Treball final de grau

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TOOLS OF DIFFERENTIAL CALCULUS TO SOLVE FUNCTIONAL EQUATIONS

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Abstract

In this work we first prove the converse of Taylor's Theorem. This allows us to prove next the Omega-lemma and the differentiability of the Evaluation map for certain Banach spaces of analytic functions. These two theorems together with the Implicit Function Theorem are applied to certain functional equations in order to prove Poincaré's Linearization Theorem and the Analytic Stable Manifold Theorem.

Resum

En aquest treball provem en primer lloc el recíproc del Teorema de Taylor. Aquest resultat ens permet provar l'Omega-lemma i la diferenciabilitat de l'aplicació Avaluació per a certs espais de Banach de funcions analítiques. Aquests dos teoremes juntament amb el Teorema de la Funció Implícita els apliquem a certes equacions funcionals per tal de provar el Teorema de Linealització de Poincaré així com el Teorema de la Varietat Estable Analítica.

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Contents

1 Introduction

Project

This work has two main objectives:

On the one hand, to become acquainted with some facts and techniques of mathematical analysis not widely known. First, the converse of Taylor's Theorem and then, a couple of theorems from global analysis, namely the so called Omega-lemma and the differentiability of the Evaluation map, both in the setting of certain Banach spaces of analytic maps. And on the other, to have a close look at the power of these tools of functional analysis when applied to certain functional equations in order to supply clear proofs of deep results such as Poincaré's Linearization Theorem and the analytic version of the Stable Manifold Theorem.

Structure

This work mainly consists of three blocks, the first of which contains the prerequisites needed in the subsequent parts. These prerequisites aim at covering the main topics of differential calculus in Banach spaces. This block consists of Sections 2 to 5 and in them one of the most relevant theorems is the converse of Taylor's Theorem. The second block is Section 6, where the background and properties of certain Banach spaces is built including the Omega-lemma and the differentiability of the Evaluation map. All these results are then applied in the third block which consists of Sections 7 and 8. To be more precise we next give a brief description of the different sections of the work.

In Section 2 we describe some basic facts on normed spaces, either real or complex, which are needed in the subsequent sections.

In Section 3, relying mainly on [2] and [4], we focus on those linear and multilinear maps between normed spaces which are also continuous. This leads us to the concept of operator norms but it turns out that on some occasions there are other interesting norms which will be most useful in our later considerations and to which we have paid special attention. A central point in dealing with the various continuous multilinear maps between normed spaces is what is called a consistent family of norms. This section also contains the fundamentals of polynomials in the general setting of Banach spaces because of their essential role in Taylor's formula.

Section 4 starts with the definition of differential or derivative of a map between Banach spaces at a point, and this needs the consideration of the Banach space of continuous linear maps. In this section we have given the basic definitions and facts concerning differentiability in dealing with maps between Banach spaces, including examples because, especially when dealing with higher derivatives, the framework required is more difficult than in the finite-dimensional situation and requires dealing with spaces of continuous multilinear maps. One of the important results in this section is Taylor's formula; we have presented the version which uses Landau's o-notation. But the most interesting result in this section is the converse to Taylor's Theorem because it represents the key point in proving most subsequent results which are essential in this work. We have been inspired by the proof occurring in

[4] but we have simplified and modified it. Its statement has been given in terms of Landau's o-notation.

Section 5 quickly revisits the Inverse and the Implicit Function Theorems because the latter will be applied in the next sections. Fortunately, the statements of these key theorems are quite close to those of the well-known case of $\mathbb{R}^n$. This section also includes comments on partial derivatives in view of their applications in proving the C^∞-differentiability character of certain maps.

After these preliminary considerations, we begin, in Section 6, with the main machinery in this work. Here and following the sketchy notes of Meyer [5] we have introduced the spaces $A_\delta(E, F)$ and we have provided an elementary proof that they are Banach. These spaces consist of generalized power series, in the sense that we are working over a Banach space rather than just working with power series involving a finite number of variables, an important classical case of course covered by $A_\delta(E, F)$. The role of δ can be thought of as a generalization of radius of multiconvergence for these series. It turns out that the functions of $A_\delta(E, F)$ (from a centered ball of E into F) are easily seen to be continuous in a neighbourhood of the origin of E, but the real aim of the beginning of this section is to prove that these functions are actually C^∞-differentiable. We have essentially followed the proof of [5] and it turns out that the key ingredient in the proof is the converse to Taylor's Theorem. At the same time some bounds on certain norms of the derivatives for functions in $A_\delta(E, F)$ are proved, since they are needed later. The second part of Section 6 is devoted to the so called Omega-lemma, which proves C^∞-differentiability of composition of functions lying in spaces of type $A_\delta(E, F)$. Here we have given an original proof because of the essential gaps found in trying to follow Meyer's proof. It has not been easy to fill these gaps. In the course of the proof we have needed not only the converse of Taylor's Theorem again but also the bounds in norm found in the previous subsection. We have also included a third subsection which deals with the C^∞-differentiability of the so called Evaluation map, because, apart from its intrinsic importance, we have made use of it in the applications given in the following sections. We have supplied a couple of original proofs, one of which has been inspired by [4].

Section 7 is devoted to our main application, which is Poincaré's linearization Theorem on the local behaviour near a fixed point of an analytic map from $\mathbb{R}^n$ or $\mathbb{C}^n$ into itself when all eigenvalues of its linear part have modulus less than one and are different from zero. We begin by establishing and proving a lemma of our own whose role is to make everything clear when it comes to applying the Implicit Function Theorem in later proofs. Following Meyer's paper [5] but trying to make clearer some obscure points in it we have applied the Implicit Function Theorem in order to reduce everything to formal algebraic computations. But in order to satisfy the assumptions required by the Implicit Function Theorem, we have needed to make use of the Omega-lemma in trying to solve a certain functional equation. The interesting fact here is that by using the properties of the Banach spaces $A_\delta(E, F)$ (in the case of $E = F = \mathbb{R}^n$ or $\mathbb{C}^n$) all problems concerning convergence of power series have been solved. We have treated both the real and the complex case, as well as both the so called resonant and non-resonant cases.

Finally in Section 8 we deal with the analytic case of the Stable Manifold Theorem concerning an analytic map with hyperbolic linear part and whose proof in this work is similar to that of Poincaré's Theorem, in the sense that a functional equation is established and then the Omega-lemma is used in order to guarantee that the Implicit Function Theorem can be applied to prove the solubility of this functional equation.

2 Normed spaces

In this section, we introduce a few facts from differential calculus in Banach spaces, either real or complex, since they are the foundation for properly understanding the results of the paper [5]. We first recall some basic facts on normed spaces.

Definition 2.1. A norm on a real or complex vector space E is a map $\| \ \| : E \to \mathbb{R}$ satisfying the following properties:

(i) For all x in E, $\|x\| \geq 0$ and equality holds if and only if $x = 0$,

(ii) $\|\lambda\, x\| = |\lambda| \cdot \|x\|$, for all $\lambda \in \mathbb{R}$ or $\mathbb{C}$ and all $x \in E$,

(iii) triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$.

Definition 2.2. A normed space is a pair $(E, \| \ \|)$ consisting of a (real or complex) vector space E and a norm $\| \ \|$ defined on it.

When there is no danger of confusion, we just speak of a normed space E, with $\| \ \|$ tacitly understood.

A normed space E may always be considered as a metric space with distance defined by $d(x, y) := \|x - y\|$, and moreover, as a topological space through the topology induced by the distance. This topological space is always Hausdorff, by virtue of (i) in Definition 2.1. Actually if we do not assume that $\|x\| = 0$ implies $x = 0$, we speak of a seminorm, a pseudodistance, and the Hausdorff condition fails to hold.

Definition 2.3. Two norms $| \ |$ and $\| \ \|$ on a vector space E are said to be equivalent if they induce the same topology on E.

The following is a well-known fact (cf. [2] I, Prop. 1.6.1, [4] Prop. 2.1.9).

Proposition 2.4. *Two norms $| \ |$ and $\| \ \|$ on a vector space E are equivalent if and only if there exist strictly positive real numbers λ and μ such that, for all x in E,*

$$\lambda\, |x| \leq \|x\| \leq \mu\, |x|.$$

Definition 2.5. A Banach space is a complete normed space, i.e. a normed space in which every Cauchy sequence is convergent.

Proposition 2.6. *If E is a finite dimensional vector space, then there exists at least a norm on E. Furthermore, any two norms in E are equivalent and E is complete with respect to any norm.*

See [4] Prop. 2.1.10.

Remarks 2.7.

(i) Any Cauchy sequence with respect to a norm in a normed space is also Cauchy with respect to any other equivalent norm. The same result obviously holds for convergent sequences.

(ii) If E is a normed space, the sum $(x, y) \mapsto x + y$ and product by scalars $(\lambda, x) \mapsto \lambda\, x$ from $E \times E$ and $\mathbb{R} \times E$ or $\mathbb{C} \times E$ into E are both continuous, so that E may be considered as a topological vector space. In fact, the addition is uniformly continuous on $E \times E$ (see [3] 5.1.5).

Examples 2.8.

(i) In $\mathbb{R}^n$ or $\mathbb{C}^n$ there are very interesting norms, namely

$$x = (x_1, \dots, x_n) \mapsto \|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \quad \text{for}\, p \geq 1$$

(which in case $p = 2$ gives the euclidean norm), and the norm

$$x = (x_1, \dots, x_n) \mapsto \max\,(|x_1|, \dots, |x_n|)$$

which corresponds to the case $p \to \infty$. As stated before, all these, as well as any other norm, are equivalent.

(ii) The sequences (x_n) of real or complex numbers such that $\sum\limits_{n=0}^{\infty} |x_n|^p < \infty$, for $p \geq 1$, constitute a vector space under componentwise addition and

$$\lambda \cdot (x_n) = (y_n), \text{ with } y_n = \lambda x_n \text{ for all } n.$$

Defining $\|(x_n)\|_p := (\sum\limits_{n=0}^{\infty} |x_n|^p)^{\frac{1}{p}}$, we actually get a Banach space, usually denoted by ℓ^p. (When $p = \infty$, ℓ^∞ is the space of all bounded sequences with $\|(x_n)\| = \sup\limits_{n \geq 0} |x_n|$).

(iii) If E and F are normed spaces then $(x, y) \mapsto \|x\|_E + \|y\|_F$ defines a norm on the vector space $E \times F$. The same occurs with $(x, y) \mapsto \max\,(\|x\|_E, \|y\|_F)$.

3 Continuous linear and multilinear maps

We first recall that if $E, F, G, \dots$ are normed vector spaces, the continuous linear maps from E into F form a vector subspace of the vector space of all linear maps from E into F. In fact, the space $L(E;F)$ of continuous linear maps from E to F can be normed by defining for $f \in L(E;F)$ its operator norm $\|f\|$ as the infimum of those real numbers K such that

$$\|f(x)\|_F \leq K\|x\|_E, \qquad \forall x \in E.$$

Actually continuity needs to be checked only at 0 thanks to linearity, and we have (cf. [2] I, Thm. 1.4.1 or [4] Def. 2.2.3)

$$\|f\| = \sup_{\|x\|_E=1} \|f(x)\|_F = \sup_{\|x\|_E\leq 1} \|f(x)\|_F = \sup_{x\neq 0} \tfrac{\|f(x)\|_F}{\|x\|_E},$$

which turns out to be finite if and only if f is continuous and this tells us that f is bounded on the unit ball of E centered at 0, this being the reason why sometimes one speaks of bounded linear maps as in [4] or [5].

Similar considerations and results hold for the case of continuous bilinear maps from $E \times F$ into G: for instance, if f belongs to the vector space $L(E, F; G)$ of continuous bilinear maps from $E \times F$ into G, its operator norm $\|f\|$ is defined as the infimum of those K such that:

$$\|f(x,y)\|_G \leq K\,\|x\|_E\,\|y\|_F, \qquad \forall (x,y) \in E \times F.$$

And we also have (cf. [4] Def. 2.2.8)

$$\|f\| = \sup_{\|x\|_E=1, \|y\|_F=1} \|f(x,y)\|_G = \sup_{\|x\|_E\leq 1, \|y\|_F\leq 1} \|f(x,y)\|_G = \sup_{x\neq 0, y\neq 0} \frac{\|f(x,y)\|_G}{\|x\|_E\|y\|_F}.$$

This is directly generalized to the case of multilinear maps and we get for instance the normed space of continuous k-linear maps $L(E_1, \cdots, E_k; F)$. Concerning notation, if $E_1 = E_2 = \cdots = E_k = E$, this space will simply be denoted by $L^k(E;F)$.

It is remarkable that the map $f \mapsto \widetilde{f}$ from $L(E,F;G)$ into $L(E; L(F;G))$ defined by $\widetilde{f}(x) : y \mapsto f(x,y)$, where $x \in E$ and $y \in F$, turns out to be an isomorphism of vector spaces which preserves norms (see [2] I, Section 1.9), i.e. $\|f\| = \|\widetilde{f}\|$, or said in other words, it is an isometry. This directly generalizes to the case (see [4] Prop. 2.2.9)

$$L(E_1, \dots, E_h, \dots, E_{h+k}; F) \simeq L(E_1, \dots, E_h; L(E_{h+1}, \dots, E_{h+k}; F)).$$

And in particular we have $L^{h+k}(E;F) \simeq L^h(E; L^k(E;F))$.

Another essential fact is that if F is assumed to be a Banach space, then $L(E;F), L(E_1, E_2; F), \dots$ are Banach spaces too (the proofs are straightforward, cf. [3] 5.7.3 and [4] Prop. 2.2.4).

The vector subspace $L^k_s(E;F)$ of $L^k(E;F)$ consisting of the symmetric k-linear maps f from E^k into F, i. e. those continuous k-linear maps f satisfying $f^\sigma = f$, where

$$f^\sigma(x_1, \ldots, x_k) := f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),$$

i.e.

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = f(x_1, \ldots, x_k),$$

for any permutation σ of $\{1, 2, \ldots, k\}$, is easily seen to be closed in $L^k(E; F)$ (see [4] Section 2.2), and consequently is a Banach space in case F is Banach.

So far we have considered the so-called operator norms in the various spaces

$$L(E_1, \ldots, E_k; F)$$

but it turns out that in the case $E = \mathbb{R}^n$ or $\mathbb{C}^n$ there are other norms which are key in our project and which we now describe:

Let us consider the norm $|x| = \max(|x_1|, \ldots, |x_n|)$ for $x = (x_1, \ldots, x_n)$ in E (this norm will be important when dealing with the expansions of analytic functions). Then if $f : E^k \longrightarrow F$ is k-linear, it is well-known that f is determined by the images $f(e_{i_1}, \ldots, e_{i_k})$, where $1 \leq i_1, i_2, \ldots, i_k \leq n$, of all the k-tuples obtained from any basis, say $\{e_1, \ldots, e_n\}$ of E: in fact, if $x_i = \sum\limits_{j=1}^{n} x_{ij} e_j$ for $1 \leq i \leq k$, by multilinearity we have

$$f(x_1, \ldots, x_k) = \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} x_{1i_1} \cdots x_{ki_k}\ f(e_{i_1}, \ldots, e_{i_k}). \tag{1}$$

Then, we associate to f the following norm

$$|f|_k := \sum_{i_1, \ldots, i_k} \|f(e_{i_1}, \ldots, e_{i_k})\|_F.$$

It is easy to see that $f \mapsto |f|_k$ is actually a norm. This norm can be seen as the corresponding one to the 1-norm of $\mathbb{R}^n$ or $\mathbb{C}^n$, i. e. $\|(x_1, \ldots, x_n)\|_1 = \sum\limits_i |x_i|$.

Proposition 3.1. *Let us consider the norm* $|(x_1, \ldots, x_n)|_E = \max(|x_1|, \ldots, |x_n|)$ *in* $E\,(= \mathbb{R}^n$ *or* $\mathbb{C}^n)$*. Then the norm* $|f|_k$ *just defined above is equivalent to the operator norm* $\|f\|$*.*

The case when F is finite-dimensional is automatic: all norms in a finite-dimensional normed space are equivalent as stated in Proposition 2.6, so the assertion becomes interesting when $\dim F = \infty$.

Proof. By (1) we have

$$\begin{aligned}
\|f(x_1,\dots,x_k)\|_F &= \left\| \sum_{i_1,\dots,i_k} f(e_{i_1},\dots,e_{i_k}) x_{1i_1}\cdots x_{ki_k} \right\|_F \\
&\le \sum_{i_1,\dots,i_k} \|f(e_{i_1},\dots,e_{i_k})\|_F \; |x_{1i_1}|\cdots|x_{ki_k}| \\
&\le \left(\sum_{i_1,\dots,i_k} \|f(e_{i_1},\dots,e_{i_k})\|_F \right) \; |x_1|_E \cdots |x_k|_E \\
&= |f|_k \; |x_1|_E \cdots |x_k|_E,
\end{aligned}$$

which proves $\|f\| \le |f|_k$. And from

$$|f|_k = \sum_{i_1,\dots,i_k} \|f(e_{i_1},\dots,e_{i_k})\|_F \le \sum_{i_1,\dots,i_k} \|f\| \cdot |e_{i_1}|_E \cdots |e_{i_k}|_E = \sum_{i_1,\dots,i_k} \|f\| = n^k \cdot \|f\|\,,$$

we finally obtain the equivalence of both norms.

□

Our next considerations will turn out to be important in dealing with the expansions of the analytic maps we will introduce in Section 6. We first begin with the following

Definition 3.2. (see [5]). A family of norms $|\ |_k$ on the vector spaces $L^k(E, F)$, for $k = 1, 2, \dots$, is said to be *consistent* (or simply, the norms $|\ |_k$ are *consistent*), if F is a Banach space and the following four properties hold:

(i) $\{L^k(E, F), |\ |_k\}$ is a Banach space,

(ii) $\|f(x_1,\dots,x_k)\|_F \le |f|_k \cdot \|x_1\|_E \cdots \|x_k\|_E$, for all $x_i \in E$,

(iii) The isomorphism of $L^{h+k}(E; F) \simeq L^h(E; L^k(E; F))$ is norm-preserving, and

(iv) $|f^\sigma|_k = |f|_k$, for any permutation σ of $\{1, 2, \dots, k\}$.

It is well-known that the family of the usual operator norms is consistent. On the other hand, not every norm in $L^k(E, F)$ satisfies (ii): for instance $f \mapsto \frac{1}{2}\|f\|$ where $\|f\|$ stands for the operator norm fails to satisfy (ii). Let us see now that the norms $|\ |_k$ just introduced are consistent.

Proposition 3.3. *The norms $|\ |_k$, for $k = 1, 2, \dots$, are consistent on the spaces $L^k(E; F)$, where $E = \mathbb{R}^n$ or $\mathbb{C}^n$ and F is a Banach space.*

Proof. Obviously (i) holds by Proposition 3.1, since $L^k(E;F)$ is Banach under the operator norm.

Concerning (ii), by multilinearity it suffices to assume $\|x_i\|_E = 1$, for all x_i, and with the notations of (1), we have

$$\begin{aligned}
\|f(x_1,\ldots,x_k)\|_F &= \left\| \sum_{i_1,\ldots,i_k} f(e_{i_1},\ldots,e_{i_k})\ x_{1i_1}\cdots x_{ki_k} \right\|_F \\
&\leq \sum_{i_1,\ldots,i_k} \|f(e_{i_1},\ldots,e_{i_k})\|_F = |f|_k\,.
\end{aligned}$$

For (iii), if $f \in L^{h+k}(E;F)$ and $\widetilde{f}$ is its corresponding element in $L^h(E;L^k(E;F))$, for any $(x_1,\ldots,x_h) \in E^h$, as

$$\widetilde{f}(x_1,\ldots,x_h) = f(x_1,\ldots,x_h,*,\ldots,*) \in L^k(E;F),$$

we have

$$\begin{aligned}
|\widetilde{f}|_h &= \sum_{i_1,\ldots,i_h} \|\widetilde{f}(e_{i_1},\ldots,e_{i_h})\|_{L^k(E;F)} \\
&= \sum_{i_1,\ldots,i_h} \left(\sum_{j_1,\ldots,j_k} \|(\widetilde{f}(e_{i_1},\ldots,e_{i_h}))(e_{j_1},\ldots,e_{j_k})\|_F \right) \\
&= \sum_{i_1,\ldots,i_h} \left(\sum_{j_1,\ldots,j_k} \|f(e_{i_1},\ldots,e_{i_h},e_{j_1},\ldots,e_{j_k}\|_F \right) \\
&= \sum_{i_1,\ldots,i_h,j_1,\ldots,j_k} \|f(e_{i_1},\ldots,e_{i_h},e_{j_1},\ldots,e_{j_k}\|_F = |f|_{h+k}
\end{aligned}$$

which proves (iii), and (iv) is obvious.

□

Next, we recall a few basic facts related to polynomials. First we assume E, F to be vector spaces over any field K of characteristic 0, but later on we will be interested in the case of normed spaces over $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.4. A homogeneous polynomial map of degree k, or just k-homogeneous polynomial from E into F is a map $\varphi : E \to F$ induced by a nonzero k-linear map $f : E^k \to F$ in the following way:

$$\varphi(x) = f(x,\ldots,x), \qquad \forall x \in E.$$

Remarks 3.5.

(i) f can be taken to be symmetric in the preceding definition: in any case,

$$g(x_1, \ldots, x_k) := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

is k-linear, symmetric and $\varphi(x) = g(x, \ldots, x)$. Usually we write $g = \mathrm{Sym}_k(f)$, so that $\mathrm{Sym}_k(f) = \frac{1}{k!} \sum\limits_{\sigma \in \mathcal{S}_k} f^\sigma$, where $f^\sigma(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.

(ii) Furthermore, g is unique, i.e., there exists just one symmetric k-linear function $g : E^k \to F$ satisfying $g(x, \ldots, x) = \varphi(x)$. One possible proof (based on [4] Prop. 2.2.11 (iii)) may be the following: for any k-tuple $(v_1, \ldots, v_k)$ of vectors of E, expanding $\varphi(x)$, with $x = t_1 v_1 + \cdots + t_k v_k$ and $t_1, \ldots, t_k$ being indeterminates, by multilinearity and symmetry of g, we have

$$\begin{aligned} \varphi(x) &= g(x, \ldots, x) = g(t_1 v_1 + \cdots + t_k v_k, \ldots, t_1 v_1 + \cdots + t_k v_k) \\ &= \sum_{\substack{\alpha_1 + \cdots + \alpha_k = k \\ \alpha_i \geq 0}} t_1^{\alpha_1} \cdots t_k^{\alpha_k}\, g(\underbrace{v_1, \ldots, v_1}_{\alpha_1}, \ldots, \underbrace{v_k, \ldots, v_k}_{\alpha_k}) \\ &= \sum_{\substack{\alpha_1 + \cdots + \alpha_k = k \\ \alpha_i \geq 0}} t_1^{\alpha_1} \cdots t_k^{\alpha_k}\, g_{\alpha_1, \ldots, \alpha_k}\,, \end{aligned}$$

a usual homogeneous polynomial (with coefficients in F) of degree k in the indeterminates $t_1, \ldots, t_k$, where we have set $g_{\alpha_1, \ldots, \alpha_k} = g(\underbrace{v_1, \ldots, v_1}_{\alpha_1}, \ldots, \underbrace{v_k, \ldots, v_k}_{\alpha_k})$.

By formally deriving with respect to t_1 we get

$$\frac{\partial}{\partial t_1} \varphi(t_1 v_1 + \cdots + t_k v_k) = \sum_{\substack{\alpha_1 + \cdots + \alpha_k = k \\ \alpha_i \geq 0}} \alpha_1\, t_1^{\alpha_1 - 1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}\, g_{\alpha_1, \ldots, \alpha_k}$$

and going on deriving with respect to $t_2, \ldots, t_k$, we eventually get

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} \varphi(t_1 v_1 + \cdots + t_k v_k) = \sum_{\substack{\alpha_1 + \cdots + \alpha_k = k \\ \alpha_i \geq 0}} \alpha_1 \cdots \alpha_k\, t_1^{\alpha_1 - 1} \cdots t_k^{\alpha_k - 1}\, g_{\alpha_1, \ldots, \alpha_k}\,.$$

As $(\alpha_1 - 1) + \cdots + (\alpha_k - 1) = \alpha_1 + \cdots + \alpha_k - k = 0$, the last sum reduces to just one term, namely $g_{1, \ldots, 1} = g(v_1, \ldots, v_k)$, which actually proves that g is uniquely determined by φ. Another proof of this uniqueness may be found in [2] I Cor. 6.3.3.

(iii) When E is finite-dimensional, taking a basis $e_1, \ldots, e_n$ in E and having a look at (1) with $x = x_1 = \cdots = x_k$, we see that a k-homogeneous polynomial is just a usual homogeneous polynomial of degree k in the n coordinates of x, with coefficients in F.

Definition 3.6. A polynomial (function) from E into F is a function $\varphi : E \to F$ defined by a finite sum of homogeneous polynomials, in which case, we can write $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_k$, for some k, where each φ_j is j-homogeneous. If $\varphi_k \neq 0$ we say φ has degree k, and if $\varphi = 0$ (and here we allow $\varphi_0, \ldots, \varphi_k$ to be zero), we set degree $(\varphi) = -\infty$.

Remarks 3.7.

(i) The expression of φ as a sum of j-homogeneous polynomials for distinct j's is unique, i.e., if $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_k = \psi_0 + \cdots + \psi_k$, where φ_j and ψ_j are assumed j-homogeneous, then $\varphi_j = \psi_j$ for all j. In other words, the summands φ_j are uniquely determined by φ.

A proof of this fact can be given by induction on k: the details can be found in [2] I, Cor. 6.3.2.

(ii) It also arises the question of continuity: the fact is that global continuity comes from continuity at the origin (this is obvious since this is the case in the context of multilinear maps and polynomials are obtained from them). Moreover, if $\dim E < \infty$, as all multilinear maps from E into F are continuous (this follows easily from (1)), we gather that all polynomial functions are continuous. If $\dim E = \infty$, then a polynomial $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_k$ is continuous if and only if each φ_j is continuous and this is so if and only if, for each j, the unique j-linear symmetric map defining φ_j is continuous (see [2] I, Thm. 6.4.1). As we will deal with continuous multilinear maps, all polynomials will, in turn, be continuous.

4 Differentiability in Banach spaces

Here we introduce some basic facts concerning differentiability of functions defined on open sets of a Banach space and taking values in another Banach space. Special attention will be devoted to Taylor's formula and specially to the converse of Taylor's Theorem.

4.1 Basic definitions and Taylor's formula

Definition 4.1. A function $f : U \to F$, where E, F are Banach spaces and U is open in E, is differentiable at $x \in U$ if and only if there exists a continuous linear map, necessarily unique, $Df(x) : E \to F$ such that, using the Landau notation, for $x + h \in U$,

$$\|f(x+h) - f(x) - Df(x)(h)\| = o(\|h\|),$$

in which case we call $Df(x)$ the (first) derivative of f at x. If $Df(x)$ exists for all $x \in U$, f is said to be differentiable on U and $x \mapsto Df(x)$ is a map from U into the Banach space $L(E; F)$. In case this latter map is continuous, we say f is of class C^1 on U and write $f \in C^1(U)$.

If $Df : U \to L(E; F)$ turns out to be also differentiable we get the second derivative of f, denoted by D^2f which is none other than the derivative of Df:

$$D^2f : U \to L(E; L(E; F)).$$

But we have a canonical norm preserving isomorphism $L(E; L(E; F)) \simeq L^2(E; F)$ into the Banach space of continuous bilinear maps from E into F, so that in what follows, we will consider D^2f as a map from U into $L^2(E; F)$.

Now a remarkable fact occurs: If $a \in U$ and $D^2f(a)$ exists, then $D^2f(a)$ is symmetric i.e., $D^2f(a)(u, v) = D^2f(a)(v, u)$, for all $(u, v) \in E^2$ (proof in [2] I, Thm. 5.1.1). We do not need here D^2f to be continuous at a. If $D^2f : U \to L^2_s(E; F)$ turns out to be continuous, f is said to be of class C^2. Similarly, the n-th derivative of f, if it exists is a map $D^nf : U \to L^n_s(E; F)$, etc.

Examples 4.2.

(i) By definition, if $f : U \to F$ is the restriction of a continuous linear map $\widetilde{f} : E \to F$, then $Df : U \to L(E; F)$ is the constant map $Df(x) = \widetilde{f}$, for all $x \in U$, and as the derivative of a constant map is zero, we see that $D^2f = 0$.

(ii) Any continuous bilinear map $f : E \times F \to G$ is differentiable and its derivative at $(a, b) \in E \times F$ is the map $Df(a, b) \in L(E \times F; G)$ defined by

$$Df(a, b)(h, k) = f(a, k) + f(h, b)$$

(see [2] I, Thm. 2.4.3, or [4] Ch.2 Ex 2.3-1). As the maps $h \mapsto f(h, b)$ and $k \mapsto f(a, k)$ are continuous and linear, we get $D^3f = 0$.

(iii) If $f : E_1 \times \cdots \times E_n \to F$ is continuous and n-linear, then f is differentiable and we have

$$\begin{aligned} &Df(a_1, \ldots, a_n)(h_1, \ldots, h_n) \\ &= f(h_1, a_2, \ldots, a_n) + f(a_1, h_2, a_3, \ldots, a_n) + \cdots + f(a_1, \ldots, h_n). \end{aligned}$$

In particular, for the map $\varphi : x \mapsto f(x, \ldots, x)$ from E into F, where f is assumed continuous and n-linear, we have $D\varphi(x) : E \to L(E; F)$ given by

$$D\varphi(x)(h) = f(h, x, \ldots, x) + f(x, h, x, \ldots, x) + \cdots + f(x, x, \ldots, x, h)$$

and in case f is symmetric the above sum is just $n\, f(x, x, \cdots, x, h)$.

Let us turn now to a simple version of Taylor's formula which will turn out to be quite useful in our later applications.

Theorem 4.3. *Let E, F be Banach spaces and U open in E. Assume $f : U \to F$ is $n - 1$ times differentiable on U and that $D^{n-1}f$ is differentiable at $a \in U$. Then whenever $a + h \in U$, we have, using the Landau notation,*

$$f(a + h) = f(a) + \frac{Df(a)}{1!} \cdot h + \frac{D^2 f(a)}{2!} \cdot h^2 + \cdots + \frac{D^n f(a)}{n!} \cdot h^n + o(\|h\|^n).$$

(Here, for instance, $D^n f(a) \cdot h^n$ is the value of $D^n f(a) : E^n \to F$ on $(h, \ldots, h)$, and $D^n f(a) \cdot (h^{n-1}, k) = D^n f(a)(h, \ldots, h, k)$, etc).

Observe that this formula approximates in F the value $f(a+h)$ by a polynomial of degree at most n.

The proof of this theorem is by induction on n (details in [2] Part I Thm. 5.6.3).

4.2 The converse of Taylor's Theorem

Using the preceding notations, we state and prove a converse to Theorem 4.3, namely the so-called converse to Taylor's Theorem (cf. [4]).

Theorem 4.4. *Given $f : U \to F$, assume there exist continuous maps*

$$\varphi_j : U \subset E \to L^j_s(E; F), \qquad j = 1, \ldots, n$$

such that, for any $a \in U$ and any h with $a + h \in U$, we have

$$f(a + h) = f(a) + \frac{\varphi_1(a)}{1!} \cdot h + \frac{\varphi_2(a)}{2!} \cdot h^2 + \cdots + \frac{\varphi_n(a)}{n!} \cdot h^n + o(\|h\|^n).$$

Then f is $\mathcal{C}^n$ in U and $\varphi_j(a) = D^j f(a)$, for all j.

Remark 4.5. The expression on the right hand side before the o-term is a polynomial function of degree $\leq n$, and it is continuous by the assumptions on the φ_j's.

Proof. We follow the lines of ([4] Supplement 2.4.B and [6]) .

We proceed by induction on n, the case $n = 1$ being obvious by the definition of derivative of a map at a point. So assume the theorem holds for $j = 1, \dots, n-1$ and let us prove that it also holds for $j = n$. This entails that we have

$$\varphi_1(a) = Df(a), \dots, \varphi_{n-1}(a) = D^{n-1}f(a)$$

and we contend that $\varphi_n(a) = D^n f(a)$, for every a in U. We will be considering, for any $a \in U$, elements h, k in U such that $a + h$ and $a + h + k$ lie in U and such that $r\|k\| \leq \|h\| \leq s\|k\|$, for some $0 < r < s$. This condition will ensure that $o(\|h\|^n) = o(\|k\|^n) = o(\|h + k\|^n)$, for all $n \geq 1$. Now we write the formula in the theorem in two different ways, namely

$$\begin{aligned} f((a+h)+k) &= f(a+h) + Df(a+h)k + \dots + \frac{1}{(n-1)!}D^{n-1}f(a+h)k^{n-1} \\ &+ \frac{1}{n!}\varphi_n(a+h)k^n + o(\|k\|^n). \end{aligned}$$

$$\begin{aligned} f(a+(h+k)) &= f(a) + Df(a)(h+k) + \dots + \frac{1}{(n-1)!}D^{n-1}f(a)(h+k)^{n-1} \\ &+ \frac{1}{n!}\varphi_n(a)(h+k)^n + o(\|h+k\|^n). \end{aligned}$$

Subtracting these two expansions and collecting terms homogeneous in k^j we get, using symmetry,

$$\begin{aligned} 0 &= f(a+h) - f(a) - Df(a)h - \frac{D^2f(a)}{2!}h^2 - \dots - \frac{D^{n-1}f(a)}{(n-1)!}h^{n-1} - \frac{\varphi_n(a)}{n!}h^n \\ &+ Df(a+h)k - Df(a)k - \frac{D^2f(a)}{2!}2(h,k) - \dots - \frac{D^{n-1}f(a)}{(n-1)!}(n-1)(h^{n-2},k) \\ &- \frac{\varphi_n(a)}{n!}n(h^{n-1},k) + \dots + \frac{D^{n-2}f(a+h)}{(n-2)!}k^{n-2} - \frac{D^{n-2}f(a)}{(n-2)!}k^{n-2} \\ &- \frac{D^{n-1}f(a)}{(n-1)!}(n-1)(h,k^{n-2}) - \frac{\varphi_n(a)}{n!} \cdot \binom{n}{2}(h^2,k^{n-2}) + \frac{D^{n-1}f(a+h)}{(n-1)!}k^{n-1} \\ &- \frac{D^{n-1}f(a)}{(n-1)!}k^{n-1} - \frac{\varphi_n(a)}{n!}n(h,k^{n-1}) + \frac{\varphi_n(a+h)}{n!}k^n - \frac{\varphi_n(a)}{n!}k^n + o(\|k\|^n). \end{aligned}$$

Calling $g_0(h)$ the 1st line above, $g_1(h)k$ the 2nd, $g_2(h)k^2$ the 3rd,…, and finally $g_n(h)k^n$ the $(n+1)$th without the o-term, we see that

$$g_0(h) + g_1(h)k + \dots + g_{n-1}(h)k^{n-1} + g_n(h)k^n = o(\|k\|^n).$$

As $g_j(0) = 0$ and the g_j are continuous for $j = 0, 1, \dots, n$, so that, in particular,

$g_n(h)k^n$ is $o(\|k\|^n)$, then we get

$$g_0(h) + g_1(h)k + \cdots + g_{n-1}(h)k^{n-1} = o(\|k\|^n).$$

We want to prove that each term in the preceding sum is actually of order $o(\|k\|^n)$. This can be achieved by taking distinct numbers $\lambda_1, \dots, \lambda_n$ and replace k by $\lambda_j k$ in the above expression. So we are led to the system of n linear equations with unknowns the vectors $g_0(h), \dots, g_{n-1}(h)k^{n-1}$ on F:

$$\begin{cases} g_0(h) + g_1(h)\lambda_1 k + \cdots + g_{n-1}(h)(\lambda_1 k)^{n-1} = o(\|k\|^n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ g_0(h) + g_1(h)\lambda_n k + \cdots + g_{n-1}(h)(\lambda_n k)^{n-1} = o(\|k\|^n) \end{cases}$$

or written otherwise

$$\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \cdot & \cdots & \cdots & \cdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} g_0(h) \\ g_1(h)k \\ \cdots \\ g_{n-1}(h)k^{n-1} \end{pmatrix} = \begin{pmatrix} o(\|k\|^n) \\ o(\|k\|^n) \\ \cdots \\ o(\|k\|^n) \end{pmatrix}.$$

As the matrix of coefficients of this system of equations is a Vandermonde matrix with nonzero determinant $\prod_{i>j} (\lambda_i - \lambda_j)$, we can multiply on the left by its inverse and get that all vectors $g_0(h), g_1(h)k, \dots, g_{n-1}(h)k^{n-1}$ are of order $o(\|k\|^n)$, since they can be expressed as linear combinations of the independent terms.

We will fix our attention to the last vector, but first observe that the restriction $r\|k\| \leq \|h\| \leq s\|k\|$ does not affect the following results because in dealing with multilinear maps (such as $D^{n-1}f(a)$ and $\varphi_n(a)$), scalars may be placed either inside or outside them without altering values.

Now $g_{n-1}(h)k^{n-1} = o(\|k\|^n) = o(\|h\|^n)$ says that

$$\left\| \frac{D^{n-1}f(a+h)}{(n-1)!}k^{n-1} - \frac{D^{n-1}f(a)}{(n-1)!}k^{n-1} - \frac{\varphi_n(a)}{(n-1)!}(h, k^{n-1}) \right\| = o(\|k\|^n),$$

or equivalently, that

$$\|D^{n-1}f(a+h) - D^{n-1}f(a) - \varphi_n(a)(h, *)\| = o(\|h\|),$$

so that $\varphi_n(a) = D(D^{n-1}f)(a) = D^n f(a)$.

$\square$

5 The Inverse and the Implicit Function Theorem

In this section we recall two of the central theorems of differential calculus in the context of Banach spaces, which are the Inverse Map Theorem and the Implicit Function Theorem, but for the latter we also need to introduce partial derivatives because of their appearance in the Implicit Function Theorem. Partial derivatives also play a relevant role concerning C^r-differentiability.

Definition 5.1. A map $f : U \to V$, where U and V are open sets in the respective Banach spaces E and F, is a C^r diffeomorphism ($r \geq 1$) if f is C^r-differentiable, f is bijective and f^{-1} is also of class C^r.

Now we are going to establish the Inverse Map Theorem:

Theorem 5.2. *With the preceding notations, if $f : U \to V$ is of class C^r, where $r \geq 1$, $a \in U$ and $Df(a) : E \to F$ is a linear isomorphism, then f is a C^r-diffeomorphism of some neighbourhood of a onto some neighbourhood of $f(a)$. Furthermore,*

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1},$$

for y in this neighbourhood of $f(a)$.

Proofs may be found in [2], or [4].

Definition 5.3. Let U be open in $E_1 \times E_2$ and $f : U \to F$. If $(a, b) \in U$ and the derivatives of the maps $x \mapsto f(x, b)$, $y \mapsto f(a, y)$ exist at a and b respectively, for $x \in E_1, y \in E_2$, they are called the partial derivatives of f at $(a, b) \in U$ and will be denoted by $D_1f(a, b) \in L(E_1; F)$ and $D_2f(a, b) \in L(E_2; F)$.

Obviously it may happen that one partial derivative exists but not the other, or that none of them exist and, of course, the preceding definition may be directly generalized to the case of any finite direct product $E_1 \times \cdots \times E_n$ of Banach spaces instead of $E_1 \times E_2$.

In this context we have the following

Proposition 5.4. *With the above notations, if $f : U \to F$ is differentiable at $(a, b) \in U$, then both partial derivatives exist and*

$D_1f(a, b)(v) = Df(a, b)(v, 0),$

$D_2f(a, b)(w) = Df(a, b)(0, w),$

$Df(a, b)(v, w) = D_1f(a, b)(v) + D_2f(a, b)(w).$

Moreover f is of class C^r on U ($r \geq 1$) if and only if both D_1f and D_2f are of class C^{r-1} on U.

Proofs may be found in [2], [3], or [4].

We end this section with one of the most fundamental theorems in Analysis, the Implicit Function Theorem:

Theorem 5.5. *Let $U \subset E$, $V \subset F$ be open in the Banach spaces E and F, and let $f : U \times V \to G$ be a C^r map ($r \geq 1$) into the Banach space G. Assume that, for $(a, b) \in U \times V$, $D_2 f(a, b) : F \to G$ is an isomorphism. Then there exist neighbourhoods U_0 of a and W_0 of $f(a, b)$ and a unique C^r map $g : U_0 \times W_0 \to V$, such that, for all $(x, w) \in U_0 \times W_0$,*

$$f(x, g(x, w)) = w.$$

Proof in [4].

Remarks 5.6.

(i) By a Theorem of Banach (see [4] 2.2.16 or [2] I Section 1.6) if $A : F \to G$ is an algebraic isomorphism which is continuous, then A^{-1} is also continuous, i.e., A is automatically a homeomorphism. As, by definition, $D_2 f(a, b)$ is a continuous map from F into G, $D_2 f(a, b)$ is actually a homeomorphism.

(ii) The Implicit Function Theorem 5.5 is very often used when w is fixed, where we may even assume $w = 0$. In this case we say that in the equation $f(x, y) = 0$, for (x, y) near (a, b), y can be locally solved in the sense that $y = g(x)$, with g as regular as f is.

6 Banach spaces of analytic functions

In this section we first introduce the Banach spaces of analytic functions we will be interested in and then we will focus on the Omega-lemma and the Evaluation map.

6.1 The spaces $A_\delta(E, F)$

Assume that E, F are Banach spaces, both real or both complex, and consider a family of norms $|\ |_n$ on the Banach spaces $L^n(E; F)$ which is consistent in the sense of Definition 3.2.

We now introduce the basic Banach spaces we will be dealing with in what follows, and it will easily be observed that the norms we will define are best adapted for the case of analytic functions in terms of the coefficients occurring in their expansions. For simplicity in what follows we will omit subscripts in the norms.

Thus, take $\delta > 0$ and consider the set $A_\delta(E, F)$ of all formal power series $f = \sum_{k=0}^{\infty} a_k$, with $a_k \in L_s^k(E; F)$, such that $\sum_{k=0}^{\infty} |a_k|\delta^k < \infty$, and define $\|f\|_\delta$ to be the finite value of the sum of the preceding series: $\|f\|_\delta = \sum_{k=0}^{\infty} |a_k|\delta^k$. It turns out that $A_\delta(E, F)$, with addition and product by scalars defined in the usual way, is a vector space, $\|\ \|_\delta$ is a norm in it, and actually $A_\delta(E, F)$ is a Banach space under $\|\ \|_\delta$, since this norm is essentially the ℓ^1 norm. Obviously it contains all polynomial functions (see Definition 3.6).

Observe that if $0 < \rho < \delta$, then

$$A_\delta(E, F) \subseteq A_\rho(E, F), \text{ and } \|f\|_\delta \geq \|f\|_\rho.$$

Let us prove, for the sake of completeness, that $A_\delta(E, F)$ is actually a Banach space.

Proposition 6.1. *$A_\delta(E, F)$ with the norm $\|\cdot\|_\delta$ is a Banach space.*

Proof. We have just mentioned that $A_\delta(E, F)$ is a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$) and now we have to prove that it is complete, and for this, the proof that ℓ^1 is Banach may easily be adapted here. It runs as follows:

Let $f_n := \sum_{k=0}^{\infty} a_k^{(n)}, n = 1, 2, \ldots,$ be a Cauchy sequence of elements in $A_\delta(E, F)$. This means that, for each n, $\sum_{k=0}^{\infty} |a_k^{(n)}|\delta^k < \infty$ and that for any given $\varepsilon > 0$ there exists an n_0, depending on ε, such that

$$\|f_p - f_q\|_\delta = \sum_{k=0}^{\infty} |a_k^{(p)} - a_k^{(q)}|\delta^k < \varepsilon, \text{ whenever } p, q \geq n_0.$$

But then, for any fixed k, $|a_k^{(p)} - a_k^{(q)}|\delta^k < \varepsilon$, which means that the sequence $\left(a_k^{(n)}\right)_{n\in\mathbb{N}}$ is Cauchy in $L_s^k(E,F)$ and therefore convergent, say to $a_k \in L_s^k(E,F)$, since $L_s^k(E,F)$ is Banach (being closed in $L^k(E,F)$ which is Banach because F is so). Now there remains to see that $f = \sum\limits_{k=0}^{\infty} a_k$ lies in $A_\delta(E,F)$ and is in fact the limit of f_n.

Observe that, for each $t \in \mathbb{N}$, if $p, q \geq n_0$,

$$\sum_{k=0}^{t} \left|a_k^{(p)} - a_k^{(q)}\right| \delta^k \leq \sum_{k=0}^{\infty} \left|a_k^{(p)} - a_k^{(q)}\right| \delta^k < \varepsilon$$

so that

$$\lim_{p\to\infty} \sum_{k=0}^{t} \left|a_k^{(p)} - a_k^{(q)}\right| \delta^k = \sum_{k=0}^{t} \left|a_k - a_k^{(q)}\right| \delta^k \leq \varepsilon.$$

As this holds, for any t, letting $t \to \infty$, we get

$$\sum_{k=0}^{\infty} \left|a_k - a_k^{(q)}\right| \delta^k \leq \varepsilon.$$

This tells us that $f - f_q \in A_\delta(E,F)$, and as $f_q \in A_\delta(E,F)$, we conclude that

$$f = (f - f_q) + f_q \in A_\delta(E,F).$$

Furthermore, the last inequality shows that $\lim\limits_{q\to\infty} f_q = f$.

$\square$

Now to any f in $A_\delta(E,F)$, with our preceding notation, we associate the map

$$\widetilde{f} : \overline{B(0,\delta)} \subset E \to F$$

from the closed ball $\overline{B(0,\delta)} = \{x \in E : |x|_E \leq \delta\}$ of E into F, defined by

$$\widetilde{f}(x) = \sum_{k=0}^{\infty} a_k(x^k),$$

where $x^k = (x, \ldots, x)$, for each k, and $x \in \overline{B(0,\delta)}$, and we will identify in the sequel $\widetilde{f}$ with its power series representation f.

This is due to the fact that $f \mapsto \widetilde{f}$ is injective, i.e. that if $\sum\limits_k a_k(x^k) = 0$ whenever $|x| \leq \delta$, then all $a_k = 0$. Otherwise take the first nonzero a_k and an $x \in E$ such that $|x| \leq \delta$ and $a_k(x^k) \neq 0$. By homogeneity we have for $|\lambda| < 1$

$$\begin{aligned} 0 &= a_k((\lambda x)^k) + a_{k+1}((\lambda x)^{k+1}) + a_{k+2}((\lambda x)^{k+2}) + \cdots \\ &= \lambda^k a_k(x^k) + \lambda^{k+1}(a_{k+1}(x^{k+1}) + \lambda(a_{k+2}(x^{k+2}) + \cdots). \end{aligned}$$

But the series in the last term is absolutely convergent for $|\lambda| < 1$, and dividing through by λ^k and letting $\lambda \to 0$ we get $a_k(x^k) = 0$, a contradiction. (Actually this generalizes Remark 3.7 (i)).

By virtue of the Weierstrass M-test, f is absolutely and uniformly convergent for $|x| \le \delta$, so that f is continuous on $\overline{B(0,\delta)}$. Furthermore, as the series defining $\|f\|_\delta$ consists of nonnegative terms, we see that each term satisfies $|a_k|\delta^k \le \|f\|_\delta$ or, equivalently, $|a_k| \le \frac{\|f\|_\delta}{\delta^k}$ (a Cauchy-type inequality), and also, obviously, $|f(x)|_F \le \|f\|_\delta$ whenever $|x|_E \le \delta$.

Now, one of our major goals here will be to show that f is not only continuous but also C^∞ on the open ball $B(0,\delta) = \{x \in E : |x|_E < \delta\}$, as is the case of the usual analytic functions of one or several variables. As is well-known these latter functions are always C^∞, but not conversely, and their derivatives are easily recognized in the coefficients of their expansions. But first we make a couple of observations, the first of which pays attention to $A_\delta(E,F)$ in case E is finite-dimensional, say $E = \mathbb{R}^n$ or $\mathbb{C}^n$:

We take a basis $e_1, \dots, e_n$ in E with $|e_i|_E = 1$, for $i = 1, \dots, n$. Then any $a_k \in L^k_s(E,F)$ may be described as follows:

If $x = \sum\limits_{j=1}^{n} x_j e_j$, then as seen in (1) we have

$$a_k(x^k) = \sum_{i_1,\dots,i_k=1}^{n} a_k(e_{i_1}, \dots, e_{i_k})\, x_{i_1} \cdots x_{i_k}$$

But using the fact that a_k is symmetric we can gather those terms in which e_1 occurs α_1 times, e_2 occurs α_2 times,..., and e_n, α_n times, (with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$) and get

$$a_k(x^k) = \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_n=k \\ \alpha_i \ge 0}} \frac{k!}{\alpha_1! \cdots \alpha_n!}\, a_k(\underbrace{e_1, \dots, e_1}_{\alpha_1}, \dots, \underbrace{e_n, \dots, e_n}_{\alpha_n})\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Here we recover the usual homogeneous polynomial expansion for a_k in the coordinates $x_1, \dots, x_n$ of x with coefficients in F and see that the norm $|a_k|$ appearing in Proposition 3.1 is just the sum of the norms (in F) of the coefficients associated with a_k. When $F = \mathbb{C}^n$, for instance, the coefficients are indeed vectors in $\mathbb{C}^n$, so that in this case $f \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ is given by a usual convergent power series in $x_1, \dots, x_n$ with coefficients in $\mathbb{C}^n$. A similar result obviously holds for $A_\delta(\mathbb{R}^n, \mathbb{R}^n)$ when $F = \mathbb{R}^n$.

The second observation is contained in the following proposition and shows that the familiar analytic functions are in the spaces just introduced when $E = \mathbb{R}^n$ or $\mathbb{C}^n$.

Proposition 6.2. *If $f : \{x \in \mathbb{C}^n : |x_i| < r\} \to F$ is analytic and bounded in norm by M, then for each $\eta \in (0,r)$, f lies in $A_\eta(\mathbb{C}^n, F)$ and $\|f\|_\eta \le M(1 - \frac{\eta}{r})^{-n}$.*

Proof. Consider the power series expansion for f when $|x_i| < r$:

$$f(x_1, \ldots, x_n) = \sum_{k_i \geq 0} a_{k_1 \cdots k_n} \, x_1^{k_1} \cdots x_n^{k_n}$$

By Cauchy's inequalities, $|a_{k_1 \cdots k_n}| \leq M r^{-(k_1 + \cdots + k_n)}$ and consequently the series

$$\|f\|_\eta = \sum_{k_i > 0} |a_{k_1 \cdots k_n}| \, \eta^{k_1 + \cdots + k_n}$$

is majorized term by term by the series

$$\sum_{k_i \geq 0} M \left(\frac{\eta}{r}\right)^{k_1 + \cdots + k_n} = M \left(\sum_{k_1 \geq 0} \left(\frac{\eta}{r}\right)^{k_1} \right) \cdots \left(\sum_{k_n \geq 0} \left(\frac{\eta}{r}\right)^{k_n} \right) =$$

$$= M \left(1 - \frac{\eta}{r}\right)^{-1} \cdots \left(1 - \frac{\eta}{r}\right)^{-1} = M \left(1 - \frac{\eta}{r}\right)^{-n}.$$

□

Remark 6.3. This proof works when the spaces $L_s^k(\mathbb{C}^n; F)$ are normed as in Proposition 3.1, but as the operator norms are dominated by them the assertion also holds in the usual situation.

Next, we give a technical lemma:

Lemma 6.4. *Let $a_k \geq 0, k = 0, 1, 2, \ldots,$ and $\delta > 0$ be such that $\sum\limits_{k=0}^{\infty} a_k \delta^k = M < \infty$. Then for any positive integer k, and ρ such that $0 < \rho < \delta$, we have*

$$\sum_{j=k}^{\infty} \frac{j!}{(j-k)!} \, a_j \rho^{j-k} \leq \frac{k! M}{(\delta - \rho)^k}.$$

Proof. The complex-valued function $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ is analytic in the disc $|z| < \delta$ and bounded by M.

Consider its expansion $g(z) = \sum\limits_{k=0}^{\infty} b_k (z - \rho)^k$ around ρ. This power series has radius of convergence $\geq \delta - \rho$ and $b_k = \frac{g^{(k)}(\rho)}{k!}$. Since M obviously bounds $|g(z)|$ on the ball centered at ρ of radius $\delta - \rho$, the Cauchy's inequalities entail $|b_k| \leq \frac{M}{(\delta - \rho)^k}$, i.e.,

$$|g^{(k)}(\rho)| \leq \frac{k! M}{(\delta - \rho)^k}.$$

But from $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ we see that

$$g^{(k)}(z) = \sum_{j=k}^{\infty} \frac{j!}{(j-k)!} \, a_j z^{j-k}.$$

Setting $z = \rho$, we get the result.

$\square$

We now come to a very important fact concerning differentiability:

Theorem 6.5. *If $f \in A_\delta(E, F)$ then, for all $i \geq 1$, there exists $D^i f$. Moreover, for $0 < \rho < \delta$ it belongs to $A_\rho(E, L_s^i(E; F))$, and*

$$\|D^i f\|_\rho \leq \frac{i!\|f\|_\delta}{(\delta - \rho)^i}.$$

In particular f is C^∞ in the disc $B(0, \delta) = \{x \in E : |x|_E < \delta\}$.

Proof. Let $f(x) = \sum_{k=0}^{\infty} a_k(x^k)$, as in our previous notations. We have

$$\infty > \|f\|_\delta = \sum_{k=0}^{\infty} |a_k|\delta^k = \sum_{k=0}^{\infty} |a_k|(\rho + (\delta - \rho))^k = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{k}{i} |a_k|\rho^{k-i}(\delta - \rho)^i$$

$$= \sum_{i=0}^{\infty}\left(\sum_{k=i}^{\infty} \binom{k}{i} |a_k|\rho^{k-i}\right)(\delta - \rho)^i, \tag{2}$$

where we have rearranged terms, since they are all positive. Let $|x| < \rho$ and $|y| < \frac{1}{2}(\delta - \rho)$. Then

$$f(x + y) = \sum_{k=0}^{\infty} a_k((x + y)^k) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \binom{k}{i} a_k(x^{k-i}, y^i)\right)$$

$$= \sum_{i=0}^{\infty}\left(\sum_{k=i}^{\infty} \binom{k}{i} a_k(x^{k-i}, *)\right)(y^i) \tag{3}$$

the last equality holding because the norms are assumed to be consistent and the last two series in (2) majorize (term by term) the last two series in (3), which implies absolute convergence.

Formula (3) suggests that the ith derivative of f at x should be

$$D^i f(x) = i!\sum_{k=i}^{\infty} \binom{k}{i} a_k(x^{k-i}, *) \tag{4}$$

and the next aim is to prove this. Let us denote the right-hand side of (4) by $\varphi_i(x)$. We then have by Lemma 6.4

$$\|\varphi_i\|_\rho = i!\sum_{k=i}^{\infty} \binom{k}{i} |a_k|\rho^{k-i} \leq i!\frac{\|f\|_\delta}{(\delta - \rho)^i} \tag{5}$$

so that $\varphi_i \in A_\rho(E, L_s^i(E; F))$.

Next, we go on with (3) using the notation of (4):

$$f(x+y) = \sum_{i=0}^{\infty} \frac{\varphi_i(x)}{i!}(y^i) = \sum_{i=0}^{n} \frac{\varphi_i(x)}{i!}(y^i) + \sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^i) \tag{6}$$

Now we estimate the last term in (6) assuming $|y| < \frac{\delta-\rho}{2}$ and bearing in mind (5) we have:

$$\begin{aligned}
\left| \sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^i) \right| &= \left| \left(\sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^{i-n-1}, *) \right)(y^{n+1}) \right| \\
&\leq \left(\sum_{i=n+1}^{\infty} \frac{\|f\|_\delta}{(\delta-\rho)^i} \left(\frac{\delta-\rho}{2} \right)^{i-n-1} \right) \cdot \|y\|^{n+1} \\
&= \frac{\|f\|_\delta}{(\delta-\rho)^{n+1}} \sum_{t=0}^{\infty} \frac{1}{2^t} \|y\|^{n+1} = \frac{2\|f\|_\delta}{(\delta-\rho)^{n+1}} \cdot \|y\|^{n+1} = o(\|y\|^n).
\end{aligned}$$

We can now apply the converse of Taylor's Theorem 4.4 and conclude that $D^i f = \varphi_i$, which lies in $A_\rho(E, L_s^i(E;F))$ and the bound for $\|D^i f\|_\rho$ is given in (5).

□

6.2 The Omega-lemma

Our next goal is to prove the so-called Omega-lemma in the spaces $A_\delta(E,F)$ which deals with composition of functions. The techniques involved follow the same pattern as that of substitution of convergent power series into convergent power series (cf. [3] Ch. IX.2 and 5.5.3). So assume $f \in A_\delta(E,F)$ and $g \in A_\eta(D,E)$ with $\|g\|_\eta \leq \delta$. Then for each $x \in D$ such that $|x| \leq \eta$, we know that $|g(x)| \leq \|g\|_\eta \leq \delta$ and thus $f(g(x))$ makes sense, and even we have $|f(g(x))| \leq \|f\|_\delta$. This suggests that $\|f \circ g\|_\eta \leq \|f\|_\delta$. But in order to establish this last inequality we need to expand $f \circ g$ as a power series of continuous symmetric multilinear maps. All this requires some explanations.

Assume $a_k \in L^k(E;F)$ and $b_j \in L^j(D;E)$. Then

$$a_k(b_{j_1}(*), \ldots, (b_{j_k}(*)) \in L^{j_1+\cdots+j_k}(D;F).$$

Indeed, if $\ell = j_1 + \cdots + j_k$, the map

$$(x_1, \ldots, x_\ell) \mapsto a_k(b_{j_1}(x_1, \ldots, x_{j_1}), \ldots, b_{j_k}(x_{\ell-j_k+1}, \ldots, x_\ell))$$

from D^ℓ into F is obviously continuous and multilinear because a_k and the b_j's are so. Observe that even in case a_k and the b_j's are symmetric, it is not clear whether the preceding map is symmetric.

Now assume $f = \sum_{k=0}^{\infty} a_k \in A_\delta(E,F)$ and $g = \sum_{j=0}^{\infty} b_j \in A_\eta(D,E)$ (with $\|g\|_\eta = \sum_j |b_j|\eta^j \le \delta$). We have, for $x \in D, |x| \le \eta$,

$$\begin{aligned}(f \circ g)(x) &= \sum_{k=0}^{\infty} a_k \left(\sum_{j_1=0}^{\infty} b_{j_1}(x^{j_1}), \dots, \sum_{j_k=0}^{\infty} b_{j_k}(x^{j_k}) \right) \\ &= \sum_{k=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} a_k \left(b_{j_1}(x^{j_1}), \dots, b_{j_k}(x^{j_k})\right),\end{aligned}$$

the last equality due to continuity and multilinearity. But by absolute convergence this can be rewritten grouping first those terms sharing a common sum $j_1+\cdots+j_k$, say equal to ℓ. But before proceeding further and as remarked above observe that $a_k\left(b_{j_1}(*), \dots, b_{j_k}(*)\right)$ generally fails to be symmetric. For instance, we can easily check this taking $k = 2, \ell = 3$, paying attention to

$$a_2\left(b_1(x_1), b_2(x_2, x_3)\right) + a_2\left(b_2(x_1, x_2) + b_1(x_3)\right)$$

and comparing this to the corresponding expression with x_1 and x_2 interchanged.

If we momentarily denote $a_k\left(b_{j_1}(*), \dots, b_{j_k}(*)\right)$ by $f = f(x_1, \dots, x_\ell)$, and f^σ stands for the map

$$(x_1, \dots, x_\ell) \mapsto f(x_{\sigma(1)}, \dots, x_{\sigma(\ell)}),$$

for any permutation σ of $\{1, 2, \dots, \ell\}$, then the symmetrized map

$$\mathrm{Sym}_\ell f = \frac{1}{\ell!} \sum_\sigma f^\sigma$$

obtained from f (cf. Remark 3.5 (i)) satisfies

$$|\mathrm{Sym}_\ell f| = \left| \frac{1}{\ell!} \sum_\sigma f^\sigma \right| = \frac{1}{\ell!} \left| \sum_\sigma f^\sigma \right| \le \frac{1}{\ell!} \sum_\sigma |f^\sigma|,$$

and recalling Definition 3.2 (iv), we get

$$|\mathrm{Sym}_\ell f| \le \frac{1}{\ell!} \sum_\sigma |f| = |f|.$$

Of course when $x_1 = \cdots = x_\ell = x$, we have $(\mathrm{Sym}_\ell f)(x, \dots, x) = f(x, \dots, x)$ so that the difference between f and $\mathrm{Sym}_\ell f$ becomes clear when $x_1, \dots, x_\ell$ are unequal.

Coming back to our previous discussion we realize that we can write

$$f \circ g = \sum_{\ell=0}^{\infty} \gamma_\ell,$$

where

$$\gamma_\ell = \sum_{k=0}^{\infty} \sum_{j_1+\cdots+j_k=\ell} \mathrm{Sym}_\ell\, a_k\left(b_{j_1}(*), \dots, b_{j_k}(*)\right)$$

is symmetric ℓ-linear and continuous. Note that when $\ell > 0$ the sum $\sum\limits_{j_1+\cdots+j_k=\ell}$ is void for $k = 0$.

As

$$
\begin{aligned}
|\mathrm{Sym}_\ell\, a_k\left(b_{j_1}(x^{j_1}), \ldots, b_{j_k}(x^{j_k}\right)| &= |a_k\left(b_{j_1}(x^{j_1}), \ldots, b_{j_k}(x^{j_k})\right)| \\
&\leq |a_k| \cdot |b_{j_1}(x^{j_1})| \cdots |b_{j_k}(x^{j_k})| \\
&\leq |a_k| \cdot |b_{j_1}| \cdot |x|^{j_1} \cdots |b_{j_k}| \cdot |x|^{j_k} \\
&= |a_k| \cdot |b_{j_1}| \cdots |b_{j_k}| \cdot |x|^\ell
\end{aligned}
$$

we see that

$$
\begin{aligned}
\|f \circ g\|_\eta &= \sum_{\ell=0}^{\infty} |\gamma_\ell| \eta^\ell \leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\substack{j_1+\cdots+j_k=\ell \\ j_i \geq 0}} |a_k||b_{j_1}| \cdots |b_{j_k}| \eta^\ell \\
&= \sum_{\ell=0}^{\infty} \left(\sum_{k=0}^{\infty} |a_k| \sum_{\substack{j_1+\cdots+j_k=\ell \\ j_i \geq 0}} |b_{j_1}| \cdots |b_{j_k}| \right) \eta^\ell \\
&= \sum_{k=0}^{\infty} |a_k| \left(\sum_{\ell=0}^{\infty} \sum_{\substack{j_1+\cdots+j_k=\ell \\ j_i \geq 0}} |b_{j_1}| \cdots |b_{j_k}| \eta^\ell \right) \\
&= \sum_{k=0}^{\infty} |a_k| \left(\sum_{j=0}^{\infty} |b_j| \eta^j \right)^k = \sum_{k=0}^{\infty} |a_k| \cdot \|g\|_\eta^k \leq \sum_{k=0}^{\infty} |a_k|\, \delta^k = \|f\|_\delta.
\end{aligned}
$$

What we have just proved may be summarized in the following

Lemma 6.6. *If $g \in A_\eta(D, E)$ with $\|g\|_\eta \leq \delta$ and $f \in A_\delta(E, F)$, then*

$$f \circ g \in A_\eta(D, F), \text{ and } \|f \circ g\|_\eta \leq \|f\|_\delta.$$

Next we prove another lemma which will be needed in the subsequent propositions.

Lemma 6.7. *Let $f \in A_\delta(E, F)$ and $g, h \in A_\eta(D, E)$ such that $\|g\|_\eta = \alpha < \delta$ and $\|h\|_\eta \leq \beta := \frac{1}{3}(\delta - \alpha)$. Then*

$$\left\| \frac{D^k f(g(*))}{k!}(h(*)^k) \right\|_\eta \leq \left\| \frac{D^k f(*)}{k!} \right\|_\alpha \cdot \|h\|_\eta^k.$$

Proof. Write $f = \sum_{k=0}^{\infty} a_k$, $g = \sum_{\ell=0}^{\infty} b_\ell$ and $h = \sum_{\ell=0}^{\infty} h_\ell$, with $a_k \in L_s^k(E;F)$, and $b_\ell, h_\ell \in L_s^\ell(D;E)$. Then, as seen in the proof of Theorem 6.5,

$$\frac{D^k f(g(x))}{k!} = \sum_{j\geq k} \binom{j}{k} a_j(g(x)^{j-k}, *)$$

and consequently,

$$\frac{D^k f(g(x))}{k!}(h(x)^k) = \sum_{j\geq k} \binom{j}{k} a_j(g(x)^{j-k}, h(x)^k).$$

Now we proceed to expand this last sum as in the arguments proving Lemma 6.6 and have that $\frac{D^k f(g(x))}{k!}(h(x)^k)$ is equal to

$$\sum_{j\geq k} \binom{j}{k} \sum_{\ell_1,\ldots,\ell_j\geq 0} a_j(b_{\ell_1}(*),\ldots,b_{\ell_{j-k}}(*), h_{\ell_{j-k+1}}(*),\ldots,h_{\ell_j}(*))x^{\ell_1+\cdots+\ell_j}$$

from which, if $\iota = \ell_1 + \cdots + \ell_{j-k} + \ell_{j-k+1} + \cdots + \ell_j$, we see that

$$\begin{aligned}
\left\| \frac{D^k f(g(*))}{k!}(h(*)^k) \right\|_\eta &\leq \sum_{j\geq k} \binom{j}{k} \sum_{\ell_1,\ldots,\ell_j\geq 0} |a_j|\cdot|b_{\ell_1}|\cdots|b_{\ell_{j-k}}|\cdot|h_{\ell_{j-k+1}}|\cdots|h_{\ell_j}|\eta^\iota \\
&= \sum_{j\geq k} \binom{j}{k} |a_j| \cdot \left(\sum_{\ell\geq 0} |b_\ell|\eta^\ell\right)^{j-k} \cdot \left(\sum_{\ell\geq 0} |h_\ell|\eta^\ell\right)^k \\
&= \sum_{j\geq k} \binom{j}{k} |a_j| \cdot \|g\|_\eta^{j-k} \cdot \|h\|_\eta^k \\
&= \sum_{j\geq k} \binom{j}{k} |a_j|\alpha^{j-k} \cdot \|h\|_\eta^k = \left\| \frac{D^k f}{k!} \right\|_\alpha \cdot \|h\|_\eta^k
\end{aligned}$$

the last equality by virtue of formula (5) of Theorem 6.5.

□

Our final aim in this subsection, as mentioned earlier, is to prove the Ω-lemma, i.e. differentiability of composition. To begin with, let us consider continuity.

Proposition 6.8. *The map $\Omega : (f,g) \mapsto f\circ g$ from $A_\delta(E,F)\times U$, where $U = \{g \in A_\eta(D,E) : \|g\|_\eta < \delta\}$, into $A_\eta(D,F)$ is continuous.*

Proof. We have by Lemma 6.6

$$\|(f+f_1)\circ g - f\circ g\|_\eta = \|f_1\circ g\|_\eta \leq \|f_1\|_\delta\,,$$

and this entails Ω is uniformly continuous in the first variable and independently of the second argument g. Now we examine what happens with the second variable.

Assume as in Lemma 6.7 that $\|g\|_\eta = \alpha < \delta$ and set $\beta := \frac{1}{3}(\delta - \alpha)$, so that $\alpha + \beta = \delta - 2\beta < \delta$. By Theorem 6.5 we know that, for $f \in A_\delta(E, F)$, its kth derivative $D^k f \in A_{\delta-2\beta}(E, L_s^k(E; F))$ and

$$\|D^k f\|_{\delta-2\beta} \leq k! \cdot \|f\|_\delta \cdot (2\beta)^{-k}.$$

Take now $h \in U$ such that $\|h\|_\eta < \beta$, which entails

$$\|g + h\|_\eta \leq \|g\|_\eta + \|h\|_\eta < \alpha + \beta < \delta.$$

From

$$f \circ (g + h)(x) - f \circ g(x) = f(g(x) + h(x)) - f(g(x)) = \sum_{k=1}^{\infty} \frac{D^k f(g(x))}{k!} (h(x)^k),$$

we have,

$$\begin{aligned}
\left\| \sum_{k=1}^{\infty} \frac{D^k f(g(*))}{k!} (h(*)^k) \right\|_\eta &\leq \sum_{k=1}^{\infty} \left\| \frac{D^k f}{k!} \right\|_\alpha \cdot \|h\|_\eta^k \\
&\leq \sum_{k=1}^{\infty} \left\| \frac{D^k f}{k!} \right\|_{\alpha+\beta} \cdot \|h\|_\eta^k \leq \sum_{k=1}^{\infty} \frac{\|f\|_\delta}{(2\beta)^k} \cdot \|h\|_\eta^k \\
&\leq \sum_{k=1}^{\infty} \frac{\|f\|_\delta}{(2\beta)^k} \cdot \beta^{k-1} \cdot \|h\|_\eta = \frac{\|f\|_\delta}{\beta} \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \|h\|_\eta \\
&= \frac{\|f\|_\delta}{\beta} \cdot \|h\|_\eta \,,
\end{aligned}$$

where the first inequality is due to Lemma 6.7, the second holds because $\alpha < \alpha + \beta = \delta - 2\beta < \delta$, the third by Theorem 6.5, and the fourth is obvious. This implies continuity with respect to the second argument. Now continuity with respect to both variables follows from

$$\begin{aligned}
|\Omega(f', g') - \Omega(f, g)| &= |\Omega(f', g') - \Omega(f, g') + \Omega(f, g') - \Omega(f, g)| \\
&\leq |\Omega(f', g') - \Omega(f, g')| + |\Omega(f, g') - \Omega(f, g)|.
\end{aligned}$$

□

Theorem 6.9. *The map Ω of the previous proposition is C^∞.*

Proof. Keeping our previous notations, we first show that Ω has continuous partial derivatives with respect to its second argument. In particular, we assume that $\|h\|_\eta$ is small enough. We have

$$\begin{aligned} f\circ(g+h)(x) &= f(g(x)+h(x)) = f(g(x)) + \sum_{k=1}^{n} \frac{D^k f(g(x))}{k!}\,(h(x)^k) \\ &\quad + \sum_{k=n+1}^{\infty} \frac{D^k f(g(x))}{k!}\,(h(x)^k) \end{aligned}$$

and the $\|\ \|_\eta$-norm of the last term, following the lines of the second part of the proof of Proposition 6.8, is bounded by

$$\frac{\|f\|_\delta}{2^n\beta^{n+1}}\cdot\|h\|_\eta^{n+1} = o(\|h\|_\eta^n).$$

As $(D^k f)\circ g$ is continuous by Lemma 6.6, we can apply the converse of Taylor's Theorem 4.4 and thus $D_2^k\Omega$, for $0\le k\le n$, does exist and equals $D^k f\circ g$. Now, since $D^k f\in A_\rho(E;L_s^k(E;F))$, for all $\rho<\delta$, by Theorem 6.5, and composition is continuous by the previous proposition, $D_2^k\Omega$ is continuous in both arguments.

With respect to the first argument, we observe that $(f,g)\mapsto\Omega(f,g)=f\circ g$ is linear (and continuous) in f, so that

$$D_1\Omega(f,g)=\Omega(*,g)$$

(cf. Example 4.2 (i)), which is continuous by Proposition 6.8. But then

$$D_1^2\,\Omega = D_1^3\,\Omega = \cdots = 0.$$

Let us check this for the most interesting case, namely that of D_1^2: As

$$D_1\Omega(f+h,g) - D_1\Omega(f,g) = \Omega(*,g)-\Omega(*,g)=0,$$

the equality holding because $D_1\Omega(f,g)=\Omega(*,g)$ as just seen above, we get that $D_1^2\,\Omega=0$ (cf. Example 4.2 (i) again).

Concerning mixed partial derivatives, as, for $k\ge1$, $D_2^k\Omega(f,g)=D^k f\circ g$, we see that $(f,g)\mapsto D_2^k\Omega(f,g)$ is continuous and linear in f, as in the case of Ω, and therefore the same reasoning as before can be applied, so that

$$D_1D_2^k\Omega(f,g)=D_2^k\Omega(*,g),\ \text{and}\ D_1^jD_2^k\Omega(f,g)=0,\ \text{for}\ j\ge2.$$

Thus Ω has continuous partial derivatives of all orders, i.e. Ω is C^∞ (cf. Proposition 5.4).

□

6.3 The Evaluation map

Another important theorem of global analysis is related to the Evaluation map. It will be useful in considering some closed subspaces of the Banach spaces $A_\delta(E,F)$, as we will see in the next two sections.

Theorem 6.10. *The Evaluation map*

$$Ev : A_\delta(E, F) \times \{x \in E : |x| < \delta\} \to F$$

defined by $Ev(f, x) = f(x)$ *is* C^∞*, and we have*

$$D_2^k Ev(f, x)(y^k) = D^k f(x)(y^k).$$

Proof. It follows the lines of the preceding proof since obviously Ev is linear in its first variable f and is continuous. In fact, continuity is proved as in Proposition 6.8: Ev is uniformly continuous with respect to f and independently of x because

$$|Ev(f + f_1, x) - Ev(f, x)|_F = |f_1(x)|_F \leq \|f_1\|_\delta$$

and Ev is continuous with respect to x because, as said at the beginning of this section, any f in $A_\delta(E, F)$ yields a continuous function on $\{x \in E : |x| \leq \delta\}$. From this, we see, as in the preceding proof, that

$$D_1 Ev(f, x) = Ev(*, x), \text{ and } D_1^2 Ev = D_1^3 Ev = \cdots = 0.$$

If we now fix the first variable f and let vary the second, i.e. x, we are just considering the function $x \mapsto f(x)$ which according to Theorem 6.5 is C^∞. This gives the result

$$D_2^k Ev(f, x) = D^k f(x),$$

as stated in the theorem. The mixed partial derivatives are dealt with in the same way as in the preceding proof, since $D_2^k Ev(f, x)$ is obviously linear in f, and continuous because of Theorem 6.5, from which follows the C^∞-differentiability of Ev. □

As a matter of fact and using the converse of Taylor's Theorem 4.4 again we end this section with an explicit expression of the kth derivative of the Evaluation map. We have

Theorem 6.11. *The* k*th derivative* $D^k Ev$ *of the Evaluation map of Theorem 6.10 at* (f, x) *is the continuous* k*-linear map from* $(A_\delta(E, F) \times E)^k$ *into* F *given by*

$$\begin{aligned} D^k Ev(f, x)((g_1, y_1), \ldots, (g_k, y_k)) &= D^k f(x)(y_1, \ldots, y_k) \\ &\quad + \sum_{i=1}^{k} D^{k-1} g_i(x)(y_1, \ldots, \widehat{y_i}, \ldots, y_k). \end{aligned}$$

Proof. Let us denote by $\varphi_k(f, x)$ the map from $(A_\delta(E, F) \times E)^k$ into F given by the right hand side of the equality in the statement of the Theorem 6.11. Obviously $\varphi_k(f, x)$ is symmetric in $(g_1, y_1), \ldots, (g_k, y_k)$, and $\varphi_k(f, x)$ is k-linear and continuous because $D^k f(x)$ and $D^{k-1} g_i(x)$ are both multilinear and continuous. Moreover, the map $(f, x) \mapsto \varphi_k(f, x)$ is continuous: as f is continuous in $\{x \in E : |x| < \delta\}$, it turns out that φ_k is continuous in the second variable, and with respect to the first, it is uniformly continuous by virtue of the bounds given in Theorem 6.5. Hence, φ_k is continuous as it is indicated in the proof of Proposition 6.8. Now we are in a situation where the converse of Taylor's Theorem can be applied. In fact, for (g, y) small,

$$
\begin{aligned}
&Ev((f,x)+(g,y)) - Ev(f,x) = Ev(f+g, x+y) - Ev(f,x) \\
&= (f+g)(x+y) - f(x) = f(x+y) - f(x) + g(x+y) \\
&= \sum_{k=1}^{n} \tfrac{D^k f(x)}{k!}(y^k) + \sum_{k=n+1}^{\infty} \tfrac{D^k f(x)}{k!}(y^k) + \sum_{k=0}^{n-1} \tfrac{D^k g(x)}{k!}(y^k) + \sum_{k=n}^{\infty} \tfrac{D^k g(x)}{k!}(y^k) \\
&= \sum_{k=1}^{n} \Big(\tfrac{D^k f(x)}{k!}(y^k) + \tfrac{D^{k-1} g(x)}{(k-1)!}(y^{k-1}) \Big) + \sum_{k=n+1}^{\infty} \tfrac{D^k f(x)}{k!}(y^k) + \sum_{k=n}^{\infty} \tfrac{D^k g(x)}{k!}(y^k).
\end{aligned}
$$

The last two series can be treated as in the last part of the proof of Theorem 6.5: the first, involving f, is of order $o(\|y\|^n)$, and the second may be bounded in norm by

$$\frac{2\|g\|_\delta}{(\delta-\rho)^{n+1}} \cdot \|y\|^n, \text{ for a suitable } \rho < \delta,$$

which tells us that it is of order $o(\|(g,y)\|^n)$, so that

$$
\begin{aligned}
Ev((f,x) \;&+\; (g,y)) - Ev(f,x) \\
&= \sum_{k=1}^{n} \frac{1}{k!}(D^k f(x)(y^k) + k\, D^{k-1} g(x)(y^{k-1})) + o(\|(g,y)\|^n) \\
&= \sum_{k=1}^{n} \frac{1}{k!} \varphi_k(f,x)(y^k) + o(\|(g,y)\|^n),
\end{aligned}
$$

and the theorem follows by virtue of the converse of Taylor's Theorem 4.4. □

7 Poincaré's Linearization Theorem

In this section we want to prove Poincaré's linearization Theorem for an analytic map from $\mathbb{R}^n$ or $\mathbb{C}^n$ into itself near a fixed point. It states that under certain conditions it is conjugate, via an analytic isomorphism, to its linear part (see [5], cf. [1] Ch.5).

We begin with the complex case i.e., that of $\mathbb{C}^n$, but first let us prove a technical lemma.

Lemma 7.1. *The subspace*

$$V = \{u \in A_\delta(\mathbb{C}^n, \mathbb{C}^n) : \, u(0) = Du(0) = 0\}$$

of $A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ is closed in $A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ and, as a result, V is a Banach space under the induced norm $\| \ \|_\delta$.

Proof. We give two proofs.

First proof. It is easy to check that the complementary set of V is open in $A_\delta(\mathbb{C}^n, \mathbb{C}^n)$: in the expansion of any u not in V some vector coefficient corresponding to zeroth or first order has to be nonzero and by definition of the norm $\| \ \|_\delta$, an $\varepsilon > 0$ can be found so that any $\widetilde{u}$ in the ball centered at u with radius ε has a nonvanishing coefficient of order < 2, i.e., the ball considered has no elements of V. More specifically, in case $u(0) = u_0$ is nonzero, it suffices to take $0 < \varepsilon < |u_0|$, since if

$$\|v - u\|_\delta = \sum_{k=0}^{\infty} |v_k - u_k| \, \delta^k < \varepsilon,$$

then, in particular, $|v_0 - u_0| < \varepsilon$, which is not possible if $v_0 = 0$, i.e., $v \notin V$. And similarly, if $u_1 = Du(0) \neq 0$, then it suffices to take $0 < \varepsilon < |u_1| \, \delta$, in which case, from $\|v - u\|_\delta < \varepsilon$, we would have $|v_1 - u_1| < \varepsilon$, which is not satisfied if $v_1 = 0$.

Second proof. The maps $u \mapsto u(0)$ and $u \mapsto Du(0)$ defined on $A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ are continuous, because the Evaluation map

$$Ev : A_\delta(\mathbb{C}^n, \mathbb{C}^n) \times \{x \in \mathbb{C}^n : |x| < \delta\} \to \mathbb{C}^n$$

is C^∞-differentiable and $D_2 Ev(f, 0) = Df(0)$, as stated in Theorem 6.10. Thus the preimages of zero of the former two maps are closed, but V is just the intersection of these two closed preimages and is therefore closed.

$\square$

Now we state Poincaré's Theorem:

Theorem 7.2. *Let $\Phi \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ be such that $\Phi(0) = 0$, and $A := D\Phi(0)$ is an $n \times n$- matrix that satisfies the following conditions:*

(i) A diagonalizes.

(ii) All eigenvalues $\lambda_1, \dots, \lambda_n$ of A are nonzero and less than one in modulus.

(iii) For all nonnegative integers $k_1, \dots, k_n$ such that $k_1 + \dots + k_n \geq 2$ and for all $j = 1, \dots, n$, one has $\lambda_j \neq \lambda_1^{k_1} \cdots \lambda_n^{k_n}$ (non-resonance condition).

Then there exist $\eta > 0$ and $\Psi \in A_\eta(\mathbb{C}^n, \mathbb{C}^n)$ such that $\Psi(0) = 0, D\Psi(0) =$ identity and $\Psi \circ \Phi \circ \Psi^{-1} = A$.

(So that Ψ is an analytic change of variables near the origin in $\mathbb{C}^n$ which by conjugation linearizes Φ).

Proof. As $\Phi \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ with $\Phi(0) = 0$, and $D\Phi(0) = A$, we can write

$$\Phi(x) = A(x) + g(x)$$

with $g \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ of order ≥ 2, i.e., with $g(0) = Dg(0) = 0$, and we contend that there exists $u \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ of order ≥ 2 such that if we take as new coordinates $\Psi(x) = w = x + u(x)$, Φ is described in these coordinates by the transformation $w \mapsto Aw$. This can be more clearly viewed in terms of the existence of a commutative diagram (actually Φ is only defined for $x \in \mathbb{C}^n$ with $|x| \leq \delta$)

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\Phi"] \arrow[d, "\Psi"] & \mathbb{C}^n \arrow[d, "\Psi"] \\ \mathbb{C}^n \arrow[r, "A"] & \mathbb{C}^n \end{tikzcd}$$

and taking elements

$$\begin{tikzcd} x \arrow[r, "\Phi"] \arrow[d, "\Psi"] & \Phi(x) \arrow[d, "\Psi"] \\ w \arrow[r, "A"] & Aw \end{tikzcd}$$

for a suitable Ψ such that $\Psi(x) = w = x + u(x)$ with $u \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ of order ≥ 2. Following the diagram we get in one direction

$$(A \circ \Psi)(x) = A(\Psi(x)) = A(x + u(x)) = Ax + Au(x)$$

and in the other

$$(\Psi \circ \Phi)(x) = \Psi(\Phi(x)) = \Psi(Ax + g(x)) = Ax + g(x) + u(Ax + g(x))$$

so that the condition of commutativity reads

$$Ax + Au(x) = Ax + g(x) + u(A(x) + g(x))$$

and cancelling the Ax terms, we are led to the following functional equation for u:

$$Au(x) - g(x) - u(Ax + g(x)) = 0$$

As all eigenvalues of A are less than one in modulus, there exists a norm on A such that $|A| < \alpha < 1$. This allows us to write

$$|Ax + g(x)| \leq |Ax| + |g(x)| \leq \alpha\delta + \|g\|_\delta \,,$$

if $|x| \leq \delta$ and leads us to introduce the function $F : U \times U_0 \to U$, where U_0 is the centered ball of radius $(1-\alpha)\delta$ in the Banach space

$$U = \{\varphi \in A_\delta(\mathbb{C}^n, \mathbb{C}^n) : \varphi(0) = D\varphi(0) = 0\}$$

(see Lemma 7.1), defined by

$$F(u, g)(x) = Au(x) - g(x) - u(Ax + g(x))$$

Observe that F is well defined and $\mathcal{C}^\infty$-differentiable by virtue of Lemma 6.6, Theorem 6.9 and the chain rule.

Now, it is clear that $F(0, 0) = 0$. Our aim is to show that given g (in our case g is given by Φ), there exists u (as a function, actually of class $\mathcal{C}^\infty$, of g) satisfying $F(u, g) = 0$ and for this we will need to apply the Implicit Function Theorem 5.5, so we have to deal with the first partial derivative of F with respect to the first argument at $(0, 0)$.

As $F(u, 0)(x) = Au(x) - u(Ax)$, we immediately see that $u \mapsto F(u, 0)$ is continuous and linear in u, and this being the case, its derivative is the same continuous linear function (defined on U), for any u, and in particular when $u = 0$. Thus

$$D_1F(0, 0)(v) = Av - v \circ A.$$

Let us denote $D_1F(0, 0)$ by L. Now, to satisfy the requirements of the Implicit Function Theorem 5.5 we need to prove that L is invertible. Let us seek a formal complex solution for v to $Lv = w$ where v and w are formal power series of order ≥ 2 say in the indeterminates $x_1, \ldots, x_n$ (with coefficients in $\mathbb{C}^n$). Recall that $Lv = Av - v \circ A$. As we have assumed that A diagonalizes, there exists an invertible complex $n \times n$-matrix P such that $P^{-1}AP = D$ =diag$(\lambda_1, \ldots, \lambda_n)$. Introducing new indeterminates $y_1, \ldots, y_n$ such that $x = Py$, where $x = (x_1, \ldots, x_n)^T$, and similarly for y, and setting

$$\nu(y) := P^{-1}v(Py), \;\; \omega(y) := P^{-1}w(Py),$$

we see that the equation $Lv = w$ may be translated into the form $\Lambda\nu = \omega$,

where
$$\Lambda\nu(y) = D\nu(y) - \nu(Dy),$$
because
$$\begin{aligned} P(D\nu - \nu \circ D) \circ P^{-1} &= PDP^{-1}P\nu \circ P^{-1} - P\nu \circ (P^{-1}PDP^{-1}) \\ &= Av - v \circ A = Lv = w = P\omega \circ P^{-1}. \end{aligned}$$

We proceed now, after diagonalization, to isolate ν in $\Lambda\nu = \omega$, but this requires working with explicit expansions for ν and ω. So let us consider the set K of integer vectors $k = (k_1, \dots, k_n)$ with $k_i \geq 0$ and abbreviate $y_1^{k_1} \cdots y_n^{k_n}$ to y^k and set $|k| := k_1 + \cdots + k_n$. With these notations, if the expansions of ν and ω are
$$\nu(y) = \sum_{|k|\geq 2} \nu_k y^k, \;\; \omega(y) = \sum_{|k|\geq 2} \omega_k y^k,$$
and $\omega_k^{(t)}$ and $\nu_k^{(t)}$, for $t = 1, 2, \dots, n$, are the respective components of the vectors ω_k and ν_k, we have for each t, that the t-component of $\Lambda\nu(y)$ is given by
$$\lambda_t \sum_{|k|\geq 2} \nu_k^{(t)} y^k - \sum_{|k|\geq 2} \nu_k^{(t)} (\lambda y)^k,$$
where $(\lambda y)^k$ means $(\lambda_1 y_1)^{k_1} \cdots (\lambda_n y_n)^{k_n} = \lambda_1^{k_1} \cdots \lambda_n^{k_n} y_1^{k_1} \cdots y_n^{k_n} = \lambda^k y^k$.

But this is just
$$\sum_{|k|\geq 2} (\lambda_t - \lambda^k) \nu_k^{(t)} y^k \qquad (t = 1, \dots, n)$$
or in vector notation, M_k standing for the matrix diag $(\lambda_1 - \lambda^k, \cdots, \lambda_n - \lambda^k) = D - \lambda^k I$,
$$\Lambda\nu(y) = \sum_{|k|\geq 2} (D - \lambda^k I)\, \nu_k\, y^k = \sum_{|k|\geq 2} M_k\, \nu_k\, y^k,$$

We have that M_k is invertible for all $k, |k| \geq 2$, in view of assumption (iii). Consequently,
$$\Lambda\nu(y) = \sum_{|k|\geq 2} M_k \nu_k y^k = \omega(y) = \sum_{|k|\geq 2} \omega_k y^k$$
and we get $\nu_k = M_k^{-1} \omega_k$, which yields a formal solution to $\Lambda\nu = \omega$ or equivalently to $Lv = w$.

But, by condition (ii), we have $\lambda^k = \lambda_1^{k_1} \dots \lambda_n^{k_n} \to 0$ when the k_j's are large and this implies that both M_k and $M_k^{-1}$are bounded, for any $k \in K$, $|k| \geq 2$. In particular, there exists a real number H such that $|M_k^{-1}| < H$, for all these k, and this implies $|\nu_k| = |M_k^{-1}\, \omega_k| < H\, |\omega_k|$, from which we see that $\nu \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$, whenever $\omega \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$, and thus $D_1 F(0,0)$ is invertible.

Now that we have just proved that $D_1 F(0,0)$ is invertible, the Implicit Function Theorem 5.5 assures us the existence of an $\varepsilon > 0$ such that u is uniquely determined by g as far as $\|g\|_\delta < \varepsilon$ and $\|u\|_\delta < \varepsilon$. In order to get rid of this restriction on g we can rescale as follows: We set $\widetilde{g}(x) := \alpha^{-1} g(\alpha x)$, for $\alpha \in (0,1]$. If $g = \sum_{k\geq 2} g_k$, with $g_k \in L_s^k(\mathbb{C}^n, \mathbb{C}^n)$, then by definition of $\| \, \|_\delta$, we have

$$\|\widetilde{g}\|_\delta = \alpha^{-1} \sum_{k\geq 2} |g_k| \, \alpha^k \delta^k = \alpha \sum_{k\geq 2} |g_k| \, \alpha^{k-2} \delta^k \leq \alpha \sum_{k\geq 2} |g_k| \, \delta^k = \alpha \|g\|_\delta$$

the last inequality holding because $0 < \alpha \leq 1$. This allows us to choose α such that $\|\widetilde{g}\|_\delta < \varepsilon$, and by the Implicit Function Theorem 5.5, for this $\widetilde{g}$ there exists a unique $\widetilde{u} \in A_\delta(\mathbb{C}^n, \mathbb{C}^n)$ such that $F(\widetilde{u}, \widetilde{g}) = 0$. Next, define $u(x) := \alpha \widetilde{u}(\alpha^{-1} x)$ for the α chosen. Now, $F(\widetilde{u}, \widetilde{g}) = 0$ means

$$A\widetilde{u}(x) - \widetilde{u}(Ax + \widetilde{g}(x)) - \widetilde{g}(x) = 0,$$

which, in terms of u and g, is

$$A\alpha^{-1} u(\alpha x) - \alpha^{-1} u(\alpha A x + g(\alpha x)) - \alpha^{-1} g(\alpha x) = 0.$$

Cancelling α^{-1}, we have under the change of variables $\alpha x = y$

$$Au(y) - u(Ay + g(y)) - g(y) = 0,$$

i.e. $F(u, g)(y) = 0$. And setting $\eta = \alpha\delta$, if $\widetilde{u} = \sum_{k\geq 2} \widetilde{u}_k$, we see that

$$\|u\|_\eta = \alpha \sum_{k\geq 2} |\widetilde{u}_k \alpha^{-k}| \eta^k = \alpha \sum_{k\geq 2} |\widetilde{u}_k| \left(\frac{\eta}{\alpha}\right)^k = \alpha \sum_{k\geq 2} |\widetilde{u}_k| \delta^k = \alpha \|\widetilde{u}\|_\delta < \infty,$$

so that $u \in A_{\alpha\delta}(\mathbb{C}^n, \mathbb{C}^n)$.

$\square$

Corollary 7.3. *Poincaré's Theorem 7.2 holds true if $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$ everywhere, i.e., if $\Phi \in A_\delta(\mathbb{R}^n, \mathbb{R}^n)$, so that in particular $A = D\Phi(0)$ has real entries, then Ψ can be found in $A_\eta(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\Psi \circ \Phi \circ \Psi^{-1} = A$.*

Remark 7.4. The corollary establishes that all transformations involved are real independently of the fact that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A may not be real.

Proof. (Keeping the notations occurring both in Poincaré's Theorem 7.2 and in its proof)

If A is real and λ is a non-real eigenvalue of A then its conjugate $\overline{\lambda}$ is also another eigenvalue of A and obviously if v is a complex eigenvector corresponding to λ, then $\overline{v}$ is an eigenvector corresponding to $\overline{\lambda}$ (since from $\lambda v = Av$ we get $\overline{\lambda}\overline{v} = \overline{\lambda v} = \overline{Av} = A\overline{v}$ because $A = \overline{A}$). These facts allow us to consider a complex basis of eigenvectors

$$\{e_1, \ldots, e_\ell, e_{\ell+1} = \overline{e_1}, \ldots, e_{2\ell} = \overline{e_\ell}, e_{2\ell+1}, \ldots, e_n\}$$

corresponding to the eigenvalues

$$\{\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1} = \overline{\lambda_1}, \ldots, \lambda_{2\ell} = \overline{\lambda_j}, \lambda_{2\ell+1}, \ldots, \lambda_n\},$$

the 2ℓ first of which are non-real and the last $n-2\ell$ are real. Let P be the complex $n \times n$-matrix whose columns are the (complex) coordinates of the e_i's in the standard basis of $\mathbb{C}^n$, i.e., the matrix such that $P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = D$, and let Q stand for the matrix which interchanges the complex conjugate vectors of the basis $\{e_1, \ldots, e_n\}$ i.e. the matrix

$$\begin{pmatrix} 0 & I_\ell & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & I_{n-2\ell} \end{pmatrix}.$$

From this, it is obvious that $QD = \overline{D}Q$ and $\overline{P} = PQ$. Now, as in the proof of the theorem, set

$$x = Py, \;\; \nu(y) = P^{-1}v(Py), \;\; \omega(y) = P^{-1}w(Py)$$

and observe that v and w are real if and only if $v(\overline{x}) = \overline{v(x)}$ and $w(\overline{x}) = \overline{w(x)}$ (since x real means $x = \overline{x}$). Let us express this in terms of ν and ω: $v(\overline{x}) = \overline{v(x)}$ in terms of $\nu = P^{-1}v \circ P$ is expressed as $P\nu \circ P^{-1} = \overline{P}\overline{\nu} \circ \overline{P^{-1}}$, but from $\overline{P} = PQ$, we see that $P\nu \circ P^{-1} = PQ\overline{\nu} \circ (Q^{-1}P^{-1})$, i.e. $\nu = Q\overline{\nu} \circ Q^{-1}$ or $Q\overline{\nu} = \nu \circ Q$ (as expected since Q interchanges conjugate basis vectors), and similarly $Q\overline{\omega} = \omega \circ Q$ or $\overline{\omega} = Q^{-1}\omega \circ Q$, which means that $\overline{\omega_k} = Q\omega_q$ for $q = kQ$, where $q = (q_1, \ldots, q_n)$ and $k = (k_1, \ldots, k_n)$.

Coming back to the proof of the theorem, we have

$$\overline{\nu_k} = \overline{M_k}^{-1}\,\overline{\omega_k} = (\overline{D} - \overline{\lambda}^k I)^{-1} Q\omega_q = Q(D - \lambda^k I)^{-1}\omega_q = Q\nu_q$$

i.e., $Q\overline{\nu} = \nu \circ Q$ so that v is real when w is real.

□

Remark 7.5. Observe that the proof of this corollary is vacuous if all eigenvalues of A are real.

Now, we will deal with the resonant case

Theorem 7.6. *Let us just drop assumption* (*iii*) *in Theorem 7.2 and keep the same notations. Then the conjugate* $\Psi \circ \Phi \circ \Psi^{-1}$ *sends* w *to* $Aw + h(w)$ *where* h *lies in the kernel of* L*, i.e.* $Ah(w) - h(Aw) = 0$*, and* h *is a polynomial.*

Proof. We follow the lines of the proof of Poincaré's Theorem 7.2 and consider the set (of resonant terms):

$$S = \left\{(j,k) : \, j \in \{1, \ldots, n\}, \, k \in K, \, |k| \geq 2, \, \lambda_j = \lambda^k\right\}.$$

By condition (ii) we see that $\lambda^k = \lambda_1^{k_1} \cdots \lambda_n^{k_n} \to 0$ as the k_j's become large. This implies that the set S is finite. Then $\nu \in \operatorname{Ker} \Lambda$ if and only if ν is of the form

$$\nu(y) = \sum_{(j,k) \in S} \alpha_{(j,k)}\, y^k\, e_j,$$

and as S is finite, $\operatorname{Ker}\Lambda$ is finite-dimensional. Here e_j stands for the jth element of the canonical basis of $\mathbb{C}^n$.

Let now π be the projection of $A_\delta(\mathbb{C}^n,\mathbb{C}^n)$ defined by

$$\pi\,\nu(y) = \sum_{(j,k)\in S} \nu_k^{(j)}\, y^k\, e_j,$$

i.e., π sends $\nu(y)$ to the finite sum of its resonant terms i.e., those where $\lambda_j - \lambda^k$ vanishes.

Obviously π is linear and $\|\pi\,\nu\|_\delta \le \|\nu\|_\delta$, which implies π is continuous.

Under these circumstances we can assure (see [3] 5.4.2) that $A_\delta(\mathbb{C}^n,\mathbb{C}^n)$ is the topological direct sum of the Banach spaces $\pi\,A_\delta(\mathbb{C}^n,\mathbb{C}^n)$ and $(I-\pi)\,A_\delta(\mathbb{C}^n,\mathbb{C}^n)$. It is clear that $\Lambda\,\pi = \pi\,\Lambda = 0$ and that Λ sends $(I-\pi)\,A_\delta(\mathbb{C}^n,\mathbb{C}^n)$ into itself. Furthermore, Λ has a (continuous) inverse on the Banach space

$$U = \{\ell \in (I-\pi)\,A_\delta(\mathbb{C}^n,\mathbb{C}^n) :\, \ell(0) = D\ell(0) = 0\}\,.$$

Let U_0 be the open (centered) ball of radius δ in the preceding Banach space,

$$V = \{h \in \pi\,A_{2\delta}(\mathbb{C}^n,\mathbb{C}^n) :\, h(0) = Dh(0) = 0\}\,,$$

$$W = \left\{g \in A_{(1-\alpha)\delta}(\mathbb{C}^n,\mathbb{C}^n) :\, g(0) = Dg(0) = 0\right\}\,,$$

and

$$Z = \{m \in A_\delta(\mathbb{C}^n,\mathbb{C}^n) :\, m(0) = Dm(0) = 0\}\,.$$

Recall that $A_{2\delta}(\mathbb{C}^n,\mathbb{C}^n) \subset A_\delta(\mathbb{C}^n,\mathbb{C}^n)$. Now, proceeding as at the beginning of the proof of Poincaré's Theorem 7.2, we are led to consider the functional equation which both u and h must satisfy:

$$F(u,h,g) = A\,u(x) - u\,(Ax + g(x)) - g(x) + h(x + u(x)) = 0,$$

for $F : U_0 \times V \times W \to Z$.

As before, and bearing in mind that by restricting C^∞-differentiable functions to subspaces we get C^∞-differentiable functions again, we have that F is well-defined and C^∞-differentiable, and as

$$F(0,0,0) = 0, \quad F(u,0,0) = A\,u - u\circ A, \text{ and } F(0,h,0) = h,$$

we see that

$$D_1F(0,0,0) = L|\,(I-\pi)\,A_\delta(\mathbb{C}^n,\mathbb{C}^n) =: \widetilde{L}, \text{ and } D_2F(0,0,0) = I,$$

and these partial derivatives clearly define a continuous homomorphism from $U\times V$ into Z, namely $(u,h) \mapsto \widetilde{L}\,u + h$. If we show this is invertible we can apply the Implicit Function Theorem 5.5 and rescale as before to conclude the proof. But invertibility is immediate: In order to uniquely solve

$$\widetilde{L}\,v + h = g$$

for $v \in (I-\pi)\,A_\delta(\mathbb{C}^n,\mathbb{C}^n)$ and $h \in \pi\,A_{2\delta}(\mathbb{C}^n,\mathbb{C}^n)$, for any given $g \in A_\delta(\mathbb{C}^n,\mathbb{C}^n)$, with all three functions of order at least two, we see that we have to take precisely $h = \pi\,g$ (which is a polynomial and therefore lies in $\pi\,A_\eta(\mathbb{C}^n,\mathbb{C}^n)$, for any $\eta > 0$) and $v = \widetilde{L}^{-1}\,(I-\pi)g$. □

Corollary 7.7. *The statement of Corollary 7.3 is also true in the case of Theorem 7.6, i.e., in the real case.*

Proof. It is the same as that of Corollary 7.3 except for the case of the resonant terms $(j,k) \in S$. Recalling that v is real if and only if $\overline{v(x)} = v(\overline{x})$, i.e., $Q\,\overline{\nu(y)} = \nu(Q\,\overline{y})$, and this obviously holds when ν is replaced by $(I-\pi)\,\nu$, there remains the case of $\pi\,\nu$, i.e. we are asking whether $Q\,\overline{\pi\,\nu(y)} = \pi\,\nu(Q\,\overline{y})$. But this is true because if A is a real matrix and $(j,k) \in S$, i.e., $\lambda_j = \lambda^k$, then $\overline{\lambda_j} = \overline{\lambda}^k$ is also a resonant term for A (recall that the set of eigenvalues of A is invariant by conjugation).

□

We end this section covering the case when A does not diagonalize. It turns out that the same conclusions as those of the preceding theorems hold.

Theorem 7.8. *If we drop condition (i) in Poincaré's Theorem 7.2, the same conclusion holds. And this also occurs in Theorem 7.6, Corollary 7.3 and Corollary 7.7.*

Proof. If A does not diagonalize, by the theory of the Jordan blocks there exists an invertible matrix P such that $P^{-1}A\,P = D + N$ with D diagonal and N nilpotent, and we can further assume N is small (this can easily be achieved by substituting the vectors of a Jordan basis by suitable multiples of them). Then with our previous changes, namely $v = P\,\nu \circ P^{-1}$, $w = P\,\omega \circ P^{-1}$, the equation $L\,v = w$, with $L\,v = A\,v - v \circ A$, is transformed into

$$\begin{aligned}
A\,v - v \circ A &= P(D+N)P^{-1}\,P\,\nu \circ P^{-1} - P\,\nu \circ [P^{-1}\,P(D+N)P^{-1}] \\
&= P(D+N)\,\nu \circ P^{-1} - P\,\nu \circ [(D+N)P^{-1}] \\
&= P[(D+N)\,\nu - \nu \circ (D+N)] \circ P^{-1} \\
&= P[D\,\nu + N\,\nu - \nu \circ (D+N)] \circ P^{-1} \\
&= P[D\,\nu - \nu \circ D + N\,\nu - \nu \circ (D+N) + \nu \circ D] \circ P^{-1} = P\,\omega \circ P^{-1}
\end{aligned}$$

i.e., $\Lambda\,\nu = \omega$, where $\Lambda = \Lambda_1 + \Lambda_2$, with

$$\Lambda_1\,\nu(y) = D\,\nu(y) - \nu\,(Dy), \qquad \Lambda_2\,\nu(y) = N\,\nu(y) - (\nu(D\,y + N\,y) - \nu\,(Dy)).$$

These expressions allow us to conclude that Λ_1 is, as seen before, invertible (in the resonant case, leaving aside the resonant terms), and Λ_2 is small. As the invertible elements in the Banach space of continuous linear maps between two Banach spaces form an open set (see [2] I, Thm. 1.7.3(a)) we see that $\Lambda = \Lambda_1 + \Lambda_2$ can be made invertible, which is all we need.

□

8 The Analytic Stable Manifold Theorem

This section deals with the analytic Stable Manifold Theorem:

Theorem 8.1. *Let C be an $n \times n$ real nonsingular matrix with k eigenvalues less than one in modulus and the other $n-k$ eigenvalues greater than one in modulus and $\Phi \in A_\delta(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\Phi(0) = 0$ and $D\,\Phi(0) = C$. Then there exists a neighbourhood N of 0 in $\mathbb{R}^n$such that the subset of those $p \in N$ satisfying $\Phi^n(p) \in N$, for all $n \in \mathbb{N}$, is a k-dimensional real analytic submanifold W^s of N. In fact, for any $p \in W^s$, we have $\lim_{n\to\infty} \Phi^n(p) = 0$.*

(We say that W^s is the stable submanifold of Φ).

Proof. The assumption on the eigenvalues of C allows a linear change of variables leading to the case when

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with A a real $k \times k$-matrix with norm $|A| = \alpha < 1$ and B a real $(n-k) \times (n-k)$-matrix such that the norm $|B^{-1}| = \beta < 1$. We will write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, so that Φ may be described as

$$\Phi : (x, y) \mapsto (x^*, y^*) = (Ax + f(x, y),\ By + g(x, y))$$

where $f \in A_\delta(\mathbb{R}^n, \mathbb{R}^k), g \in A_\delta(\mathbb{R}^n, \mathbb{R}^{n-k})$, with both f and g of order ≥ 2. Let us seek a change of variables of the form $\xi = x, \eta = y - h(x)$ for some $h \in A_\delta(\mathbb{R}^k, \mathbb{R}^{n-k})$ of order ≥ 2 such that the ξ-axis is Φ-invariant. Then we will show that the ξ-axis, i.e. the graph of h, is W^s.

In the new variables $\{\xi, \eta\}, \Phi$ may be described as

$$(\xi, \eta) \mapsto (\xi^*, \eta^*) = (A\,\xi + f^*(\xi, \eta), B\,\eta + g^*(\xi, \eta))$$

with both f^* and g^* of order ≥ 2.

The relationship between f, g and f^*, g^* comes from the fact that if (x, y) corresponds to (ξ, η) then (x^*, y^*) must correspond to (ξ^*, η^*), i.e.

$$\xi^* = x^*, \text{ and } \eta^* = y^* - h(x^*).$$

Paying attention only to the last equation, note it can be rewritten as

$$B\,\eta + g^*(\xi, \eta) = B\,y + g(x, y) - h(A\,x + f(x, y))$$

and after substituting x by ξ and y by $\eta + h(x) = \eta + h(\xi)$, we arrive at

$$B\,\eta + g^*(\xi, \eta) = B(\eta + h(\xi)) + g(\xi, \eta + h(\xi)) - h(A\,\xi + f(\xi, \eta + h(\xi))),$$

or, cancelling $B\,\eta$ in both members,

$$g^*(\xi,\eta) = B(h(\xi)) + g(\xi,\eta+h(\xi)) - h(A\,\xi + f(\xi,\eta+h(\xi))).$$

Observe that the ξ-axis $\eta = 0$ is invariant under Φ if and only if $\eta = 0$ implies $\eta^* = 0$. As $\eta^* = B\,\eta + g^*(\xi,\eta)$ we realize that this is exactly the same as requiring $g^*(\xi,0) = 0$, or using the above expression for $g^*(\xi,\eta)$, that h has to satisfy the following functional equation

$$B(h(\xi)) + g(\xi,h(\xi)) - h(A\,\xi + f(\xi,h(\xi))) = 0.$$

Let $F(h,f,g)(\xi)$ stand for the left hand side of this functional equation, so that F can be seen as a map from $U_0 \times V_0 \times W$ into U, where U_0 is the centered ball of radius δ in the Banach space

$$U = \{h \in A_\delta(\mathbb{R}^k,\mathbb{R}^{n-k}) : h(0) = Dh(0) = 0\},$$

V_0 is the centered open ball of radius $(1-\alpha)\delta$ of the Banach space

$$V = \{f \in A_\delta(\mathbb{R}^n,\mathbb{R}^k) : f(0) = Df(0) = 0\}$$

and W is the Banach space

$$W = \{g \in A_\delta(\mathbb{R}^n,\mathbb{R}^{n-k}) : g(0) = Dg(0) = 0\}.$$

By Lemma 6.6, Theorem 6.9 and the chain rule, $F : U_0 \times V_0 \times W \to U$ is well-defined and C^∞-differentiable.

Obviously $F(0,0,0) = 0$ and as $F(h,0,0) = B\,h - h \circ A$ is linear in h, we have that the first partial derivative $D_1F(0,0,0)$ of F at $(0,0,0)$ is the continuous linear map $\Lambda : U \to U$ defined by $\Lambda(\ell) = B\,\ell - \ell \circ A$. The map Λ is invertible: in fact,

$$\Lambda^{-1}(\ell)(\xi) = B^{-1}\,\ell(\xi) + B^{-2}\,\ell(A\,\xi) + B^{-3}\,\ell(A^2\,\xi) + \cdots = \sum_{s=0}^{\infty} B^{-s-1}\,\ell(A^s\,\xi).$$

Formally

$$\begin{aligned}\Lambda^{-1}(\Lambda\ell) &= B^{-1}(B\ell - \ell\circ A) + B^{-2}(B\ell - \ell\circ A)\circ A + \cdots \\ &= (\ell - B^{-1}\ell\circ A) + (B^{-1}\ell\circ A - B^{-2}\,\ell\circ A^2) + \cdots = \ell,\end{aligned}$$

and

$$\begin{aligned}\Lambda(\Lambda^{-1}\ell) &= B(B^{-1}\ell + B^{-2}\ell\circ A + B^{-3}\ell\circ A^2 + \cdots) \\ &\quad -(B^{-1}\ell + B^{-2}\ell\circ A + B^{-3}\ell\circ A^2 + \cdots)\circ A \\ &= \ell + B^{-1}\ell\circ A + B^{-2}\ell\circ A^2 + \cdots - (B^{-1}\ell\circ A + B^{-2}\ell\circ A^2 + \cdots) = \ell.\end{aligned}$$

But the series $\sum_{s=0}^{\infty} B^{-s-1}\,\ell\circ A^s$ obviously converges in U, since by Lemma 6.6,

$$\sum_{s=0}^{\infty} \|B^{-s-1}\,\ell\circ A^s\|_\delta \le \sum_{s=0}^{\infty} \beta^{s+1}\,\|\ell\|_\delta = \frac{\beta}{1-\beta}\|\ell\|_\delta\,.$$

This also shows that $\|\Lambda^{-1}\| \leq \frac{\beta}{1-\beta}$. Now we can apply the Implicit Function Theorem 5.5 and conclude that there exists an $\varepsilon > 0$ and a unique $h \in U_0$ which is a C^∞-function of (f, g), for $f \in V_0, \|f\|_\delta < \varepsilon$ and $g \in W, \|g\|_\delta < \varepsilon$, satisfying $F(h, f, g) = 0$. But we would like to solve $F(h, f, g) = 0$ without the assumptions that f and g have to be small in norm. This can be done by rescaling as we have done in the proof of the Poincaré's Theorem 7.2, and we eventually see that there exists an $\alpha > 0$ and an $h \in A_{\alpha\delta}(\mathbb{R}^k, \mathbb{R}^{n-k})$ satisfying $F(h, f, g) = 0$. So we have shown there is a change of variables

$$x = \xi, \ y = \eta - h(\xi)$$

such that the ξ-axis is Φ-invariant. Considering now Φ^{-1} instead of Φ, we see there is a change of variables leaving the η-axis invariant. After making both changes of variables we consider Φ again but in the new variables: As before,

$$\Phi(\xi, \eta) = (\xi^*, \eta^*) = (A\xi + f^*(\xi, \eta), B\eta + g^*(\xi, \eta)),$$

with $f^*(0,0) = Df^*(0,0) = 0$ and $g^*(0,0) = Dg^*(0,0) = 0$, but now, by the Φ-invariance of the axes, we also have $f^*(0, \eta) = 0$, $g^*(\xi, 0) = 0$.

By the Mean Value Theorem(see [4] 2.4.8) there is a neighbourhood N of the origin in $\mathbb{R}^n$ and a $\vartheta \in (0,1)$ such that

$$|A\xi + f^*(\xi, \eta)| < \vartheta\,|\xi|, \text{ and } |B\eta + g^*(\xi, \eta)| > \vartheta^{-1}|\eta|, \text{ for all } (\xi, \eta) \in N.$$

Let us justify the former inequality. So consider the map

$$(\xi, \eta) \mapsto \xi^* = A\xi + f^*(\xi, \eta).$$

As $f^*(0, \eta) = 0$, we have by the Mean Value Theorem

$$|A\,\xi + f^*(\xi, \eta)| \leq |A\,\xi| + |f^*(\xi, \eta) - f^*(0, \eta)| \leq |A| \cdot |\xi| + \sup_{0 \leq t \leq 1} |Df^*(t\,\xi, \eta)| \cdot |\xi| < \vartheta_1 |\xi|,$$

for some $\vartheta_1 \in (0,1)$, since $|A| = \alpha < 1$ and $Df^*(t\,\xi, \eta)$ is near 0 if (ξ, η) is near the origin, because $Df^*(0,0) = 0$. The same reasoning applies to the map $(\xi, \eta) \mapsto \eta^* = B\eta + g^*(\xi, \eta)$ to get near the origin $|B\,\eta + g^*(\xi, \eta)| > \vartheta_2^{-1}|\eta|$, for some $\vartheta_2 \in (0,1)$ and, of course, by replacing ϑ_1 and ϑ_2 by $\max(\vartheta_1, \vartheta_2)$, we can assume $\vartheta_1 = \vartheta_2$.

Let (ξ_0, η_0) be in N and set $(\xi_n, \eta_n) = \Phi^n(\xi_0, \eta_0)$ where Φ^n stands for the nth iterate of Φ. Then the preceding inequalities given by the Mean Value Theorem imply that $|\xi_n| < \vartheta^n(\xi_0)$ and $|\eta_n| > \vartheta^{-n}|\eta_0|$ as long as $(\xi_n, \eta_n) \in N$. So, if $\eta_0 = 0$, and ξ_0 is small, i.e., $(\xi_0, 0) \in N$, we obtain

$$\lim_{n \to \infty} (\xi_n, \eta_n) = \lim_{n \to \infty} (\xi_n, 0) = 0.$$

However, if $\eta_0 \neq 0$, then given any compact set K of N, an n can be found such that $(\xi_n, \eta_n) \notin K$, and this concludes the proof of the theorem.

□

9 Conclusions

In this work we have dealt with some essential topics in Mathematical Analysis which do not seem to be widely known, namely the converse of Taylor's Theorem, the Omega-lemma and the Evaluation map. In this work we have introduced and elaborated in detail the proofs of these important theorems of Global Analysis in the case of the Banach spaces $A_\delta(E, F)$ since these are those we have been interested in, because with them we have supplied nice proofs of the deep theorems of Poincaré concerning linearization of certain analytic maps and of the Analytic Stable Manifold Theorem. Guided by the sketchy paper of Meyer [5], we have relied on good standard books such as Arnold [1], Cartan [2], Dieudonné [3], and Marsden [4]. In the end we realize that functional analysis, in our case concerning the key spaces $A_\delta(E, F)$, turns out to be useful not only in itself but also in important applications as those considered in this work.

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