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GENERALIZATIONS OF THE HEXAGRAMME MYSTIQUE

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“Bees... by virtue of a certain geometrical forethought... know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each.”

Pappus of Alexandria

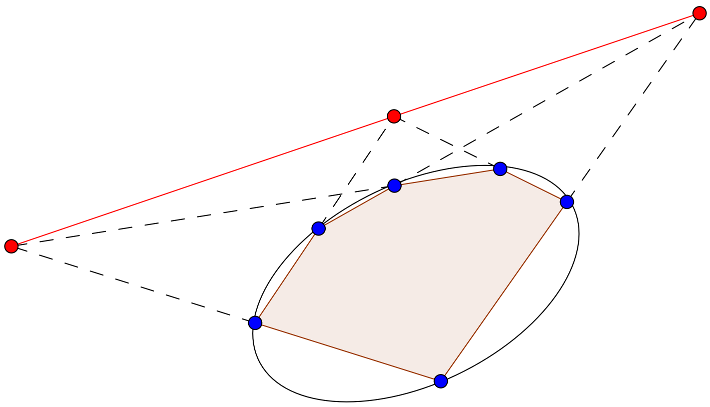
CONTENTS

1. INTRODUCTION

The history of projective geometry is a very complex one. Most of the more formal developments on the subject were made in the 19th century as a result of the movement away from the geometry of Euclid. If one digs a little deeper, however, one can see that the basic concepts upon which this branch of geometry is based can be traced back as far as the fourth century, where a theorem of Pappus of Alexandria appears as *Proposition 139* of *Book VII* of the *Mathematical Collection.* These very early discoveries along with Euclid's Elements are the building blocks for the foundations that were laid down by the projective geometers of the 17th century. It is here that the history of the subject becomes more interesting. Great strides were made in the 17th century, but for some reason projective geometry did not become popular among mathematicians until the 19th century. From this moment, very important results on this subject were made by great mathematicians as Max Noether or David Hilbert.

In particular, the base of these notes is the study of the theory of plane algebraic curves. Willing to know more about the geometry behind the plane algebraic curves, I began to work with the *Algebraic Curves* of William Fulton [1]. Introducing myself with the algebraic sets and its ideals, and with its properties as well, I venture on the theory of intersection of plane algebraic curves, studying them on the affine plane and on the projective plane. To doing so, I had to apprehend so importants results such that the intersection number at points on curves, the *Bézout's Theorem* or the *Max Noether Fundamental Theorem.* As an application, I proved some problems of the algebraic geometry, from the classics to the most contemporary, begining with the *Pappu's Theorem* and ending with the addition on the *Elliptic Law.* Moreover, I state some ideas of plane algebraic curves from a more modern point of view, talking about the divisors on smooth curves and the concepts that derive from them.

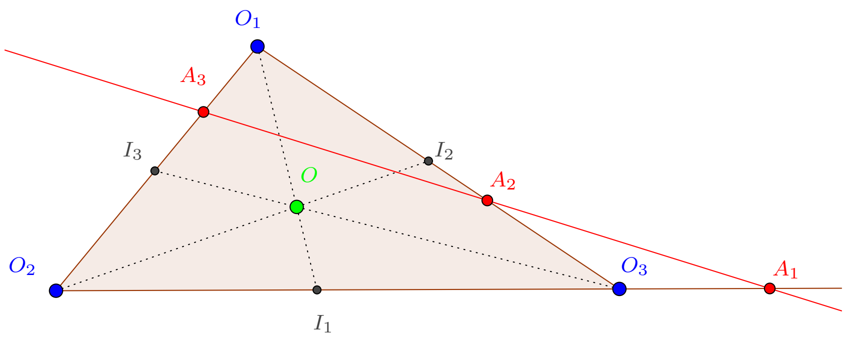
Once with all this baggage, I had to come back some years ago. In 1640's, Blaise Pascal discovered a remarkable property of a hexagon inscribed in a conic, the *Pascal's Theorem* (or *Hexagrammum Mysticum Theorem*). This result states that given a non necessarily regular (or even convex) hexagon inscribed in a conic section, the three pairs of the continuations of opposite sides meet on a straight line, called the Pascal line.



Its generalization have a glorious history, it has been a subject of active and exciting research. As generalizations of the Pascal's Theorem we have the *Chasles-Cayley-Bacharach Theorem* and the *Cayley-Bacharach theorems* in various versions. An article written by *Zhongxuan Luo* [7], states one more generalization of the *Pascal's Theorem*, called the *Pascal's Type Theorem*, using a tool which the author defines as an invariant of plane algebraic curves. This article consists basically of an extension of the cross ratio and a result attributed to Menelaus d'Alexandria, called the *Menelaus' Theorem*, which stablishes the sufficient and necessary condition for which three different points laying each one on a different side of a triangle (or on the extensions of them) are collinear. With modern lenguage, the theorem could be stated as following: given a triangle $O_1O_1O_3$ and three different points $A_1 \in O_2O_3$, $A_2 \in O_3O_1$, $A_3 \in O_1O_2$ distinct from the vertexs, A_1, A_2 and A_3; then O_1, O_2, O_3 are collinear if and only if

$$(O_2, O_3, I_1, A_1)(O_3, O_1, I_2, A_2)(O_1, O_2, I_3, A_3) = -1,$$

where I_1, I_2, I_3 are the projection from a point not in any side of the triangle and a vertex, over its opposite side.



At this point appears our first aim on these notes. It was interesting to see if that final theorem of the article could be proved using only the *Max Noether Fundamental Theorem*. So that, my work on this article consisted of two parts: one of them was to understand the article and all its way to get the proof of the *Pascal's Type Theorem*, and the other was to make my own proof using the tools which I suppose to get on [1]. However, when I began to read the paper, together with my advisor, we realized that the article lacked stringency and basis, since the author worked on the projective plane as he was on the affine plane. Moreover, the theory which the author had developed had some gaps, i.e., it needed some concepts and results that gave it consistency. Also, most of the proofs were maid using the theory of B-splines and the theory of duality, which were not the goal of these notes.

Therefore, although it wasn't the target of the thesis, I first had to rewrite all the article: giving sense to the concepts by changing some of the notions to give them coherence, finding news results to supply all the gaps, and proving rigorously all the results. Fortunately, doing so we develop an interesting theory that with simple calculations of the cross ratio, gets some importants results such that the addition on the *Elliptic Law* or the *Pascal's Type Theorem*. That is because once fixed a reference, then that $3n$ points of the projective plane lie on a plane algebraic curve of degree n is related with the number $(-1)^n$, thus many problems that seems to have a very difficult proof or that seems to need so many knowledgs on plane algebraic curves, are easly proved with a simple calculation.

At this point, the logical question was to ask ourselves if there was more extensions or generalizations related with the *Pascal's Theorem*. Since we know about the *Chasles-Cayley-Bacharach Theorem*, for which *Pascal's Theorem* is a particular case, we sought for researches that were about this subject. So that, we found an article [8] of three great mathematicians: *David Eisenbud*, *Mark Green* and *Joe Harris*, which indeed was about of what we were looking forward. From a more geometric point of view, that article begins with an study of the conditions on the coefficients of polynomials imposed by sets of points. Then, step by step, extends the *Chasles-Cayley-Bacharach Theorem* using the theory of linear series to get a theorem attributed to *Bacharach*, which it is generalized to the projective space. Finally, explains one of the modern developments arising from it, a series of conjectures about the linear conditions imposed by a set of points in projective space on the forms that vanish on them. Then, the most interesting of this article was to understand the geometric methods which they use (e.g. the Hilbert Function) and reference results relative current.

To finish, thanks are due to Joan Carles Naranjo for leading me and watching that I did not deviate much from the way by giving me advice of the best method on each case. Moreover, he gave me a great range of bibliography and helped me to understand some concepts which were far from me knowledges.

Thereby, I will begin introducting some concepts on the affine space. One important thing is that in all the thesis I will suppose $k = \overline{k}$ to be an algebraically closed field. Moreover, I will denote $\mathbb{A}^n = \mathbb{A}^n(k)$ and $\mathbb{P}^n = \mathbb{P}^n(k)$.

2. AFFINE ALGEBRAIC CURVES

2.1. Affine Varieties

In this section, there are some useful results which basically foment the bijection between algebraic sets and their ideals. All of them have been collected from subjects of the degree of mathematics in the *Universitat de Barcelona*, such that *Commutative Algebra* and *Algebraic Varieties*. Because of that, I will skip most of the proofs.

First, I introduce the notions of algebraic sets and its ideals, and I state some important properties of them.

Definition 2.1.1. Let $F \in k[X_1, \dots, X_n]$, a point $P = (a_1, a_2, \dots, a_n) \in \mathbb{A}^n$ is called a zero of F if $F(P) = 0$. If F is not a constant, the set of zeros of F is called the **hypersurface** defined by F, and is denoted by $V = V(F)$.

Definition 2.1.2. Let $S \subseteq k[X_1, \dots, X_n]$ be any set of polynomials, we let

$$V(S) = \{P \in \mathbb{A}^n;\ F(P) = 0 \text{ for all } F \in S\} = \bigcap_{F \in S} V(F).$$

A subset $X \subset \mathbb{A}^n$ is an **affine algebraic set**, or simply an **algebraic set**, if $X = V(S)$, for some $S \subseteq k[X_1, \dots, X_n]$.

Properties 2.1.3. Let $S \subseteq k[X_1, \dots, X_n]$, I, J ideals of $k[X_1, \dots, X_n]$:

(i) $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal of $k[X_1, \dots, X_n]$ generated by S.

(ii) If $I \subset J$, then $V(J) \subset V(I)$.

(iii) $V(FG) = V(F) \cup V(G)$ for any polynomials $F, G \in k[X_1, \dots, X_n]$ and $V(I) \cup V(J) = V(\{FG;\ F \in I, G \in J\})$, so any finite union of algebraic sets is an algebraic set.

Definition 2.1.4. For any subset X of $\mathbb{A}^n$, we call

$$I(X) = \{F \in k[X_1, ..., X_n];\ \text{every } P \in X \text{ is a zero of } F\}$$

the **ideal** of X.

Properties 2.1.5. Let $X, Y \subset \mathbb{A}^n$:

(i) If $X \subset Y$, then $I(Y) \subset I(X)$.

(ii) $S \subset I(V(S))$ for any set S of polynomials and $X \subset V(I(X))$.

(iii) $V(S) = V(I(V(S)))$ for any set S of polynomials and $I(X) = I(V(I(X)))$. So if V is an algebraic set, $V = V(I(V))$, and if I is the ideal of an algebraic set, $I = I(V(I))$.

(iv) $I(X)$ is a radical ideal.

Theorem 2.1.6. (**Hilbert Basis Theorem**) If R is a Noetherian ring, then $R[X_1, \dots, X_n]$ is a Noetherian ring.

Corollary 2.1.7. Every algebraic set is the intersection of a finite number of hypersurfaces.

Proof. Due to k is a field, k is Noetherian. Hence, $k[X_1, \dots, X_n]$ is Noetherian. Therefore, every ideal I in $k[X_1, \dots, X_n]$ is finitely generated. Then, $I = (F_1, \dots, F_r)$ and

$$V(I) = V(F_1, \dots, F_r) = V(F_1) \cap \dots \cap V(F_r).$$

$\square$

Definition 2.1.8. An algebraic set $V \subset \mathbb{A}^n$ is **reducible** if $V = V_1 \cup V_2$, where V_1, V_2 are algebraic sets in $\mathbb{A}^n$, and $V_i \neq V$, $i = 1, 2$. Otherwise, V is **irreducible**. An irreducible affine algebraic set $V \subset \mathbb{A}^n(k)$ is called **affine variety**.

Proposition 2.1.9. An algebraic set V is irreducible if and only if $I(V)$ is prime.

Proof. ($\Rightarrow$) Suppose V is irreducible. Let $FG \in I(V)$, where F and G are homogeneous polynomials. Then,

$$FG \in I(V) \Rightarrow \forall P \in V,\ FG(P) = 0 \Rightarrow \forall P \in V,\ F(P)G(P) = 0 \Rightarrow$$

$$\Rightarrow \forall P \in V,\ F(P) = 0 \text{ or } G(P) = 0 \Rightarrow \forall P \in V,\ P \in V(F) \text{ or } P \in V(G).$$

Therefore, $V \subseteq V(F) \cup V(G)$. As V is irreducible, $V \subseteq V(F)$ or $V \subseteq V(G)$, thus, $F \in I(V)$ or $G \in I(V)$ and $I(V)$ is prime.

($\Leftarrow$) Suppose $I(V)$ is prime and $V \subseteq Z_1 \cup Z_2$, where Z_1, Z_2 closed algebraic sets. If $V \nsubseteq Z_1$ and $V \nsubseteq Z_2$, then, $I(Z_1) \nsubseteq I(V)$ and $I(Z_2) \nsubseteq I(V)$. Therefore, $\exists F \in I(Z_1)$, $G \in I(Z_2)$ such that $F, G \notin I(V)$ but $FG \in I(Z_1 \cup Z_2) \subseteq I(V)$.

But $I(V)$ is prime, so we reach a contradiction with supposing $V \nsubseteq Z_1$ and $V \nsubseteq Z_2$. Thus, V is irreducible. $\square$

Theorem 2.1.10. Let V be an algebraic set in $\mathbb{A}^n$. Then, there are unique irreducible algebraic sets $V_1, \dots, V_m$ such that $V = V_1 \cup \dots \cup V_m$ and $V_i \nsubseteq V_j$ for all $i \neq j$.

Definition 2.1.11. The V_i are called the **irreducible components** of V and $V = V_1 \cup \dots \cup V_m$ is the **decomposition** of V into irreducible components.

Theorem 2.1.12. (**Weak Hilbert's Nullstellensatz**) If I is a proper ideal in $k[X_1, \dots, X_n]$, then $V(I) \neq \emptyset$.

Theorem 2.1.13. (**Hilbert's Nullstellensatz**) Let I be an ideal in $k[X_1, \dots, X_n]$. Then, $I(V(I)) = \text{rad}(I)$.

Corollary 2.1.14. If I is a prime ideal, then $V(I)$ is irreducible. Moreover, $I = rad(I) = I(V(I))$. Hence, there is a one-to-one correspondence between prime ideals and irreducible algebraic sets. The maximal ideals correspond to points.

Corollary 2.1.15. Let I be an ideal in $k[X_1, \dots, X_n]$. Then, $V(I)$ is a finite set if and only if $k[X_1, \dots, X_n]/I$ is a finite dimensional vector space over k.

Definition 2.1.16. Let $V \in \mathbb{A}^n$ be a nonempty variety, then $I(V)$ is a prime ideal in $k[X_1, ..., X_n]$, so $k[X_1, ..., X_n]/I(V)$ is a domain. We let

$$\Gamma(V) = k[X_1, ..., X_n]/I(V),$$

and call it the **coordinate ring** of V.

Definition 2.1.17. Let V be a nonempty variety in $\mathbb{A}^n$, $\Gamma(V)$ its coordinate ring. Since $\Gamma(V)$ is a domain, we may form its quotient field. This field is called the **field of rational functions** on V, and is written $k(V)$. An element of $k(V)$ is a **rational function** on V. If f is a rational function on V, and $P \in V$, we say that f is defined at P if for some $g, h \in \Gamma(V)$, $f = g/h$, and $h(P) \neq 0$.

Definition 2.1.18. Let $P \in V$. We define $\mathcal{O}_P(V)$ to be the set of rational functions on V that are defined at P. It is easy to verify that $\mathcal{O}_P(V)$ forms a subring of $k(V)$ containing $\Gamma(V)$. The ring $\mathcal{O}_P(V)$ is called the **local ring** of V at P. The set of points $P \in V$ where a rational function f is not defined is called the **pole set** of f.

Definition 2.1.19. The ideal $\mathfrak{m}_P(V) = \{f \in \mathcal{O}_P(V);\ f(P) = 0\}$ is called the **maximal ideal** of V at P. It is the kernel of the evaluation homomorphism $f \mapsto f(P)$ of $\mathcal{O}_P(V)$ onto k, so $\mathcal{O}_P(V)/\mathfrak{m}_P(V)$ is isomorphic to k.

Proposition 2.1.20. Let I be an ideal in $k[X_1, ..., X_n]$ (k algebraically closed), and suppose $V(I) = \{P_1, ..., P_N\}$ is finite. Then, there is a natural isomorphism of $k[X_1, ..., X_n]/I$ with

$$\prod_{i=1}^{N} \mathcal{O}_{P_i}(\mathbb{A}^n)/I\mathcal{O}_{P_i}(\mathbb{A}^n).$$

Proof. (see [1], §2.9 *Proposition 6*).

Corollary 2.1.21. $\dim_k(k[X_1, \ldots, X_n]/I) = \sum_{i=1}^{N} \dim_k(\mathcal{O}_{P_i}/I\mathcal{O}_{P_i})$.

Proposition 2.1.22. Let V be a variety in $\mathbb{A}^n$, $I = I(V) \subset k[X_1, \ldots, X_n]$, $P \in V$, and let J be an ideal of $k[X_1, \ldots, X_n]$ that contains I. Let J' be the image of J in $\Gamma(V)$. Then, there is a natural homomorphism φ from $\mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n)$ to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$, and it holds that φ is an isomorphism. In particular,

$$\mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n) \cong \mathcal{O}_P(V).$$

Proof. Let $\pi : \Gamma(\mathbb{A}^n) = k[X_1, \ldots, X_n] \longrightarrow \Gamma(V)$ be the canonical map. Then, $J' = \pi(J)$. As $k(V) \subset k(\mathbb{A}^n)$, we can extend π to the quotient field of the rings

$$\pi : k(\mathbb{A}^n) \longrightarrow k(V).$$

Let f be a rational function in $k(V)$ that is defined at P, then $f \in \mathcal{O}_P(\mathbb{A}^n)$. So we can restrict the previous map to

$$\pi\Big|_{\mathcal{O}_P(\mathbb{A}^n)} : \mathcal{O}_P(\mathbb{A}^n) \longrightarrow \mathcal{O}_P(V).$$

Take now the natural map $\sigma : \mathcal{O}_P(V) \longrightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. Then,

$$\Phi = \sigma \circ \pi\Big|_{\mathcal{O}_P(\mathbb{A}^n)} : \mathcal{O}_P(\mathbb{A}^n) \longrightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$$

is a well-difined map. Moreover, it is an epimorphism.

Let's find now the $ker(\Phi)$. If $f \in \mathcal{O}_P(\mathbb{A}^n)$ is such that $\Phi(f) = \overline{0}$, then, $\Phi(f) \in J'\mathcal{O}_P(V)$, hence, by the definiton of our morphism, $f \in J\mathcal{O}_P(\mathbb{A}^n)$. Conversly, for all $g \in J\mathcal{O}_P(\mathbb{A}^n)$, $\Phi(g) = \overline{0}$.

Therefore, $ker(\Phi) = J\mathcal{O}_P(\mathbb{A}^n)$ and by the *First Isomorphism's Theorem*, exists an isomoprhism φ from $\mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n)$ to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. □

To end this section, let's see a some algebraic results and concepts that will play an important role on these notes.

Definition 2.1.23. A ring R satisfying

(1) R is Noetherian and local, and the maximal ideal is principal,

(2) there is an irreducible element $t \in R$ such that every nonzero $z \in R$ may be written uniquely in the form $z = ut^n$, u a unit in R, n a nonnegative integer,

is called a **discrete valuation ring**, written *DVR*. The irreducible element t is called a **uniformizing parameter** for R; any other uniformizing parameter is of the form ut, u a unit in R. Let K be the quotient field of R. Then (when t is fixed) any nonzero element $z \in K$ has a unique expression $z = ut^n$, where u is a unit in R, $n \in \mathbb{Z}$. The exponent n is called the **order** of z, and is written $n = ord(z)$; we define $ord(0) = \infty$.

Definition 2.1.24. Let $F = \sum a_i X^{(i)} \in k[X_1, ..., X_n]$, where $a_i \in k$ and $X^{(i)}$ are monomials of degree i. We call F **homogeneous**, or a **form**, of degree d, if all coefficients a_i are zero except for monomials of degree d (i.e., $\forall \lambda \in k \setminus \{0\}$, $P = (a_1, \dots, a_n) \in \mathbb{A}^n$, it is satisfied that $F(\lambda a_1, \dots, \lambda a_n) = \lambda^d F(a_1, \dots, a_n)$). Any polynomial F has a unique expression $F = F_0 + F_1 + ... + F_d$, where F_i is a form of degree i. If $F_d \neq 0$, then d is the **degree** of F, written $\deg(F)$.

Definition 2.1.25. If $F \in k[X_1, ..., X_{n+1}]$ is a form, we define $F_* \in k[X_1, ..., X_n]$ by setting $F_* = F(X_1, ..., X_n, 1)$. Conversely, for any polynomial $f \in k[X_1, ..., X_n]$ of degree d, write $f = f_0 + f_1 + ... + f_d$, where f_i is a form of degree i, and define $f^* \in R[X_1, ..., X_{n+1}]$ by setting

$$f^* = X_{n+1}^d f(X_1/X_{n+1}, ..., X_n/X_{n+1})$$

then, f^* is a form of degree d. These processes are often described as **dehomogenizing** and **homogenizing** polynomials with respect to X_{n+1}.

Proposition 2.1.26. If $F \neq 0$, $F \in k[X_1, \dots, X_{n+1}]$, and r is the highest power of X_{n+1} that divides F, then $X_{n+1}^r (F_*)^* = F$.

Proof. Let d be the degree of F and let $F = X_{n+1}^r G$, where $0 \leq r \leq d$, is the highest power of X_{n+1} that divides F, and G is an homogenus polynomial of degree $d - r \geq 0$.

Case 1: Suppose $r = 0$ (i.e. X_{n+1} doesn't divide F), then, $f = F_* = F(X_1, ..., X_n, 1) \in k[X_1, \dots, X_n]$ is a polynomial of degree d. So, $f = f_0 + f_1 + ... + f_d$, where f_i is a form of degree

$\deg(f_i) = i$. Therefore,

$$f^* = X_{n+1}^d \sum_{i=0}^{d} f_i(X_1/X_{n+1}, ..., X_n/X_{n+1}) = \sum_{i=0}^{d} X_{n+1}^{d-i} f_i(X_1, ..., X_n) = F.$$

Case 2: Suppose now that $r > 0$, then,

$$F_* = F(X_1, ..., X_n, 1) = G(X_1, ..., X_n, 1) = G_*,$$

where G is still a polynomial of degree $d - r \geq 0$. Since X_{n+1} doesn't divide G, as we see in the *Case 1*,

$$(F_*)^* = (G_*)^* = X_{n+1}^{d-r} G(X_1/X_{n+1}, ..., X_n/X_{n+1}) = G.$$

Thus,

$$X_{n+1}^r (F_*)^* = X_{n+1}^r (G_*)^* = X_{n+1}^r G = F.$$

□

Definition 2.1.27. A **finite exact sequence** (or simply exact sequence) of finite-dimensional vector spaces $\{V_i\}_{1 \leq i \leq n}$, $n \geq 2$, is a sequence of maps

$$\varphi_i : V_{i+1} \longrightarrow V_i,$$

$i = 1, \ldots, n-1$, which satisfies

$$Im(\varphi_i) = Ker(\varphi_{i-1}).$$

Moreover, φ_{n-1} is injective and φ_1 is surjective. It is usually denoted by

$$0 \to V_n \xrightarrow{\varphi_{n-1}} V_{n-1} \xrightarrow{\varphi_{n-2}} \cdots \xrightarrow{\varphi_1} V_1 \to 0.$$

When $n = 3$, it is known as **short exact sequence**.

Proposition 2.1.28. Let

$$0 \to V_n \xrightarrow{\varphi_{n-1}} V_{n-1} \xrightarrow{\varphi_{n-2}} \cdots \xrightarrow{\varphi_1} V_1 \to 0$$

be an exact sequence of finite-dimensional vector spaces, $n \geq 2$. Then

$$\sum_{i=1}^{n} (-1)^i \dim(V_i) = 0.$$

Proof. Let's see it by induction. If $n = 2$, then, φ_1 is bijective. Hence,

$$\dim(V_2) = \dim(Im(\varphi_1)) = \dim(V_1).$$

If $n = 3$, then φ_2 is injective and φ_1 is surjective. Hence, $\dim(V_1) = \dim(Im(\varphi_1))$ and $\dim(V_3) = \dim(Im(\varphi_2))$. As V_2 is a vector space, $\dim(V_2) = \dim(Im(\varphi_1)) + \dim(ker(\varphi_1))$. Moreover, due to the exactness of the sequence, $Im(\varphi_2) = ker(\varphi_1)$. Therefore,

$$\dim(V_3) - \dim(V_2) + \dim(V_1) = \dim(Im(\varphi_1)) - [\dim(Im(\varphi_1)) + \dim(ker(\varphi_1))] + \dim(Im(\varphi_2)) = 0.$$

Suppose now that the sentence is true for every exact sequence with length k, such that $k < n$. Let

$$0 \to V_n \xrightarrow{\varphi_{n-1}} V_{n-1} \xrightarrow{\varphi_{n-2}} \dots \xrightarrow{\varphi_1} V_1 \to 0$$

be an exact sequence of length n, then, we can deduce the following sequences:

$$0 \to V_n \xrightarrow{\varphi_{n-1}} V_{n-1} \xrightarrow{\varphi_{n-2}} \dots \xrightarrow{\varphi_3} V_3 \xrightarrow{\varphi_2} Im(\varphi_2) \to 0,$$

$$0 \to Im(\varphi_2) \hookrightarrow V_2 \xrightarrow{\varphi_1} V_1 \to 0.$$

The first sequence is obviosuly exact, because φ_2 is surjective over its image. The second sequence is also exact, due to the inclusion of $Im(\varphi_2)$ over V_1 is injective, and φ_1 is surjective by hypothesis.

Therefore, there are two exact sequences of length $n-1$ and 3 respectively. Hence, $\dim(Im(\varphi_2)) - \sum_{i=3}^{n}(-1)^{i+1}\dim(V_i) = 0$ and $\dim(V_1) - \dim(V_2) + \dim(Im(\varphi_{n-2})) = 0$. Thus,

$$\dim(V_1) - \dim(V_2) + \sum_{i=3}^{n}(-1)^{i+1}\dim(V_i) =$$

$$= \sum_{i=1}^{n}(-1)^{i+1}\dim(V_i) = 0.$$

□

2.2. Affine Plane Curves and the Intersection Number

From this section, there are shown the new results that I have been gathering for my thesis. So we start talking about the affine plane curves. We have seen that an hypersurface correspond to a set of zeros of a nonconstant polynomial $F \in k[X_1, \dots, X_n]$ without multiple factors, where F is determined up to multiplication by a nonzero constant. An hypersurface in the coordinate ring $k[X, Y]$ is called an **affine plane curve**. For some purposes, it is useful to allow $F \in k[X, Y]$ to have multiple factors, so we modify our definition slightly:

Definition 2.2.1. We say that two polynomials $F, G \in k[X, Y]$ are **equivalent** if $F = \lambda G$ for some nonzero $\lambda \in k$. We define an **affine plane curve** to be an equivalence class of nonconstant polynomials under this equivalence relation. The **degree** of a curve is the degree of a defining polynomial for the curve. Observe that we admit non-reduced algebraic sets.

Observation 2.2.2. We often will ignore this equivalence distinction, and say, e.g., "the plane curve F", where $F \in k[X, Y]$.

Observation 2.2.3. If F is irreducible, $V(F)$ is a variety in $\mathbb{A}^2$. We will usually write $\Gamma(F)$, $k(F)$, and $\mathcal{O}_P(F)$ instead of $\Gamma(V(F))$, $k(V(F))$, and $\mathcal{O}_P(V(F))$.

Definition 2.2.4. Let F be a curve, $P \in F$. The point P is called a **simple point** of F if either derivative $F_X(P) \neq 0$ or $F_Y(P) \neq 0$. A point that isn't simple is called **multiple** (or **singular**). A curve with only simple points is called a **nonsingular curve**.

Example 2.2.5. If the curve is F^k, $k \geq 2$, all points are singular.

Definition 2.2.6. Let F be any curve, $P = (a, b)$. Let T be a translation that takes $(0, 0)$ to P, i.e., $T(x, y) = (x+a, y+b)$. Then, $F^T = F(X+a, Y+b)$. Write $F^T = G_m + G_{m+1} + \cdots + G_n$, where G_i is a form in $k[X, Y]$ of degree i and $G_m \neq 0$. We define m to be the **multiplicity** of F at P, denoted by $m_P(F) := m$. If $G_m = \prod L_i^{r_i}$, $L_i = \alpha_i X + \beta_i Y$, the lines $\alpha_i(X - a) + \beta_i(Y - b)$ are defined to be the **tangent lines** to F at P, and r_i is the **multiplicity** of the tangent.

Observation 2.2.7. *P is a simple point in a irreducible curve F if and only if $m_p(F) = 1$. Note the importance of allowing multiple factors of the polynomial F on the definition of the multiplicity.*

Definition 2.2.8. Suppose P is a simple point on an irreducible curve F. We let ord_P^F be the **order** function on $k(F)$ defined by the DVR $\mathcal{O}_P(F)$. If $G \in k[X, Y]$ is another curve, and g is its image in $\mathcal{O}_P(F)$, then $ord_P^F(G) = \dim_k(\mathcal{O}_P(F)/(g))$. If L is any line through P, then $ord_P^F(L) = 1$ if L is not tangent to F at P, and $ord_P^F(L) > 1$ if L is tangent to F at P.

Definition 2.2.9. We say that F and G **intersect properly** at P if F and G have no common component that passes through P.

Definition 2.2.10. Two curves F and G are said to **intersect transversally** at P if P is a simple point both on F and on G, and if the tangent line to F at P is different from the tangent line to G at P.

Proposition 2.2.11. *Let $F, G \in k[X, Y]$ be polynomials with no common factors. Then $V(F, G) = V(F) \cap V(G)$ is a finite set of points.*

Proof. F and G have no common factors in $k[X, Y] \cong k[X][Y]$, so they also have no common factors in $k(X)[Y]$. Since $k(X)[Y]$ is a PID ($k(X)$ is a field), $(F, G) = (1)$ in $k(X)[Y]$. So, exists $C, D \in k(X)[Y]$ such that $CF + DG = 1$. Then, there is a nonzero $H \in k[X]$ such that $HC = A \in k[X][Y]$ and $HD = B \in k[X][Y]$. Therefore, $AF + BG = H$. If $(a, b) \in V(F, G)$, then $H(a) = 0$. But H has only a finite number of zeros. Thus, $V(F, G)$ has only a finite number of points in the X-coordinates. Since the same reasoning applies to the Y-coordinates, there can be only a finite number of points. □

Proposition 2.2.12. *Two plane curves F and G with no common components intersect in a finite number of points.*

Proof. It is a direct consequence of the *Proposition 2.2.11.* □

Definition 2.2.13. Let F and G be plane curves, $P \in \mathbb{A}^2$. We want to define the **intersection number** of F and G at P, which it will be denoted by $I(P, F \cap G)$. We shall first list seven properties we want this intersections number to have, and then we will prove that there is only one possible definition that satisfies such properties.

Our first requirements are:

(1) $I(P, F \cap G)$ is a nonnegative integer for any F, G, and P such that F and G intersect properly at P. If F and G do not intersect properly at P, $I(P, F \cap G) = \infty$.

(2) $I(P, F \cap G) = 0$ if and only if $P \notin F \cap G$. That is to say, $I(P, F \cap G)$ depends only on the components of F and G that pass through P. Moreover, $I(P, F \cap G) = 0$ if F or G is a nonzero constant.

(3) If T is an affine change of coordinate on $\mathbb{A}^2$, $Q \in \mathbb{A}^2$ and $T(Q) = P$, then $I(P, F \cap G) = I(Q, F^T \cap G^T)$.

(4) $I(P, F \cap G) = I(P, G \cap F)$, i.e, the intersection number is symmetric.

(5) $I(P, F \cap G) = 1$ when F and G meet transversally at P. More generally, $I(P, F \cap G) \geq m_P(F)m_P(G)$, with equality occuring if and only if F and G have not tangent lines in common at P.

(6) If $F = \prod F_i^{r_i}$ and $G = \prod G_j^{s_j}$, where F_i and G_j are forms in $k[X, Y]$ of degree i and j repectively and $r_i,\ s_i \in Z_{\geq 0}$, then, $I(P, F \cap G) = \sum_{i,\text{ j}} r_i s_j I(P, F_i \cap G_j)$.

(7) $I(P, F \cap G) = I(P, F \cap (G + AF))$ for any $A \in k[X, Y]$.

Theorem 2.2.14. There is a unique intersection number $I(P, F \cap G)$ defined for all plane curves F, G, and all points $P \in \mathbb{A}^2$, satisfaying properties (1)-(7). It is given by the formula

$$I(P, F \cap G) = \dim_k(\mathcal{O}_P(\mathbb{A}^2)/(F, G)).$$

Proof of uniqueness. Assume we have a number $I(P, F \cap G)$ defined for all F, G, and P, satisfaying (1)-(7). We will give a constructive procedure in order to compute $I(P, F \cap G)$ using only these seven properties, which is stronger than the required uniqueness. We may suppose that $I(P, F \cap G)$ is finite (by (1)), i.e, F and G have no common component that passes through P. Let T be a translation that takes $(0, 0)$ to P, by (3), we can suppose $P = (0, 0)$.

If $I(P, F \cap G) = 0$, then $P \notin F \cap G$, so for this case it has been already calculated. Thus, let's proceed by induction. Assume $I(P, F \cap G) = n > 0$, and $I(P, F \cap G)$ can be calculated whenever $I(P, F \cap G) < n$.

Let $F(X, 0), G(X, 0) \in k[X]$ be two homogeneus polynomials of degree r, s respectively, where r or s is taken to be zero if the polynomial vanishes. We may suppose $r \leq s$ due to the symmetry of the intersection number.

Case 1: $r = 0$. Then Y divides F, so $F = YH$, and by (6),

$$I(P, F \cap G) = I(P, Y \cap G) + I(P, H \cap G).$$

If $G(X, 0) = X^m(a_0 + a_1X + \cdots + a_sX^{s-m})$, $a_0 \neq 0$, then $G(X, Y) = G(X, 0) + YA(X, Y)$, where $A(X, Y) \in k[X, Y]$. So, by (7),

$$I(P, Y \cap G) = I(P, Y \cap (G(X, 0) + YA)) = I(P, Y \cap G(X, 0)).$$

Moreover, by (6),

$$I(P, Y \cap G(X, 0)) = I(P, Y \cap X^m) + I(P, Y \cap (a_0 + a_1X + \cdots + a_sX^{s-m})).$$

Due to $P \notin (a_0 + a_1X + \cdots + a_sX^{s-m})$, then, by (2),

$$I(P, Y \cap (a_0 + a_1X + \cdots + a_sX^{s-m})) = 0.$$

Besides, as X^m and Y have no tangent lines in common, by (5),

$$I(P, Y \cap X^m) = \deg(Y) \cdot \deg(X^m) = 1 \cdot m = m.$$

Since $P \in G$ and, by (1), $m > 0$, we get that $I(P, H \cap G) = n - m < n$ and by induction, it can be computed. Consequently, $I(P, F \cap G)$ can be calculated.

Case 2: $r > 0$. Let's multiply F and G by constants that make $F(X, 0)$ and $G(X, 0)$ monic (as $F, G \in k[X, Y]$, and k is a field). Let $H = G - X^{s-r}F$. Then, by (7),

$$I(P, F \cap G) = I(P, F \cap (H + X^{s-r}F)) = I(P, F \cap H),$$

and $\deg(H(X, 0)) = t < s$. Repeating this process (interchanging the order of F and H if $t < r$) a finite number of times, we eventually reach a pair of curves A, B that fall under *Case 1*, and with $I(P, F \cap G) = I(P, A \cap B)$. This concludes the proof. □

Proof of existence. (Idea) Define $I(P, F \cap G)$ to be $\dim_k(\mathcal{O}_P(\mathbb{A}^2)/(F, G))$. We must show that properties $(1) - (7)$ are satisfied.

As $(F, G) = (G, F) = (F, G + AF)$ for any $A \in k[X, Y]$, properties (4) and (7) are clearly satisfied. Moreover, due to an affine change of coordinates on $\mathbb{A}^2$ is an isomorphism, (3) is obvious. Besides, as $\dim_k(\mathcal{O}_P(\mathbb{A}^2)/(F, G))$ depends only on the ideal $\mathcal{O}_P(\mathbb{A}^2)$ generated by F and G, property (2) is, as well, direct.

Then, we may assume that $P = (0, 0)$ and that all the components of F and G pass through P.

For seeing (1): if F and G have no common components, by *Proposition 2.2.12*, $I(P, F \cap G)$ is finite. If F and G have a common component H, then $(F, G) \subset (H)$, so $I(P, F \cap G) \geq \dim_k(\mathcal{O}_P(\mathbb{A}^2)/(H))$. Thus, it's sufficient to see that $\mathcal{O}_P(\mathbb{A}^2)/(H) \cong \mathcal{O}_P(H)$, due to $\Gamma(H) \subset \mathcal{O}_P(H)$ and $\Gamma(H)$ is infinite-dimensional (as $V(F, G)$ is infinite dimensional).

To prove (6), it is enough to show that $I(P, F \cap GH) = I(P, F \cap G) + I(P, F \cap H)$ for any F, G, H. If F and GH have no common components (since is clear otherwise) then the idea is to see that the following sequence

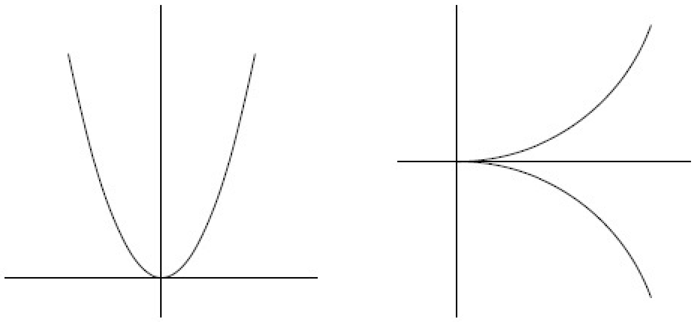
$$0 \to \mathcal{O}_P(\mathbb{A}^2)/(F, H) \xrightarrow{\psi} \mathcal{O}_P(\mathbb{A}^2)/(F, GH) \xrightarrow{\varphi} \mathcal{O}_P(\mathbb{A}^2)/(F, G) \to 0$$

is a short exact sequence.

In order to see all the details of the proof, see [1], §5.3. □

Now, let's see an example of the intersection number between two curves that intersect properly at $P = (0, 0)$:

Example 2.2.15. Given the curves $A = Y - X^2$, $C = Y^2 - X^3$ and the point $P = (0, 0)$, let us calculate $I(P, A \cap C)$.



$A = Y - X^2$ $\qquad\qquad$ $C = Y^2 - X^3$

Figure 1: Intersection number of curves A and C

First, by (7),

$$I(P, A \cap C) = I(P, A \cap (C - AX)) = I(P, A \cap Y(Y - X)).$$

Then, using the property (6),

$$I(P, A \cap Y(Y - X)) = I(P, A \cap Y) + I(P, A \cap (Y - X)).$$

Next, choosing the plane curves $H = A - Y$ and $G = A - (Y - X)$, by (7) and (4),

$$I(P, A \cap Y) + I(P, A \cap (Y - X)) = I(P, Y \cap A) + I(P, (Y - X) \cap A) =$$

$$= I(P, Y \cap H) + I(P, (Y - X) \cap G) = I(P, Y \cap (-X^2)) + I(P, (Y - X) \cap X(1 - X)).$$

Finnaly, using the properties (5) and (2),

$$I(P, A \cap C) = I(P, Y \cap (-X^2)) + I(P, (Y - X) \cap X(1 - X)) = m_P(Y)m_P(-X^2)+$$

$$+I(P, (Y - X) \cap X) + I(P, (Y - X) \cap (1 - X)) = 1 \cdot 2 + I(P, Y \cap X) + 0 =$$

$$= 2 + m_P(Y)m_P(X) = 2 + 1 \cdot 1 = 3.$$

Proposition 2.2.16. There are two more properties of the intersection number:

(8) If P is a simple point on F, then $I(P, F \cap G) = ord_P^F(G)$ (where $ord_P^F(G)$ is the order function on $k(F)$ defined by the DVR $\mathcal{O}_P(F)$).

(9) If G and F have no components in common, then

$$\sum_P I(P, F \cap G) = \dim_k(k[X, Y]/(F, G)).$$

Proof. (8) We may assume that F is irreducible. Let g be the image of G at $\mathcal{O}_P(F)$. Then, $ord_P^F(G) = \dim_k(\mathcal{O}_P(F)/(g))$. Since $\mathcal{O}_P(F)/(g)$ is isomoprhic to $\mathcal{O}_P(\mathbb{A}^2)/(F, G)$ (*Proposition 2.1.22*),

$$ord_P^F(G) = \dim_k(\mathcal{O}_P(F)/(g)) = \dim_k(\mathcal{O}_P(\mathbb{A}^2)/(F, G)) = I(P, F \cap G).$$

□

(9) Due to G and F have no common components, then they intersect in a finite number of points (*Proposition 2.2.12*). So, let $I = (F, G) \subset k[X, Y]$ be the ideal of the two curves, then $V(I) = \{P_1, \ldots, P_N\}$ is finite.

We know by *Proposition 2.1.20* that there exists a natural isomoprhism of $k[X, Y]/I$ with

$$\prod_{i=1}^{N} \mathcal{O}_{P_i}(\mathbb{A}^n)/I\mathcal{O}_{P_i}(\mathbb{A}^n).$$

Therefore, $\dim_k(k[X, Y]/I) = \sum_{i=1}^{N} \dim_k(\mathcal{O}_{P_i}(\mathbb{A}^n)/I\mathcal{O}_{P_i}(\mathbb{A}^n)) = \sum_{i=1}^{N} I(P_i, F \cap G)$. $\square$

3. PROJECTIVE ALGEBRAIC CURVES

3.1. Projective Varieties

In this section we develop the idea of algebraic sets in $\mathbb{P}^n$. Since the concepts and most of the proofs are enterely similar than the affine algebraic sets, I will skipe most of them.

Definition 3.1.1. A point $P \in \mathbb{P}^n$ is said to be a **zero** of a polynomial $F \in k[X_0, \ldots, X_n]$ if $F(a_0, \ldots, a_n) = 0$, for every choice of homogeneous coordinates $(a_0, \ldots, a_n)$ for P.

Definition 3.1.2. For any set S of polynomials in $k[X_0, ..., X_n]$, we let

$$V(S) = \{P \in \mathbb{P}^n; \text{P is a zero of each } F \in S\},$$

which is called an **algebraic set** in $\mathbb{P}^n$, or a **projective algebraic set**. If $I = \langle S \rangle$ is the ideal generated by S, $V(I) = V(S)$.

Definition 3.1.3. For any set $X \subset \mathbb{P}^n$, we let

$$I(X) = \{F \in k[X_0, ..., X_n]; \text{ every } P \in X \text{ is a zero of } F\}.$$

The ideal $I(X)$ is called the **ideal** of X. An ideal $I \subset k[X_0, ..., X_n]$ is called **homogeneous** if for every $F = \sum_{i=0}^{m} F_i \in I$, F_i a form of degree i, we have also $F_i \in I$. For any set $X \subset P^n$, $I(X)$ is a homogeneous ideal.

Observation 3.1.4. The projective algebraic sets and the ideals of projective sets, satisfies the same properties as in the affine case (*Properties 2.1.3* and *Properties 2.1.5*).

Proposition 3.1.5. An ideal $I \subset k[X_0, ..., X_n]$ is homogeneous if and only if it is generated by a finite set of forms.

Definition 3.1.6. An algebraic set $V \subset \mathbb{P}^n$ is **irreducible** if it is not the union of two smaller algebraic sets. An irreducible algebraic set in $\mathbb{P}^n$ is called a **projective variety**. Any projective algebraic set can be written uniquely as a union of projective varieties, wich are called its **irreducible components**.

Proposition 3.1.7. An algebraic set $V \subset \mathbb{P}^n$ is irreducible if and only if $I(V)$ is prime.

Theorem 3.1.8. (**Projective Nullstellensatz**) Let $I \subset k[X_0, ..., X_n]$ be an homogeneous ideal, then:

(1) $V(I) = \emptyset$ if and only if $(X_0, \ldots, X_n) = rad(I)$.

(2) If $V(I) \neq \emptyset$, $I(V(I)) = rad(I)$.

Definition 3.1.9. Let V be a nonempty projective variety in $\mathbb{P}^n$. Then, $I(V)$ is a prime ideal, so the residue ring

$$\Gamma(V) = k[X_0, ..., X_n]$$

is a domain. It is called the **homogeneous coordinate ring** of V.

Definition 3.1.10. We call the **homogeneous function field** of a nonempty projective variety V the quotient field of $\Gamma(V)$, which is denoted by $k(V)$. In contrast with the case of affine varieties, f/g ($f, g \in \Gamma(V)$) defines a function, at least where g is not zero, if f, g are both forms of the same degree. In that case, we say that f/g is defined at $P \in V$ if $g(P) \neq 0$. Moreover, if f/g, $f'/g' \in k(V)$,

$$f/g \sim f'/g' \Leftrightarrow fg' - f'g \in I(V).$$

Definition 3.1.11. We let

$$\mathcal{O}_P(V) = \{h = f/g \in k(V);\ h \text{ is defined at P}\},$$

be the **local ring** of V at P. We define its maximal ideal as

$$\mathfrak{m}_P(V) = \{h = f/g \in k(V);\ g(P) \neq 0, f(P) = 0\}.$$

Proposition 3.1.12. Let $U_0 = \{(a_0, \ldots, a_n) \in \mathbb{P}^n;\ a_0 \neq 0\}$, we consider $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ by means of the map $\varphi_0 : \mathbb{A}^n \longrightarrow U_0$ such that $\varphi_0(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$. Let $V \subset \mathbb{A}^n$ be an algebraic set, $I = I(V) \subset k[X_1, \ldots, X_n]$. Let $I^* = \{F^*;\ F \in I\} \subset k[X_0, ..., X_n]$. This I^* is an homogeneous ideal, and we define $V^* = V(I^*) \subset \mathbb{P}^n$. Then, it is satisfied:

(i) $\varphi_0(V) = V^* \cap U_0$, and $(V^*)_* = V(\{G_*;\ G \in I^*\} = V$.

(ii) If $V \subset W \subset \mathbb{A}^n$, then $V^* \subset W^* \subset \mathbb{P}^n$.

(iii) If V is irreducible in $\mathbb{A}^n$, then V^* is irreducible in $\mathbb{P}^n$.

(iv) If $V = \cup_i V_i$ is the irreducible decomposition of V in $\mathbb{A}^n$, then $V^* = \cup_i V_i^*$ is the irreducible decomposition of V^* in $\mathbb{P}^n$.

(v) If $V \subsetneq \mathbb{A}^n$ is not empty, then no component of V^* lies in or contains $H_\infty = \mathbb{P}^n \setminus U_0$.

3.2. Projective Plane Curves and the Intersection Number

Let $F \in k[X_0, ..., X_n]$ be an homogeneus polynomial, we refer to the projective set $V(F)$ as a hypersurface on $\mathbb{P}^n$. Note that in the definition of projective sets, we are not allowing multiple factors on F. So, we deffine a **projective plane curve** to be a hypersurface in $\mathbb{P}^2$ except that, as with affine curves, we want to allow multiple components:

Definition 3.2.1. We say that two nonconstant forms F, $G \in k[X,Y,Z]$ are **equivalent** if there is a nonzero $\lambda \in k$ such that $G = \lambda F$. A **projective plane curve** is an equivalence class of forms. The **degree** of a curve is the degree of a defining form. Curves of degree 1, 2, 3 and 4 are called **lines**, **conics**, **cubics**, and **quartics** respectively.

Definition 3.2.2. Two curves F and G are said to be **projectively equivalent** if there is a projective change of coordinates T such that $G = F^T$. Everything we will say about curves will be the same for two projectively equivalent curves.

Definition 3.2.3. If $F \in k[X_0, ..., X_n]$ is a projective curve, then we define the **multiplicity** of F at $P \in U_i = \{(a_0, ..., a_n) \in \mathbb{P}^n;\ a_i \neq 0\}$ as $m_P(F) = m_P(F_*)$, where F_* is a dehomogenization of F respect to X_i. The multiplicity is independent of the choice of U_i, and invariant under projective change of coordinates.

Definition 3.2.4. Let F, G be projective plane curves, $P \in \mathbb{P}^2$. We define the **intersection number** $I(P, F \cap G)$ to be $\dim_k(\mathcal{O}_P(\mathbb{P}^2)/(F_*, G_*))$. This is independent of the way that F_* and G_* are formed, and it satisfies properties (1)-(9) of the intersection number in the affine case. In (3), however, T should be a projective change of coordinates, and in (7), A should be a form of degree $\deg(A) = \deg(G) - \deg(F)$.

Definition 3.2.5. We say that a line L is **tangent** to a curve F at P if $I(P, F \cap L) > m_p(F)$. A point P in F is an **ordinary multiple point** of F if F has $m_p(F)$ distinct tangents at P.

Proposition 3.2.6. Two projective plane curves F and G with no common components intersect in a finite number of points.

Proof. Using that $k(X_1, X_2)[X_3]$ is a DIP, this proof is an extension of the proof of the *Proposition 2.2.11.* □

3.3. Bézout's Theorem

The projective plane was constructed so that any two distinct lines would intersect at one point. The famous theorem of Bézout tells us that much more is true. Of course, here the hypothesis of $k = \overline{k}$ is crucial.

Theorem 3.3.1. (**Bézout's Theorem**) Let F and G be projective plane curves of degree m and n respectively. Assume F and G have no common component. Then

$$\sum_P I(P, F \cap G) = mn.$$

Proof. By *Proposition 3.2.6*, as F and G have no common component, $F \cap G$ is finite. So we may assume, by a projective change of coordinates if necessary, that none of the points in $F \cap G$ is on the line at infinity $Z = 0$.

Then, $\sum_P I(P, F \cap G) = \sum_P I(P, F_* \cap G_*) = \dim_k k[X,Y]/(F_*, G_*)$, by property (9) of intersection number. Let

$$\gamma_* = k[X,Y]/(F_*, G_*),\ \gamma = k[X,Y,Z]/(F,G),\ R = k[X,Y,Z],$$

and let γ_d (resp. R_d) be the vector space of forms of degree d in γ (resp. R). The theorem will be proved if we can show that $\dim_k \gamma_* = \dim_k \gamma_d$ and $\dim_k \gamma_d = mn$ for some large d.

Step 1: $\dim_k \gamma_d = mn$ for all $d \geq m+n$.

Let $\pi : R \to \gamma$ be the natural map (or canonical map), let $\phi : R \oplus R \to R$ be defined by $\phi(A,B) = AF + BG$, and let $\psi : R \to R \oplus R$ be defined by $\psi(C) = (GC, -FC)$. Using the fact that F and G have no common factors, it is not difficult to check the exactness of the following sequence:

$$0 \to R \xrightarrow{\psi} R \oplus R \xrightarrow{\phi} R \xrightarrow{\pi} \gamma \to 0.$$

(i) π, by the definition of the canonical map, it is an epimorphism.

(ii) $ker(\pi) = Im(\phi)$:

($\subseteq$) Let $H \in ker(\pi)$, then, $\pi(H) = \bar{0}$, so, $H \in (F,G)$. Therefore, $H = AF + BG = \phi(A,B)$, where $A, B \in R$. Thus, $H \in Im(\phi)$.

($\supseteq$) Let $H \in Im(\phi)$, then, $H = AF + BG$, for some $A, B \in R$. So, $\pi(H) = \pi(AF + BG) = \pi(AF) + \pi(AG) = \bar{0}$. Thus, $H \in ker(\pi)$.

(iii) $ker(\phi) = Im(\psi)$:

($\subseteq$) Let $(A,B) \in ker(\phi)$, then, $\phi(A,B) = 0$, so, $AF = -BG$. Due to F and G have no common factors, thus, $A = GC$ and $B = -FC$, where $C \in R$. Therefore, $(A,B) = (GC, -FC) = \psi(C)$, and $(A,B) \in Im(\psi)$.

($\supseteq$) Let $(A,B) \in Im(\psi)$, then, exists $C \in R$ such that $\psi(C) = (GC, -FC) = (A,B)$. So, $\phi(A,B) = \phi(GC, -FC) = (GC)F + (-FC)G = 0$. Thus, $(A,B) \in ker(\phi)$.

(iv) ψ injective:

Let $C_1, C_2 \in R$ such that $\psi(C_1) = \psi(C_2)$. Then, $(GC_1, -FC_1) = (GC_2, -FC_2)$ if and only if for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$,

$$\begin{cases} GC_1 &= \lambda GC_2 \\ FC_1 &= \mu FC_2 \end{cases} \Leftrightarrow \begin{cases} C_1 &= \lambda C_2 \\ C_1 &= \mu C_2 \end{cases}$$

Therefore, $C_1 = \lambda C_2$, $\lambda = \mu$ and ψ is injective.

Now, if we restrict these maps to the forms of various degrees, we get the following exact sequence:

$$0 \to R_{d-m-n} \xrightarrow{\psi} R_{d-m} \oplus R_{d-n} \xrightarrow{\phi} R_d \xrightarrow{\pi} \gamma_d \to 0.$$

Since $R = k[X,Y]$, $\dim_k R_{d-\lambda} = \binom{d+2-\lambda}{2}$, and it follows from *Proposition 2.1.28* that

$$\dim_k \gamma_d = \dim_k(R_d) - \dim_k(R_{d-m} \oplus R_{d-n}) + \dim_k(R_{d-m-n}) = \binom{d+2}{2} - \binom{d+2-m}{2}$$

$$-\binom{d+2-n}{2} + \binom{d+2-m-n}{2} = \frac{(d+2)(d+1)}{2} - \frac{(d+2-m)(d+1-m)}{2}$$

$$-\frac{(d+2-n)(d+1-n)}{2} + \frac{(d+2-m-n)(d+1-m-n)}{2} = mn,$$

if $d \geq m+n$.

Step 2: The map $\alpha : \gamma \to \gamma$ defined by $\alpha(\overline{H}) = \overline{ZH}$ (where the bar denotes residue modulo (F, G)) is one-to-one:

We must show that if $ZH = AF + BG$, then $H = A'F + B'G$ for some A', B'. For any $J \in k[X, Y, Z]$, denote $J_0 = J(X, Y, 0)$. Since F, G, and Z have no common zeros, F_0 and G_0 are relatively prime forms in $k[X, Y]$.

If $ZH = AF + BG$, then $A_0F_0 = -B_0G_0$, so $B_0 = F_0C$ and $A_0 = -G_0C$ for some $C \in k[X, Y]$. Let $A_1 = A + CG$, $B_1 = B - CF$. Since $(A_1)_0 = (B_1)_0 = 0$, we have $A_1 = ZA'$, $B_1 = ZB'$ for some A', B'. Then, since $ZH = A_1F + B_1G$, it follows that $H = A'F + B'G$, as claimed.

Step 3: Let $d \geq m + n$, and choose $A_1, ..., A_{mn} \in R_d$ whose residues in γ_d forms a basis for γ_d. Let $A_{i*} = A_i(X, Y, 1) \in k[X, Y]$, and let a_i be the residue of A_{i*} in γ_*. We will se that $a_1, ..., a_{mn}$ forms a basis for γ_*.

First notice that if $d \geq m + n$, the map α of *Step 2* restricts to an isomorphism from γ_d onto γ_{d+1}, due to $\dim_k(\gamma_d) = \dim_k(\gamma_{d+1}) = mn$ and a one-to-one linear map of vector spaces of the same dimension is an isomorphism. Then, it follows that the residues of $Z^rA_1, ..., Z^rA_{mn}$ form a basis for γ_{d+r} for all $r \geq 0$.

Now let's see that the a_i generate γ_*. If $h = \overline{H} \in \gamma_*$, $H = b_1F_* + b_2G_* \in k[X, Y]$, by *Proposition 2.1.26*, some Z^NH^* is a form of degree $d + r$, so $Z^NH^* = \sum_{i=1}^{mn} \lambda_i Z^r A_{i*} + BF + CG$ for some $\lambda_i \in k$, B, $C \in k[X, Y, Z]$. Then $H = (Z^NH^*)_* = \sum \lambda_i A_{i*} + B_*F_* + C_*G_*$, so $h = \sum \lambda_i a_i$, as desired.

Finally, we will see that the a_i are independent. If $\sum \lambda_i a_i = 0$, then $\sum \lambda_i A_{i*} = BF_* + CG_*$. Therefore, by *Proposition 2.1.26*,

$$Z^r \sum \lambda_i A_i = Z^s B^* F + Z^t C^* G,$$

for some r, s, t. But then $\sum \lambda_i \overline{Z^rA_i} = 0$ in γ_{d+r}, and the $\overline{Z^rA_i}$ form a basis, so each $\lambda_i = 0$.

Therefore,

$$\sum_P I(P, F \cap G) = \sum_P I(P, F_* \cap G_*) = \dim_k k[X, Y]/(F_*, G_*) = \dim_k k[X, Y]/(F, G) = mn.$$

This ends the proof. □

Corollary 3.3.2. If F and G have no common component, then

$$\sum_P m_P(F)m_P(G) \leq \deg(F) \cdot \deg(G).$$

Proof. By Bézout's Theorem, $\sum_P I(P, F \cap G) = \deg(F) \cdot \deg(G)$. Using the property (5) of the intersection number, $\deg(F) \cdot \deg(G) = \sum_P I(P, F \cap G) \geq \sum_P m_P(F)m_P(G)$. □

Corollary 3.3.3. If F and G meet in mn distinct points, $m = \deg(F)$, $n = \deg(G)$, then these points are all simple points on F and on G, and they have not a component in common.

Corollary 3.3.4. If two curves F and G of degrees m and n respectively, have more than mn points in common, then they have a common component.

Theorem 3.3.5. (**Pascal's Theorem**) Given an hexagon inscribed in a conic section, the three pairs of the continuations of the opposite sides meet on a straight line (called the **Pascal line**).

Proof. Let $P_1, \ldots, P_6$ be the six vertex of the hexagon and let Q be the conic section that have the hexagon inscribed. Suppose F is the cubic homogeneous polynomial that defines the union of the three lines P_1P_2, P_3P_4, P_5P_6 (consisting on three sides of the hexagon that are not adjacents between them) and G is the cubic homogeneous polynomial that defines the union of the other three lines P_2P_3, P_4P_5, P_6P_1. Let's denote by C_1 and C_2 the cubics defined by F and G respectively. From *Bézout's Theorem* follows that the two cubics intersects in 9 points (C_1 and C_2 have no common factors), so let $P_1, P_2, P_3, P_4, P_5, P_6, R_1, R_2, R_3$ be these nine points.

Let P be another point of the conic, different from P_i $(i = 1, \ldots, 6)$, and let $\lambda, \mu \in \mathbb{C}$ such that the polynomial $H = \mu G + \lambda F$ vanishes on P. Let C be the cubic defined by H. Then, C has seven points in common with the conic. Since $\deg(Q) \cdot \deg(C) = 6 < 7$, we conclude, by *Corollary 3.3.4*, that the conic and the cubic C have a common component. For degree reasons and because and hexagon have no three collinear point, the only possibility of this is that the cubic C is reducible and contains the conic as a factor. Therefore, $C = Q \cup L$, where L is a line.

Finally, as $P_1, P_2, P_3, P_4, P_5, P_6, R_1, R_2, R_3$ lie on C_1 and C_2, then lie on C. Thus, as R_1, R_2, R_3 are not in Q, then $R_1, R_2, R_3 \in L$. $\square$

3.4. Max Noether's Fundamental Theorem

Max Noether's Theorem is a useful tool in order to prove some important results in algebraic geometry, from the classics to the more contemporary. This theorem is concerned with the following situation: suppose F, G are curves with no common factors, and H is another curve satisfying $G \cap F \subset H \cap F$ counted with multiplicity, i.e, if P has multiplicity m in $G \cap F$, then it has mutiplicity $m' \geq m$ in $H \cap F$. So, when is there a curve B such that B intersects with F in the points of $H \cap F$ that are not in $G \cap F$?

Let's see first some definitions that will make easier and more understandable the writting of the statement:

Definition 3.4.1. A **zero-cycle** on $\mathbb{P}^2$ is a formal sum $\sum_{P \in \mathbb{P}^2} n_P P$, where n_P's are integers, and all but a finite number of them are zero. The **degree** of a zero cycle $\sum n_P P$ is defined to be $\sum n_P$. The zero cycle is **positive** if each $n_p \geq 0$. We say that $\sum n_P P$ is **bigger** than $\sum r_P P$, and write $\sum n_P P \geq \sum r_P P$, if each $n_p \geq r_p$.

Definition 3.4.2. Let F, G be projective plane curves of degrees m, n respectively, with no common components. We define the **intersection cycle** $F \cdot G$ by

$$F \cdot G = \sum_{P \in \mathbb{P}^2} I(P, F \cap G) P.$$

Observation 3.4.3. Bézout's Theorem says that $F \cdot G$ is a positive zero-cycle of degree mn.

Properties 3.4.4. (1) $F \cdot G = G \cdot F$.

(2) $F \cdot GH = F \cdot G + F \cdot H$.

(3) $F \cdot (G + AF) = F \cdot G$, if A is a form and $\deg(A) = \deg(G) - \deg(F)$.

Definition 3.4.5. Let $P \in \mathbb{P}^2$, F, G curves with no common component through P, H another curve. We say that **Noether's Conditions are satisfied at** P (with respect to F, G, and H), if $H_* \in (F_*, G_*) \subset \mathcal{O}_P(\mathbb{P}^2)$, i.e., if there are a, $b \in \mathcal{O}_P(\mathbb{P}^2)$ such that $H_* = aF_* + bG_*$.

Theorem 3.4.6. (**Max Noether's Fundamental Theorem**) Let F, G, H be projective plane curves. Assume F and G have no common components. Then there is an equation $H = AF + BG$ (with A, B forms of degree $\deg(H) - \deg(F)$, $\deg(H) - \deg(G)$ respectively) if and only if Noether's conditions are satisfied at every $P \in F \cap G$.

Proof. ($\Rightarrow$) If $H = AF + BG$, then $H_* = A_*F_* + B_*G_*$ at any P.

($\Leftarrow$) By *Proposition 3.2.6*, as F and G have no common components, they intersect in a finite number of points. Then, we may assume, by a projective change of coordinate if necessary, that none of the points in $F \cap G$ is on the line at infinity $Z = 0$. That means $V(F, G, Z) = 0$.

We may take $F_* = F(X, Y, 1)$, $G_* = G(X, Y, 1)$, $H_* = H(X, Y, 1)$. *Noether's conditions* say that the residue of H_* in $\mathcal{O}_P(\mathbb{P}^2)/(F_*, G_*)$ is zero for each $P \in F \cap G$.

It follows from *Proposition 2.1.20* that the residue of H_* in $k[X, Y]/(F_*, G_*)$ is zero, i.e., $H_* = aF_* + bG_*$, $a, b \in k[X, Y]$.

Moreover, let r_1 and r_2 be the highest power of Z that divides F and G respectively, and let $r = r_1 + r_2$. By *Proposition 2.1.26*,

$$Z^s H = Z^r (H_*)^* = Z^r (aF_*)^* + Z^r (bG_*)^* = AF + BG,$$

for $s \leq r$, $A = Z^{r_2}(a)^*$, $B = Z^{r_1}(b)^*$.

Besides, we see in the *step 2* of the proof of the *Bézout's Theorem* that

$$\alpha : k[X, Y, Z]/(F, G) \to k[X, Y, Z]/(F, G)$$

defined by $\alpha(\overline{H}) = \overline{ZH}$ is one-to-one. So the multiplication by Z on $k[X, Y, Z]/(F, G)$ is one-to-one. Then,

$$H = A'F + B'G$$

for some A', B'.

If $A' = \sum A'_i$, $B' = \sum B'_i$, A'_i, B'_i forms of degree i, as F and G have no common components, then

$$H = A'_s F + B'_t G,$$

with $s = \deg(H) - \deg(F)$, $t = \deg(H) - \deg(G)$. $\square$

Max Noether's Fundamental Theorem is also known as $Af + B\varphi$ **Theorem**. Of course, the usefulness of this theorem depends on finding criteria that ensure that Noether's conditions are satisfied at P.

Proposition 3.4.7. Let F, G, H be plane curves, $P \in F \cap G$. Then Noether's conditions are satified at P if P is a simple point on F and $I(P, H \cap F) \geq I(P, G \cap F)$.

Proof. $I(P, H \cap F) \geq I(P, G \cap F)$ implies that $ord_P^F(H) \geq ord_P^F(G)$ (*Property (9)* of the intersection number), so

$$\overline{H}_* \in (\overline{G}_*) \subset \mathcal{O}_P(F).$$

Since $\mathcal{O}_P(F)/(\overline{G}_*) \cong \mathcal{O}_P(P^2)/(F_*, G_*)$ (*Proposition 2.1.22*), the residue of H_* is zero in $\mathcal{O}_P(P^2)/(F_*, G_*)$. $\square$

Corollary 3.4.8. If all the points of $F \cap G$ are simple points of F, and $H \cdot F \geq G \cdot F$, then there is a curve B such that $B \cdot F = H \cdot F - G \cdot F$.

Proof. If $H \cdot F \geq G \cdot F$, then, $\forall P \in F \cap G$, $P \in H \cap G$. Moreover, by the intersection cycles properties, $\forall P \in F \cap G$, $I(P, H \cap F) \geq I(P, G \cap F)$. So, by *Proposition 3.4,7*, Noether's conditions are satified at every $P \in F \cap G$. Therefore, using Max Noether's Theorem, there is an equation $H = AF + BG$ (with A, B forms of degree $\deg(H) - \deg(F)$, $\deg(H) - \deg(G)$ respectively).

Finally,

$$H \cdot F = (AF + BG) \cdot F = BG \cdot F = B \cdot F + G \cdot F.$$

$\square$

Observation 3.4.9. B has to be a curve of degree $\deg(B) = \deg(H) - \deg(G)$.

3.5. Applications of $Af + B\varphi$ Theorem

We indicate in this section a few of the many interesting consequences of the *Max Noether's Theorem*. We will begin with the proof of one of the classical algebraic geometry problems, the *Pappus Theorem*. It appears as *Proposition 139* of Book VII of the *Mathematical Collection* of Pappus d'Alexandria in the fourth century A.D.

Theorem 3.5.1. (**Pappus theorem**) Let L_1, L_2 be two lines and $P_1, P_2, P_3 \in L_1$, $Q_1, Q_2, Q_3 \in L_2$ six points (none of these points in $L_1 \cap L_2$). Let L_{ij} be the line between P_i and Q_j. For each i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, let $R_k = L_{ij} \cdot L_{ji}$ $(i \neq j)$. Then R_1, R_2 and R_3 are collinear.

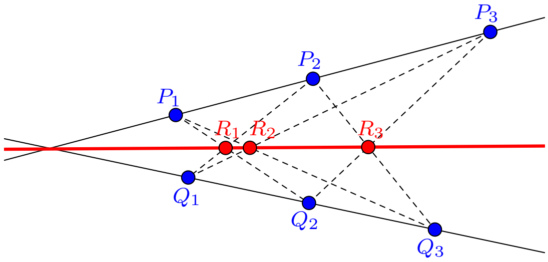


Figure 2: Pappus theorem

Proof. Let Q the conic formed by L_1 and L_2. Let C_1, C_2 be cubics, where $C_1 = L_{12}L_{13}L_{23}$ and $C_2 = L_{21}L_{31}L_{32}$. Then, $C_1 \cdot C_2 = P_1 + P_2 + P_3 + Q_1 + Q_2 + Q_3 + R_1 + R_2 + R_3 \geq C_1 \cdot Q = P_1 + P_2 + P_3 + Q_1 + Q_2 + Q_3$. By *Corollary 3.4.8*, there is a curve L such that $L \cdot C_1 = C_2 \cdot C_1 - Q \cdot C_1$, satisfying $\deg(L) = \deg(C_2) - \deg(Q) = 1$. Therefore, R_1, R_2 and R_3 are collinear. $\square$

In *Theorem 3.3.5*, we saw a proof of one implication of the *Pascal's Theorem* using the Bézout's Theorem. Now, with the Max Noether's Theorem, we will see the reciprocal as well.

Theorem 3.5.2. (**Pascal's Theorem**) An hexagon with vertices $P_1, P_2, \ldots, P_6$ is inscribed in an irreducible conic, if and only if, the opposite sides meet in three collinear points Q_1, Q_2, Q_3.

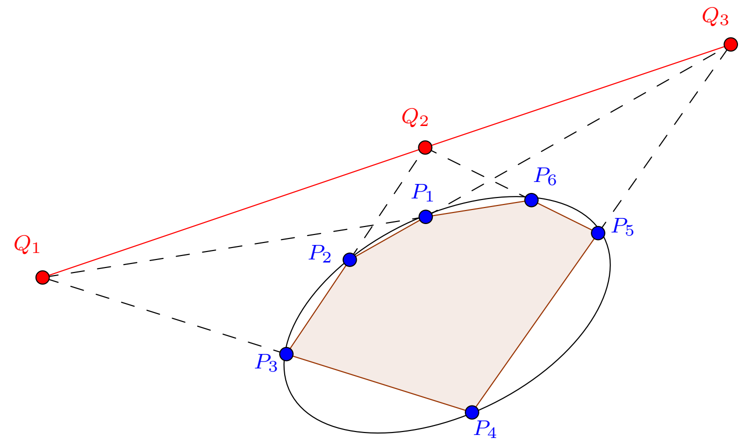


Figure 3: Pascal's Theorem

Proof. Let C_1 be the union of three non adjacent sides of the hexagon, and let C_2 be the union of the other three sides. Then, C_1 and C_2 are cubics. Suppose Q is the conic which has the hexagon inscribed. Then,

$$C_1 \cdot C_2 \geq C_1 \cdot Q.$$

Using the *Corollary 3.4.8*, as all the points on the conic are simple, there is a curve L such that $L \cdot C_1 = C_2 \cdot C_1 - Q \cdot C_1$, satisfying $\deg(L) = \deg(C_2) - \deg(Q) = 1$. Thus, the opposite sides meet in collinear points. $\square$

Observation 3.5.3. *Pappu's Theorem* becomes a special case of *Pascal's Theorem* if we allow a conic to be degenerated, i.e, the union of the two lines L_1 and L_2. Then, the six points P_i must be taken with $P_1, P_2, P_3 \in L_1$ and $P_4, P_5, P_6 \in L_2$, and $P_i \neq L_1 \cap L_2$ for every i.

The following theorem is known as *Cayley-Bacharach Theorem*. However, in a papers written by the mathematicians David Eisenburd, Mark Green and Joe Harris, the name of the theorem is atributted to Chasles, so in order to be impartial, I will denote the theorem as *Chasles-Cayley-Bacharach Theorem*.

Theorem 3.5.4. (**Chasles-Cayley-Bacharach Theorem**) Let C_1, C_2 be cubic plane curves meeting in nine points $P_1, \ldots, P_9$. If C is any cubic containing $P_1, \ldots, P_8$, then C contains P_9 as well.

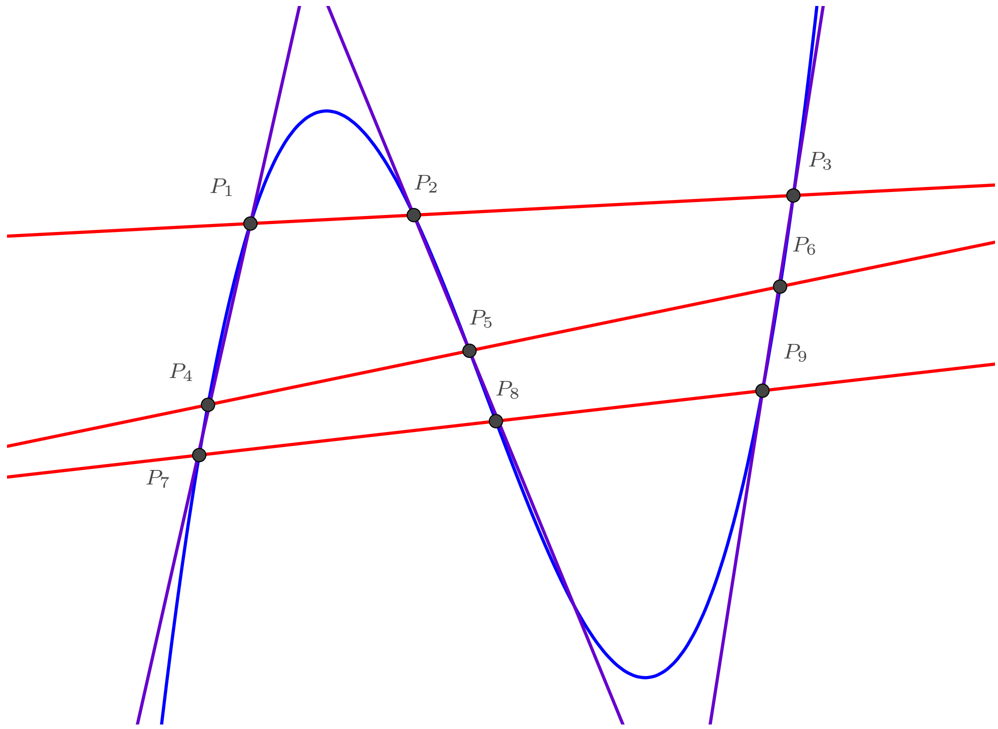


Figure 4: Chasles-Cayley-Bacharach Theorem

Proof. Suppose that

$$C_1 \cdot C = P_1 + \cdots + P_8 + R.$$

Let L be a line through P_9 that doesn't pass through R. Then,

$$L \cdot C_1 = P_9 + S_1 + S_2.$$

Consequently,

$$LC_2 \cdot C_1 = L \cdot C_1 + C_2 \cdot C_1 = P_1 + \cdots + P_9 + R + S_1 + S_2.$$

As $C_2 \cdot C_1 < LC_2 \cdot C_1$, using *Corollary 3.4.8*, there is a curve L_1 such that

$$L_1 \cdot C_1 = LC_2 \cdot C_1 - C_2 \cdot C_1 = R + S_1 + S_2$$

and $\deg(L_1) = \deg(LC_2) - \deg(C_2) = 1$.

So, L_1 is a line passing through S_1 and S_2. Since two points spans a line, $L = L_1$ and $R = P_9$. $\square$

Observation 3.5.5. *Pascal's Theorem* follows if we take the cubics C_1 and C_2 to be the triangles formed by alternate edges of the hexagon, $C_1 = P_1P_2 \cup P_3P_4 \cup P_5P_6$ and $C_2 = P_1P_6 \cup P_2P_3 \cup P_4P_5$, and take C to be the union of the conic Q and the line R_1R_2, where $R_1 = P_1P_2 \cap P_4P_5$ and $R_2 = P_3P_4 \cap P_6P_1$. The point $R_3 = P_5P_6 \cap P_2P_3$ also lies on $C_1 \cap C_2$, so *Chasles-Cayley-Bacharach Theorem* says it must lie on C, i.e, it must lie on the union of Q and R_1R_2. Since it does not lie on Q, it must lie on R_1R_2.

A point P of a nonsingular cubic C is called a **flex** if $I(P, C \cap L) = 3$, where L is the tangent of C at P. It is known that a nonsingular cubic has always 9 different flexes. The following theorem tells us much more:

Theorem 3.5.6. A line joining two different flexes of a cubic passes through a third flex.

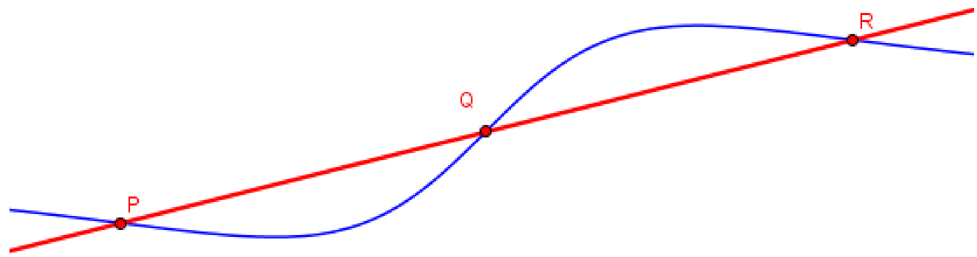


Figure 5: Three flexes of a cubic

Proof. Let C be the cubic and let P, Q be the two flexes of the cubic. By *Bézout's Theorem*, the line PQ and the cubic meet in three points. Let R be the third point where they meet, then,

$$C \cdot PQ = P + Q + R.$$

Let L be the tangent of C at R, hence, $C \cdot L = 2R + S$. Let now L_1 be the tangent of C at P and L_2 be the tangent of C at Q. Then, $C \cdot L_1 = 3P$ i $C \cdot L_2 = 3Q$.

Let C_1 be the union of L, L_1, L_2 and let $2PQ$ be the two times union of PQ then

$$C \cdot C_1 = 3P + 3Q + 2R + S \geq 2P + 2Q + 2R = 2PQ \cdot C.$$

Therefore, using *Corollary 3.4.8*, exists a curve B such that $\deg(B) = \deg(C_1) - \deg(2PQ) = 1$ and

$$B \cdot C = P + Q + S.$$

Thus, $B = PQ$, $R = S$ and R is a flex. $\square$

Observation 3.5.7. Notice the importance of $k = \overline{k}$, because since the number of flexes in a cubic is finite, the theorem is false in the Real Euclidean plane (**Sylveser-Gallai Theorem**):

Theorem 3.5.8. (**Sylveser-Gallai Theorem**) Given a finite number of points in the Euclidean plane, either all the points are collinear, or there is a line which contains exactly two points.

Proof. Suppose by contradiction that we have a finite set of points not all collinear but such that at least each three points are collinear. Call it S. Let's define a *connecting line* to be a line which contains at least three points of S. Let's choose the pair (P, L) where $P \in S$ and L is a connecting line which not contains P, satisfying that they have the smallest positive distance apart among all point-line pairs. That is to say that if there are another pair (Q, M) such that $Q \in S$ and M is a connecting line that not contains Q, then $dist(Q, M) \geq dist(P, L)$.

By hypothesis, the connecting line L goes through at least three points of S. So dropping a perpendicular M' from P to L, there must be at least two points on one side of the perpendicular (allowing that one could be exactly on the intersection of the perpendicular with L). Of those two points, call P_1 the point closer to the perpendicular, and call the other point P_2. Draw the line L' connecting P to P_2. By hypothesis, L' is a connecting line because we suppose that a line through two points of S passes through a third. Moreover, $dist(P_1, L') \leq dist(M' \cap L, L') \leq dist(P, L)$ (the hypotenuse of a right triangle T is always grater or equal than the two other sides of T).

Therefore, we have found a pair (P_1, L') which have smaller distance than (P, L). Thus, there cannot be a smallest positive distance between point-line pairs, i.e, every point must be distance 0 from every line. In other words, every point must lie on the same line if each connecting line has at least three points.

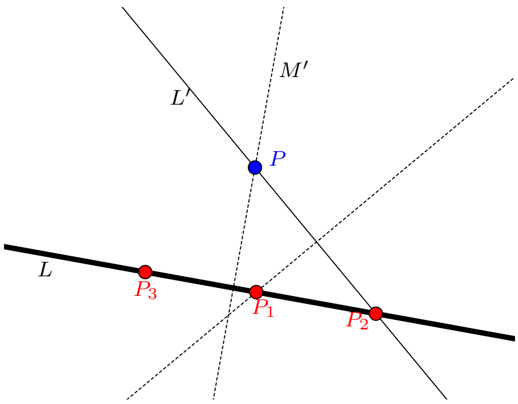


Figure 6: Sylveser-Gallai Theorem

□

Finally, an important result on the elliptic curves theory, its structure as an abelian group, it can be easly proved by the *Max Noether's Fundamental Theorem*.

Definition 3.5.9. (**Addition on a cubic**) Let C be a nonsingular cubic. For any two points P, $Q \in C$, there is a unique line L such that $L \cdot C = P + Q + R$, for some $R \in C$. (If $P = Q$, L is the tangent to C at P). Define $\varphi : C \times C \to C$ by setting $\varphi(P, Q) = R$. This φ is like an addition on C, but there is no identity. To remedy this, choose a point O on C. Then define an addition $\oplus$ on C as follows:

$$P \oplus Q = \varphi(O, \varphi(P, Q)).$$

Theorem 3.5.10. *C, with the opperation $\oplus$, forms an abelian group, with the point O being the identity.*

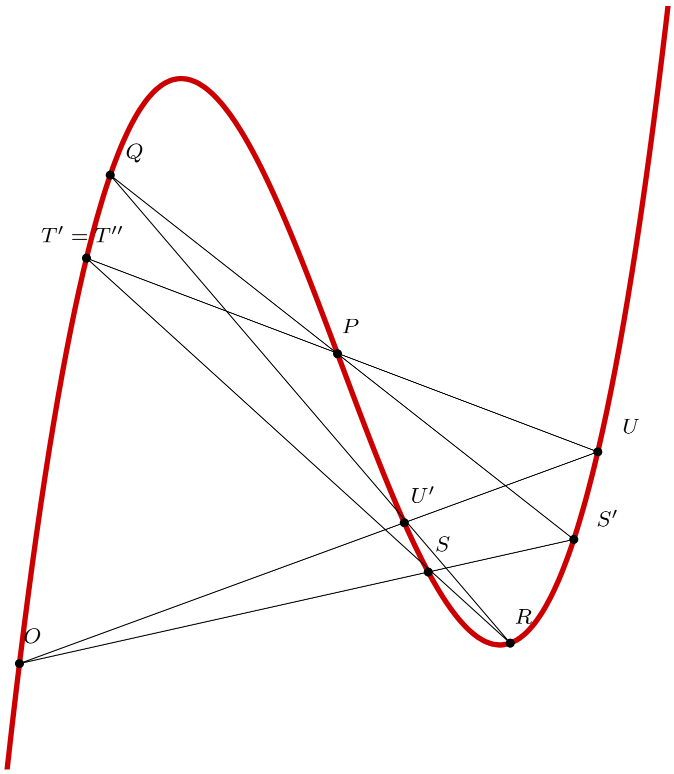


Figure 7: Addition on a cubic

Proof of associativity. Suppose $P, Q, R \in C$. Let $L_1 \cdot C = P + Q + S'$, $L_2 \cdot C = S + R + T'$ and $M_1 \cdot C = O + S' + S$. Also, let $M_2 \cdot C = Q + R + U'$, $L_3 \cdot C = O + U' + U$ and $M_3 \cdot C = P + U + T''$. Since

$$(P \oplus Q) \oplus R = \varphi(O, \varphi(P, Q)) \oplus R = \varphi(O, S') \oplus R = S \oplus R = \varphi(O, \varphi(S, R)) = \varphi(O, T'),$$

and

$$P \oplus (Q \oplus R) = P \oplus \varphi(O, \varphi(Q, R)) = P \oplus \varphi(O, U') = P \oplus U = \varphi(O, \varphi(P, U)) = \varphi(O, T''),$$

it suffices to show that $T' = T''$.

Let $C' = L_1 L_2 L_3$, $C'' = M_1 M_2 M_3$ be cubics. Then

$$C' \cdot C = P + Q + S' + S + R + O + U' + U + T'$$

and

$$C'' \cdot C = P + Q + S' + S + R + O + U' + U + T''.$$

So, we have two cubics that met another cubic in eight equal points. Then, by *Chasles-Cayley-Bacharach Theorem*, they meet also at the ninth point. Therefore, $T' = T''$. □

3.6. Theory of Curves

On this section, I will give a brief introduction on Theory of Curves. The most of the results form a part of the subject *Algebraic Varieties*, so I will not lay much in the details and I will skip most of the proofs.

Definition 3.6.1. If $X \subset \mathbb{P}^2$ is a nonsingular algebraic plane curve of degree d, then we define a **divisor** D on X as a formal linear combination, with integer coefficients, of the points of X, i.e.,

$$D = \sum_{P_i \in X} n_i P_i$$

where $n_i \in \mathbb{Z}$. By the **degree** of D we will mean the sum of its coefficients. The divisor is called **effective** if all of its coefficients are nonnegative. For a homogeneous polynomial F we will define its divisor as

$$(F) = \sum_{Q_i \in C} m_{Q_i}(F) Q_i.$$

Then, any rational function $f \in k(X)$ determines a divisor $(f) := (f)_0 - (f)_\infty$ on X called **principal divisor**, where

$$(f)_0 = \sum_{\substack{P_i \in C \\ f(P_i) = 0}} m_{P_i}(f) P_i$$

and

$$(f)_\infty = \sum_{\substack{P_i \in C \\ \frac{1}{f(P_i)} = 0}} m_{P_i}(1/f) P_i.$$

Two divisors D and D' are said to be **linearly equivalent**, written $D \sim D'$, if they differ by the divisor of a rational function.

Theorem 3.6.2. If X is a smooth projective curve, then every principal divisor on C has degree 0.

Observation 3.6.3. If C and C' are plane curves defined by homogeneous polynomials F and G, with no common components, then the *intersection cycle* (or **divisor cut**) $C \cdot C' := F \cdot G = \sum_{P \in \mathbb{P}^2} I(P, F \cap G) P$ is an effective divisor. Moreover, if X is a nonsingular plane curve defined by the homogeneous polynomial H, not contained neither in C nor in C', and we suppose that C and C' satisfies $\deg(C) = \deg(C')$, then the divisors $F \cdot H$ and $G \cdot H$ are linearly equivalent: their differ by the principal divisor defined by the rational function F/G restricted to X, i.e.,

$$(F/G) = (F/G)_0 - (F/G)_\infty = (F) - (G) = \sum_{P_i \in X} m_{P_i}(F) P_i - \sum_{Q_i \in C} m_{Q_i}(G) Q_i =$$

$$= \sum_{P \in \mathbb{P}^2} I(P, F \cap H) P - \sum_{Q \in \mathbb{P}^2} I(Q, G \cap H) Q = F \cdot H - G \cdot H.$$

In this context, we may give a new version of the *Bézout's Theorem* using this concepts, which we will use in the last chapter of these notes:

Theorem 3.6.4. (**Bézout's Theorem**) Let X be a nonsingular algebraic plane curve X of degree d. If C is a plane curve of degree e not containing any component of X, the degree of the divisor cut on X by C is $d \cdot e$.

Proof. As X is a nonsingular curve, for each P in $C \cap X$, $I(P, X \cap C) = m_P(C)$. Moreover, since C and X have no component in common, the two curves meet in $d \cdot e$ points (counted with multiplicity), thus the degree of $C \cdot X$ is

$$\sum_{P \in C \cap X} I(P, C \cap X) = \sum_{P \in C \cap X} m_P(C) = d \cdot e.$$

□

Definition 3.6.5. Let $X \subset \mathbb{P}^2$ be a nonsingular curve, the **divisor class group** (or **Picard group**) of X is defined by

$$Cl(X) := Div(X)/\sim,$$

where $Div(X)$ is the group of the divisors on X.

Now I will introduce the **regular and rational differential forms** on a smooth curve, in order to define the **canonical divisor** and to state the **Riemann-Roch Theorem** but without proof.

A **Riemann surface** is a 1-dimensional complex manifold. The group of divisors on a compact Riemann surface X is the free abelian group on the points of X. On a compact Riemann surface, the degree of a principal divisor is zero; that is, the number of zeros of a meromorphic function is equal to the number of poles, counted with multiplicity. As a result, the degree is well-defined on linear equivalence classes of divisors.

We shall now define regular and rational differential forms on a smooth curve X. First, for each open set $U \subset X$, we consider the vector space

$$\phi(U) := \left\{ \varphi : U \to \cup_{P \in U} \frac{\mathfrak{m}_P}{\mathfrak{m}_P^2};\ \varphi(x) \in \frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} \right\}$$

where $\mathfrak{m}_P$ is the maximal ideal of X at P. For a regular function $f \in \mathcal{O}_X(U) := \{f \in k(X);\ f \text{ is regular on } U\}$, we define an element $df \in \phi(U)$ by

$$df(P) := f - f(P) \bmod(\mathfrak{m}_P^2).$$

Definition 3.6.6. An element $\varphi \in \Phi(U)$ is called a **regular differential form** on U if for every point $P \in U$, there are a variety V and regular functions $f_1, \ldots, f_l, g_1, \ldots, g_l \in \mathcal{O}_X(V)$ such that

$$\varphi\Big|_V = \sum_{i=1}^{l} f_i dg_i.$$

We shall now define rational differential forms. For this we consider pairs (U, ω), where $U \subset X$ is open and nonempty, and w is a regular differential form on U. We define an equivalence relation by

$$(U, \omega) \sim (U', \omega') \Leftrightarrow \omega\Big|_V = \omega'\Big|_V$$

for some nonempty open $V \subset U$.

Definition 3.6.7. A **rational differential form** on X is an equivalence class of pairs (U, ω), where U is a nonempty open set in X and ω is a regular differential form on U.

From now, we will denote by $H = \sum_{i=1}^{d} P_i$ the divisor cut on a nonsingular plane curve X of degree d by a line $L \subset \mathbb{P}^2$.

Definition 3.6.8. We call the **canonical divisor** class, denoted by K_X, as the divisor of zeros minus the divisor of poles of any rational funtion on X. Abusing notation in a traditional way, we shall also denote by K_X any divisor in this class. Now, for a plane curve it is not hard to compute K_X by writting down a specific rational differential form. The result is a special case of what algebraic geometers call the **adjunction formula**, and it tells us that

$$K_X \sim (d-3) \cdot H = \sum_{i=1}^{d} (d-3) P_i,$$

i.e., K_X is the representator of the class consisting on divisors linearly equivalent to $X \cdot C$, where C is any curve of degree $d-3$.

Observation 3.6.9. Notice that if $H_j = \sum_{i=1}^{d} P_i$ is the divisor cut on a nonsingular plane curve X of degree d by a line $L_j = V(f_j)$ $(j = 1, 2)$, then $H_1 - H_2 = (f_1/f_2)$. Thus, $K_X \sim (d-3) \cdot H_1 \sim (d-3) \cdot H_2$, so not depend on the line choosen. Moreover, if C is any curve of degree e defined by the homeogeneous polynomial F and L is a line defined by the homogeneous polynomial G. Then, $L^e \cdot X - C \cdot X = (F/G)$. Thus,

$$e \cdot H \sim L^e \cdot X \sim C \cdot X,$$

i.e., for any curve C of degree d, C is linearly equivalent to $e \cdot H$.

Definition 3.6.10. The **genus** of X is the integer g that satisfies $\deg(K_X) = 2g - 2$.

Observation 3.6.11. For a nonsingular plane curve X, if the ground field is $\mathbb{C}$, then g is equal to the topological genus of the compact Riemann surface C. Thus, since $\mathbb{P}^1$ is homeomorphic to a sphere, if C is isomorphic to the projective line, then $g = 0$.

Now, we define a partial ordering on divisors on curves as follows. For divisors $D_1 = \sum n_P P$ and $D_2 = \sum m_P P$, we write

$$D_1 \geq D_2 \Leftrightarrow n_P \geq m_P \text{ for all } P \in X.$$

Definition 3.6.12. For any divisor D on X, we shall denote by $L(D)$ the vector space of rational functions f such that D plus the divisor of f is effective. We denote the vector space dimension of $L(D)$ by $l(D)$. For example, the only rational functions with no zeros or poles are the constant functions, so $l(0) = 1$.

Observation 3.6.13. If D is a divisor on X such that $deg(D) < 0$, then, $deg(-D) > 0$ and if there is $f \in L(D)$ such that D plus the divisor of f is effective, then f would have more zeros than poles on X, but the degree of a principal divisor is always zero. Thus, $L(D) = \{0\}$ and $l(D) = 0$.

Proposition 3.6.14. If $D_1 \sim D_2$, then $L(D_1) \cong L(D_2)$. Hence, for every divisor D, exists an effective divisor D' such that $L(D) \cong L(D')$ (if $L(D) \neq 0$, then $\exists f \in L(D)$ such that $D' := (f) + D \geq 0$ and $D' \sim D$).

Observation 3.6.15. By *Bézout's Theorem*, the degree of the divisor $(d-3)\cdot H$ is $d(d-3)$, so we get $2g-2 = d(d-3) \Rightarrow g = (d-1)(d-2)/2$.

Definition 3.6.16. Let D be a divisor, we define a **complete linear system** (or **complete linear serie**) as

$$\Big|D\Big| := \{D' \geq 0;\ D' \sim D\} = \{D_f := D + (f);\ 0 \neq f \in L(D)\}.$$

Proposition 3.6.17. There is a natural bijection between the complete linear system $\Big|D\Big|$ and the projective space $\mathbb{P}(L(D))$.

Proof. For any nonzero $f \in L(D)$, let $D_f := (f) + D$. Then, $D_f \geq 0$ and $D_f \sim D$. Moreover, for $\lambda \in k^*$, we have $(\lambda f) = (\lambda) + (f) = (f)$ (so $\mathbb{P}(L(D))$ it's well defined). Thus, we obtain a map

$$\begin{array}{ccc} \mathbb{P}(L(D)) & \to & \Big|D\Big|, \\ f & \mapsto & D_f. \end{array}$$

Let's see that is surjective. Suppose that $D' \geq 0$ and $D' \sim D$. Let f be a rational function with $(f) = -D + D'$, since $D' \geq 0$, it follows that $f \in L(D)$. To show injectivity, suppose that f and g are rational functions with $(f) = (g)$. Then, f/g is an everywhere regular function. By *Theorem 2.1.6*, f/g is constant, i.e, $f = \mu g$ for some $\mu \in k^*$. ☐

Theorem 3.6.18. (**Riemann-Roch Theorem**) For any divisor D of degree n on a nonsingular plane curve X of degree d,

$$l(D) = n - g + 1 + l(K_X - D).$$

Example 3.6.19. For $D = K_X$, $l(K_X) = (2g-2) - g + 1 + l(0) = g$.

Observation 3.6.20. We could also define the genus of a nonsingular curve X as $g := l(K_X)$.

4. A GENERALIZATION OF PASCAL'S THEOREM

From now until the end of this chapter, we suppose to be on the projective complex plane $\mathbb{CP}^2$. Moreover, we set a projective reference $R = \{A_1, A_2, A_3; O\}$, so that $A_1 = (1 : 0 : 0)$, $A_2 = (0 : 1 : 0)$, $A_3 = (0 : 0 : 1)$ and $O = (1 : 1 : 1)$. Observe that then $A_{2,3} = OA_1 \cap A_2A_3 = (0 : 1 : 1)$, $A_{3,1} = OA_2 \cap A_3A_1 = (1 : 0 : 1)$, $A_{1,2} = OA_3 \cap A_1A_2 = (1 : 1 : 0)$.

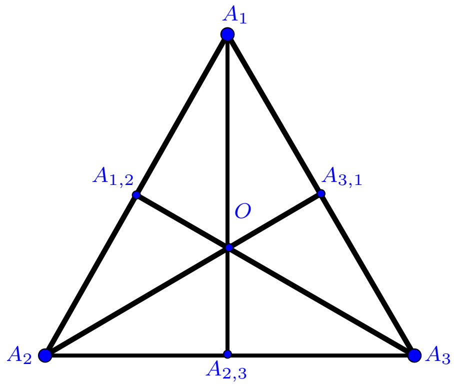


Figure 8: Projective reference

4.1. An Extension of the Menelaus' Theorem

Using a natural extension of the *Menelaus' Theorem* and the cross ratio, on this section we develop some concepts and find some results which we will use to prove some of the problems of algebraic geometry in a simple way. First, we introduce the notion of *characteristic ratio*, which will play an important role in all this chapter.

Definition 4.1.1. Let $P_1, \ldots, P_k \in A_iA_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$, where $P_r \neq A_i, A_j$, $\forall r \in \{1, \cdots, k\}$. We define the **characteristic ratio** of $P_1, \cdots, P_k$ with respect to the reference R as

$$[A_i, A_j; P_1, \ldots, P_k]_R = \prod_{r=1}^{k} (A_i, A_j, A_{i,j}, P_r),$$

where $(A_i, A_j, A_{i,j}, P_r)$ denotes the *cross ratio*.

Example 4.1.2. Let $P \in A_2A_3$, then, with the notation above, $P = (0 : \lambda : 1)$. So,

$$[A_2, A_3; P]_R = (A_2, A_3, A_{2,3}, P) = \frac{\lambda}{1} = \lambda.$$

In this context, we may give a different version of the *Menelaus' Theorem* using the *characteristic ratio*.

Proposition 4.1.3. (**Menelaus' Theorem**) Let $P_1 \in A_2A_3$, $P_2 \in A_3A_1$ and $P_3 \in A_1A_2$ be three simple and different points such that $P_i \neq A_j$ $(\forall i, j \in \{1, 2, 3\})$. Then,

$$[A_2, A_3; P_1]_R[A_3, A_1; P_2]_R[A_1, A_2; P_3]_R = -1,$$

if and only if, P_1, P_2 and P_3 are collinear.

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1)$, $P_2 = (1 : 0 : b_1)$ and $P_3 = (c_1 : 1 : 0)$. A necessary and suficient condition to the points of being alligned is that

$$\begin{vmatrix} 0 & 1 & c_1 \\ a_1 & 0 & 1 \\ 1 & b_1 & 0 \end{vmatrix} = 0 \Leftrightarrow a_1 b_1 c_1 = -1.$$

Therefore,

$$[A_2, A_3; P_1]_R [A_3, A_1; P_2]_R [A_1, A_2; P_3]_R =$$

$$= (A_2, A_3, A_{2,3}, P_1)(A_3, A_1, A_{3,1}, P_2)(A_1, A_2, A_{1,2}, P_3) = \frac{a_1}{1}\frac{b_1}{1}\frac{c_1}{1} = -1,$$

if and only if, P_1, P_2 and P_3 are collinear. □

A natural question is if there is an equivalent of *Menelaus' Theorem* but with conics. The following theorem gives conditions for which six different points lie on a conic.

Theorem 4.1.4. Let $P_1, P_2 \in A_2A_3$, $P_3, P_4 \in A_3A_1$, $P_5, P_6 \in A_1A_2$ be six different points such that $P_i \neq A_j$ ($i \in \{1, 2, 3, 4, 5, 6\}$ and $j \in \{1, 2, 3\}$). Then, $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a conic, if and only if,

$$[A_2, A_3; P_1, P_2]_R [A_3, A_1; P_3, P_4]_R [A_1, A_2; P_5, P_6]_R = 1.$$

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1), P_2 = (0 : a_2 : 1), P_3 = (1 : 0 : b_1), P_4 = (1 : 0 : b_2), P_5 = (c_1 : 1 : 0), P_6 = (c_2 : 1 : 0)$. Let $F(X, Y, Z) = AX^2 + BXY + CXZ + DY^2 + EYZ + GZ^2$ be a general conic. We know that 5 points determine a conic, so we will see what condition imposes the sixth point over the conics.

We want to see for which coefficients F vanishes on $P_1, \dots, P_6$, i.e., $F(0, a_i, 1) = F(1, 0, b_i) = F(c_i, 1, 0) = 0$, for $i = 1, 2$. Therefore, with an easy calculation, we get the following conditions on the coefficents of F:

$$\begin{cases} G &= -Da_1^2 - Ea_1, \\ E &= -D(a_1 + a_2), \\ A &= -Gb_1^2 - Cb_1, \\ C &= -G(b_1 + b_2), \\ D &= -Ac_1^2 - Bc_1, \\ B &= -A(c_1 + c_2). \end{cases}$$

So, it follows that $G = Da_1a_2$, $A = Gb_1b_2 = Da_1a_2b_1b_2$ and $D = Ac_1c_2 = Da_1a_2b_1b_2c_1c_2$.

Notice that if $D = 0$ in such conditions, then $F = 0$. Thus, if $D \neq 0$, F vanishies on $P_1, \dots, P_6$ if and only if $a_1a_2b_1b_2c_1c_2 = 1$ if and only if

$$[A_2, A_3; P_1, P_2]_R [A_3, A_1; P_3, P_4]_R [A_1, A_2; P_5, P_6]_R = (A_2, A_3, A_{2,3}, P_1)(A_2, A_3, A_{2,3}, P_2)$$

$$(A_3, A_1, A_{3,1}, P_3)(A_3, A_1, A_{3,1}, P_4)(A_1, A_2, A_{1,2}, P_5)(A_1, A_2, A_{1,2}, P_6) =$$

$$= \frac{a_1}{1}\frac{a_2}{1}\frac{b_1}{1}\frac{b_2}{1}\frac{c_1}{1}\frac{c_2}{1} = 1.$$

□

Now, we see a concept that will help us with the notation. Moreover, I will state some properties deriving from it.

Definition 4.1.5. Let $\sigma_{i,j} : A_iA_j \to A_iA_j$ be the involution on the projective line A_iA_j that keeps fixed $A_{i,j}$ and swaps A_i with A_j (i.e, $\sigma(A_i) = A_j$ and $\sigma(A_j) = A_i$). We call $P \in A_iA_j$ to be the **characteristic mapping** point of $Q \in A_iA_j$ with respecte A_iA_j, if $Q = \sigma_{i,j}(P)$.

Observation 4.1.6. If P is the **characteristic mapping** point of Q with respecte A_iA_j, then

$$[A_i, A_j; P, Q]_R = (A_i, A_j, A_{i,j}, P)(A_i, A_j, A_{i,j}, Q) = 1.$$

Proof. Let's restrict in the projective line A_iA_j. Then, we can suppose $A_{i,j} = (1 : 1)$, $A_i = (1 : 0)$ and $A_j = (0 : 1)$. So, the involution that keeps fixed $A_{i,j}$ and swaps A_i with A_j is defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Therefore, if $P = (\lambda : \mu) \in A_iA_j$, then $Q = \sigma_{i,j}(P) = (\mu : \lambda)$. Thus,

$$[A_i, A_j; P, Q]_R = (A_i, A_j, A_{i,j}, P)(A_i, A_j, A_{i,j}, Q) = \frac{\lambda\mu}{\mu\lambda} = 1.$$

□

Notice that taking $A_i, A_j, A_{i,j}$ as a projective reference on the line A_iA_j, if $P = (\lambda : \mu) \in A_iA_j$, then, $\sigma_{i,j}(P) = (\mu : \lambda)$.

Proposition 4.1.7. Any three points $P_1 \in A_2A_3$, $P_2 \in A_3A_1$, $P_3 \in A_1A_2$ are collinear if and only if $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$, $\sigma_{1,2}(P_3)$ are collinear.

Proof. With the notation above, we can suppose that $P_1 = (0 : a : 1)$, $P_2 = (1 : 0 : b)$ and $P_3 = (c : 1 : 0)$. Then, $\sigma_{2,3}(P_1) = (0 : 1 : a)$, $\sigma_{3,1}(P_2) = (b : 0 : 1)$ and $\sigma_{1,2}(P_3) = (1 : c : 0)$. So, P_1, P_2, P_3 are collinear if and only if

$$[A_2, A_3; P_1]_R[A_3, A_1; P_2]_R[A_1, A_2; P_3]_R = abc = -1.$$

Moreover, $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$, $\sigma_{1,2}(P_3)$ are collinear if and only if

$$[A_2, A_3; \sigma_{2,3}(P_1)]_R[A_3, A_1; \sigma_{3,1}(P_2)]_R[A_1, A_2; \sigma_{1,2}(P_3)]_R = \frac{1}{abc} = -1.$$

Thus, P_1, P_2, P_3 are collinear if and only if $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$, $\sigma_{1,2}(P_3)$ are collinear. □

Proposition 4.1.8. Let $P_1, P_2 \in A_2A_3$, $P_3, P_4 \in A_3A_1$, $P_5, P_6 \in A_1A_2$ be any six distinct points. Then, $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a conic if and only if their characteristic mappings lie on a conic as well.

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1), P_2 = (0 : a_2 : 1), P_3 = (1 : 0 : b_1), P_4 = (1 : 0 : b_2), P_5 = (c_1 : 1 : 0), P_6 = (c_2 : 1 : 0)$. Then, by *Theorem 4.1.4*, the six points lie on a conic, if and only if,

$$[A_2, A_3; P_1, P_2]_R[A_3, A_1; P_3, P_4]_R[A_1, A_2; P_5, P_6]_R = a_1a_2b_1b_2c_1c_2 = 1.$$

Due to the images of the points by the characteristic mapping are $\sigma_{2,3}(P_1) = (0 : 1 : a_1)$, $\sigma_{2,3}(P_2) = (0 : 1 : a_2)$, $\sigma_{3,1}(P_3) = (b_1 : 0 : 1)$, $\sigma_{3,1}(P_4) = (b_2 : 0 : 1)$, $\sigma_{1,2}(P_5) = (1 : c_1 : 0)$, $\sigma_{1,2}(P_6) = (1 : c_2 : 0)$, then, the six characteristic mappings lie on a conic, if and only if

$$[A_2, A_3; \sigma_{2,3}(P_1), \sigma_{2,3}(P_2)]_R[A_3, A_1; \sigma_{3,1}(P_3), \sigma_{3,1}(P_4)]_R[A_1, A_2; \sigma_{1,2}(P_5), \sigma_{1,2}(P_6)]_R =$$

$$= \frac{1}{a_1a_2b_1b_2c_1c_2} = 1 \Leftrightarrow a_1a_2b_1b_2c_1c_2 = 1.$$

Thus, the six images by the characteristic mapping lie on a conic, if and only if, $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a conic. $\square$

Theorem 4.1.9. Let $l_1 = A_2A_3$, $l_2 = A_3A_1$, $l_3 = A_1A_2$ and let $P_i \in l_i$ such that $P_i \neq A_j$ for $i, j \in \{1, 2, 3\}$. Then, it is satisfied that the points P_1, P_2, P_3, $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$, $\sigma_{1,2}(P_3)$, lie on a conic.

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1)$, $P_2 = (1 : 0 : b_1)$, $P_3 = (c_1 : 1 : 0)$. So, their characteristic mapping are $\sigma_{2,3}(P_1) = (0 : 1 : a_1)$, $\sigma_{3,1}(P_2) = (b_1 : 0 : 1)$, $\sigma_{1,2}(P_3) = (1 : c_1 : 0)$.

By *Theorem 4.1.4* it is enough to see that

$$[A_2, A_3; \sigma_{2,3}(P_1), P_1]_R[A_3, A_1; \sigma_{3,1}(P_2), P_2]_R[A_1, A_2; \sigma_{1,2}(P_3), P_3]_R = 1,$$

i.e, $\frac{1}{a_1}\frac{a_1}{1}\frac{1}{b_1}\frac{b_1}{1}\frac{1}{c_1}\frac{c_1}{1} = 1$, which is obvious. $\square$

Once we have seen the *Menelaus' Theorem* and its extension to the conics, we will see a natural extension of that theorem for curves of arbitrary degree n. The following theorem, gives us conditions for which $3n$ different points lie on an algebraic curve of degree n.

Theorem 4.1.10. (**Menelaus' Type Theorem**) Let $l_1 = A_2A_3$, $l_2 = A_3A_1$, $l_3 = A_1A_2$ and let $P_1^{(l_1)}, \ldots, P_n^{(l_1)} \in l_1$, $P_1^{(l_2)}, \ldots, P_n^{(l_2)} \in l_2$, $P_1^{(l_3)}, \ldots, P_n^{(l_3)} \in l_3$, $3n$ different and simple points, satisfying $P_i^{(l_k)} \neq A_1, A_2, A_3$, for every $i \in \{1, \ldots, n\}$ and $k \in \{1, 2, 3\}$. Then, it is satisfied that

$$[A_2, A_3; P_1^{(l_1)}, \ldots, P_n^{(l_1)}]_R[A_3, A_1; P_1^{(l_2)}, \ldots, P_n^{(l_2)}]_R[A_1, A_2; P_1^{(l_3)}, \ldots, P_n^{(l_3)}]_R = (-1)^n,$$

if and only if, the $3n$ points lie on an algebraic curve of degree n.

Proof. With the notation above, we can suppose that $P_i^{(l_1)} = (0 : a_i : 1)$, $P_i^{(l_2)} = (1 : 0 : b_i)$, $P_i^{(l_3)} = (c_i : 1 : 0)$, for every $i = 1, \ldots, n$. Note that the cas $n = 1$ and $n = 2$ have been already done in *Theorem 4.1.3* and *Theorem 4.1.4* respectively. So let's proved for $n \geq 3$.

Let $\mathbb{C}[X, Y, Z]_n = \{F \in \mathbb{C}[X, Y, Z]; F \text{ homogeneus polynomial of degree } n\}$ and let $E_n := \mathbb{C}[X, Y, Z]_n/(XYZ)$. We deffine the map

$$\varphi : E_n \to E_n/(X) \times E_n/(Y) \times E_n/(Z),$$

where $\varphi([F(X, Y, Z)]) = ([F(0, Y, Z)], [F(X, 0, Z)], [F(X, Y, 0)])$.

Then, φ is defined and is lineal. Let's see that is injective:

If $[F] \in ker(\varphi)$, $\varphi([F(X, Y, Z)]) = ([F(0, Y, Z)], [F(X, 0, Z)], [F(X, Y, 0)]) = ([0], [0], [0])$. Therefore, X, Y and Z divides F. Thus, $[F] = [0]$.

Therefore, φ is an isomorphism over its image. If we define

$$M_n := \left\{ \left(\left[\sum_{i+j=n} B_{i,j} Y^i Z^j \right], \left[\sum_{i+j=n} C_{i,j} X^i Z^j \right], \left[\sum_{i+j=n} D_{i,j} X^i Y^j \right] \right) ; \; B_{n,0} = D_{0,n}, \right.$$

$$B_{0,n} = C_{0,n}, C_{n,0} = D_{n,0}\} \subset E_n/(X) \times E_n/(Y) \times E_n/(Z),$$

the inclusion $Im(\varphi) \subset M_n$ is obvious and

$$\dim_{\mathbb{C}}(E_n) = 3n = \dim_{\mathbb{C}}(E_n/(X)) + \dim_{\mathbb{C}}(E_n/(Y)) + \dim_{\mathbb{C}}(E_n/(Z)) - 3 = \dim_{\mathbb{C}}(M_n).$$

Since $\dim_{\mathbb{C}}(Im(\varphi)) = \dim_{\mathbb{C}}(E_n)$ and $Im(\varphi) \subseteq M_n$, it folows that $Im(\varphi) = M_n$.

Hence, there exists a curve of degree n passing through $P_i^{(l_1)}, P_i^{(l_2)}, P_i^{(l_3)}$ $(i = 1, \ldots, n)$, if and only if,

$$\left(\left[\lambda_1 \prod_{i=1}^{n} (a_i Z - Y) \right], \left[\lambda_2 \prod_{i=1}^{n} (b_i X - Z) \right], \left[\lambda_3 \prod_{i=1}^{n} (c_i Y - X) \right] \right) \in Im(\varphi),$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0\}$. According with the deffinition of $Im(\varphi)$, it is necessary that

$$\begin{cases} \lambda_1 \prod_{i=1}^{n} a_i &= (-1)^n \lambda_2, \\ (-1)^n \lambda_1 &= \lambda_3 \prod_{i=1}^{n} c_i, \\ \lambda_2 \prod_{i=1}^{n} b_i &= (-1)^n \lambda_3, \end{cases}$$

what it follows $\prod_{i=1}^{n} a_i b_i c_i = (-1)^n$.

Therefore, there exists a curve of degree n passing through $P_i^{(l_1)}, P_i^{(l_2)}, P_i^{(l_3)}$ $(\forall i = 1, \ldots, n)$, if and only if,

$$[A_2, A_3; P_1^{(l_1)}, \ldots, P_n^{(l_1)}]_R [A_3, A_1; P_1^{(l_2)}, \ldots, P_n^{(l_2)}]_R [A_1, A_2; P_1^{(l_3)}, \ldots, P_n^{(l_3)}]_R =$$

$$= \prod_{i=1}^{n} a_i b_i c_i = (-1)^n.$$

□

We will see on the next section that *Menelaus' Type Theorem* has many applications on problems of intersection of two plane curves when one of them is the union of three general lines and the other is an arbitrary curve of degree n. Due to the simplicity of computation of the cross ratio, we will find an easy proofs for all of its applications.

Finally, to end this section we see one more concept which will be useful in order to state the final theorem of this chapter, the *Pascal's Type Theorem*.

Definition 4.1.11. Let $l_1 = A_2A_3$, $l_2 = A_3A_1$ and $l_3 = A_1A_2$. Following the notation as [7], we define the **pascal mapping** as $\Psi = (\sigma_{2,3} \times \sigma_{3,1} \times \sigma_{1,2}) \circ \Phi$, where

$$\Phi : (l_1 \setminus \{A_2, A_3\})^2 \times (l_2 \setminus \{A_3, A_1\})^2 \times (l_3 \setminus \{A_1, A_2\})^2 \to l_1 \times l_2 \times l_3,$$

is a map satisfying,

$$\Phi((P_1, P_2), (P_3, P_4), (P_5, P_6)) = (P_1P_2 \cap P_4P_5, P_3P_4 \cap P_6P_1, P_5P_6 \cap P_2P_3).$$

If we denote $Q_1 = P_1P_2 \cap P_4P_5$, $Q_2 = P_3P_4 \cap P_6P_1$ and $Q_3 = P_5P_6 \cap P_2P_3$, then,

$$\Psi((P_1, P_2), (P_3, P_4), (P_5, P_6)) = (\sigma_{2,3}(Q_1), \sigma_{3,1}(Q_2), \sigma_{1,2}(Q_3)).$$

Theorem 4.1.12. Let $P_1, P_2 \in A_2A_3$, $P_3, P_4 \in A_3A_1$ and $P_5, P_6 \in A_1A_2$ be six different points such that are all different of A_i $(i = 1, 2, 3)$. Then, $P_1, \ldots, P_6$ lie on a conic, if and only if, the three images of the pascal mapping applied to $P_1, \ldots, P_6$ are collinear.

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1), P_2 = (0 : a_2 : 1), P_3 = (1 : 0 : b_1), P_4 = (1 : 0 : b_2), P_5 = (c_1 : 1 : 0)$ and $P_6 = (c_2 : 1 : 0)$. Hence, $Q_1 = P_1P_2 \cap P_4P_5 = (0 : -1 : b_2c_1)$, $Q_2 = P_3P_4 \cap P_6P_1 = (a_1c_2 : 0 : -1)$ and $Q_3 = P_5P_6 \cap P_2P_3 = (-1 : b_1a_2 : 0)$.

Notice that by *Proposition 4.1.7*, it is just necessary and sufficient seeing that $P_1, \ldots, P_6$ lie on a conic, if and only if, Q_1, Q_2, Q_3 are collinear, which is obvious, because $P_1, \ldots, P_6$ lie on a conic, if and only if,

$$[A_2, A_3; P_1, P_2]_R [A_3, A_1; P_3, P_4]_R [A_1, A_2; P_5, P_6]_R = a_1a_2b_1b_2c_1c_2 = 1,$$

and Q_1, Q_2, Q_3 are collinear, if and only if,

$$[A_2, A_3; P_1]_R [A_3, A_1; P_2]_R [A_1, A_2; P_3]_R = \frac{-1}{b_2c_1}\frac{-1}{a_1c_2}\frac{-1}{b_1a_2} = -1.$$

□

4.2. Applications of the Characteristic Ratio

In this section, there is a collection of some problems of algebraic geometry. All of them can be proved by the *Max Noether's Fundamental Theorem* (which I have already done with some of them). However, it is interesting to prove them using the results about the characteristic ratio that we just saw.

By the moment, we have seen two different proofs about the *Pascal's Theorem*; one using the *Bézout's Theorem* and the other from *$Af + B\varphi$ Theorem*. Now, we will prove it again in order to see the easiness of the use of the characteristic ratio and the powerful of the *Menelaus' Type Theorem*.

Theorem 4.2.1. (Pascal's Theorem) Let $P_1, P_2 \in A_2A_3$, $P_3, P_4 \in A_3A_1$ and $P_5, P_6 \in A_1A_2$ six different points such that not three points are collinear. Suppose that $P_i \neq A_j$ $(i \in \{1, 2, 3, 4, 5, 6\}$ and $j \in \{1, 2, 3\})$. Then, the six points lie on a conic, if and only if, the three points $Q_1 = P_1P_2 \cap P_4P_5$, $Q_2 = P_2P_3 \cap P_5P_6$, $Q_3 = P_3P_4 \cap P_6P_1$ are collinear.

Proof. With the notation above, we can suppose that $P_1 = (0 : a_1 : 1), P_2 = (0 : a_2 : 1), P_3 = (1 : 0 : b_1), P_4 = (1 : 0 : b_2), P_5 = (c_1 : 1 : 0), P_6 = (c_2 : 1 : 0)$. The points lie on a conic, if and only if,

$$[A_2, A_3; P_1, P_2]_R [A_3, A_1; P_3, P_4]_R [A_1, A_2; P_5, P_6]_R = a_1a_2b_1b_2c_1c_2 = (-1)^2 = 1.$$

Moreover, the points

$$Q_1 = P_1P_2 \cap P_4P_5 = (0 : -1 : b_2c_1)$$
$$Q_2 = P_2P_3 \cap P_5P_6 = (-1 : a_2b_1 : 0)$$
$$Q_3 = P_3P_4 \cap P_6P_1 = (c_2a_1 : 0 : -1)$$

are collinear if and only if

$$\begin{vmatrix} 0 & -1 & c_2a_1 \\ -1 & a_2b_1 & 0 \\ b_2c_1 & 0 & -1 \end{vmatrix} = -a_2b_1c_2a_1b_2c_1 + 1 = 0 \Leftrightarrow a_1a_2b_1b_2c_1c_2 = 1.$$

$\square$

As we have seen with *Max Noether's Theorem*, just with a simple computing, we can prove:

Theorem 4.2.2. A line joining two flexes of a cubic passes through a third flex.

Proof. Let P_1, P_2 be the two flexes of a cubic Γ_3 and let's define $P_3 = \Gamma_3 \cap P_1P_2$. Suppose l_1, l_2, l_3 are three tangents passing through P_1, P_2, P_3, respectively, where $l_1 = A_2A_3$, $l_2 = A_3A_1$ and $l_3 = A_1A_2$. As P_1, P_2 are flexes, they have multiplicity 3 at the intersection of its tangent with the cubic.

We just can ensure that P_3 has multiplicity two at the intersection of l_3 with the cubic. By the *Bézout's Theorem*, l_3 intersects in a third point with the cubic. Let Q be that point. With the notation above, $P_1 = (0 : a_1 : 1), P_2 = (1 : 0 : b_1), P_3 = (c_1 : 1 : 0)$, $Q = (c_2 : 1 : 0)$.

Hence, P_1, P_2, P_3 are collinear, if and only if,

$$\begin{vmatrix} 0 & 1 & c_1 \\ a_1 & 0 & 1 \\ 1 & b_1 & 0 \end{vmatrix} = 0 \Leftrightarrow a_1b_1c_1 = -1.$$

Moreover, using that P_1, P_2, P_3 are collinear, P_1, P_2, P_3 and Q lie on a cubic (each point with its multiplicity at the intersection of l_i with Γ_3, $i = 1, 2, 3$), if and only if,

$$[A_2, A_3; P_1, P_1, P_1]_R[A_3, A_1; P_2, P_2, P_2]_R[A_1, A_2; P_3, P_3, Q]_R =$$

$$= (a_1b_1c_1)^2(a_1b_1c_2) = (a_1b_1c_2) = -1.$$

Now, a sufficient and necessary condition of P_1, P_2, Q being alligned is that

$$\begin{vmatrix} 0 & 1 & c_2 \\ a_1 & 0 & 1 \\ 1 & b_1 & 0 \end{vmatrix} = a_1b_1c_2 + 1 = 0,$$

if and only if, $a_1b_1c_2 = -1$.

Then, P_1, P_2, Q are collinear, therefore, P_3, $Q \in P_1P_2$. Due to they are the third intersection point of P_1P_2 with Γ_3, it follows that $P_3 = Q$. Thus, P_3 has multiplicity 3 at the intersection of Γ_3 with l_3 and, consequently, P_3 is a flex. $\square$

Now we see two results of the residual intersections between curves: one between a line with a cubic and the other between a conic with a cubic.

Theorem 4.2.3. If a line cuts a cubic in three distinct points, the residual intersections of the tangents at these points are collinear.

Proof. Let Γ_3 be that cubic, and let L be the line that cuts the cubic in three distinct points P_1, P_2, P_3. Let $l_1 = A_2A_3$, $l_2 = A_3A_1$, $l_3 = A_1A_2$ be the tangents at P_1, P_2, P_3, respectively. As l_1, l_2, l_3 are lines, by the *Bézout's Theorem*, they intersect with Γ_3 in three points. Due to the lines l_1, l_2, l_3 are tangents at P_1, P_2, P_3 respectively, these points have multiplicity 2 at the intersection of its tangent with Γ_3. So, let $\Gamma_3 \cap l_1 = \{P_1, Q_1\}$, $\Gamma_3 \cap l_2 = \{P_2, Q_2\}$, $\Gamma_3 \cap l_3 = \{P_3, Q_3\}$ the six intersections of the cubic with these three lines.

With the notations above, $P_1 = (0 : a_1 : 1), Q_1 = (0 : a_2 : 1), P_2 = (1 : 0 : b_1), Q_2 = (1 : 0 : b_2), P_3 = (c_1 : 1 : 0), Q_3 = (c_2 : 1 : 0)$. Hence, $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a cubic (P_i with multiplicity 2, $i = 1, 2, 3$), if and only if,

$$[A_2, A_3; P_1, P_1, Q_1]_R[A_3, A_1; P_2, P_2, Q_2]_R[A_1, A_2; P_3, P_3, Q_3]_R =$$

$$= (a_1)^2a_2(b_1)^2b_2(c_1)^2c_2 = -1.$$

Besides, a sufficient and necessary condition of P_1, P_2, P_3 being collinear is that

$$\begin{vmatrix} 0 & 1 & c_1 \\ a_1 & 0 & 1 \\ 1 & b_1 & 0 \end{vmatrix} = 0 \Leftrightarrow a_1b_1c_1 = -1.$$

Therefore, P_1, P_2, P_3 lie at the intersection of a cubic with a line, if and only if, $(a_1)^2a_2(b_1)^2 b_2(c_1)^2c_2 = a_2b_2c_2 = -1$.

Thus, Q_1, Q_2, Q_3 are collinear, if and only if,

$$\begin{vmatrix} 0 & 1 & c_2 \\ a_2 & 0 & 1 \\ 1 & b_2 & 0 \end{vmatrix} = 0 \Leftrightarrow a_2b_2c_2 = -1,$$

if and only if, P_1, P_2, P_3 lie at the intersection of a cubic with a line. □

Theorem 4.2.4. If a conic is tangent to a cubic at three distinct points, the residual intersections of the tangents at these points are collinear.

Proof. Let Γ_2 and Γ_3 be the conic and the cubic respectively. Let $\Gamma_2 \cap \Gamma_3 = \{P_1, P_2, P_3\}$. Suppose l_1, l_2, l_3 be the tangents at P_1, P_2, P_3 respectively, where $l_1 = A_2A_3$, $l_2 = A_3A_1$ and $l_3 = A_1A_2$. Then, with the notation above, $P_1 = (0 : a_1 : 1)$, $P_2 = (1 : 0 : b_1)$, $P_3 = (c_1 : 1 : 0)$. Due to a line cuts a cubic in three points and P_1, P_2, P_3 have multiplicity 2 at the intersection of the cubic and the tangents, there exists $Q_1 = (0 : a_2 : 1)$, $Q_2 = (1 : 0 : b_2)$, $Q_3 = (c_2 : 1 : 0)$ such that $l_1 \cap \Gamma_3 = \{P_1, Q_1\}$, $l_2 \cap \Gamma_3 = \{P_2, Q_2\}$, $l_3 \cap \Gamma_3 = \{P_3, Q_3\}$.

Then, P_1, P_2, P_3 lie on a conic, if and only if,

$$[A_2, A_3; P_1, P_1]_R[A_3, A_1; P_2, P_2]_R[A_1, A_2; P_3, P_3]_R = (a_1b_1c_1)^2 = 1.$$

Moreover, $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a cubic, if and only if,

$$[A_2, A_3; P_1, P_1, Q_1]_R[A_3, A_1; P_2, P_2, Q_2]_R[A_3, A_1; P_3, P_3, Q_3]_R = (a_1b_1c_1)^2(a_2b_2c_2) =$$

$$= (a_2b_2c_2) = -1.$$

Therefore, due to the fact that

$$\begin{vmatrix} 0 & 1 & c_2 \\ a_2 & 0 & 1 \\ 1 & b_2 & 0 \end{vmatrix} = a_2c_2b_2 + 1 = 0,$$

consequently, Q_1, Q_2, Q_3 are collinear. $\square$

We have already proved the *Chasles-Cayley-Bacharach Theorem* for any two cubics meeting in nine different points. Now, we will see a reduced version, restricting in the case when one of the cubics is the union of three general lines, which is just what we will use in order to prove the addition of the opperation $\oplus$ of the abelian group of a cubic using this news concepts.

Theorem 4.2.5. (**Chasles-Cayley-Bacharach Type Theorem**) Let C_1, $C_2 = l_1l_2l_3$ be two different cubics, where $l_1 = A_2A_3$, $l_2 = A_3A_1$, $l_3 = A_1A_2$. If C_1, C_2 meet in nine points $P_1, \ldots, P_9$ different of A_1, A_2, A_3 such that $P_1, P_2, P_3 \in l_1$, $P_4, P_5, P_6 \in l_2$, $P_7, P_8, P_9 \in l_3$, and C is any cubic containing $P_1, \ldots, P_8$, then C contains P_9 as well.

Proof. Suppose that $C \cap C_1 = \{P_1, \ldots, P_8, Q\}$. With the notation above, $P_1 = (0 : a_1 : 1)$, $P_2 = (0 : a_2 : 1)$, $P_3 = (0 : a_3 : 1)$, $P_4 = (1 : 0 : b_1)$, $P_5 = (1 : 0 : b_2)$, $P_6 = (1 : 0 : b_2)$, $P_7 = (c_1 : 1 : 0)$, $P_8 = (c_2 : 1 : 0)$, $P_9 = (c_3 : 1 : 0)$, $Q = (c : 1 : 0)$.

Hence, $P_1, \ldots, P_9$ lie on a cubic, if and only if,

$$[A_2, A_3; P_1, P_2, P_3]_R[A_3, A_1; P_4, P_5, P_6]_R[A_1, A_2; P_7, P_8, P_9]_R = a_1a_2a_3b_1b_2b_3c_1c_2c_3 = -1.$$

Similarly, $P_1, \ldots, P_8, R$ lie on a cubic, if and only if,

$$[A_2, A_3; P_1, P_2, P_3]_R[A_3, A_1; P_4, P_5, P_6]_R[A_1, A_2; P_7, P_8, Q]_R = a_1a_2a_3b_1b_2b_3c_1c_2c = -1.$$

Since $a_1a_2a_3b_1b_2b_3c_1c_2c = -1 = a_1a_2a_3b_1b_2b_3c_1c_2c_3$, it follow that $c = c_3$. Therefore, $P_9 = Q$. $\square$

Theorem 4.2.6. A cubic C, with the opperation $\oplus$, forms an abelian group, with the point O being the identity.

Proof. With the same notation of the proof given in the *Theorem 3.5.10*, it is a consequence of the *Chasles-Cayley-Bacharach Type Theorem.* $\square$

4.3. Pascal's Type Theorem

In this section, we achieve the first motivation of this thesis: can be *Pascal's Type Theorem* proved by $Af + B\varphi$ *Theorem*? A priori, it is not a trivial theorem, but the answer is affirmative. However, we will give an alternative proof using the characteristic ratio and the *Menelaus' Type Theorem.*

Theorem 4.3.1. (**Pascal's Type Theorem.**) Let $l_1 = A_2A_3$, $l_2 = A_3A_1$, $l_3 = A_1A_2$ and let $\{P_i^{(l_1)}\}_{i=1}^n$, $\{P_i^{(l_2)}\}_{i=1}^n$, $\{P_i^{(l_3)}\}_{i=1}^n$, n points on each line, different of A_1, A_2, A_3. Then, those $3n$ points lie on an algebraic curve of degree n, if and only if, the $3(n-1)$ points consisting of the three points determined by the pascal mapping applied to any six points among those $3n$ (no three points of which are collinear) as well as the remaining $3(n-2)$ points, lie on an algebraic curve of degree $n-1$.

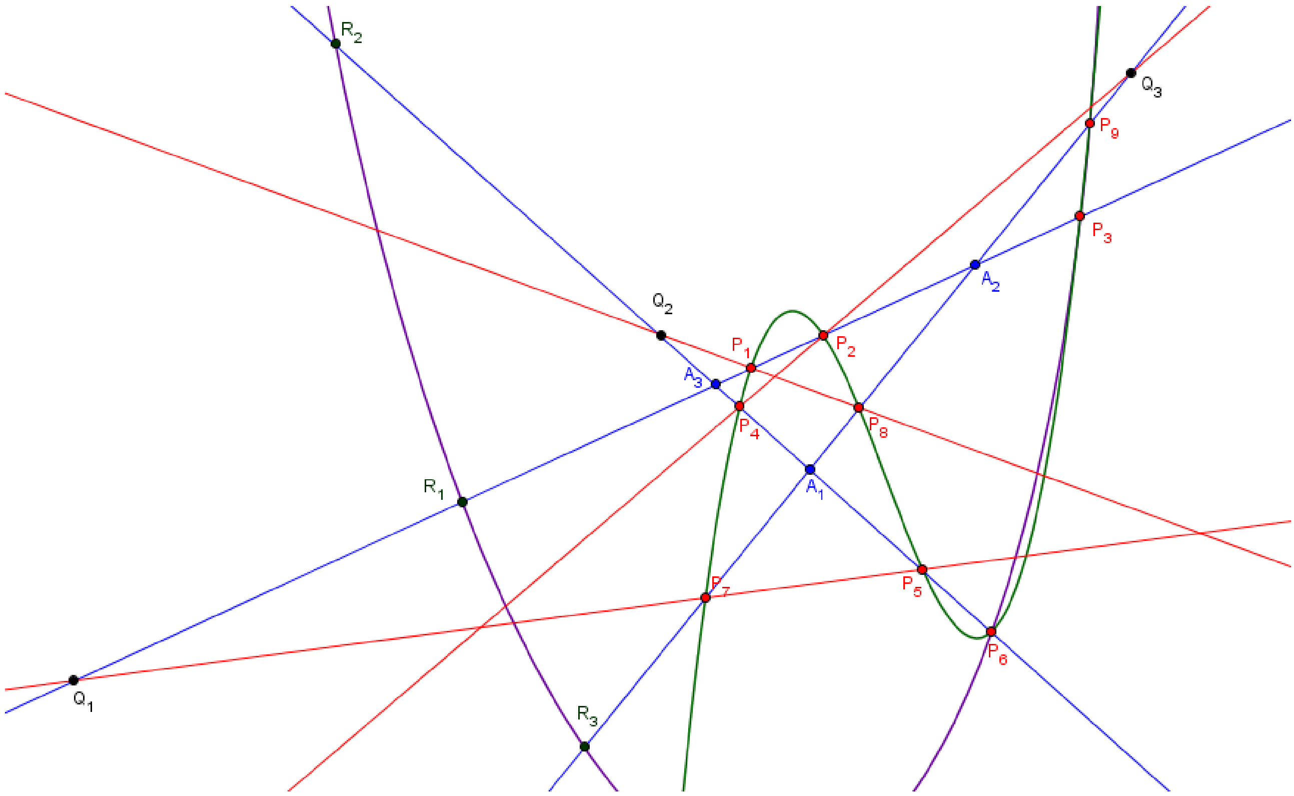


Figure 9: Pascal's Type Theorem for n = 3

Proof by the Max Noether's Fundamental Theorem. Without loss of generality, let's choose the points $P_1^{(l_1)}$, $P_2^{(l_1)}$, $P_1^{(l_2)}$, $P_2^{(l_2)}$, $P_1^{(l_3)}$, $P_2^{(l_3)}$, in order to apply the Pascal mapping. Let's denote

$$\Phi((P_1^{(l_1)}, P_2^{(l_1)}), (P_1^{(l_2)}, P_2^{(l_2)}), (P_1^{(l_3)}, P_2^{(l_3)})) = (Q_1, Q_2, Q_3).$$

Then,

$$\Psi((P_1^{(l_1)}, P_2^{(l_1)}), (P_1^{(l_2)}, P_2^{(l_2)}), (P_1^{(l_3)}, P_2^{(l_3)})) = (\sigma_{2,3}(Q_1), \sigma_{3,1}(Q_2), \sigma_{1,2}(Q_3)).$$

We know by *Theorem 4.1.9* that $\sigma_{2,3}(Q_1)$, $\sigma_{3,1}(Q_2)$, $\sigma_{1,2}(Q_3)$ and Q_1, Q_2, Q_3 lie on a conic. Let Γ_2 be that conic.

Let's deffine the cubic $C = (P_2^{(l_1)} P_1^{(l_2)})(P_2^{(l_2)} P_1^{(l_3)})(P_2^{(l_3)} P_1^{(l_1)})$ generated by the three opposites sides, different of l_1, l_2, l_3, of the hexagon deffined by the six points $P_1^{(l_1)}$, $P_2^{(l_1)}$, $P_1^{(l_2)}$, $P_2^{(l_2)}$, $P_1^{(l_3)}$, $P_2^{(l_3)}$.

($\Rightarrow$) Suppose that $\{P_i^{(l_1)}\}_{i=1}^n$, $\{P_i^{(l_2)}\}_{i=1}^n$, $\{P_i^{(l_3)}\}_{i=1}^n$ lie on an algebraic curve Γ_n of degree n.

Let $\Gamma_{n+2} = \Gamma_2 \Gamma_n$ be an algebraic curve of degree $n+2$. Then,

$$\Gamma_{n+2} \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_i^{(l_1)} + P_i^{(l_2)} + P_i^{(l_3)}) + Q_1 + Q_2 + Q_3 + \sigma_{2,3}(Q_1) + \sigma_{3,1}(Q_2) + \sigma_{1,2}(Q_3),$$

and

$$C \cdot l_1 l_2 l_3 = \sum_{i=1}^{2} (P_i^{(l_1)} + P_i^{(l_2)} + P_i^{(l_3)}) + Q_1 + Q_2 + Q_3.$$

So,

$$\Gamma_{n+2} \cdot l_1 l_2 l_3 - C \cdot l_1 l_2 l_3 = \sum_{i=3}^{n} (P_i^{(l_1)} + P_i^{(l_2)} + P_i^{(l_3)}) + \sigma_{2,3}(Q_1) + \sigma_{3,1}(Q_2) + \sigma_{1,2}(Q_3).$$

Using *Corollary 3.4.8*, exists a curve Γ of degree $\deg(\Gamma_{n+2}) - \deg(C) = n + 2 - 3 = n - 1$ such that, $\Gamma \cdot l_1 l_2 l_3 = \Gamma_{n+2} \cdot l_1 l_2 l_3 - C \cdot l_1 l_2 l_3$. $\square$

($\Leftarrow$) Suppose that $\{P_i^{(l_1)}\}_{i=3}^n$, $\{P_i^{(l_2)}\}_{i=3}^n$, $\{P_i^{(l_3)}\}_{i=3}^n$, $\sigma_{2,3}(Q_1)$, $\sigma_{3,1}(Q_2)$, $\sigma_{1,2}(Q_3)$, lie on an algebraic curve Γ'_{n-1} of degree $n-1$.

Let $\Gamma'_{n+2} = C\Gamma'_{n-1}$ be an algebraic curve of degree $n+2$. Then,

$$\Gamma'_{n+2} \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_i^{(l_1)} + P_i^{(l_2)} + P_i^{(l_3)}) + Q_1 + Q_2 + Q_3 + \sigma_{2,3}(Q_1) + \sigma_{3,1}(Q_2) + \sigma_{1,2}(Q_3),$$

and

$$\Gamma_2 \cdot l_1 l_2 l_3 = Q_1 + Q_2 + Q_3 + \sigma_{2,3}(Q_1) + \sigma_{3,1}(Q_2) + \sigma_{1,2}(Q_3).$$

So,

$$\Gamma'_{n+2} \cdot l_1 l_2 l_3 - \Gamma_2 \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_i^{(l_1)} + P_i^{(l_2)} + P_i^{(l_3)}).$$

Using *Corollary 3.4.8*, exists a curve Γ' of degree $\deg(\Gamma'_{n+2}) - \deg(\Gamma_2) = n + 2 - 2 = n$, such that, $\Gamma' \cdot l_1 l_2 l_3 = \Gamma'_{n+2} \cdot l_1 l_2 l_3 - \Gamma_2 \cdot l_1 l_2 l_3$. $\square$

Proof by the Menelaus' Type Theorem. With the notation above, we can suppose $P_i^{(l_1)} = (0 : a_i : 1), P_i^{(l_2)} = (1 : 0 : b_i), P_i^{(l_3)} = (c_i : 1 : 0)$ for every $i = 1, \ldots, n$.

Whitout loss of generality, let's choose the points $P_1^{(l_1)}$, $P_2^{(l_1)}$, $P_1^{(l_2)}$, $P_2^{(l_2)}$, $P_1^{(l_3)}$, $P_2^{(l_3)}$, to apply the pascal mapping. So, $\Phi((P_1^{(l_1)}, P_2^{(l_1)}), (P_1^{(l_2)}, P_2^{(l_2)}), (P_1^{(l_3)}, P_2^{(l_3)})) = (Q_1, Q_2, Q_3)$, where

$$Q_1 = P_1P_2 \cap P_4P_5 = (0 : -1 : c_1b_2),$$

$$Q_2 = P_3P_4 \cap P_6P_1 = (c_2a_1 : 0 : -1),$$

$$Q_3 = P_5P_6 \cap P_2P_3 = (-1 : a_2b_1 : 0).$$

Then, $\Psi((P_1^{(l_1)}, P_2^{(l_1)}), (P_1^{(l_2)}, P_2^{(l_2)}), (P_1^{(l_3)}, P_2^{(l_3)})) = (R_1, R_2, R_3)$, where,

$$R_1 = \sigma_{2,3}(Q_1) = (0 : c_1b_2 : -1),$$

$$R_2 = \sigma_{3,1}(Q_2) = (-1 : 0 : c_2a_1),$$

$$R_3 = \sigma_{1,2}(Q_3) = (a_2b_1 : -1 : 0).$$

Now, $\{P_i^{(l_1)}\}_{i=1}^n$, $\{P_i^{(l_2)}\}_{i=1}^n$, $\{P_i^{(l_3)}\}_{i=1}^n$ lie on an algebraic curve of degree n, if and only if,

$$[A_2, A_3; P_1^{(l_1)}, \ldots, P_n^{(l_1)}]_R [A_3, A_1; P_1^{(l_2)}, \ldots, P_n^{(l_2)}]_R [A_1, A_2; P_1^{(l_3)}, \ldots, P_n^{(l_3)}] =$$

$$= \prod_{i=1}^{n} (A_2, A_3, A_{2,3}, P_i^{(l_1)})(A_3, A_1, A_{3,1}, P_i^{(l_2)})(A_1, A_2, A_{1,2}, P_i^{(l_3)}) = \prod_{i=1}^{n} \frac{a_i}{1} \frac{b_i}{1} \frac{c_i}{1} =$$

$$= \prod_{i=1}^{n} a_i b_i c_i = (-1)^n.$$

Moreover, $\{P_i^{(l_1)}\}_{i=3}^n$, $\{P_i^{(l_2)}\}_{i=3}^n$, $\{P_i^{(l_3)}\}_{i=3}^n$, R_1, R_2, R_3, lie on an algebraic curve of degree $n-1$, if and only if,

$$[A_2, A_3; P_3^{(l_1)}, \ldots, P_n^{(l_1)}, R_1]_R [A_3, A_1; P_3^{(l_2)}, \ldots, P_n^{(l_2)}, R_2]_R [A_1, A_2; P_1^{(l_3)}, \ldots, P_n^{(l_3)}, R_3]_R =$$

$$= (A_2, A_3, A_{2,3}, R_1)(A_3, A_1, A_{3,1}, R_2)(A_1, A_2, A_{1,2}, R_3) \prod_{i=3}^{n} (A_2, A_3, A_{2,3}, P_i^{(l_1)})$$

$$\prod_{i=3}^{n} (A_3, A_1, A_{3,1}, P_i^{(l_2)}) \prod_{i=3}^{n} (A_1, A_2, A_{1,2}, P_i^{(l_3)}) = \frac{c_1 b_2}{-1} \frac{c_2 a_1}{-1} \frac{a_2 b_1}{-1} \prod_{i=3}^{n} \frac{a_i}{1} \frac{b_i}{1} \frac{c_i}{1} =$$

$$= - \prod_{i=1}^{n} a_i b_i c_i = (-1)^{n-1}.$$

Observe that the two expressions are equivalent. Thus, $\{P_i^{(l_1)}\}_{i=1}^n$, $\{P_i^{(l_2)}\}_{i=1}^n$, $\{P_i^{(l_3)}\}_{i=1}^n$ lie on an algebraic curve of degree n, if and only if, $\{P_i^{(l_1)}\}_{i=3}^n$, $\{P_i^{(l_2)}\}_{i=3}^n$, $\{P_i^{(l_3)}\}_{i=3}^n$, R_1, R_2, R_3, lie on an algebraic curve of degree $n-1$. $\square$

Hence, a theorem that seems to have a difficult proving or seems to need so much knowledges in algebraic geometry, it is easy proved with a simply theory that does not requiere so much familiarity with a lot of concepts. Taking a good reference and choosing correctly the coordinates of the points, all jointly with an easy computation, it leads us to the solution without going further neither in the properties of the curve nor the points.

5. CAYLEY-BACHARACH THEOREM AND CONJECTURES

In all the thesis, we have been asking about the polynomials functions that vanish on a set of points. On this section, we will study it from a new point of view.

Suppose that Γ is a set of γ distinct points in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Let $f \in \mathbb{R}[X_0, \dots, X_n]$, then the condition that $f(P) = 0$ for $P \in \Gamma$ becomes a non trivial linear conditions on the coefficients on f. Thus, the vanishing of f on Γ is ensured by γ linear conditions on these coefficients of f.

We are interested in the space of polynomials of degree at most a given number d. In this case, the γ conditions are generally not independent. As a trivial example, consider the three conditions imposed by three collinear points in $\mathbb{R}^2$, and choose $d \leq 1$. A linear polynomial vanishing on any two of the points vanishes on the line joining them and hence, automatically vanishes on the third point. Thus, the three points impose only two independent conditions on the polynomials of degree ≤ 1. Conversely, following the same example, if the three points are not collinear, then it doesn't exist a polynomial of degree ≤ 1 passing through them, so the three points impose three independent conditions on the polynomials of degree ≤ 1.

In general, if λ of the γ conditions imposed by Γ suffice to imply all of them, and λ is the least such number, then we say that Γ **imposes λ independent conditions** on polynomials of degree $\leq d$. Since $\lambda \leq \gamma$, we concentrate on the difference $\gamma - \lambda$, "the failure of Γ to impose independent conditions on polynomials of degree $\leq d$". There are of course many variants of this question, perhaps, the most useful is to take points in projective space and to ask about *homogeneous forms of degree d* instead of *polynomials of degree almost d*.

5.1. Chasles, Cayley and Bacharach

Let k be an algebraically closed field. We will talk about polynomials of degree d meaning homogeneous polynomials of degree d.

Definition 5.1.1. If $\Gamma = \{P_1, \dots, P_m\} \subset \mathbb{P}^2$ is any collection of m distinct points, we shall say that Γ **imposes l conditions** on polynomials of degree d if in the vector space of polynomials of degree d in $\mathbb{P}^2$ the subspace of thoses vanishing at $P_1, \dots, P_m$ has codimension l, or equivanlently if in the projective space of curves of degree d, the subspace of those containing Γ has codimension l. If $l = m$, then we say that Γ **imposes independent conditions** on polynomials of degree d. Conversely, we say that the points of Γ **fail to impose independent conditions** on polynomials of degree d if Γ imposes $l < m$ conditions on such polynomials, and we write $m - l$ as the failure of Γ to impose independent conditions on polynomials of degree d. We denote the number of conditions imposed by Γ on forms of degree d by $h_\Gamma(d)$, and call $h_\Gamma(d)$ the **Hilbert function** of Γ.

Since the *Hilbert function* is studied on the *Alegbraic Varieties* subject, I will used without entering in further details.

Observation 5.1.2. (1) Let $\Gamma \subset \mathbb{P}^2$ and $I(\Gamma) = \{f_1, \dots, f_r\}$, then

$$h_\Gamma(d) = \dim_k((k[X_0, X_1X_2]/I(\Gamma))_d) = \binom{d+2}{2} - \dim_k(I(\Gamma)_d).$$

(2) If $\Gamma = \mathbb{P}^2$, $h_\Gamma(d) = \dim_k((k[X_0, X_1, X_2])_d) = \binom{d+2}{2}$.

Example 5.1.3. Suppose fixed a projective reference $R = \{P_1, P_2, P_3; P_4\}$ in $\mathbb{P}^2$. Let $\Gamma = \{P_1, P_2, P_3, P_4, P_5, P_6\}$, where $P_5 = (1:1:0)$ and $P_6 = (1:2:1)$. We want to know how many conditions imposes Γ on polynomials of degree 2.

A conic is determined by 5 points, so at least Γ imposes 5 independent conditions on polynomials of degree 2. Moreover, the conic passing through $P_1, \dots, P_5$ is $F(X, Y, Z) = XZ - YZ$, and $F(P_6) = -1 \neq 0$, hence Γ imposes independent conditions on polynomials of degree 2 (there is no conic that vanishing on 5 points of Γ, vanishes on the sixth).

Moreover, since $I(\{P_1, \dots, P_6\})_2 = \{F \in k[X_0, X_1, X_2]_2;\ F(P_i) = 0, \forall i \in \{1, \dots, 6\}\} = (0)$, Γ imposes

$$h_\Gamma(2) = dim_k((k[X, Y, Z]/I(\Gamma))_2) = \binom{2+2}{2} - 0 = 6$$

independent conditions on polynomials of degree 2.

Proposition 5.1.4. Let $\Gamma = \{P_1, \dots, P_n\} \subset \mathbb{P}^2$ be any collection of $n \leq 2d+2$ distinct points. The points of Γ fail to impose independent conditions on curves of degree d if and only if, either $d+2$ of the points of Γ are collinear or $n = 2d+2$ and Γ is contained in a conic.

Proof. ($\Leftarrow$) Suppose that $d+2$ points of Γ lie on a line L and let C be any curve of degree d containing Γ. If L and C have no common components, by the *Bézout's Theorem*, the two curves intersect in $d \cdot 1 = d$ points (counted with multiplicity). Since L and C meet in $d+2$ different points, then, by *Corollary 3.3.4*, the two curves have a component in common (that is, the equations defining them have a common factor). As a line is irreducible, C must contain L. Hence, if F is the defining polynomial of degree 1 of L, then, the polynomial of degree d that defines C must be $G = FH$, where H is an polynomial of degree $d-1$.

Thus, the subset of curves of degree d that contains L has the same dimension as the set of curves of degree $d-1$, which is $\binom{2+(d-1)}{2}$. Therefore, the codimension of the set of curves of degree d containing L is

$$\binom{2+d}{2} - \binom{d+1}{2} = d+1.$$

The remaining $n-(d+2)$ points of Γ impose at most $n-d-2$ conditions on curves of degree d. Then, Γ imposes at most $d+1+n-d-2 = n-1$ conditions, so the points of Γ fail to impose independent conditions on curves of degree d.

Suppose now that $n = 2d+2$ and Γ is contained in a conic Q. Let C' be any curve of degree d containing Γ. If Q and C' have no common components, by the *Bézout's Theorem*, the two curves intersect in $d \cdot 2 = 2d$ points (counted with multiplicity). Since Q and C' meet in $2d+2$ different points, then, by *Corollary 3.3.4*, the two curves have a component in common.

If Q is irreducible, then Q is contained in the curve of degree d. Conversely, if Q is the product of two lines L_1, L_2, then L_1 lie on m points of Γ and L_2 lie on the remaining $n-m$ points. Notice that if $m \neq d+1$, then either $m \geq d+2$ or $n-m \geq d+2$, i.e. at least $d+2$ points of Γ are collinear, and we are in the first case, so we are done. Thus, we can suppose that each line lie on $d+1$ different points of Γ. With the same arguments as below, we see that L_1 and L_2 are contained in C'. Hence, we conclude that Q is contained in C'.

Therefore, any curve of degree d containing Γ must contain Q. Let F' be the defining polynomial of degree 2 of Q, then the polynomial of degree d that defines C' must be $G' = F'H'$, where H' is a polynomial of degree $d-2$. Thus, the subset of curves of degree d that contains Q has the same dimension as the set of curves of degree $d-2$, which is $\binom{2+(d-2)}{2}$. Hence, the codimension of the set of curves of degree d containing Q is

$$\binom{2+d}{2} - \binom{d}{2} = 2d+1.$$

Then, Γ imposes at most $2d+1$ conditions and $2d+1 < 2d+2 = n$, so the points of Γ fail to impose independent conditions on curves of degree d.

($\Rightarrow$) For the more serious direction, we do induction first on the degree d and second on the number n of points. By the induction hypothesis on the number n of points we may assume that any proper subset of Γ does impose independent conditions on curves of degree d. Therefore, if Γ fails to impose independent condition and we denote by Γ' the set of all but one of the points of Γ, then $h_\Gamma(d) = h_{\Gamma'}(d)$ (i.e., $\dim_k(I(\Gamma)_d) = \dim_k(I(\Gamma')_d)$) because Γ' imposes independent conditions. So, since $I(\Gamma)_d \subseteq I(\Gamma')_d$, it follows that $I(\Gamma)_d = I(\Gamma')_d$. Thus, the statement that Γ itself fails to impose independent conditions amounts to saying that any plane curve of degree d containing all but one of the points of Γ contains Γ.

To start the inductions, we note first that the *Proposition 5.1.4* is trivial for $d=1$: any set Γ of $n \leq 4$ points in the plane fails to impose independent conditions on lines if and only if $n = 3 = d+2$ and the points of Γ are all collinear (then the three points impose 2 independent conditions), or $n = 4 > 3 = \binom{2+1}{2} = \dim_k(k[X_0, X_1, X_2])_1$ (and of course the four points lie on a conic).

Second, for arbitrary d, we will see that for $n \leq d+1$, the points of Γ always impose independent conditions on curves of degree d. To doing so, it suffices to find a curve of degree d containing all but one point of Γ. If we take the union of general lines L_i through P_i for $i = 1, \ldots, n-1$ but not for P_j with $j \neq i$, and an arbitrary plane curve C of degree $d-n+1$ not passing through P_n, we get a curve $C' = (\bigcup_i L_i) \bigcup C$ of degree d such that contains all but one of the points of Γ. So in this case, the result is trivial without imposing any conditions on the points of Γ. Hence, we will suppose $n > d+1$ and d arbitrary.

Suppose first that Γ contains $d+1$ points lying on a line L. Assume that no further point of Γ lies on L, and let Γ' be the complementary set of $n-d-1 \leq (2d+2)-d-1 = d+1$ points of Γ. If Γ' imposes independent conditions on curves of degree $d-1$, then we can find a curve X of degree $d-1$ containing all but any one point of Γ', and then the union $X \cup L$ would be a curve of degree d containing all but one point of Γ. Since we are supposing that Γ fails to impose independent conditions on curves of degree d, we reach a contradiction. Therefore, Γ' fails to impose independent conditions on polynomials of degree $d-1$, and by induction, either $(d-1)+2 = d+1$ of the points of Γ' lie on a line M or either $n-d-1 = 2(d-1)+2 = 2d$ and Γ' is contained in a conic. In the first case, Γ' has at least $d+1$ points, and since $d+1 \leq n-d-1 \leq (2d+2)-d-1 = d+1$, implies that Γ' has exactly $d+1$ points, so $n-d-1 = d+1$. Hence, $n = 2d+2$ and Γ lies on the conic $L \cup M$. In the second case, $n-d-1 = 2d$ implies that $n = 3d+1 \leq 2d+2$, hence, $d \leq 1$ and this case is already done.

Next, suppose only that some line L' contains $l \geq 3$ points of Γ. By the same argument as in the last paragraph, the remaining $n-l$ points of Γ must fail to impose independent conditions on curves of degree $d-1$. Let Γ' be the complementary set of $n-l$ points, by induction, either $(d-1)+2=d+1$ of the points of Γ' lie on a line M' or $n-l=2(d-1)+2=2d$ and Γ' is contained in a conic. In the first case, since $d+1$ points of Γ are collinear we are back in the case considered in the preceding paragraph. In the second case, $n-l=2d$ implies that $n=2d+l\leq 2d+2$, hence, $3\leq l\leq 2$, which is absurd.

We are now done except in the case where Γ contains no three collinear points. Choose any three points $P_1,P_2,P_3\in\Gamma$, and let $\Gamma'=\Gamma\setminus\{P_1,P_2,P_3\}$. If for any $i\in\{1,2,3\}$ the points of $\Gamma'\cup\{P_i\}$ impose independent conditions on curves of degree $d-1$, we can find a curve C of degree $d-1$ containing Γ' but not P_i, and the union of this curve with the line joining the two points of the set $\{P_1,P_2,P_3\}\setminus P_i$ is a curve of degree d containing all but exactly one point of Γ. Hence, Γ would not fail to impose independent conditions on polynomials of degree d. Thus, we may assume that $\Gamma'\cup\{P_i\}$ fails to impose independent conditions on curves of degree $d-1$. Since it cannot contain $d+1$ collinear points (just in the case $d=1$ which is already done), we have by induction $n-2=2(d-1)+2$ ($\Rightarrow n=2d+2$) and $\Gamma'\cup\{P_i\}$ is contained in a conic. Note that in the case $d=2$ we are done, since six points fail to impose independent conditions on conics if and only if they lie all on a conic. On the other hand, if $d\geq 3$, then Γ' contains at least five points ($n-3=(2d+2)-3\geq 5$), no three collinear, and since five points determine a conic, there may be at most one conic containing Γ'. Notice that we have 3 conics C_1,C_2 and C_3 containing $\Gamma'\cup\{P_1\}$, $\Gamma'\cup\{P_2\}$ and $\Gamma'\cup\{P_3\}$ respectively. Since the three contains Γ', they must be equal to a single conic curve Q, which then contains all of Γ. □

Theorem 5.1.5. (**Chasles-Cayley-Bacharach Theorem**) If $\Gamma=\{P_1,\dots,P_9\}$ is the intersection of two plane cubics C_1 and C_2, and $\Gamma'=\Gamma\setminus P_i$ is any subset of Γ omitting one point, then Γ and Γ' impose the same number of conditions on cubics.

Proof. We shall prove it without making any assumptions about the smoothness or the irreducibility of C_1 and C_2. Moreover, we shall actually prove the stronger statement, that Γ and Γ' each impose exactly eight conditions on cubics; that is, that the eight points of Γ' impose independent conditions on cubics and the $P_i\in\Gamma$ imposes a condition dependent of these eight.

Part of this is obvious: the nine points of Γ visibly fail to impose independent conditions on cubics, since the 10-dimensional vector space of cubic polynomials

$$(k[X_0,X_1,X_2])_3=\langle X_0^3,X_0^2X_1,X_0^2X_2,X_1^3,X_1^2X_0,X_1^2X_2,X_2^3,X_2^2X_0,X_2^2X_0,X_0X_1X_2\rangle$$

contains at least a two-dimensional subspace of polynomials vanishing on Γ, $\langle F_1,F_2\rangle$, such that F_1 and F_2 are the definig polynomials of C_1 and C_2 respectively. So Γ imposes at most $10-2=8$ conditions on cubics.

Now we will see that Γ' impose exactly 8 independent conditions on cubics. Suppose that Γ' fails to impose independent conditions on cubics. We know that Γ' consists of 8 points of Γ, so taking $n=8$ and $d=3$ on the *Proposition 5.1.4*, either $3+2=5$ of the points of Γ' lie on a line L or Γ' is contained in a conic Q.

Suppose that 5 of the points of Γ' lie on a line L. Then, by the *Bézout's Theorem*, if C_1 and C_2 have no component in common with L respectively, then C_1 and C_2 must intersect with L in 3 points (counted with multiplicity) respectively. Hence, since $\sum_P I(C_1\cap L,P)=\sum_P I(C_2\cap L,P)=5$, C_1 and C_2 must contain L as a component, but we have supposed that C_1 and C_2 have no component in common.

Similarly, if Γ' is contained in a conic Q, since a conic meets a cubic in 6 points (counted with mutiplicity) if they have no components in common, we get that C_1 and C_2 contains Q as a component, but we have supposed that C_1 and C_2 have no component in common.

Thus, Γ and Γ' impose 8 independent conditions on cubics. $\square$

In 1843, Arthur Cayley, when he was twenty-two, he published a note stating an extension of *Chasles-Cayley-Bacharach Theorem* to the case of intersection of curves of degree higher than 3. The basis of this extension was again the idea of counting conditions imposed by sets of points. His first observation was:

Proposition 5.1.6. (**Cayley Theorem**) If two curves X_1 and X_2 of degrees d and e respectively meet in a collection Γ of $d \cdot e$ points, then for any λ, the number $h_\Gamma(\lambda)$ of conditions imposed by Γ on forms of degree λ is independent of the choice of the curves X_1 and X_2; it can be written down explicity as

$$h_\Gamma(\lambda) = \binom{\lambda+2}{2} - \binom{\lambda+2-d}{2} - \binom{\lambda+2-e}{2} + \binom{\lambda+2-d-e}{2},$$

where the binomial coefficent $\binom{a}{2}$ is to be interpreted as 0 if $a < 2$.

Proof. If the two curves meet exactly in a collection of $d \cdot e$ points, then the two curves have not a component in common. Let F and G be the defining polynomials of X_1 and X_2 respectively such that $\deg(F) = d$ and $\deg(G) = e$. By hypothesis, F and G have no factor in common.

Let $R = k[X, Y, Z]$, on the *Step 1* of the *Bézout's Theorem's* proof, we saw that the following sequence

$$0 \to R_{\lambda-d-e} \xrightarrow{\psi} R_{\lambda-d} \oplus R_{\lambda-e} \xrightarrow{\phi} R_\lambda \xrightarrow{\pi} (R/(F,G))_\lambda \to 0,$$

was exact, where

$$\phi(A, B) = AF + BG, \ \forall A, B \in R_{\lambda-d} \oplus R_{\lambda-e},$$
$$\psi(C) = (GC, -FC), \ \forall C \in R_{\lambda-d-e},$$

and π is the natural map.

Thus, for $\lambda \geq d + e - 2$,

$$h_\Gamma(\lambda) = \dim_k((R/(F,G))_\lambda) = \dim_k(R_\lambda) - \dim_k(R_{\lambda-d} \oplus R_{\lambda-e}) + \dim_k(R_{\lambda-d-e}) =$$
$$= \binom{\lambda+2}{2} - \binom{\lambda+2-d}{2} - \binom{\lambda+2-e}{2} + \binom{\lambda+2-d-e}{2}.$$

We have the same identity when $\lambda < d + e - 2$ interpreting the binomial coefficent $\binom{a}{2}$ as 0 if $a < 2$. $\square$

Observation 5.1.7. *Cayley Theorem* is equivalent to the *Max Noether Fundamental Theorem*: any curve X passing through all the points of the intersection of X_1 and X_2 is defined by a polynomial that is the linear combination (with polynomials coefficients) of the polynomials defining X_1 and X_2. That is, if $F = 0$, $G = 0$ and $H = 0$ are the equations of X_1, X_2 and X respectively, then there exists polynomials A and B such that $H = AF + BG$.

Arthur Cayley went on to conclude that if Γ' is any subset of $h_\Gamma(\lambda)$ points of Γ such that imposes independent conditions on forms of degree λ, then a form of degree λ vanishing on the points of Γ' must vanish on Γ. Observe that the case $d = e = \lambda = 3$ is *Chasles-Cayley-Bacharach Theorem*. Unfortunately, this is false in general. For example, let L be a line on the projective plane, and let $\Gamma' = \{P_1, P_2, P_3\}$ be three points on L. Let X_1 and X_2 be two nonsingular cubic curves containing Γ' and let Γ be the set consisting on the nine points where X_1 and X_2 meet. Hence, X_1 and X_2 have no components in common and we have by *Proposition 5.1.6* that $h_\Gamma(1) = \binom{2+1}{2} = 3$ (that is, Γ does not lie on any lines). By the Arthur Cayley statement, if the points of Γ' impose independent conditions on lines, every line containing them would contain all of Γ. This is of course nonsense: the points of Γ' do lie on the line L, and none of the other points of X_i lie on L, or L would be a component of X_i $(i = 1, 2)$ and X_1, X_2 would meet in more than nine points.

Proposition 5.1.8. Let X_1, X_2 be two nonsingular cubic curves and let Γ be the set consisting on the nine points where X_1 and X_2 meet. Let Γ' be the subset of Γ consisting in any three points of it. Then, that Γ' imposes dependent conditions on lines is equivalent to the statement that the "residual" set $\Gamma'' := \Gamma - \Gamma'$ imposes dependent conditions on conics.

Proof. $(\Rightarrow)$ Let L be a line that contains Γ', and let Q be a conic containing five points of Γ''. Then, by *Chasles-Cayley-Bacharach Theorem*, since the cubic $C = L \cup Q$ contains all but one point of Γ (call it P), it has to contains all of them. Hence, since L is not a component of neither X_1 nor X_2, the point P must lie on the conic.

$(\Leftarrow)$ Similarly, we construct a cubic with the union of a conic that contains Γ'' and a line that contains two points of Γ' and since the conic is not a component of neither X_1 nor X_2, it follows that the three points of Γ' lies on such line. $\square$

Observation 5.1.9. *Pascal's Theorem* is a particular case of *Proposition 5.1.8*.

It may be illuminating to see the ideas of the linear series in order to correct Cayley's error. A paper written by Alexander von Brill and Max Noether, which appeared in 1874, helped Bacharach to state an important result which we will see at the end of this section. Thus, the aim from now will be to proof what we call the **Bacharach Theorem**. First, we will see one of the central results proved by Brill and Noether, but first let's make an observation which is a consequence of the *Max Noether Fundamental Theorem*:

Observation 5.1.10. If there is a curve Y with equation H such that $Y \cdot X \geq C \cdot X$, as X is nonsingular, then by the *$Af + B\varphi$ Theorem*, H must be written as $H = AF + BG$, where F and G are the equations of C and X respectively, and A, B are forms of degree $\deg(H) - \deg(F)$ and $\deg(H) - \deg(G)$ respectively. Since $(H - BG) \cdot G = H \cdot G$ (*Properties 3.4.4 (3)*), it follows that the curve Y' with equation $H - BG$ meets X in the same way that Y does; that is, $Y' \cdot X = Y \cdot X$. Moreover, Y' has equation $H - BG = AF$, so Y' contains C as a component.

Now we can state the result of Brill and Noether. The original version takes the curve X to be irreducible, but the difference is mostly a matter of how the definitions are formulated:

Theorem 5.1.11. (**Brill-Noether Theorem**) If X is a plane curve, then the linear series cut on X by plane curves of any degree d is complete.

Proof. First notice that for every curves C_1 and C_2 of the same degree with equations F_1 and F_2, then if we restrict on X,
$$(F_1/F_2) = C_1 \cdot X - C_2 \cdot X.$$
Thus, $C_1 \cdot X \sim C_2 \cdot X$. Hence, it suffices to see that given a plane curve C of degree d not containing any component of X and a divisor D linearly equivalent to $C \cdot X$, there is a plane curve C' of degree d not containing any component of X such that $C' \cdot X = D$.

So that, given a divisor D linearly equivalent to $C \cdot X$, then exists a rational function φ such that $C \cdot X - D = (\varphi)$. The rational function φ must be expressed as $\varphi = f/g$ where f and g are polynomials in three variable of the same degree. Let Y and Z be the curves defined by f and g respectively, we have
$$C \cdot X - D = (\varphi) = \left(\frac{f}{g}\right) = (f) - (g) = Y \cdot X - Z \cdot X.$$
Multiplying together the equations of C and Z, we get the equation of a curve Z' such that $Z' \cdot X = C \cdot X + Z \cdot X = D + Y \cdot X$. In particular, Z' contains $Y \cdot X$, hence, using the *Observation 5.1.10* that we have done above, there is a curve Z'' containing Y such that $Z'' \cdot X = Z' \cdot X$. If h is the equation of Z'', then, since Z'' contains Y, it follows that f divides h. If we write C' as the curve defined by h/f, then $C' \cdot X = Z'' \cdot X - Y \cdot X = Z' \cdot X - Y \cdot X = D$. Moreover, $\deg(C') = \deg(h) - \deg(f) = (\deg(C) + \deg(g)) - \deg(f) = \deg(C) = d$. □

Given this, the proof of *Chasles-Cayley-Bacharach Theorem*, and even its generalization becomes trivial:

Corollary 5.1.12. If X is a plane curve of degree $e \geq 3$ and C, C' are plane curves of some degree d meeting X in divisors D and D' that differ by at most one smooth point, say, $D - D' = P - Q$, then $P = Q$.

Proof. First notice that if F and F' are the polynomials that define C and C' respectively. Then, $D = \sum_{P \in X} m_P(F) = (F)$ and $D' = \sum_{P \in X} m_P(F') = (F')$, where m_P denotes the multiplicity at P on X. Then, $P - Q = D - D' = (F) - (F') = (F/F')$ and thus $P \sim Q$.

Now, let L be a general line that passes through P. By the **Brill-Noether Theorem** there is some line L' that cuts out the divisor $L \cdot X - P + Q$, i.e., $L' \cdot X = L \cdot X - P + Q$ and such that $L' \cdot X \sim L \cdot X$. Since $e \geq 3$, the effective divisor $L \cdot X - P$ contains at least two points which spans L. Hence, since L' and X meet in e points (counted with multiplicity), it follows that $L' \cdot X$ contains at least two points that spans L. Thus, $L = L'$ and $P = Q$. □

Observation 5.1.13. *Chasles-Cayley-Bacharach Theorem* is the special case when $d = e = 3$.

In fact a much stronger statement, still short of the full version of *Bacharach Theorem*, can be derived by related methods and will play a role in the sequel, so we pause to examine it. To doing so, let's see first some interesting results:

As in the *Theory of Curves* section, we will denote by $H = \sum_{i=i}^{d} P_i$ the divisor cut on a nonsingular plane curve X of degree d by a line $L \subset \mathbb{P}^2$. Now, we will state a surprising property of the divisor $(d-3) \cdot H = \sum_{i=i}^{d} (d-3)P_i \sim K_X$.

Proposition 5.1.14. Let $X \subset \mathbb{P}^2$ be a nonsingular plane curve of degree d, and let P be a point of X. Every effective divisor linearly equivalent to $(d-3) \cdot H + P$ actually contains P.

Proof. Since $K_X \sim (d-3)\cdot H$, we will us that $L(K_X) \cong L((d-3)\cdot H)$. Let P be a point of X. The degree of the divisor $K_X + P$ is $(2g-2)+1$, so by the *Riemann-Roch Theorem* we have

$$l(K_X + P) = (2g-1) - g + 1 + l(-P) = g + l(-P).$$

Clearly, the degree of $-P$ is -1, so no effective divisor can be equivalent to $-P$ because if there is $0 \neq f \in K(X)$ such that the divisor defined by (f) plus $-P$ is effective, then, f would have more zeros than poles and $0 = \deg((f)) = \deg((f)_0) - \deg((f)_\infty) > 0$. Thus, $l(-P) = 0$. Therefore, $l(K_X + P) = g = l(K_X)$.

Observe now that if D is any effective divisor linearly equivalent to K_X, then $D+P$ is an effective divisor linearly equivalent to $K_X + P$. Then, for every effective divisor D linearly equivalent to K_X, let $f \in L(D) \cong L(K_X)$, so $(f) + D \geq 0 \Rightarrow (f) = D' - D$, where $D' \geq 0$. Hence, let $D'' := D' + P \Rightarrow (f) = D'' - D - P \Rightarrow (f) + D + P = D'' \geq 0$, so $f \in L(D+P) \cong L(K_X + P)$.

Since $l(D) = l(K_X) = l(K_X + P) = l(D+P)$ and $L(D) \subseteq L(D+P)$, they must be equal. So, if $P \notin D$ and $0 \neq f \in L(D+P)$, then $(f) + D + P \geq 0$, so since $D + P \geq 0$, f has P as a pole and thus $(f) + D \not\geq 0$, i.e, $L(D) \subsetneq L(D+P)$. Thus, $P \in D$. $\square$

Theorem 5.1.15. Let X_1, $X_2 \subset \mathbb{P}^2$ be plane curves of degrees d and e respectively, meeting in a collection of $d\cdot e$ distinct points $\Gamma = \{P_1, \dots, P_{de}\}$. If $C \subset \mathbb{P}^2$ is any plane curve of degree $d+e-3$ containing all but one point of Γ, then C contains all of Γ.

Proof. We will see the result under the assumption that X_1 is nonsingular. Suppose $C \subset \mathbb{P}^2$ is a plane curve of degree $d+e-3$ containing all of $X_1 \cap X_2$ except for the point P_{de}. We can write the divisor cut on X_1 by C as

$$C \cdot X_1 = P_1 + \cdots + P_{de-1} + Q_1 + \cdots + Q_{d(d-3)+1}.$$

Let f be the defining polynomial of a line $L \subset \mathbb{P}^2$ such that $H = X \cdot L$, and let g and h be the defining polynomials of X_2 and C respectively. Since

$$X_1 \cdot X_2 - X_1 \cdot L^e = (g) - (f^e) = \left(\frac{g}{f^e}\right), \text{ with } \deg(g) = e = \deg(f^e)$$

and

$$C \cdot X_1 - X_1 \cdot L^{d+e-3} = (h) - (f^{d+e-3}) = \left(\frac{h}{f^{d+e-3}}\right), \text{ with } \deg(h) = d+e-3 = \deg(f^{d+e-3}),$$

we have

$$X_1 \cdot X_2 = P_1 + \cdots + P_{de} \sim e \cdot H = X_1 \cdot L^e$$

and

$$C \cdot X_1 \sim (d+e-3)\cdot H = X_1 \cdot L^{d+e-3}.$$

Thus, we can rewrite the equation as

$$(d+e-3)\cdot H \sim e\cdot H - P_{de} + Q_1 + \cdots + Q_{d(d-3)+1}$$

or, equivalently,

$$Q_1 + \cdots + Q_{d(d-3)+1} \sim (d-3)\cdot H + P_{de} \sim K_X + P_{de}.$$

By *Proposition 5.1.15*, we get that the divisor $Q_1 + \cdots + Q_{d(d-3)+1} = C \cdot X_1 - P_1 - \cdots - P_{de-1}$ contains P_{de}. As $P_1 + \cdots + P_{de-1}$ don't include P_{de}, necessarily, $C \cdot X_1$ contains P_{de}. In particular, $P_{de} \in C$. □

The version of *Bézout's Theorem* on the Theory of Curves' section says that by counting each point with an appropiate multiplicity, we may regard the intersection of C and X as a divisor on X of degree $d \cdot k$. The fact we want is a corollary of *Brill-Noether Theorem*:

Corollary 5.1.16. Let X be a nonsingular plane curve, and let C be any plane curve not containing any component of X. Let D be any effective divisor on X. The family of plane curves containing D cuts out on X the complete linear series of divisors linearly equivalent to $C \cdot X - D$.

Proof. First note that if C' is a plane curve of the same degree than C, then $C' \cdot X \sim C \cdot X$, hence, $C' \cdot X - D \sim C \cdot X - D$, for any divisor D on X.

Now, let D' be an effective divisor such that $D' \sim C \cdot X - D$, then $D' + D \sim C \cdot X$. Using the *Brill-Noether Theorem*, exists a plane curve C'' of the same degree than the curve C satisfying

$$C'' \cdot X = D' + D.$$

Therefore,

$$C'' \cdot X - D = D' \sim C \cdot X - D.$$

□

Observation 5.1.17. Since for any divisor D on a nonsingular plane curve X, $\left|D\right| \cong \mathbb{P}(L(D))$, it follows that

$$\dim_k(\left|D\right|) = \dim_k(\mathbb{P}(L(D))) = l(D) - 1.$$

We shall exploit the previous corollary to express the number of conditions imposed on forms of degree m by a set of points of X in terms that are accessible to the *Riemann-Roch Theorem*. The precise result is the following:

Proposition 5.1.18. Let X be a smooth plane curve of degree d, and let $\Lambda \subset X$ be a set of λ points, regarded as a divisor on X. The number of linear conditions imposed by Λ on forms of degree m is equal to $l(m \cdot H) - l(m \cdot H - \Lambda)$. In particular, the "failure of λ to impose independent conditions on forms of degree m" is $\lambda - [l(m \cdot H) - l(m \cdot H - \Lambda)]$.

Proof. First observe that if X is a smooth plane curve, then X must be irreducible, because if there exists two plane curves X_1 and X_2 such that $X = X_1 \cup X_2$, then the multiplicity of the points at $X_1 \cap X_2$ is greater or equal than 2 on X. Once knowing that:

The "number of linear conditions imposed by Λ on forms of degree m" is the dimension t of the vector space of forms of degree m modulo those vanishing on Λ, i.e.,

$$t = h_\Lambda(m) = \dim_k((k[X_0, X_1, X_2]/I(\Lambda))_m).$$

The number $l(m \cdot H) = \dim_k(\left|m \cdot H\right|) + 1$ is the dimension of the vector space $L(m \cdot H)$. Using the *Corollary 5.1.16* (applied in the case when $D = 0$) we get that $\dim_k(\left|m \cdot H\right|)$ is the dimension of the space of forms of degree m modulo those vanishing on X (since X is irreducible, if a curve have no common component with X means that doesn't contains all of X). Hence,

$$l(m \cdot H) = \dim_k((k[X_0, X_1, X_2]/I(X))_m) + 1.$$

Similarly, the number $\dim_k(\left|m \cdot H - \Lambda\right|)$ is by the *Corollary 5.1.16*, the dimension of the space of forms of degree m vanishing on Λ modulo those vanishing on all of X, i.e.,

$$l(m \cdot H - \Lambda) = \dim_k((I(\Lambda)/I(X))_m) + 1.$$

Thus,

$$t = \dim_k((k[X_0, X_1, X_2]/I(\Lambda))_m) = \dim_k(k[X_0, X_1, X_2]_m) - \dim_k(I(\Lambda)_m) =$$

$$= \dim_k(k[X_0, X_1, X_2]_m) - \dim_k(I(\Lambda)_m) + \dim_k(I(X)_m) - \dim_k(I(X)_m) =$$
$$= l(m \cdot H) - 1 - [l(m \cdot H - \Lambda) - 1] = l(m \cdot H) - l(m \cdot H - \Lambda).$$

The "failure of Λ to impose independent conditions on forms of degree m" is simply the number of points λ of Λ (the maximal number of conditions that Λ could impose) minus the number of conditions actually imposed, i.e.,

$$\lambda - [l(m \cdot H) - l(m \cdot H - \Lambda)].$$

□

Finally, using *Proposition 5.1.18* follows the *Bacharach Theorem*:

Theorem 5.1.19. (**Bacharach Theorem**) Let X_1, $X_2 \subset \mathbb{P}^2$ be plane curves of degrees d and e respectively, intersecting in $d \cdot e$ points $\Gamma = X_1 \cap X_2 = \{P_1, \ldots, P_{de}\}$, and suppose that Γ is the disjoint union of subsets Γ' and Γ''. Set $s = d + e - 3$. If $m \leq s$ is a nonnegative integer, then the dimension of the vector space of forms of degree m vanishing on Γ' modulo those containing all of Γ is equal to the failure of Γ'' to impose independent conditions on forms of degree $s - m$.

Proof. We will suppose that X_1 is nonsingular. As before, denote by H the hyperplane divisor on X_1; we shall consider Γ, Γ', and Γ'' as divisors on X_1 as well (where $\Gamma = \Gamma' + \Gamma''$). Suppose that the number of points on Γ'' is γ.

The dimension of the vector space of forms of degree m vanishing on Γ' modulo those containing all of Γ is equal to $\dim_k(I(\Gamma')_m) - \dim_k(I(\Gamma)_m) = \dim_k((k[X_0, X_1, X_2])_m) - \dim_k(I(\Gamma)_m) - [\dim_k((k[X_0, X_1, X_2])_m) - \dim_k(I(\Gamma')_m)] = h_\Gamma(m) - h_{\Gamma'}(m)$. By *Theorem 5.1.18*,

$$h_\Gamma(m) - h_{\Gamma'}(m) = [l(m \cdot H) - l(m \cdot H - \Gamma)] - [l(m \cdot H) - l(m \cdot H - \Gamma')] =$$

$$= l(m \cdot H - \Gamma') - l(m \cdot H - \Gamma).$$

Using the *Riemann-Roch Theorem* and the equivalence $\Gamma = X_1 \cdot X_2 \sim e \cdot H$, we get

$$l(m \cdot H - \Gamma') - l(m \cdot H - \Gamma) = l((m-e) \cdot H + \Gamma'') - l((m-e) \cdot H) = \deg((m-e) \cdot H + \Gamma'') + 1 - g +$$

$$+ l(K_X - ((m-e) \cdot H + \Gamma'')) - [\deg((m-e) \cdot H) + 1 - g + l(K_X - (m-e) \cdot H)] =$$
$$= (m-e)d + \gamma + 1 - g + l((d+e-3-m) \cdot H - \Gamma'') - (m-e)d - 1 + g - l((d+e-3-m) \cdot H)).$$

Since $s = d + e - 3$, the dimension of the vector space of forms of degree m vanishing on Γ' modulo those containing all of Γ is equal to

$$\gamma - [l((s-m) \cdot H) - l((s-m) \cdot H - \Gamma'')] = \gamma - h_{\Gamma''}(s-m),$$

which is the failure of Γ'' to impose independent conditions on curves of degree $s - m$. $\square$

Observation 5.1.20. Note that if we choose $m = s$ and let Γ'' be a simple point $P = P_{de}$, then $\gamma - [l((s-m) \cdot H) - l((s-m) \cdot H - \Gamma'')] = 1 - 1 = 0$. Hence, by *Theorem 5.1.19*, the dimension of the vector space of forms of degree $m = d+e-3$ vanishing on Γ' modulo those containing all of Γ is 0, i.e., Γ and Γ' impose the same conditions on forms of degree $d+e-3$. That is, if $C \subset \mathbb{P}^2$ is any other plane curve of degree $d + e - 3$ containing all but one point of Γ, then contains Γ. Therefore, *Theorem 5.1.15* is a consequence of the *Bacharach Theorem*. Thus, the conclusion is that there are no hypersurfaces of degree m containing $\Gamma \setminus \{P\}$ except those containing Γ. The previous theorem says further that any curve of degree $s-1 = d+e-4$ containing all but two points of Γ contains Γ (because two points impose independent conditions on polynomials of degree $s - 1$). Moreover, there exists a curve of degree $s - 1$ containing all but three points $P, Q, R \in \Gamma$ but not containing Γ if and only if P, Q and R are collinear (if the three points are collinear, the failure to impose independent conditions is 1, hence the dimension of the vector space of forms of degree m vanishing on Γ' modulo those containing all of Γ is also 1), and so on.

There is an immediate generalization of *Theorem 5.1.19* to a statement about the transverse intersection of n hypersurfaces X_i of degrees d_i in $\mathbb{P}^n$. Before seeing it, let's stop to see the idea of a result involving the canonical divisor on hypersurfaces (which I will not define):

Proposition 5.1.21. If $X_1, \dots, X_r$ are nonsingular hypersurfaces meeting transversaly in $\mathbb{P}^n$ of degrees $d_1, \dots, d_r$ respectively, then $K_{X_1 \cap \cdots \cap X_r} \sim (\sum_{i=1}^{r} d_i - n - 1) \cdot H$.

Proof. Since the knowledgs needed for that proposition are far from these notes, I will give just the idea of the proof. Moreover, I have not defined the canonical divisor on hypersurfaces, for that I want to insist that this is just an idea of the proof, due to the following theorem only needs the consequence of this result, not the concepts involved on them, since I will just us the notion of the canonical divisor on a nonsingular curve.

By the adjunction formula, for the nonsingular hypersurface X_1 it is satisfied

$$K_{X_1} \sim (d_1 - n - 1) \cdot H$$

(notice that a plane curve X is an hypersurface on the plane, and if $\deg(X) = d$, we have $K_X \sim (d-3) \cdot H = (d-2-1) \cdot H$).

Then, for the transversal intersection of two nonsingular hypersurfaces X_1, X_2 (which is nonsingular as well) the adjunction formula says that

$$K_{X_1 \cap X_2} \sim (d_1 + d_2 - n - 1) \cdot H.$$

Doing an induction of the number of hypersurfaces, we get that

$$K_{X_1 \cap \cdots \cap X_r} \sim \left(\sum_{i=1}^{r} d_i - n - 1 \right) \cdot H.$$

$\square$

Theorem 5.1.22. (Generalization of Bacharach Theorem) Let $X_1, \ldots, X_n$ by hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \ldots, d_n$ respectively, meeting trasnversaly, and suppose that the intersection $\Gamma = X_1 \cap \cdots \cap X_n$ is the disjoint union of subsets Γ' and Γ''. Set $s = \sum_{i=1}^{n} d_i - n - 1$. If $m \leq s$ is a nonnegative integer, then the dimension of the family of forms of degree m containing Γ' modulo those containing all of Γ is equal to the failure of Γ'' to impose independent conditions on forms of "complementary" degree $s - m$.

Proof. (In the case where $X_1, \ldots, X_n$ are nonsingular). Let H be the hyperplane divisor on X. Then, $\Gamma = X_n \cdot X \sim d_n \cdot H$. Since the transversal intersection of $n-1$ nonsingular hypersurfaces defines a nonsingular plane curve, we have that $X = X_1 \cap \cdots \cap X_n$ is a nonsingular plane curve. Moreover, by *Proposition 5.1.21* we have that

$$K_X \sim \left(\sum_{i=1}^{n-1} d_i - n - 1 \right) \cdot H \sim (s - d_n) \cdot H.$$

Hence, replacing e and $d-3$ in the proof of *Theorem 5.1.19* with d_n and $s - d_n$ respectively, we get that the dimension of the family of forms of degree m containing Γ' modulo those containing all of Γ is equal to the failure of Γ'' to impose independent conditions on forms of degree $s - m$. $\square$

Observation 5.1.23. If $n = 2$, $d_1 = d$ and $d_2 = e$, then $s = d + e - 3$ and it clearly follows *Bacharach Theorem*.

5.2. Conjectures

The statement of the *Theorem 5.1.22* represents a good understanding on the extensions of the *Chasles-Cayley-Bacharach Theorem*, which express the result as a relationship between the Hilbert functions of residual subsets of sets of points. What I would like to propose instead the *Theorem 5.1.22* are some conjectures which you can find on [8] which are a new extension of the *Chasles-Cayley-Bacharach Theorem* that takes a different direction over the others toward a collection of inequalities on the Hilbert Function of a set of points of a complete intersection.

On the original paper [8] the conjecctures are stated by the theory of schemes. For the more expert, I will say that they used the scheme theory in order to avoid the hypothesis that the hypersurfaces intersect transversaly and replaced with the weaker one that they intersect in isolated points; that is, in modern lenguage, that it is not necessary to assume that the scheme $\Gamma = X_1 \cap \cdots \cap X_n$ (where $X_1, \ldots, X_n$ are hypersurfaces of $\mathbb{P}^n$) is reduced, only that is zero-dimensional. There is an interesting book of *Joe Harris* and *David Eisenbud*, called *The Geometry of Schemes* [6], which introduce the reader to that theory. Since at this point I don't have much time to introduce myself to that theory, I will suppose that the hypersurfaces intersect transversaly and I will avoid the schemes. So that, first I give some concepts and some results necessaries for the conjectures. Due to the idea of this section is only to see one possible future study of the theory developed on these notes, and the many of the results require some knowledge which at this moment I don't have, I will skip most of the proofs.

Definition 5.2.1. A projective algebraic variety V is a **complete intersection** if its vanishing ideal can be generated by $n - \dim(V)$ homogeneous polynomials in $k[X_0, \ldots, X_n]$. That is, if

V has dimension m, there should exist $n-m$ homogeneous polynomials $F_1(X_0,\dots,X_n),\dots,$ $F_m(X_0,\dots,X_n)$ such that $I(V)=\langle F_0,\dots,F_m\rangle$.

Observation 5.2.2. Note that if $I(V)=\langle F_0,\dots,F_m\rangle$, then $V=V(I(V))=\bigcap_{i=1}^m V(F_i)$. Since $V(F_i)=H_i$ defines an hypersurface in $\mathbb{P}^n$, if V is a complete intersection, $V=\bigcap_{i=1}^m H_i$ for some hypersurfaces $H_i\subset\mathbb{P}^n$.

Theorem 5.2.3. (**General Bézout's Theorem**) Let X and $Y\subset\mathbb{P}^n$ be varieties of dimension d and e with $d+e\geq n$, and suppose they intersect transversaly. Then,

$$\deg(X\cap Y)=\deg(X)\cdot\deg(Y).$$

In particular, if $d+e=n$, this says that $X\cap Y$ will consist of $\deg(X)\cdot\deg(Y)$ points.

Observation 5.2.4. If $\Gamma\subset\mathbb{P}^n$ is a complete intersection of n hypersurfaces $X_1,\dots,X_n$ meeting transversaly of degrees $d_1,\dots,d_n$, then, since $X=X_1\cap\cdots\cap X_{n-1}$ is a plane curve, by an easy induction on n and using the *General Bézout's Theorem* it follows that

$$\deg(X)=\prod_{i=1}^{n-1}\deg(X_i)=\prod_{i=1}^{n-1}d_i.$$

Thus, since $\dim(X)+\dim(X_n)=1+(n-1)=n$, $X_1\cap\cdots\cap X_n=X\cap X_n$ will consist of $\prod_{i=1}^n d_i$ points.

Now, I can state the conjectures. Although I am not proving any of them, I will see that indeed the first and the second are equivalent.

Conjecture 5.2.5. Let Γ be a complete intersection of n quadrics in $\mathbb{P}^n$ that intersect transversaly (i.e., Γ consist on 2^n points). If $X\subset\mathbb{P}^n$ is any hypersurface of degree $k\in\{1,\dots,n\}$ containing a subset Γ_0 that include at least $2^n-2^{n-k}+1$ points of Γ, then X contains all of Γ.

Note that this conjecture is sharp, if true: for any $k<n$, we can find a complete intersection $\Gamma\subset\mathbb{P}^n$ containing a complete intersection $\Omega\subset\mathbb{P}^{n-k}\subset\mathbb{P}^n$; then by *Therorem 5.1.22*, the residual set of points Γ_0 to Ω in Γ will then lie on a hypersurface of degree k not containing Γ. On the article, the authors also make the further conjecture that if X is a hypersurface of degree k that contains exactly 2^n-2^{n-k} points of Ω, then the residual set of points to $X\cap\Omega$ in Ω is a complete intersection of quadrics in a subspace $\mathbb{P}^{n-k}$.

Conjecture 5.2.6. Let Γ be any subset of a complete intersection of n quadrics in $\mathbb{P}^n$ that intersect transversaly. If Γ fails to impose independent conditions on hypersurfaces of degree m, then Γ contains at least 2^{m+1} points.

Here again this statement is sharp, if true: a complete intersection in $\mathbb{P}^{m+1}$ provides examples of equality for each m. Moreover, on the article, the authors conjecture further that equality holds if and only if Γ is itself a complete intersection of quadrics in $\mathbb{P}^{m+1}$.

Theorem 5.2.7. For all n, the following are equivalent:

(i) *Conjecture 5.2.5* for all k.

(ii) *Conjecture 5.2.6* for all m.

Proof. (ii) $\Rightarrow$ (i). Assume that *Conjecture 5.2.6* is true for a given value of m. Let's take $k = n - m - 1$ and let Γ be a complete intersection of n quadrics meeting transversaly. Let X be any hypersurface of degree k not containing Γ and let $\Gamma' = \Gamma \setminus \Gamma_0$, where $\Gamma_0 = X \cap \Gamma$. Since each quadric has degree 2, we have $m = n-k-1 \leq s = \sum_{i=1}^{n} 2-n-1 = n-1$. Moreover, exists an hypersurface of degree k containing Γ_0 but not containing Γ, hence, $h_{\Gamma_0}(k) - h_{\Gamma}(k) > 0$. Therefore, by *Theorem 5.1.22*, Γ' must fail to impose independent conditions on hypersurfaces of degree $s - k = n - 1 - k = m$. Thus, from *Conjecture 5.2.6* it follows that the number of points on Γ' is greater or equal than $2^{m+1} = 2^{n-k}$. Therefore, the number of points of Γ_0 is less or equal than $2^n - 2^{n-k}$.

(i) $\Rightarrow$ (ii). Now assume that *Conjecture 5.2.5* is true for all k. Let Γ be any subset of points of a complete intersection of n quadrics meeting transversaly, and suppose that Γ fail to impose independent conditions on hypersurfaces of degree $m = n - k - 1$. Take Ω to be a set of points that contains Γ, and let $\Gamma' \subset \Omega$ be the residual set of points to Γ. Since each quadric has degree 2, we have $m = n-k-1 \leq s = \sum_{i=1}^{n} 2-n-1 = n-1$. By *Theorem 5.1.22*, $h_{\Gamma'}(m) - h_{\Omega}(m) > 0$, hence, exists a hypersurface X of degree $s - m = n - 1 - m = k$ such that X contains Γ' but not contains Ω. It follows then by *Conjecture 5.2.5* that the number of point of Γ' is less or equal than $2^n - 2^{n-k}$. Thus, the number of points of Γ is greater or equal than $2^{n-k} = 2^{m+1}$.□

On the original paper, and with its original hypothesis, the *Conjecture 5.2.5* and *Conjecture 5.2.6* are proved for $n \leq 7$. To doing so, the authors prove that the *Conjecture 5.2.6* is true when $m \leq 4$ and correspondingly *Conjecture 5.2.5* when $n \leq 7$.

To end this chapter, I will formulate a general version of the *Conjecture 5.2.5* that does not require quadrics. Once again, to adapt to these notes, I skip the theory of schemes replacing it by the hypothesis of hypersurfaces meeting transversaly.

Conjecture 5.2.8. Let Γ be any subset of a complete intersection of n hypersurfaces of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ that intersect transversaly in a projective space $\mathbb{P}^n$. If Γ fails to impose independent conditions on hypersurfaces of degree m, then Γ contains at least $e \cdot d_s \cdot d_{s+1} \cdot \cdots \cdot d_n$ points, where e and s are defined by the relations

$$\sum_{i=s}^{n} (d_i - 1) \leq m + 1 < \sum_{i=s-1}^{n} (d_i - 1)$$

and

$$e = m + 1 - \sum_{i=s+1}^{n} (d_i - 1).$$

Observation 5.2.9. If we restrict these hypersurfaces to be quadrics, then the s that satisfies such relation is $s = n - m$. Moreover,

$$\sum_{i=s+1}^{n} (d_i - 1) = \sum_{i=n-m+1}^{n} 1 = m,$$

then $e = 1$ and it follows *Conjecture 5.2.6*. Indeed, it may it turn be translated, by an argument generalizing *Theorem 5.2.7*, into a statement analogous to *Conjecture 5.2.5*.

6. CONCLUSIONS

From a more objective point of view, I can ensure that working with these notes have exceeded my expectations. Not only I have achieved to get what motivates me to begin on this subject (proving the *Pascal Type Theorem* using the *Max Noether Fundamental Theorem*), but I have had the oportunity to introduce myself to the theory of plane algebraic curves. Begining with a classical problem, the *Hexagramme Mystique*, I have had to learn and understand some results related with plane algebraic curves in order to study its generalizations.

Moreover, the fact of having to rewrite the article [1], forces myself to be able of make definitions or writing the results with the best coherence and rigor with which I could. I know that writting may not be difficult (in the strict literally meaning), however, to make it understandable for the reader require to the writer to read it more than twice to give the nod. A big part of the work of a mathematician is to make articles about its results or its knowledges over a subject, hence I could say that in some way these notes give to me the first approach into the real world of a mathematician (even the complex one, since to explain something on the best way require that you first have apprehended it). Making some research by reading articles of a great range of distinct mathematicians, each one with its own style and with its own strengths, and trying to follow its studies, is the best way of preparing myself to my mathematical career.

By the other side, I will say that which I really regret is not have had more time in order to deepen in some aspects of these notes. I have encountered with some subjects that I would have liked to study and to be introduced to them as well. For example, an interesting theory with a lot of applications in the algebraic geometry is the theory of schemes, objects which enlarges the concept of algebraic varieties and with which there are multiple extensions of the *Pascal Theorem*. Besides, what it could be so interesting is to see particular cases of the conjectures that I have stated with my hypothesis, and ilustrated with examples where the conjectures work.

Finally, for further reading, I recommend two books (a part from those which are in the bibliography) both written by William Fulton. The first, *"Adjoints and Max Noether's Fundamentalsatz"*, which gives an exposition of the theory of adjoints and conductors for curves on nonsingular surfaces (which may be regarded as a ninth chapter of [1]); and the second, *"Introduction to Intersection Theory in Algebraic Geometry"*, which extends most of the results on these notes of plane algebraic curves to hypersurfaces in $\mathbb{P}^n$.

7. BIBLIOGRAPHY

[1] William Fulton: *ALGEBRAIC CURVES: An Introduction to Algebraic Geometry*, Reverté, s.a., 1971.

[2] Gerd Fischer: *Plane Algebraic Curves*, Student Mathematical Library, Volum 15, American Mathematical Society, 2001.

[3] R.J. Walker: *Algebraic Curves*, Princeton: Princeton University Press, 1950.

[4] M.F. Atiyah; I.G. Macdonald: *Introduction to Commutative Algebra*, Universty of Oxford, Addison-Wesley Series in Mathematics, 1969.

[5] C.G. Gibson: *Elementary geometry of algebraic curves*, Cambridge University Press, 1998.

[6] David Eisenbud; Joe Harris: *The Geometry of Schemes*, Springer, 2000.

[7] Zhongxuan Luo: *An Invariant of Algebraic Curves from the Pascal Theorem*, School of Software, School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China, September 5, 2007.

[8] David Eisenbud; Mark Green; Joe Harris: *CAYLEY-BACHARACH THEOREMS AND CONJECTURES*, American Mathematical Society, 1996.

[9] Terence Tao: *A partial converse to Bézout's Theorem*, 25 September, 2012 https://terrytao.wordpress.com/tag/bezouts-theorem/