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Schottky Groups in Valuation Rings

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Abstract

Given a complete field K, and a valuation over K, we can construct a "tree of balls", where the vertex are the open balls obtained from a subset $\mathcal{L}$ of $\mathbb{P}^1(K)$ seen as a topological space, and the edges are obtained from the paths between elements of the subset of $\mathbb{P}^1(K)$. In order to define the open balls we need to give a topology. It comes from our valuation and gives the property that or two balls does not intersect or one is contained in the other.

Moreover given a Schottky group Γ acting on the tree of balls we will see that we obtain a finite tree. In order to see that we will see first that this tree of balls of a subset $\mathcal{L}$ of $\mathbb{P}^1(K)$ is locally finite. We will see that the subset $\mathcal{L}$ has to be compact in order to guarantee the finiteness of the resultant tree. Other result will consist on see that the closure of the limit points of a Schottky group, $\mathcal{L}_\Gamma$, is equal to the closure of the orbit of some point, which by definition of Schottky group will guarantee that this set is compact so we will be able to apply the previous theory. In order to define a Schottky group we will consider that it has to be topologically nilpotent in order to extend the non-Archimedian results to any totally ordered group as a image of our valuation. We also will see a characterisation of hyperbolic matrices and we will consider some example of the graph $\mathcal{L}_\Gamma/\Gamma$

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1 Introduction

In the Section 2 we will see elementary properties of valuations and valuation rings, where a valuation is not defined in the usual sense. We also will see the usual definition and its relation with our definition.

In the Section 3 we give properties of a distance obtained from a valuation and that allow us to define a graph called the tree of balls, obtained from the topology of our field K and a subset $\mathcal{L}$ of $\mathbb{P}^1(K)$. We prove that this graph is in fact a tree. Moreover, we use the compactness of $\mathcal{L}$ to prove that the number of vertices between any two vertices is finite and that $T(\mathcal{L})$ is locally finite. There are a subsection which talks about rays in order to study the non isolated points of $\mathcal{L}$.

In the Section 4 we will study the hyperbolic elements of $PGL_2(K)$ arriving to a characterisation result that says when a matrix is hyperbolic studying its trace and determinant (Proposition 4.8); to prove this result we use the completeness of K and the Hensel's Lemma. Moreover we have other result (Theorem 4.9), which says that, supposing that the matrix is diagonalitzable then it is hyperbolic or has finite order if an only if the closure of the orbit of a point is compact for any point in $\mathbb{P}^1(K)$. Then we will define Schottky group Γ and we will see that the closure of the set of points of $\mathcal{P}^1(K)$ fixed by some element of the Schottky group, denoted by $\mathcal{L}_\Gamma$, is perfect and compact. So from this result we can apply what we know from the Section 3 arriving to the fact that $T(\mathcal{L}_\Gamma)/\Gamma$ if finite. Finally we see some example of $T(\mathcal{L}_\Gamma)/\Gamma$.

2 Valuations

2.1 Definition and properties

First of all we define what is a valuation.

Definition 2.1. A surjective map $v : K \to \Delta \cup \{0\}$ from a field K to $\Delta \cup \{0\}$, where Δ is a non trivial ordered group, is called a **valuation** of K if satisfies

- $|xy| = |x| \cdot |y|$
- $|x + y| \leq \max\{|x|, |y|\}$
- $|x| = \infty \Leftrightarrow x = 0$.

Note that this definition is very similar to the standard definition of the norm but instead of having the usual triangular inequality we have this strong inequality.

Note that for all $\delta \in \Delta$ exists $\epsilon \in \Delta$ such that $\epsilon < \delta$. Since Δ is non trivial. If $\delta > 1$, we can take $\epsilon = 1$; if $\delta < 1$, we can take $\epsilon = \delta^2$; and if $\delta = 1$, since Δ is non trivial, there exists $1 \neq \delta' \in \Delta$, and we take $\epsilon = \delta'$ or $\epsilon = \delta'^{-1}$, depending if $\delta' < 1$ or $\delta' > 1$.

And a property that can be deduced immediately from the definition is the next.

Proposition 2.2. *If $|.|$ is a valuation of a field K and if $a, b \in K$ are such that $|a| \neq |b|$ then $|a+b| = \max\{|a|, |b|\}$.*

Proof. We know that $|a+b| \leq \max\{|a|, |b|\}$. Now suppose that $|b| < |a|$ (the other case is analogue). Now we have

$$|a| = v((a+b) - b) \leq \max\{|a+b|, |b|\}$$

but since $|b| < |a|$ it implies that $|a| \leq |a+b|$. And we also have that $|a| = \max\{|a+b|, |b|\}$, so $\max\{|a+b|, |b|\} \leq |a+b|$ and with $|a+b| \leq \max\{|a|, |b|\}$ it implies that $|a+b| = \max\{|a|, |b|\}$. $\square$

Moreover from this definition is easy to see that $|1_K| = 1$ considering that $|x| = |x \cdot 1_K| = |x| \cdot |1_k|$, for a non zero element x.

Definition 2.3. We say that $|.|$ is a **non Archimedian absolute value** if it is a valuation and $\Delta_0 \hookrightarrow \mathbb{R}_{\geq 0}$ as ordered groups.

Definition 2.4. Let K be a field with a valuation $|.|$ we define **ring of integers** of K with respect $|.|$ as

$$\mathcal{O}_{|.|} = \{x \in K \mid |x| \leq 1\}.$$

Remark 2.5. Note that $\mathcal{O}_{|.|}$ is a domain whose field of fractions is K.

Notation 2.6. For simplify we will call $\mathcal{O}_{|.|}$ as $\mathcal{O}$.

Definition 2.7. Let A be a domain. We say that A is a valuation ring if for all $x \in Q(A)$ or $x \in A$ or $x^{-1} \in A$. Where $Q(A)$ denotes the field of fractions of A.

Remark 2.8. Note that our $\mathcal{O}$ is of valuation because if $|x| > 1$ then $|x^{-1}| \leq 1$.

We define the following map and we will claim that is a valuation $|.|$:

$$\begin{aligned} |.| : K &\longrightarrow \Delta_0 \\ z &\longmapsto \overline{z} \end{aligned}$$

where $\Delta_0 = (K^*/R^*) \cup \{0\}$.

In fact this proposition will be one implication of a next theorem.

Proposition 2.9. *The map $|.|$ defined as above is a valuation where $a \leq b$ if and only if $ab^{-1} \in R$.*

Proof. Since send an element to its equivalence calss is a morphism only remains to be prove the following. We have to see that we have a total order. let $a, b \in K^*$. If $\overline{a} \leq \overline{b} \Leftrightarrow \overline{a}\overline{b}^{-1} \leq 1 \Leftrightarrow \overline{ab^{-1}} \leq 1$ $ab^{-1} \in R$. Note that since R is a valuation then if $ab^{-1} \notin R$ then $(ab^{-1})^{-1} = ba^{-1} \in R$, which means that $b \leq a$. So we have a totally order.

It only remains to prove that $\overline{a+b} \leq \max\{\overline{a}, \overline{b}\}$, and it is the same that $\overline{\frac{a}{b}+1} \leq \max\{\frac{\overline{a}}{\overline{b}}, 1\} = 1$. We are supposing that $\overline{a} \leq \overline{b}$ and $b \neq$, if not we divide by a. And the case where $a = 0$ or $b = 0$ is clear. And the previous inequality is equivalent to say that if $\overline{\frac{a}{b}} \leq 1$ then $\overline{\frac{a}{b}+1} \leq 1$, and it the same that if $\frac{a}{b} \in R$ then $\frac{a}{b}+1 \in R$, and it is true because R is a ring. □

2.2 Complete Valuations

Definition 2.10. Recall first that given a sequence $\{\gamma_n\}_n$ of elements of Δ we say that this sequence **tends to zero** or **has limit zero** and we denote it by $\gamma_n \to 0$ if and only if $\forall \delta \in \Delta$, exist a n_0 such that $\forall n \geq n_0$ we have $\gamma < \delta$.

From this definition of limit over the elements of the ordered set Δ we can define the limit of elements of K.

Definition 2.11. Let $\{a_n\}_n$ a sequence of elements in K, and let $a \in K$, we say that it **tends to zero** or **has limit zero**, and denote it by $a_n \to a$ or $\lim_{n\to\infty} a_n = a$ if and only if $|a_n - a| \to 0$. We also say that it is a **Cauchy sequence** if $|a_{n+1} - a_n| \to 0$.

Definition 2.12. We say that K **is complete respect** $|.|$ if every Cauchy sequence has limit.

Proposition 2.13. *The limit is unique*

Proof. Suppose that $a_n \to a$ and $a_n \to b$. Hence $|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| \to 0$, so $a = b$. □

Proposition 2.14. *If a sequence has limit then is a Cauchy sequence.*

Proof. Call its limit a, then

$$|a_{n+1} - a_n| = |a_{n+1} - a + a - a_n| \leq \max\{|a_{n+1} - a|, |a_n - a|\},$$

and now $\forall \delta \in \Delta$, exist n_0 such that $\forall n \geq n_0$ one has $|a_n - a| < \delta$. So the previous terms is bounded by the $\max\{\delta, \delta\}$ so $|a_{n+1} - a_n| < \delta$, so it is a Cauchy sequence. □

2.3 Properties of valuations

Proposition 2.15. *Let A be a valuation ring, then for all different ideals I and J of A either $I \subset J$ or $J \subset I$.*

Proof. If $\forall x \in I$ then $x \in J$ then $I \subset J$. If there are $x \in I$ so that $x \notin J$, then for all $y \in J$ different from zero we have $\frac{x}{y} \notin A$, it is true because if not we would have something of J multiplied by something of A, so be definition of ideal x would be in J what is false . And since A is a valuation ring and $\frac{x}{y} \in Q(A)$ we have $\frac{y}{x} \in A$ and $y = x(\frac{y}{x}) \in I$, so $J \subset I$. □

This property allows us to order the ideals, so in other words we will have a maximal ideal or what is the same, a valuation ring is local. Since $\mathcal{O}_K$ is a valuation ring it has a maximal ideal, which is the following:

$$\mathfrak{m}_K = \{x \in K \mid |x| < 1\}.$$

Moreover, the invertible elements of $\mathcal{O}_K$ are exactly $\mathcal{O}_K^* = \mathcal{O}_K/\mathfrak{m}_K = \{x \in K \mid |x|_v = 1\}$ which is denoted by k which is called the residue field. Other property of the valuation rings is the following:

Proposition 2.16. *In a valuation ring any finitely generated ideal is principal.*

Proof. We will prove it by induction. First we consider the case $I = (\alpha_1, \alpha_2)$ for different $\alpha_1, \alpha_2 \in R$. Consider two ideals (α_1) and (α_2). Since we are in a valuation ring, either $\alpha_1 \subset \alpha_2$ or $\alpha_2 \subset \alpha_1$, or what is the same either $I = (\alpha_2)$ either $I = (\alpha_1)$. So I is principal. An ideal I with n generators will be of the form $I = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) = (\alpha_i, \alpha_n)$ for some i because $I' = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ is, by induction, $I' = (\alpha_i)$ for some i, so using the same argument than before I is principal. $\square$

Definition 2.17. If $\Delta_0 \cong \mathbb{Z}$ we say that the valuation is **discrete**, if Δ_0 is isomorphic to a dense subroup of $\mathbb{R}_{\geq 0}$ the valuation is called **dense**.

We also denote $\sqrt{\Delta_0}$ by $\{a \in \mathbb{R}_{\geq 0} \mid a^n \in \Delta_0,\ \text{for some } n \geq 1\}$.

Lemma 2.18. *Let K be a field such that $char(K) = 0$. Then*

$$|.|\big|_{\mathbb{Q}} = \begin{cases} \textit{trivial} & \textit{if and only if } char(\mathcal{O}/\mathfrak{m}_K) = 0, \\ |.|_p^{\epsilon} & \textit{if and only if } char(\mathcal{O}/\mathfrak{m}_K) = p. \end{cases}$$

Proof. We know that for any integer n, $|n| \leq 1$. Then if $|n| = 1$ for all integer n then $|.|\big|_{\mathbb{Q}}$ is trivial. If not, let $p \in \mathbb{Z}_{>0}$ be the minimal with $|p| < 1$. We can call it p because it is prime. If not $p = ab$ for a and b distinct than p and 1. Then in this case we can write $1 > |p| = |a||b|$ which means that $|a| < 1$ or $|b| < 1$, which contradicts the minimality of p.

Take $m \in \mathbb{Z}$ coprime with p, then there exists $a, b \in \mathbb{Z}\backslash\{0\}$ such that $am + bp = 1$ so

$$|a||m| = |am| = |1 - bp| = |1| = 1,$$

hence $|m| = 1$.

So for any n we have $|n| = |p|^{v_p(n)}$, where $v_p(n)$ denotes the p-adic valuation in the additive sense. $\square$

2.4 Equivalent definition of discrete valuation

Now we define what is a valuation in an other way. In this case we will call additive valuation, and denote it by $v(.)$.

Definition 2.19. We say that $v(.) : K \to \Delta_0$ is an **additive valuation** if:

- $v(x \cdot y) = v(x) + v(y)$
- $v(x + y) \geq \min\{v(x), v(y)\}$
- $v(x) = 0 \Leftrightarrow x = 0$.

Example 2.20. The main example of discrete valuation is the p-adic valuation, v_p, which is defined as follows:

Let p be a prime number, and $a \in \mathbb{Q}$ then $v(a) = ord_p a$ that denotes the highest power of p which divides a, i.e. v(a)=n where $a = p^n \cdot \frac{b}{q}$ where $p \not| \, b, q$.

In this case we define the valuation $|.|$ of $a \in \mathbb{Q}$, $|a|_p$ as

$$|a|_p = \begin{cases} p^{-v(a)} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Note that one can consider Cauchy sequences in $\mathbb{Q}$ with limit out of $\mathbb{Q}$ so one can complete this field with respect this norm, and in this case be obtain what is called the **p-adic completion of** $\mathbb{Q}$ and we denote it by $\mathbb{Q}_p$ (or $\mathbb{Q}_{v_p}$). It has all the rational numbers and the limits of the Cauchy sequences

Other example can be the following.

Example 2.21. Let k be a field and consider $k(x)$. Then one has that

$$\begin{aligned} v : k(x) &\longrightarrow \mathbb{Z} \\ \frac{p(x)}{q(x)} &\longmapsto v\left(\frac{p(x)}{q(x)}\right) = -deg(p(x)) + deg(q(x)) \end{aligned}$$

is a discrete valuation and

$$\begin{aligned} |.| : k(x) &\longrightarrow \mathbb{Z} \\ \frac{p(x)}{q(x)} &\longmapsto \left|\frac{p(x)}{q(x)}\right| = e^{deg(p(x)) - deg(q(x))} \end{aligned}$$

is its absolute value.

If one see the Example (2.20) we have an explicit way to pass from an additive valuation to a valuation, in this case with $p^{(-.)}$. If our image are not the real numbers or any subgroup of it we also can pass from one valuation to the other defining properly an operation which inverts the order, as this exponential does in the example.

2.5 Topologically nilpotent elements

Definition 2.22. We say that $q \in K$ is topologically nilpotent if $\lim_{n\to\infty} q^n = 0$ (recall the Definition 2.11).

Lemma 2.23. *Let $q \in \mathcal{O}_K$, $q \neq 0$. Then q is not topologically nilpotent if and only if exists $\delta, \delta' \in \Delta$ such that $\delta \leq |q^n| \leq \delta'$, for all $n \in \mathbb{Z}$.*

If the image of the valuation is defined over the reals is equivalent to say that q is topologically nilpotent than say that $|q| < 1$. But if not is not enough to say that $|q| < 1$ in order to guarantee that q tends to zero. We will se it in an example:

Example 2.24. Let $\Delta = (\mathbb{R}_{\geq 0})^2_{lex} = \{(a, b) \mid a, b \in \mathbb{R}_{\geq 0}\}$ where the order is defined by

$$(a,b) \leq (a',b') \Leftrightarrow \begin{cases} a < a' \\ \text{ or} \\ a = a' \text{ and } b < b'. \end{cases}$$

Now let $\gamma = (1, \epsilon)$ which satisfies $\gamma < (1, 1)$ if $\epsilon < 1$. Now we consider the sequence $\gamma_n = (1, \epsilon^n)$. And we can see that it does not tend to zero (recall that tend to zero means that $\forall \delta \in \Delta$, exist a n_0 such that $\forall n \geq n_0$ we have $\gamma < \delta$). For example take $\delta = (\frac{1}{2}, 1)$, then $\delta < (1, \epsilon^n)$ for all n.

Note that if our valuation is discrete then be topologically nilpotent is the same than $|q| < 1$.

2.6 Hensel's Lemma

In this section we will follow Conrad [1] in order to prove the Hensel's Lemma first we need a lemma about polynomials which are true in any commutative ring.

Lemma 2.25. *Let R be a commutative ring and let $f(x) \in R[x]$. Then we have the following equalities:*

1. *$f(x + y) = f(x) + f'(x)y + zy^2$ for some $z(x, y) \in R[x, y]$.*
2. *Exists $g(x, y) \in R[x, y]$ such that $f(x) - f(y) = (x - y)g(x, y)$.*

Proof. 1. We apply the Newton Binomial. Let

$$f(x) = \sum a_n x^n,$$

then

$$\begin{aligned}
f(x+y) &= \sum a_n(x+y)^n = \\
&= \sum a_n \left(x^n + nx^{n-1}y + \left(\sum_{i=0}^{n-2} \binom{n}{i} x^i y^{n-i-2} \right) y^2 \right) = \\
&= \sum a_n x^n + \left(\sum n a_n x^{n-1} \right) y + y^2 \left(\sum a_n \left(\sum_{i=0}^{n-2} \binom{n}{i} x^i y^{n-i-2} \right) \right) = \\
&= f(x) + f'(x)y + y^2 \left(\sum a_n \left(\sum_{i=0}^{n-2} \binom{n}{i} x^i y^{n-i-2} \right) \right).
\end{aligned}$$

2. We have

$$\begin{aligned}
f(x) - f(y) &= \sum a_n(x^n - y^n) = \sum a_n(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = \\
&= (x-y) \sum a_n(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).
\end{aligned}$$

□

Lemma 2.26. *[Hensel's Lemma] Let $f(x) \in \mathcal{O}_K[x]$ and $a \in \mathcal{O}_K$. Let $t = \frac{f(a)}{f'(a)^2}$ topologically nilpotent. Then there is a unique $\alpha \in K$ such that $f(\alpha) = 0$ and $|\alpha - a| < |f'(a)|$. Moreover*

1. $|\alpha - a| = |f(a)/f'(a)| < |f'(a)|$,
2. $|f'(\alpha)| = |f'(a)|$.

Proof. We define a sequence $\{a_n\}$ in K by $a_1 = a$ and

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} \tag{1}$$

for $n \geq 1$. Set $t = |f(a)/f'(a)^2| < 1$. We will show by induction on n the following:

1. $|a_n| \in \mathcal{O}_K$,
2. $|f'(a_n)| = |f'(a_1)|$,
3. $|f(a_n)| \leq |f'(a_1)|^2 t^{2^{n-1}}$.

For $n = 1$ there are nothing to check. Then by Lemma 2.25

$$x, y \in \mathcal{O}_K \text{ then } f(x+y) = f(x) + f'(x)y + zy^2, \text{ for some } z \in \mathcal{O}_K. \tag{2}$$

and let $f(x) = \sum_{i=1}^{n} b_i x^i$, since

$$f(x) - f(y) = \sum_{i=1}^{m} b_i(x^i - y^i)$$

we have

$$x, y \in \mathcal{O}_K \text{ then } |f(x) - f(y)| = |x - y| g(x, y) \leq |x - y| \tag{3}$$

for some $g(x, y) \in \mathcal{O}_K[x, y]$ i.e. $|g(x, y)| \leq 1$.

Now we assume that 1, 2, 3 are true for n. To prove 1 for $n+1$ first we note that by 2 for n means that $f'(a_n) \neq 0$, so it only remains to prove that $|\frac{f(a_n)}{f'(a_n)}| \leq 1$. Using 2 and 3 for n we have

$$\left| \frac{f(a_n)}{f'(a_n)} \right| = \left| \frac{f(a_n)}{f'(a_1)} \right| \leq |f(a_1)| t^{2^{n-1}} \leq 1.$$

So 1 is proved. To prove 2 we use that 3 for n implies $|f(a_n)| < |f'(a_1)|^2$, recall $t < 1$, so by equation 3 applied to $f'(x)$ we have

$$|f'(a_{n+1}) - f'(a_n)| \leq |a_{n+1} - a_n| = \frac{|f(a_n)|}{|f'(a_n)|} = \frac{|f(a_n)|}{|f'(a_1)|} < |f'(a_1)|,$$

and hence $|f'(a_{n+1})| = |f'(a_1)| = |f'(a_n)|$. Finally, to prove 3, we use 2 with $x = a_n$ and $y = -f(a_n)/f'(a_n)$, which gives us

$$f(a_{n+1}) = f(a_n) + f'(a_n) \left(-\frac{f(a_n)}{f'(a_n)} \right) + z \left(\frac{f(a_n)}{f'(a_n)} \right)^2 = z \left(\frac{f(a_n)}{f'(a_n)} \right)^2,$$

where $z \in \mathcal{O}_K$. And by 3 for n,

$$|f(a_{n+1})| \leq |\frac{f(a_n)}{f'(a_n)}|^2 = \frac{|f(a_n)|}{|f'(a_1)|} \leq |f(a_1)|^2 t^{2^n}.$$

Now we have the induction. It remains to found the roots. Using 2 and 3 we have the following inequality:

$$|a_{n+1} - a_n| = \left| \frac{f(a_n)}{f'(a_n)} \right| = \frac{|f(a_n)|}{|f'(a_1)|} \leq |f'(a_1)| t^{2^{n-1}}, \tag{4}$$

which means that $\{a_n\}$ is a Cauchy sequence, because t is topologically nilpotent, and since K is complete, it has a limit, call it α, and by 1 $|\alpha| \leq 1$ (i.e. $\alpha \in \mathcal{O}_K$). Now considering 2 and 3 when n tends to ∞ we obtain

$$|f'(\alpha)| = |f'(a_1)| = |f'(a)| \text{ and } f(\alpha) = 0.$$

Now we will show $|\alpha - a| = |f(a)/f'(a)|$. We see that $|a_n - a| = |f(a)/f'(a)|$ for all $n \geq 2$, and it will be valid for its limit. For $n = 2$ there is nothing to check because it follows from the definition using $a_1 = a$. Using equation 4, for any $n \geq 2$

$$|a_{n+1} - a_n| \leq |f'(a_1)| t^{2^{n-1}} \leq |f'(a_1)| t^2 < |f'(a_1)| t = |f'(a)| t = \left| \frac{f(a)}{f'(a)} \right|.$$

And finally if $|a_n - a| = |f(a)/f'(a)|$ we have $|a_{n+1} - a_n| < |a_n - a|$, so

$$|a_{n+1} - a| = |(a_{n+1} - a_n) + (a_n - a)| = |a_n - a| = \left|\frac{f(a)}{f'(a)}\right|.$$

To finish the proof we have to see that α is the only root of $f(x)$ in the set $\{x \in \mathcal{O}_K \mid |x-a| < |f'(a)|\}$. So we assume that $f(\beta) = 0$ and $|\beta - a| < |f'(a)|$. Since $|\alpha - a| < |f'(a)|$ it implies $|\beta - \alpha| < |f'(a)|$. Now we write $\beta = \alpha + h$, hence $h \in \mathcal{O}_K$. And using equation 2,

$$0 = f(\beta) = f(\alpha + h) = f(\alpha) + f'(\alpha)h + zh^2 = f'(\alpha)h + zh^2$$

for some $z \in \mathcal{O}_K$. Now if $h \neq 0$ then $f'(\alpha) = -zh$, so

$$|f'(\alpha)| \geq |h| = |\beta - \alpha| < |f'(a)|,$$

but $f'(\alpha)| = |f'(a)|$, contradiction. Thus $h = 0$ and $\beta = \alpha$.

□

Example 2.27. Let $f(x) = x^2 - 7 \in \mathbb{Q}_3[x]$. It has two roots in $\mathbb{Z}_3$:

$$\alpha_1 = 1 + 3 + 3^2 + 2 \cdot 3^4 + 2 \cdot 3^7 + 3^8 + 3^9 + \ldots,$$
$$\alpha_2 = 2 + 3 + 3^2 + 2 \cdot 3^3 + 2 \cdot 3^5 + 2 \cdot 3^6 + 3^8 + 3^9 + \ldots.$$

Starting with $a_1 = 1$, for which $|f(a_1)/f'(a_1)^2|_3 = 1/3$ ($||_3$ denotes the 3-adic norm), Newton recursion has limit α where $|\alpha - a_1|_3 < |f'(a_1)|_3 = 1$, so $\alpha \cong a_1 \mod 3$. Thus a_n tends to α_1. For example

$$a_4 = \frac{977}{368} = 1 + 3 + +3^2 + 2 \cdot 3^4 + 2 \cdot 3^7 + 3^8 + 3^9 + 3^10 + \ldots,$$

which has the first eight 3-adic digits equals than α_1 ($|\alpha_1 - a_n|$ is bounded by $(1/3)^{2^{n-1}}$).

3 The space of Balls $T(\mathcal{L})$

3.1 Properties of the distance defined from $|.|$

If we do not say the opposite, from now a valuation is a valuation $|.|$ (with the multiplicative notation).

Definition 3.1. Let K be a field and let $|.|$ its valuation, then we can define a distance. For $p_1, p_2 \in K$, we define

$$d(p_1, p_2) = |p_1 - p_2|.$$

Proposition 3.2. *For $p_1, p_2, p_3 \in K$ then or $d(p_1, p_2) = d(p_1, p_2)$, or $d(p_1, p_2) = d(p_1, p_3)$ or $d(p_1, p_3) = d(p_2, p_3)$. Moreover, the third distance will be less or equal than the other two.*

Proof. We can suppose that the three points are different. If not, suppose for example that $p_1 = p_2$, then clearly $d(p_1, p_3) = d(p_2, p_3)$. If all are different we can define $a = p_1 - p_2$ and $b = p_2 - p_3$. If $|a| = |b|$ we are done. If not we use the Proposition 2.2 and we have

$$d(p_1, p_3) = |p_1 - p_3| = |p_1 - p_2 + p_2 - p_3| = |a + b| = \max\{|a|, |b|\} = \max\{d(p_1, p_2), d(p_2, p_3)\}. \tag{5}$$

So if $|a| > |b|$ then $d(p_1, p_3) = d(p_1, p_2)$ and if $|a| < |b|$ then $d(p_1, p_3) = d(p_2, p_3)$.

Now by (5) one distance is equal to the maximum of the other two, so if the third is not equal, then necessarily has to be less than the others (if not $d(p_1, p_3)$ should be this equal and the smaller distance will be the other).

□

This proposition is equivalent to say that all the triangles are isosceles.

Definition 3.3. Let Δ_0 be a totally ordered group. Let $p \in K$, $r \geq 0$, $r \in \Delta_0$, we define **the closed ball** with center p and radius r as $B(p, r) = \{y \in K \mid |y - p| \leq r\}$.

Proposition 3.4. *Let $p_1, p_2 \in K$ and $r_1, r_2 \in \Delta_0$, $r_1, r_2 \geq 0$ such that $B(p_1, r_1) \cap B(p_2, r_2) \neq \emptyset$ then*

$$B(p_1, r_1) \cap B(p_2, r_2) = \begin{cases} B(p_1, r_1) & \text{if } r_1 \leq r_2, \\ B(p_2, r_2) & \text{if } r_2 \leq r_1. \end{cases}$$

Proof. Suppose that $r_1 \leq r_2$ (the other case is analogue). Since $B(p_1, r_1) \cap B(p_2, r_2) \neq \emptyset$ we can consider $q \in B(p_1, r_1) \cap B(p_2, r_2)$. Since $q \in b(p_1, r_1)$ it means that $|p_1 - q| \leq r_1$. Now in order to prove that $B(p_1, r_1) \subset B(p_2, r_2)$ we take an element, call it $q_1 \in B(p_1, r_1)$ and we have to see that $|q_1 - p_2| \leq r_2$:

$$\begin{aligned} |q_1 - p_2| &= |q_1 - p_1 + p_1 - q + q - p_2| \leq \max\{|q_1 - p_1 + p_1 - q|, |q - p_2|\} \\ &\leq \max\{\max\{|q_1 - p_1|, |p_1 - q|\}, |q - p_2|\} = \max\{r_1, r_1, r_2\} = r_2. \end{aligned}$$

□

So it means that or two balls do not intersect or one is into the other.

Remark 3.5. Note that if $q \in B(p, r)$ then $B(q, r) = B(p, r)$.

Definition 3.6. We define the **space of balls** $T(K) = \{B(p, r) \mid p \in K, r > 0\}$. We also define $\overline{T(K)} = T(K) \cup K \cup \infty$.

Definition 3.7. We define the path from $p \in K$ to ∞ as $\pi(p, \infty) = \{B(p, r) \mid r > 0\}$

Remark 3.8. Note that $\pi(p, \infty) \cong \Delta$.

Definition 3.9. We have that the intersection is

$$\pi(p,\infty)\cap\pi(q,\infty)=\{B(p,r)\mid r\geq d(p,q)\}\cong\Delta_{\geq d(p,q)}.$$

and we also have

$$\pi(p,q)=\pi(p,B(p,d(p,q)))\cup\pi(q,B(p,d(p,q))).$$

Notation 3.10. We denote $\widetilde{K^3}:=\{(p_1,p_2,p_3)\in K^3/p_1\neq p_2\neq p_3\neq p_1\}/\sim=(K^3\backslash\Lambda)/\sim$, where Λ are the points (p_1,p_2,p_3) such that $p_1\neq p_2\neq p_3\neq p_1$.

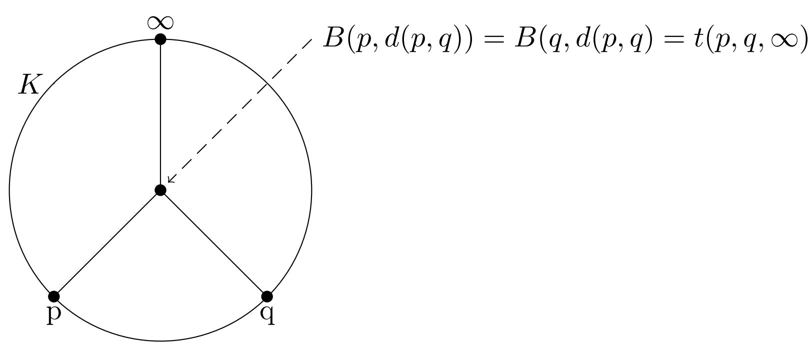


Figure 1: T(K) with some points and paths.

For all different elements of $T(K)$ there are a path between these elements.

Definition 3.11. We define $t:\widetilde{K^3}\to T(K)$ as $t(p_1,p_2,p_3):=B(p_i,\rho)$ where ρ is the smallest distance between the three points and p_i is one of the two elements that gives this smallest distance. We also have $t(p_1,p_2,\infty):=B(p_1,|p_1-p_2|)$.

3.2 Topology of K given by its valuation

In this section we will give the topology of K in order to define a notion of compactness. Then we will see that if $\mathcal{L}$ is compact then there are a finite number of vertex between two given vertex.

Recall that $|.|$ is a valuation from the field K to a totally ordered group Δ_0 which includes a minimal element call it 0. To define a topology we have to give a base of opens. First we will define what we call **basic open** $U\in\mathbb{P}^1(K)=K\cup\{\infty\}$ that will consist in

$$U=\begin{cases}B(p,\rho) & \text{for } p\in K,\rho\in\Delta,\\ B^0(p,\rho)^c=\{z\in K\mid |z-p|\geq\rho\}\cup\{\infty\}.\end{cases}$$

The base of opens is formed by finite intersections of basic opens.

Notation 3.12. We denote an annulus as $C(p,\rho_1,\rho_2) = \{z \in K \mid \rho_1 \le |z-p| \le \rho_2\}$.

Lemma 3.13. *The opens of the base are of the form $C(p,\rho_1,\rho_2)$ where $\rho_1 \in \Delta_0$, $\rho_2 \in \Delta \cup \{\infty\}$ and $\rho_1 \le \rho_2$.*

Proof. If it is a basic open then or $B(p,\rho) = C(p,0,\rho)$ or $B^0(p,\rho)^c = C(p,\rho,\infty)$.

If not, then it is a finite intersection of basic opens. But the intersection of two balls $B(p_1,\rho_1)$, $B(p_2,\rho_2)$ or is the empty set or is the smallest of this two balls. The intersection of $B^0(p_1,\rho_1)^c$ and $B^0(p_2,\rho_2)^c$ is

$$B^0(p_1,\rho_1)^c \cap B^0(p_2,\rho_2)^c = (B^0(p_1,\rho_1) \cup B^0(p_2,\rho_2))^c$$

and it the complementary of the ball with the biggest ρ_i. So we only have to see what is $B(p_1,\rho_1) \cap B^0(p_2,\rho_2)^c$. By the remark 3.5 we only have to prove it for $B(p,\rho_1) \cap B^0(p,\rho_2)^c$ for $p \in \{p_1,p_2\}$ and it is

$$\begin{aligned}&\{z \in K \mid |z-p| \le \rho_1\} \cap (\{z \in K \mid |z-p| \ge \rho_2\} \cup \{\infty\}) = \\ &= \{z \in K \mid \rho_1 \le |z-p| \le \rho_2\} = C(p,\rho_1,\rho_2).\end{aligned}$$

□

We know how are the opens of the base, so an open is a union (finite or not) of annulus. Now its time to define the notion of compactness:

Definition 3.14. We say that $\mathcal{L} \in \mathbb{P}^1(K)$ is **compact** if given a covering $\mathcal{L} \subset \bigcup_{i\in I} C(p_i,\rho_i,\rho_i')$, of annulus of $\mathbb{P}^1(K)$, exists $J \subset I$ finite with $\mathcal{L} \subset \bigcup_{j\in J} C(p_j,\rho_j,\rho_j')$.

Before to prove the main result we have this Lemma.

Lemma 3.15. *Let $\mathcal{L}$ be a compact such that $\mathcal{L} \subset \bigcup_{i\in I} B_i$ with $\mathcal{L} \cap B_i \ne \emptyset$ for all $i \in I$ and $B_i \cap B_j = \emptyset$ for all $i,j \in I$ different then $|I| < \infty$ (the union is finite).*

Proof. We have $\bigcup_{i\in I} \mathcal{L} \cap B_i \subset \mathcal{L}$ so, since $\mathcal{L}$ is compact exist a finite set J with $\bigcup_{j\in J} K \cap B_j \subset \mathcal{L}$. Now we consider an $i \in I\backslash J$. We take an $x \in \mathcal{L} \cap B_i$. But we have $x \in \mathcal{L}$ and since the B_i are disjoint $x \notin \bigcup_{j\in J} \mathcal{L} \cap B_j \subset \mathcal{L}$. Contradiction, we can not consider any element of $I\backslash J$, so $I = J$ and it means that the initial union is also finite as we claimed. □

3.3 The graph $T(\mathcal{L})$ is a tree

Definition 3.16. Let $\mathcal{L} \subset \mathbb{P}^1(K)$ be with at least three elements. We define

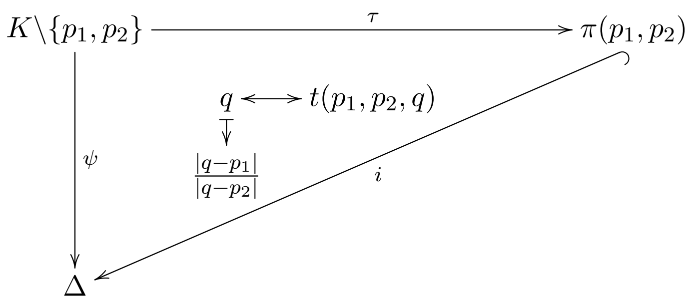
$$T(\mathcal{L}) = \bigcup_{\substack{p_1,p_2\in\mathcal{L} \\ p_1\ne p_2}} \pi(p_1,p_2)$$

and the vertices of $T(\mathcal{L})$ as

$$\mathrm{V}(T(\mathcal{L})) = \{t(p_1,p_2,p_3) \mid p_1,p_2,p_3 \in \mathcal{L} \text{ different }\}.$$

Note that $\mathrm{V}(T(\mathcal{L})) \subset T(\mathcal{L})$.

To define an edge we need to give an order between to elements of a path. In order to do that we will see that we have an injective map from $\pi(p_1, p_2)$ to Δ, which is a totally ordered group:



Proposition 3.17. *The map i is injective.*

Proof. We will see that

$$\begin{aligned} i(t(p_1, p_2, q)) = i(t(p_1, p_2, q')) &\Leftrightarrow t(p_1, p_2, q) = t(p_1, p_2, q') \\ \psi(q) = \psi(q') &\Leftrightarrow \tau(q) = \tau(q') \\ \frac{|q-p_1|}{|q-p_2|} = \frac{|q'-p_1|}{|q'-p_2|} &\Leftrightarrow t(p_1, p_2, q) = t(p_1, p_2, q'). \end{aligned}$$

$\Leftarrow$) Suppose that

$$|p_1 - q| = |p_2 - q| \tag{6}$$
$$|p_1 - q'| = |p_2 - q'| \tag{7}$$

are both true. In this case $t(p_1, p_2, q) = B(p_1, |p_1 - p_2|)$ and $t(p_1, p_2, q') = B(p_1, |p_1 - p_2|)$ and the two bigger sides are equals, i.e. $|p_1 - q| = |p_2 - q|$ and $|p_1 - q| = |p_2 - q|$ so

$$\frac{|p_1 - q|}{|p_2 - q|} = 1 = \frac{|p_1 - q'|}{|p_2 - q'|}.$$

Now we see that if (6) or (7) is not satisfied then necessary both are not satisfied: suppose that $|p_1 - p_2| = |p_1 - q| > |p_2 - q|$, and we also have $B(p_2, |p_2 - q|) = B(p_2, |p_1 - p_2|)$, which is not possible. So we can suppose that

$$\begin{aligned} |p_1 - p_2| = |p_1 - q| > |p_2 - q| \\ |p_1 - p_2| = |p_1 - q'| > |p_2 - q'|. \end{aligned}$$

So the numerators are equals. We only have to see that $|p_2 - q| = |p_2 - q'|$, and it is true by hypothesis, because they are the radius of two equal balls with the same center.

$\Rightarrow$) We only have to prove that the smallest distances between p_1, p_2, q and p_1, p_2, q' are equal. We have

$$\frac{|p_1 - q|}{|p_2 - q|} = \lambda = \frac{|p_1 - q'|}{|p_2 - q'|}.$$

If $\lambda = 1$, it means that the third edge of both parts are $|p_1 - p_2|$. If $\lambda > 1$ we have

$$|p_1 - p_2| = |q - p_1| \leq |q - p_2|$$
$$|p_1 - p_2| = |q' - p_1| \leq |q' - p_2|,$$

so

$$\frac{|p_1 - q|}{|p_2 - q|} = \frac{|p_1 - p_2|}{|p_2 - q|} = \frac{|p_1 - p_2|}{|p_2 - q'|} = \frac{|p_1 - q'|}{|p_2 - q'|}$$

and hence $|p_2 - q| = |p_2 - q'|$. And finally if $\lambda < 1$

$$|p_1 - p_2| = |q - p_2| \leq |q - p_1|$$
$$|p_1 - p_2| = |q' - p_2| \leq |q' - p_1|,$$

so

$$\frac{|p_1 - q|}{|p_2 - q|} = \frac{|p_1 - q|}{|p_2 - p_1|} = \frac{|p_1 - q'|}{|p_2 - p_1|} = \frac{|p_1 - q'|}{|p_2 - q'|}$$

and hence $|p_1 - q| = |p_1 - q'|$. $\square$

Remark 3.18. Given p_1, p_2 different points of $\mathcal{L}$. Let $v_1, v_2 \in \pi(p_1, p_2)$ different we can order these two points. Recall that $\pi(p_1, p_2) = \{t(p_1, p_2, q) \mid q \in \mathcal{P}^1(K) \backslash \{p_1, p_2\}\}$. So if $v_1, v_2 \in \pi(p_1, p_2)$ there exist $q_1, q_2 \in \mathcal{P}^1(K) \backslash \{p_1, p_2\}\}$ different such that $v_1 = t(p_1, p_2, q_1)$ and $v_2 = t(p_1, p_2, q_2)$. So we can say that

$$v_1 \leq v_2 \Leftrightarrow \frac{|q_1 - p_1|}{|q_1 - p_2|} \leq \frac{|q_2 - p_1|}{|q_2 - p_2|}.$$

So it will allow us to order the vertex t and t' when they share two of its three points i.e. they are in the same path $\pi(p_1, p_2)$ for some $p_1, p_2 \in \mathcal{L}$. Later we will see in Lemma (3.31) that for all t, t' there exists $p_1, p_2 \in \mathcal{L}$ such that $t, t' \in \pi(p_1, p_2)$. Hence this order relation will be useful on the next pages.

Definition 3.19. Given p_1, p_2 different points of $\mathcal{L}$. Let $v_1, v_2 \in \pi(p_1, p_2) \cap \mathrm{V}(T(\mathcal{L}))$ different, with $v_1 < v_2$. Then we define

$$[v_1, v_2] = \{w \in \pi(p_1, p_2) \mid v_1 \leq w \leq v_2\}$$

and

$$(v_1, v_2) = \{w \in \pi(p_1, p_2) \mid v_1 < w < v_2\}.$$

Definition 3.20. Given $v_1, v_2 \in \mathrm{V}(T(\mathcal{L}))$. We say that $[v_1, v_2]$ is an **edge** of $T(\mathcal{L})$ and that v_1, v_2 are **the ends of an edge** if

$$(v_1, v_2) \cap \mathrm{V}(T(\mathcal{L})) = \emptyset.$$

Definition 3.21. We define the set of edges as

$$\mathrm{E}(\mathcal{L}) = \{[v_1, v_2] \mid v_1, v_2 \in \mathrm{V}(T(\mathcal{L})) \text{ which are an edge}\}.$$

From now our aim is prove that the path is well defined in order to see that is a tree. First we will see in the case that given t and t' that gives two balls such that one is contained on the other the path is well defined. Then we will see that for any t and t' the path is well defined. We will see this considering a ball such that contains the balls of t and t'.

Lemma 3.22. *Let $t, t' \in V(T(\mathcal{L}))$ then $[t, t']$ is well defined, i.e. do not depend of the path.*

Proof. We start considering the case where the vertices corresponds to a balls such that one is contained in the other: $t = B(p, \rho) < B(p, \rho') = t'$. We can suppose that $[t, t'] \subset \pi(x, y)$ for some $x, y \in \mathcal{L}$. We can suppose that $p = x$, and since belong to the path $\pi(x, y)$ they are of the form $t = t(x, y, q)$ and $t' = t(x, y, q')$. We have to see that let $\overline{t} = t(x, y, \overline{q})$ and $\overline{t} \in (t, t')$ then $\overline{t} = B(x, \overline{\rho})$, for $\rho < \overline{\rho} < \rho'$. But since $\overline{t} \in (t, t')$

$$\left|\frac{x-q}{y-q}\right| < \left|\frac{x-\overline{q}}{y-\overline{q}}\right| < \left|\frac{x-q'}{y-q'}\right|$$

and since $|x - q'| \leq |y - q'| = |x - y|$ we have

$$\left|\frac{x-q'}{y-q'}\right| \leq 1$$

so

$$\left|\frac{x-\overline{q}}{y-\overline{q}}\right| < 1$$

which means that $|x - \overline{q}| \leq |y - \overline{q}| = |x - y|$, hence $\overline{t} = B(x, |x - \overline{q}|) = B(p, |x - \overline{q}|)$ and

$$\rho = |x - q| < |x - \overline{q}| < |x - q'| = \rho'$$

so we call $\overline{\rho} = |x - \overline{q}|$.

Now, if $t = B(p, \rho)$ and $t' = B(p', \rho')$ do not intersect, then $|p - p'| > \rho, \rho'$, and, if we denote $t'' := B(p, |p - p'|) = B(p', |p - p'|)$, we have that $t < t'' < t'$ in $\pi(p, p')$, and $[t, t'] = [t, t''] \cup [t'', t']$, and we are reduced to the previous case. □

Corollary 3.23. *If $\mathcal{L}$ is finite then the graph $T(\mathcal{L})$ is a tree.*

Proof. We have seen that for any two vertices of the graph there is a unique way $[v, v']$ to pass from one to the other, but if $\mathcal{L}$ is finite, $\mathrm{V}(T(\mathcal{L}))$ is also finite, hence $[v, v']$ can be subdivided in a finite number o edges. So $T(\mathcal{L})$ is a tree. □

Example 3.24 (Idea of the structure of $T(\mathcal{L})$)**.** We know that in order to have a graph with some element we need to ask for $\mathcal{L}$ has at least three elements. In this case we can define a single vertex $t(p_1, p_2, p_3)$ where p_i are the points of $\mathcal{L}$.

If we add other point can happen two things; we obtain other t different from the previous, or we obtain the same t. Since $T(\mathcal{L})$ is a tree, in the first case we are adding one edge and one vertex. In the second case we do not add anything. So we can claim that if the number of points of $\mathcal{L}$ is finite then the number of vertex is bounded by $\#\mathcal{L} - 2$ and the number of edges is bounded by $\#\mathcal{L} - 3$.

3.4 Finiteness of $[v_1, v_2] \cap \mathbf{V}(T(\mathcal{L}))$

In order to show that $T(\mathcal{L})$ is a tree in general, we have to prove that any two vertices are joined by a finite number of edges, hence that $[v_1, v_2] \cap \mathrm{V}(T(\mathcal{L}))$ is finite.

Theorem 3.25. *Let $\mathcal{L} \in \mathbb{P}^1(K)$ compact. Then for $v_1, v_2 \in V(T(\mathcal{L}))$ one has that $[v_1, v_2] \cap V(T(\mathcal{L}))$ is finite.*

Proof. We can suppose that $v_1, v_2 \in \pi(p, \infty)$ for some $p \in K$. If not one can consider $[v_1, v_2] = [v_1, t(p_1, p_2, \infty)] \cup [t(p_1, p_2, \infty), v_2]$ (we have that $t(p_1, p_2, \infty) \in \mathrm{V}(\mathcal{L} \cup \infty) \supset \mathrm{V}(T(\mathcal{L})))$. We also can suppose that $\infty \in \mathcal{L}$ because if $\mathcal{L}$ is compact then $\mathcal{L} \cup \infty$ is also compact (the converse is not true).

If $v_1 = v_2$ then $[v_1, v_1] \cap \mathrm{V}(T(\mathcal{L})) = v_1$. So if $v_1 \neq v_2$ then $v_1 = B(p, \rho_1) \subsetneq B(p, \rho_2) = v_2$. Now

$$(v_1, v_2) \cap \mathrm{V}(T(\mathcal{L})) = \{B(p, \rho) \mid \rho_1 \leq \rho \leq \rho_2\}$$

where $B(p, \rho) = t(p, q, \infty)$ for some $q \in \mathcal{L}$. We want to see if there are a finite number of q. Each q is in $C(p, \rho, \rho)$ for $\rho_1 \leq \rho \leq \rho_2$. So we want to see how many different ρ there are satisfying this properties. We have

$$\mathcal{L} \subset B(p, \rho_1) \cup B^0(p, \rho_2)^c \cup \bigcup_{\rho_1 < \rho < \rho_2} C(p, \rho, \rho)$$

with $C(p, \rho, \rho) \cap \mathcal{L} \neq \emptyset$. And also each annulus is disjoint, so we can apply Lemma 3.15 and it means that we have a finite number of ρ so $(v_1, v_2) \cap \mathrm{V}(T(\mathcal{L}))$ is finite and $[v_1, v_2] \cap \mathrm{V}(T(\mathcal{L}))$ is also finite. □

Corollary 3.26. *The graph $T(\mathcal{L})$ is a tree.*

Proof. We need to show it is connected. But given two vertices v and v', we have $[v, v'] = \{v = v_0 < v_1 < v_2 < .. < v_n < v_{n+1} = v'\} = [v, v_1] \cup \cdots \cup [v_n.v']$ for some $n \geq 0$, and any of these $[v_i, v_{i+1}]$ are edges. Clearly this is the unique path from v to v', hence it is a tree. □

3.5 Locally finiteness of $T(\mathcal{L})$

Definition 3.27. We define the **edges** of a vertex as $E_v(T(\mathcal{L})) = \{[v,w] \mid (v,w) \cap \mathrm{V}(T(\mathcal{L})) = \emptyset\}$.

Notation 3.28. Is also usual denote $E_v(T(\mathcal{L}))$ as $\mathrm{Star}_{T(\mathcal{L})}(v)$.

Now we will see that our tree can be infinite but it also will be locally finite in the following sense:

Definition 3.29. We say that a tree is **locally finite** if $E_v(T(\mathcal{L}))$ is finite for all $v \in \mathrm{V}(T)$.

Remark 3.30. From now we will suppose that $\infty \in \mathcal{L}$. We can do that because if not, we can apply an automorphism (see Section 4.1) in our field such that sends any element of $\mathcal{L}$ to ∞, and by the definition of $T(\mathcal{L})$ if this three is locally finite, the tree obtained applying the automorphism will be also locally finite. Specifically we should talk about the three obtained applying the automorphism instead of $T(\mathcal{L})$, but we will do an abuse of notation calling it $T(\mathcal{L})$.

Lemma 3.31. *Take $v = t(p_1, p_2, p_3)$ and $w = t(q_1, q_2, q_3)$ then there exist $i, j \in \{1, 2, 3\}$ such that $v, w \in \pi(p_i, q_j)$.*

Proof. We can suppose that

$$|p_1 - p_3| = |p_2 - p_3| \geq |p_1 - p_2| \text{ and}$$
$$|q_1 - q_3| = |q_2 - q_3| \geq |q_1 - q_2|.$$

If not, we change the sub-indexes. So $v = B(p_1, |p_1 - p_2|) = B(p_2, |p_1 - p_2|)$ and $w = B(q_1, |q_1 - q_2|) = B(q_2, |q_1 - q_2|)$. If

$$|q_1 - p_1| = |q_1 - p_2| \tag{8}$$

and

$$|q_1 - p_1| = |q_2 - p_1|, \tag{9}$$

then we have that $v = t(p_1, p_2, q_1)$ and $w = t(q_1, q_2, p_1)$ as we want. If one of these two inequalities does not hold we can suppose

$$|q_1 - p_2| > |q_1 - p_1| \tag{10}$$

if not, change the sub-indexes. By (10) we have $|q_1 - p_2| = |p_1 - p_2|$, because they form a triangle and we have seen that always there are two equal sides and the third is less or equal than the others. We also have

$$|q_2 - p_2| > |q_2 - p_1|$$

and with this inequality and using the same argument than before we have $|q_2 - p_2| = |p_1 - p_2|$. So

$$|q_1 - p_2| = |q_2 - p_2|.$$

Hence or $v = t(p_1, p_2, q_1)$ or $w = t(q_1, q_2, p_1)$. We can suppose without lose generality $w = t(q_1, q_2, p_1)$ and we will found that our v is of the form that we claim.

At this moment we know that one of (8) or (9) does not hold, we have said that in fact, (9) is satisfied, but (8) can be satisfied or not so we consider the following two cases.

- $\underline{|q_1 - p_1| = |q_1 - p_2|}$. Then $|p_1 - p_2| \leq |q_1 - p_1|$ which means that $v = t(p_1, p_2, q_1)$.
- $\underline{|q_1 - p_1| > |q_1 - p_2|}$. Then $|q_1 - p_2| < |q_1 - p_1| = |p_1 - p_2| \leq |p_1 - p_3| = |p_2 - p_3| = |q_1 - q_3|$ so $|p_1 - q_1| \leq |q_1 - p_3| = |p_1 - p_3|$ which implies that $v = t(p_1, p_3, q_1)$.

□

Lemma 3.32. *Take $v_1 = (p, \infty, p_1)$ and $v_2 = t(p, \infty, p_2)$ with $p, p_1, p_2 \in \mathcal{L}$. Take also $w = t(q_1, q_2, q_3) \in \pi(v_1, v_2)$ with $q_1, q_2, q_3 \in \mathcal{L}$ then $w = t(p, \infty, p_3)$ for some $p_3 \in \mathcal{L}$.*

Proof. Since $T(\mathcal{L})$ is a tree, there is a unique path from v_1 to v_2 which consists in

$$[v_1, v_2] = \{t(p, \infty, q) \mid p_1 \leq q \leq p_2 \text{ for } q \in \mathcal{L}\},$$

where $\leq$ is the order defined in Remark 3.18. So if $w \in \pi(v_1, v_2)$ with $q_1, q_2, q_3 \in \mathcal{L}$ then $w = t(p, \infty, p_3)$ for some $p_3 \in \mathcal{L}$. □

Theorem 3.33. *$T(\mathcal{L})$ is locally finite.*

Proof. As in 3.25 we will construct a covering of our compact set $\mathcal{L}$ satisfying the conditions of the Lemma 3.15 and it will imply that the set $E_v(T(\mathcal{L}))$ is finite.

Recall that $E_v(T(\mathcal{L})) = \mathrm{Star}_{T(\mathcal{L})}(v) = \{w \in \mathrm{V}(T(\mathcal{L})) \mid (v, w) \cap \mathrm{V}(T(\mathcal{L})) = \emptyset\}$. Our v is of the form $B(p, \rho)$ (call it B) and the elements w of $\mathrm{Star}_{T(\mathcal{L})}(v)$ correspond to $B(p_w, \rho_w)$, and by notation we call it B_w. We also define the following set

$$\mathcal{L}' = \mathcal{L} \setminus \bigcup_{w \in \mathrm{Star}_{T(\mathcal{L})}(v)} B_w.$$

So our claim is that

$$\mathcal{L} \subset \bigcup_{w \in \mathrm{Star}_{T(\mathcal{L})}(v)} B_w \cup \bigcup_{p' \in \mathcal{L}'} B(p', \delta) \tag{11}$$

satisfies the hypothesis of Lemma 3.15 for some $\delta < \rho$.

For a given w can happen three options:

1. $B \cap B_w = B$,

2. $B \cap B_w = B_w$,

3. $B \cap B' = \emptyset$.

Note that the first case only can happen with a unique B_w: we can write $B_w = B(p, \rho')$. Recall that $\rho' > \rho$. Suppose that there are other $B_{w'}$ such that $B'' \cap B = B$. Then we consider $B' \cap B''$ which is or B' or B'', so or $w \in [v, w'] = \{v, w'\}$ or $w' \in [v, w] = \{v, w\}$ which implies in both cases that $w = w'$

We will see later that the third option is not possible. We will denote $\mathrm{Star}_{T(\mathcal{L})}(v)$ by $\mathrm{Star}(v)$.

And now we define

$$V_v = \{w \in \mathrm{Star}(v) \mid B \cap B' = B\}$$

and

$$V_v' = \{w \in \mathrm{Star}(v) \mid B \cap B' = B'\}.$$

We have that $\forall w, w' \in \mathrm{Star}(v)$ different, $B_w \cap B_{w'} = \emptyset$: let $w, w' \in V_v'$, if $B_w \cap B_{w'} = B_w$ then $[w, v] \in w'$ so $w' = w$. If $w' \in V_v$ then $B \cap B_w = B_w$ so $B^c \cap B_w = \emptyset$ and since $B_{w'} \subset B^c$ we have $B_{w'} \cap B_w = \emptyset$.

Notice that $B \cap B' = \emptyset$ is not possible. Fist we denote $B = B(p, \rho) = v$ and $B' = B(p', \rho') = v'$. Now take B'' as the ball that corresponds to $t(p, p', \infty)$ so $B'' = B(p, |p - p'|) = B(p', |p - p'|)$, and we define $\rho = |p - p'|$. Note that $p, p' \in B(p, |p - p'|)$ so $p \in B'' \cap B$ so this two balls are not disjoint, so one is contained in the other. If the intersection is B'' then we have that $p' \in B''$ so $p' \in B'$, which means that the intersection between B and B' is not empty. On the other hand if B is contained in B'' it means that $|p - p'| > \rho$. Now the ball B'' corresponds to some $v'' \in \pi[p, \infty] \cap \pi[p', \infty]$. And since $[v', v''] \cap [v'', v] = \{v''\}$ it means that $v'' \in [v', v]$ so $v'' = v$ or $v' = v$. This last step is true because $B(\rho', \alpha) \in [v', v'']$ and $B(\rho, \beta) \in [v'', v]$ for $\rho' \leq \alpha \leq \rho''$ and $\rho \leq \beta \leq \rho''$. So we can write $\mathrm{Star}(v) = V_v \cup V_v'$, as a disjoint union.

Finally $\forall p' \in \mathcal{L}'$ ad $\forall r \in \mathcal{L}$ different from p' one has that $|r - p'| \geq \rho$. Because if $|r - p'| < \rho$ then $v \neq v' = t(p', r, \infty) \in \mathrm{V}(T(\mathcal{L}))$ which implies that $[v', v] \cap \mathrm{Star}(v) = \{w\}$ hence $p' \in B_w$ which is not possible.

Summarising, take $\delta < \rho$. Is true that $B(p', \delta) \cap \mathcal{L} = \{p'\}$ and

$$\mathcal{L} \subset \bigcup_{w \in \mathrm{Star}_{T(\mathcal{L})}(v)} B_w \cup \bigcup_{p' \in \mathcal{L}'} B(p', \delta)$$

with the balls disjoint in $\mathcal{L}$ and we can not skip any ball so by Lemma 3.15, $\mathrm{Star}(v)$ is finite, so $T(\mathcal{L})$ is locally finite. □

3.6 Rays

Definition 3.34. Let $\mathcal{T}$ be a tree. A **ray** on $\mathcal{T}$ is an infite progression in the following sense: is a sequence $v_0, v_1, \ldots$ such that $[v_i, v_{i+1}]$ is an edge and $v_i \neq v_j \ \forall i \neq j$.

Given a progression $v_0, v_1, \ldots, v_n, \ldots$ of distinct vertices we say they generate a ray if the progression formed by the ordered set $\bigcup_{i \geq 0} \mathrm{V}([v_i, v_{i+1}])$ is a ray, which we call the ray generated by the v_n's. And we denote $\mathrm{Rays}(T(\mathcal{L}))$ the set of rays of $T(\mathcal{L})$.

An immediate property is the following:

Lemma 3.35. *Let $[v_0, v_1]$ and $[v_1, v_2]$ vertex with $v_0 \neq v_2$. Each vertex corresponds to a ball denoted B_0, B_1 and B_2 respectively. Then if $B_0 \cap B_2 = \emptyset$ we have that B_1 contains B_0 and B_2; and if $B_0 \cap B_2 \neq \emptyset$ either $B_2 \subset B_1 \subset B_0$ or $B_1 \subset B_1 \subset B_2$.*

Proof. If B_0 and B_2 are disjoint but they are linked to the same vertex then they must be in the ball corresponding to this vertex. If $B_0 \cap B_2 \neq \emptyset$, let p in the intersection. Since they are linked to B_1, p is also in B_1, so the possibilities are either $B_2 \subset B_1 \subset B_0$ or $B_1 \subset B_1 \subset B_2$. □

And a direct consequence from this fact is that set $(v_0, v_1, \ldots)$ be a ray then exists $m \geq 0$ such that either $B_i \subset B_{i+1}$ for all $i \geq m$ or $B_{i+1} \subset B_i$ for all $i \geq m$. In fact, once you found two balls in the sequence such that $B_{j+1} \subset B_j$ then $B_{i+1} \subset B_i$ for all $i \geq j$.

Definition 3.36. We say that $\mathcal{L}$ is **perfect** if does not have isolated points in the sense that $\forall x \in \mathcal{L}$, $\exists \{x_i\}_i$ with $x_i \in \mathcal{L}$ and $x_i \neq x_j$ for all $i \neq j$ such that $\lim_{i \mapsto \infty} x_i = x$.

Proposition 3.37. *If, given $m \geq 0$, $B_{i+1} \subset B_i$ for all $i \geq m$, where the balls corresponds to the vertex of a ray defined as above then*

$$\bigcap_{i \geq m} (B_i \cap \mathcal{L}) = \{p\} \ , \ \textit{where } p \in \mathcal{L}. \tag{12}$$

Proof. For any i, $B_i \cap \mathcal{L} \neq \emptyset$, where $B_i = t(p, p', p'')$ for $p, p', p'' \in \mathcal{L}$ and two of them are in B_i. In fact $(B_i \cap \mathcal{L}) \backslash (B_{i+1} \cap \mathcal{L}) \neq \emptyset$, so since $B_i \cap \mathcal{L}$ are closed in $\mathcal{L}$ and non empty

$$\bigcap_{i \geq m} B_i \cap \mathcal{L} \neq \emptyset.$$

Now we have to see that in the intersection there is a unique points. Suppose p_1, p_2 different in $\bigcap_{i \geq m} B_i \cap \mathcal{L}$. Then we take $p_3 \in B_m \cap \mathcal{L}$ different from p_1 and p_2. Let $v = t(p_1, p_2, p_3) \in \mathrm{V}(T(\mathcal{L}))$, note that $v_i \in [v_m, v]$ for any $i \geq m$ because $B_m \supset B_i \supset B(p_1, |p_1, p_2|) = v$, but it is not possible because we know that $\#[v_m, v] \cap \mathrm{V}(T(\mathcal{L})) < \infty$. So $\bigcap_{i \geq m} B_i \cap \mathcal{L} = \{p\}$. □

Conversely, a sequence of nested balls $B_i \supset B_{i+1}$ for all $i > 0$ generate a ray if they intersection $\bigcap (B_i \cap L)$ is a point.

Theorem 3.38. *Let $\mathcal{L}$ be a compact subset of $P^1(K)$. Then there is a well defined map*

$$\Psi : Rays(T(\mathcal{L})) \to \mathcal{L}$$

whose image is the set of non isolated points.

Proof. If $r = (v_i)$ with corresponding balls verifying $B_{i+1} \subset B_i$ for some i, then $B_{i+1} \subset B_i$ for all $i > m$ and Proposition 3.37 implies that $\bigcap_{i \geq m}(B_i \cap L) = \{p\}$. We define then $\Phi(r) = p$. If not, so $B_{i+1} \supset B_i$ for all i, we take $\Phi(r) = \infty$.

If $p \neq \infty$ is in $\mathrm{Im}(\Psi)$, so $p = \Psi(r)$ with $r = (v_i)$, then $v_i = B_i = B(p_{v_i}, r_{v_i})$ for some $p_{v_i} \neq p$, $|p - p_{v_i}| = r_{v_i}$. Since $B_j \subset B_i$, for all $j > i > m$, for some m, we have $p_{v_i} \neq p_{v_j}$ and $|p - p_{v_i}| = r_{v_i} \to 0$ when i tends to ∞ by Proposition 3.37.

Moreover, any non-isolated point $x \in L$ is in the image of Φ, since, if $x_i \in L$, $x_i \neq x_j$ for $i \neq j$ and $\lim x_i = x$, then $v_i := t(x_1, x_i, x)$ for $i > 1$ large enough generate a ray. To show this, suppose $x \neq \infty$ (the case $x = \infty$ is done by an analogous argument). Then for i large enough, v_i corresponds to a ball B_i around x and $B_i \supset B_{i+1}$ since x_i converge to x. □

Corollary 3.39. *If $\mathcal{L}$ is compact and perfect, then $T(\mathcal{L})$ is a locally finite tree with all vertices of valence strictly bigger than 2 and Ψ is surjective.*

Proof. Let v a vertex of $T(\mathcal{L})$, corresponding to a ball $B(p, \delta) = t(p, p', p'')$ for some $p, p', p'' \in \mathcal{L}$ with $\delta = |p - p'|$. Take $\epsilon < \delta$ in Δ. Then, since $\mathcal{L}$ is perfect, $B(p, \epsilon) \cap \mathcal{L}$ contains another point $r \in \mathcal{L}$, and $t(p, r, p'') \neq t(p, p', p'')$. Similarly, there exists $p' \neq r' \in \mathcal{L} \cap B(p', \epsilon)$ and moreover $B(p', \epsilon) \cap B(p, \epsilon) = \emptyset$, and even with p'', with some changes in the case that $p'' = \infty$. So we have vertices v', v'' and v''' connected with disjoint paths to v, which means v has valence 3 or larger. □

4 Schottky Groups

4.1 The subgroup $PGL_2(K)$

Definition 4.1. Given a field K complete by a non-Archimedian absolute value, we define

$$PGL_2(K) = GL_2(K)/K^* Id.$$

Remember that $GL_2(K) = \{A \in M_2(K) \mid \det(A) \neq 0\}$. And the equivalence relation consists in the following. Take

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $A \sim A' \Leftrightarrow \exists x \in K^*$ such that

$$a = xa', b = xb', c = xc' \text{ and } d = xd',$$

i.e. $A = xA'$.

Note that we have the following bijection:

$$A \leftrightsquigarrow \varphi$$

where

$$\begin{aligned}\varphi : \mathbb{P}^1(K) &\longrightarrow \mathbb{P}^1(K)\\ x &\longmapsto \frac{ax+b}{cx+d}\end{aligned}$$

where $\mathbb{P}^1(K) = K \cup \{\infty\}$. So $\mathrm{Aut}(\mathbb{P}^1(K)) \cong PGL_2(K)$.

Notation 4.2. Since $\mathrm{Aut}(\mathbb{P}^1(K)) \cong PGL_2(K)$, we will denote φ as a matrix.

Now, an important fact which says that with few information of φ, it is determined:

Proposition 4.3. *Given P_0, P_1 and $P_\infty \in \mathbb{P}^1(K)$ different, then exist a unique automorphism φ such that $\varphi(0) = P_0$, $\varphi(1) = P_1$ and $\varphi(\infty) = P_\infty$.*

Proof. Recall that

$$\varphi(z) = \frac{az+b}{cz+d} \tag{13}$$

with $ad - bc \neq 0$. We replace the values of z in the function and we obtain the following three equations:

$$\begin{aligned}p_0 = \varphi(0) &= \frac{b}{d}\\ p_1 = \varphi(1) &= \frac{a+b}{d+c}\\ p_\infty = \varphi(\infty) &= \frac{a}{c}.\end{aligned}$$

From this we can isolate and obtain

$$\begin{aligned}a &= p_\infty \frac{p_0 - p_1}{p_1 - p_\infty} d\\ b &= p_0 d\\ c &= \frac{p_0 - p_1}{p_1 - p_\infty} d.\end{aligned}$$

Note that we have a one degree of freedom (we can choose any d), but adding it in to the formula (13) we have

$$\begin{aligned}\varphi(z) = \frac{az+b}{cz+d} &= \frac{p_\infty(\frac{p_0-p_1}{p_1-p_\infty})dz + p_0 d}{(\frac{p_0-p_1}{p_1-p_\infty})dz + d}\\ &= \frac{p_\infty(\frac{p_0-p_1}{p_1-p_\infty})z + p_0}{(\frac{p_0-p_1}{p_1-p_\infty})z + 1} = \frac{p_\infty(p_0-p_1)z + p_0(p_1-p_\infty)}{(p_0-p_1)z + p_1 - p_\infty}.\end{aligned}$$

And using that p_0, p_1 and p_∞ are different, the denominators does not give problems and the three conditions are satisfied, so φ is determined. $\square$

Now we can study the fixed points of φ. A first result can be

Remark 4.4. Let φ defined as above. Then φ has exactly 2 fixed points if and only if A diagonalises in K, where A is the corresponding matrix of φ of $PGL_2(K)$.

As a example we can consider

Example 4.5. If we take $\mathcal{L} = \{q^n : n \in \mathbb{Z}\} \cup \{0, \infty\}$, then $\mathcal{L}$ acts in $T(\mathcal{L})$, and the quotient is a graph $T(\mathcal{L})/\mathcal{L}$.

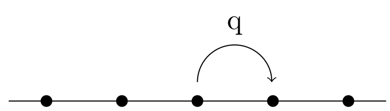


Figure 2: The graph $T(\mathcal{L})$.

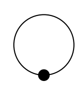


Figure 3: The graph $T(\mathcal{L})/\mathcal{L}$.

We also have other example that will give as some intuition.

Example 4.6.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It does not diagonalises and $\varphi(x) = x + 1$, which has only one fixed point: ∞.

4.2 Hyperbolic matrices

Definition 4.7. Let $A \in PGL_2(K)$ we say that is **hyperbolic** if there exists $P \in PGL_2(K)$ such that $PAP^{-1} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, where $q \in K$ is topologically nilpotent.

In order to work with any element of $PGL_2(K)$ (without be like $Ax = gx$ for some $g \in K$), we need to identify if a matrix is conjugated to a topologically nilpotent matrix without finding explicitly the conjugate matrix. We will do that considering the trace and the determinant of the matrix. We will see that q is nilpotent if and only if $\frac{\det(A)}{\mathrm{Tr}(A)^2}$ is topologically nilpotent.

In order to prove that if $A \in PGL_2(K)$ with $\frac{\det(A)}{\mathrm{Tr}(A)^2}$ topologically nilpotent then A is hyperbolic we will need a version of Hensel's Lemma proved in Section 2.6. In fact the proof is over $\mathbb{Z}_p$ but it is also valid in our case. We will use the completeness of K, because from the trace and the determinant we will have to found a root of a polynomial and we will do that using the Newton's Method.

Proposition 4.8. *Let K be a complete field. Consider $A \in PGL_2(K)$, then $\frac{\det(A)}{Tr(A)^2}$ topologically nilpotent if and only if A is hyperbolic.*

Proof. $\Leftarrow$) If A is hyperbolic its characteristic polynomial is of the form

$$(x - \alpha)(x - \beta) = x^2 - ax + b$$

where $a = \mathrm{Tr}(A)$ and $b = \det(A)$ and also $\frac{\alpha}{\beta}$ is topologically nilpotent. From this we have to see that $\frac{b}{a^2}$ is also topologically nilpotent, and it is true because

$$\frac{b}{a^2} = \frac{\frac{\alpha}{\beta}}{(\frac{\alpha}{\beta} + 1)^2},$$

and $|(\frac{\alpha}{\beta}) + 1| = 1$ since $|\frac{\alpha}{\beta}| < 1$, hence $|\frac{b}{a^2}| = |\frac{\alpha}{\beta}|$.

Now we see the other implication, which is not so simple because we have to guarantee that the roots are in K, here we will use the completeness of K.

$\Rightarrow$) We take A a representative with coefficients in $\mathcal{O}_K$. Let $a = \mathrm{Tr}(A)$ and $b = \det(A)$. Then the matrix A has a characteristic polynomial of the form $f(x) = x^2 - ax + b$. We have to found the roots of this polynomial. We have $f(0) = b$, $f'(0) = -a$ and by hypothesis $t = \frac{|f(0)|}{|f'(0)|^2}$ is topologically nilpotent. Hence by Hensel's Lemma exists $\alpha \in \mathcal{O}_K$ with $f(\alpha) = 0$. So exist $\beta \in \mathcal{O}_K$ sucht that $f(x) = (x - \alpha)(x - \beta)$. note that $\beta \in \mathcal{O}_K$ because α and $\alpha + \beta \in \mathcal{O}_K$.

Moreover $\frac{\alpha}{\beta} = 1$ is not possible because it is equivalent to say that

$$|\alpha| = |\beta|$$

and then

$$\left|\frac{\alpha\beta}{(\alpha + \beta)^2}\right| = \frac{|\alpha|^2}{|\alpha + \beta|^2} < 1$$

so $|\alpha| = |\alpha + \beta|$ and thus $|\beta| = |\alpha| = |\alpha + \beta|$ which is a contradiction.

So summarising we have that or $|\frac{\alpha}{\beta}| < 1$ or $|\frac{\beta}{\alpha}| < 1$, so one of these is is equal to $\left|\frac{\det(A)}{\mathrm{Tr}(A)^2}\right|$ and so it is topologically nilpotent.

□

Theorem 4.9. *Let K be complete. Suppose $\gamma \in PGL_2(K)$ diagonalizable. Let $\Gamma = \langle\gamma\rangle$ be the subgroup generated by γ. For any $p \in \mathbb{P}^1(K)$, let $\Gamma_p = \{\gamma^n p \mid n \in \mathbb{Z}\}$ the orbit of p. Then*

$$\overline{\langle\gamma\rangle p} \text{ is compact for all } p \in \mathbb{P}^1(K) \Leftrightarrow \begin{cases} \gamma \text{ is hyperbolic,} \\ \gamma \text{ is of finite order.} \end{cases}$$

Proof. $\Leftarrow$) First note that if γ is of finite order it means that $\langle\gamma\rangle p$ is finite, is a set of a finite number of points, so it is compact. So it remains to be proved that is γ is hyperbolic then $\overline{\langle\gamma\rangle p}$ is compact.

We have that our hyperbolic matrix γ is conjugated to

$$\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

with q topologically nilpotent. The closure of the orbit of γ will have the same structure. So we have $\gamma(p) = qp$, $p \in K$. Note that

$$\langle\gamma\rangle p = \begin{cases} \{p\} \text{ if } p = 0 \text{ or } p = \infty, \\ \{q^n p \mid n \in \mathbb{Z} \text{ if } p \neq 0, \infty\}. \end{cases}$$

We have to see that the second case is also compact. First note that $\overline{\langle\gamma\rangle p} = \{q^n p \mid n \in \mathbb{Z}\} \cup \{0, \infty\}$ because $\lim_{n\to\infty} q^n p = 0$ and $\lim_{n\to-\infty} q^n p = \infty$. Let $\overline{\langle\gamma\rangle p} = \bigcup_{i\in I} B_i$ be a covering of open balls. Since 0 is in the covering we have an open ball that contains it, and must be of the form $B = B(0, \rho)$. By the same reasoning there are a ball that contains ∞ like $B' = B^0(0, \rho')^c$.

Our claim is that $\overline{\langle\gamma\rangle p} \backslash (B \cup B')$ is finite. It is true because

$$|q^n p| \xrightarrow[n\to\infty]{} 0 \Rightarrow q^n p \in B(0, \rho), \text{ for all } n \geq n_0$$

and

$$|q^n p|^{-1} \xrightarrow[n\to-\infty]{} 0 \Rightarrow q^n p \in B^0(0, \rho')^c, \text{ for all } n \geq n_0'.$$

$\Leftarrow$) We can reduce to consider the case where

$$\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

with $|q| \leq 1$ and q not topologically nilpotent. Since the matrix is well defined up to constants it also covers the case that $|q| > 1$ and q^{-1} is not topologically nilpotent. The case $|q| > 1$ and q^{-1} is topologically nilpotent means that our matrix is hyperbolic so it has not to be considered here.

Since q is not topologically nilpotent we have that, by Lemma 2.23,

$$|q^n| > \delta > 0$$

for some δ. We only have to found a point such that its orbit is not compact, for example we consider 1. Hence $\gamma^n(1) = q^n$. So, for $n > m$,

$$|q^n - q^m| = |q|^m|q^{n-m} - 1| = |q|^m|1| = |q^m|\delta.$$

It means that any point is not close to other, so there are not Cauchy successions, so there are not limit points. So

$$\overline{\{\gamma^n(1) \mid n \in \mathbb{Z}\}} = \{\gamma^n(1) \mid n \in \mathbb{Z}\} = \{q^n \mid n \in \mathbb{Z}\},$$

and it can be covered by

$$\bigcup_{n\in\mathbb{Z}} B(q^n, \delta)$$

which satisfies

$$B(q^n, \delta) \cap \overline{\{\gamma^n(1) \mid n \in \mathbb{Z}\}} = \{q^n\}$$

and are pairwise disjoint, so we can not remove any ball which means that is not compact. Since $|q|^n > \delta$ for all $n \in \mathbb{Z}$ then $|q|^{-n} > \delta^{-1}$ for all $n \in \mathbb{Z}$, so

$$\overline{\{\gamma^n(1) \mid n \in \mathbb{Z}\}} \subset B(0, \delta^{-1}).$$

□

Note that in the proof of the theorem we have shown the following result.

Corollary 4.10. *Let $\gamma \in PGL_2(K)$ be an hyperbolic matrix. Let $p \in \mathbb{P}^1(K)$ such that $\gamma(p) \neq p$. Then $\lim_{n\to\infty} \gamma^n(p)$ and $\lim_{n\to-\infty} \gamma^n(p)$ exists and are the two fixed points by γ.*

Proof. By conjugation, reduce the case

$$\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

with q topologically nilpotent. In this case $\lim_{n\to\infty} |q^n||p| = |0|$ so $\lim_{n\to\infty} \gamma^n p = 0$ and the fixed points are 0 and ∞. □

Theorem 4.11. *Let K be a complete field and algebraically closed and either $char(K) = p > 0$ or $char(\mathcal{O}_K)/\mathfrak{m}_K = 0$. Then for $\gamma \in PGL_2(K)$*

$$\overline{\langle\gamma\rangle p} \text{ is compact for all } p \in \mathbb{P}^1(K) \Leftrightarrow \begin{cases} \gamma \text{ is hyperbolic,} \\ \gamma \text{ is of finite order.} \end{cases}$$

Proof. ⇐) Is the same that Theorem 4.9. ⇒). Since K is algebraically closed we can suppose that either γ is diagonalizable or is conjugate to

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $a \in K \backslash \{0\}$. If A diagonalises we use the Theorem 4.9. If not we can consider

$$A = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

And we take the orbit of 0, $\langle \gamma \rangle 0 = \{na \mid n \in \mathbb{Z}\}$. If $\mathrm{char}(K) = p > 0$ the γ has finite order, in fact p. We can suppose that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\mathcal{O}_K/\mathfrak{m}_K) = 0$. Then $|n| = 1$ for all $n \in \mathbb{Z}$ so $|na| = |a|$ for all $n \in \mathbb{Z}$. Moreover if $n \neq m$ then $|na - ma| = |n - m||a|$. So let $\epsilon < |a|$ and consider the balls $B(na, \epsilon)$ for $n \in \mathbb{Z}$. Note that the balls are disjoint and moreover the closure of this set is itself because there are not Cauchy sequences so $\overline{\langle A \rangle} 0 = \langle A \rangle 0 = \bigcup_{m \in \mathbb{Z}} B(na, \epsilon)$, and it is a infinite union of disjoints balls so $\overline{\langle A \rangle} 0$ is not compact.

□

Remark 4.12. We need to ask for either $\mathrm{char}(K) = p > 0$ or $\mathrm{char}(\mathcal{O}_K)/\mathfrak{m}_K = 0$ because the result is not true if $\mathrm{char}(K) = 0$ and $\mathrm{char}(\mathcal{O}_K/\mathfrak{m}_K) = p > 0$, because in this case, by Lemma 2.18 $|.||_{\mathbb{Q}} = |.|_p^{\epsilon}$, where $|.|_p^{\epsilon}$ denotes the p-adic absolute value. In this case the closure of the orbit of 0 for

$$A = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

is $\mathbb{Z}_p$, the p-adic integers, which is compact. And the same argument work for any $p \in \mathbb{P}^1(K)$.

4.3 Schottky groups

Definition 4.13. Let

$$\mathrm{Fix}(\Gamma) = \{p \in \mathbb{P}^1(K) \mid \exists \gamma \in \Gamma, \gamma \neq \mathrm{Id} \text{ with } \gamma(p) = p\}$$

and we define $\mathcal{L}_\Gamma = \overline{\mathrm{Fix}(\Gamma)}$ as the closure of $\mathrm{Fix}(\Gamma)$.

Lemma 4.14. *Let γ be hiperbolic then $t(p_1, p_2, p_3) \neq t(\gamma p_1, \gamma p_2, \gamma p_3)$.*

Proof. We can suppose that $\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, for $|q| < 1$. So $t(p_1, p_2, p_3)$ corresponds (reordering if is necessary) to $B(p_1, |p_1 - p_2|)$ but also $t(qp_1, qp_2, qp_3)$ corresponds (reordering if is necessary) to $B(qp_1, |q||p_1 - p_2|)$and since $|q| > 1$ one has $|q||p_1 - p_2| > |p_1 - p_2|$ so $B(p_1, |p_1 - p_2|) \neq B(qp_1, |q||p_1 - p_2|)$ as we want. □

Definition 4.15. Let $\Gamma \subseteq PGL_2(K)$ a subgroup. We say that it is a **Schottky group** if

- Γ is finitely generated
- every element of Γ different from the identity is hyperbolic
- $\overline{\Gamma_p}$ (the closure of the orbit of p) is compact for all $p \in \mathbb{P}^1(K)$.
- Γ is not cyclic, i.e. has rank bigger or equal than 2.

In the definition of Schottky group we impose that is not cyclic, if not we have this example.

Remark 4.16. Note that a Schottky group is torsion free (if $\gamma^n = id$ then $\gamma = id$).

Remark 4.17. Note that a finitely generated but not cyclic subgroup of a Schottky group is Schottky.

The following two lemmas will help us to prove the next lemma.

Lemma 4.18. *If q and $r \in K^*$ topologically nilpotent, the subgroup Γ that they generate does not have torsion elements and $q^m = r^n$ for some n and $m \in \mathbb{Z} \setminus \{0\}$, then Γ is cyclic.*

Proof. We can suppose m and n are coprime, since, if $s = \gcd(n, m)$, $n = sn'$, $m = sm'$, then $(q^{n'}r^{-m'})^s = q^n r^{-m} = 1$ so $q^{n'}r^{-m'} = 1$ since Γ has no elements of finite order. But now, there exists $a, b \in \mathbb{Z}$ with $am + bn = 1$ and we have $(q^b r^a)^m = q$ and $(q^b r^a)^n = r$, so q and r belong to the subgroup generated by $q^b r^a$. □

Lemma 4.19. *If q and $r \in K^*$ are topologically nilpotent, the subgroup Γ that they generate does not have torsion elements and $|q|^m \neq |r|^n$ for all n and $m \in \mathbb{Z} \setminus \{0\}$, then $\mathcal{W} := \overline{\{q^n r^m \; : \; (n, m) \in \mathbb{Z}^2\}}$ is not compact.*

Proof. We will suppose that $|r| > |q|$. We will show that (W) contains infinitely many isolated points. Consider

$$W := \mathcal{W} \cap \{x \; : \; 1 \geq |x| > |q|\}$$

Now, for any $x \in W$, take the ball $B(x, |q|)$. Observe that for any $y \in B(x, |q|)$, $|y| = |(y - x) + x| = \max(|y - x|, |x|) = |x|$, since $|y - x| \leq q < |x|$. But by hypothesis no two elements in $\mathcal{W}$ have the same valuation, hence $\mathcal{W} \cap B(x, |q|) = \{x\}$ for any $x \in W$.

But the set W contains an infinite number of points, since, that for any $m \geq 1$, there exists $f(m) \geq 0$ such that $|r|^{f(m)} \leq |q|^m \geq |r|^{f(m)+1}$, hence $x = q^n r^{-f(n)}$ is in W, since $1 \geq |q|^n |r|^{-f(n)} \geq |r| > |q|$.

Then

$$\mathcal{W} = \overline{\{q^n r^m \; : \; (n, m) \in \mathbb{Z}^2\}} = \bigcup_{w \in W} B(w, |q|) \cup B(0, |q|) \cup B^0(0, 1)^c$$

and no ball can be removed. □

Lemma 4.20. *Let Γ be a Schottky group. Then, for any $id \neq \gamma \in \Gamma$, there exists $\tau \in \Gamma$ such that there exists $p \in \mathbb{P}^1(K)$ whit $\tau(p) = p$ and $\gamma(p) \neq p$.*

Proof. Suppose it is false for some $\gamma \in \Gamma$. This means there exists $\tau \in \Gamma$ such that $\tau^m \neq \gamma^n$ for all $n, m \in \mathbb{Z}$, but with the same fixed points.

To see this, observe that if $\tau^m = \gamma^n$ for some n and $m \in \mathbb{Z} \setminus \{0\}$, then they have the same fixed points. These is because the fixed points of γ and of γ^n are the same for any $n \in \mathbb{Z} \setminus \{0\}$.

So we can suppose $\gamma(x) = qx$ and $\tau(x) = rx$ for some q and $r \in K^*$ topologically nilpotent, and $q^m = r^n$.

So, by Lemma 4.18, there should exists γ and τ that they do not belong to a cyclic subgroup of Γ but with the same fixed points. We take the subgroup generated by γ and τ described as above, which must be a Schottky group, and we will find a contradiction.

Now, if $q^m \neq r^n$ for all n and $m \in \mathbb{Z} \setminus \{0\}$, but $|q|^m = |r|^n$ for some n and $m \in \mathbb{Z} \setminus \{0\}$, then $q^m r^{-n}$ is not 1 but has valuation 1. Hence $\gamma^m \tau^{-n} \in \Gamma$ and it is not hyperbolic. Hence we can suppose $|q|^m \neq |r|^n$ for all n and $m \in \mathbb{Z} \setminus \{0\}$. But Lemma 4.19 gives us a contradiction. □

Lemma 4.21. *Suppose Γ is a Schottky group. Consider $p \in \mathcal{L}_\Gamma$. Then $\overline{\Gamma p} = \mathcal{L}_\Gamma$.*

Proof. If p is fixed by $\gamma \neq 1 \in \Gamma$, then $\alpha(p)$ is fixed by $\alpha^{-1}\gamma\alpha$ for any $\alpha \in \Gamma$. So $\overline{\Gamma p} \subset \mathcal{L}_\Gamma$.

Now, if p' is another point in $\mathcal{L}_\Gamma$, with $\gamma(p') \neq p'$, and fixed by some $\alpha \in \Gamma$, by the previous lemma, then $\alpha(p) \neq p$ and hence $\alpha^n(p) \to p'$ for $n \to \pm\infty$. So $p' \in \overline{\Gamma p}$. So all points fixed by some $\alpha \in \Gamma$, except may be the other point different from p fixed by γ, are in $\overline{\Gamma p}$, which imply that its closure, which is $\mathcal{L}_\Gamma$, is contained in $\overline{\Gamma p}$.

Finally, if $p \in L_\Gamma$ is in the limit of points p_n fixed by some $\gamma_n \in \Gamma$, then any point in $\overline{\Gamma p}$ is limit of points in $\overline{\Gamma p_n} = \mathcal{L}_\Gamma$, so is in $\mathcal{L}_\Gamma$. The reverse inclusion is also clear. □

Lemma 4.22. *Suppose Γ is a Schottky group. Then the set $\mathcal{L}_\Gamma$ is perfect and compact.*

Proof. It is compact since, by the previous lemma, $\mathcal{L}_\Gamma = \overline{\Gamma_p}$ for some $p \in \mathcal{L}_\Gamma$, and $\overline{\Gamma_p}$ is compact by definition of Schottky group.

Let p be fixed by $\gamma \in \Gamma$. Let $p' \in \mathcal{L}$ not fixed by γ (for example, fixed by some γ' not contained in the subgroup generated by γ, that it exists because Γ is not cyclic). Then $\gamma^n(p') \to p$ when $n \to \infty$ or when $n \to -\infty$. Hence no point fixed by some $\gamma \neq 1$ in Γ is isolated, so the same is true for the points in the closure. □

4.4 The tree $\mathcal{T}_\Gamma$ and the graph $\mathcal{T}_\Gamma/\Gamma$.

Definition 4.23. We denote $\mathcal{T}_\Gamma = T(\mathcal{L}_\Gamma)$.

Theorem 4.24. *$\mathcal{T}_\Gamma$ is a locally finite tree. The graph Γ acts on $\mathcal{T}_\Gamma$ freely and $G_\Gamma := \mathcal{T}_\Gamma/\Gamma$ is a finite graph.*

We will show the theorem along the section. The first part of the result is a consequence of Lemma 4.21 and the results of Section 3. The group acts freely because of Lemma 4.14 which says that for all $\gamma \in \Gamma$ different from the identity and for all $v \in \mathrm{V}(\mathcal{T}_\Gamma)$, $\gamma(v) \neq \gamma$.

So we can take $G_\Gamma = \mathcal{T}_\Gamma/\Gamma$ the quotient and the quotient map

$$\mathcal{T}_\Gamma \to \mathcal{T}_\Gamma/\Gamma$$

is the universal cover (see Chapter 1 of [4]).

We will dedicate the rest of the section to the proof that G_Γ is finite.

Definition 4.25. Let $B_\Gamma \subset \Gamma$ a finite set of generators such that if $\gamma \in B_\Gamma$ then $\gamma^{-1} \in B_\Gamma$. And also $id \in B_\Gamma$. Now, for a given vertex $\omega \in \mathcal{T}_\Gamma$ we define $S_\omega = \{\gamma\omega \mid \gamma \in B_\Gamma\}$. Note that S_ω is finite. Moreover we define $\mathcal{T}_{S_\omega} = \bigcup_{v_1,v_2 \in S_\omega}[v_1, v_2] = \bigcup_{\gamma \in B_\Gamma}[\omega, \gamma\omega]$, and note that it is the minimal finite subtree that contains S_ω. Finally we define

$$\mathcal{T}_{B_\Gamma,\omega} = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{T}_{S_\omega}).$$

Remark 4.26. T_{S_ω} is connected.

Note that $\mathcal{T}_{B_\Gamma,\omega} \subset \mathcal{T}_\Gamma$. Now a Lemma that we will use to prove the connectedness of $\mathcal{T}_{B_\Gamma,\omega}$.

Lemma 4.27.

1. *$\forall \gamma \in \Gamma$, $[\omega, \gamma\omega] \subset \mathcal{T}_{B_\Gamma,\omega}$,*

2. *$\forall \gamma \neq \gamma' \in \Gamma$, $[\gamma\omega, \gamma'\omega] \subset \mathcal{T}_{B_\Gamma,\omega}$.*

Proof. 1) Since $\gamma \in \Gamma$, then $\gamma = \gamma_1\gamma_2 \dots \gamma_n$, where $\gamma_i \in B_\Gamma$. Then

$$[\omega, \gamma\omega] \subset [\omega, \gamma_1\omega] \cup [\gamma_1\omega, \gamma_1, \gamma, 2\omega] \cup \dots \cup [\gamma_1\gamma_2 \dots \gamma_{n-1}\omega, \gamma_1\gamma_2 \dots \gamma_n\omega]$$

and also each

$$[\gamma_1 \dots \gamma_i\omega, \gamma_1 \dots \gamma_{i+1}\omega] \subset (\gamma_1 \dots \gamma_i)(\mathcal{T}_{B_\Gamma,\omega}) = \mathcal{T}_{B_\Gamma,\omega}.$$

1) $\Rightarrow$ 2) We can devide the path as follows $[\gamma\omega, \gamma'\omega] \subset [\gamma\omega, \omega] \cup [\omega, \gamma'\omega] \subset \mathcal{T}_{B_\Gamma,\omega}$. Or we also can argue that $[\gamma\omega, \gamma'\omega] = \gamma[\omega, \gamma^{-1}\gamma'\omega] \subset \gamma(\mathcal{T}_{B_\Gamma,\omega}) = \mathcal{T}_{B_\Gamma,\omega}$. □

Proposition 4.28. *$\mathcal{T}_{B_\Gamma,\omega}$ is connected.*

Proof. Let $v_1, v_2 \in \mathcal{T}_{B_\Gamma,\omega}$, then $v_1 = \gamma_1(\omega_1)$ and $v_2 = \gamma_2(\omega_2)$ for some $\omega_1, \omega_2 \in T_{B_\Gamma,\omega}$. Since $T_{B_\Gamma,\omega}$ is connected then there exist paths $[\omega_1, \omega]$ and $[\omega_2, \omega]$ in $T_{B_\Gamma,\omega}$ and from this one has $\gamma_1[\omega_1, \omega] = [v_1, \gamma_1(\omega)]$ and $\gamma_2[\omega_2, \omega] = [v_2, \gamma_2(\omega)]$ contained in $\mathcal{T}_{B_\Gamma,\omega}$. And also by Lemma 4.27 we have $[\gamma_1(\omega), \gamma_2(\omega)] \subset \mathcal{T}_{B_\Gamma,\omega}$, so

$$[v_1, v_2] \subset [v_1, \gamma_1(\omega)] \cup [\gamma_1(\omega), \gamma_2(\omega)] \cup [\gamma_2(\omega), v_2] \subset \mathcal{T}_{B_\Gamma,\omega}.$$

□

Corollary 4.29. *$\mathcal{T}_{B_{\Gamma,\omega}}$ is a subtree.*

Proof. We have $\mathcal{T}_{B_{\Gamma,\omega}} \subset \mathcal{T}_\Gamma$, so we have a connected subset of a tree, so it is a subtree. □

Lemma 4.30. *Suppose Γ is a Schottky group. Let $\mathcal{T}' \subset \mathcal{T}_\Gamma$ be a non-empty subtree which is invariant by Γ. Then $\mathcal{T}' = \mathcal{T}_\Gamma$.*

Proof. First, $\mathcal{T}'$ is infinite since it contains infinite vertices of the form $\gamma(v)$, for some $v \in \mathcal{T}'$ and $\gamma \in \Gamma$.

Let $\mathcal{L}'$ the image of $\mathcal{T}'$ with respect to the map

$$\Psi : \mathrm{Rays}(T(\mathcal{L}_\Gamma)) \to \mathcal{L}_\Gamma.$$

Clearly $\mathcal{L}'$ is invariant by Γ, and non-empty (it is infinite, so it has some ray). Take $p \in \mathcal{L}'$. Then $\Gamma p \subset \mathcal{L}'$. Thus $\overline{\mathcal{L}'} = \mathcal{L}_\Gamma$.

First, observe that for any x and $y \in \mathcal{L}'$, all the points of the form $t(x, y, z)$, for $z \in \mathcal{L}$, are in fact in $\mathcal{T}'$. To show this, observe that $x \in \mathcal{L}'$ implies that there exists a ray r in $\mathcal{T}'$ such that intersects all other rays of $T(\mathcal{L})$ that go to x for Ψ in and infinite set of vertices of $\mathcal{T}'$. Concretely, it intersects the ray $[t(x, y, z), x]$ in some vertex v_x of $\mathcal{T}'$. The same happens for y, so $[t(x, y, z), y]$ contains a vertex v_y of $\mathcal{T}'$. But $t(x, y, z) \in [v_x, v_y] \subset \mathcal{T}'$ since $\mathcal{T}'$ is a tree, hence connected.

But $\mathcal{L}'$ is closed. Effectively, suppose we have a progression of distinct points $p_n \in \mathcal{L}'$ such that $p_n \to p \in \mathcal{L}$ when $n \to \infty$. Then the vertices $v_i := t(p_1, p_2, p_i)$ for $i > 2$ are in $\mathcal{T}'$, and they generated a ray r. Then $\Psi(r) = p$, and hence $p \in \mathcal{L}'$. So $\overline{\mathcal{L}'} = \mathcal{L}'$ and we are done. □

Theorem 4.31. *Let $\mathcal{T}_{B_{\Gamma,\omega}}$ defined as above, then $\mathcal{T}_{B_{\Gamma,\omega}} = \mathcal{T}_\Gamma$*

Proof. We have $\mathcal{T}_{B_{\Gamma,\omega}}$ is invariant by Γ by definition and it is a subtree by Corollary 4.29, so $\mathcal{T}_{B_{\Gamma,\omega}} = \mathcal{T}_\Gamma$. □

Now $\mathcal{T}_{B_{\Gamma,\omega}}/\Gamma$ is finite since

$$\mathrm{V}(\mathcal{T}_{S_\omega}) \to \mathrm{V}(\mathcal{T}_{B_{\Gamma,\omega}}/\Gamma) = \mathrm{V}(\mathcal{T}_\Gamma/\Gamma)$$

and $\mathrm{V}(\mathcal{T}_{S_\omega})$ is finite. But $\mathcal{T}_\Gamma$ is locally finite hence $\mathcal{T}_\Gamma/\Gamma$ also, so it is finite.

4.5 Examples of G_Γ

Recall that $\mathrm{V}(\mathcal{T}_{S_\omega}) \to \mathrm{V}(\mathcal{T}_{B_{\Gamma,\omega}}/\Gamma) = \mathrm{V}(\mathcal{T}_\Gamma/\Gamma)$ is the universal covering, so $\pi_1(G_\Gamma, v) \cong \Gamma$, and moreover, all vertices of $\mathcal{T}_\Gamma$ have valence greater or equal than 3, by Corollary 3.39. This implies some relations between the number of generators g of Γ, which is the genus of G_Γ, and $\#\mathrm{V}(G_\Gamma)$.

Proposition 4.32. *Let g be the genus of G_Γ, i.e. the rank of the generators, then*

$$2(g-1) \geq \# V(G_\Gamma).$$

Proof. We have a general result that says

$$g = \#\mathrm{E}(G_\Gamma) - \#\mathrm{V}(G_\Gamma) + 1$$

and by the fact that each edge starts in a vertex and ends in other, which can be the same, and that the valency is at least 3, we deduce

$$3\#\mathrm{V}(G) \leq 2\#\mathrm{E}(G).$$

Combining this information we obtain

$$\#\mathrm{V}(G_\Gamma) = \#\mathrm{E}(G_\Gamma) - g + 1 \geq \frac{3}{2}\#\mathrm{V}(G_\Gamma) - g + 1,$$

so

$$g - 1 \geq \frac{1}{2}\#\mathrm{V}(G_\Gamma)$$

which implies

$$2(g-1) \geq \#\mathrm{V}(G_\Gamma).$$

□

Corollary 4.33. *When the genus of the graph is two, which is our case, the maximum number of vertices is three*

Proof. Apply the previous proposition for $g = 2$. □

Consider the following case:

$$\Gamma = \langle \gamma_1, \gamma_2 \rangle.$$

In this case G_Γ can be only or 2 wedges of S^1 or two vertices that shares three edges. Note that each case the vertices has valence at least 3: in the first has valence 4, two loops, and the second both vertices has valence 3. We following result says that no more possiblities are allowed, see the graph G_Γ of Figure 4 of the Appendix for the second case. The G_Γ for the first case is a wedge of two S^1.

5 Conclusion

In the Section 2 we have seen properties of valuations and valuation rings, where a valuation is not defined in the usual sense. We also have seen the usual definition and its relation with our definition.

In the Section 3 we give properties of a distance obtained from a valuation and that allow us to define a graph called the tree of balls, obtained from the topology of our K field and a subset $\mathcal{L}$ of $\mathbb{P}^1(K)$. We prove that this graph is in fact a tree. Moreover, we use the compactness of $\mathcal{L}$ to prove that the number of vertices between any two vertices is finite and that $T(\mathcal{L})$ is locally finite. There are a subsection which talks about rays in order to study the non isolated points of $\mathcal{L}$.

In the Section 4 we study the hyperbolic elements of $PGL_2(K)$ arriving to a characterisation result that says when a matrix is hyperbolic studying its trace and determinant (Proposition 4.8); to prove this result we use the completeness of K and the Hensel's Lemma. Moreover we have other result (Theorem 4.9), which says that, supposing that the matrix is diagonalitzable it is hyperbolic or has finite order if an only if the closure of the orbit of a point is compact for any point in $\mathbb{P}^1(K)$. Then we define Schottky group Γ and we see that the closure of the set of points of $\mathcal{P}^1(K)$ fixed by some element of the Schottky group, denoted by $\mathcal{L}_\Gamma$, is perfect and compact. So from this result we can apply what we know from the Section 3 arriving to the fact that $T(\mathcal{L}_\Gamma)/\Gamma$ if finite. Finally we see some example of $T(\mathcal{L}_\Gamma)/\Gamma$.

In the work there are results extended from the non Archimedian case where we have to pay attention in the following fact: in the non Archimedian case is enough to say that $|q| < 1$, for $q \in \mathbb{P}^1(K)$ in order to prove the results. But in our case Δ is any totally ordered group, hence in order to prove the results we need q topologically nilpotent.

Moreover to prove the Hensel's Lemma and use it in the work we need K to be complete, in order to guarantee that the roots of the characteristic polynomial of a matrix are in the field. We also need to use completeness in the part where we talk about rays and isolated points.

5.1 Future questions

- Can we translate the construction of Gerritzen and van der Put of Schottky groups done in [2]?
- In the definition of Schottky group can we put conditions on a finite number of $\gamma \in \Gamma$ and of $p \in \mathbb{P}^1(K)$? So we can prove a subgroup of $PGL_2(K)$ is a Schottky groups by checking a finite number of conditions?

References

[1] Keith Conrad. Hensel's lemma. `http://www.math.uconn.edu/~kconrad/blurbs/gradnumthy/hensel.pdf`.

[2] Marius van der Put Lothar Gerritzen. *Schottky groups and Mumford curves*. Springer-Verlag, 1980.

[3] Jean-Pierre Serre. *Trees*. Springer-Verlag, 1980.

[4] M. J. Dunwoody Warren Dicks. *Groups acting on graphs*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989.

A Picture of non degenerate case of $\mathcal{T}_\Gamma$

Here we see in a picture how is the non degenerate graph $T(\Gamma)$ with genus 2 and how G_Γ, the graph obtained when Γ acts on $T(\Gamma)$.

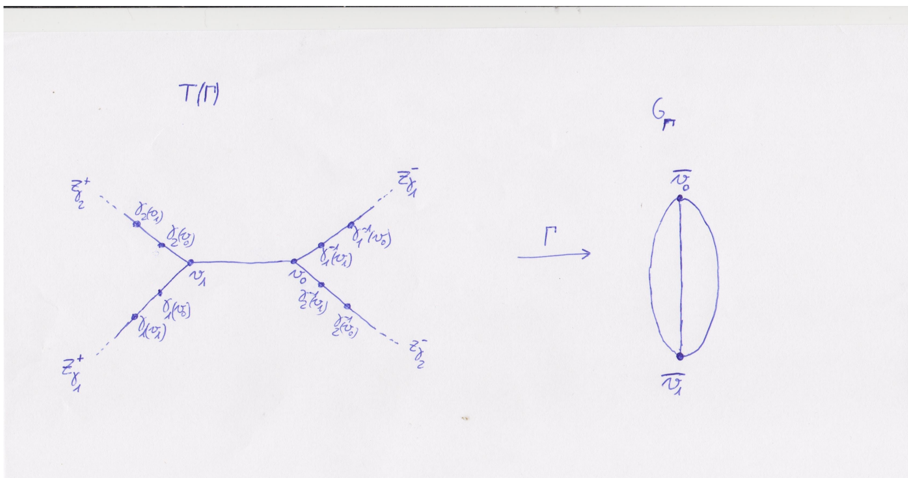


Figure 4: