Universitat de Barcelona

Master's Final Project

Pointwise convergence of Fourier series. Carleson's theorem.

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Abstract

In this project we study the pointwise convergence of Fourier series. Our main goal is the proof of Carleson's theorem, which states, roughly speaking, that the Fourier series of any periodic and square integrable function converges to the function almost everywhere. The proof will be based on that presented in the article *Pointwise convergence of Fourier series*, by Charles Fefferman (see [4]). The structure and the notations will be similar to those of the article, but the proofs and the concepts will be explained in much more detail.

In Chapter 1 we revise the history of Fourier series until the proof of Carleson's theorem by Fefferman [1] [3]. We also explain the structure of the project in detail. In Chapter 2 we relate the convergence problem of Fourier series to the boundedness of an operator. In the third chapter, using dyadic grids, we decompose the mentioned operator in simpler operators. In the fourth chapter we handle some technicalities concerning the dyadic grids chosen. In Chapter 5 we give the intuition for the proof of Carleson's theorem and we specify the main goal. In the sixth chapter the main lemmas of the project are proved, which give as a consequence the proof of Carleson's theorem in the seventh chapter.

Keywords: Fourier series, pointwise convergence, Carleson's theorem.

Mathematics Subject Classification: 42A20.

Chapter 1

Introduction

In the first quarter of the 19th century, Joseph Fourier published works on the heat diffusion equation. He solved the problem of the temperature distribution at any given time from the distribution at the starting time. To that end, he invented the separation of variables method. In order to describe the solution, he needed to write the function given by the initial data as a sum of a trigonometric series. In general, if f is the function, which is assumed to be 2π-periodic, the problem consists on finding a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \tag{1.1}$$

such that its sum coincides with $f(x)$ at each x. Fourier correctly stated that

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx. \tag{1.2}$$

The trigonometric series (1.1) with coefficients (1.2) is called the *Fourier series* of f. Fourier was convinced that (1.1) always converged to f. However, in his time, the concept of function was not even well-defined: for instance, Fourier considered that every function could be expressed as a power series. Moreover, he never asked himself the possibility of defining the coefficients from (1.2), since for him every function was always integrable, in the sense that there is always an area between the function and the abscissa axis. Thus, Fourier did not give a result on the convergence of (1.1), but left a problem about the representation of a function as a trigonometric series.

The trials to solve the problem immediately appeared. In 1829, Dirichlet gave the first correct result on the convergence of (1.1): if a bounded function is piecewise continuous and piecewise monotone, then its Fourier series converges at each point to the midpoint of the lateral limits of the function. Dirichlet did not try to find a general result, but sufficient conditions that guarantee the convergence of the series. He started the *convergence criteria*.

A great portion of the mathematics developed from that time went towards the proof of convergence criteria: for instance, the Riemann integral (1855) or the Lebesgue integral (1902) owed part of their success to the application to Fourier series.

At the beginning of the 20th century, Hilbert spaces were born: the concept of orthogonality, the ℓ^2 and L^2 spaces, etc. Frigyes Riesz and Ernst Fischer, in 1907, proved that the Fourier series of a function in $L^2([-\pi, \pi])$ converges in the L^2-norm to it. Thus, the space L^2 represented the correct setting to study Fourier series.

Then, new questions on convergence in the L^p spaces and a.e. convergence began to arise. In 1913, Nikolai Lusin conjectured that every function in $L^2([-\pi,\pi])$ had an a.e. convergent Fourier series. Banach and Steinhauss proved in 1918 that there is no convergence in the L^1-norm, and Marcel Riesz (the younger brother of Frigyes Riesz) showed that there is convergence in the L^p-norm for $1 < p \leq \infty$. Kolmogorov, in 1923, gave an example of a function in $L^1([-\pi,\pi])$ with an a.e. divergent Fourier series. Soon after, in 1926, he proved the divergence at every point.

From that moment, Lusin's conjecture was trying to be proved or refuted. In 1959, Calderon showed that, if the Fourier series of every function in $L^2([-\pi,\pi])$ converged a.e., then

$$|\{x : \sup_n |S_n f(x)| > y\}| \leq C\frac{\|f\|_2}{y^2}.$$

Then many people started to think that Lusin's conjecture should be false. This is why Lennart Carleson started looking for a counterexample to Lusin's conjecture. However, he realized that no counterexample could exist, and that he should make every effort to prove the convergence. In 1966, Carleson managed to prove what was an authentic surprise for everybody: Lusin's conjecture is true [2]. From that moment, the fact that the Fourier series of a function in $L^2([-\pi,\pi])$ converges a.e. was known as *Carleson's theorem.*

The next year, Richard Hunt extended the a.e. convergence to $L^p([-\pi,\pi])$ for $1 < p \leq \infty$ [5]. This supposed the culmination of a problem that had started a century and a half before. In some way, the big questions about the convergence of Fourier series had been solved.

The proof of Carleson was difficult to understand. Charles Fefferman, in 1973, proved Carleson's theorem again [4]. His methods were distinct to those of Carleson, but again difficult. There is a recent proof of Carleson's theorem (from 2000) by Lacey and Thiele, based on the ideas of Fefferman [6].

In this work, our goal is to study the proof done by Fefferman in 1973 [4]. The work is structured in six more chapters.

In the second chapter, we will relate the problem of convergence to the boundedness of an operator T^0, defined as

$$T^0 f(x) = \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)\, dy, \quad x \in [0, 2\pi],$$

where the integral is understood in the principal value sense and N is a bounded and measurable function with constant sign on $[0, 2\pi]$.

In Chapter 3, we will consider dyadic grids in $[0, 2\pi]$ and $\mathbb{R}$ and we will define pairs of intervals of the form $p = [\omega, I]$, where ω and I are dyadic intervals in $\mathbb{R}$ and $[0, 2\pi]$ respectively, such that the length of ω is the inverse of the length of I. In this way, we will decompose

$$T^0 = \sum_p T_p,$$

where T_p will be constructed using the dyadic grids and will be bounded in L^2. The key will be to arrange the pairs in the correct way to transfer the boundedness of the T_p's to T^0.

In Chapter 4, we will deal with some technicalities concerning the dyadic intervals ω in $\mathbb{R}$. We will consider only central dyadic intervals ω (the dyadic intervals that verify that their double is contained in their grandfather), and the corresponding pairs will be called admissible. Using admissible pairs we will construct another operator $T = \sum_p T_p$, where p runs through the admissible pairs. We will see that the boundedness of T implies the boundedness of T^0.

In Chapter 5, we will sketch the proof of the boundedness of T, rearranging the admissible pairs in a suitable way. The idea will be to define an order relation between the pairs and to build larger and larger sets of admissible pairs $\mathcal{P}$ such that $\sum_{p\in\mathcal{P}} T_p$ is bounded. Also, in this Chapter 5 we will prove the Orthogonality Lemma, which will have its application at the end of the work.

In Chapter 6 we will prove lemmas concerning the boundedness of $\sum_{p\in\mathcal{P}} T_p$ for sets of pairs $\mathcal{P}$ with a determined structure: no two pairs in $\mathcal{P}$ are comparable, $\mathcal{P}$ is a *tree*, it is a *row*, it is a *forest*, etc.

In Chapter 7 we will obtain, as a consequence of the lemmas, that T is bounded from L^2 to $L^{p,\infty}$, $0 < p < 2$, and by interpolation this will result in the boundedness of T from L^2 to L^1. This will prove Carleson's theorem.

Chapter 2

Preliminaries

We start by recalling some basic definitions. Given $f \in L^1(\mathbb{T})$, we define the n-th Fourier coefficient of f as

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

We define the N-th partial sum of the Fourier series of f as

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

We say that f is the sum of its Fourier series at a point x_0 if $\lim_{N\to\infty} S_N f(x_0) = f(x_0)$.

Our goal is to prove Carleson's theorem:

Theorem 2.1 (Carleson's theorem) *Let* $f \in L^2(\mathbb{T})$. *Then*

$$\lim_{N\to\infty} S_N f(x) = f(x) \text{ a.e.} \tag{2.1}$$

Remark 2.1 *In this work all integrals will be computed with respect to the Lebesgue measure.*

Let $f \in L^2(\mathbb{T})$. Suppose that there is a constant $C > 0$ so that

$$\left\| \sup_{N\geq 1} |S_N f(\cdot)| \right\|_1 \leq C \|f\|_2 \tag{2.2}$$

for all $f \in L^2(\mathbb{T})$. Let $\epsilon > 0$. Since $C^\infty(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, there exists $\varphi \in C^\infty(\mathbb{T})$ such that $\|f - \varphi\|_2 < \epsilon$. Now use the following trivial bounds:

$$\begin{aligned}
\left\| \limsup_n |S_n f(\cdot) - f(\cdot)| \right\|_1 &= \left\| \limsup_n |S_n f(\cdot) - S_n\varphi(\cdot) + S_n\varphi(\cdot) - \varphi(\cdot) + \varphi(\cdot) - f(\cdot)| \right\|_1 \\
&\leq \left\| \limsup_n \left(|S_n f - S_n\varphi| + |S_n\varphi - \varphi| + |\varphi - f|\right) \right\|_1 \\
&\leq \left\| \limsup_n |S_n f - S_n\varphi| + \limsup_n |S_n\varphi - \varphi| + |\varphi - f| \right\|_1 \\
&\leq \left\| \limsup_n |S_n(f - \varphi)| \right\|_1 + \left\| \limsup_n |S_n\varphi - \varphi| \right\|_1 + \|\varphi - f\|_1.
\end{aligned}$$

Since $\varphi \in C^\infty(\mathbb{T})$,

$$\left\| \limsup_n |S_n \varphi - \varphi| \right\|_1 = 0.$$

On the other hand, by Cauchy-Schwarz inequality,

$$\|f - \varphi\|_1 \leq (2\pi)^{\frac{1}{2}} \|f - \varphi\|_2 < (2\pi)^{\frac{1}{2}} \epsilon.$$

Finally, by assumption (2.2),

$$\left\| \limsup_n |S_n(f - \varphi)| \right\|_1 \leq \left\| \sup_{n \geq 1} |S_n(f - \varphi)| \right\|_1 \leq C\|f - \varphi\|_2 < C\epsilon.$$

Therefore, as $\epsilon > 0$ is arbitrary,

$$\left\| \limsup_n |S_n f(\cdot) - f(\cdot)| \right\|_1 = 0,$$

which is equivalent to (2.1). Thus, our goal will be to show (2.2) for every $f \in L^2(\mathbb{T})$.

Notice that, although we have given a direct proof of the implication

$$(2.2)\, \forall f \in L^2(\mathbb{T}) \;\Rightarrow\; (2.1)\, \forall f \in L^2(\mathbb{T}),$$

this fact is easily deduced from the theory of maximal operators. Consider the linear operators

$$\begin{aligned} T_n : L^2(\mathbb{T}) &\to \{\text{measurable functions on } \mathbb{T}\} \\ f &\mapsto S_n f. \end{aligned}$$

Consider the *maximal operator*

$$T^* f(x) := \sup_{N \geq 1} |T_N f(x)|.$$

Inequality (2.2) is equivalent to the fact that

$$T^* : L^2(\mathbb{T}) \to L^1(\mathbb{T})$$

is bounded. By theory from the course of Harmonic Analysis, this implies that T^* is continuous in measure at zero, therefore the set

$$E = \{f \in L^2(\mathbb{T}) : f \text{ satisfies (2.1)}\}$$

is closed, so $E = L^2(\mathbb{T})$ because (2.1) holds for functions that belong to the dense space $C^\infty(\mathbb{T})$.

To sum up, we need to show (2.2) for every $f \in L^2(\mathbb{T})$. Take a large and arbitrary $M \in \mathbb{N}$. Suppose we prove that, for some $C > 0$ independent of M,

$$\left\| \sup_{1 \leq N \leq M} |S_N f(\cdot)| \right\|_1 \leq C \, \|f\|_2 \tag{2.3}$$

for all $f \in L^2(\mathbb{T})$. Since

$$\left\{ \sup_{1 \leq N \leq M} |S_N f(\cdot)| \right\}_{M=1}^{\infty}$$

is an increasing sequence of nonnegative measurable functions, the Monotone Convergence Theorem applies:

$$\left\|\sup_{N\geq 1}|S_N f(\cdot)|\right\|_1 = \left\|\lim_{M\to\infty}\sup_{1\leq N\leq M}|S_N f(\cdot)|\right\|_1 = \lim_{M\to\infty}\left\|\sup_{1\leq N\leq M}|S_N f(\cdot)|\right\|_1 \leq C\|f\|_2,$$

so (2.2) holds for all $f \in L^2(\mathbb{T})$. Thus, our objective will be to prove that there exists $C > 0$ such that for any $M \in \mathbb{N}$ (2.3) holds for all $f \in L^2(\mathbb{T})$.

Fix a large $M^* \in \mathbb{N}$ for the rest of this work. Given $x \in [0, 2\pi]$, choose $n(x) \in \mathbb{N}$ as the least number such that

$$\sup_{1\leq N\leq M^*}|S_N f(x)| = |S_{n(x)} f(x)|.$$

Our goal becomes

$$\|S_{n(\cdot)} f(\cdot)\|_1 \leq C||f||_2, \tag{2.4}$$

where C does not depend either on M^* or f.

Notice that, for (2.4) to make sense, we need to assure the measurability of $n : [0, 2\pi] \to \{1, \ldots, M^*\}$. This is what we proceed to show in the following lines. Consider the sets

$$E_1 = \left\{x \in [0, 2\pi] : \sup_{1\leq N\leq M^*}|S_N f(x)| = |S_1 f(x)|\right\},$$

$$E_2 = \left\{x \in [0, 2\pi]\backslash E_1 : \sup_{1\leq N\leq M^*}|S_N f(x)| = |S_2 f(x)|\right\},$$

$$\cdots$$

$$E_{M^*} = \left\{x \in [0, 2\pi]\backslash(E_1 \cup \ldots \cup E_{M^*-1}) : \sup_{1\leq N\leq M^*}|S_N f(x)| = |S_{M^*} f(x)|\right\}.$$

Each $|S_N f|$ is measurable, so $\sup\{|S_N f| : 1 \leq N \leq M^*\}$ is also measurable. Then we can use the following result: "if A is measurable in $\mathbb{R}$ and $F, G : A \to \mathbb{R}$ are measurable functions, then the set $\{x \in A : F(x) = G(x)\}$ is measurable". From this it follows that the previous sets $E_1, \ldots, E_{M^*}$ are measurable. To finish, note that the function n can be written as a linear combination of the characteristic functions $\mathbb{1}_{E_j}$: if $x \in E_j$, then $n(x) = j$, and since $\{E_j\}_{j=1}^{M^*}$ is a partition of $[0, 2\pi]$,

$$n(x) = \sum_{j=1}^{M^*} j\,\mathbb{1}_{E_j}(x).$$

This equality shows that n is measurable, as wanted [1].

To demonstrate (2.4), we express $S_{n(\cdot)} f(\cdot)$ in an alternative way, making use of the representation of the partial Fourier sums by means of a convolution with the Dirichlet

[1] I posted and answered a question on the Internet site MathOverflow concerning the measurability of n: `http://mathoverflow.net/questions/229415/pointwise-convergence-of-fourier-series-feffermans-article`

kernel. We start by estimating the difference

$$
\begin{aligned}
S_{n(x)}f(x) &- \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-y)\frac{\sin(n(x)y)}{y}\,dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\underbrace{\left(\frac{\sin\left(\left(n(x)+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} - 2\frac{\sin(n(x)y)}{y}\right)}_{h_{n(x)}(y)}\,dy. \qquad (2.5)
\end{aligned}
$$

Now we will show that, in terms of $\|\cdot\|_1$, the right-hand side term in (2.5) is negligible. For that purpose, we demonstrate that $h_{n(x)}$ is bounded with a bound independent of n. Start by making the following bounds:

$$
\begin{aligned}
|h_{n(x)}(y)| &= \left|\frac{\sin\left(\left(n(x)+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} - 2\frac{\sin(n(x)y)}{y}\right| \\
&= \left|\frac{\sin(n(x)y)\cos\left(\frac{y}{2}\right)+\cos(n(x)y)\sin\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)} - 2\frac{\sin(n(x)y)}{y}\right| \\
&\leq |\sin(n(x)y)|\left|\frac{\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)} - \frac{1}{y/2}\right| + |\cos(n(x)y)| \\
&\leq \left|\frac{\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)} - \frac{1}{y/2}\right| + 1 = \left|\frac{y/2}{\sin(y/2)}\right|\left|\frac{1}{y/2}\left(\cos\left(\frac{y}{2}\right) - \frac{\sin\left(\frac{y}{2}\right)}{y/2}\right)\right| + 1.
\end{aligned}
$$

Note that this last bound does not depend on $n(x)$. By means of Taylor developments or l'Hôpital's rule, one can check that

$$
\lim_{y\to 0}\frac{1}{y/2}\left(\cos\left(\frac{y}{2}\right) - \frac{\sin\left(\frac{y}{2}\right)}{y/2}\right) = 0.
$$

Then

$$
\left|\frac{y/2}{\sin(y/2)}\right|\left|\frac{1}{y/2}\left(\cos\left(\frac{y}{2}\right) - \frac{\sin\left(\frac{y}{2}\right)}{y/2}\right)\right| \in \mathcal{C}([-\pi,\pi]),
$$

so $|h_{n(x)}| \leq K$, for some constant K independent of n. Now it is easy to check that the right-hand side term of (2.5) does not contribute anything when estimating $\|S_{n(\cdot)}(\cdot)\|_1$:

$$
\begin{aligned}
\|S_{n(\cdot)}f(\cdot)\|_1 &\leq \frac{1}{\pi}\left\|\int_{-\pi}^{\pi} f(\cdot-y)\frac{\sin(n(\cdot)y)}{y}\,dy\right\|_1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} f(x-y)h_{n(x)}(y)\,dy\right|dx \\
&\leq \frac{1}{\pi}\left\|\int_{-\pi}^{\pi} f(\cdot-y)\frac{\sin(n(\cdot)y)}{y}\,dy\right\|_1 + \frac{K}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f(x-y)|\,dx\right)dy \\
&\leq \frac{1}{\pi}\left\|\int_{-\pi}^{\pi} f(\cdot-y)\frac{\sin(n(\cdot)y)}{y}\,dy\right\|_1 + K\|f\|_1 \\
&\leq \frac{1}{\pi}\left\|\int_{-\pi}^{\pi} f(\cdot-y)\frac{\sin(n(\cdot)y)}{y}\,dy\right\|_1 + (2\pi)^{\frac{1}{2}}K\|f\|_2.
\end{aligned}
$$

Hence, (2.4) will follow if we prove that there exists $C>0$ independent of n so that

$$
\left\|\int_{-\pi}^{\pi} f(\cdot-y)\frac{\sin(n(\cdot)y)}{y}\,dy\right\|_1 \leq C\|f\|_2 \qquad (2.6)
$$

for all $f \in L^2(\mathbb{T})$. Actually, it is sufficient to demonstrate (2.6) for $f \in C^\infty(\mathbb{T})$. Let us show it. Assume that (2.6) holds for all functions in $C^\infty(\mathbb{T})$. Let $f \in L^2(\mathbb{T})$. We want to show that (2.6) is also satisfied by f. Take $\{f_n\}_{n=1}^\infty \subseteq C^\infty(\mathbb{T})$ with $\lim_n \|f - f_n\|_2 = 0$. We have that

$$\begin{aligned}
&\left\| \int_{-\pi}^{\pi} f(\cdot - y)\frac{\sin(n(\cdot)y)}{y}\, dy - \int_{-\pi}^{\pi} f_n(\cdot - y)\frac{\sin(n(\cdot)y)}{y}\, dy \right\|_1 \\
&\qquad \le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x-y) - f_n(x-y)| \left|\frac{\sin(n(x)y)}{y}\right| dy\, dx \\
&\qquad \le \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(x-y) - f_n(x-y)|\, dy\right) |n(x)|\, dx \\
&\qquad \le 2\pi M^* \|f - f_n\|_1 \le 2\pi M^* (2\pi)^{\frac{1}{2}} \|f - f_n\|_2.
\end{aligned}$$

Then

$$\begin{aligned}
\left\| \int_{-\pi}^{\pi} f(\cdot - y)\frac{\sin(n(\cdot)y)}{y}\, dy \right\|_1 &\le 2\pi M^*(2\pi)^{\frac{1}{2}} \|f - f_n\|_2 + \left\| \int_{-\pi}^{\pi} f_n(\cdot - y)\frac{\sin(n(\cdot)y)}{y}\, dy \right\|_1 \\
&\le 2\pi M^*(2\pi)^{\frac{1}{2}} \|f - f_n\|_2 + C\|f_n\|_2 \overset{n\to\infty}{\longrightarrow} C\|f\|_2,
\end{aligned}$$

and (2.6) holds for $f \in L^2(\mathbb{T})$.

Thus, it is sufficient to demonstrate (2.6) for $f \in C^\infty(\mathbb{T})$.

Fix $f \in C^\infty(\mathbb{T})$. We can decompose the integral of (2.6) using principal values:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x-y)\frac{\sin(n(x)y)}{y}\, dy &= \lim_{\epsilon\to 0^+} \int_{\epsilon\le|y|\le\pi} f(x-y)\frac{\sin(n(x)y)}{y}\, dy \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2i}\left(\int_{\epsilon\le|y|\le\pi} \frac{e^{in(x)y}}{y} f(x-y)\, dy - \int_{\epsilon\le|y|\le\pi} \frac{e^{-in(x)y}}{y} f(x-y)\, dy \right).
\end{aligned}$$

Does the limit of each of those integrals exist? Yes, due to the fact that $f \in C^\infty(\mathbb{T})$. For instance, for the first integral, note that

$$\int_{\epsilon\le|y|\le\pi} \frac{e^{in(x)y}}{y} f(x-y)\, dy \underbrace{=}_{y\leftarrow -y} \int_{\epsilon\le|y|\le\pi} \frac{e^{-in(x)y}}{-y} f(x+y)\, dy,$$

so

$$\begin{aligned}
\int_{\epsilon\le|y|\le\pi} \frac{e^{in(x)y}}{y} f(x-y)\, dy &= \frac{1}{2}\int_{\epsilon\le|y|\le\pi} \frac{e^{in(x)y} f(x-y) - e^{-in(x)y} f(x+y)}{y}\, dy \\
&= \frac{1}{2}\int_{\epsilon\le|y|\le\pi} f(x-y)\frac{e^{in(x)y} - e^{-in(x)y}}{y}\, dy \\
&\quad -\frac{1}{2}\int_{\epsilon\le|y|\le\pi} e^{-in(x)y}\frac{f(x+y) - f(x-y)}{y}\, dy. \qquad (2.7)
\end{aligned}$$

The integrands appearing in (2.7) belong to $L^1([-\pi,\pi], dy)$. Indeed, for the first one

$$\begin{aligned}
\left| f(x-y)\frac{e^{in(x)y} - e^{-in(x)y}}{y} \right| &\le 2\left| f(x-y)\frac{\sin(n(x)y)}{y} \right| \\
&\le 2|n(x)||f(x-y)| \le 2M^*|f(x-y)| \in L^1([-\pi,\pi], dy), \quad (2.8)
\end{aligned}$$

and for the second one the key is the existence of

$$\lim_{y\to 0} \frac{f(x+y)-f(x-y)}{y} = \frac{1}{2}f'(x),$$

which gives

$$e^{-in(x)y}\frac{f(x+y)-f(x-y)}{y} \in L^1([-\pi,\pi],dy). \tag{2.9}$$

Then there exists the principal value

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi} &\frac{e^{in(x)y}}{y}f(x-y)\,dy\\ &= \frac{1}{2}\int_{-\pi}^{\pi} f(x-y)\frac{e^{in(x)y}-e^{-in(x)y}}{y}\,dy - \frac{1}{2}\int_{-\pi}^{\pi} e^{-in(x)y}\frac{f(x+y)-f(x-y)}{y}\,dy.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\int_{-\pi}^{\pi} &f(x-y)\frac{\sin(n(x)y)}{y}\,dy\\ &= \frac{1}{2i}\left(\lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{in(x)y}}{y}f(x-y)\,dy - \lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{-in(x)y}}{y}f(x-y)\,dy\right). \end{aligned}\tag{2.10}$$

From (2.10), it follows that to prove (2.6) it suffices to demonstrate that

$$\begin{aligned}\left\|\lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{in(x)y}}{y}f(x-y)\,dy\right\|_1 &\le C||f||_2,\\ \left\|\lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{-in(x)y}}{y}f(x-y)\,dy\right\|_1 &\le C||f||_2\end{aligned}\tag{2.11}$$

for all $f\in C^\infty(\mathbb{T})$. Let $N:[0,2\pi]\to\mathbb{R}$ be any bounded measurable function that is either negative on $[0,2\pi]$ or positive on $[0,2\pi]$. Then (2.11) will follow if we prove that there exists $C>0$ independent of N such that

$$\left\|\lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{iN(x)y}}{y}f(x-y)\,dy\right\|_1 \le C||f||_2 \tag{2.12}$$

for every $f\in C^\infty(\mathbb{T})$.

This is our last reduction to prove (2.1). We will refer to the left-hand side of (2.12) as a new operator:

$$\begin{aligned}T^0 : (C^\infty(\mathbb{T}),\|\cdot\|_2) &\to \{\text{measurable functions on } [0,2\pi]\}\\ f &\mapsto \left[T^0(f):[0,2\pi]\to\mathbb{C},\, T^0(f)(x) = \lim_{\epsilon\to 0^+}\int_{\epsilon\le|y|\le\pi}\frac{e^{iN(x)y}}{y}f(x-y)\,dy\right].\end{aligned}$$

Equivalently to (2.12), our objective will be to show that there is a $C>0$ independent of N such that

$$\|T^0 f\|_1 \le C\|f\|_2 \tag{2.13}$$

for all $f\in C^\infty(\mathbb{T})$. Notice that in this case T^0 will be defined in a more "natural" way, in the sense that $T^0: L^2(\mathbb{T})\to L^1([0,2\pi])$ will be well-defined. Indeed, if $g\in L^2(\mathbb{T})$, take

$\{g_n\}_{n=1}^{\infty} \subseteq C^{\infty}(\mathbb{T})$ such that $\lim_n \|g - g_n\|_2 = 0$. Then $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{T})$, therefore $\{T^0 g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^1([0, 2\pi])$ by (2.13), and it is possible to set

$$T^0(g) := \|\cdot\|_1 - \lim_n T^0(g_n).$$

This definition is well-set, in the sense that $T^0(g)$ does not depend on the sequence chosen: if $\lim_n \|g - g_n^{(1)}\|_2 = 0$ and $\lim_n \|g - g_n^{(2)}\|_2 = 0$, then

$$\|T^0(g_n^{(1)}) - T^0(g_n^{(2)})\|_1 = \|T^0(g_n^{(1)} - g_n^{(2)})\|_1 \leq C\|g_n^{(1)} - g_n^{(2)}\|_2 \overset{n\to\infty}{\longrightarrow} 0.$$

In summary, our goal will be to prove (2.13) for all $f \in C^{\infty}(\mathbb{T})$.

For technical reasons, instead of working with functions defined on $\mathbb{T}$ (that is, with 2π-periodic functions on the whole real line), we will work with $(1 \leq q < \infty)$

$$L_r^q = \{f : \mathbb{R} \to \mathbb{C} : \text{support}(f) \subseteq [-4\pi, 6\pi],\ f \in L^q([-4\pi, 6\pi])\}$$

and

$$C_r^{\infty} = \{\varphi : \mathbb{R} \to \mathbb{C} : \varphi \in C^{\infty}(\mathbb{R}),\ \text{support}(\varphi) \subseteq [-4\pi, 6\pi]\}$$

(the subscript r stands for restricted). These technical reasons refer to the fact that we will multiply by five intervals contained in $[0, 2\pi]$, and it will be convenient to have them contained in support(f), with support(f) being an interval of finite Lebesgue measure. Note that L_r^q is a Banach space with norm $\|f\|_q = \int_{\mathbb{R}} |f|^q$, which is equal to the norm in $L^q(\mathbb{R})$.

Thus, for the rest of this work our goal will be to prove (2.13) for all $f \in C_r^{\infty}$ (T^0 is well-defined on C_r^{∞}, and the proof of this fact is the same as that for $C^{\infty}(\mathbb{T})$).

Chapter 3

Decomposition of the operator T^0

We need to reduce and make simpler our problem (2.13). What we will do is to decompose our operator T^0 in simpler operators.

For that purpose, we first decompose the $1/y$ term that appears in the integral expression of $T^0(f)$. Let us see that there exists an odd $C^\infty(\mathbb{R})$ function $\psi^{(0)}$, supported in $]-2\pi, 2\pi[\setminus[-\pi/2, \pi/2]$, such that $|\psi^{(0)}| < 1$, $\psi^{(0)}|_{\mathbb{R}^+} \geq 0$, $\psi^{(0)}|_{\mathbb{R}^-} \leq 0$ and

$$\frac{1}{z} = \sum_{j=0}^{\infty} \psi_j(z), \quad \forall z \in [-\pi, \pi]\setminus\{0\}, \tag{3.1}$$

where $\psi_j(z) = 2^j \psi^{(0)}(2^j z)$. The infinity smoothness and the bounded support makes us think on a Uryshon function. Consider an even Uryshon function $\varphi \in C^\infty(\mathbb{R})$ with support$(\varphi) \subseteq]-2\pi, 2\pi[$, $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ for all $x \in [-\pi, \pi]$. Let $\psi(x) = \varphi(x) - \varphi(2x)$. Note that $0 \leq \psi \leq 1$, support$(\psi) \subseteq]-2\pi, 2\pi[\setminus[-\pi/2, \pi/2]$, $\psi \in C^\infty(\mathbb{R})$ and ψ is even. In addition, for all $x \in [-\pi, \pi]$

$$\sum_{k=0}^{\infty} \psi(2^k x) \underbrace{=}_{\substack{\text{telescopic}\\ \text{sum}}} \varphi(x) = 1.$$

Since we would like something as $1/x$, we consider

$$\psi^{(0)}(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{\psi(x)}{x}, & \text{if } x \neq 0. \end{cases}$$

Then $\psi^{(0)}$ is C^∞, because ψ is 0 around 0. Moreover, it is odd (because ψ is even), it has support in $]-2\pi, 2\pi[\setminus[-\pi/2, \pi/2]$ and $|\psi^{(0)}| \leq 1/(\pi/2) = 2/\pi < 1$. Finally, (3.1) holds:

$$\forall z \in [-\pi, \pi]\setminus\{0\} \quad \sum_{j=0}^{\infty} 2^j \psi^{(0)}(2^j z) = \sum_{j=0}^{\infty} 2^j \frac{\psi(2^j z)}{2^j z} = \frac{1}{z} \sum_{j=0}^{\infty} \psi(2^j z) = \frac{1}{z}.$$

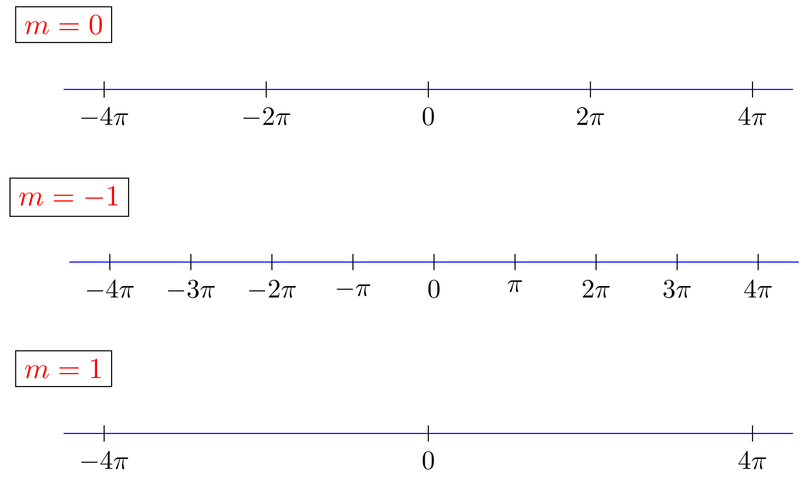
Once we have decomposed the $1/y$ term that appears in the integral expression of $T^0(f)$, we decompose the domain $[0, 2\pi]$ by means of *dyadic intervals.* Given $P \in \mathbb{Z}$, a dyadic interval of order $m \in \mathbb{Z}$ is defined as

$$[2\pi \cdot P \cdot 2^m, 2\pi \cdot (P+1) \cdot 2^m[\,.$$

A *dyadic grid* of order $m \in \mathbb{Z}$ is the family of dyadic intervals

$$\{[2\pi \cdot P \cdot 2^m, 2\pi \cdot (P+1) \cdot 2^m[\,\}_{P \in \mathbb{Z}}.$$

Here we present some examples of dyadic grids:



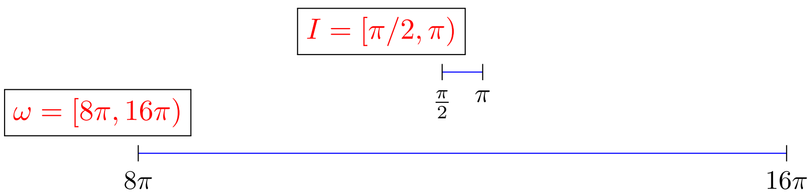
Notice that, if I and J are dyadic intervals of orders m and l respectively, with $m \neq l$, then either $I \subseteq J$, or $J \subseteq I$ or $I \cap J = \emptyset$. This property is one of the key facts concerning dyadic intervals.

Given dyadic intervals $\omega \subseteq \mathbb{R}$ and $I \subseteq [0, 2\pi]$, we will say that $[\omega, I]$ is a *pair* if

$$1 \leq \frac{|\omega|}{2\pi} = \frac{2\pi}{|I|}$$

($|\cdot|$ represents the length of the interval computed with respect to the Lebesgue measure).

Example of a pair $[\omega, I]$



We will denote the set of all pairs by $\mathcal{B}$. Given $[\omega, I] \in \mathcal{B}$, we define

$$E(\omega, I) = \{x \in I : N(x) \in \omega\} \subseteq [0, 2\pi].$$

Let, for $k \in \mathbb{N} \cup \{0\}$,

$$\mathcal{B}_k = \{[\omega, I] \in \mathcal{B} : |I| = 2\pi \cdot 2^{-k} \text{ and } |\omega| = 2\pi \cdot 2^k\}.$$

For each $k \in \mathbb{N} \cup \{0\}$ we have a partition

$$\{E(\omega, I) : [\omega, I] \in \mathcal{B}_k\}$$

of $[0, 2\pi]$. The partition is finite, because N is bounded.

Thus, we can proceed to decompose T^0. Given $[\omega, I] \in \mathcal{B}_k$, define a new operator

$$T_{[\omega,I]}f(x) = \int_{-\pi}^{\pi} e^{iN(x)y}\psi_k(y)f(x-y)\,dy \cdot \mathbb{1}_{E(\omega,I)}(x)$$

for $x \in \mathbb{R}$. With this new operator $T_{[\omega,I]}$ there is no need to go down until C_r^∞, since it is well-defined from L_r^2 to L_r^2. Indeed, let $f \in L_r^2$ and denote by $p = [\omega, I]$. At a first step, we bound

$$|T_pf(x)| \leq \int_{-\pi}^{\pi} |\psi_k(y)||f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x).$$

Now we distinguish two cases, depending on whether $k = 0$ or $k > 0$:

- Case $k = 0$. We have $\psi_k = \psi^{(0)}$, $I = [0, 2\pi[$ and

$$\begin{aligned} |T_pf(x)| &\leq \int_{-\pi}^{\pi} |\psi^{(0)}(y)||f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x) \underbrace{\leq}_{|\psi^{(0)}|<1} \int_{-\pi}^{\pi} |f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x) \\ &\leq \int_{\mathbb{R}} |f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x) = \|f\|_1 \cdot \mathbb{1}_{E(p)}(x) \underbrace{\leq}_{\text{Cauchy-Schwarz}} \sqrt{10\pi}\|f\|_2 \cdot \mathbb{1}_{E(p)}(x). \end{aligned}$$

  Now apply $\|\cdot\|_2$:

$$\|T_pf\|_2 \leq \sqrt{20}\pi \left(\frac{|E(p)|}{2\pi}\right)^{\frac{1}{2}} \|f\|_2.$$

- Case $k > 0$. Recall that support$(\psi_k) \subseteq]-2\pi/2^k, 2\pi/2^k[\subseteq [-\pi, \pi]$. Then we can write

$$\begin{aligned} |T_pf(x)| &\leq \int_{-\frac{2\pi}{2^k}}^{\frac{2\pi}{2^k}} |\psi_k(y)||f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x) \underbrace{\leq}_{|\psi^{(0)}|<1} 2^k \int_{-\frac{2\pi}{2^k}}^{\frac{2\pi}{2^k}} |f(x-y)|\,dy \cdot \mathbb{1}_{E(p)}(x) \\ &= 2^k \int_{x-\frac{2\pi}{2^k}}^{x+\frac{2\pi}{2^k}} |f(y)|\,dy \cdot \mathbb{1}_{E(p)}(x) \leq 2^k \int_{I^3} |f(y)|\,dy \cdot \mathbb{1}_{E(p)}(x), \end{aligned} \tag{3.2}$$

  where I^3 is formed by adding I to the left and to the right of I. Apply $\|\cdot\|_2$:

$$\begin{aligned} \|T_pf\|_2 &\leq \frac{2\pi}{|I|} \int_{I^3} |f(y)|\,dy \cdot |E(p)|^{\frac{1}{2}} \underbrace{\leq}_{\text{Cauchy-Schwarz}} \frac{2\pi}{|I|}\|f\|_2|I^3|^{\frac{1}{2}}|E(p)|^{\frac{1}{2}} \\ &\underbrace{=}_{|I^3|=3|I|} \sqrt{3} \cdot 2\pi \left(\frac{|E(p)|}{|I|}\right)^{\frac{1}{2}} \|f\|_2. \end{aligned}$$

Thus,

$$T_p : L_r^2 \to L_r^2$$

is well-defined, linear and continuous, with norm

$$\|T_p\|_2 \leq C \left(\frac{|E(p)|}{|I|}\right)^{\frac{1}{2}}. \tag{3.3}$$

In connection with this expression, we will denote

$$A_0(p) = \frac{|E(p)|}{|I|}. \tag{3.4}$$

For $k \in \mathbb{N} \cup \{0\}$, denote

$$T_k f(x) = \sum_{p \in \mathcal{B}_k} T_p f(x) = \begin{cases} \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy, & x \in [0, 2\pi], \\ 0, \quad x \notin [0, 2\pi] \end{cases}$$

(remember that $\{E(p) : p \in \mathcal{B}_k\}$ is a partition of $[0, 2\pi]$). The sum $\sum_{p \in \mathcal{B}_k}$ is finite independently on x and f, because N is bounded.

Notice that

$$T_k : L^2_r \to L^2_r \tag{3.5}$$

is well-defined, linear and continuous, because

$$|T_k f(x)| \leq \int_{-\pi}^{\pi} |\psi_k(y)||f(x-y)|\, dy \leq 2^k \|f\|_1 \leq C \cdot 2^k \cdot \|f\|_2 \Rightarrow \|T_k f\|_2 \leq C \cdot 2^k \cdot \|f\|_2.$$

Now we compute

$$\sum_{k=0}^{\infty} T_k f(x).$$

Intuitively, one should be able to interchange $\sum_{k=0}^{\infty}$ and $\int_{-\pi}^{\pi}$ and, using (3.1), arrive at

$$\sum_{k=0}^{\infty} T_k f(x) = T^0 f(x) \tag{3.6}$$

for all $f \in C^\infty_r$ and $x \in \mathbb{R}$ (remember: every time we work with T^0 we need to go down to C^∞_r; $T^0 f$ is understood to be 0 outside of $[0, 2\pi]$). We prove (3.6) formally. Let $f \in C^\infty_r$ and $x \in \mathbb{R}$. By (2.7), the principal value defining $T^0 f$ can be expressed as an integral:

$$T^0 f(x) = \frac{1}{2} \int_{-\pi}^{\pi} f(x-y) \frac{e^{iN(x)y} - e^{-iN(x)y}}{y}\, dy - \frac{1}{2} \int_{-\pi}^{\pi} e^{-iN(x)y} \frac{f(x+y) - f(x-y)}{y}\, dy. \tag{3.7}$$

By (3.1),

$$\begin{aligned} T^0 f(x) = & \frac{1}{2} \int_{-\pi}^{\pi} f(x-y)(e^{iN(x)y} - e^{-iN(x)y}) \left(\sum_{k=0}^{\infty} \psi_k(y) \right) dy \\ & - \frac{1}{2} \int_{-\pi}^{\pi} e^{-iN(x)y} (f(x+y) - f(x-y)) \left(\sum_{k=0}^{\infty} \psi_k(y) \right) dy. \end{aligned}$$

By Lebesgue's theorems, in order to interchange $\sum_{k=0}^{\infty}$ and $\int_{-\pi}^{\pi}$, we have to check

$$\begin{aligned} (I) = & \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} |f(x-y)(e^{iN(x)y} - e^{-iN(x)y}) \psi_k(y)|\, dy \\ \underbrace{=}_{\text{known}} & \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} |f(x-y)(e^{iN(x)y} - e^{-iN(x)y}) \psi_k(y)|\, dy \underbrace{<}_{\substack{\text{to} \\ \text{check}}} \infty \end{aligned}$$

and

$$(II) = \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} |e^{-iN(x)y}(f(x+y) - f(x-y))\psi_k(y)|\, dy$$

$$\underbrace{=}_{\text{known}} \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} |e^{-iN(x)y}(f(x+y) - f(x-y))\psi_k(y)|\, dy \underbrace{<}_{\substack{\text{to}\\ \text{check}}} \infty.$$

From (3.1) and the fact that $\psi^{(0)}|_{\mathbb{R}^+} \geq 0$ and $\psi^{(0)}|_{\mathbb{R}^-} \leq 0$, we have

$$\sum_{k=0}^{\infty} |\psi_k(y)| = \begin{cases} \frac{1}{y}, \text{ if } 0 < y \leq \pi \\ -\frac{1}{y}, \text{ if } -\pi \leq y < 0 \end{cases} = \frac{1}{|y|}, \quad \forall y \in [-\pi, \pi]\backslash\{0\}.$$

Then

$$(I) = \int_{-\pi}^{\pi} |f(x-y)||e^{iN(x)y} - e^{-iN(x)y}| \left(\sum_{k=0}^{\infty} |\psi_k(y)| \right) dy$$

$$= \int_{-\pi}^{\pi} |f(x-y)| \left| \frac{e^{iN(x)y} - e^{-iN(x)y}}{y} \right| dy \underbrace{<}_{\substack{\text{by}\\ (2.8)}} \infty$$

as well as

$$(II) \leq \int_{-\pi}^{\pi} |f(x+y) - f(x-y)| \left(\sum_{k=0}^{\infty} |\psi_k(y)| \right) dy = \int_{-\pi}^{\pi} \left| \frac{f(x+y) - f(x-y)}{y} \right| dy \underbrace{<}_{\substack{\text{by}\\ (2.9)}} \infty.$$

This justifies the interchange of $\sum_{k=0}^{\infty}$ and $\int_{-\pi}^{\pi}$. Then

$$\begin{aligned} T^0 f(x) &= \frac{1}{2} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} f(x-y)(e^{iN(x)y} - e^{-iN(x)y})\psi_k(y)\, dy \\ &\quad - \frac{1}{2} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} e^{-iN(x)y}(f(x+y) - f(x-y))\psi_k(y)\, dy \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} (f(x-y)e^{iN(x)y} - f(x+y)e^{-iN(x)y})\psi_k(y)\, dy \\ &\underbrace{=}_{\substack{\psi_k\\ \text{odd}}} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} e^{iN(x)y} f(x-y)\psi_k(y)\, dy = \sum_{k=0}^{\infty} T_k f(x) \end{aligned} \tag{3.8}$$

and (3.6) is proved.

Could we express (3.6) in a natural way as

$$T^0 f(x) = \sum_{p \in \mathcal{B}} T_p f(x), \tag{3.9}$$

for $f \in C_r^{\infty}$ and $x \in \mathbb{R}$? To answer this question, we need to study some sort of unconditional convergence.

Remark 3.1 *Given an integer $k \geq 0$, let A_k be a finite set. Denote $A = \cup_{k=0}^{\infty} A_k$. Consider complex numbers a_p, $p \in A$. Suppose that*

$$\sum_{k=0}^{\infty} \sum_{p \in A_k} |a_p| < \infty.$$

Then

$$\sum_{n \in A} a_n$$

does not depend on the ordering of A and

$$\sum_{n \in A} a_n = \sum_{k=0}^{\infty} \sum_{p \in A_k} a_p.$$

In the notation of the remark, take $A_k = \mathcal{B}_k$, $A = \mathcal{B}$ and $a_p = T_p f(x)$. Since

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{p \in \mathcal{B}_k} |T_p f(x)| &= \sum_{k=0}^{\infty} \left| \int_{-\pi}^{\pi} e^{iN(x)y} f(x-y) \psi_k(y)\, dy \right| \\
&\underbrace{\leq}_{\substack{\text{by} \\ (3.8)}} \frac{1}{2} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} |f(x-y)| |e^{iN(x)y} - e^{-iN(x)y}| |\psi_k(y)|\, dy \\
&+ \frac{1}{2} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} |f(x+y) - f(x-y)| |\psi_k(y)|\, dy \\
&= \int_{-\pi}^{\pi} |f(x-y)| \left| \frac{e^{iN(x)y} - e^{-iN(x)y}}{y} \right| dy \\
&+ \int_{-\pi}^{\pi} \left| \frac{f(x+y) - f(x-y)}{y} \right| dy < \infty,
\end{aligned} \tag{3.10}$$

the remark applies:

$$\sum_{p \in \mathcal{B}} T_p f(x)$$

does not depend on the ordering of $\mathcal{B}$, converges absolutely and

$$\sum_{p \in \mathcal{B}} T_p f(x) = \sum_{k=0}^{\infty} \sum_{p \in \mathcal{B}_k} T_k f(x) = T^0 f(x),$$

so (3.9) is justified. This is the decomposition of T^0 we were looking for.

Now we are going to make a small change on the decomposition of T^0. Remember that we are working with ψ_k's, $k \geq 0$. We do not like the case $k = 0$. Why? Because this is a special case (look at the bound of $\|T_p\|_2$), in the sense that support$(\psi_k) \subseteq [-\pi, \pi]$ for $k \geq 1$ but support$(\psi_0) =$ support$(\psi^{(0)}) \nsubseteq [-\pi, \pi]$. If we do not act, this will cause future problems. What we are going to do then is to delete in some way ψ_0. By (3.6), we have

$$T^0 f(x) = T_0 f(x) + \sum_{k=1}^{\infty} T_k f(x),$$

and since by (3.5) $T_0 : L^2_r \to L^2_r$ is bounded, what we just have to do is to bound

$$\left\| \sum_{k=1}^{\infty} T_k f \right\|_1$$

by $C\|f\|_2$ for all $f \in C^\infty_r$. With some abuse of notation, we will denote

$$T^0 f(x) = \sum_{k=1}^{\infty} T_k f(x)$$

and

$$T^0 f(x) = \sum_{p \in \mathcal{B}} T_p f(x),$$

where $\mathcal{B}$ stands for all the pairs $[\omega, I]$ with $|I| = 2\pi \cdot 2^{-k}$, $k \geq 1$. To sum up, we will always assume $k \geq 1$ and the case $k = 0$ will not have to be considered.

In this case, the T_p's will have a more easy and adequate form to handle:

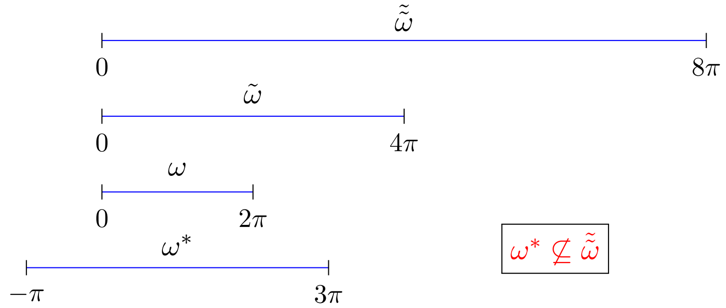
$$T_p f(x) = ((e^{iN(x)\cdot} \psi_k(\cdot)) * f)(x) \cdot \mathbb{1}_{E(p)}(x),$$

where $*$ stands for the convolution in $\mathbb{R}$.

Chapter 4

Unfortunate technicalities

For any dyadic interval $\omega \subseteq \mathbb{R}$, let $\tilde{\omega}$ be the next larger dyadic interval containing ω ($\tilde{\omega}$ is called the *father* of ω) and let ω^* be the double of ω (that is, ω^* is the interval with the same center as ω but twice its length). We would like to say that $\omega^* \subseteq \tilde{\tilde{\omega}}$ ($\tilde{\tilde{\omega}}$ is called the *grandfather* of ω), but that is not true in general:



We will say that ω is *central* if $\omega^* \subseteq \tilde{\tilde{\omega}}$. We will say that $[\omega, I] \in \mathcal{B}$ is *admissible* if ω is central. Since we are interested on central intervals, we define a new operator

$$Tf(x) = \sum_{\substack{p \in \mathcal{B} \\ p \text{ admissible}}} T_p f(x)$$

for $f \in C_r^\infty$ and $x \in [0, 2\pi]$. That is, T is as T^0 but just considering the admissible pairs. Notice that the sum defining T is unconditionally convergent because $\sum_{p \in \mathcal{B}}$ is.

We will see that, in order for (2.13) to hold, it suffices to show that there exists $C > 0$ independent of N such that

$$\|Tf\|_1 \leq C\|f\|_2 \tag{4.1}$$

for all $f \in C_r^\infty$.

Which should be the problem when trying to prove that (4.1) implies (2.13)? The problem should rely on the fact that we do not control the non-admissible pairs when working with T. However, this may have a solution. When we have the dyadic grid on $\mathbb{R}$, there are central and non-central intervals. Our dyadic grid is centered around 0. If we translate our dyadic grid, there will be new central and non-central intervals. We hope that the central intervals of the new translated dyadic grids will give information about the non-central intervals of our usual dyadic grid centered at 0. Let us put this intuition into practice.

Denote our usual dyadic grid on $\mathbb{R}$ centered at 0 as $\mathcal{G}_0$. We translate it by $\xi \in \mathbb{R}$, in order to have a new dyadic grid on $\mathbb{R}$ called $\mathcal{G}_\xi$. We have $\bar{\omega} \in \mathcal{G}_\xi$ if and only if $\bar{\omega} = \omega + \xi$, $\omega \in \mathcal{G}_0$. Now, in order to construct a new "interesting" operator T_ξ on the new dyadic grid, we follow the same procedure as in the decomposition of T^0. Denote by

$$\mathcal{B}_k(\xi) = \{[\bar{\omega}, I] : \bar{\omega} \in \mathcal{G}_\xi, |\bar{\omega}| = 2\pi \cdot 2^k, |I| = 2\pi \cdot 2^{-k}\}$$

(note that $\mathcal{B}_k(0)$ is our usual $\mathcal{B}_k$). If $p = [\bar{\omega}, I] \in \mathcal{B}_k(\xi)$, we define

$$T_p f(x) = \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \cdot \mathbb{1}_{E(p)}(x),$$

where

$$E(p) = \{x \in I : N(x) \in \bar{\omega}\} = \{x \in I : N(x) - \xi \in \omega\}.$$

Thus, we are considering the same operators as usual but with a translated dyadic grid in $\mathbb{R}$. For $k \geq 0$, define an operator

$$T_{k,\xi} f(x) = \sum_{p \in \mathcal{B}_k(\xi)} T_p f(x) = \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy.$$

Now focus on admissible pairs:

$$\sum_{\substack{p \in \mathcal{B}_k(\xi) \\ p \text{ admissible}}} T_p f(x) = \alpha_k(N(x), \xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy,$$

where

$$\alpha_k(N(x), \xi) = \begin{cases} 1, & \text{if the unique dyadic interval } \bar{\omega} \text{ in } \mathcal{G}_\xi \text{ of length } 2\pi \cdot 2^k \text{ containing } \\ & N(x) \text{ is central,} \\ 0, & \text{if the unique dyadic interval } \bar{\omega} \text{ in } \mathcal{G}_\xi \text{ of length } 2\pi \cdot 2^k \text{ containing } \\ & N(x) \text{ is not central.} \end{cases}$$

Notice that

$$\alpha_k(N(x), \xi) = \alpha_k(N(x) - \xi, 0) =: \alpha_k(N(x) - \xi). \tag{4.2}$$

Hence, $\alpha_k(N(x), \cdot)$ is like an infinite step function on $\mathbb{R}$, more concretely, it is an infinite sum of characteristic functions over intervals.

Let

$$\mathcal{B}(\xi) = \bigcup_{k=0}^{\infty} \mathcal{B}_k(\xi)$$

(note that $\mathcal{B}(0)$ is our usual $\mathcal{B}$). This is the set of pairs in the new dyadic grid $\mathcal{G}_\xi$.

From all this we have the new operator T_ξ on the dyadic grid $\mathcal{G}_\xi$:

$$T_\xi f(x) := \sum_{\substack{p \in \mathcal{B}(\xi) \\ p \text{ admissible}}} T_p f(x) = \sum_{k=0}^{\infty} \alpha_k(N(x), \xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy.$$

The fact that the sum is unconditionally convergent is proved exactly equal as in the case of $\mathcal{B} = \mathcal{B}(0)$.

As we explained previously, we can assume without loss of generality that $\mathcal{B}(\xi)$ does not contain the pairs corresponding to $k = 0$ and then denote (abusing of notation)

$$T_\xi f(x) := \sum_{\substack{p \in \mathcal{B}(\xi) \\ p \text{ admissible}}} T_p f(x) = \sum_{k=1}^{\infty} \alpha_k(N(x), \xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy.$$

Once we have the operators created on the new dyadic grid $\mathcal{G}_\xi$ for all $\xi \in \mathbb{R}$, do these operators provide any kind of information about T^0, that is, about the non-admissible pairs of $\mathcal{B}$? The answer is yes, and, more specifically, $T^0 f(x)$ is some sort of average of the $T_\xi f(x)$'s. This will be due to the fact that half of the dyadic intervals in $\mathcal{G}_0$ are central.

Lemma 4.1 *For all $f \in C_r^\infty$ and for all $x \in \mathbb{R}$ we have*

$$T^0 f(x) = \lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} T_\xi f(x)\, d\xi.$$

Proof. We first need to proof the measurability of $T_{\cdot} f(x)$ on $\mathbb{R}$. Write

$$T_\xi f(x) = \sum_{k=1}^{\infty} \alpha_k(N(x) - \xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy.$$

Only $\alpha_k(N(x)-\xi)$ depends on ξ. Since α_k is an infinite sum of characteristic functions over intervals, it is measurable on $\mathbb{R}$, and since $\alpha_k(N(x)-\cdot)$ is just a translation of a measurable function, it is measurable on $\mathbb{R}$. Then $T_{\cdot} f(x)$ is a pointwise limit of measurable functions, therefore it is measurable on the real line.

Now act intuitively:

$$\begin{aligned}
&\lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} T_\xi f(x)\, d\xi \\
&\quad = \lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} \left[\sum_{k=1}^{\infty} \alpha_k(N(x) - \xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right] d\xi \\
&\quad = \sum_{k=1}^{\infty} \left[\left(\lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} \alpha_k(N(x) - \xi)\, d\xi \right) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right].
\end{aligned}$$

If half of the dyadic intervals on $\mathcal{G}_0$ were central, then in $[-M, M]$ there would be M central dyadic intervals, therefore we could demonstrate that

$$\lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} \alpha_k(N(x) - \xi)\, d\xi = 1. \tag{4.3}$$

Then we would have

$$\lim_{M \to \infty} \frac{1}{M} \int_{-M}^{M} T_\xi f(x)\, d\xi = \sum_{k=1}^{\infty} \left[\int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right] = T^0 f(x),$$

which is what we want.

This is the intuition, but there are several things to prove. We have to show that, for any $M \in \mathbb{R}^+$, we can interchange $\sum_{k=1}^{\infty}$ and $\int_{-M}^{M}$. We also have to prove that we can interchange $\lim_{M\to\infty}$ and $\sum_{k=1}^{\infty}$. Finally, we will need to check (4.3) by analyzing the structure of the central dyadic intervals on $\mathcal{G}_0$.

Let us see that, for any $M \in \mathbb{R}^+$, we can interchange $\sum_{k=1}^{\infty}$ and $\int_{-M}^{M}$. Apply Lebesgue's theorems:

$$\sum_{k=1}^{\infty} \int_{-M}^{M} \left| \alpha_k(N(x)-\xi) \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right| d\xi$$
$$= \sum_{k=1}^{\infty} \left| \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right| \int_{-M}^{M} \underbrace{\alpha_k(N(x)-\xi)}_{\leq 1}\, d\xi$$
$$\leq 2M \sum_{k=1}^{\infty} \left| \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right| \underbrace{<}_{\substack{\text{by} \\ (3.10)}} \infty.$$

Then the swap between $\sum_{k=1}^{\infty}$ and $\int_{-M}^{M}$ is justified.

Let us check that we can exchange $\lim_{M\to\infty}$ and $\sum_{k=1}^{\infty}$. We apply the Dominated Convergence Theorem for series. Since for all $M > 0$

$$\frac{1}{M} \left| \int_{-M}^{M} \underbrace{\alpha_k(N(x)-\xi)}_{\leq 1}\, d\xi \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right|$$
$$\leq 2 \left| \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right|$$

and

$$\sum_{k=1}^{\infty} \left| \int_{-\pi}^{\pi} e^{iN(x)y} \psi_k(y) f(x-y)\, dy \right| \underbrace{<}_{\substack{\text{by} \\ (3.10)}} \infty,$$

the swap between $\lim_{M\to\infty}$ and $\sum_{k=1}^{\infty}$ is fully justified.

Finally, we require a proof for (4.3). We first plot the graph of $\alpha_k(\cdot)$, that is, we first do a study on the location of the central dyadic intervals on $\mathcal{G}_0$. Consider a dyadic interval on $\mathcal{G}_0$ of the form

$$\omega = [2\pi \cdot 2^k \cdot m, 2\pi \cdot 2^k \cdot (m+1)[,$$

for an integer m. We want to analyze if α_k is 0 or 1 on ω. Write

$$\tilde{\omega} = [2\pi \cdot 2^{k+1} \cdot \tilde{m}, 2\pi \cdot 2^{k+1} \cdot (\tilde{m}+1)[.$$

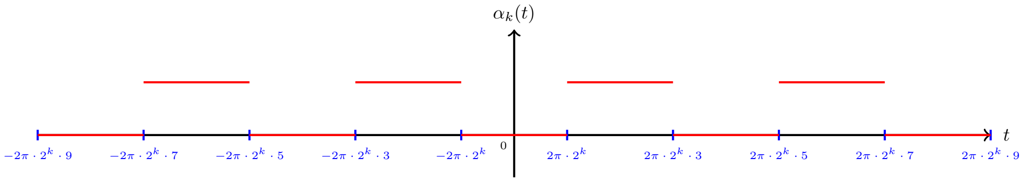
Note that $\omega \subseteq \tilde{\omega}$ if and only if $2\tilde{m} \leq m \leq 2\tilde{m}+1$. Write

$$\tilde{\tilde{\omega}} = [2\pi \cdot 2^{k+2} \cdot \tilde{\tilde{m}}, 2\pi \cdot 2^{k+2} \cdot (\tilde{\tilde{m}}+1)[.$$

We need again $2\tilde{\tilde{m}} \leq \tilde{m} \leq 2\tilde{\tilde{m}}+1$, that is, $4\tilde{\tilde{m}} \leq m \leq 4\tilde{\tilde{m}}+3$. We have

$$\omega^* = \left[2\pi \cdot 2^k \cdot \left(m - \frac{1}{2}\right), 2\pi \cdot 2^k \cdot \left(m + \frac{3}{2}\right)\right[.$$

Then $\omega^* \subseteq \tilde{\tilde{\omega}}$ if and only if $m \in \{4\tilde{\tilde{m}}+1, 4\tilde{\tilde{m}}+2\}$, that is, if and only if m is congruent with 1 or 2 modulo 4. Since there are four classes modulo 4, half of the dyadic intervals on $\mathcal{G}_0$ are central. The graph of α_k for a fixed $k \in \mathbb{N} \cup \{0\}$ is the following:

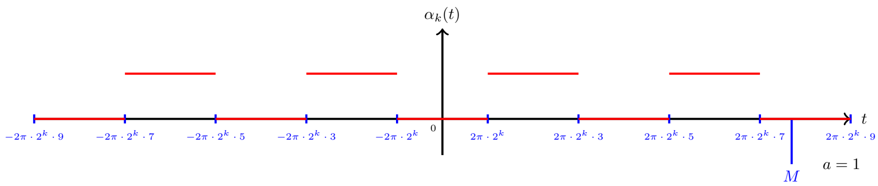


Once we know the shape of $\alpha_k(\cdot)$, we can prove (4.3). We will start by estimating

$$\frac{1}{M}\int_{-M}^{M}\alpha_k(\xi)\,d\xi.$$

Let $m = m(M)$ be the unique integer with $2\pi\cdot 2^k\cdot m \le M < 2\pi\cdot 2^k\cdot(m+1)$ (that is, we are locating M in a dyadic interval). We distinguish cases depending on whether m is congruent with 1 or 2 modulo 4 or is congruent with 0 or 3 modulo 4, that is to say, we distinguish cases depending on whether the dyadic interval containing M is central or not:

- Case $m \in \{4a+3, 4a+4\}$ for some $a \in \mathbb{Z}$ (that is, M belongs to a non-central dyadic interval):



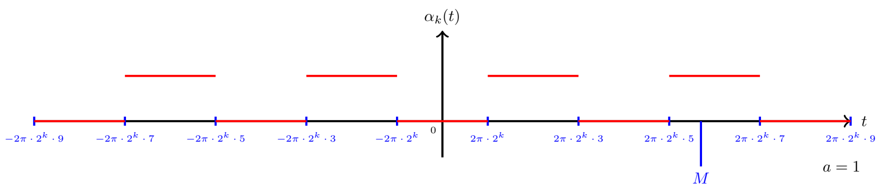
  There are $4a+4$ squares of basis $2\pi\cdot 2^k$ and height 1 in $[-M, M]$ (in the above example there are 8 squares). Then

$$\int_{-M}^{M}\alpha_k(\xi)\,d\xi = (4a+4)\cdot 2\pi\cdot 2^k,$$

  which gives the following estimation for $(1/M)\int_{-M}^{M}\alpha_k(\xi)\,d\xi$:

$$\frac{m}{m+1} \le \frac{(4a+4)\cdot 2\pi\cdot 2^k}{2\pi\cdot 2^k\cdot(m+1)} < \frac{1}{M}\int_{-\pi}^{\pi}\alpha_k(\xi)\,d\xi \le \frac{(4a+4)\cdot 2\pi\cdot 2^k}{2\pi\cdot 2^k\cdot m} \le \frac{m+1}{m}.$$

- Case $m \in \{4a+1, 4a+2\}$ for some $a \in \mathbb{Z}$ (that is, M belongs to a central dyadic interval):



Then the area $\int_{-M}^{M} \alpha_k(\xi)d\xi$ is greater or equal than the area of $4a$ squares of basis $2\pi \cdot 2^k$ and height 1 and less or equal than the area of $4a+4$ squares of basis $2\pi \cdot 2^k$ and height 1. That is,

$$4a \cdot 2\pi \cdot 2^k \leq \frac{1}{M} \int_{-M}^{M} \alpha_k(\xi)d\xi \leq (4a+4) \cdot 2\pi \cdot 2^k.$$

This gives the desired estimation on $(1/M) \int_{-M}^{M} \alpha_k(\xi)d\xi$:

$$\frac{m-2}{m+1} \leq \frac{4k \cdot 2\pi \cdot 2^k}{2\pi \cdot 2^k \cdot (m+1)} < \frac{1}{M} \int_{-M}^{M} \alpha_k(\xi)d\xi \leq \frac{(4k+4) \cdot 2\pi \cdot 2^k}{2\pi \cdot 2^k \cdot m} \leq \frac{m+3}{m}.$$

Hence, for all $M > 0$ we have established

$$\frac{m(M)-2}{m(M)+1} < \frac{1}{M} \int_{-M}^{M} \alpha_k(\xi)d\xi \leq \frac{m(M)+3}{m(M)}. \tag{4.4}$$

Write $(1/M) \int_{-M}^{M} \alpha_k(N(x)-\xi)d\xi$ as

$$\frac{1}{M} \int_{-M}^{M} \alpha_k(N(x)-\xi)d\xi = \frac{1}{M} \int_{-M+N(x)}^{M+N(x)} \alpha_k(\xi)d\xi$$

$$= \frac{1}{M-N(x)} \int_{-(M-N(x))}^{M-N(x)} \alpha_k(\xi)d\xi \cdot \frac{M-N(x)}{M} + \frac{1}{M} \int_{M-N(x)}^{M+N(x)} \alpha_k(\xi)d\xi.$$

From (4.4) and $0 \leq \alpha_k \leq 1$ one has

$$\left(\frac{m(M-N(x))-2}{m(M-N(x))+1}\right)\left(\frac{M-N(x)}{M}\right) < \frac{1}{M} \int_{-M+N(x)}^{M+N(x)} \alpha_k(\xi)d\xi \leq \frac{m(M-N(x))+3}{m(M-N(x))} + \frac{2N(x)}{M}.$$

If $M \to \infty$ (which is equivalent to $m(M-N(x)) \to \infty$), then

$$\frac{1}{M} \int_{-M+N(x)}^{M+N(x)} \alpha_k(\xi)d\xi \to 1,$$

and (4.3) holds, as wanted. This completes the proof of the lemma.

□

Thus, from the T_ξ's, we can recover T^0.

Suppose that (4.1) holds for all $f \in C_r^\infty$. Then for the same C of (4.1) it holds

$$\|T_\xi f\|_1 \leq C\|f\|_2 \tag{4.5}$$

for all $f \in C_r^\infty$ and for every $\xi \in \mathbb{R}$. Why? Because T and T_ξ are really the same operator, but on different (translated) grids. For the skeptic reader, substitute in all the future proofs T by T_ξ and $\mathcal{G}$ by $\mathcal{G}_\xi$ and check that everything is completely equal. From (4.5) it is easy to show (2.13):

$$\|T^0 f\|_1 = \int_0^{2\pi} |T^0 f(x)|\,dx \underbrace{=}_{\substack{\text{Lemma}\\ 4.1}} \int_0^{2\pi} \left(\lim_{M\to\infty} \frac{1}{M} \left| \int_{-M}^{M} T_\xi f(x)\,d\xi \right| \right) dx$$

$$\underbrace{\leq}_{\text{Fatou}} \liminf_{M\to\infty} \frac{1}{M} \int_0^{2\pi} \left| \int_{-M}^{M} T_\xi f(x)\,d\xi \right| dx \leq \liminf_{M\to\infty} \frac{1}{M} \int_0^{2\pi} \int_{-M}^{M} |T_\xi f(x)|\,d\xi\,dx$$

$$\leq \liminf_{M\to\infty} \frac{1}{M} \int_{-M}^{M} \|T_\xi f(\cdot)\|_1\,d\xi \underbrace{\leq}_{\substack{\text{by}\\ (4.5)}} 2C\|f\|_2.$$

Thus, from now on we will work with T, that is, with admissible pairs, and our objective will be to prove (4.1). Henceforth, "pair" will mean "admissible pair" and $\mathcal{B}$ will refer to the set of admissible pairs.

Remark 4.1 *There is a fact concerning T which will be extremely useful for the future proofs. The sum defining T takes into account the admissible pairs. On the other hand, the function $|N|$ is bounded by an $M^* > 0$. Assume $N > 0$ (the case $N < 0$ is analogous). Take K^* as the least natural number verifying $M^* < 2\pi \cdot 2^{K^*}$. Suppose $x \in E(p)$, where $p = [\omega, I] \in \mathcal{B}$ is an admissible pair with $|\omega| = 2\pi \cdot 2^k$ and $k \geq K^*$. Then it holds $\omega = [0, 2\pi \cdot 2^k[$, which is non central. Hence, all pairs $p = [\omega, I] \in \mathcal{B}$ with $x \in E(p)$ for some x verify $|\omega| < 2\pi \cdot 2^{K^*}$. This shows that the sum defining T is finite, independently on the evaluated f and x (however, the number of terms in the sum obviously depends on the bound for N, but this will not cause any problem).*

From now on, our set of pairs $\mathcal{B}$ will be our latest $\mathcal{B}$ but having deleted all the pairs that do not contribute in $\sum_{p\in\mathcal{B}} T_p$. We have then a finite $\mathcal{B}$. This will be very useful to interchange operators concerning sums and integrals without applying Lebesgue's convergence results. Moreover, T is then well-defined on L^2_r, and we will not have to deal with C^∞_r anymore. Our goal will be to prove (4.1) for every $f \in L^2_r$.

Note also that we can assume that $\omega \subseteq [0,\infty[$ for all $[\omega,\cdot] \in \mathcal{B}$ (if $N > 0$) or $\omega \subseteq]-\infty, 0[$ for all $[\omega,\cdot] \in \mathcal{B}$ (if $N < 0$). Otherwise, the characteristic function $\mathbb{1}_{E(\omega,\cdot)}$ appearing in the sum defining T would be 0. This assumption on $\mathcal{B}$ will be essential for the future Lemma 6.4.

We state and prove a property of nested central dyadic intervals which will be essential in the proof of the future Main Lemma 6.6.

Lemma 4.2 *Suppose that $\omega_0 \subsetneq \omega_1 \subsetneq \ldots \subsetneq \omega_{M+1}$, with $\omega_0, \ldots, \omega_{M+1}$ central, $M \geq 1$. Then*

$$d(\partial\omega_{M+1}, \omega_0) \geq 2^{\frac{M}{2}-2}|\omega_0|.$$

Proof. By centrality, $\omega_0^* \subseteq \tilde{\tilde{\omega_0}} \subseteq \omega_2$, $\omega_2^* \subseteq \tilde{\tilde{\omega_2}} \subseteq \omega_4$, etc. Hence,

$$\omega_0^{*\overset{\lfloor (M+1)/2 \rfloor)}{\cdots}*} \subseteq \omega_{M+1}.$$

Then

$$\begin{aligned} d(\partial\omega_{M+1}, \omega_0) &\geq \frac{|\omega_0|}{2} + |\omega_0| + 2|\omega_0| + \ldots + 2^{\lfloor (M+1)/2 \rfloor - 2}|\omega_0| \\ &= \left(\frac{1}{2} + 2^{\lfloor (M+1)/2 \rfloor - 1} - 1 \right) |\omega_0| = \left(2^{\lfloor (M+1)/2 \rfloor - 1} - \frac{1}{2} \right) |\omega_0|. \end{aligned}$$

If M is odd, then $2^{\lfloor (M+1)/2 \rfloor -1} - 1/2 = 2^{(M+1)/2-1} - 1/2 = 2^{(M-1)/2} - 1/2 \geq 2^{M/2-2}$ for $M \geq 1$. If M is even, then $2^{\lfloor (M+1)/2 \rfloor -1} - 1/2 = 2^{M/2-1} - 1/2 \geq 2^{M/2-2}$ for every $M \geq 2$. □

Chapter 5

Sketch of the proof of the basic estimate (4.1)

We know that, for all $f \in L^2_r$ and a.e. $x \in \mathbb{R}$,

$$Tf(x) = {\sum_{k\geq 1}}' \sum_{p\in\mathcal{B}_k} T_p f(x) = {\sum_{k\geq 1}}' {\sum_{n\geq 0}}' \sum_{\substack{p\in\mathcal{B}_k \\ 2^{-n-1}<A_0(p)\leq 2^{-n}}} T_p f(x)$$

($\sum'$ indicates that $\sum$ is finite). The a.e. comes from the fact that, if $A_0(p) = 0$, then $|E(p)| = 0$, so $\mathbb{1}_{E(p)} = 0$ a.e. and $T_p f = 0$ a.e.. As we are dealing with norms in the Lebesgue spaces, this a.e. will not have any significance. Thus, from now on, the "a.e." will be omitted.

We can interchange the sums:

$$Tf(x) = {\sum_{n\geq 0}}' {\sum_{k\geq 1}}' \sum_{\substack{p\in\mathcal{B}_k \\ 2^{-n-1}<A_0(p)\leq 2^{-n}}} T_p f(x). \tag{5.1}$$

Define

$$\mathcal{P}_n = {\bigcup_{k\geq 1}}' \{p \in \mathcal{B}_k : 2^{-n-1} < A_0(p) \leq 2^{-n}\}$$

(again, $\cup'$ represents a finite union). Then

$$\sum_{p\in\mathcal{P}_n} T_p f(x) = {\sum_{k\geq 1}}' \sum_{\substack{p\in\mathcal{B}_k \\ 2^{-n-1}<A_0(p)\leq 2^{-n}}} T_p f(x). \tag{5.2}$$

Thus, using (5.1) and (5.2), we can express T as

$$Tf(x) = {\sum_{n\geq 0}}' \sum_{p\in\mathcal{P}_n} T_p f(x), \tag{5.3}$$

for any $f \in L^2_r$ and $x \in \mathbb{R}$.

Given a subset of pairs $\mathcal{P} \subseteq \mathcal{B}$, denote

$$T^{\mathcal{P}} f(x) = \sum_{p\in\mathcal{P}} T_p f(x),$$

which is a finite sum independently on $f \in L^2_r$ and $x \in \mathbb{R}$. With this new notation, we have a compact expression for T:

$$Tf(x) = \sum_{n\geq 0}{}' T^{\mathcal{P}_n} f(x). \tag{5.4}$$

Let us see why this decomposition of T is useful. We know, by (3.3), that for all $p \in \mathcal{P}_n$

$$\|T_p\|_2 \leq CA_0(p)^{1/2}.$$

More or less, the idea would be to transmit this bound to $T^{\mathcal{P}_n}$:

$$\|T^{\mathcal{P}_n}\|_2 \leq C \max_{p\in\mathcal{P}_n} A_0(p)^{1/2} \leq C2^{-\frac{n}{2}}.$$

In this case,

$$\|Tf\|_1 = \int_0^{2\pi} |Tf(x)|\,dx = \int_0^{2\pi} \left|\sum_{n\geq 0}{}' T^{\mathcal{P}_n} f(x)\right| dx \leq \int_0^{2\pi} \left(\sum_{n\geq 0}{}' |T^{\mathcal{P}_n} f(x)|\right) dx$$
$$= \sum_{n\geq 0}{}' \int_0^{2\pi} |T^{\mathcal{P}_n} f(x)|\,dx = \sum_{n\geq 0}{}' \|T^{\mathcal{P}_n} f\|_1 \leq C \sum_{n\geq 0}{}' \|T^{\mathcal{P}_n} f\|_2 \leq C \sum_{n=0}^{\infty} 2^{-\frac{n}{2}},$$

so (4.1) would hold.

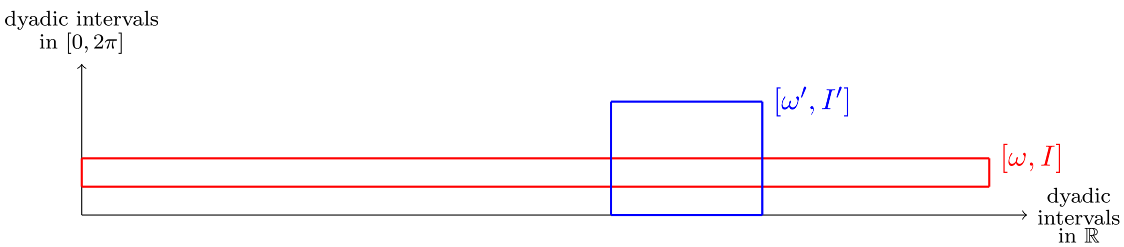
In the next chapter, we will make a systematic study of the sets $\mathcal{P} \subseteq \mathcal{B}$ for which, if $A_0(p) \leq \delta$ for all $p \in \mathcal{P}$, then

$$\|T^{\mathcal{P}}\|_2 \leq C\delta^{1/2} \tag{5.5}$$

(that is, we transmit the bound of $\|T_p\|_2$, $p \in \mathcal{P}$, to $\|T^{\mathcal{P}}\|_2$). We will start with small subsets of $\mathcal{B}$ verifying (5.5) and we will construct larger and larger $\mathcal{P}$'s satisfying (5.5). For that purpose, we need to specify what we mean by a "small" pair and a "large" pair, that is to say, we need to establish an order relation in $\mathcal{B}$. The order relation is the following: given $[\omega, I], [\omega', I'] \in \mathcal{B}$, we will write

$$[\omega, I] < [\omega', I']$$

if and only if $I \subseteq I'$ and $\omega' \subseteq \omega$.



In the following chapter we will build up the $\mathcal{P}$'s verifying (5.5).

Remark 5.1 *Pictures as the last one will be very common through the rest of the project. A pair $[\omega, I]$ is represented with a rectangle: the horizontal lines represent ω and the vertical lines I. This type of pictures allow representing the order relation between the pairs, without paying attention on lengths, scales or admissibility (unless we want to put a specific example).*

We prove here a lemma that will be used in the future Main Lemma 6.6:

Lemma 5.1 (Orthogonality Lemma) *Let $\{A_i\}_{i=1}^n$, $n \in \mathbb{N}$, be bounded operators on a Hilbert space H. Assume:*

(a) $A_i^* A_j = 0$ for $i \neq j$,

(b) $\|A_i A_j^*\| \leq \frac{M^2}{|i-j|^2+1}$.

Then

$$\left\| \sum_{i=1}^n A_i \right\| \leq C \cdot M,$$

where $C = (\pi \coth \pi)^{1/2}$.

Proof. Call $T = \sum_{i=1}^n A_i$. As $\|T\|^2 = \|T^*T\|$, one has by induction that $\|T\|^{2m} = \|(T^*T)^m\|$. We have $T^*T = \sum_{i=1}^n \sum_{j=1}^n A_i^* A_j$, so

$$(T^*T)^m = \sum_{1 \leq i_1, \ldots, i_{2m} \leq n} A_{i_1}^* A_{i_2} \cdots A_{i_{2m-1}}^* A_{i_{2m}}.$$

Hence,

$$\begin{aligned} \|T\|^{2m} &\leq \sum_{1 \leq i_1, \ldots, i_{2m} \leq n} \|(A_{i_1}^* A_{i_2}) \cdots (A_{i_{2m-1}}^* A_{i_{2m}})\| \\ &\underbrace{=}_{\text{(a)}} \sum_{1 \leq j_1, \ldots, j_m \leq n} \|A_{j_1}^* (A_{j_1} A_{j_2}^*) \cdots (A_{j_{m-1}} A_{j_m}^*) A_{j_m}\|. \end{aligned}$$

Now we use $\|A_{j_1}^*\| \leq M$, $\|A_{j_m}\| \leq M$ and hypothesis (b):

$$\|T\|^{2m} \leq M^{2m} \sum_{1 \leq j_1, \ldots, j_m \leq n} \frac{1}{(j_1 - j_2)^2 + 1} \cdots \frac{1}{(j_{m-1} - j_m)^2 + 1}.$$

Now make the change of variables $j_1 - j_2 = t_1, j_2 - j_3 = t_2, \ldots, j_{m-1} - j_m = t_{m-1}, j_m = t_m$. Then

$$\|T\|^{2m} \leq M^{2m} \sum_{-n \leq t_1, \ldots, t_m \leq n} \frac{1}{t_1^2 + 1} \cdots \frac{1}{t_{m-1}^2 + 1} = 2n \cdot M^{2m} \left(\sum_{j=-n}^n \frac{1}{j^2 + 1} \right)^{m-1}.$$

Then

$$\|T\| \leq (2n)^{\frac{1}{2m}} \cdot M \left(\sum_{j=-n}^n \frac{1}{j^2 + 1} \right)^{\frac{m-1}{2m}}.$$

Let $m \to \infty$:

$$\|T\| \leq M \left(\sum_{j=-n}^n \frac{1}{j^2 + 1} \right)^{\frac{1}{2}} \leq C \cdot M,$$

where we use $\sum_{j=-\infty}^{\infty} \frac{1}{j^2+1} = \pi \coth \pi$.

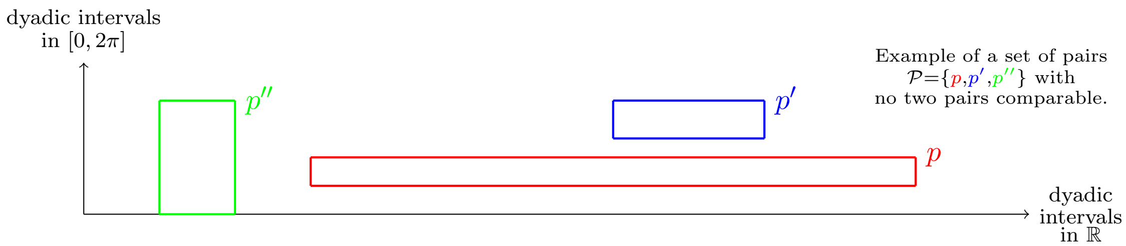
$\square$

Chapter 6

Main sequence of lemmas

In the following lemmas we will build up $\mathcal{P}$'s verifying (5.5). These lemmas are rather technical and of substantial difficulty, so we advise the reader to study them with patience.

We will start in the first two lemmas by considering subsets $\mathcal{P}$ with the property that no two of its pairs are comparable under $<$. The key of this assumption is that it implies that the $E(p)$'s are pairwise disjoint for p's in $\mathcal{P}$. Indeed, assume that $E(p) \cap E(p') \neq \emptyset$ for some distinct pairs $p = [\omega, I], p' = [\omega', I'] \in \mathcal{P}$. Then $I \cap I' \neq \emptyset$ and $\omega \cap \omega' \neq \emptyset$. Assume without loss of generality that $|I| \leq |I'|$. This gives $|\omega'| \leq |\omega|$. Since the intervals are dyadic, $I \subseteq I'$ and $\omega' \subseteq \omega$, so $p < p'$, which is a contradiction.



Instead of working with the A_0 expression defined in (3.4), we will work with

$$A([\omega, I]) = \sup_{\substack{p'=[\omega',I'] \in \mathcal{B} \\ I \subseteq I'}} \frac{|E(p')|}{|I'|} \left(\frac{d(\omega, \omega') + |\omega|}{|\omega|} \right)^{-2000}$$

for technical reasons.

Remark 6.1 *Throughout the lemmas we will make use of maximal operators in order to prove the boundedness of our operators. In concrete, we will deal with two maximal operators:*

- *Let* $\{I_j\}_{j=1}^{J}$ *be a partition of* $[0, 2\pi]$. *Define the maximal operator*

$$M_0 f(x) = \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) \mathbb{1}_{I_j}(x).$$

As it happens with the Hardy-Littlewood maximal operator, $M_0 : L^1_r \to L^{1,\infty}_r$ *and* $M_0 : L^q_r \to L^q_r$, $1 < q < \infty$, *where* $L^{1,\infty}_r = \{h : \mathbb{R} \to \mathbb{C} :$ *support*$(h) \subseteq [-4\pi, 6\pi]$, $\sup_{t>0} t|\{x \in \mathbb{R} : |h(x)| > t\}| < \infty\}$.

- *The second maximal operator is*

$$f^*_q(x) = \sup_{x\in I} \left(\frac{1}{|I|} \int_I |f(y)|^q \, dy \right)^{\frac{1}{q}}$$

for $x \in [0, 2\pi]$ *(and* 0 *outside). It holds* $\|f^*_q\|_2 \leq C_q \|f\|_2$ *for all* $1 \leq q < 2$.

When $q = 1$*, we will denote* $f^* = f^*_1$.

Remark 6.2 *We will use the maximal operators to bound convolutions: if* $f \in L^1_{loc}(\mathbb{R})$ *and* $0 \leq \varphi \in L^1(\mathbb{R})$ *is radially decreasing, then*

$$|(f * \varphi)(x)| \leq C\|\varphi\|_1 \left(\sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| \, dy \right) \leq C\|\varphi\|_1 f^*(x).$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(y)|f(x-y)| \, dy = \sum_{m=-\infty}^{\infty} \int_{\{2^m \leq |y| \leq 2^{m+1}\}} \varphi(y)|f(x-y)| \, dy \\
&\leq \sum_{m=-\infty}^{\infty} \varphi(2^m) \int_{\{2^m \leq |y| \leq 2^{m+1}\}} |f(x-y)| \, dy \leq \sum_{m=-\infty}^{\infty} \varphi(2^m) \frac{2^{m+2}}{2^{m+2}} \int_{-2^{m+1}}^{2^{m+1}} |f(x-y)| \, dy \\
&\leq \sum_{m=-\infty}^{\infty} \varphi(2^m) \frac{2^{m+2}}{2^{m+2}} \int_{x-2^{m+1}}^{x+2^{m+1}} |f(y)| \, dy \leq \left(\sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| \, dy \right) \sum_{m=-\infty}^{\infty} 2^{m+2} \varphi(2^m) \\
&\leq 8 \left(\sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| \, dy \right) \sum_{m=-\infty}^{\infty} \int_{\{2^{m-1} \leq |y| \leq 2^m\}} \varphi(y) \, dy \\
&= 8\|\varphi\|_1 \left(\sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| \, dy \right) \leq 8\|\varphi\|_1 f^*(x).
\end{aligned}$$

In general, we will deal with a φ *of the form*

$$\varphi(x) = \frac{a}{x^2 + a^2}, \quad a > 0.$$

In this case, $\|\varphi\|_1 = \pi$ *does not depend on* a.

In fact, this procedure of relating a convolution with a maximal operator is more or less a particular case of the following theorem (or better said, of the idea of its proof): "Let $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $0 \leq \varphi \in L^1(\mathbb{R}^n)$*. Denote* $\varphi_t(x) = (1/t^n)\varphi(x/t)$, $t > 0$*, and define* $T_t f(x) = (f * \varphi_t)(x)$*. If* φ *is radially decreasing, then* $T^* f(x) = \sup_{t>0} |T_t f(x)| \leq C_n \|\varphi\|_1 M f(x)$*, where* M *denotes the Hardy-Littlewood maximal operator". To bound convolutions, use this theorem with* $t = 1$.

Lemma 6.1 *Let* $\mathcal{P}$ *be a set of pairs, no two of which are comparable under* $<$*. Let* $0 < \eta < 1/4^{4004}$, $n_0 \in \mathbb{R}$ *and* $0 < \epsilon < 1/5000$*. Suppose that* $A(p) \leq \eta$ *and* $d(n_0, \omega) \leq \eta^{-\epsilon}|\omega|$ *for all* $p = [\omega, I] \in \mathcal{P}$*. Then* $T^{\mathcal{P}} : L^q_r \to L^q_r$ *has norm*

$$\|T^{\mathcal{P}}\|_q \leq C_q \eta^{\frac{1-5000\epsilon}{q}}, \quad \forall q > 1.$$

Proof. We start with some technicalities. Call $\delta = 4^{2002}\eta$. Then for all $p = [\omega, I] \in \mathcal{P}$ we have $d(n_0, \omega) \leq \eta^{-\epsilon}|\omega| = (1/\eta)^{\epsilon}|\omega| = 4^{2002\epsilon}\delta^{-\epsilon}|\omega| < \delta^{-2\epsilon}|\omega|$ (because $(4^{2002}\delta)^{\epsilon} = (4^{4004}\eta)^{\epsilon} < 1$).

The idea to prove the lemma is to bound $T^{\mathcal{P}}$ by a suitable maximal operator. Let $\{I_j\}_{j=1}^{J}$ be a partition of $[0, 2\pi]$. Consider $\{E_j\}_{j=1}^{J}$ such that $E_j \subseteq I_j$ for all $j = 1, \dots, J$, with

$$\frac{|E_j|}{|I_j|} \leq 2\delta^{1-5000\epsilon}.$$

Define

$$Mf(x) = \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) \mathbb{1}_{E_j}(x)$$

for $x \in [0, 2\pi]$ (and outside of $[0, 2\pi]$ is 0). Define the maximal operator

$$M_0 f(x) = \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) \mathbb{1}_{I_j}(x)$$

for $x \in [0, 2\pi]$ (and outside of $[0, 2\pi]$ is 0). By Remark 6.1, $M_0 : L^q_r \to L^q_r$ for $q > 1$, so we can estimate $\|Mf\|_q$:

$$\begin{aligned}
\|Mf\|_q &= \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) |E_j|^{1/q} \\
&\leq 2^{1/q}\delta^{\frac{1-5000\epsilon}{q}} \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) |I_j|^{1/q} = 2^{1/q}\delta^{\frac{1-5000\epsilon}{q}} \|M_0 f\|_q \\
&\leq C_q \delta^{\frac{1-5000\epsilon}{q}} \|f\|_q \leq C_q 4^{2002/q} \eta^{\frac{1-5000\epsilon}{q}} \|f\|_q = C_q \eta^{\frac{1-5000\epsilon}{q}} \|f\|_q. \qquad (6.1)
\end{aligned}$$

Thus, to prove the lemma, it suffices to dominate $|T^{\mathcal{P}} f(x)|$ by $|Mf(x)|$ for all $x \in [0, 2\pi]$ and $f \in L^q_r$, with the E_j's and I_j's to be constructed.

We make a partition $\{I_j\}_{j=1}^{J}$, $J \in \mathbb{N}$, as follows. We will say that a dyadic interval $I \subseteq [0, 2\pi]$ is *good* if

$$\frac{|\{x \in I : |N(x) - n_0| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2 \cdot |I|^{-1}\}|}{|I|} > \delta^{1-5000\epsilon}.$$

Since $N(\cdot)$ is bounded by an $M > 0$ in $[0, 2\pi]$, we have $|N(x) - n_0| \leq M + |n_0|$ for all $x \in [0, 2\pi]$, so if the length of I satisfies $M + |n_0| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2/|I|$, then $\{x \in I : |N(x) - n_0| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi \cdot |I|^{-1}\} = I$. Thus, for any sufficiently small I (more specifically, $|I| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2/(M + |n_0|)$), I is good. To construct the partition, start with $[0, 2\pi]$. If it is good, our partition is $\{[0, 2\pi]\}$. Otherwise, subdivide it into two halves: $[0, \pi]$ and $[\pi, 2\pi]$. If any of them is good, keep it for the partition; if not, subdivide it again into two halves. This process will end up in a finite number of steps (since once we consider intervals of length less than $\delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2/(M + |n_0|)$, we will have surely a good dyadic interval), and we obtain a partition $\{I_j\}_{j=1}^{J}$ of maximal dyadic intervals (in the sense that if we take an interval I_j of the partition and we then take the smallest bigger dyadic interval $\tilde{I}_j$ containing it, then $\tilde{I}_j$ will not be good).

Define $\tilde{E}_j = \{x \in \tilde{I}_j : |N(x) - n_0| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2 \cdot |\tilde{I}_j|^{-1}\}$ (we will show in part (a) below that I_j is never the whole $[0, 2\pi]$, therefore $\tilde{I}_j \subseteq [0, 2\pi]$ for all $j = 1, \dots, J$

and the definition of $\tilde{E}_j$ makes sense). By maximality, $\tilde{I}_j$ is not good, therefore $|\tilde{E}_j| \leq \delta^{1-5000\epsilon}|\tilde{I}_j| = 2\delta^{1-5000\epsilon}|I_j|$. Define $E_j = \{x \in I_j : |N(x) - n_0| \leq \delta^{-2\epsilon} \cdot 2 \cdot 4\pi^2 \cdot |\tilde{I}_j|^{-1}\}$. Then $E_j \subseteq I_j$ and $|E_j| \leq |\tilde{E}_j| \leq 2\delta^{1-5000\epsilon}|I_j|$.

Consider the corresponding

$$Mf(x) = \sum_{j=1}^{J} \left(\sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy \right) \mathbb{1}_{E_j}(x).$$

We show that there is a $C > 0$ such that $|T^{\mathcal{P}} f(x)| \leq C|Mf(x)|$ for all $x \in [0, 2\pi]$ and $f \in L^q_r$, since in this case the lemma will be a consequence of (6.1). This amounts to proving that

$$|T^{\mathcal{P}} f(x)| \leq C \sup_{I_j \subseteq I} \frac{1}{|I|} \int_I |f(y)|\, dy, \quad \forall x \in E_j, \tag{6.2}$$

$$T^{\mathcal{P}} f(x) = 0, \quad \forall x \in I_j \backslash E_j. \tag{6.3}$$

Recall that $T^{\mathcal{P}} f(x) = \sum_{p \in \mathcal{P}} T_p f(x)$. As $p \not< p'$ for all distinct $p, p' \in \mathcal{P}$, it follows that the $E(p)$'s are pairwise disjoint. Then, for each $x \in [0, 2\pi]$, the sum $\sum_{p \in \mathcal{P}} T_p f(x)$ contains a single term:

$$|T^{\mathcal{P}} f(x)| = \max_{\substack{p=[\omega,I] \in \mathcal{P} \\ x \in E(p)}} |T_p f(x)| \underbrace{\leq}_{\substack{\text{by} \\ (3.2)}} \max_{\substack{p=[\omega,I] \in \mathcal{P} \\ x \in E(p)}} \frac{2\pi}{|I|} \int_{I^3} |f(y)|\, dy. \tag{6.4}$$

We will show:

(a) $[\omega, I] \in \mathcal{P}$, $I \cap I_j \neq \emptyset \Rightarrow \tilde{I}_j \subseteq I$.

(b) $[\omega, I] \in \mathcal{P}$, $I \cap I_j \neq \emptyset \Rightarrow I_j \cap E(\omega, I) \subseteq E_j$.

Notice that, from (a), if some I_j is $[0, 2\pi]$, then for any pair $[\omega, I] \in \mathcal{P}$ it holds $I \cap I_j \neq \emptyset$, so $\tilde{I}_j \subseteq I \subseteq [0, 2\pi]$, and we arrive at a contradiction. Hence, none of the I_j's is $[0, 2\pi]$, as we stated previously when defining the $\tilde{E}_j$'s.

Using (a) and (b), both (6.2) and (6.3) easily follow. Let $x \in [0, 2\pi]$. We distinguish two cases:

- Case $x \in I_j \backslash E_j$. Suppose by contradiction that $x \in E(p)$ for some $p = [\omega, I] \in \mathcal{P}$. Then $x \in I$, so $I \cap I_j \neq \emptyset$. By (b), $I_j \cap E(p) \subseteq E_j$, which implies $x \in E_j$, which is a contradiction. Hence, x does not lie in any of the $E(p)$'s, so $T^{\mathcal{P}} f(x) = 0$ and (6.3) follows.

- Case $x \in E_j$. If $x \in E(p)$, $p = [\omega, I] \in \mathcal{P}$, then $x \in I \cap I_j$, so by (a) $\tilde{I}_j \subseteq I$. In particular, $I_j \subseteq I^3$, therefore

$$\frac{1}{|I^3|} \int_{I^3} |f(y)|\, dy \leq Mf(x).$$

  Then by (6.4) $|T^{\mathcal{P}} f(x)| \leq 6\pi Mf(x)$, and (6.2) holds with $C = 6\pi$.

Thus, to finish the proof of the lemma, we just need to verify (a) and (b).

Proof of (a). We have $I \cap I_j \neq \emptyset$, where both I and I_j are dyadic intervals. Then either $I \subseteq I_j$ or $I_j \subsetneq I$, that is to say, either $I \subseteq I_j$ or $\tilde{I}_j \subseteq I$. Assume by contradiction that $I \subseteq I_j$. Consider the dyadic intervals ω satisfying $|\omega| = 4\pi^2/|I_j|$ and $d(n_0, \omega) \leq 2\delta^{-2\epsilon}|\omega|$.

As $d(n_0,\omega) \le 2\delta^{-2\epsilon}\cdot 4\pi^2/|I_j|$, there is a finite number of such ω's, call it K_j. Let ω' with $|\omega'| = 4\pi^2/|I_j|$, $d(n_0,\omega') \le 2\delta^{-2\epsilon}\cdot 4\pi^2/|I_j|$ and

$$|E(\omega', I_j)| = \max\left\{|E(\omega, I_j)| : |\omega| = \frac{4\pi^2}{|I_j|},\ d(n_0,\omega) \le 2\delta^{-2\epsilon}\frac{4\pi^2}{|I_j|}\right\}.$$

Then

$$\delta^{1-5000\epsilon}|I_j| < |\{x \in I_j : |N(x) - n_0| \le 2\delta^{-2\epsilon}\cdot 4\pi^2 \cdot |I_j|^{-1}\}| \le K_j|E(\omega', I_j)|,$$

which gives

$$\frac{|E(\omega', I_j)|}{|I_j|} > \frac{\delta^{1-5000\epsilon}}{K_j}.$$

We can, in fact, bound $K_j \le 2(2\delta^{-2\epsilon}+1) \le 16\delta^{-2\epsilon}$, therefore

$$\frac{|E(\omega', I_j)|}{|I_j|} > \frac{\delta^{1-4998\epsilon}}{16}.$$

By hypothesis, we have

$$\frac{d(\omega, n_0)}{|\omega|} \le \delta^{-2\epsilon}, \quad \frac{d(\omega', n_0)}{|\omega'|} \le 2\delta^{-2\epsilon},$$

where $[\omega, I] \in \mathcal{P}$. Since $I \subseteq I_j$, then $|I| \le |I_j|$, and taking inverses $|\omega'| \le |\omega|$. We have

$$\begin{aligned}\left(\frac{d(\omega,\omega') + |\omega|}{|\omega|}\right)^{-2000} &\ge \left(\frac{d(\omega, n_0) + d(\omega', n_0) + |\omega|}{|\omega|}\right)^{-2000}\\ &= \left(\frac{d(\omega, n_0)}{|\omega|} + \frac{|\omega'|}{|\omega|}\frac{d(\omega', n_0)}{|\omega'|} + 1\right)^{-2000}\\ &> (3\delta^{-2\epsilon}+1)^{-2000} > 4^{-2000}\delta^{4000\epsilon}.\end{aligned}$$

Then, since $I \subseteq I_j$,

$$A(\omega, I) \ge \frac{|E(\omega', I_j)|}{|I_j|}\left(\frac{d(\omega,\omega') + |\omega|}{|\omega|}\right)^{-2000} > \frac{\delta^{1-998\epsilon}}{4^{2002}} > \frac{\delta}{4^{2002}} = \eta,$$

and this is a contradiction.

Proof of (b). We have that $I_j \cap E(\omega, I) = \{x \in I_j : N(x) \in \omega\}$, since by (a) $I_j \subseteq I$. If $N(x) \in \omega$, then

$$|N(x) - n_0| \le |N(x) - \xi_{n_0}| + \underbrace{|\xi_{n_0} - n_0|}_{d(\omega, n_0)} \le |\omega| + \delta^{-2\epsilon}|\omega| = (1+\delta^{-2\epsilon})|\omega| < 2\delta^{-2\epsilon}|\omega|$$

(ξ_{n_0} is the nearest endpoint of ω to n_0). Thus,

$$I_j \cap E(\omega, I) \subseteq \{x \in I_j : |N(x) - n_0| < 2\delta^{-2\epsilon}|\omega|\}.$$

By (a), $\tilde{I}_j \subseteq I$, so $|\tilde{I}_j| \le |I|$, that is, $|\omega| = 4\pi^2/|I| \le 4\pi^2/|\tilde{I}_j|$. This implies

$$I_j \cap E(\omega, I) \subseteq \{x \in I_j : |N(x) - n_0| < 2\delta^{-2\epsilon}\cdot 4\pi^2 \cdot |\tilde{I}_j|^{-1}\} = E_j,$$

so (b) is proved.

$\square$

Remark 6.3 *We know that $T_p : L^2_r \to L^2_r$ is a bounded operator, so we can compute its adjoint operator $T_p^* : L^2_r \to L^2_r$. Let $f, g \in L^2_r$. Then*

$$\begin{aligned}(T_p f, g) &= \int_{\mathbb{R}} T_p f(x)\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{iN(x)y}\psi_k(y) f(x-y)\,dy\right) \mathbb{1}_{E(p)}(x)\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{iN(x)(x-y)}\psi_k(x-y) f(y)\,dy\right) \mathbb{1}_{E(p)}(x)\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{iN(x)(x-y)}\psi_k(x-y) \mathbb{1}_{E(p)}(x)\overline{g(x)}\,dx\right) f(y)\,dy \\ &= \int_{\mathbb{R}} \left(\overline{\int_{E(p)} e^{-iN(x)(x-y)}\psi_k(x-y) g(x)\,dx}\right) f(y)\,dy.\end{aligned}$$

Suppose $y \notin I^3$. If, given $x \in E(p)$, we had $x - y \in [-2\pi \cdot 2^{-k}, 2\pi \cdot 2^k]$, then it would hold $y \in [x-2\pi\cdot 2^{-k}, x+2\pi\cdot 2^{-k}] \subseteq I^3$, which is a contradiction. Then, $x-y \notin [-2\pi\cdot 2^{-k}, 2\pi\cdot 2^k]$, which implies $\psi_k(x-y) = 0$. Then

$$(T_p f, g) = \int_{\mathbb{R}} \left(\overline{\int_{E(p)} e^{-iN(x)(x-y)}\psi_k(x-y) g(x)\,dx}\right) \mathbb{1}_{I^3}(y) f(y)\,dy.$$

Thus,

$$T_p^* g(y) = \int_{E(p)} e^{-iN(x)(x-y)}\psi_k(x-y) g(x)\,dx \cdot \mathbb{1}_{I^3}(y),$$

or, with our usual variables,

$$T_p^* f(x) = \int_{E(p)} e^{-iN(y)(y-x)}\psi_k(y-x) f(y)\,dy \cdot \mathbb{1}_{I^3}(x) \tag{6.5}$$

for all $f \in L^2_r$ and $x \in \mathbb{R}$.

A useful inequality for the adjoint operator, using the fact that $|\psi_k| \leq 2^k$, is

$$|T_p^* f(x)| \leq \frac{C}{|I|} \int_{E(p)} |f(y)|\,dy \cdot \mathbb{1}_{I^3}(x) \tag{6.6}$$

(here $C = 2\pi$).

Lemma 6.2 *Let $\mathcal{P}$ be a set of pairs, no two of which are comparable under $<$. Assume that $A(p) \leq \delta$ for all $p \in \mathcal{P}$ ($0 < \delta \leq 1$). Then $T^{\mathcal{P}} : L^2_r \to L^2_r$ has norm*

$$\|T^{\mathcal{P}}\|_2 \leq C_\eta \delta^{\frac{1}{4}-\eta}, \quad \forall \eta > 0. \tag{6.7}$$

Proof. We will deal with the case $0 < \delta < 1/4^{4004}$ until further notice. This bound will allow us to apply Lemma 6.1.

Fix $f \in L^2_r$. We have

$$\|T^{\mathcal{P}} f\|_2^2 = \|T^{\mathcal{P}^*} f\|_2^2 = \left\| \sum_{p\in\mathcal{P}} T_p^* f \right\|_2^2 = \left(\sum_{p\in\mathcal{P}} T_p^* f, \sum_{p'\in\mathcal{P}} T_{p'}^* f \right) = \sum_{p,p'\in\mathcal{P}} \int_{\mathbb{R}} T_p^* f(x) \overline{T_{p'}^* f(x)}\, dx$$

$$\leq \left| \sum_{\substack{[\omega,I],[\omega',I']\in\mathcal{P}\\ |I|\leq|I'|}} \int_{\mathbb{R}} T^*_{[\omega,I]} f(x) \overline{T^*_{[\omega',I']} f(x)}\, dx \right| + \left| \sum_{\substack{[\omega,I],[\omega',I']\in\mathcal{P}\\ |I'|\leq|I|}} \int_{\mathbb{R}} T^*_{[\omega,I]} f(x) \overline{T^*_{[\omega',I']} f(x)}\, dx \right|.$$

We want to estimate both terms. Let us look at the first:

$$\sum_{\substack{[\omega,I],[\omega',I']\in\mathcal{P}\\ |I|\leq|I'|}} \int_{\mathbb{R}} T^*_{[\omega,I]} f(x) \overline{T^*_{[\omega',I']} f(x)}\, dx = \sum_{p'=[\omega',I']} \int_{\mathbb{R}} \overline{T^*_{[\omega',I']} f(x)} \left[\sum_{\substack{p=[\omega,I]\\ |I|\leq|I'|}} T_p^* f(x) \right] dx$$

$$= \sum_{p'\in\mathcal{P}} \int_{\mathbb{R}} \overline{T^*_{p'} f(x)} \left[\sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right] dx \tag{6.8}$$

$$+ \sum_{p'\in\mathcal{P}} \int_{\mathbb{R}} \overline{T^*_{p'} f(x)} \left[\sum_{p\in\mathcal{B}(p')} T_p^* f(x) \right] dx, \tag{6.9}$$

where

$$\mathcal{A}(p') = \left\{ p = [\omega, I] \in \mathcal{P},\, |I| \leq |I'| :\, d(\omega,\omega') \leq \frac{1}{2}\delta^{-\epsilon}|\omega| \text{ and } I \subseteq (I')^5 \right\},$$

$$\mathcal{B}(p') = \left\{ p = [\omega, I] \in \mathcal{P},\, |I| \leq |I'| :\, d(\omega,\omega') > \frac{1}{2}\delta^{-\epsilon}|\omega| \text{ or } I \not\subseteq (I')^5 \right\}.$$

We estimate both (6.8) and (6.9). We start with the terms of (6.8):

$$\left| \int_{\mathbb{R}} \overline{T^*_{p'} f(x)} \left[\sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right] dx \right| \leq \int_{\mathbb{R}} |T^*_{p'} f(x)| \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right| dx$$

$$\underbrace{\leq}_{\substack{\text{by}\\(6.6)}} \left(\frac{C}{|I'|} \int_{E(p')} |f(y)|\, dy \right) \int_{(I')^3} \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right| dx$$

$$\underbrace{\leq}_{C\leftarrow 5C} \left(\frac{C}{|(I')^5|} \int_{E(p')} |f(y)|\, dy \right) \int_{(I')^5} \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right| dx$$

$$\underbrace{\leq}_{\substack{\text{Hölder}\\ q>1,\, \frac{1}{q}+\frac{1}{q'}=1}} \left(\frac{C}{|(I')^5|} \int_{E(p')} |f(y)|\, dy \right) |(I')^5|^{\frac{1}{q'}} \left(\int_{(I')^5} \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right|^q dx \right)^{\frac{1}{q}}$$

$$= C \left(\int_{E(p')} |f(y)|\, dy \right) \left(\frac{1}{|(I')^5|} \int_{(I')^5} \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\int_{E(p')} |f(y)|\, dy \right) \left(\frac{1}{|(I')^5|} \int_{\mathbb{R}} \left| \sum_{p\in\mathcal{A}(p')} T_p^* f(x) \right|^q dx \right)^{\frac{1}{q}},$$

Notice that, by definition of T_p^*, we have $T_p^* f = T_p^*(\mathbb{1}_{E(p)} f)$ for all $f \in L_r^2$, and since $E(p) \subseteq I \subseteq (I')^5$ if $p \in \mathcal{A}(p')$, it holds that $T_p^* f = T_p^*(\mathbb{1}_{(I')^5} f)$, so

$$\begin{aligned}
\left(\int_{\mathbb{R}} \left| \sum_{p \in \mathcal{A}(p')} T_p^* f(x) \right|^q dx \right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}} \left| \sum_{p \in \mathcal{A}(p')} T_p^*(\mathbb{1}_{(I')^5} f)(x) \right|^q dx \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}} \left| T^{\mathcal{A}(p')^*}(\mathbb{1}_{(I')^5} f)(x) \right|^q dx \right)^{\frac{1}{q}} \\
&\underbrace{=}_{\frac{1}{q}+\frac{1}{q'}=1} \sup_{\substack{g \in L_r^{q'} \\ \|g\|_{q'}=1}} \left| \int_{\mathbb{R}} T^{\mathcal{A}(p')^*}(\mathbb{1}_{(I')^5} f)(x) g(x)\, dx \right|. \qquad (6.10)
\end{aligned}$$

Take $1 < q < 2$. This implies $q' > 2$, so if $g \in L_r^{q'}$, then $g \in L_r^2$, so we can evaluate the adjoint operator on g:

$$\begin{aligned}
(6.10) &= \sup_{\substack{g \in L_r^{q'} \\ \|g\|_{q'}=1}} \left| \int_{\mathbb{R}} \mathbb{1}_{(I')^5}(x) f(x) T^{\mathcal{A}(p')} g(x)\, dx \right| \\
&\underbrace{\leq}_{\text{Hölder}} \sup_{\substack{g \in L_r^{q'} \\ \|g\|_{q'}=1}} \left(\int_{(I')^5} |f(x)|^q\, dx \right)^{\frac{1}{q}} \|T^{\mathcal{A}(p')} g\|_{q'}.
\end{aligned}$$

Note that the set of pairs $\mathcal{A}(p')$ satisfies the hypotheses of Lemma 6.1. Indeed, let $n_0 =$ midpoint of ω'. Then for all $p = [\omega, I] \in \mathcal{A}(p')$ we have $d(n_0, \omega) = d(\omega, \omega') + |\omega'|/2 \leq (1/2) \cdot \delta^{-\epsilon} |\omega| + |\omega|/2 = (1/2) \cdot (\delta^{-\epsilon} + 1)|\omega| \leq \delta^{-\epsilon}|\omega|$ $(\delta \leq 1)$. Thus, by Lemma 6.1,

$$\|T^{\mathcal{A}(p')} g\|_{q'} \leq C_{q'} \delta^{\frac{1-5000\epsilon}{q'}}$$

(recall that $\|g\|_{q'} = 1$). This gives

$$\left(\int_{\mathbb{R}} \left| \sum_{p \in \mathcal{A}(p')} T_p^* f(x) \right|^q dx \right)^{\frac{1}{q}} \leq C_{q'} \delta^{\frac{1-5000\epsilon}{q'}} \left(\int_{(I')^5} |f(x)|^q\, dx \right)^{\frac{1}{q}}.$$

Hence, we have an estimate for (6.8):

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{A}(p')} T_p^* f(x) \right] dx \right| \\
&\qquad \leq C_{q'} \delta^{\frac{1-5000\epsilon}{q'}} \left(\int_{E(p')} |f(y)|\, dy \right) \left(\frac{1}{|(I')^5|} \int_{(I')^5} |f(x)|^q\, dx \right)^{\frac{1}{q}}.
\end{aligned}$$

Consider the maximal operator

$$f_q^*(x) = \sup_{x \in I} \left(\frac{1}{|I|} \int_I |f(y)|^q\, dy \right)^{\frac{1}{q}}$$

for $x \in [0, 2\pi]$ (and 0 outside). By Remark 6.1, $\|f_q^*\|_2 \leq C_q \|f\|_2$ for all $1 \leq q < 2$.

Using this maximal operator, we can make a new bound for the terms of (6.8):

$$\left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{A}(p')} T_p^* f(x) \right] dx \right| \leq C_{q'} \delta^{\frac{1-5000\epsilon}{q'}} \int_{E(p')} |f(y)| f_q^*(y)\, dy, \tag{6.11}$$

for $1 < q < 2$.

To estimate the addends of the second term (6.9), we write

$$\left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{B}(p')} T_p^* f(x) \right] dx \right| = \left| \left(\sum_{p \in \mathcal{B}(p')} T_p^* f, T_{p'}^* f \right) \right| = \left| \sum_{p \in \mathcal{B}(p')} \left(T_p^* f, T_{p'}^* f \right) \right|$$
$$= \left| \sum_{p \in \mathcal{B}(p')} \left(T_{p'} T_p^* f, f \right) \right| = \left| \left(\sum_{p \in \mathcal{B}(p')} T_{p'} T_p^* f, f \right) \right| = \left| \int_{\mathbb{R}} \overline{f(x)} \left[\sum_{p \in \mathcal{B}(p')} T_{p'} T_p^* f(x) \right] dx \right|.$$

As $\text{support}(T_{p'}(T_p^* f)) \subseteq E(p')$ by definition of $T_{p'}$, the last expression is

$$= \left| \int_{E(p')} \overline{f(x)} \left[\sum_{p \in \mathcal{B}(p')} T_{p'} T_p^* f(x) \right] dx \right| \leq \int_{E(p')} |f(x)| \sum_{p \in \mathcal{B}(p')} |T_{p'} T_p^* f(x)|\, dx. \tag{6.12}$$

It would be good to estimate $|T_{p'} T_p^* f(x)|$ for $p \in \mathcal{B}(p')$. In fact, let us see that

$$T_{p'} T_p^* f \equiv 0 \ \text{ if } \ I \not\subseteq (I')^5, \tag{6.13}$$

$$|T_{p'} T_p^* f(x)| \leq \frac{C_\epsilon \delta^{10}}{|(I')^5|} \int_{E(p)} |f(y)|\, dy \ \text{ if } \ d(\omega, \omega') > \delta^{-\frac{\epsilon}{2}} |\omega|. \tag{6.14}$$

We have

$$\begin{aligned} T_{p'} T_p^* f(x) &= \int_{\mathbb{R}} e^{iN(x)y} \psi_{k'}(y) T_p^* f(x-y)\, dy \cdot \mathbb{1}_{E(p')}(x) \\ &= \int_{\mathbb{R}} e^{iN(x)y} \psi_{k'}(y) \left(\int_{E(p)} e^{-iN(z)(z-x+y)} \psi_k(z-x+y) f(z)\, dz \right) dy \cdot \mathbb{1}_{E(p')}(x) \\ &= \int_{E(p)} f(z) e^{iN(z)(x-z)} \left(\int_{\mathbb{R}} e^{i(N(x)-N(z))y} \psi_{k'}(y) \psi_k(z-x+y)\, dy \right) dz \cdot \mathbb{1}_{E(p')}(x). \end{aligned} \tag{6.15}$$

Call $\phi_z(y) = \psi_{k'}(y) \psi_k(z-x+y)$. Bound

$$|T_{p'} T_p^* f(x)| \leq \int_{E(p)} |f(z)| \left| \int_{\mathbb{R}} e^{i(N(x)-N(z))y} \phi_z(y)\, dy \right| dz \cdot \mathbb{1}_{E(p')}(x),$$

and note that the inside integral is a Fourier transform[1]:

$$\widehat{\phi_z}(N(z) - N(x)) = \int_{\mathbb{R}} e^{i(N(x)-N(z))y} \phi_z(y)\, dy.$$

[1]If $F \in L^1(\mathbb{R})$, we define its Fourier transform as $\hat{F}(\xi) = \int_{\mathbb{R}} F(x) e^{-ix\xi}\, dx$. The inverse Fourier transform theorem says that if F is Schwartz on $\mathbb{R}$ or $F, \hat{F} \in L^1(\mathbb{R})$, then $F(x) = 1/(2\pi) \int_{\mathbb{R}} \hat{F}(\xi) e^{ix\xi}\, d\xi$. We stress this definition because there are some others accepted.

Suppose $d(\omega,\omega') > (1/2)\delta^{-\epsilon}|\omega|$. Let $x \in E(p')$ and $z \in E(p)$. Then $N(x) \in \omega'$ and $N(z) \in \omega$. Therefore,

$$|N(x) - N(z)| \geq d(\omega,\omega') > \frac{1}{2}\delta^{-\epsilon}|\omega| = \frac{1}{2}\delta^{-\epsilon}\cdot 2\pi \cdot 2^k.$$

We want to estimate $|\widehat{\phi_z}(N(z)-N(x))|$. Can we use the previous estimation of $|N(z) - N(x)|$? Yes, because the Fourier transform of the derivative relates a point ξ with its image $\widehat{\phi_z}(\xi)$ $(m \in \mathbb{N})$:

$$|\widehat{\phi_z^{(m)}}(\xi)| = \int_{\mathbb{R}} \phi_z^{(m)}(y)e^{-iy\xi}\,dy \underbrace{=}_{\text{parts}} i\xi\int_{\mathbb{R}} \phi_z^{(m-1)}(y)e^{-iy\xi}\,dy = \ldots = (i\xi)^m\widehat{\phi_z}(\xi),$$

which implies

$$|\widehat{\phi_z}(\xi)| = \frac{|\widehat{\phi_z^{(m)}}(\xi)|}{|\xi|^m} \leq \frac{\|\phi_z^{(m)}\|_1}{|\xi|^m}.$$

We center on the estimation of $\|\phi_z^{(m)}\|_1$ in order to bound $\widehat{\phi_z}(N(z)-N(x))$. By Leibniz's rule,

$$\phi_z^{(m)}(y) = \sum_{j=0}^{m}\binom{m}{j}\psi_{k'}^{(j)}(y)\psi_k^{(m-j)}(z-x+y).$$

Then

$$\begin{aligned}|\phi_z^{(m)}(y)| &\leq \sum_{j=0}^{m}\binom{m}{j}2^{k'(j+1)}2^{k(m-j+1)}|(\psi^{(0)})^{(j)}(2^{k'}y)||(\psi^{(0)})^{(m-j)}(2^k(z-x+y))| \\ &\leq C_m\cdot 2^{k'}\cdot 2^k\cdot\sum_{j=0}^{m}2^{k'j}2^{k(m-j)}|(\psi^{(0)})^{(j)}(2^{k'}y)||(\psi^{(0)})^{(m-j)}(2^k(z-x+y))|.\end{aligned}$$

Now integrate:

$$\begin{aligned}\|\phi_z^{(m)}\|_1 &\leq C_m\cdot 2^{k'}\cdot 2^k\cdot\sum_{j=0}^{m}2^{k'j}2^{k(m-j)}\int_{\mathbb{R}}|(\psi^{(0)})^{(j)}(2^{k'}y)||(\psi^{(0)})^{(m-j)}(2^k(z-x+y))|\,dy \\ &\underbrace{\leq}_{\substack{\text{derivatives}\\ \text{of } \psi^{(0)}\\ \text{bounded}}} C_m\cdot 2^{k'}\cdot 2^k\cdot\sum_{j=0}^{m}2^{k'j}2^{k(m-j)}\int_{\mathbb{R}}|(\psi^{(0)})^{(m-j)}(2^k(z-x+y))|\,dy \\ &\underbrace{\leq}_{\substack{|I|\leq|I'|\\ \text{so } k'\leq k}} C_m\cdot 2^{k'}\cdot 2^k\cdot 2^{km}\cdot\sum_{j=0}^{m}\int_{\mathbb{R}}|(\psi^{(0)})^{(m-j)}(2^k(z-x+y))|\,dy \\ &= C_m\cdot 2^{k'}\cdot 2^{km}\cdot\sum_{j=0}^{m}\int_{\mathbb{R}}|(\psi^{(0)})^{(m-j)}(u)|\,du \underbrace{\leq}_{\substack{\text{derivatives}\\ \text{of } \psi^{(0)}\\ \text{bounded}}} C_m\cdot 2^{k'}\cdot 2^{km}.\end{aligned}$$

Thus,

$$|\widehat{\phi_z}(N(z)-N(x))| \leq \frac{\|\phi_z^{(m)}\|_1}{|N(z)-N(x)|^m} \leq C_m\delta^{\epsilon m}\frac{2^{k'}2^{km}}{2^{km}} = C_m\delta^{\epsilon m}2^{k'}.$$

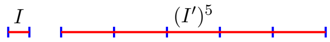
Choose $m \in \mathbb{N}$ such that $\epsilon m > 10$ and take the corresponding $C_m = C_\epsilon$ (m depends on ϵ). Then

$$|\widehat{\phi_z}(N(z) - N(x))| \le C_\epsilon \delta^{10} 2^{k'}.$$

Then

$$|T_{p'}T_p^* f(x)| \le C_\epsilon \delta^{10} 2^{k'} \int_{E(p)} |f(z)|\, dz \cdot \mathbb{1}_{E(p')}(x) = C_\epsilon \frac{\delta^{10}}{|I'|} \int_{E(p)} |f(z)|\, dz \cdot \mathbb{1}_{E(p')}(x),$$

and (6.14) follows. It remains to see (6.13) under the assumption of $I \not\subseteq (I')^5$. Let $x \in E(p') \subseteq I'$ and $z \in E(p) \subseteq I$. By the properties of dyadic intervals, $I \cap (I')^5 = \emptyset$, so there are two intervals I' between x and z:



Then $|x - z| > 2|I'| = 2\pi \cdot 2^{-k'+1}$. In the inside integral of (6.15), we have $y \in \text{support}(\psi_{k'}) \subseteq [-2\pi \cdot 2^{-k'}, 2\pi \cdot 2^{-k'}]$, so $|y| \le 2\pi \cdot 2^{-k'}$. Therefore $|z-x+y| \ge |x-z| - |y| > 2\pi \cdot 2^{-k'+1} - 2\pi \cdot 2^{-k'} = 2\pi \cdot 2^{-k'} \ge 2\pi \cdot 2^{-k}$ (recall: $|I| \le |I'|$). This means that $z - x + y \notin \text{support}(\psi_k)$, so the inside integral of (6.15) is 0 and (6.13) is proved.

With (6.13) and (6.14) we can continue bounding from (6.12):

$$\begin{aligned}
\int_{E(p')} |f(x)| \sum_{p \in \mathcal{B}(p')} |T_{p'}T_p^* f(x)|\, dx &= \underbrace{\le}_{\substack{\text{by} \\ (6.13), \\ (6.14)}} C_\epsilon \int_{E(p')} |f(x)| \frac{\delta^{10}}{|(I')^5|} \sum_{\substack{p \in \mathcal{B}(p') \\ I \subseteq (I')^5}} \int_{E(p)} |f(y)|\, dy\, dx \\
&\underbrace{=}_{\text{disjoint}} C_\epsilon \int_{E(p')} |f(x)| \frac{\delta^{10}}{|(I')^5|} \int_{\cup\{E(p):\, p \in \mathcal{B}(p'),\, I \subseteq (I')^5\}} |f(y)|\, dy\, dx \\
&\underbrace{\le}_{E(p) \subseteq I} C_\epsilon \delta^{10} \int_{E(p')} |f(x)| \left(\frac{1}{|(I')^5|} \int_{(I')^5} |f(y)|\, dy \right) dx \\
&\le C_\epsilon \delta^{10} \int_{E(p')} |f(x)| f_1^*(x)\, dx.
\end{aligned}$$

This last expression gives the bound for the addends of (6.9):

$$\sum_{p' \in \mathcal{P}} \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{B}(p')} T_p^* f(x) \right] dx \le C_\epsilon \delta^{10} \int_{E(p')} |f(x)| f_1^*(x)\, dx. \tag{6.16}$$

Thus, using (6.11) and (6.16),

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{P},\, |I| \le |I'|} T_p^* f(x) \right] dx \right| \\
&\qquad \le C_{q'} \delta^{\frac{1-5000\epsilon}{q'}} \int_{E(p')} |f(y)| f_q^*(y)\, dy + C_\epsilon \delta^{10} \int_{E(p')} |f(x)| f_1^*(x)\, dx.
\end{aligned}$$

We want to mix in some way these two last addends. Regarding the δ's, note that $\delta^{10} \le \delta^{(1-5000\epsilon)/q'}$ because $(1 - 5000\epsilon)/q' \le 10$. Concerning the maximal operators, it is trivial that $f_1^*(x) \le f_q^*(x)$. Therefore,

$$\left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{P},\, |I| \le |I'|} T_p^* f(x) \right] dx \right| \le C_{q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \int_{E(p')} |f(y)| f_q^*(y)\, dy.$$

Hence, summing over $p' \in \mathcal{P}$,

$$\begin{aligned}
&\left| \sum_{p' \in \mathcal{P}} \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{P},\, |I| \le |I'|} T_p^* f(x) \right] dx \right| \\
&\qquad \le \sum_{p' \in \mathcal{P}} \left| \int_{\mathbb{R}} \overline{T_{p'}^* f(x)} \left[\sum_{p \in \mathcal{P},\, |I| \le |I'|} T_p^* f(x) \right] dx \right| \\
&\qquad \le C_{q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \int_{\cup\{E(p')\colon p' \in \mathcal{P}\}} |f(y)| f_q^*(y)\, dy \\
&\qquad \le C_{q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \int_0^{2\pi} |f(y)| f_q^*(y)\, dy \\
&\qquad \le C_{q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \|f\|_2 \|f_q^*\|_2 \underbrace{\le}_{\text{proved}} C_{q,q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \|f\|_2^2.
\end{aligned}$$

The estimation of the second term

$$\sum_{\substack{[\omega,I],[\omega',I'] \in \mathcal{P} \\ |I'| \le |I|}} \int_{\mathbb{R}} T_{[\omega,I]}^* f(x) \overline{T_{[\omega',I']}^* f(x)}\, dx$$

is analogous, interchanging the roles of I and I'.

Thus,

$$\sum_{p,p' \in \mathcal{P}} \int_{\mathbb{R}} T_p^* f(x) \overline{T_{p'}^* f(x)}\, dx \le C_{q,q',\epsilon}\, \delta^{\frac{1-5000\epsilon}{q'}} \|f\|_2^2.$$

Let $\eta > 0$. Choose $q = q(\eta)$ and $\epsilon = \epsilon(\eta)$ so that

$$\frac{1}{2} - 2\eta < \frac{1-5000\epsilon}{q'}.$$

Indeed, if $\eta \ge 1/4$, the left-hand side of the inequality is nonpositive and there is nothing to do. If $0 < \eta < 1/4$, write $q' = 2/(1-\nu)$, $0 < \nu < 1$, and the problem reduces to finding ν and ϵ with $\nu/2 + 5000\epsilon/q' < 2\eta$, which is obviously possible by first taking $0 < \nu < 1$ with $\nu/2 < \eta$, then taking the corresponding q' and finally choosing $0 < \epsilon < \min\{1/5000, \eta q'/5000\}$. For those q, q' and ϵ, let $C_\eta = C_{q,q',\epsilon}$. Then

$$\sum_{p,p' \in \mathcal{P}} \int_{\mathbb{R}} T_p^* f(x) \overline{T_{p'}^* f(x)}\, dx \le C_\eta \delta^{\frac{1}{2}-2\eta} \|f\|_2^2,$$

and since $\eta > 0$ is arbitrary, this holds for all $\eta > 0$, as wanted.

Now we deal with the case $1/4^{4004} \le \delta \le 1$. This is by far the easiest case. We will just show that there is a $C > 0$ such that for any set of pairs $\mathcal{P}$ for which no two pairs are comparable it holds $\|T^{\mathcal{P}}\|_2 \le C$. This is enough, as in such a case there is a constant $D > 0$ for which $\|T^{\mathcal{P}}\|_2 \le D/4^{4004} \le D(1/4^{4004})^{1/4-\eta} \le D\delta^{1/4-\eta}$ for all $\eta > 0$.

Thus, our goal is to prove that $\|T^{\mathcal{P}}\|_2 \le C$. Let $p = [\omega, I] \in \mathcal{P}$. By (3.2) and the fact that the $E(\cdot)$'s are disjoint, for all $x \in E(p)$

$$|T^{\mathcal{P}} f(x)| = |T_p f(x)| \le \frac{2\pi}{|I|} \int_{I^3} |f(y)|\, dy.$$

Use the maximal operator

$$f_1^*(x) = \sup_{x \in J} \frac{1}{|J|} \int_J |f(y)|\, dy$$

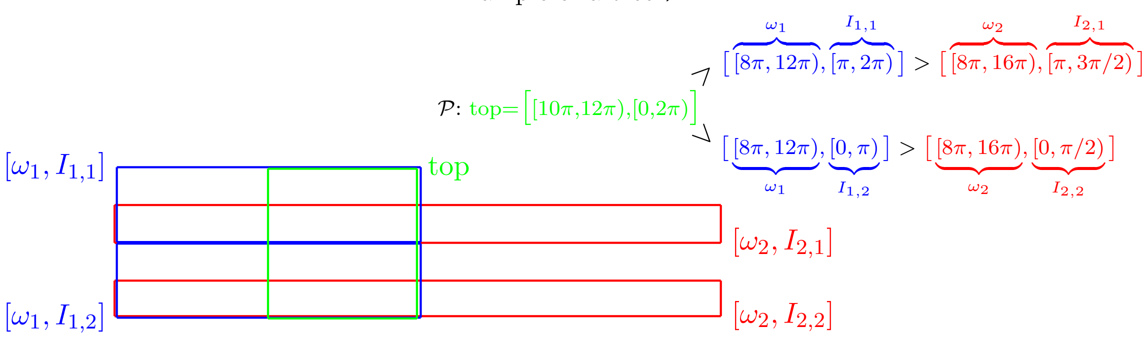
to conclude that $|T^{\mathcal{P}} f(x)| \le C \cdot f_1^*(x)$. This inequality holds for all $x \in [0, 2\pi]$, therefore $\|T^{\mathcal{P}} f\|_2 \le \|f_1^*\|_2 \le C\|f\|_2$, as desired.

□

We keep dealing with larger sets:

Definition 6.1 *A tree $\mathcal{P}$ with top $p^0 = [\omega^0, I^0]$ is a set of pairs with the properties:*
(a) if $p < p' < p''$, with p' admissible and $p, p'' \in \mathcal{P}$, then $p' \in \mathcal{P}$;
(b) $p < p^0$ for every $p \in \mathcal{P}$.

Example of a tree $\mathcal{P}$:



Lemma 6.3 *Let $\mathcal{P}$ be a tree with top $p^0 = [\omega^0, I^0]$. Suppose that $A(p) \le \delta$ for all $p \in \mathcal{P}$ $(0 < \delta < 1)$. Then $\|T^{\mathcal{P}}\|_2 \le C\delta^{\frac{1}{2}}$ (as usual, the norm $\|\cdot\|_2$ is understood in L^2_r).*

Proof. Let us see how $T^{\mathcal{P}}$ really looks like. We are going to prove that

$$T^{\mathcal{P}} f(x) = \sum_{\substack{K_0(x) \le k \le K_1(x) \\ k \in J}} (e^{iN(x)\cdot}\psi_k(\cdot)) * f(x)$$

or $T^{\mathcal{P}} f(x) = 0$, depending on x. $K_0(x)$ and $K_1(x)$ are finite functions of x and J is a set of positive integers.

Pick $\xi_0 \in \omega^0$. Let $I \subseteq [0, 2\pi]$ with $|I| = 2\pi \cdot 2^{-k}$. Let $\omega_I = \omega(k)$ be the dyadic interval in $\mathbb{R}$ of length $2\pi \cdot 2^k$ containing ξ_0. If $[\omega, I] \in \mathcal{P}$, then $[\omega, I] < [\omega^0, I^0]$, so $\xi_0 \in \omega^0 \subseteq \omega$, therefore $\omega = \omega_I$. Thus, $\mathcal{P}$ consists entirely of pairs $[\omega_I, I]$. Let $J = \{k \ge 1 : \omega(k)$ is central$\}$. Let

$$\begin{aligned} A(x) &= \{k \ge 1 : [\omega_I, I] \in \mathcal{P},\ x \in I,\ |I| = 2\pi \cdot 2^{-k}\} \\ &= \{k \in J : [\omega_I, I] \in \mathcal{P},\ x \in I,\ |I| = 2\pi \cdot 2^{-k}\} \end{aligned}$$

($\mathcal{P}$ consists of admissible pairs) and

$$B(x) = \{k \geq 1 : N(x) \in \omega(k)\}.$$

Then

$$T^{\mathcal{P}} f(x) = \sum_{k \in A(x) \cap B(x)} (e^{iN(x) \cdot} \psi_k(\cdot)) * f(x)$$

(it could be $A(x) \cap B(x) = \emptyset$). Fixed x, all the pairs $[\omega_I, I]$ with $x \in I$ can be comparable, so by definition (a) of tree $A(x) = \{k \in J : K(x) \leq k \leq K'(x)\}$. Since $B(x)$ clearly has the form $\{k \geq K''(x)\}$ (because if $N(x)$ and ξ_0 are in a dyadic interval, they will be in all larger dyadic intervals containing the previous one), it follows that $A(x) \cap B(x)$ has the form $\{k \in J : K_0(x) \leq k \leq K_1(x)\}$ or $\emptyset$. This gives the stated form for $T^{\mathcal{P}}$.

If $\xi_0 \geq 0$ and $N < 0$, then $E(\omega_I, I) = \emptyset$ for all $[\omega_I, I] \in \mathcal{P}$, so $T^{\mathcal{P}} f \equiv 0$ for all $f \in L^2_r$. Thus, we can assume that either $\xi_0 < 0$ and $N < 0$ or $\xi_0 \geq 0$ and $N > 0$. In this case, if I is any dyadic interval in $[0, 2\pi]$ which is large enough, ω_I will be of the form $[a, 0[$ $(a < 0)$ if $\xi_0 < 0$ (and $N < 0$) and $[0, a[$ $(a > 0)$ if $\xi_0 \geq 0$ (and $N > 0$), with $|a|$ large, and since N is bounded, $E(\omega_I, I) = I$. Thus, we can consider a partition of I^0, $\{I_j\}_{j=1}^J$, defined by means of the maximal dyadic subintervals of I^0 for which

$$\frac{|E(\omega_I, I)|}{|I|} > \delta$$

(start with I^0; if it does not satisfy the bound, divide it into two halves; if some of the halves satisfies the bound, keep it for the partition, otherwise divide it again into two new halves; etc. The process is finite since for every sufficiently small interval I it holds $E(\omega_I, I) = I$ by the reasoning above).

Set

$$\tilde{E}_j = E(\omega_{\tilde{I}_j}, \tilde{I}_j) = \{x \in \tilde{I}_j : N(x) \in \omega_{\tilde{I}_j}\}$$

and

$$E_j = \{x \in I_j : N(x) \in \omega_{\tilde{I}_j}\}.$$

By (α) below, $\tilde{I}_j \subseteq I^0$, and by maximality $|\tilde{E}_j|/|\tilde{I}_j| \leq \delta$, therefore $|E_j| \leq |\tilde{E}_j| \leq \delta |\tilde{I}_j| = 2\delta |I_j|$. Let us see that:
(α) $p = [\omega, I] \in \mathcal{P}$, $I \cap I_j \neq \emptyset \Rightarrow \tilde{I}_j \subseteq I$;
(β) $p = [\omega, I] \in \mathcal{P}$, $I \cap I_j \neq \emptyset \Rightarrow E(p) \cap I_j \subseteq E_j$.

Proof of (α). As $I \cap I_j \neq \emptyset$, either $I \subseteq I_j$ or $\tilde{I}_j \subseteq I$. Suppose by contradiction that $I \subseteq I_j$. We have, by construction of the partition, $|E(\omega_{I_j}, I_j)|/|I_j| > \delta$. On the other hand, $\omega = \omega_I$, so $\xi_0 \in \omega_{I_j} \cap \omega$, and since $|I| \leq |I_j|$, necessarily $\omega_{I_j} \subseteq \omega$, therefore $d(\omega_{I_j}, \omega) = 0$. This gives (using $I \subseteq I_j$)

$$A(p) \geq \underbrace{\frac{|E(\omega_I, I)|}{|I|}}_{>\delta} \underbrace{\left(\frac{d(\omega, \omega_{I_j}) + |\omega|}{|\omega|} \right)^{-2000}}_{=1} > \delta,$$

which is a contradiction.

Proof of (β). By (α), $E(p) \cap I_j = \{x \in I_j : N(x) \in \omega = \omega_I\}$. As $\xi_0 \in \omega \cap \omega_{\tilde{I}_j}$, we have $\omega_I \cap \omega \cap \omega_{\tilde{I}_j} \neq \emptyset$. Since $\tilde{I}_j \subseteq I$ by (α), $\omega_I \subseteq \omega_{\tilde{I}_j}$. Then $E(p) \cap I_j \subseteq \{x \in I_j : N(x) \in \omega_{\tilde{I}_j}\} = E_j$ and (β) is proved.

Write

$$T^{\mathcal{P}}f(x) = \Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} \psi_k\Bigg) * f(x) + \Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} (e^{iN(x)\cdot}\psi_k(\cdot) - \psi_k)\Bigg) * f(x). \qquad (6.17)$$

Suppose that the sum is nonempty. Recall

$$\begin{aligned} A(x)\cap B(x) &= \{k\in J:\ [\omega_I, I]\in\mathcal{P},\ x\in I,\ N(x)\in\omega_I,\ |I| = 2\pi\cdot 2^{-k}\}\\ &= \{k\in J:\ K_0(x)\le k\le K_1(x)\}.\end{aligned}$$

Then $x\in E(\omega_{I_x}, I_x)$ for some $[\omega_{I_x}, I_x]\in\mathcal{P}$ with $|I_x| = 2\pi\cdot 2^{-K_0(x)}$. As $N(x), \xi_0\in\omega_{I_x}$, we have $|N(x)-\xi_0|\le|\omega_{I_x}| = 2\pi\cdot 2^{K_0(x)}$. With this, we can bound the second sum of the right-hand side of (6.17): call $g(t) = e^{-i\xi_0 t}f(t)\in L^2_r$, then

$$\begin{aligned}
&\left|\Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} (e^{iN(x)\cdot}\psi_k(\cdot)-\psi_k)\Bigg) * f(x)\right|\\
&= \left|\Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} (e^{i(N(x)-\xi_0)\cdot}\psi_k(\cdot)-\psi_k)\Bigg) * g(x)\right|\\
&\underbrace{\le}_{|f|=|g|} \Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} |e^{i(N(x)-\xi_0)\cdot}-1||\psi_k|\Bigg) * |f|(x)\\
&= 2\Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} \left|\sin\left(\frac{(N(x)-\xi_0)y}{2}\right)\right||\psi_k|\Bigg) * |f|(x)\\
&\underbrace{\le}_{|\sin x|\le|x|} |N(x)-\xi_0|\Bigg[|\cdot|\Bigg(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} |\psi_k(\cdot)|\Bigg)\Bigg] * |f|(x)\\
&\le |N(x)-\xi_0|\Bigg[|\cdot|\Bigg(\sum_{k=K_0(x)}^{\infty} |\psi_k(\cdot)|\Bigg)\Bigg] * |f|(x)\\
&\le |N(x)-\xi_0|\Bigg(|\cdot|\frac{1}{|\cdot|}\mathbb{1}_{[-2\pi\cdot 2^{-K_0(x)}, 2\pi\cdot 2^{-K_0(x)}]}(\cdot)\Bigg) * |f|(x)\\
&= |N(x)-\xi_0|\int_{-2\pi\cdot 2^{-K_0(x)}}^{2\pi\cdot 2^{-K_0(x)}} |f(x-y)|\,dy \le 2\pi\cdot 2^{K_0(x)}\int_{-2\pi\cdot 2^{-K_0(x)}}^{2\pi\cdot 2^{-K_0(x)}} |f(x-y)|\,dy\\
&\le C\frac{1}{|I_x|}\int_{-2\pi\cdot 2^{-K_0(x)}}^{2\pi\cdot 2^{-K_0(x)}} |f(x-y)|\,dy \le C\frac{1}{|I_x|}\int_{(I_x)^3} |f(y)|\,dy.
\end{aligned}$$

On the hand, if $x\in I_j$, then $x\in I_j\cap I_x$, and by (α) above $\tilde{I}_j\subseteq I_x$. Let, as in Lemma 6.1,

$$M_0 f(x) = \sum_{j=1}^{J}\left(\sup_{I_j\subseteq I}\frac{1}{|I|}\int_I |f(y)|\,dy\right)\mathbb{1}_{I_j}(x).$$

If $x\in I_j$, then $I_j\subseteq I_x\subseteq (I_x)^3$, so

$$C\frac{1}{|I_x|}\int_{(I_x)^3} |f(y)|\,dy = C\frac{1}{|(I_x)^3|}\int_{(I_x)^3}|f(y)|\,dy \le CM_0 f(x).$$

Thus, the final bound for the second sum of the right-hand side of (6.17) is the following:

$$\left|\left(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}} (e^{iN(x)\cdot}\psi_k(\cdot)-\psi_k)\right)*f(x)\right|\le CM_0f(x).$$

We go on the estimation of the first sum of the right-hand side of (6.17).
Let

$$R(x,y)=\sum_{1\le k\le K_1(x)}\psi_k(y),\quad x,y\in\mathbb{R},$$

$$R(y)=\sum_{1\le k\le K^*}\psi_k(y),\quad y\in\mathbb{R},$$

where K^* (see Remark 4.1) depends only on the function N and satisfies $|\omega|\le 2\pi\cdot 2^{K^*}$ for all $[\cdot,\omega]\in\mathcal{B}$ (don't worry, the constants that will appear will not depend on K^*).

It will be convenient not to have J on the sum, since we will need at some moment to have consecutive numbers in the subscript of the sum. Then we apply some sort of trick:

$$\begin{aligned}\left|\left(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}}\psi_k\right)*f(x)\right| &= \left|\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}}\int_{\mathbb{R}}\psi_k(y)f(x-y)\,dy\right|\\ &\le \sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}}\int_{\mathbb{R}}\underbrace{|\psi_k(y)|}_{\psi_k(y)\mathrm{sign}(y)}\,|f(x-y)|\,dy\\ &\le \sum_{1\le k\le K_1(x)}\int_{\mathbb{R}}\psi_k(y)\mathrm{sign}(y)|f(x-y)|\,dy\\ &= \int_{\mathbb{R}}R(x,y)\mathrm{sign}(y)|f(x-y)|\,dy.\end{aligned}$$

Then, if we define $g(x,y)=\mathrm{sign}(x-y)|f(y)|$,

$$\left|\left(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}}\psi_k\right)*f(x)\right|\le (R(x,\cdot)*g(x,\cdot))(x).$$

Let us see that

$$\left|R(x,y)-2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}R(x,y+z)\,dz\right|\le C\frac{2^{-K_1(x)}}{y^2+2^{-2K_1(x)}}\tag{6.18}$$

for all $y\in\mathbb{R}$. We distinguish two cases:

- Case $16\pi\cdot 2^{-K_1(x)}\le|y|$. We start with the trivial bound

$$\begin{aligned}&\left|R(x,y)-2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}R(x,y+z)\,dz\right|\\ &\le 2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}\sum_{1\le k\le K_1(x)}|\psi_k(y)-\psi_k(y+z)|\,dz.\end{aligned}$$

It is important to demonstrate that the number of terms in the sum does not depend on $K_1(x)$, and this will be done using the bound $16\pi \cdot 2^{-K_1(x)} \leq |y|$ and the fact that $\text{support}(\psi_k) \subseteq \{\pi/2 \cdot 2^{-k} \leq |t| \leq 2\pi \cdot 2^{-k}\}$.

We have $16\pi \cdot 2^{-K_1(x)} = 2\pi \cdot 2^{-K_1(x)+3} \leq |y|$. There is a natural number k_0 such that $\pi/2 \cdot 2^{-k_0} \leq |y| \leq 2\pi \cdot 2^{-k_0}$. By the definition of the support of the ψ_k's, it holds $\psi_k(y) = 0$ for all $k \notin \{k_0 - 1, k_0, k_0 + 1\}$.

On the other hand, $2\pi \cdot 2^{-K_1(x)+3} \leq |y| \leq 2\pi \cdot 2^{-k_0}$, so $k_0 \leq K_1(x) - 3$.

Let $|z| \leq 2\pi \cdot 2^{-K_1(x)}$. Then $|z| \leq 2\pi \cdot 2^{-k_0-3}$, that is, $z \in [-2\pi \cdot 2^{-k_0-3}, 2\pi \cdot 2^{-k_0-3}]$. Thus,

$$\begin{aligned}\text{if } y > 0 \Rightarrow y + z \in & \left[\frac{\pi}{2} \cdot 2^{-k_0}, 2\pi \cdot 2^{-k_0}\right] + [-2\pi \cdot 2^{-k_0-3}, 2\pi \cdot 2^{-k_0-3}] \\ = & \left[\frac{\pi}{4} \cdot 2^{-k_0}, \frac{9\pi}{4} \cdot 2^{-k_0}\right];\end{aligned}$$

$$\begin{aligned}\text{if } y < 0 \Rightarrow y + z \in & \left[-2\pi \cdot 2^{-k_0}, -\frac{\pi}{2} \cdot 2^{-k_0}\right] + [-2\pi \cdot 2^{-k_0-3}, 2\pi \cdot 2^{-k_0-3}] \\ = & \left[-\frac{9\pi}{4} \cdot 2^{-k_0}, -\frac{\pi}{4} \cdot 2^{-k_0}\right].\end{aligned}$$

We want to analyze for which k's it is verified

$$\text{support}(\psi_k) \cap \left(\left[-\frac{9\pi}{4} \cdot 2^{-k_0}, -\frac{\pi}{4} \cdot 2^{-k_0}\right] \cup \left[\frac{\pi}{4} \cdot 2^{-k_0}, \frac{9\pi}{4} \cdot 2^{-k_0}\right]\right) \neq \emptyset.$$

This last condition implies $\pi/2 \cdot 2^{-k} \leq 9\pi/4 \cdot 2^{-k_0}$ and $2\pi \cdot 2^{-k} \geq \pi/4 \cdot 2^{-k_0}$, which are equivalent to $2^{-k} \leq 9/2 \cdot 2^{-k_0}$ and $2^{-k_0} \leq 8 \cdot 2^{-k} = 2^{-k+3}$. The first condition implies $2^{-k} \leq 8 \cdot 2^{-k_0} = 2^{-k_0+3}$, that is, $k_0 - 3 \leq k$. The second one is equivalent to $k - 3 \leq k_0$. We conclude that $k_0 - 3 \leq k \leq k_0 + 3$. Hence, $\psi_k(y + z) = 0$ for all $k \notin \{k_0 - 3, \ldots, k_0 + 3\}$.

Thus,

$$\begin{aligned}&\left|R(x,y) - 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} R(x, y+z)\, dz\right| \\ &\leq 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} \sum_{1 \leq k \leq K_1(x)} |\psi_k(y) - \psi_k(y+z)|\, dz \\ &= 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} \sum_{k=k_0-3}^{k_0+3} |\psi_k(y) - \psi_k(y+z)|\, dz \\ &= 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} \sum_{k=k_0-3}^{k_0+3} 2^k |\psi^{(0)}(2^k y) - \psi^{(0)}(2^k y + 2^k z)|\, dz.\end{aligned}$$

By the mean value theorem, $\psi^{(0)}(2^k y) - \psi^{(0)}(2^k y + 2^k z) = (\psi^{(0)})'(2^k \xi_{y,z}) 2^k z$, where $\xi_{y,z} \in [y, y+z]$. We have $|\xi_{y,z}| \geq \min\{|y|, |y+z|\} \geq \min\{|y|, |y| - |z|\} = |y| - |z| \geq$

$|y| - 2\pi \cdot 2^{-K_1(x)} \geq |y| - (2/3)|y| = |y|/3$. Since $(\psi^{(0)})'$ is Schwartz, there is a constant $C > 0$ with $|(\psi^{(0)})'(t)| \leq C/t^2$ for all $t \in \mathbb{R}$. Then

$$|\psi^{(0)}(2^k y) - \psi^{(0)}(2^k y + 2^k z)| \leq \frac{C}{2^{2k} y^2} 2^k |z| = \frac{C}{2^k y^2} |z|.$$

Hence,

$$\begin{aligned}
&\left| R(x,y) - 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} R(x, y+z)\, dz \right| \\
&\leq 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} \sum_{k=k_0-3}^{k_0+3} 2^k \left(\frac{C}{2^k y^2} |z| \right) dz \\
&= 2^{K_1(x)-1} \underbrace{\left(2 \int_0^{2\pi \cdot 2^{-K_1(x)}} z\, dz \right)}_{C \cdot 2^{-2K_1(x)}} \frac{C}{y^2} \underbrace{\left(\sum_{k=k_0-3}^{k_0+3} 1 \right)}_{=7} \leq C \frac{2^{-K_1(x)}}{y^2}.
\end{aligned}$$

Finally, use the fact that

$$\frac{2^{-K_1(x)}}{y^2} \leq \left(1 + \frac{1}{16^2 \pi^2}\right) \frac{2^{-K_1(x)}}{y^2 + 2^{-2K_1(x)}}$$

(this is equivalent to $16\pi \cdot 2^{-K_1(x)} \leq |y|$).

- Case $0 \leq |y| \leq 16\pi \cdot 2^{-K_1(x)}$. We have

$$\begin{aligned}
&\left| R(x,y) - 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} R(x, y+z)\, dz \right| \\
&\leq \sum_{1 \leq k \leq K_1(x)} |\psi_k(y)| + 2^{K_1(x)-1} \int_{-2\pi \cdot 2^{-K_1(x)}}^{2\pi \cdot 2^{-K_1(x)}} \left(\sum_{1 \leq k \leq K_1(x)} |\psi_k(y+z)| \right) dz \\
&\leq \sum_{1 \leq k \leq K_1(x)} 2^k + 2\pi \sum_{1 \leq k \leq K_1(x)} 2^k \\
&\leq 2^{K_1(x)+1} + 2\pi \cdot 2^{K_1(x)+1} \leq C \cdot 2^{K_1(x)}.
\end{aligned}$$

Finally, use the fact that

$$2^{K_1(x)} \leq (1 + 16^2 \pi^2) \frac{2^{-K_1(x)}}{y^2 + 2^{-2K_1(x)}}$$

(this is a consequence of $|y| \leq 16\pi \cdot 2^{-K_1(x)}$).

Decompose

$$\begin{aligned}
|(R(x,\cdot) * g(x,\cdot))(x)| \leq & \left| \int_{\{|y| \leq 2\pi \cdot 2^{-K_1(x)}\}} R(x,y) g(x, x-y)\, dy \right| \\
& + \left| \int_{\{|y| \geq 2\pi \cdot 2^{-K_1(x)}\}} R(x,y) g(x, x-y)\, dy \right|.
\end{aligned}$$

To estimate the first integral, note that $|g(s,t)| = |f(t)|$. Thus, the module allows us to avoid the dependence on x. We have

$$\left|\int_{\{|y|\le 2\pi\cdot 2^{-K_1(x)}\}} R(x,y)g(x,x-y)\,dy\right| \le \int_{\{|y|\le 2\pi\cdot 2^{-K_1(x)}\}} |R(x,y)||f(x-y)|\,dy$$
$$\le 2^{K_1(x)+1}\int_{\{|x-y|\le 2\pi\cdot 2^{-K_1(x)}\}} |f(y)|\,dy.$$

Suppose that $x \in I_j$. By definition of $K_1(x)$, $x \in E(\omega_{I(x)}, I(x))$ for some $[\omega_{I(x)}, I(x)] \in \mathcal{P}$, $|I(x)| = 2\pi\cdot 2^{-K_1(x)}$. Since $x \in I(x)\cap I_j$, by (α) it holds $\tilde{I}_j \subseteq I(x)$, so $|\tilde{I}_j| \le |I(x)| = 2\pi\cdot 2^{-K_1(x)}$. Then $I_j \subseteq [x - 2\pi\cdot 2^{-K_1(x)}, x + 2\pi\cdot 2^{-K_1(x)}]$, so by definition of the operator M_0,

$$2^{K_1(x)+1}\int_{\{|x-y|\le 2\pi\cdot 2^{-K_1(x)}\}} |f(y)|\,dy \le CM_0f(x).$$

Thus,

$$\left|\int_{\{|y|\le 2\pi\cdot 2^{-K_1(x)}\}} R(x,y)g(x,x-y)\,dy\right| \le CM_0f(x).$$

To estimate $|(R(x,\cdot) * g(x,\cdot))(x)|$, it remains to bound

$$\int_{\{|y|\ge 2\pi\cdot 2^{-K_1(x)}\}} R(x,y)g(x,x-y)\,dy = \int_{\mathbb{R}} R(x,y)G(x,x-y)\,dy = (R(x,\cdot) * G(x,\cdot))(x),$$

where

$$G(x,y) := g(x,y)\cdot \mathbb{1}_{]-\infty, x-2\pi\cdot 2^{-K_1(x)}[\,\cup\,]x+2\pi\cdot 2^{-K_1(x)},\infty[}(y).$$

We will use the result given in (6.18). Write

$$\left|(R(x,\cdot) * G(x,\cdot))(x) - 2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}} [R(x,\cdot) * G(x,\cdot)](x+z)\,dz\right|$$
$$= \left|\int_{\mathbb{R}} G(x,x-y)R(x,y)\,dy - 2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}\int_{\mathbb{R}} R(x,y+z)G(x,x-y)\,dy\,dz\right|$$
$$\le \int_{\mathbb{R}}\left|R(x,y) - 2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}} R(x,y+z)\,dz\right||G(x,x-y)|\,dy$$
$$\le \int_{\mathbb{R}} \frac{2^{-K_1(x)}}{y^2 + 2^{-2K_1(x)}}|G(x,x-y)|\,dy = \int_{\{|y|\ge 2\pi\cdot 2^{-K_1(x)}\}} \frac{2^{-K_1(x)}}{y^2 + 2^{-2K_1(x)}}|g(x,x-y)|\,dy$$
$$= \int_{\{|y|\ge 2\pi\cdot 2^{-K_1(x)}\}} \frac{2^{-K_1(x)}}{y^2 + 2^{-2K_1(x)}}|f(x-y)|\,dy$$
$$\underbrace{\le}_{\substack{\text{Remark}\\ 6.2}} C\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-y)|\mathbb{1}_{\{|y|\ge 2\pi\cdot 2^{-K_1(x)}\}}(y)\,dy = C\sup_{h\ge 2\pi\cdot 2^{-K_1(x)}}\frac{1}{2h}\int_{-h}^{h}|f(x-y)|\,dy$$
$$\underbrace{\le}_{\substack{\text{as}\\ \text{before}}} CM_0f(x).$$

On the other hand, using the fact that

$$|R(x,y)| = \sum_{1\le k\le K_1(x)} |\psi_k(y)| \le \sum_{1\le k\le K^*} |\psi_k(y)| = |R(y)|$$

(remember that $\psi_k(y) = \text{sign}(y)|\psi_k(y)|$), we obtain

$$\begin{aligned}
&\left|2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}[R(x,\cdot)*G(x,\cdot)](x+z)\,dz\right| \\
&\qquad \le 2^{K_1(x)-1}\int_{-2\pi\cdot 2^{-K_1(x)}}^{2\pi\cdot 2^{-K_1(x)}}(|R|*|f|)(x+z)\,dz \underbrace{\le}_{\substack{\text{as}\\ \text{before}}} CM_0(|R|*|f|)(x).
\end{aligned}$$

To sum up,

$$\text{first sum in (6.17)} = \left|\left(\sum_{\substack{K_0(x)\le k\le K_1(x)\\ k\in J}}\psi_k\right)*f(x)\right| \le CM_0(|R|*|f|)(x) + M_0(f)(x).$$

Combining our estimates for the sums in (6.17), we obtain

$$|T^{\mathcal{P}}f(x)| \le CM_0f(x) + CM_0(|R|*|f|)(x).$$

On the other hand, $T^{\mathcal{P}}f$ is supported in

$$\bigcup_{p\in\mathcal{P}}E(p) = \bigcup_{j=1}^{J}\left(\bigcup_{p\in\mathcal{P}}(E(p)\cap I_j)\right) \subseteq \bigcup_{j=1}^{J}E_j,$$

where the last inclusion is reasoned as follows: given $j \in \{1,\ldots,J\}$ and $p = [\omega, I] \in \mathcal{P}$, either $I \cap I_j = \emptyset$, in which case $E(p) \cap I_j = \emptyset \subseteq E_j$, or $I \cap I_j \ne \emptyset$, which implies $E(p) \cap I_j \subseteq E_j$ by (β).

Thus, $T^{\mathcal{P}}f(x) = 0$ for all $x \in I_j \backslash E_j$ and $j = 1,\ldots,J$, so

$$|T^{\mathcal{P}}f(x)| \le CMf(x) + CM(|R|*|f|)(x)$$

for all x. Hence,

$$\begin{aligned}
\|T^{\mathcal{P}}f\|_2 &\le C\|Mf\|_2 + C\|M(|R|*|f|)\|_2 \underbrace{\le}_{\substack{\text{as in}\\ \text{Lemma 6.1}}} C\delta^{\frac{1}{2}}(\|f\|_2 + \||R|*|f|\|_2) \\
&\underbrace{\le}_{\text{Young}} C\delta^{\frac{1}{2}}(\|f\|_2 + \|R\|_2\|f\|_1) \underbrace{=}_{\text{Plancherel}} C\delta^{\frac{1}{2}}(\|f\|_2 + \|\widehat{R}\|_2\|f\|_1) \\
&\underbrace{\le}_{\substack{\text{compact}\\ \text{support}}} C\delta^{\frac{1}{2}}(\|f\|_2 + C\|\widehat{R}\|_\infty\|f\|_1) \underbrace{\le}_{\text{Hölder}} C\delta^{\frac{1}{2}}(\|f\|_2 + C\|\widehat{R}\|_\infty\|f\|_2).
\end{aligned}$$

To finish the proof of the lemma, it remains to demonstrate that $\widehat{R}$ is bounded. We have

$$\widehat{R}(\xi) = \sum_{1\le k\le K^*}\widehat{\psi_k}(\xi) = \sum_{1\le k\le K^*}\widehat{\psi^{(0)}}\left(\frac{\xi}{2^k}\right).$$

Now, in the notation of the construction of $\psi^{(0)}$, we can write

$$\psi^{(0)}(x) = \frac{\varphi(x)-\varphi(2x)}{x}.$$

We would like to apply transforms, put $\widehat{(\varphi/x)}(\xi)$ and $(\widehat{\varphi(2x)/x})(\xi)$ and in some way obtain a telescoping sum to compute $\widehat{R}(\xi)$. However, things are not so easy, because $\varphi/x \notin L^1$ (because $\varphi \equiv 1$ in $[-\pi,\pi]$). However, we can turn to principal values, which allow splitting the integral that defines the transform:

$$\begin{aligned}\widehat{\psi^{(0)}}(\xi) = &\int_{\mathbb{R}} \frac{\varphi(x)-\varphi(2x)}{x} e^{-ix\xi}\,dx = \lim_{\epsilon\to 0+}\int_{|x|\geq\epsilon} \frac{\varphi(x)-\varphi(2x)}{x} e^{-ix\xi}\,dx \\ = &\lim_{\epsilon\to 0+}\int_{|x|\geq\epsilon} \frac{\varphi(x)}{x} e^{-ix\xi}\,dx - \lim_{\epsilon\to 0+}\int_{|x|\geq\epsilon} \frac{\varphi(2x)}{x} e^{-ix\xi}\,dx.\end{aligned}$$

Note that, if $a \in \{1,2\}$,

$$\begin{aligned}\int_{|x|\geq\epsilon} \frac{\varphi(ax)}{x} e^{-ix\xi}\,dx &= \int_{\epsilon}^{\infty} \frac{\varphi(ax)}{x} e^{-ix\xi}\,dx + \int_{-\infty}^{-\epsilon} \frac{\varphi(ax)}{x} e^{-ix\xi}\,dx \\ &\underbrace{=}_{\varphi \text{ even}} \int_{\epsilon}^{\infty} \frac{\varphi(ax)}{x} e^{-ix\xi}\,dx - \int_{\epsilon}^{\infty} \frac{\varphi(ax)}{x} e^{ix\xi}\,dx \\ &= -2i\int_{\epsilon}^{\infty} \varphi(ax)\frac{\sin(x\xi)}{x}\,dx.\end{aligned}$$

Hence,

$$\begin{aligned}\widehat{\psi^{(0)}}(\xi) = &-2i\int_0^{\infty} \varphi(x)\frac{\sin(x\xi)}{x}\,dx + 2i\int_0^{\infty} \varphi(2x)\frac{\sin(x\xi)}{x}\,dx \\ &= 2i\int_0^{\infty} (\varphi(2x)-\varphi(x))\frac{\sin(x\xi)}{x}\,dx\end{aligned}$$

(these integrals exist as Lebesgue integrals). This gives

$$\begin{aligned}\widehat{\psi^{(0)}}\left(\frac{\xi}{2^k}\right) &= 2i\int_0^{\infty} (\varphi(2x)-\varphi(x))\frac{\sin(x\xi/2^k)}{x}\,dx \\ &\underbrace{=}_{y=x/2^k} 2i\int_0^{\infty} (\varphi(2^{k+1}y)-\varphi(2^k y))\frac{\sin(y\xi)}{y}\,dy.\end{aligned}$$

We have achieved our telescopic sum:

$$\widehat{R}(\xi) = \sum_{k=1}^{K^*} \widehat{\psi^{(0)}}\left(\frac{\xi}{2^k}\right) = 2i\int_0^{\infty} (\varphi(2^{K^*+1}y)-\varphi(2y))\frac{\sin(y\xi)}{y}\,dy.$$

To bound uniformly $\widehat{R}$, we use the following trick: given any $a > 0$ and $\xi \in \mathbb{R}$,

$$\begin{aligned}
\int_0^\infty \varphi(ay)\frac{\sin(y\xi)}{y}\,dy &= \operatorname{sign}(\xi)\int_0^\infty \varphi(ay)\frac{\sin(y|\xi|)}{y}\,dy = \frac{\operatorname{sign}(\xi)}{2}\int_{\mathbb{R}} \varphi(ay)\frac{\sin(y|\xi|)}{y}\,dy \\
&= \frac{\operatorname{sign}(\xi)}{2}\int_{\mathbb{R}} \varphi(ay)\int_0^{|\xi|}\cos(yt)\,dt\,dy = \frac{\operatorname{sign}(\xi)}{2}\int_0^{|\xi|}\left(\int_{\mathbb{R}}\varphi(ay)\cos(yt)\,dy\right)dt \\
&= \frac{\operatorname{sign}(\xi)}{2}\int_0^{|\xi|}\left(\int_{\mathbb{R}}\varphi(ay)\frac{e^{iyt}+e^{-iyt}}{2}\,dy\right)dt \\
&= \frac{\operatorname{sign}(\xi)}{4}\int_0^{|\xi|}\left(\int_{\mathbb{R}}\varphi(ay)e^{iyt}\,dy + \int_{\mathbb{R}}\varphi(ay)e^{-iyt}\,dy\right)dt \\
&= \frac{\operatorname{sign}(\xi)}{4}\int_0^{|\xi|}\frac{1}{a}\left(\int_{\mathbb{R}}\varphi(y)e^{iy\frac{t}{a}}\,dy + \int_{\mathbb{R}}\varphi(y)e^{-iy\frac{t}{a}}\,dy\right)dt \\
&= \frac{\operatorname{sign}(\xi)}{4}\int_0^{|\xi|}\frac{1}{a}\left(\widehat{\varphi}\left(-\frac{t}{a}\right)+\widehat{\varphi}\left(\frac{t}{a}\right)\right)dt = \frac{\operatorname{sign}(\xi)}{4}\int_0^{|\xi|/a}(\widehat{\varphi}(-t)+\widehat{\varphi}(t))\,dt \\
&\underbrace{=}_{\varphi\text{ even}} \frac{\operatorname{sign}(\xi)}{2}\int_0^{|\xi|/a}\widehat{\varphi}(t)\,dt.
\end{aligned}$$

Define, for $s \geq 0$, $F(s) = \int_0^s \widehat{\varphi}(t)\,dt$. Since F is continuous on $[0,\infty[$ and

$$\lim_{s\to\infty} F(s) = \int_0^\infty \widehat{\varphi}(t)\,dt = \frac{1}{2}\int_{\mathbb{R}}\widehat{\varphi}(t)\,dt = \frac{\varphi(0)}{2} = \frac{1}{2},$$

F is bounded: there exists $C > 0$ such that $|F(s)| \leq C$ for all $s \geq 0$. Then

$$\left|\int_0^\infty \varphi(ay)\frac{\sin(y\xi)}{y}\,dy\right| \leq C$$

for all $a > 0$ and $\xi \in \mathbb{R}$. We conclude that $|\widehat{R}(\xi)| \leq 4C$ for all $\xi \in \mathbb{R}$, and the lemma is proved.

□

Corollary 6.1 *If $\mathcal{P}$ is a tree, then $\|T^{\mathcal{P}}\|_2 \leq C$ (C does not depend on the tree $\mathcal{P}$).*

Proof. The proof is more or less a consequence of the above proof. We just need to simplify it. In the previous proof, we essentially bound $T^{\mathcal{P}}$ by means of a maximal operator. Then one needs to construct a partition $\{I_j\}_{j=1}^J$ of I^0 and $\{E_j\}_{j=1}^J$ verifying (α) and (β) above. For the E_j's, just take $E_j = I_j$ and $\tilde{E}_j = \tilde{I}_j$. As ($\beta$) is trivial (because $E_j = I_j$), it just remains to construct $\{I_j\}_{j=1}^J$ such that (α) holds. Since $\mathcal{P}$ is finite, we may consider $d = \min\{|I| : [\omega, I] \in \mathcal{P}\}$. Take $\{I_j\}_{j=1}^J$ as the dyadic partition of I^0 with $|I_j| = d/2$ for all $j = 1, \ldots, J$. If $p = [\omega, I] \in \mathcal{P}$ and $I \cap I_j \neq \emptyset$, then either $\tilde{I}_j \subseteq I$ or $I \subseteq I_j$, and since $|I_j| = d/2 < d \leq |I|$, necessarily $\tilde{I}_j \subseteq I$, so (α) holds. With this partition $\{I_j\}_{j=1}^J$, just copy the proof of Lemma 6.3.

Notice that we do not use $A(p) \leq 1$, because $A(p) \leq 1$ always holds by definition of $A(p)$. Thus, this corollary is Lemma 6.3 with $\delta = 1$.

□

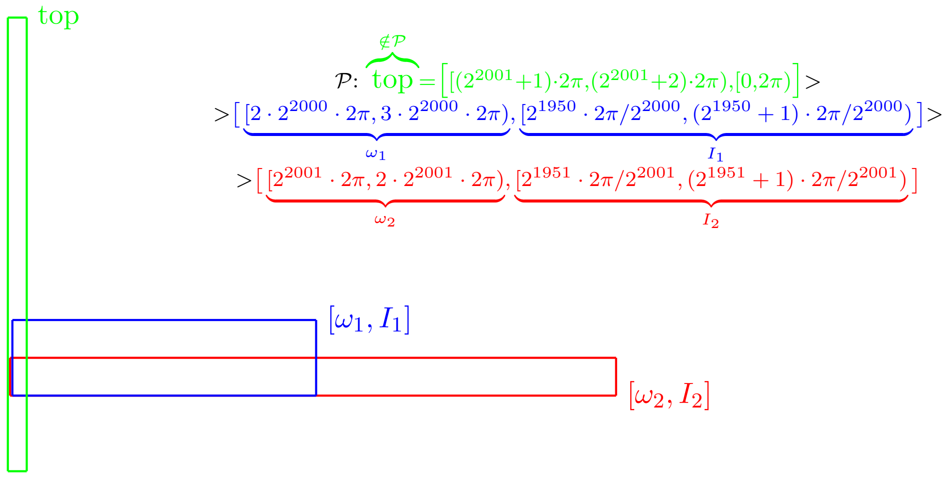
Remark 6.4 *From Lemma 6.3, we see that (5.5) holds for trees.*

Fix $0 < \delta \leq 1$ and $K > 0$ large (for instance, a good choice is $K > 10$).

Definition 6.2 *A tree $\mathcal{P}$ with top $p^0 = [\omega^0, I^0]$ is normal if for $[\omega, I] \in \mathcal{P}$ we have $|I| \leq (\delta^{1000}/K^2)|I^0|$ and $d(I, \partial I^0) > 3(\delta^{100}/K^2)|I^0|$.*

To make an intuitive idea of what a normal tree is, it is a tree such that the I's are small compared to I^0 and are contained far from the endpoints of I^0.

Example of a normal tree $\mathcal{P}$ with $\delta = 1/2$ and $K = 100$:



We have $|I_1|=2\pi/2^{2000}<(1/2^{1000}\cdot 10000)|I^0|$ and $d(I_1,\partial I^0)=2\pi/2^{50}>3\cdot 2\pi/(2^{100}\cdot 10000)$.
Also $|I_2|=2\pi/2^{2001}<(1/2^{1000}\cdot 10000)|I^0|$ and $d(I_2,\partial I^0)=2\pi/2^{50}>3\cdot 2\pi/(2^{100}\cdot 10000)$.

An example of a non-normal tree is given in the picture immediately after the definition of tree: with that same notation, $d(I_{1,1}, \partial I^0) = 0$.

A key fact of normal trees is that, for any $f \in L^2_r$, $T^{\mathcal{P}^*} f$ lives in I^0. An intuition for this is that, as we said, the I's are small and contained in I^0 far from ∂I^0, therefore I^3, which is the support of $T_p^* f$ ($p = [\omega, I]$), will be contained in I^0. We prove this formally. The finite sum of the adjoints is the adjoint of the sum:

$$T^{\mathcal{P}^*} f(x) = \sum_{\substack{p=[\omega,I]\in\mathcal{P} \\ |I|=2\pi\cdot 2^{-k}}} \int_{E(p)} e^{-iN(y)(y-x)} \psi_k(y-x) f(y)\, dy \cdot \mathbb{1}_{I^3}.$$

Then

$$\text{support}(T^{\mathcal{P}^*} f) \subseteq \bigcup_{[\omega,I]\in\mathcal{P}} I^3.$$

Since

$$d(I^5, \partial I^0) = d(I, \partial I^0) - 2|I| > 3\frac{\delta^{100}}{K^2}|I^0| - 2\frac{\delta^{1000}}{K^2}|I^0| \geq 3\frac{\delta^{100}}{K^2}|I^0| - 2\frac{\delta^{100}}{K^2}|I^0| = \frac{\delta^{100}}{K^2}|I^0|,$$

this gives $\text{support}(T^{\mathcal{P}^*} f) \subseteq I^0$, as desired.

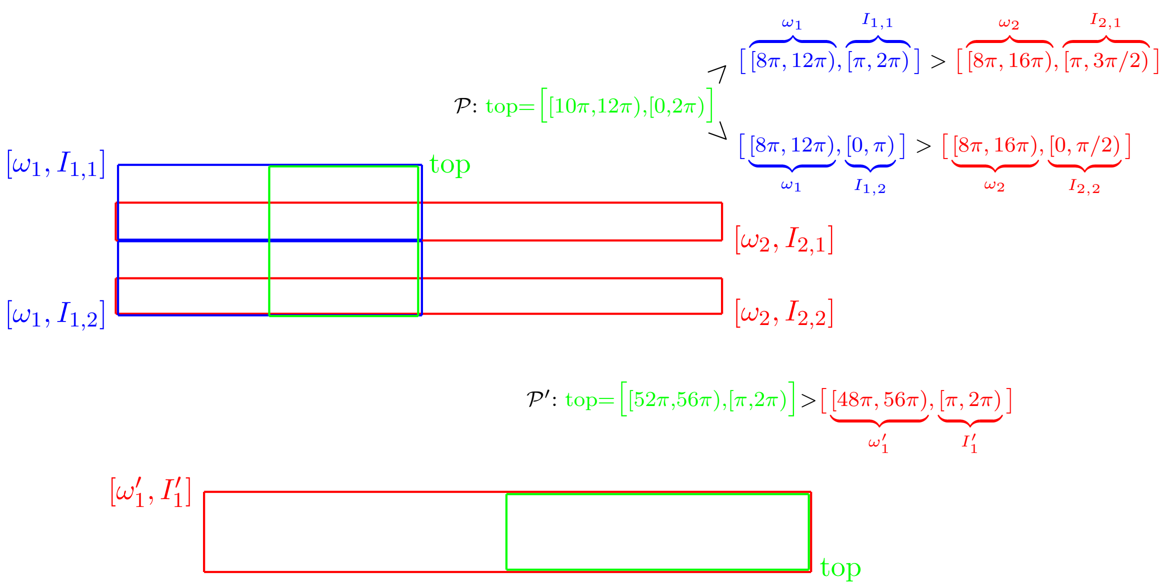
In fact, more can be said about $\text{support}(T^{\mathcal{P}^*} f)$: $\text{support}(T^{\mathcal{P}^*} f) \subseteq \{x \in I^0 : d(x, \partial I^0) > (\delta^{100}/K^2)|I^0|\}$. That is, the support of $T^{\mathcal{P}^*} f$ is not only contained in I^0, it is a set of points in I^0 far from its endpoints.

The choice of I^5 when computing $d(I^5, \partial I^0)$ will be necessary to prove Lemma 6.5.

Definition 6.3 *Two trees, $\mathcal{P}$ with top $[\omega^0, I^0]$, and $\mathcal{P}'$ with top $[\omega^1, I^1]$, are separated if either $I^0 \cap I^1 = \emptyset$ or:*
(α) $[\omega, I] \in \mathcal{P}$, $I \subseteq I^1 \Rightarrow d(\omega, \omega^1) > \frac{1}{\delta}|\omega|$,
(β) $[\omega', I'] \in \mathcal{P}'$, $I' \subseteq I^0 \Rightarrow d(\omega', \omega^0) > \frac{1}{\delta}|\omega'|$.

This condition is stronger than saying that no two $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ are comparable under $<$.

Example of two separated trees $\mathcal{P}$ and $\mathcal{P}'$ with $\delta = 1/2$:



In the following lemma we study condition (b) of the Orthogonality lemma for two separated trees.

Lemma 6.4 *Let $\mathcal{P}$ and $\mathcal{P}'$ be separated trees with tops $[\omega^0, I^0]$ and $[\omega^1, I^0]$. Then*

$$\|T^{\mathcal{P}'}T^{\mathcal{P}^*}\|_2 \leq C_M \delta^M, \quad \forall M > 0.$$

Proof. First of all, note that the statement of the lemma is equivalent to

$$|(T^{\mathcal{P}^*}f, T^{\mathcal{P}'^*}g)| \leq C_M \delta^M \|f\|_2 \|g\|_2 \tag{6.19}$$

for all $f, g \in L^2_r$. Indeed, $\|T^{\mathcal{P}'}T^{\mathcal{P}^*}\|_2 \leq C_M \delta^M$ if and only if $\|T^{\mathcal{P}'}T^{\mathcal{P}^*}f\|_2 \leq C_M \delta^M \|f\|_2$ for every $f \in L^2_r$, but

$$\|T^{\mathcal{P}'}T^{\mathcal{P}^*}f\|_2 = \sup_{\substack{g \in L^2_r \\ \|g\|_2 = 1}} |(T^{\mathcal{P}'}T^{\mathcal{P}^*}f, g)| = \sup_{\substack{g \in L^2_r \\ \|g\|_2 = 1}} |(T^{\mathcal{P}^*}f, T^{\mathcal{P}'^*}g)| = \sup_{\substack{g \in L^2_r \\ g \neq 0}} \frac{|(T^{\mathcal{P}^*}f, T^{\mathcal{P}'^*}g)|}{\|g\|_2}.$$

Then $\|T^{\mathcal{P}'}T^{\mathcal{P}^*}f\|_2 \leq C_M \delta^M \|f\|_2$ if and only if (6.19) holds for all $g \in L_2^r$, as desired. Then the proof of (6.19) for all $f, g \in L^2_r$ will be our goal in the rest of the demonstration of this lemma.

Let $d = \min\{|I| : [\omega, I] \in \mathcal{P}\}$ and $d' = \min\{|I'| : [\omega', I'] \in \mathcal{P}'\}$. Consider $\varphi \in C_c^\infty(\mathbb{R})$ such that
(i) $\text{support}(\varphi) \subseteq \{x \in \mathbb{R} : |x| \leq \delta^{\frac{1}{2}} d\}$ and $\|\varphi\|_1 \leq C_M$.
(ii) $|\hat{\varphi}(\xi)| \leq C_M(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{-2M}$ for all $\xi \in \mathbb{R}$ (ξ_0 is the midpoint of ω^0).
(iii) $|\hat{\varphi}(\xi) - 1| \leq C_M(\delta^{\frac{1}{2}} d|\xi - \xi_0|)^{2M}$ for all $\xi \in \mathbb{R}$.

To construct such a φ, look at the coming Remark 6.5. Analogously, pick a φ' corresponding to $\mathcal{P}'$ ($\xi_1 =$ midpoint of ω^1).

Let us see that from the fact that $\mathcal{P}$ and $\mathcal{P}'$ are separated, one can deduce that

$$|\hat{\varphi}(\xi)\widehat{\varphi'}(-\xi)| \leq C_M \delta^M, \ \ \forall \xi \in \mathbb{R}. \tag{6.20}$$

Take $[\omega, I] \in \mathcal{P}$ such that $|I| = d$. By definition of top, $I \subseteq I^0$, therefore by (α) in the definition of separated tress we have $d(\omega, \omega^1) > 4\pi^2/(\delta d)$. By definition of top again, $\omega^0 \subseteq \omega$, so $d(\omega^0, \omega^1) \geq d(\omega, \omega^1) > (4\pi^2)/(\delta d)$. Taking $[\omega', I'] \in \mathcal{P}'$ such that $|I'| = d'$, we obtain as before that $d(\omega^0, \omega^1) > 4\pi^2/(\delta d')$. To sum up, $d(\omega^0, \omega^1) > 4\pi^2/(\delta d'')$, $d'' = \min\{d, d'\}$. On the other hand, note that, if $\xi \in \mathbb{R}$, then $|\xi - \xi_0| > 4\pi^2/(2\delta d'')$ or $|\xi - \xi_1| > 4\pi^2/(2\delta d'')$ (indeed, if there is $\xi \in \mathbb{R}$ satisfying $|\xi - \xi_0| \leq 4\pi^2/(2\delta d'')$ and $|\xi - \xi_1| \leq 4\pi^2/(2\delta d'')$, then $d(\omega^0, \omega^1) \leq |\xi_0 - \xi_1| \leq |\xi - \xi_0| + |\xi - \xi_1| \leq 4\pi^2/(\delta d'')$, which is not true).

Suppose that our function N is positive (the case $N < 0$ is analogous). Then $\xi_0, \xi_1 > 0$ by the last part of Remark 4.1 (if $N < 0$, then $\xi_0, \xi_1 < 0$). Suppose that $\xi_0 < \xi_1$ (the case $\xi_1 < \xi_0$ is completely analogous). Let

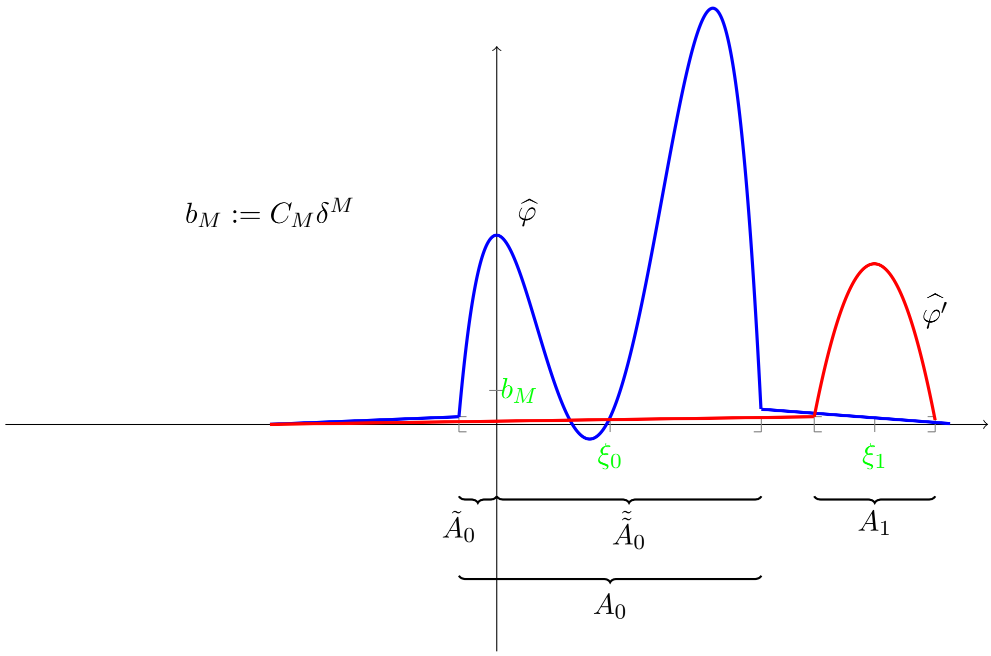
$$A_0 = [\xi_0 - 4\pi^2/(2\delta d''), \xi_0 + 4\pi^2/(2\delta d'')],$$

$$A_1 = [\xi_1 - 4\pi^2/(2\delta d''), \xi_1 + 4\pi^2/(2\delta d'')].$$

Pick $\xi \notin A_0$, that is, $|\xi - \xi_0| > 4\pi^2/(2\delta d'') > 1/(2\delta d)$. By (ii), $|\hat{\varphi}(\xi)| \leq C_M(\delta^{1/2} d|\xi - \xi_0|)^{-2M} \leq C_M(\delta^{1/2}/(2\delta))^{-2M} = C_M \delta^M$. By (i), $|\widehat{\varphi'}(-\xi)| \leq \|\varphi'\|_1 \leq C_M$. Then

$$|\hat{\varphi}(\xi)\widehat{\varphi'}(-\xi)| \leq C_M \delta^M.$$

Now take $\xi \in A_0$. By (i), $|\widehat{\varphi}(\xi)| \leq \|\varphi\|_1 \leq C_M$. Note that $-\xi \notin A_1$, which is a direct consequence of the fact that $0 < \xi_0 < \xi_1$. Indeed, call $\tilde{A}_0 = A_0 \cap]-\infty, 0[$ (which could be empty) and let $\tilde{\tilde{A}}_0 = A_0 \backslash \tilde{A}_0$. As $-\tilde{\tilde{A}}_0 \subseteq]-\infty, 0]$ and $]-\infty, 0] \cap A_1 = \emptyset$, we just have to deal with $\tilde{A}_0$. As ξ_0 is the midpoint of A_0 and $\xi_0 \notin \tilde{A}_0$, then $-\tilde{A}_0 \subseteq \tilde{\tilde{A}}_0$, so $-\tilde{A}_0 \cap A_1 = \emptyset$. Then we have proved that $-A_0 \cap A_1 = \emptyset$, therefore $-\xi \notin A_1$, as claimed.



As $-\xi \notin A_1$, $|-\xi-\xi_1| > 4\pi^2/(2\delta d'') > 1/(2\delta d')$. By (ii),

$$|\widehat{\varphi'}(-\xi)| \le C_M(\delta^{1/2}d|-\xi-\xi_1|)^{-2M} \le C_M(\delta^{1/2}/(2\delta))^{-2M} = C_M\delta^M.$$

This gives $|\hat{\varphi}(\xi)\widehat{\varphi'}(-\xi)| \le C_M\delta^M$. Thus, (6.20) is true.

Now define, for $f \in L^2_r$,

$$\epsilon(f) = T^{\mathcal{P}^*}f - \tilde{\varphi} * (T^{\mathcal{P}^*}f)$$

and

$$\epsilon'(f) = T^{\mathcal{P}'^*}f - \tilde{\varphi'} * (T^{\mathcal{P}'^*}f),$$

where $\tilde{\varphi}(x) := \varphi(-x)$, $\tilde{\varphi'}(x) := \varphi'(-x)$ and $*$ stands for the convolution on $\mathbb{R}$. We want to show that

$$\|\epsilon\|_2 \le C_M\delta^M \tag{6.21}$$

and

$$\|\epsilon'\|_2 \le C_M\delta^M. \tag{6.22}$$

where the norm $\|\cdot\|_2$ is taken, as usual, in the sense of L^2_r (ϵ and ϵ' have their image with support contained in $[-4\pi, 6\pi]$). We will prove (6.21), and (6.22) will follow by analogy. Notice that, given $g \in L^2_r$,

$$(\tilde{\varphi} * (T^{\mathcal{P}^*}f), g) = \int_{\mathbb{R}} (\tilde{\varphi} * (T^{\mathcal{P}^*}f))(x)\overline{g(x)}\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\varphi}(x-y)T^{\mathcal{P}^*}f(y)\,dy\,\overline{g(x)}\,dx$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{\varphi}(x-y)\overline{g(x)}\,dx\right) T^{\mathcal{P}^*}f(y)\,dy = \int_{\mathbb{R}} \left(\overline{\int_{\mathbb{R}} \varphi(y-x)g(x)\,dx}\right) T^{\mathcal{P}^*}f(y)\,dy$$
$$= (T^{\mathcal{P}^*}f, \varphi * g) = (f, T^{\mathcal{P}}(\varphi * g)).$$

Hence, by duality,

$$\begin{aligned}\|\epsilon(f)\|_2 &= \sup_{\substack{g\in L^2_r\\ \|g\|_2=1}} (T^{\mathcal{P}^*}f - \tilde{\varphi} * (T^{\mathcal{P}^*}f), g) = \sup_{\substack{g\in L^2_r\\ \|g\|_2=1}} \{(T^{\mathcal{P}^*}f, g) - (\tilde{\varphi} * (T^{\mathcal{P}^*}f), g)\}\\ &= \sup_{\substack{g\in L^2_r\\ \|g\|_2=1}} \{(T^{\mathcal{P}^*}f, g) - (f, T^{\mathcal{P}}(\varphi * g)) = \sup_{\substack{g\in L^2_r\\ \|g\|_2=1}} (f, T^{\mathcal{P}}g - T^{\mathcal{P}}(\varphi * g))\\ &\le \|f\|_2 \sup_{\substack{g\in L^2_r\\ \|g\|_2=1}} \|T^{\mathcal{P}}g - T^{\mathcal{P}}(\varphi * g)\|_2.\end{aligned}$$

Thus, it suffices to show that for all $f \in L^2_r$ we have

$$\|T^{\mathcal{P}}f - T^{\mathcal{P}}(\varphi * f)\|_2 \le C_M \delta^M \|f\|_2. \tag{6.23}$$

Write, using convolutions,

$$\begin{aligned}&|T^{\mathcal{P}}f(x) - T^{\mathcal{P}}(\varphi * f)(x)|\\ &= \Bigg| \sum_{\substack{p=[\omega,I]\in\mathcal{P}\\ |I|=2\pi\cdot 2^{-k},\, x\in E(p)}} ((e^{iN(x)\cdot}\psi_k(\cdot)) * f)(x) - \sum_{\substack{p=[\omega,I]\in\mathcal{P}\\ |I|=2\pi\cdot 2^{-k},\, x\in E(p)}} ((e^{iN(x)\cdot}\psi_k(\cdot)) * (\varphi * f))(x)\Bigg|\\ &\le \sum_{\substack{p=[\omega,I]\in\mathcal{P}\\ |I|=2\pi\cdot 2^{-k},\, x\in E(p)}} \Big|[(e^{iN(x)\cdot}\psi_k(\cdot)) * f](x) - [((e^{iN(x)\cdot}\psi_k(\cdot)) * \varphi) * f](x)\Big|\\ &\le \Bigg(\sum_{\substack{p=[\omega,I]\in\mathcal{P}\\ |I|=2\pi\cdot 2^{-k},\, x\in E(p)}} \Big|e^{iN(x)\cdot}\psi_k(\cdot) - (e^{iN(x)\cdot}\psi_k(\cdot)) * \varphi\Big| \Bigg) * |f|(x).\end{aligned}$$

If $p = [\omega, I] \in \mathcal{P}$ and $x \in E(p)$, then $N(x) \in \omega$, and since $\xi_0 \in \omega^0 \subseteq \omega$, we have

$$|N(x) - \xi_0| \le |\omega| = 2\pi \cdot 2^k. \tag{6.24}$$

This bound will be essential to deal with $|e^{iN(x)\cdot}\psi_k(\cdot) - (e^{iN(x)\cdot}\psi_k(\cdot)) * \varphi|$. Indeed, call $h(t) = e^{iN(x)t}\psi_k(t)$. Then

$$\begin{aligned}|h(y) - (h * \varphi)(y)| &= C \left| \int_{\mathbb{R}} (\hat{h}(\xi) - \widehat{(h * \varphi)}(\xi)) e^{iy\xi}\, d\xi \right|\\ &\le C \int_{\mathbb{R}} |\hat{h}(\xi) - \hat{h}(\xi)\hat{\varphi}(\xi)|\, d\xi = C \int_{\mathbb{R}} |\hat{h}(\xi)||1 - \hat{\varphi}(\xi)|\, d\xi.\end{aligned}$$

By (iii), $|1 - \hat{\varphi}(\xi)| \le C_M(\delta^{1/2} d|\xi - \xi_0|)^{2M}$. On the other hand,

$$|\hat{h}(\xi)| = |\widehat{\psi_k}(\xi - N(x))| = \left| \widehat{\psi^{(0)}} \left(\frac{\xi - N(x)}{2^k} \right) \right|.$$

Then

$$\begin{aligned}
|h(y)-(h*\varphi)(y)| &\le C_M(\delta^{\frac{1}{2}}d)^{2M}\int_{\mathbb{R}}|\xi-\xi_0|^{2M}\left|\widehat{\psi^{(0)}}\left(\frac{\xi-N(x)}{2^k}\right)\right|d\xi \\
\underbrace{=}_{(\xi-N(x))/2^k=z} &\quad C_M(\delta^{\frac{1}{2}}d)^{2M}2^k\int_{\mathbb{R}}|2^kz+N(x)-\xi_0|^{2M}|\widehat{\psi^{(0)}}(z)|\,dz \\
\underbrace{=}_{\substack{(a+b)^n\le 2^{n-1}(a^n+b^n),\\ a,b\ge 0}} &\quad C_M(\delta^{\frac{1}{2}}d)^{2M}2^k\left(\int_{\mathbb{R}}|N(x)-\xi_0|^{2M}|\widehat{\psi^{(0)}}(z)|\,dz+2^{2kM}\int_{\mathbb{R}}z^{2M}|\widehat{\psi^{(0)}}(z)|\,dz\right) \\
\underbrace{\le}_{\substack{\widehat{\psi^{(0)}}\\ \text{Schwartz},\\ (6.24)}} &\quad C_M(\delta^{\frac{1}{2}}d)^{2M}2^k(2^{2kM}+2^{2kM}) \\
&= C_M(\delta^{\frac{1}{2}}\cdot 2^kd)^{2M}\cdot 2^k.
\end{aligned}$$

We can be sharper in the estimate of $|h(y)-(h*\varphi)(y)|$ by studying its support. As $\text{support}(h)\subseteq[-2\pi\cdot 2^{-k},2\pi\cdot 2^{-k}]$ and $\text{support}(\varphi)\subseteq[-\delta^{1/2}d,\delta^{1/2}d]\subseteq[-2\pi\cdot 2^{-k},2\pi\cdot 2^{-k}]$, we have

$$\text{support}(h-h*\varphi)\subseteq[-2\pi\cdot 2^{-k+1},2\pi\cdot 2^{-k+1}],$$

therefore

$$|h(y)-(h*\varphi)(y)|\le C_M(\delta^{\frac{1}{2}}\cdot 2^kd)^{2M}\cdot 2^k\cdot \mathbb{1}_{[-2\pi\cdot 2^{-k+1},2\pi\cdot 2^{-k+1}]}(y).$$

This gives

$$\sup_{x\in\mathbb{R}}\sum_{\substack{p=[\omega,I]\in\mathcal{P}\\ |I|=2\pi\cdot 2^{-k},\,x\in E(p)}}\left|e^{iN(x)y}\psi_k(y)-[(e^{iN(x)\cdot}\psi_k(\cdot))*\varphi](y)\right|\le\mathcal{R}(y),$$

where

$$\mathcal{R}(y)=\sum_{k\ge 1,\,d\le 2\pi\cdot 2^{-k}}C_M(\delta^{\frac{1}{2}}\cdot 2^kd)^{2M}\cdot 2^k\cdot \mathbb{1}_{[-2\pi\cdot 2^{-k+1},2\pi\cdot 2^{-k+1}]}(y),$$

therefore

$$|T^{\mathcal{P}}f(x)-T^{\mathcal{P}}(\varphi*f)(x)|\le(\mathcal{R}*|f|)(x).$$

In order for (6.23) to hold, we need $\|\mathcal{R}\|_1\le C_M\delta^M$, because $\|\mathcal{R}*|f|\|_2\le\|\mathcal{R}\|_1\|f\|_2$. We have

$$\begin{aligned}
\|\mathcal{R}\|_1 &\le C_Md^{2M}\delta^M\sum_{k=1}^{\lfloor\log_2(2\pi/d)\rfloor}(4^M)^k\le C_Md^{2M}\delta^M\frac{4^{M(\lfloor\log_2(2\pi/d)\rfloor+1)}}{4^M-1} \\
&\le C_Md^{2M}\delta^M\frac{4^M}{4^M-1}2^{2M\lfloor\log_2(2\pi/d)\rfloor},
\end{aligned}$$

and since $d^{2M}2^{2M\lfloor\log_2(2\pi/d)\rfloor}=(d\cdot 2^{\lfloor\log_2(2\pi/d)\rfloor})^{2M}\le(2\pi)^{2M}$, it follows $\|\mathcal{R}\|_1\le C_M\delta^M$. As a consequence, (6.23) is true and so are (6.21) and (6.22), as wanted.

From both (6.21) and (6.22) the proof of (6.19) is easy. Write

$$\begin{aligned}
(T^{\mathcal{P}^*}f,T^{\mathcal{P}'^*}g) &= (\epsilon(f)+\tilde{\varphi}*(T^{\mathcal{P}^*}f))+(\epsilon'(g)+\tilde{\varphi}'*(T^{\mathcal{P}'^*}g)) \\
&= (\tilde{\varphi}*(T^{\mathcal{P}^*}f),\tilde{\varphi}'*(T^{\mathcal{P}'^*}g))+(\epsilon(f),\tilde{\varphi}'*(T^{\mathcal{P}'^*}g)) \\
&\quad +(\tilde{\varphi}*(T^{\mathcal{P}^*}f),\epsilon'(g))+(\epsilon(f),\epsilon'(g)).
\end{aligned}$$

We have

$$\left|(\epsilon(f), \tilde{\varphi'} * (T^{\mathcal{P}'^*} g))\right| \leq \|\epsilon(f)\|_2 \|\tilde{\varphi'} * (T^{\mathcal{P}'^*} g)\|_2 \leq \|\epsilon(f)\|_2 \|\tilde{\varphi'}\|_1 \|T^{\mathcal{P}'^*} g\|_2 \leq C_M \delta^M \|f\|_2 \|g\|_2,$$

$$\left|(\tilde{\varphi} * (T^{\mathcal{P}^*} f), \epsilon'(g))\right| \leq \|\tilde{\varphi} * (T^{\mathcal{P}^*} f)\|_2 \|\epsilon'(g)\|_2 \leq \|\tilde{\varphi}\|_1 \|T^{\mathcal{P}^*} f\|_2 \|\epsilon'(g)\|_2 \leq C_M \delta^M \|f\|_2 \|g\|_2,$$

$$|(\epsilon(f), \epsilon'(g))| \leq \|\epsilon(f)\|_2 \|\epsilon'(g)\|_2 \leq C_M \delta^{2M} \|f\|_2 \|g\|_2 \leq C_M \delta^M \|f\|_2 \|g\|_2$$

(we have used Lemma 6.3 with $\delta = 1$). For the remaining term we will make use of (6.20). First write

$$\begin{aligned}
(\tilde{\varphi} * (T^{\mathcal{P}^*} f), \tilde{\varphi'} * (T^{\mathcal{P}'^*} g)) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{\varphi}(x-y) T^{\mathcal{P}^*} f(y)\, dy \right) \left(\int_{\mathbb{R}} \tilde{\varphi'}(x-z) \overline{T^{\mathcal{P}'^*} g(z)}\, dz \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{\varphi}(x-y) \tilde{\varphi'}(x-z)\, dx \right) T^{\mathcal{P}^*} f(y)\, dy \right) \overline{T^{\mathcal{P}'^*} g(z)}\, dz \\
&\underbrace{=}_{x=z+u} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{\varphi}(z+u-y) \tilde{\varphi'}(u)\, du \right) T^{\mathcal{P}^*} f(y)\, dy \right) \overline{T^{\mathcal{P}'^*} g(z)}\, dz \\
&\underbrace{=}_{u \leftarrow -u} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{\varphi}(z-u-y) \varphi'(u)\, du \right) T^{\mathcal{P}^*} f(y)\, dy \right) \overline{T^{\mathcal{P}'^*} g(z)}\, dz \\
&= \left((\varphi' * \tilde{\varphi}) * (T^{\mathcal{P}^*} f), T^{\mathcal{P}'^*} g \right).
\end{aligned}$$

Then

$$\begin{aligned}
|(\tilde{\varphi} * (T^{\mathcal{P}^*} f), \tilde{\varphi'} * (T^{\mathcal{P}'^*} g))| &\underbrace{\leq}_{\text{C-S}} \|(\varphi' * \tilde{\varphi}) * (T^{\mathcal{P}^*} f)\|_2 \|T^{\mathcal{P}'^*} g\|_2 \\
&\underbrace{=}_{\text{Plancherel}} \|\widehat{(\varphi' * \tilde{\varphi})} \widehat{T^{\mathcal{P}^*} f}\|_2 \|T^{\mathcal{P}'^*} g\|_2 \\
&\leq \|\widehat{\varphi' * \tilde{\varphi}}\|_\infty \|\widehat{T^{\mathcal{P}^*} f}\|_2 \|T^{\mathcal{P}'^*} g\|_2 \underbrace{\leq}_{(6.20)} C_M \delta^M \|\widehat{T^{\mathcal{P}^*} f}\|_2 \|T^{\mathcal{P}'^*} g\|_2 \\
&\underbrace{=}_{\text{Plancherel}} C_M \delta^M \|T^{\mathcal{P}^*} f\|_2 \|T^{\mathcal{P}'^*} g\|_2 \underbrace{\leq}_{\text{Corollary 6.1}} C_M \delta^M \|f\|_2 \|g\|_2.
\end{aligned}$$

It follows (6.19), and we are done.

□

Remark 6.5 *The idea to construct the function φ from Lemma 6.4 is the following: for (i) and (ii) to hold, it suffices to take a bump function, and (ii) will hold because the Fourier transform of a Schwartz function is Schwartz. In order to have (iii), the idea will be to have $\varphi(\xi_0) = 1$ (which is achieved by having $\int \varphi = 1$ plus a modulation) and the Taylor development of φ of order $2M$, which is achieved by imposing the first $2M$ moments of φ to be zero.*

We start the formal construction of φ. Suppose for the moment that there is $\phi \in C_c^\infty(\mathbb{R})$ with support $\subseteq [-1, 1]$, $\int_{\mathbb{R}} \phi(x) dx = 1$ and $\int_{-1}^{1} x^k \phi(x) dx = 0$ for all $k = 1, \ldots, 2M-1$. Call $a = \delta^{\frac{1}{2}} d > 0$. Let

$$\varphi(x) = \frac{e^{ix\xi_0}}{a} \phi\left(\frac{x}{a}\right).$$

Then $\varphi \in C_c^\infty(\mathbb{R})$ and support$(\varphi) \subseteq [-a, a]$, so (i) holds. For (ii), use the fact that $\phi \in C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}$ ($\mathcal{S}$ is the Schwartz space on $\mathbb{R}$) implies $\hat{\phi} \in \mathcal{S}$, so by definition of $\mathcal{S}$ there exists $C_M > 0$ such that $|\hat{\phi}(t)| \le C_M t^{-2M}$ for all $t \in \mathbb{R}$. As

$$\hat{\varphi}(\xi) = \frac{1}{a}\int_{\mathbb{R}} \phi\left(\frac{x}{a}\right) e^{-ix(\xi-\xi_0)}\,dx = \int_{\mathbb{R}} \phi(x) e^{-iax(\xi-\xi_0)}\,dx = \hat{\phi}(a(\xi-\xi_0)),$$

we have directly (ii). Finally, for (iii), use the Taylor development of $\hat{\phi}$: since

$$\hat{\phi}^{(n)}(\xi) = \int_{\mathbb{R}} (-ix)^n \phi(x) e^{-ix\xi}\,dx,$$

we have $\hat{\phi}(0) = 1$ and $\hat{\phi}^{(n)}(0) = 0$ for $n = 1, \ldots, 2M-1$, and since $\phi^{(2M)}$ is bounded on $\mathbb{R}$, $|\hat{\phi}(\xi) - 1| \le C_M \xi^{2M}$. Changing to φ, $|\hat{\varphi}(\xi) - 1| \le C_M (a(\xi-\xi_0))^{2M}$, which is (iii).

Hence, to have φ, we need to construct the assumed ϕ. The condition on the moments of ϕ makes us think on some sort of orthogonality in the Hilbert space L^2. If $M = 1$, just take a usual bump function $\phi \in C_c^\infty(\mathbb{R})$ with support$(\phi) \subseteq [-1, 1]$, ϕ even and $\int_{\mathbb{R}} \phi(x)dx = 1$ (since ϕ is even, the required condition for the $2M-1 = 1$ moment holds). Suppose $M \ge 2$. Take $0 \ne g \in L^2([1/4, 1/2])$ such that $\int_{\mathbb{R}} x^k g(x)dx = 0$ for all $k = 0, \ldots, 2M-2$ and $\int_{\mathbb{R}} g(x)/(x+1/4)\,dx \ne 0$. We can find such a function g because $\langle 1, x, \ldots, x^{2M-2}\rangle < \langle 1/(x+1/4), 1, x, \ldots, x^{2M-2}\rangle < L^2([1/4, 1/2])$ and because of the property saying "let $F \le H$, H Hilbert, then F is dense if and only if $F^\perp = \{0\}$". Take any $h \in C_c^\infty(\mathbb{R})$ with support$(h) \subseteq [1/4, 1/2]$. Let $G = g * h$. Then for all $k = 0, \ldots, 2M-2$

$$\begin{aligned}\int_{\mathbb{R}} x^k G(x)\,dx &= \int_{\mathbb{R}} x^k \left(\int_{\mathbb{R}} g(x-y)h(y)\,dy\right) dx = \int_{\mathbb{R}} h(y)\left(\int_{\mathbb{R}} x^k g(x-y)\,dx\right) dy \\ &= \int_{\mathbb{R}} h(y)\left(\int_{\mathbb{R}} (x+y)^k g(x)\,dx\right) dy = \sum_{l=0}^{k} \binom{k}{l} \int_{\mathbb{R}} y^{k-l} h(y) \left(\int_{\mathbb{R}} x^l g(x)\,dx\right) dy = 0.\end{aligned}$$

Also, $G \in C_c^\infty(\mathbb{R})$ and support$(G) \subseteq$ support(g) + support$(h) \subseteq [1/2, 1]$. Let $\phi(x) = G(x)/x \in C_c^\infty(\mathbb{R})$, with support$(\phi) \subseteq [1/2, 1]$. Then for all $k = 1, \ldots, 2M-1$ the k-th moments of ϕ are 0: $\int_{\mathbb{R}} x^k \phi(x)dx = \int_{\mathbb{R}} x^{k-1} G(x)dx = 0$. It remains to see that $\int_{\mathbb{R}} \phi(x)dx \ne 0$. We will use our freedom when choosing h. Suppose that for all $h \in C_c^\infty(\mathbb{R})$ with support$(h) \subseteq [1/4, 1/2]$ we have $\int_{\mathbb{R}} \phi(x)dx = 0$ for the corresponding ϕ. Then

$$\begin{aligned}0 &= \int_{\mathbb{R}} \frac{(g*h)(x)}{x}\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} h(y)g(x-y)\,dy \frac{1}{x}\,dx \\ &= \int_{\mathbb{R}} h(y)\left(\int_{\mathbb{R}} \frac{g(x)}{x+y}\,dx\right) dy = \int_{1/4}^{1/2} h(y)\left(\int_{1/4}^{1/2} \frac{g(x)}{x+y}\,dx\right) dy\end{aligned}$$

for all $h \in C_c^\infty(\mathbb{R})$ with support in $[1/4, 1/2]$. Note that

$$y \mapsto \int_{1/4}^{1/2} \frac{g(x)}{x+y}\,dx$$

is $C^\infty([1/4, 1/2])$. Consider $\{h_n\}_{n=1}^\infty \subseteq C_c^\infty(\mathbb{R})$ with support contained in $[1/4, 1/2]$ and

$$\lim_n h_n(y) = \int_{1/4}^{1/2} \frac{g(x)}{x+y}\,dx$$

for all $y \in (1/4, 1/2)$ (for each n, take a Uryshon function $\rho_n \in C_c^\infty(\mathbb{R})$ with $0 \le \rho_n \le 1$, $\rho_n|_{[1/4+1/n, 1/2-1/n]} = 1$ and support in $(1/4, 1/2)$, and define $h_n = \rho_n \cdot \int_{1/4}^{1/2} g(x)/(x+y)dx$). By dominated convergence,

$$0 = \int_{1/4}^{1/2} \left(\int_{1/4}^{1/2} \frac{g(x)}{x+y} dx \right)^2 dy,$$

which gives

$$\int_{1/4}^{1/2} \frac{g(x)}{x+y} dx = 0$$

for all $y \in [1/4, 1/2]$. Put $y = 1/4$:

$$0 = \int_{1/4}^{1/2} \frac{g(x)}{x + 1/4} dx,$$

which is a contradiction by the construction of g. Thus, there exists $h \in C_c^\infty(\mathbb{R})$ with support(h) $\subseteq [1/4, 1/2]$ so that the ϕ constructed is $C_c^\infty(\mathbb{R})$, with support in $[1/2, 1]$, $\int_{\mathbb{R}} \phi(x)dx \neq 0$ and $\int_{\mathbb{R}} x^k \phi(x)dx = 0$ for all $k = 1, \ldots, 2M-1$. Now change ϕ by $\phi / \int_{\mathbb{R}} \phi$ to have the sought ϕ. Then the existence of φ is fully justified.

Definition 6.4 *A row is a union $\mathcal{P} = \cup_k \mathcal{P}^k$ of normal trees $\mathcal{P}^k$ with tops $[\omega_0^k, I_0^k]$, where the I_0^k's are pairwise disjoint.*

Note that the union is finite, because we are considering $\mathcal{B}$ finite.

In the following lemma we extend Lemma 6.4 to a union of normal trees.

Lemma 6.5 *Let $\mathcal{P}$ be a row as above, let $\mathcal{P}'$ be a normal tree with top $[\omega_0', I_0']$, and suppose that, for each k, $I_0^k \subseteq I_0'$ and $\mathcal{P}^k, \mathcal{P}'$ are separated. Then $\|T^{\mathcal{P}'} T^{\mathcal{P}^*}\|_2 \le C_M \delta^M$ (any $M > 0$).*

Proof. We will prove

$$\sum_k |(T^{\mathcal{P}'^*} g, T^{\mathcal{P}^{k*}} f)| \le C_M \delta^M \|f\|_2 \|g\|_2. \tag{6.25}$$

We examine each term $(T^{\mathcal{P}'^*} g, T^{\mathcal{P}^{k*}} f)$. Write the tree $\mathcal{P}'$ as $\mathcal{P}' = \mathcal{P}_k' \cup \mathcal{P}_k'' \cup \mathcal{P}_k'''$, where

$$\mathcal{P}_k'' = \{[\omega', I'] \in \mathcal{P}' : |I'| > (\delta^{1000}/K^2)|I_0^k|\},$$

$$\mathcal{P}_k' = \{[\omega', I'] \in \mathcal{P}' : |I'| \le (\delta^{1000}/K^2)|I_0^k|,\ I' \subseteq I_0^k,\ d((I')^5, \partial I_0^k) > (\delta^{100}/K^2)|I_0^k|\},$$

$$\mathcal{P}_k''' = \{\text{all other } p' \in \mathcal{P}'\}.$$

Let us see that, if $\tilde{\mathcal{P}}$ is a normal tree with top $[\cdot, I_0^k]$, then

$$\text{support}(T^{\mathcal{P}_k'''^*} g) \cap \text{support}(T^{\tilde{\mathcal{P}}^*} f) = \emptyset, \quad \forall f, g \in L_r^2.$$

Pick $p''' = [\omega''', I'''] \in \mathcal{P}_k'''$ and $p' = [\omega, I'] \in \tilde{\mathcal{P}}$. By definition of the adjoint operator, it suffices to see that $(I''')^3 \cap (I')^3 = \emptyset$. Let us write first what I''' and I' verify. By definition of $\mathcal{P}_k'''$,

$$|I'''| \le \frac{\delta^{1000}}{K^2} |I_0^k| \quad \& \quad \left[I''' \not\subseteq I_0^k \text{ or } d((I''')^5, \partial I_0^k) \le \frac{\delta^{100}}{K^2} |I_0^k| \right].$$

The definition of normal tree implies

$$I' \subseteq I_0^k, \quad d((I')^5, \partial I_0^k) > \frac{\delta^{100}}{K^2}|I_0^k|.$$

Suppose that $d((I''')^5, \partial I_0^k) \leq (\delta^{100}/K^2)|I_0^k|$. This gives directly $(I')^5 \cap (I''')^5 = \emptyset$. Suppose that $d((I''')^5, \partial I_0^k) > (\delta^{100}/K^2)|I_0^k|$. Then $I''' \not\subseteq I_0^k$. Since $I_0^k \not\subseteq I'''$ (because $|I'''| < |I_0^k|$), we have $I_0^k \cap I''' = \emptyset$. In fact, $I_0^k \cap (I''')^5 = \emptyset$: as $d((I''')^5, \partial I_0^k) > (\delta^{100}/K^2)|I_0^k| > 0$, it is not possible for $(I''')^5$ to intersect I_0^k. On the other hand, as $\tilde{\mathcal{P}}$ is normal, we saw that $(I')^5 \subseteq I_0^k$. Hence, $(I')^5 \cap (I''')^5 = \emptyset$ again, and we are done (we have proved the stronger fact $(I')^5 \cap (I''')^5 = \emptyset$ rather than $(I''')^3 \cap (I')^3 = \emptyset$ because we will need it later on).

Write each term of the sum in (6.25) as

$$\begin{aligned}(T^{\mathcal{P}'^*} g, T^{\mathcal{P}^{k*}} f) &= (T^{\mathcal{P}_k'^*} g, T^{\mathcal{P}^{k*}} f) + (T^{\mathcal{P}_k''^*} g, T^{\mathcal{P}^{k*}} f) + (T^{\mathcal{P}_k'''^*} g, T^{\mathcal{P}^{k*}} f) \\ &= (T^{\mathcal{P}_k'^*} g, T^{\mathcal{P}^{k*}} f) + (T^{\mathcal{P}_k''^*} g, T^{\mathcal{P}^{k*}} f),\end{aligned}$$

where the last term disappears because $\mathcal{P}^k$ is a normal tree with top $[\omega_0^k, I_0^k]$. On the other hand, we can substitute f and g by $f\mathbb{1}_{I_0^k}$ and $g\mathbb{1}_{I_0^k}$, because in the definition of adjoint operator we integrate over $E(\text{pair})$, which is a subset of the second component of the top. Thus,

$$\begin{aligned}(T^{\mathcal{P}'^*} g, T^{\mathcal{P}^{k*}} f) &= (T^{\mathcal{P}'^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})) \\ &= (T^{\mathcal{P}_k'^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})) + (T^{\mathcal{P}_k''^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})).\end{aligned}$$

We estimate first $(T^{\mathcal{P}_k'^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k}))$. Note that $\mathcal{P}_k'$ is a tree with top $[\omega_k', I_0^k]$, for some ω_k' containing ω_0'. But $\mathcal{P}_k'$ is also a tree with top $[\omega_0', I_0']$. Notice that $\mathcal{P}_k'$, with top $[\omega_0', I_0']$, and $\mathcal{P}^k$ are separated trees (because $\mathcal{P}'$ and $\mathcal{P}^k$ are separated). The proof of Lemma 6.4 also applies in this case (the only problematic step is the proof of (6.20), since for it one uses the fact that the second components of the tops are equal, but the proof in this case is actually the same because $[\omega_k', I_0^k]$ is also a top for $\mathcal{P}_k'$). Hence,

$$|(T^{\mathcal{P}_k'^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k}))| \leq C_M \delta^M \|f\|_{L^2(I_0^k)} \|g\|_{L^2(I_0^k)}.$$

For each $\mathcal{P}^k$, take as in Lemma 6.4 a φ_k and an ϵ_k. Using the definition of ϵ_k, which gives $T^{\mathcal{P}^{k*}} f = \epsilon_k(f) + \varphi_k * (T^{\mathcal{P}^{k*}} f)$, we obtain:

$$\begin{aligned}(T^{\mathcal{P}_k''^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})) &= (T^{\mathcal{P}_k''^*}(g\mathbb{1}_{I_0^k}), \varphi_k * (T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k}))) + (T^{\mathcal{P}_k''^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) \\ &= (g\mathbb{1}_{I_0^k}, T^{\mathcal{P}_k''}(\varphi_k * (T^{\mathcal{P}^{k*}} f\mathbb{1}_{I_0^k}))) + (T^{\mathcal{P}_k''^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) \\ &= (g\mathbb{1}_{I_0^k}, T^{\mathcal{P}_k''}(\varphi_k * (T^{\mathcal{P}^{k*}} f\mathbb{1}_{I_0^k}))) + (T^{\mathcal{P}'^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) \\ &\quad -(T^{\mathcal{P}_k'^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) - (T^{\mathcal{P}_k'''^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})).\end{aligned}$$

Now, $(T^{\mathcal{P}_k'''^*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) = 0$. Indeed, to prove this, note that

$$(T^{\mathcal{P}_k'''^*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(g\mathbb{1}_{I_0^k})) = 0$$

because of the disjoint supports (proved above with $\tilde{\mathcal{P}} = \mathcal{P}^k$). On the other hand, if $d_k = \min\{|I| : [\omega, I] \in \mathcal{P}^k\}$ and $p = [\omega, I] \in \mathcal{P}^k$, then

$$\text{support}(\varphi_k * (T_p^*(g\mathbb{1}_{I_0^k}))) \subseteq [-\delta^{1/2}d_k, \delta^{1/2}d_k] + I^3 \subseteq [-|I|, |I|] + I^3 = I^5.$$

As $(I''')^3 \cap I^5 = \emptyset$ for all $[\cdot, I'''] \in \mathcal{P}_k'''$ (look at the above proof), we have

$$(T^{\mathcal{P}_k'''*}(g\mathbb{1}_{I_0^k}), \varphi_k * (T_p^*(g\mathbb{1}_{I_0^k}))) = 0.$$

This gives $(T^{\mathcal{P}_k'''*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) = 0$, as desired. Then

$$\begin{aligned}(T^{\mathcal{P}_k''*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})) &= (g\mathbb{1}_{I_0^k}, T^{\mathcal{P}_k''}(\varphi_k * (T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})))) + (T^{\mathcal{P}'*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) \\ &\quad - (T^{\mathcal{P}_k'*}(g\mathbb{1}_{I_0^k}), \epsilon_k(f\mathbb{1}_{I_0^k})) = A + B - C.\end{aligned}$$

Both B and C are easy to estimate:

$$|B| \leq \|T^{\mathcal{P}'*}(g\mathbb{1}_{I_0^k})\|_2 \|\epsilon_k(f\mathbb{1}_{I_0^k})\|_2 \leq C_M\delta^M \|f\|_{L^2(I_0^k)} \|g\|_{L^2(I_0^k)},$$

$$|C| \leq \|T^{\mathcal{P}_k'*}(g\mathbb{1}_{I_0^k})\|_2 \|\epsilon_k(f\mathbb{1}_{I_0^k})\|_2 \leq C_M\delta^M \|f\|_{L^2(I_0^k)} \|g\|_{L^2(I_0^k)},$$

where we have used Lemma 6.3 with $\delta = 1$ and (6.21). It remains to bound $|A|$. We will show that for every $F \in L_r^2$

$$|T^{\mathcal{P}_k''}(\varphi_k * F)(x)| \leq C_M\delta^M(\Phi_k * |F|)(x), \tag{6.26}$$

where

$$\Phi_k(y) = \frac{|I_0^k|\delta^{1000}/K^2}{y^2 + (|I_0^k|\delta^{1000}/K^2)^2}.$$

Put $F = T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k}) \in L_r^2$. Note that F lives in I_0^k, because $\mathcal{P}^k$ is a normal tree with top $[\omega_0^k, I_0^k]$. Then

$$\begin{aligned}|A| &\leq (|g|\mathbb{1}_{I_0^k}, |T^{\mathcal{P}_k''}(\varphi_k * (T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})))|) \leq \int_{\mathbb{R}} |g(x)|(\Phi_k * |F|)(x)\,dx \\ &= \int_{\mathbb{R}} |g(x)| \int_{I_0^k} \Phi_k(x-y)|F(y)|\,dy\,dx \underbrace{=}_{\Phi_k \text{ even}} \int_{I_0^k} \left(\int_{\mathbb{R}} \Phi_k(y-x)|g(x)|\,dx\right)|F(y)|\,dy \\ &\underbrace{\leq}_{\substack{\text{Remark}\\ 6.2}} C\int_{I_0^k} g^*(y)|F(y)|\,dy \leq C\|g^*\|_{L^2(I_0^k)} \|T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k})\|_2 \leq C_M\delta^M \|f\|_{L^2(I_0^k)} \|g^*\|_{L^2(I_0^k)},\end{aligned}$$

where in the last inequality we have used again Lemma 6.3 with $\delta = 1$.

From the estimates for A, B and C, we get:

$$|(T^{\mathcal{P}_k''*}(g\mathbb{1}_{I_0^k}), T^{\mathcal{P}^{k*}}(f\mathbb{1}_{I_0^k}))| \leq C_M\delta^M \|f\|_{L^2(I_0^k)}(\|g^*\|_{L^2(I_0^k)} + \|g\|_{L^2(I_0^k)}),$$

which also gives the bound for each addend of (6.25):

$$|(T^{\mathcal{P}'*}g, T^{\mathcal{P}^{k*}}f)| \leq C_M\delta^M \|f\|_{L^2(I_0^k)}(\|g^*\|_{L^2(I_0^k)} + \|g\|_{L^2(I_0^k)}).$$

Now, summing over k and using the fact that the I_0^k's are disjoint,

$$\begin{aligned}
|(T^{\mathcal{P}'^*}g, T^{\mathcal{P}^*}f)| &\leq C_M\delta^M \sum_k \|f\|_{L^2(I_0^k)}(\|g^*\|_{L^2(I_0^k)} + \|g\|_{L^2(I_0^k)}) \\
&\leq C_M\delta^M \left(\sum_k \|f\|^2_{L^2(I_0^k)}\right)^{\frac{1}{2}} \left[\left(\sum_k \|g^*\|^2_{L^2(I_0^k)}\right)^{\frac{1}{2}} + \left(\sum_k \|g\|^2_{L^2(I_0^k)}\right)^{\frac{1}{2}}\right] \\
&\leq C_M\delta^M\|f\|_2(\|g\|_2 + \|g^*\|_2) \leq C_M\delta^M\|f\|_2\|g\|_2,
\end{aligned}$$

where we have used the estimates for the maximal operators done in Lemma 6.2.

Thus, it remains to prove (6.26). Write

$$|T^{\mathcal{P}''_k}(\varphi_k * F)(x)| = \left| \sum_{\substack{[\omega', I'] \in \mathcal{P}''_k \text{ such that} \\ x\in E(\omega',I'),\ |I'|=2\pi\cdot 2^{-j}\geq(\delta^{1000}/K^2)|I_0^k|}} (e^{iN(x)\cdot}\psi_j(\cdot)) * (\varphi_k * F)(x)\right|.$$

It suffices to show that

$$\left| \sum_{\substack{[\omega', I'] \in \mathcal{P}''_k \text{ such that} \\ x\in E(\omega',I'),\ |I'|=2\pi\cdot 2^{-j}\geq(\delta^{1000}/K^2)|I_0^k|}} (e^{iN(x)\cdot}\psi_j(\cdot)) * \varphi_k(z)\right| \tag{6.27}$$

is bounded by $C_M\delta^M\Phi_k(z)$ for all $z \in \mathbb{R}$. We distinguish two cases:

- Case $|z| \leq |I_0^k|\delta^{1000}/K^2$. In this case $K^2/(\delta^{1000}|I_0^k|) \leq 2\Phi_k(z)$. Using the fact that $|\psi_j| \leq 2^j$, support$(\varphi_k) \subseteq [-d_k\delta^{1/2}, d_k\delta^{1/2}]$ and $|\varphi_k(t)| \leq C_M|t|^{2M-1}$, we have:

$$\begin{aligned}
(6.27) \leq \sum_{2^j\leq 2\pi\frac{K^2}{\delta^{1000}|I_0^k|}} 2^j \int_{-\delta^{1/2}}^{\delta^{1/2}} |\varphi_k(t)|\,dt &\leq C_M \sum_{2^j\leq 2\pi\frac{K^2}{\delta^{1000}|I_0^k|}} 2^j \int_{-\delta^{1/2}}^{\delta^{1/2}} |t|^{2M-1}\,dt \\
= C_M\delta^M \sum_{2^j\leq 2\pi\frac{K^2}{\delta^{1000}|I_0^k|}} 2^j = C_M\delta^M \frac{K^2}{\delta^{1000}|I_0^k|} &\leq C_M\delta^M\Phi_k(z).
\end{aligned}$$

- Case $|z| \geq |I_0^k|\delta^{1000}/K^2$. In this case, $|I_0^k|\delta^{1000}/(z^2K^2) \leq 2\Phi_k(z)$. Fix $[\omega', I'] \in \mathcal{P}''_k$ such that $x \in E(\omega', I')$, $|I'| = 2\pi\cdot 2^{-j} \geq \delta^{1000}|I_0^k|/K^2$. We have:

$$\begin{aligned}
|(e^{iN(x)\cdot}\psi_j(\cdot)) * \varphi_k(z)| &\leq 2^j \int_{-d_k\delta^{1/2}}^{d_k\delta^{1/2}} |\varphi_k(t)|\,dt \leq C_M 2^j \int_{-d_k\delta^{1/2}}^{d_k\delta^{1/2}} |t|^{2M-1+4000}\,dt \\
&\underbrace{\leq}_{d_k\leq|I_0^k|} C_M|I_0^k|^{2M+4000}\delta^{M+2000}2^j.
\end{aligned}$$

  Due to the fact that $\mathcal{P}'$ is normal, we can upper-bound K^2: by definition of normal tree, $|I'| \leq \delta^{1000}|I'_0|/K^2$, whence $K^2 \leq \delta^{1000}|I'_0|/|I'| = C2^j\delta^{1000}|I'_0|$. On the other hand, support$((e^{iN(x)\cdot}\psi_j(\cdot)) * \varphi_k) \subseteq$ support(ψ_j) + support(φ_k). We know that support$(\psi_j) \subseteq \{t : |t| \leq 2\pi\cdot 2^{-j}\}$ and support$(\varphi_k) \subseteq \{t : |t| \leq d_k\delta^{1/2}\}$. As $\mathcal{P}^k$ is normal, $d_k \leq \delta^{1000}|I_0^k|/K^2 \leq 2^{-j}$, so support$((e^{iN(x)\cdot}\psi_j(\cdot)) * \varphi_k) \subseteq \{t : |t| \leq$

$2\pi\cdot 2^{-j+1}\}$. Thus, we may assume that $|z|\le C\cdot 2^{-j}$, consequently $2^{2j}\le C/z^2$. From $K^2\le C2^j\delta^{1000}|I_0'|$ we obtain $K^4\le C2^{2j}\delta^{2000}|I_0'|^2\le C\delta^{2000}|I_0'|^2/z^2\le C\delta^{2000}/z^2$ (because $|I_0'|\le 2\pi$). Therefore $K^2\le C\delta^{2000}/(z^2K^2)$. We can conclude:

$$\begin{aligned}(6.27) &\le C_M|I_0^k|^{2M+4000}\delta^{M+2000}\sum_{2^j\le C\frac{\delta^{2000}}{z^2K^2}\frac{1}{\delta^{1000}|I_0^k|}} 2^j\\ &= C_M|I_0^k|^{2M+4000}\delta^{M+2000}\frac{\delta^{2000}}{z^2K^2}\frac{1}{\delta^{1000}|I_0^k|}\le C_M\delta^M\frac{\delta^{1000}|I_0^k|}{z^2K^2}\le C_M\delta^M\Phi_k(z).\end{aligned}$$

□

Corollary 6.2 *Let $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2\cup\cdots$ and $\mathcal{P}'=\mathcal{P}_1'\cup\mathcal{P}_2'\cup\cdots$ be rows, with tops $[\omega_k^0,I_k^0]$ for $\mathcal{P}_k$ and $[\omega_k^1,I_k^1]$ for $\mathcal{P}_k'$. Suppose that each I_k^0 is contained in some $I_{k'}^1$, with $\mathcal{P}_k$ and $\mathcal{P}_{k'}'$ separated. Then*

$$\|T^{\mathcal{P}'}T^{\mathcal{P}*}\|_2\le C_M\delta^M,\quad \forall M>0.$$

Proof. Let $f\in L_r^2$. By definition of row, the I_k^1's are pairwise disjoint, therefore we can write

$$f=\sum_k f_k+F,$$

where $f_k=f\cdot\mathbb{1}_{I_k^1}$ and $F=f-\sum_k f_k$. Note that $\|f\|_2^2=\sum_k\|f_k\|_2^2+\|F\|_2^2$.

At a first step, note that, given $g\in L_r^2$, $T^{\mathcal{P}_k'}g$ has its support contained in I_k^1, by definition of $T^{\mathcal{P}_k'}g$. Since $\{I_k^1\}_k$ are pairwise disjoint, $(T^{\mathcal{P}_k'}g,T^{\mathcal{P}_l'}g)=0$ for $k\ne l$. We can then apply Pythagoras:

$$\|T^{\mathcal{P}'}T^{\mathcal{P}*}f\|_2^2=\left\|\sum_k T^{\mathcal{P}_k'}T^{\mathcal{P}*}f\right\|_2^2=\sum_k\|T^{\mathcal{P}_k'}T^{\mathcal{P}*}f\|_2^2.$$

Let $Q_k=\cup\{\mathcal{P}_j:\ I_j^0\subseteq I_k^1\}$. By hypothesis,

$$\mathcal{P}=\bigcupdot_k Q_k$$

($\cupdot$ denotes disjoint union). Note that, maybe, $Q_k=\emptyset$ for some k.

If $Q_k\ne\emptyset$, then $T^{\mathcal{P}_k'}T^{\mathcal{P}*}=T^{\mathcal{P}_k'}T^{Q_k*}$ (indeed, if $l\ne k$, then, given $g\in L_r^2$, $T^{Q_l*}g$ lives in I_l^1 by normality and definition of Q_l, and $T^{\mathcal{P}_k'*}g$ lives in I_k^1 by normality, therefore $T^{\mathcal{P}_k'}T^{Q_l*}=0$ since $I_l^1\cap I_k^1=\emptyset$). Note that, if $Q_k=\emptyset$, then $T^{\mathcal{P}_k'}T^{\mathcal{P}*}=0$. We can write

$$\|T^{\mathcal{P}'}T^{\mathcal{P}*}f\|_2^2=\sum_k\|T^{\mathcal{P}_k'}T^{\mathcal{P}*}f\|_2^2=\sum_k\|T^{\mathcal{P}_k'}T^{Q_k*}f\|_2^2.$$

Now use the definition of the adjoint operator:

$$\begin{aligned}T^{Q_k*}f&=\sum_{p\in Q_k}T_p^*f=\sum_{\substack{p=[\omega,I]\in Q_k\\|I|=2\pi\cdot 2^{-m}}}\int_{E(p)}e^{-iN(y)(y-x)}\psi_m(y-x)f(y)\,dy\\ &\underbrace{=}_{E(p)\subseteq I\subseteq I_k^1}\sum_{\substack{p=[\omega,I]\in Q_k\\|I|=2\pi\cdot 2^{-m}}}\int_{E(p)}e^{-iN(y)(y-x)}\psi_m(y-x)\underbrace{f(y)\cdot\mathbb{1}_{I_k^1}(y)}_{f_k(y)}\,dy=T^{Q_k*}f_k.\end{aligned}$$

Hence,

$$\|T^{\mathcal{P}'}T^{\mathcal{P}*}f\|_2^2 = \sum_k \|T^{\mathcal{P}'_k}T^{Q_k*}f_k\|_2^2.$$

By Lemma 6.5 (thanks to the construction of Q_k the hypotheses of Lemma 6.5 are satisfied), $\|T^{\mathcal{P}'_k}T^{Q_k*}f_k\|_2 \leq C_M\delta^M\|f_k\|_2$. Now sum over k:

$$\|T^{\mathcal{P}'}T^{\mathcal{P}*}f\|_2^2 \leq C_M\delta^{2M}\sum_k \|f_k\|_2^2 \leq C_M\delta^{2M}\|f\|_2^2,$$

and we are done.

□

Corollary 6.3 *Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots$ be a row, with tops $[\omega_k^0, I_k^0]$ for $\mathcal{P}^k$. If $A(p) \leq \delta$ for all $p \in \mathcal{P}$, then $\|T^{\mathcal{P}}\|_2 \leq C\delta^{1/2}$.*

Proof. Write, similarly to the previous corollary, $f = \sum_k f_k + F$, where $f_k = f \cdot \mathbb{1}_{I_k^0}$ and $F = f - \sum_k f_k$ (we can do this because $\{I_k^0\}_k$ are pairwise disjoint). As $\mathcal{P}_k$ is a normal tree, $T^{\mathcal{P}_k}{}^*f$ has its support in I_k^0, so $T^{\mathcal{P}_k}{}^*f$ and $T^{\mathcal{P}_l}{}^*f$ have disjoint supports for $k \neq l$, which implies their orthogonality. Then, by Pythagoras theorem,

$$\|T^{\mathcal{P}^*}f\|_2^2 = \left\|\sum_k T^{\mathcal{P}_k}{}^*f\right\|_2^2 = \sum_k \|T^{\mathcal{P}_k}{}^*f\|_2^2.$$

We have

$$\begin{aligned} T^{\mathcal{P}_k}{}^*f(x) &= \sum_{\substack{p=[\omega,I]\in\mathcal{P}_k \\ |I|=2\pi\cdot 2^{-j}}} \int_{E(p)} e^{-iN(y)(y-x)}\psi_j(y-x)f(y)\,dy \\ &\underbrace{=}_{E(p)\subseteq I\subseteq I_k^0} \sum_{\substack{p=[\omega,I]\in\mathcal{P}_k \\ |I|=2\pi\cdot 2^{-j}}} \int_{E(p)} e^{-iN(y)(y-x)}\psi_j(y-x)\underbrace{f(y)\cdot\mathbb{1}_{I_k^0}(y)}_{f_k(y)}\,dy = T^{\mathcal{P}_k}{}^*f_k(x). \end{aligned}$$

Thus,

$$\|T^{\mathcal{P}^*}f\|_2^2 = \sum_k \|T^{\mathcal{P}_k}{}^*f_k\|_2^2.$$

Now apply Lemma 6.3 and Corollary 6.1: $\|T^{\mathcal{P}_k}{}^*f_k\|_2 \leq C\delta^{1/2}\|f_k\|_2$. We arrive at

$$\|T^{\mathcal{P}^*}f\|_2^2 \leq C\delta\sum_k \|f_k\|_2^2 \leq C\delta\|f\|_2^2,$$

as wanted.

□

Remark 6.6 *From Corollary 6.3, we see that (5.5) holds for rows.*

Lemma 6.6 (Main Lemma) *Let $\{\mathcal{P}_j\}_j$ be a family of trees with tops $[\omega_j^0, I_j^0]$. Assume that $[\omega_j^0, I_j^0] \in \mathcal{P}_j$ for each j and that:*

(a) $A(p) \leq \delta$ for all $p \in \mathcal{P}_j$ and for all j.

(b) Two pairs belonging to two different trees are not comparable.

(c) No point of $[0,2\pi]$ *belongs to more than* $K\delta^{-20}$ *of the* I_j^0*'s.*
Then there is a set $F\subseteq[0,2\pi]$ *with* $|F|\le C\delta^{80}/K$ *such that*

$$\left\|\sum_j T^{\mathcal{P}_j}f\right\|_{L^2(F^c)}\le C_\eta(\log K)\delta^{\frac14-\eta}\|f\|_2$$

for all $f\in L_r^2$ *and* $\eta>0$.

Proof. We will take

$$F=\bigcup_j\underbrace{\{x\in I_j^0:\,d(x,\partial I_j^0)\le 3\frac{\delta^{100}}{K^2}|I_j^0|\}}_{F_j}.$$

To see that $|F|\le C\delta^{80}/K$ we will use (c):

$$\begin{aligned}|F|\le\sum_j|F_j|&\le 2\cdot 3\frac{\delta^{100}}{K^2}\sum_j|I_j^0|=6\frac{\delta^{100}}{K^2}\sum_j\int_0^{2\pi}\mathbb{1}_{I_j^0}(x)\,dx\\&=6\frac{\delta^{100}}{K^2}\int_0^{2\pi}\left(\sum_j\mathbb{1}_{I_j^0}(x)\right)dx=6\frac{\delta^{100}}{K^2}\int_0^{2\pi}(\text{number of }I_j^0\text{ containing }x)\,dx\\&\underbrace{\le}_{\text{(c)}}6\frac{\delta^{100}}{K^2}K\delta^{-20}2\pi\le C\frac{\delta^{80}}{K},\end{aligned}$$

as desired.

Call $\mathcal{P}=\cup_j\mathcal{P}_j$. Fix $M=\log(K^{10000}/\delta^{10000})$, where $\log=\log_2$. We claim that $M\le C_\epsilon(\log K)\delta^{-\epsilon}$ for all $0<\epsilon<1$, where $C_\epsilon=(1/\epsilon)\cdot 10000$. Indeed, first of all rewrite the desired inequality as follows:

$$\log\left(\frac{K}{\delta}\right)\le\frac{1}{\epsilon}(\log K)\delta^{-\epsilon}\Leftrightarrow\log\left(\frac1\delta\right)\le\left(\frac1\epsilon\left(\frac1\delta\right)^\epsilon-1\right)\log K\Leftrightarrow\frac1\delta\le K^{\left(\frac1\epsilon\left(\frac1\delta\right)^\epsilon-1\right)}.$$

Define $F(a)=K^{\frac1\epsilon a^\epsilon-1}-a$. We have $F'(a)=a^{\epsilon-1}(\log K)K^{\frac1\epsilon a^\epsilon-1}-1$ and $F''(a)=(\log K)[(\epsilon-1)a^{\epsilon-2}K^{\frac1\epsilon a^\epsilon-1}+(a^{\epsilon-1})^2(\log K)K^{\frac1\epsilon a^\epsilon-1}]$. Note that $F''(a)\ge0$ if and only if $a^\epsilon\log K\ge1-\epsilon$, which is true for $a\ge1$. Then F' is increasing in $[1,\infty[$, so $F'(a)\ge F'(1)=(\log K)K^{\frac1\epsilon-1}-1\ge\log K-1\ge0$, that is, F is increasing in $[1,\infty[$ and $F(a)\ge F(1)=K^{\frac1\epsilon-1}-1\ge0$ for all $a\ge1$. Put $a=1/\delta$ to have finally $M\le C_\epsilon(\log K)\delta^{-\epsilon}$.

Let

$$\mathcal{P}^+=\{p\in\mathcal{P}:\text{ there are no ascending sequences }p\precneqq p_1\precneqq\ldots\precneqq p_M,\text{ all }p_j\in\mathcal{P}\}.$$

Put

$$\mathcal{P}^+_{(1)}=\{p\in\mathcal{P}:\text{ there is no }p_1\in\mathcal{P}\text{ with }p\precneqq p_1\},$$

$$\begin{aligned}\mathcal{P}^+_{(i)}=\{p\in\mathcal{P}:\ &\text{there are }p_1,\ldots p_{i-1}\in\mathcal{P}\text{ with }p\precneqq p_1\precneqq\ldots\precneqq p_{i-1},\text{ and for}\\&\text{every such a sequence, }p_{i-1}\text{ is not strictly smaller than }p\text{ for any }p\in\mathcal{P}\},\end{aligned}$$

for $i = 2, \ldots, M$. Then

$$\mathcal{P}^+ = \dot{\bigcup_{i=1}^{M}} \mathcal{P}^+_{(i)},$$

and, moreover, no two distinct pairs living in $\mathcal{P}^+_{(i)}$ are comparable for all $i = 1, \ldots, M$. Indeed, suppose that $p \lneqq p'$ for $p, p' \in \mathcal{P}^+_{(i)}$. By definition of $\mathcal{P}^+_{(i)}$, there are $p'_1, \ldots, p'_{i-1} \in \mathcal{P}$ with $p' \lneqq p'_1 \lneqq \ldots \lneqq p'_{i-1}$, so $p \lneqq p' \lneqq p'_1 \lneqq \ldots \lneqq p'_{i-1}$ contradicts $p \in \mathcal{P}^+_{(i)}$.

Since $A(p) \leq \delta$ for all $p \in \mathcal{P}^+_{(i)}$ (hypothesis (a)), by Lemma 6.2

$$\|T^{\mathcal{P}^+_{(i)}}\|_2 \leq C_{\eta'} \delta^{\frac{1}{4} - \eta'}$$

for all $\eta' > 0$. Then

$$\|T^{\mathcal{P}^+}\|_2 \leq \sum_{i=1}^{M} \|T^{\mathcal{P}^+_{(i)}}\|_2 \leq M \cdot C_{\eta'} \delta^{\frac{1}{4} - \eta'} \leq C_\epsilon C_{\eta'} (\log K) \delta^{\frac{1}{4} - \eta' - \epsilon},$$

for all $\eta' > 0$ and $0 < \epsilon < 1$. Given any $\eta > 0$, choose $\eta' = \eta/2$ and $\epsilon < \min\{1, \eta/2\}$ to obtain

$$\|T^{\mathcal{P}^+}\|_2 \leq C_\eta (\log K) \delta^{\frac{1}{4} - \eta}.$$

Thus, it is enough to prove the lemma for $\mathcal{P}^0 = \mathcal{P} \backslash \mathcal{P}^+$. Similarly to $\mathcal{P}^+$, define

$$\mathcal{P}^- = \{p \in \mathcal{P} : \text{there are no ascending sequences } p_1 \lneqq \ldots \lneqq p_M \lneqq p, \text{ all } p_j \in \mathcal{P}^0\}.$$

As above,

$$\|T^{\mathcal{P}^-}\|_2 \leq C_\eta (\log K) \delta^{\frac{1}{4} - \eta}.$$

Hence, it is enough to prove the lemma for $\mathcal{P}^\sharp = \mathcal{P}^0 \backslash \mathcal{P}^-$. Write $\mathcal{P}^\sharp = \cup_j \mathcal{P}^0_j$, where $\mathcal{P}^0_j = \mathcal{P}_j \cap \mathcal{P}^\sharp$ is a tree with top $[\omega^0_j, I^0_j]$. Let us study $\mathcal{P}^0_j$. We claim that:

(i) $[\omega, I] \in \mathcal{P}^0_j$ implies $|I| \leq (\delta/K)^{10000} |I^0_j|$;

(ii) For $j \neq j'$, $\mathcal{P}^0_j$ and $\mathcal{P}^0_{j'}$ are separated. In fact, (α) and (β) from the definition of separated trees hold with δ replaced by $K^{-4000}\delta$.

Proof of (i): if $[\omega, I] \in \mathcal{P}^0_j$, then $[\omega, I] \notin \mathcal{P}^+$, so $[\omega, I] < p_1 \lneqq \ldots \lneqq p_M = [\omega_M, I_M]$ for certain $p_i \in \mathcal{P}$. By hypothesis (b), all p_i's have to belong to the same tree, say $\mathcal{P}_j$, so by definition of top $[\omega_M, I_M] < [\omega^0_j, I^0_j]$. If we denote by $[\omega_i, I_i] = p_i$ for $i = 1, \ldots, M$, we have $|I| \leq (1/2)|I_1| \leq \ldots \leq (1/2^M)|I^0_j| \leq (\delta/K)^{10000} |I^0_j|$, as wanted.

Proof of (ii): We will prove condition (α) of the definition of separated trees. Suppose that $[\omega, I] \in \mathcal{P}^0_j$, $I \subseteq I^0_{j'}$. Since $[\omega, I] \notin \mathcal{P}^-$, we have $[\omega_1, I_1] = p_1 \lneqq p_2 \lneqq \ldots \lneqq p_M \lneqq [\omega, I]$ for certain $p_i \in \mathcal{P}$. Again, by hypothesis (b), all p_i's must live in the same tree, and it has to be $\mathcal{P}_j$ because $\mathcal{P}^0_j \subseteq \mathcal{P}_j$. We have $I_1 \subseteq I \subseteq I^0_{j'}$. As $[\omega_1, I_1] \in \mathcal{P}_j$ and $[\omega^0_{j'}, I^0_{j'}] \in \mathcal{P}_{j'}$ and no two pairs from distinct trees are comparable, necessarily $\omega_1 \cap \omega^0_{j'} = \emptyset$. We have $\omega \subsetneqq \omega_M \subsetneqq \ldots \subsetneqq \omega_1$, so by Lemma 4.2, $d(\omega, \omega^0_{j'}) \geq 2^{(M-1)/2-2} |\omega| = 2^{-5/2} (K/\delta)^{5000} |\omega| \geq (K/\delta)^{4000} |\omega|$, and (ii) is proved.

Decompose $\mathcal{P}^0_j = \mathcal{P}^\sharp_j \cup \mathcal{P}^\flat_j$, where

$$\mathcal{P}^\sharp_j = \{[\omega, I] \in \mathcal{P}^0_j : I \not\subseteq F_j\}$$

and

$$\mathcal{P}_j^\flat = \{[\omega, I] \in \mathcal{P}_j^0 : I \subseteq F_j\}.$$

Given $f \in L^2_r$, $T^{\mathcal{P}_j^\flat} f$ is supported in F_j, therefore

$$T^{\mathcal{P}^\sharp} f = \sum_j T^{\mathcal{P}_j^\sharp} f \text{ on } F^c,$$

so the lemma reduces to proving

$$\left\| \sum_j T^{\mathcal{P}_j^\sharp} \right\|_2 \leq C_\eta (\log K) \delta^{\frac{1}{4}-\eta}$$

for all $\eta > 0$.

Note that $\mathcal{P}_j^\sharp$ is a normal tree: given $[\omega, I] \in \mathcal{P}_j^\sharp$, by (i) we know $|I| \leq \delta^{10000}/K^{10000}|I_j^0|$, and since $I \not\subseteq F_j$, $d(I, \partial I_0) > 3(\delta^{100}/K^2)|I_j^0|$.

Take $\cup_j \mathcal{P}_j^\sharp$ and decompose it as a union of at most $K\delta^{-20}$ rows, $R_1, \ldots, R_{K\delta^{-20}}$. Let us see that this decomposition does exist. Let $I^1, I^2, \ldots$ be the maximal dyadic intervals among $\{I_j^0\}_j$. For each I^s, pick $[\omega_{j(s)}^0, I_{j(s)}^0]$ with $I_{j(s)}^0 = I^s$. Set $R_1 = \cup_s \mathcal{P}_{j(s)}^\sharp$. This R_1 is indeed a row: it is a union of normal trees with $\{I^s\}_s$ disjoint by maximality. Delete all the $\mathcal{P}_{j(s)}^\sharp$'s from $\{\mathcal{P}_j^\sharp\}_j$, and perform the same procedure to construct R_2. The maximal dyadic intervals taken to form R_2 are contained in the previous $I^1, I^2, \ldots$, therefore, by (c), we will not use more than $K\delta^{-20}$ steps to construct the R_i's. Denote $R_i = \mathcal{P}_1^{(i)} \cup \mathcal{P}_2^{(i)} \cup \ldots$, where $\mathcal{P}_k^{(i)}$ is a normal tree with top $[\omega_k^{(i)}, I_k^{(i)}]$.

As

$$\left\| \sum_j T^{\mathcal{P}_j^\sharp} \right\|_2 = \left\| \sum_i T^{R_i} \right\|_2,$$

we prepare to apply the Orthogonality Lemma to the operators T^{R_i}:

(A) $\|T^{R_i}\|_2 \leq C\delta^{1/2}$ by hypothesis (a) and Corollary 6.3.

(B) $\|T^{R_i} T^{R_{i'}*}\|_2 \leq C_q K^{-q} \delta^q$ for $i > i'$ and $q > 0$ by (ii) and Corollary 6.2, because each $I_k^{(i)}$ is contained in some $I_l^{(i')}$ by construction via maximal intervals. Since for any bounded operators H and G on a Hilbert space it holds $(G \circ H)^* = H^* \circ G^*$, we have $\|T^{R_i} T^{R_{i'}*}\|_2 \leq C_q K^{-q} \delta^q$ for $i \neq i'$. Note that $|i - i'|^2 + 1 \leq K^2\delta^{-40} + 1 \leq 2K^2\delta^{-40}$, so $K^{-q}\delta^q \leq \delta/(2K^2\delta^{-40}) \leq \delta/(|i-i'|^2+1)$ taking $q \geq 41$. Then condition (b) of the Orthogonality Lemma holds.

(C) $T^{R_i*} T^{R_{i'}} = 0$ for $i \neq i'$, because $T^{R_i} f$ and $T^{R_{i'}} g$ have disjoint supports for $f, g \in L^2_r$: if $p \in R_i$ and $p' \in R_{i'}$, then $p \in \mathcal{P}_j^\sharp \subseteq R_i$ and $p' \in \mathcal{P}_{j'}^\sharp \subseteq R_{i'}$, certain $j \neq j'$ (distinct by construction of the R_k's). By hypothesis (b), p and p' are not comparable, therefore $E(p) \cap E(p') = \emptyset$. Thus, condition (a) of the Orthogonality Lemma holds.

By the Orthogonality Lemma,

$$\left\| \sum_j T^{\mathcal{P}_j^\sharp} \right\|_2 = \left\| \sum_i T^{R_i} \right\|_2 \leq C\delta^{1/2} \leq C\delta^{1/4} \leq C\delta^{1/4} (\log K) \delta^{-\eta}$$

for every $\eta > 0$. This finishes the proof of the Main Lemma.

$\square$

Corollary 6.4 *Let $\mathcal{P}$ be a set of pairs. Suppose that:*

(a) $A(p) \leq \delta$ for all $p \in \mathcal{P}$.

(b) If $p, p'' \in \mathcal{P}$ and $p < p' < p''$, with p' admissible, then $p' \in \mathcal{P}$.

(c) If $p, p', p'' \in \mathcal{P}$ and $p < p'$, $p < p''$, then $p' < p''$ or $p'' < p'$.

(d) For any $x \in [0, 2\pi]$, there are at most $K\delta^{-20}$ mutually incomparable $[\omega_i, I_i] \in \mathcal{P}$ with $x \in I_i$.

Then there is a set $F \subseteq [0, 2\pi]$ with $|F| \leq C\delta^{80}/K$ and

$$\|T^{\mathcal{P}} f\|_{L^2(F^c)} \leq C_\eta (\log K) \delta^{\frac{1}{4}-\eta} \|f\|_2$$

for all $f \in L^2_r$ and $\eta > 0$.

Proof. Let $\{[\omega_j^0, I_j^0]\}_j$ be the maximal pairs in $\mathcal{P}$, and let $\mathcal{P}_j = \{p \in \mathcal{P} : p < [\omega_j^0, I_j^0]\}$. By hypothesis (b), $\mathcal{P}_j$ is a tree with top $[\omega_j^0, I_j^0]$. Moreover, $[\omega_j^0, I_j^0] \in \mathcal{P}_j$.

Note that $\mathcal{P} = \cup_j \mathcal{P}_j$. Indeed, suppose that there is a $p \in \mathcal{P}$ with $p \nleq [\omega_j^0, I_j^0]$ for all j. Then p is not maximal (if it were, it would be an $[\omega_j^0, I_j^0]$), therefore there is a $p' \in \mathcal{P}$ with $p \lneqq p'$. This p' cannot be maximal by supposition, so there is another another p'' with $p \lneqq p' \lneqq p''$. Continuing in this way, and by the finiteness of the number of pairs, we arrive at $p \lneqq$ some maximal pair, which contradicts our supposition.

Suppose that $p \in \mathcal{P}_j$ and $p' \in \mathcal{P}_{j'}$ with $j \neq j'$ and $p < p'$. Then $p < p' < [\omega_{j'}^0, I_{j'}^0]$ and $p < [\omega_j^0, I_j^0]$, so by (c) $[\omega_{j'}^0, I_{j'}^0] < [\omega_j^0, I_j^0]$ or $[\omega_j^0, I_j^0] < [\omega_{j'}^0, I_{j'}^0]$, which is impossible by maximality. Hence, no two pairs coming from two distinct trees are comparable.

Thus, the union of trees $\mathcal{P}$ satisfies (a) and (b) from the Main Lemma. For (c), suppose by contradiction that $x \in [0, 2\pi]$ belongs to more than $K\delta^{-20}$ of the I_j^0's. Then $\{[\omega_j^0, I_j^0]\}_j$ are mutually incomparable and x belongs to more than $K\delta^{-20}$ of the I_j^0's. This contradicts hypothesis (d).

Hence, $\mathcal{P}$ satisfies all the hypotheses from the Main Lemma, so as a consequence

$$\|T^{\mathcal{P}} f\|_{L^2(F^c)} = \left\| \sum_j T^{\mathcal{P}_j} f \right\|_{L^2(F^c)} \leq C_\eta (\log K) \delta^{\frac{1}{4}-\eta} \|f\|_2$$

for every $f \in L^2_r$ and $\eta > 0$.

$\square$

Definition 6.5 *A set $\mathcal{P}$ satisfying (a)-(d) above is called a forest.*

An example of a forest is the tree drawn immediately after the definition of tree with $N \equiv 70$ in $[0, 2\pi]$, $\delta = 1/2$ and $K = 100$. Indeed, (b) and (c) are directly checked, (d) holds because $K\delta^{-20} = 104,857,600$ is really big, and (a) also holds because $A(p) = 0$, since $E(p) = \emptyset$ for every p of our picture due to the fact that $N \equiv 70$.

Chapter 7

Proof of the pointwise convergence

Our goal is to prove (4.1): $\|Tf\|_1 \le C\|f\|_2$ for all $f \in L^2_r$. Decompose, more or less as in Chapter 5, $\mathcal{B} = \cup_n \mathcal{P}_n$, where

$$\mathcal{P}_n = \{p \in \mathcal{B} : 2^{-n-1} < A(p) \le 2^{-n}\}.$$

Then $T = \sum_n T^{\mathcal{P}_n}$ (the sum is finite because $\mathcal{B}$ is finite).

Let $\{\bar{p}_k = [\bar{\omega}_k, \bar{I}_k]\}_k$ be the set of maximal pairs with

$$\frac{|E(\omega, I)|}{|I|} \ge 2^{-n-1}. \tag{7.1}$$

Denote, similarly to the Main Lemma,

$$\mathcal{P}_n^+ = \{p \in \mathcal{P}_n : \text{there are no ascending chains } p \gneqq p_1 \gneqq \ldots \gneqq p_{n+6},\ \text{all } p_i \in \mathcal{P}_n\}.$$

Similarly to the Main Lemma, we set

$$\mathcal{P}_{n1}^+ = \{p \in \mathcal{P}_n : \text{there is no } p_1 \in \mathcal{P}_n \text{ with } p \gneqq p_1\},$$

$$\mathcal{P}_{ni}^+ = \{p \in \mathcal{P}_n : \text{there are } p_1, \ldots, p_{i-1} \in \mathcal{P}_n \text{ with } p \gneqq p_1 \gneqq \ldots \gneqq p_{i-1}, \text{ and for every such a sequence, } p_{i-1} \text{ is not strictly smaller than } p \text{ for any } p \in \mathcal{P}_n\},$$

for $i = 2, \ldots, n+6$. We have $\mathcal{P}_n^+$ expressed as a disjoint union: $\mathcal{P}_n^+ = \mathcal{P}_{n1}^+ \cup \mathcal{P}_{n2}^+ \cup \ldots \cup \mathcal{P}_{n(n+6)}^+$. Moreover, as in the Main Lemma, no two pairs from the same $\mathcal{P}_{ni}^+$ are comparable, for all $i = 1, \ldots, n+6$. Since $A(p) \le 2^{-n}$, by Lemma 6.2

$$\|T^{\mathcal{P}_{ni}^+}\|_2 \le C_\eta (2^{-n})^{\frac{1}{4}-\eta} = C_\eta 2^{-n\left(\frac{1}{4}-\eta\right)}$$

for every $\eta > 0$. Then

$$\|T^{\mathcal{P}_n^+}\|_2 \le \sum_{i=1}^{n+6} \|T^{\mathcal{P}_{ni}^+}\|_2 \le C_\eta (n+6) 2^{-n\left(\frac{1}{4}-\eta\right)}. \tag{7.2}$$

Now consider $\mathcal{P}_n^0 = \mathcal{P}_n \backslash \mathcal{P}_n^+$. We claim that every $p = [\omega, I] \in \mathcal{P}_n^0$ satisfies $p < \bar{p}_j$ for some j. The proof of the claim is presented in what follows. First, let $p = [\omega, I] \in \mathcal{P}_n$. As $A(p) > 2^{-n-1}$ by definition of $\mathcal{P}_n$, there is a $p' = [\omega', I']$ with $I \subseteq I'$ and

$$2^{-n-1} < \frac{|E(\omega', I')|}{|I'|} \left(\frac{d(\omega, \omega') + |\omega|}{|\omega|}\right)^{-2000}.$$

Since $(d(\omega,\omega')+|\omega|)/|\omega|\geq 1$, we have

$$2^{-n-1}<\frac{|E(\omega',I')|}{|I'|}\left(\frac{d(\omega,\omega')+|\omega|}{|\omega|}\right)^{-2000}\leq\frac{|E(\omega',I')|}{|I'|}.$$

Also,

$$\left(\frac{d(\omega,\omega')+|\omega|}{|\omega|}\right)^{2000}<2^{n+1}\frac{|E(\omega',I')|}{|I'|}\leq 2^{n+1},$$

therefore $d(\omega,\omega')<(2^{(n+1)/2000}-1)|\omega|$. Notice that p' must satisfy $p'<\bar{p}_j$ for some j. Otherwise, in particular $p'\neq\bar{p}_j$ for all j, that is, p' is not a maximal pair verifying (7.1), so there exists p'' satisfying (7.1) and $p'\lneqq p''$; if p'' is not maximal we can continue and by finiteness of the set of pairs the process will end up at some moment, which says that p' is strictly less than a maximal pair satisfying (7.1), and this is a contradiction. We have $I\subseteq I'\subseteq\bar{I}_j$ and $\bar{\omega}_j\subseteq\omega'$. This second condition gives us $d(\omega,\bar{\omega}_j)\leq d(\omega,\omega')+|\omega'|\leq d(\omega,\omega')+|\omega|\leq 2^{(n+1)/2000}|\omega|$. Now let $p=[\omega,I]\in\mathcal{P}_n^0$. This condition says $p\notin\mathcal{P}_n^+$, that is, there is a chain $p\lneqq p_1\lneqq p_2\lneqq\ldots\lneqq p_{n+6}=[\omega_{n+6},I_{n+6}]$, certain $p_i\in\mathcal{P}_n$. As $p_{n+6}\in\mathcal{P}_n$, the same procedure above can be applied to p_{n+6} instead of the old p to conclude that there exists a $\bar{p}_j$ such that $I_{n+6}\subseteq\bar{I}_j$ and $d(\omega_{n+6},\bar{\omega}_j)\leq 2^{(n+1)/2000}|\omega_{n+6}|$. Then we have a chain $\omega_{n+6}\subsetneqq\ldots\subsetneqq\omega_1\subsetneqq\omega$ with $d(\omega_{n+6},\bar{\omega}_j)\leq 2^{(n+1)/2000}|\omega_{n+6}|<2^{(n+5)/2-2}|\omega_{n+6}|$ (this is why we work with subscripts until $n+6$: to have this last inequality for all $n\geq 0$). There exists $\xi\in\bar{\omega}_j$ with $d(\xi,\omega_{n+6})<2^{(n+5)/2-2}|\omega_{n+6}|$. By Lemma 4.2, $\xi\in\omega$, which implies that $\omega\cap\bar{\omega}_j\neq\emptyset$, and this together with the fact that $I\subseteq\bar{I}_j$ yields $\bar{\omega}_j\subseteq\omega$. We have then demonstrated that $p<\bar{p}_j$, and the claim is done.

The $\bar{p}_j$'s are not pairwise comparable, which tells us that $\{E(\bar{p}_j)\}_j$ are mutually disjoint. Hence,

$$\sum_j|E(\bar{p}_j)|\leq|[0,2\pi]|=2\pi.$$

Since $\bar{p}_j$ verifies (7.1), we obtain that

$$\begin{aligned}\int_0^{2\pi}(\text{number of }\bar{I}_j\text{ containing }x)\,dx&=\int_0^{2\pi}\left(\sum_j\mathbb{1}_{\bar{I}_j}(x)\right)dx\\&=\sum_j|\bar{I}_j|\leq 2^{n+1}\sum_j|E(\bar{p}_j)|\leq C2^{n+1}.\end{aligned}$$

Let

$$G_n=\{x\in[0,2\pi]:x\text{ is contained in more than }K2^{2n}\text{ of the }\bar{I}_j\}.$$

Since

$$G_n=\bigcup_{m=1}^{\infty}\{x\in[0,2\pi]:\sum_j\mathbb{1}_{\bar{I}_j}(x)=K2^{2n}+m\}$$

is measurable, we can express its Lebesgue measure as an integral over the G_n, so

$$\begin{aligned}C2^{n+1}&\geq\int_0^{2\pi}(\text{number of }\bar{I}_j\text{ containing }x)\,dx\\&\geq\int_{G_n}(\text{number of }\bar{I}_j\text{ containing }x)\,dx\geq K2^{2n}|G_n|.\end{aligned}$$

Hence,

$$|G_n| \leq C\frac{1}{K\cdot 2^n}.$$

We decompose again $\mathcal{P}_n^0$. Write

$$\mathcal{P}_n^\sharp = \{[\omega, I] \in \mathcal{P}_n^0 : I \not\subseteq G_n\}.$$

If $[\omega, I] \in \mathcal{P}_n^0 \backslash \mathcal{P}_n^\sharp$, $T_{[\omega,I]}f$ lives on $I \subseteq G_n$, therefore

$$T^{\mathcal{P}_n^\sharp} f(x) = T^{\mathcal{P}^0{}_n} f(x), \quad \forall x \in G_n^c \quad \forall f \in L_r^2. \tag{7.3}$$

From now on, consider only the $\bar{p}_j$'s such that $\bar{I}_j \not\subseteq G_n$. We have that $\mathcal{P}_n^\sharp$ is a set of pairs for which:

- $A(p) \leq 2^{-n}$,
- every $p \in \mathcal{P}_n^\sharp$ satisfies $p < \bar{p}_j \in \mathcal{P}_n^\sharp$ for some j,
- no $x \in [0, 2\pi]$ belongs to more than $K2^{2n}$ of the $\bar{I}_j$'s.

We decompose $\mathcal{P}_n^\sharp$ as a disjoint union of at most $M = 2n + \log K + 1$ forests, $\mathcal{P}_{n0} \cup \mathcal{P}_{n1} \cup \ldots \cup \mathcal{P}_{n(M-1)}$, and we will apply the Corollary of the Main Lemma to each of these forests. Let us see that this decomposition is certainly possible. Let $B(p) =$ (number of j's for which $p < \bar{p}_j$), for each $p \in \mathcal{P}_n^\sharp$. Then $1 \leq B(p) \leq 2^{2n}K < 2^M$. Define

$$\mathcal{P}_{ns} = \{p \in \mathcal{P}_n^\sharp : 2^s \leq B(p) < 2^{s+1}\},$$

$s = 0, \ldots, M-1$. Then $\mathcal{P}_n^\sharp = \mathcal{P}_{n0} \cup \mathcal{P}_{n1} \cup \ldots \cup \mathcal{P}_{n(M-1)}$, and we just need to check that each $\mathcal{P}_{ns}$ is a forest. From the second and third previous points, condition (d) of the definition of forest holds. From the first point, (a) also holds. It remains to check (b) and (c). Condition (b) is easy. Indeed, suppose that $p, p'' \in \mathcal{P}_{nu}$ and $p < p' < p''$ with p' being admissible. As $p < p'$ and $p \in \mathcal{P}_n^\sharp$, it follows that $p' \in \mathcal{P}_n^\sharp$. Also $2^u \leq B(p'') \leq B(p') \leq B(p) < 2^{u+1}$, which gives condition (b). It remains condition (c), which requires a little bit more work. Suppose that $p, p', p'' \in \mathcal{P}_{nu}$ with $p < p'$ and $p < p''$, but p' and p'' not comparable. We want to arrive at a contradiction. Write $B(p') = s$ and $B(p'') = t$. Then $p' < \bar{p}_{j_1}, \ldots, \bar{p}_{j_s}$ and $p'' < \bar{p}_{k_1}, \ldots, \bar{p}_{k_t}$. Now, $\bar{p}_{j_l} \neq \bar{p}_{k_m}$ (otherwise we would have $p = [\omega, I] < p' = [\omega', I'] < p''' = [\omega''', I''']$ and $p = [\omega, I] < p'' = [\omega'', I''] < p''' = [\omega''', I''']$, where $p''' = \bar{p}_{j_l}$, so $I \subseteq I', I''$ and $\omega''' \subseteq \omega', \omega''$, therefore $I' \cap I'' \neq \emptyset \neq \omega' \cap \omega''$ and $p' < p''$ or $p'' < p'$). Then $p < \bar{p}_{j_1}, \ldots, \bar{p}_{j_s}, \bar{p}_{k_1}, \ldots, \bar{p}_{k_t}$, so $B(p) \geq s + t = B(p') + B(p'')$, but in such a case $B(p) \geq B(p') + B(p'') \geq 2^u + 2^u = 2^{u+1}$, which contradicts $p \in \mathcal{P}_{nu}$. Thus, (c) holds.

Since $\mathcal{P}_{ns}$ is a forest, by the Corollary of the Main Lemma there is a set $F_{ns} \subseteq [0, 2\pi]$ with $|F_{ns}| \leq C \cdot 2^{-80n}/K$ and

$$\|T^{\mathcal{P}_{ns}} f\|_{L^2(F_{ns}^c)} \leq C_\eta (\log K) 2^{-n\left(\frac{1}{4}-\eta\right)} \|f\|_2.$$

Then

$$\|T^{\mathcal{P}_n^\sharp}\|_{L^2(F_n^c)} \leq \sum_{s=0}^{2n+\log K} \|T^{\mathcal{P}_{ns}} f\|_{L^2(F_{ns})} \leq C_\eta (n+6)(\log K)^2 2^{-n\left(\frac{1}{4}-\eta\right)} \|f\|_2,$$

where

$$F_n = \bigcup_{s=0}^{2n+\log K} F_{ns}$$

and

$$|F_n| \le \sum_{s=0}^{2n+\log K} |F_{ns}| \le C\frac{(n+6)\log K}{2^{80n}K}.$$

By (7.3),

$$\|T^{\mathcal{P}_n^0} f\|_{L^2(E_n^c)} \le C_\eta (n+6)(\log K)^2 2^{-n\left(\frac{1}{4}-\eta\right)} \|f\|_2, \tag{7.4}$$

where $E_n = F_n \cup G_n$ and

$$|E_n| \le |F_n| + |G_n| \le C\frac{(n+6)\log K}{2^n K}.$$

As $T^{\mathcal{P}_n} = T^{\mathcal{P}_n^0} + T^{\mathcal{P}_n^+}$, both estimates (7.2) and (7.4) give

$$\|T^{\mathcal{P}_n} f\|_{L^2(E_n^c)} \le C_\eta (n+6)(\log K)^2 2^{-n\left(\frac{1}{4}-\eta\right)} \|f\|_2.$$

Put $\eta = 1/8$:

$$\|T^{\mathcal{P}_n} f\|_{L^2(E_n^c)} \le C(n+6)(\log K)^2 2^{-n/8} \|f\|_2.$$

Sum over n:

$$\|Tf\|_{L^2(E^c)} \le C(\log K)^2 \|f\|_2,$$

where

$$E = \bigcup_{n=0}^{\infty} E_n$$

and

$$|E| \le \sum_{n=0}^{\infty} |E_n| \le C\frac{\log K}{K}.$$

For $\alpha > 0$, we estimate $|\{|Tf| > \alpha\}|$ as in the proof of Chebychev's inequality:

$$\begin{aligned}
|\{|Tf| > \alpha\}| &= \int_{\{|Tf|>\alpha\}} 1 = \int_{\{x\in E:\, |Tf(x)|>\alpha\}} 1\,dx + \int_{\{x\in E^c:\, |Tf(x)|>\alpha\}} 1\,dx \\
&\le \int_E 1\,dx + \int_{E^c} \frac{|Tf(x)|^2}{\alpha^2}\,dx = |E| + \frac{\|Tf\|^2_{L^2(E^c)}}{\alpha^2} \\
&\le C(\log K)^4 \frac{\|f\|_2^2}{\alpha^2} + C\frac{\log K}{K}.
\end{aligned}$$

We would like to show that

$$|\{|Tf| > \alpha\}| \le C_p \left(\frac{\|f\|_2}{\alpha}\right)^p$$

for all $0 < p < 2$, that is, $T : L_r^2 \to L_r^{p,\infty}$ bounded for all $0 < p < 2$, and then apply the result on interpolation of operators given by the following lemma to arrive at $T : L_r^2 \to L_r^p$ bounded for every $0 < p < 2$, which proves estimate (4.1) and Carleson's theorem.

Lemma 7.1 *Let $0 < p_0 < p < p_1 < \infty$ and write $p = (1-\alpha)p_0 + \alpha p_1$, for certain $\alpha \in (0,1)$. Then for any measurable function $g : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ we have*

$$\|g\|_p \le C_{p,p_1,p_0} \left(\|g\|_{L^{p_0,\infty}}^{p_0(1-\alpha)} \|g\|_{L^{p_1,\infty}}^{p_1\alpha} \right)^{\frac{1}{p}}.$$

As a consequence, for $g = Tf$, $\Omega = [-4\pi, 6\pi]$ and $0 < p_0 < p < p_1 < 2$, we obtain

$$\|Tf\|_p \le C_{p,p_1,p_0} \left(\|Tf\|_{L^{p_0,\infty}}^{p_0(1-\alpha)} \|Tf\|_{L^{p_1,\infty}}^{p_1\alpha} \right)^{\frac{1}{p}} \le C_{p,p_1,p_0} \left(\|f\|_2^{p_0(1-\alpha)} \|f\|_2^{p_1\alpha} \right)^{\frac{1}{p}} = C_{p,p_1,p_0} \|f\|_2.$$

Proof. We will denote the distribution function of g by $\lambda_g : (0,\infty) \to [0,\infty]$, where $\lambda_g(t) = |\{x \in \Omega : |g(x)| > t\}|$.

Call $B_0 = \|g\|_{L^{p_0,\infty}}$ and $B_1 = \|g\|_{L^{p_1,\infty}}$. The idea of the proof is to show that

$$\lambda_g(t) \le \frac{B_{p,p_1,p_0}}{t^p} \min\left\{ \frac{t_0}{t}, \frac{t}{t_0} \right\}^{\eta_{p,p_1,p_0}}, \tag{7.5}$$

where $B_{p,p_1,p_0} = B_0^{p_0(1-\alpha)} B_1^{p_1\alpha}$, and then conclude by using the equality

$$\|g\|_p^p = p \int_0^\infty t^{p-1} \lambda_g(t)\, dt.$$

In order to prove (7.5), the trick is to consider the t_0 satisfying

$$\frac{B_0^{p_0}}{t_0^{p_0}} = \frac{B_1^{p_1}}{t_0^{p_1}}.$$

Consider an $\epsilon > 0$ which we will specify later on. Then

$$\frac{B_0^{p_0(1-\alpha)}}{t^{p_0(1-\alpha)}} \left(\frac{t}{t_0}\right)^{-p_0\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha+\epsilon} \left(\frac{B_0^{p_0}}{t_0^{p_0}}\right)^{-\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha+\epsilon} \left(\frac{B_1^{p_1}}{t_0^{p_1}}\right)^{-\epsilon},$$

$$\frac{B_1^{p_1\alpha}}{t^{p_1\alpha}} \left(\frac{t}{t_0}\right)^{p_1\epsilon} = \left(\frac{B^{p_1}}{t^{p_1}}\right)^{\alpha-\epsilon} \left(\frac{B_1^{p_1}}{t_0^{p_1}}\right)^{\epsilon},$$

whence

$$\frac{B_{p,p_1,p_0}}{t^p} \left(\frac{t}{t_0}\right)^{(p_1-p_0)\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha+\epsilon} \left(\frac{B^{p_1}}{t^{p_1}}\right)^{\alpha-\epsilon} \underbrace{\ge}_{\substack{\text{if} \\ 0<\epsilon<\alpha}} \lambda_g(t)^{1-\alpha+\epsilon} \lambda_g(t)^{\alpha-\epsilon} = \lambda_g(t).$$

On the other hand,

$$\frac{B_0^{p_0(1-\alpha)}}{t^{p_0(1-\alpha)}} \left(\frac{t_0}{t}\right)^{-p_0\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha-\epsilon} \left(\frac{B_0^{p_0}}{t_0^{p_0}}\right)^{\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha-\epsilon} \left(\frac{B_1^{p_1}}{t_0^{p_1}}\right)^{\epsilon},$$

$$\frac{B_1^{p_1\alpha}}{t^{p_1\alpha}} \left(\frac{t_0}{t}\right)^{p_1\epsilon} = \left(\frac{B_1^{p_1}}{t^{p_1}}\right)^{\alpha+\epsilon} \left(\frac{B_1^{p_1}}{t_0^{p_1}}\right)^{-\epsilon},$$

whence

$$\frac{B_{p,p_1,p_0}}{t^p} \left(\frac{t_0}{t}\right)^{(p_1-p_0)\epsilon} = \left(\frac{B_0^{p_0}}{t^{p_0}}\right)^{1-\alpha-\epsilon} \left(\frac{B_1^{p_1}}{t^{p_1}}\right)^{\alpha+\epsilon} \underbrace{\ge}_{\substack{\text{if} \\ 0<\epsilon<1-\alpha}} \lambda_g(t)^{1-\alpha-\epsilon} \lambda_g(t)^{\alpha+\epsilon} = \lambda_g(t).$$

This proves (7.5) for $\eta_{p,p_1,p_0} = (p_1 - p_0)\epsilon$ for all $0 < \epsilon < \min\{\alpha, 1-\alpha\}$.

To conclude, note that

$$\begin{aligned}
\|g\|_p^p &= p\int_0^\infty t^{p-1}\lambda_g(t)\,dt \le p\cdot B_{p,p_1,p_0}\cdot\int_0^\infty \frac{1}{t}\min\left\{\frac{t_0}{t},\frac{t}{t_0}\right\}^{(p_1-p_0)\epsilon} dt \\
&= p\cdot B_{p,p_1,p_0}\cdot\left[\int_0^{t_0}\frac{1}{t}\left(\frac{t}{t_0}\right)^{(p_1-p_0)\epsilon} dt + \int_{t_0}^\infty \frac{1}{t}\left(\frac{t_0}{t}\right)^{(p_1-p_0)\epsilon} dt\right] \\
&= p\cdot B_{p,p_1,p_0}\cdot\frac{1}{t_0^{(p_1-p_0)\epsilon}}\int_0^{t_0}\frac{1}{t^{1-(p_1-p_0)\epsilon}}\,dt + p\cdot B_{p,p_1,p_0}\cdot t_0^{(p_1-p_0)\epsilon}\int_{t_0}^\infty \frac{1}{t^{1+(p_1-p_0)\epsilon}}\,dt \\
&= p\cdot B_{p,p_1,p_0}\cdot\frac{1}{t_0^{(p_1-p_0)\epsilon}}\cdot\frac{t_0^{(p_1-p_0)\epsilon}}{(p_1-p_0)\epsilon} + p\cdot B_{p,p_1,p_0}\cdot t_0^{(p_1-p_0)\epsilon}\frac{t_0^{-(p_1-p_0)\epsilon}}{(p_1-p_0)\epsilon} \\
&= \underbrace{\frac{2p}{(p_1-p_0)\epsilon}}_{C_{p,p_1,p_0}} B_{p,p_1,p_0}.
\end{aligned}$$

□

Thus, if we demonstrate that

$$|\{|Tf| > \alpha\}| \le C_\epsilon\left(\frac{\|f\|_2}{\alpha}\right)^{2-\epsilon} \tag{7.6}$$

for all $0 < \epsilon < 2$, we will be done. For simplicity, call $A = |\{|Tf| > \alpha\}|$ and $B = \|f\|_2/\alpha$. We will show that

$$(\log K)^4 B^2 + \frac{\log K}{K} \le C_\epsilon B^{2-\epsilon} \tag{7.7}$$

holds for a $K > 10$ (which will depend on f and α). Inequality (7.7) is in principle difficult to show, because on the left-hand side we have a term depending on B and the other one not. We will see that a good choice is obtained by taking $K > 10$ such that

$$(\log K)^4 B^2 = \frac{\log K}{K}, \tag{7.8}$$

because in such a case the two addends from the left-hand side of (7.7) are equal (note the similarity of this idea and the choice of t_0 in the proof of the last lemma). Can we obtain $K > 10$ satisfying (7.8)? Well, if $B^2 < (\log 10)/(10\cdot(\log 10)^4)$, as $\lim_{K\to\infty}(\log K)/(K\cdot(\log K)^4) = 0$, by the Weierstrass intermediate value theorem we can choose such a K. What about B being big, in the sense that $B^2 \ge (\log 10)/(10\cdot(\log 10)^4)$? Then we merely notice that $A \le 2\pi$, so we need $2\pi \le C_\epsilon B^{2-\epsilon}$, which is achieved if $2\pi \le C_\epsilon\cdot[(\log 10)/(10\cdot(\log 10)^4)]^{2-\epsilon}$, so just define $C_\epsilon = 2\pi[(\log 10)/(10\cdot(\log 10)^4)]^{-(2-\epsilon)}$.

Thus, we may assume $B^2 < (\log 10)/(10\cdot(\log 10)^4)$. Take $K > 10$ verifying (7.8). In such a case

$$(\log K)^4 B^2 + \frac{\log K}{K} = 2B^2(\log K)^4.$$

As $K(\log K)^3 = 1/B^2$, we have $K \le 1/B^2$, therefore

$$(\log K)^4 B^2 + \frac{\log K}{K} = 2B^2(\log K)^4 \le 2B^2\left(\log\frac{1}{B^2}\right)^4.$$

That is $\leq C_\epsilon B^2 \cdot B^{-\epsilon}$ if and only if

$$B^\epsilon \left(\log \frac{1}{B^2}\right)^4 \leq C_\epsilon,$$

and this can be achieved for all $\epsilon > 0$ because B is bounded by the number $(\log 10)/(10 \cdot (\log 10)^4)$ and

$$\lim_{x \to 0^+} x^\epsilon \left(\log \frac{1}{x^2}\right)^4 = 0.$$

Hence, inequalities (7.7) and (7.6) hold and by previous comments Carleson's theorem follows.

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List of notations

$\mathbb{N}$: Set of natural numbers: $1, 2, 3, \ldots$.
$\mathbb{Z}$: Set of integers: $\ldots, -3, -2-1, 0, 1, 2, 3, \ldots$.
$\mathbb{R}$: Set of real numbers.
$\mathbb{R}^+$: Set of positive real numbers.
$\mathbb{C}$: Set of complex numbers.
$\emptyset$: Empty set.
log, $\log_2$: Both mean logarithm in base 2.
$\overline{a}$: Conjugate of $a \in \mathbb{C}$.
$|a|$: Modulus of $a \in \mathbb{C}$.
$A \cup B$: Union of the sets A and B.
$A \uplus B$: Union of the disjoint sets A and B.
$\cup'$: Finite union.
$A \cap B$: Intersection of the sets A and B.
$A \backslash B$: Set of points in A but not in B.
A^c: Complementary of a set A in $\mathbb{R}$, that is, $\mathbb{R} \backslash A$.
$[a, b]$: Closed interval.
$[a, b[$, $[a, b)$: Semiopen interval.
$d(a, A)$, $d(A, B)$: Distance from a point a to a set A, distance between two sets A and B.
$\overline{A}$: Closure of the set A in $\mathbb{R}$ with the usual distance.
∂A: Boundary of the set A in $\mathbb{R}$ with the usual distance.
$\tilde{I}$: Given a dyadic interval, the next larger dyadic interval that contains I.
I^*: The double of an interval I: interval with its same center but twice its length.
I^3: The triple of an interval I: add I to the right and to the left of I.
I^5: Five times an interval I: add twice I to the right and to the left of I.
$|I|$: Length of an interval, that is, the distance between its endpoints.
$[\omega, I]$: A pair of dyadic intervals, that is: $\omega \subseteq \mathbb{R}$, $I \subseteq [0, 2\pi]$ and $|\omega|/(2\pi) = (2\pi)/|I|$.
$\langle f_1, \ldots, f_n \rangle$: Vector space spanned by a set of vectors $f_1, \ldots, f_n$.
$A \leq B$: The vector space A is a vector subspace of B.
$\ker f$: Kernel of the linear map f.
range$f$: Image of f.
$A \oplus B$: Direct sum of the vector spaces A and B.
sign: The sign function.
$\mathbb{1}_E$: Characteristic function on the set E.
$\lfloor a \rfloor$: Integer part of $a \in \mathbb{R}$.
$\sum'$: A finite sum.
C: Any positive constant (its value is not important).
C_a: Any positive constant (its value is not important) depending on a.

$\mathbb{T}$: Unit circle.
$L^q(\mathbb{T})$: Functions from $\mathbb{R}$ to $\mathbb{C}$, 2π-periodic, in $L^q([0, 2\pi])$.
$C^\infty(\mathbb{T})$: Functions from $\mathbb{R}$ to $\mathbb{C}$, 2π-periodic, in $C^\infty(\mathbb{R})$.
C_c^∞: Functions from $\mathbb{R}$ to $\mathbb{C}$, with compact support, in $C^\infty(\mathbb{R})$.
$\mathcal{S}$: Schwartz class in $\mathbb{R}$.
L_r^q: Functions from $\mathbb{R}$ to $\mathbb{C}$, with support in $[-4\pi, 6\pi]$, in $L^q(\mathbb{R})$.
C_r^∞: Functions from $\mathbb{R}$ to $\mathbb{C}$, with support in $[-4\pi, 6\pi]$, in $C^\infty(\mathbb{R})$.
$L^1_{\text{loc}}(\mathbb{R}^n)$: Space of functions from $\mathbb{R}^n$ to $\mathbb{C}$ that are integrable over compact sets.
$\|f\|_\infty$: Supremum of f.
$\|f\|_q$: In Chapter 2, $\int_0^{2\pi} |f|^q$; from Chapter 3, $\int_{\mathbb{R}} |f|^q$.
$\hat{f}$: Fourier transform in $\mathbb{T}$: $\hat{f}(k) = 1/(2\pi)\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$; or in $\mathbb{R}$: $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$.
S_nf: n-th partial sum of the Fourier series: $S_nf(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}$.
$|A|$: Lebesgue measure of the set A.
$\lambda_f(t)$: Distribution function of f at t: $\lambda_f(t) = |\{|f| > t\}|$.
$L^{q,\infty}$: Weak L^q space: functions f from $\mathbb{R}$ to $\mathbb{C}$ such that $\|f\|_{q,\infty} := \sup_{t>0} t\lambda_f(t)^{1/q} < \infty$.
(f, g): Inner product between f and g: $\int_{\mathbb{R}} fg$; or inner product in a general Hilbert space.
$f * g$: Convolution in $\mathbb{R}$: $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$.
a.e.: Almost everywhere, almost every (with respect to the Lebesgue measure).
$L^1(A, dy)$: Set of functions in $L^1(A)$ with respect to the variable y.
$A \perp B$: The sets A and B are orthogonal in a pre-Hilbert space.
$A^\perp$: Orthogonal of a set A in a pre-Hilbert space.
T^*: Adjoint operator of a bounded operator T between two Hilbert spaces.