_\lambda\| \leq C$ and $f^k_\lambda(\sigma) = 0$ for all $\sigma \in \Sigma_k \setminus \{0\}$. Moreover, it holds that

$$\int_{\mathbb{C}} |f^k_\lambda(z)|^2 e^{-(1-\epsilon)|z|^2} dz < \infty.$$

Now, we can define $g_\lambda = \chi_{\lambda,k} f^k_\lambda + u$, where $\chi_{\lambda,k}$ is a cutoff function around λ such that $g_\lambda(z) = 0$ if $d(z, \lambda) > 2C/\sqrt{k}$ and $g_\lambda(z) = 1$ if $d(z, \lambda) < C/\sqrt{k}$ and u is the solution to the equation $\bar{\partial} u = \bar{\partial} \chi_{\lambda,k} f^k_\lambda(z)$ provided by the Hörmander theorem in our setting.

Finally, if we define $p_\lambda = g_\lambda \left(\frac{K_{\epsilon \frac{k}{2}}(z,\lambda)}{K_{\epsilon \frac{k}{2}}(\lambda,\lambda)} \right)^2$, this polynomial has a property similar to the condition of the Lemma 5.8. □

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