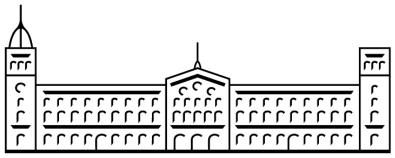

Bachelor Thesis

DEGREE IN MATHEMATICS

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The Michelson System

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Abstract

The Michelson system is a three dimension autonomous ODE system that arises in the context of studying the travelling waves solutions of the Kuramoto-Sivashinsky equation. Nonetheless, this system has its own relevance in itself, as it possesses some rich dynamics. In particular, the Michelson system is an interesting system to study since it is non-Hamiltonian, volume preserving and has a time reversing symmetry. In this work we will study the main properties of the system from a theoretical and a numerical point of view.

More precisely, in the theoretical part we introduce the properties mentioned above and study the equilibrium points stability and their invariant manifolds. Moreover, some results on the existence of some type of orbits are also given. For the numerical part, we implement an algorithm to integrate orbits and give detailed methods to find periodic orbits. However, the main result of this block is the implementation of an algorithm to find the heteroclinic orbits of the Michelson system.

Structure

This work structured in four sections. The first is just an introduction to the Michelson system deriving it from the context of the Kuramoto-Sivashinsky equation.

In the the second part the main properties of the system are introduced in a general sense and then they we relate them to the Michelson system. Such properties are the analyticity of solutions, the preservation of volume and the existence of a time reversing symmetry. This part is more on line of general ODE theory.

The third part has a more dynamical systems approach. This part is completely focused on the Michelson systems, and we study the stability of equilibrium points and invariant manifolds, and the existence of some types of orbits using index theory and perturbation theory.

For the forth and last part, we explain algorithms for numerical experimentations. The first two are implemented and consists of a custom made Taylor method to integrate the solutions of the Michelson system and a method to locate heteroclinic orbits using first hit maps. At last we explain how to explicitly use Poincaré maps to locate periodic orbits. Still this last method could not be implemented.

Finally, there are four appendices containing information on notation, useful material related to some of the proofs of the system, and the C code of all the programmes designed in this project.

Contents

1. INTRODUCTION

1.1. The Kuramoto Sivashinky Equation. The Michelson system arises from the context of finding travelling waves solutions of the Kuramoto-Shivashinsky equation (abbreviated as KSe hereafter). Thus it is only natural that we start by introducing this equation. The KSe in its most general form is defined as

$$u(\boldsymbol{x},t)_t + \nabla^4 u(\boldsymbol{x},t) + \nabla^2 u(\boldsymbol{x},t) + u(\boldsymbol{x},t)\nabla u(\boldsymbol{x},t) = 0.$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Nonetheless, the most popular and deeply studied version of the KSe is in the one dimensional spacial case with periodic boundary conditions. Under this conditions the equation is

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0, \tag{1.1}$$

for which $u = u(x,t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u(x+L,t) = u(x,t)$. Note that the subscripts indicate the partial derivative with respect to the variable indicated.

The KSe was first obtained by Kuramoto and Tsuzuki in modelling turbulence for a reaction-diffusion system for a Belouzov-Zabotinski chemical reaction. Independently, Sivashinsky derived the same equation in the context of instabilities in laminar flame fronts. This equation also appear in models of perturbations of liquid films falling down an inclined plane. However in this last case the equation appear as

$$\mathcal{U}_t + \mathcal{U}_{xxxx} + \mathcal{U}_{xx} + \frac{1}{2}(\mathcal{U}^2)_x = 0 \tag{1.2}$$

Note that this equation is related to (1.2) by integration, $U(x,t) = \displaystyle\int_0^x u(\xi,t)d\xi$.

Besides its relevance in modelling several physical phenomenon, the KSe is regarded as a paradigm of low-dimension dynamics PDEs and as one of the simplest models exhibiting spatio-temporal chaos.

To conclude this brief introduction on the KSe, we just state that there are many PDEs which hasve a similar structure to the KSe. In particular, they are of the form

$$\partial_t u + B(u,u) + Au = 0 \tag{1.3}$$

where A denotes a linear operator and B represents a non linear term. Comparing with (1.1), $A = \partial_x^2 + \partial_x^3 + \partial_x^4$ and $B(u,u) = uu_x$. Some other important known equations with same structure are the viscous Burgers equation (1.4) (the one dimension Navier-Stokes equation), the Korteweg-de Vries equation (1.5), the Kawahara equation (1.6), and the Benney-Lin equation (1.7).

$$u_t + uu_x + u_{xx} = 0, \tag{1.4}$$

$$u_t + uu_x + u_{xxx} = 0, \tag{1.5}$$

$$u_t + uu_x + u_{xxx} + u_{xxxxx} = 0, \tag{1.6}$$

$$u_t + uu_x + u_{xx} + u_{xxx} + u_{xxxx} + u_{xxxxx} = 0. \tag{1.7}$$

1.2. The Michelson system. The first issue is to tackle is to derive the Michelson system from the KSe. Hence we attempt to find the travelling waves solutions of the which are of the form $u(\psi) = u(x - vt)$. Replacing this ansatz into (1.1) gives the following ODE,

$$-vu_{\psi} + u_{\psi\psi\psi\psi} + u_{\psi\psi} + uu_{\psi} = 0.$$

Integrating once the previous equation

$$-vu + u_{\psi\psi\psi} + u_{\psi} + \frac{1}{2}u^2 = 0,$$

were we have set to zero the integration constant. Then considering linear change of variables $u = y_1 - v$, $\psi = x$ we obtain

$$y_1''' + y_1' + \frac{1}{2}y_1^2 = -\frac{1}{2}v^2. \tag{1.8}$$

which is already the Michelson system. This system is name after Michelson which was the one to first derive this system in [6].

The first time the Michelson derived this system was done in a complete different context. He was trying to find the steady state solutions (solutions such that $u_t = 0$) of the KSe in the form (1.2). According to numerical experiments, he assumed that the steady solutions had to be of the form $u(x,t) = -c^2t + y_0(x)$ where c was the speed of the wave. Therefore replacing his ansatz in (1.2) he obtained

$$\frac{\mathrm{d}^4 y_0}{\mathrm{d}x^4} + \frac{\mathrm{d}^2 y_0}{\mathrm{d}x^2} + \frac{1}{2}\left(\frac{\mathrm{d}y_0}{\mathrm{d}x}\right)^2 = c^2.$$

Considering the change of variables variables $y_1 = y_0'$ gives

$$y_1''' + y_1' + \frac{1}{2}y_1^2 = c^2, \tag{1.9}$$

where in this ODE equation the spatial component $x \in \mathbb{R}$ takes the role of the traditional time variable in ODE theory. Note that (1.8) and (1.9) are the same equation for $v = \sqrt{2}c$.

The Michelson system properly speaking is the resulting system after the change of variables $y_1' = y_2$ and $y_2' = y_3$, to render the equation a first order ODE system. Thus,

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = c^2 - y_2 - \dfrac{y_1^2}{2}. \end{cases} \tag{1.10}$$

Note that with this notation, the subindex help in bookkeeping the order of the derivative of y_0 that each variable correspond. Also take into account that according derivation of Michelson, the solutions of the system and in particular $y_0 = \int_0^x y_1(z)\mathrm{d}z$ yield waves of the KSe that move at constant speed.

2. GENERAL PROPERTIES OF THE MICHELSON SYSTEM

This objective of this section is to discuss some general characteristics of the flow and solutions for the Michelson system. By general we mean that these results rely on generic properties of vectors fields, hence they are applicable to any other ODE systems withholding the same conditions.

More precisely, it will be shown that the solutions exist, are unique and analytic, and that the flow is volume preserving and has a time reversing symmetry. Besides giving definitions, we examine how this properties succeed in organizing the dynamics of the system with some indirect results.

2.1. Existence, Uniqueness & Analyticity of the Solutions. In the upcoming part we are essentially devoted to prove the Cauchy theorem, which essentially states that if the vector field of an ODE system is analytic, then the solution to any initial value problem exists, is unique and analytic [9]. Note that this theorem directly applies to the Michelson system, which has an analytic vector field, thus leading to the existence, uniqueness and analyticity of its solutions.

The Cauchy theorem is similar to Picard-Lindelöf theorem. This latter theorem require Lipschitz continuity in the vector field to prove existence an uniqueness of solutions. Note that analyticity is a stronger condition than Lipschitz continuity (i.e. analyticity implies Lipschitz continuity), thus the existence and uniqueness of solutions in the Cauchy theorem may be seen as consequence of the Picard-Lindelöf theorem. Nevertheless, in our proof we are not assuming existence and uniqueness which are proven independently.

Regarding this theorem, bear in mind that it is the key to section 4 and sets the foundations to the numerical integration methods for the solutions, which is by the way the basic tool in all the numerical experiments and simulations performed.

Before getting started, we shall make some comments concerning the outline of the theorems and proofs that will be seen. For starters, the upcoming theorems are formulated in the case of autonomous systems for simplicity. Generalizing these results to the non-autonomous case is not complicated since a non autonomous system can be transformed into an autonomous one by considering the time (recall that we are considering x to be the time variable) as a variable of the phase space an the equation $\mathrm{d}x/\mathrm{d}x = 1$. Second, the main theorem is proven in two steps, first locally for complex variables, then it is extended globally to maximal solutions for real variables. This approach was taken to avoid the complications of taking Riemann surfaces, since the Michelson system is real valued. Finally, point out that the line of proof chosen is the method of majorants. A shorter alternative proof may be obtained considering Picard operators acting on C^ω spaces with an appropriate metric and applying the fixed point theorem. However, the line of proof chosen has been successful in generalizing the local version of this theorem in EDP theory in the known as Cauchy-Kovalevskaya theorem.

Note: The following proofs, require the use of formal power series in several variables. In order to keep the notation as simple and short as possible, we resort to multi-indices. It may be useful to check A.1 to keep up with such nomenclature.

► 2.1.1. Definition (Majorant). Let $f(\boldsymbol{x})$, $g(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$, with $\boldsymbol{x} \in \mathbb{C}^n$, be two formal series. Say each is given explicitly by

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}}, \qquad\qquad g(\boldsymbol{x}) = \sum_{\boldsymbol{l}} b_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}},$$

where $\boldsymbol{l} \in \mathbb{N}^n$ are multi-indices, and $a_{\boldsymbol{l}} \in \mathbb{C}$, $b_{\boldsymbol{l}} \in \mathbb{R}_+$ are the coefficients. Then, $g(\boldsymbol{x})$ is said to be a *majorant* of $f(\boldsymbol{x})$, symbolically written as $\boldsymbol{f} \prec \boldsymbol{g}$, if

$$|a_{\boldsymbol{l}}| \leq b_{\boldsymbol{l}}, \qquad\qquad \forall \boldsymbol{l} \in \mathbb{N}^n,$$

where, from now on, $|\cdot|$ stands for the modulus of a complex number.

► 2.1.2. Formula. Let $g(\boldsymbol{x}) \in \mathbb{C}[[\boldsymbol{x}]]$, with $\boldsymbol{x} \in \mathbb{C}^n$, be the following formal series restricted to the open poly-disk $\boldsymbol{D_1}(\boldsymbol{0}) = \{\boldsymbol{z} \in \mathbb{C}^n \mid |z_i| < 1,\ 1 \leq i \leq n\}$, (i.e. the Cartesian product of disks in the complex space),

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}}, \tag{2.1}$$

then, the previous series is equivalent to

$$g(\boldsymbol{x}) = \left(\prod_{m=0}^{n} \frac{1}{(1 - x_m)} \right). \tag{2.2}$$

Proof: Majorly the proof consists in expanding the expression (2.1) from the multi-index notation as follows

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}} = \sum_{(l_1,\ldots,l_n)} x_1^{l_1} \ldots\, x_n^{l_n} = \sum_{l_1} x^{l_1} \left(\sum_{l_2} x^{l_2} \ldots \left(\sum_{l_n} x^{l_n} \right) \right).$$

Note that the series is restricted to $\boldsymbol{D_1}(\boldsymbol{0})$, which imply that for each component $x_i \in D_1(0)$ and consequently $|x_i| < 1$. Hence, each summation is a convergent geometrical series, and by inductively replacing the value of each sum $1/(1 - x_i)$ starting with the inner parenthesis (2.2) is obtained. ■

► 2.1.3. Formula (Composition of formal series). Let $\boldsymbol{f}(\boldsymbol{y}) \in \mathbb{C}[[\boldsymbol{y}]]$, with $\boldsymbol{y} \in \mathbb{C}^n$, be a formal series, and $\boldsymbol{y}(x) \in \mathbb{C}^n[[x]]$, with $x \in \mathbb{C}$, another series. Say each is given by

$$f(\boldsymbol{y}) = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \boldsymbol{y}^{\boldsymbol{l}}, \qquad\qquad \boldsymbol{y}(x) = \sum_{q} \boldsymbol{c}_q x^q,$$

where $a_{\boldsymbol{l}} \in \mathbb{C}$ and $\boldsymbol{c}_q \in \mathbb{C}^n$. Then the composition of series $(\boldsymbol{f} \circ \boldsymbol{y})(x)$ can be expressed as

$$f(\boldsymbol{y}) = \sum_{q} \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \sum_{q_1 + \ldots + q_n = q} \left(\sum_{q_1} d_{q_1}^{(1)} \right) \ldots \left(\sum_{q_n} d_{q_n}^{(n)} \right) x^q, \tag{2.3}$$

where the coefficients are

$$d_0^{(j)} = c_{j,0}^{l_j}, \qquad d_q^{(j)} = \frac{1}{q \cdot c_{j,0}} \sum_{p=0}^{q} (p \cdot l_j - q + p)\ c_{j,p}^{l_j}\ d_{q-p}^{(j)}. \tag{2.4}$$

Proof: Directly replacing directly one series into the other gives

$$f(\boldsymbol{y}) = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \boldsymbol{y}^{\boldsymbol{l}} = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}}\ y_1^{l_1} ...\ y_n^{l_n} = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \left(\sum_q \boldsymbol{c}_{1,q} x^q \right)^{l_1} ... \left(\sum_q \boldsymbol{c}_{n,q} x^q \right)^{l_n}.$$

Using the formula for the powers of series with one indeterminate for each power gives

$$f(\boldsymbol{y}) = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \left(\sum_{q_1} d_{q_1}^{(1)} x^{q_1} \right) ... \left(\sum_{q_n} d_{q_n}^{(1)} x^{q_n} \right),$$

where each power would have coefficients

$$d_0^{(j)} = c_{j,0}^{l_j}, \qquad d_q^{(j)} = \frac{1}{q \cdot c_{j,0}} \sum_{p=0}^{q} (p \cdot l_j - q + p)\ c_{j,p}^{l_j}\ d_{q-p}^{(j)}.$$

Thus summing all possible combinations of $q_1, ..., q_n$ yield

$$f(\boldsymbol{y}) = \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \sum_q \sum_{q_1 + ... + q_n = q} \left(\sum_{q_1} d_{q_1}^{(1)} \right) ... \left(\sum_{q_n} d_{q_n}^{(1)} \right) x^q. \qquad \blacksquare$$

► 2.1.4. Lemma (Cauchy Estimates). Let $\boldsymbol{f}(\boldsymbol{x}) \in C^w(\boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{\xi}))$ be an analytic function defined on $\boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{\xi}) = D_{r_1}(\xi_1) \times ... \times D_{r_n}(\xi_n) \subseteq \mathbb{C}^n$, an open poly-disk. Suppose $\boldsymbol{f}(\boldsymbol{x})$ is bounded in the poly-disk by a real positive constant M, that is $\sup_{\boldsymbol{x} \in \boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{\xi})} |\boldsymbol{f}(\boldsymbol{x})| < M$. Then all the coefficients of the corresponding Taylor series of $\boldsymbol{f}(\boldsymbol{x})$ in a neighbourhood of $\boldsymbol{\xi}$, say it is

$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} (\boldsymbol{x} - \boldsymbol{\xi})^{\boldsymbol{l}},$$

are bounded by

$$|\boldsymbol{a}_{\boldsymbol{l}}| \leq \frac{M}{r_1^{l_1} ...\ r_n^{l_n}}. \tag{2.5}$$

Proof: From expression (A.5) in the appendix, the coefficients of the Taylor series are given by

$$\boldsymbol{a}_{\boldsymbol{l}} = \frac{1}{l_1! \ ...\ l_n!}\ \partial_{\boldsymbol{x}}^{\boldsymbol{l}} \boldsymbol{f}(\boldsymbol{\xi}). \tag{2.6}$$

The major theorem of complex analysis asserts that holomorphic functions are analytic and vice-versa. Hence $\boldsymbol{f}(\boldsymbol{x})$ is holomorphic and it is possible to consider the Cauchy integral formula in several variables for a smaller concentric poly-disk $\boldsymbol{D}_{\boldsymbol{\rho}}(\boldsymbol{\xi}) = D_{\rho_1}(\xi_1) \times ... \times D_{\rho_n}(\xi_1)$, where $\rho_i < r_i$. Therefore $\forall x \in \boldsymbol{D}_{\boldsymbol{\rho}}(\boldsymbol{\xi})$,

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{1}{(2\pi i)^n} \oint_{\partial D_{\rho_1}(\xi_1)} \cdots \oint_{\partial D_{\rho n}(\xi_n)} \frac{\boldsymbol{f}(z_1, ..., z_n)}{(z_1 - x_1)...(z_n - x_n)}\ \mathrm{d}z_1 ... \mathrm{d}z_n.$$

Differentiating the previous formula with respect to $\partial_{\boldsymbol{x}}^{\boldsymbol{l}}$, gives the known as Cauchy differentiation formula, for the case of several variables and partial derivatives,

$$\begin{aligned}\partial_{\boldsymbol{x}}^{\boldsymbol{l}}\boldsymbol{f}(\boldsymbol{x}) &= \frac{1}{(2\pi i)^n}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho n}(\xi_n)}\boldsymbol{f}(z_1,...,z_n)\ \left[\partial_{x_1}^{l_1}\left(\frac{1}{z_1-x_1}\right)...\,\partial_{x_n}^{l_n}\left(\frac{1}{z_n-x_n}\right)\right]\mathrm{d}z_1...\mathrm{d}z_n\\ &= \frac{1}{(2\pi i)^n}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}\frac{l_1!\,...\,l_n!\,\boldsymbol{f}(z_1,...,z_n)}{(z_1-x_1)^{l_1+1}...(z_n-x_n)^{l_n+1}}\,\mathrm{d}z_1...\mathrm{d}z_n.\end{aligned} \tag{2.7}$$

Comparing this last formula (2.7) with (2.6), the coefficients of the Taylor series can be computed as

$$\boldsymbol{a}_{\boldsymbol{l}} = \frac{1}{(2\pi i)^n}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}\frac{\boldsymbol{f}(z_1,...,z_n)}{(z_1-\xi_1)^{l_1+1}...(z_n-\xi_n)^{l_n+1}}\,\mathrm{d}z_1...\mathrm{d}z_n.$$

Taking modulus in the previous expression for the coefficients, using the triangle inequality (1), that $|z_i-\xi_i|=\rho_i$ (2) and $\sup_{\boldsymbol{x}\in\boldsymbol{D_r}(\boldsymbol{\xi})}|\boldsymbol{f}(\boldsymbol{x})|<M$ (3), it follows that

$$\begin{aligned}|\boldsymbol{a}_{\boldsymbol{l}}| &= \left|\frac{1}{(2\pi i)^n}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}\frac{\boldsymbol{f}(z_1,...,z_n)}{(z_1-\xi_1)^{l_1+1}...(z_n-\xi_n)^{l_n+1}}\,\mathrm{d}z_1...\mathrm{d}z_n\right|\\ &\overset{(1)}{\leq}\left|\frac{1}{(2\pi i)^n}\right|\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}\frac{|\boldsymbol{f}(z_1,...,z_n)|}{|z_1-\xi_1|^{l_1+1}...|z_n-\xi_n|^{l_n+1}}\,\mathrm{d}z_1...\mathrm{d}z_n\\ &\overset{(2)}{\leq}\frac{1}{(2\pi)^n\ \rho_1^{l_1+1}...\rho_n^{l_n+1}}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}|\boldsymbol{f}(z_1,...,z_n)|\,\mathrm{d}z_1...\mathrm{d}z_n\\ &\overset{(3)}{\leq}\frac{M}{(2\pi)^n\ \rho_1^{l_1+1}...\rho_n^{l_n+1}}\oint_{\partial D_{\rho_1}(\xi_1)}\cdots\oint_{\partial D_{\rho_n}(\xi_n)}\mathrm{d}z_1...\mathrm{d}z_n\\ &= \frac{M\ (2\pi\rho_1)...(2\pi\rho_n)}{(2\pi)^n\ \rho_1^{l_1+1}...\rho_n^{l_n+1}} = \frac{M}{\rho_1^{l_1}...\rho_n^{l_n}}.\end{aligned}$$

The previous inequality is valid for $0<\rho_i<r_i$. Considering the limit $\rho_i\to r_i$ yields (2.5). ■

► 2.1.5. Lemma. Let $\mathrm{d}\boldsymbol{y}/\mathrm{d}x=\boldsymbol{f}(\boldsymbol{y})$ and $\mathrm{d}\boldsymbol{y}/\mathrm{d}x=\boldsymbol{g}(\boldsymbol{y})$ be two initial value problems, with same initial conditions $\boldsymbol{y}(0)=\boldsymbol{0}$. Suppose additionally that $\boldsymbol{f}(\boldsymbol{y}),\boldsymbol{g}(\boldsymbol{y})\in C^\omega(\mathcal{U})$ are analytic in a open domain $\mathcal{U}\subseteq\mathbb{C}^n$ and the Taylor series of $\boldsymbol{g}$ have real positive coefficients. If the corresponding Taylor series of the vector fields majorate, this is $\boldsymbol{f}\prec\boldsymbol{g}$, then the formal solutions of the systems (i.e. solutions as a formal series convergent or not) also majorate, $\boldsymbol{y}_f\prec\boldsymbol{y}_g$. Moreover, if $\boldsymbol{y}_g$ converges in some domain $\mathcal{V}\subseteq\mathbb{C}^n$, then $\boldsymbol{y}_f$ converges too in $\mathcal{V}$.

Proof: First consider the Taylor series expansion of both vector fields in $\mathcal{U}$,

$$\boldsymbol{f}(\boldsymbol{y})=\sum_{\boldsymbol{l}}\boldsymbol{a}_{\boldsymbol{l}}\boldsymbol{y}^{\boldsymbol{l}},\qquad \boldsymbol{g}(\boldsymbol{y})=\sum_{\boldsymbol{l}}\boldsymbol{b}_{\boldsymbol{l}}\boldsymbol{y}^{\boldsymbol{l}}. \tag{2.8}$$

where $b_{k,\boldsymbol{l}}\in\mathbb{R}_+$ since it is a majorant series. Then generate two general expressions for formal series,

$$\boldsymbol{y}_f(x)=\sum_q\boldsymbol{\alpha}_q x^q,\qquad \boldsymbol{y}_g(x)=\sum_q\boldsymbol{\beta}_q x^q. \tag{2.9}$$

and compute their derivatives,

$$\boldsymbol{y}'_f(x) = \sum_q (q+1)\boldsymbol{\alpha}_{q+1}x^q, \qquad \boldsymbol{y}'_g(x) = \sum_q (q+1)\boldsymbol{\beta}_{q+1}x^q. \tag{2.10}$$

The point is to make these two series $\boldsymbol{y}_f$ and $\boldsymbol{y}_g$ formal solutions of the first and second system receptively. If the series are to be solutions, then each must satisfy that $\mathrm{d}\boldsymbol{y}_f/\mathrm{d}x = \boldsymbol{f}(\boldsymbol{y}_f)$ and $\mathrm{d}\boldsymbol{y}_g/\mathrm{d}x = \boldsymbol{g}(\boldsymbol{y}_g)$. This way it is possible to determine the relation between coefficients.

The right hand side of the systems are the composition of the vector fields Taylor series (2.8) with the formal solutions (2.9), which is a composition of series. Thus using formula (2.3),

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{y}_{\boldsymbol{f}}(x)) &= \sum_q \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} c^{(1)}_{q_1}\right) \ldots \left(\sum_{q_n} c^{(n)}_{q_n}\right) x^q, \\
\boldsymbol{g}(\boldsymbol{y}_{\boldsymbol{g}}(x)) &= \sum_q \sum_{\boldsymbol{l}} \boldsymbol{b}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} d^{(1)}_{q_1}\right) \ldots \left(\sum_{q_n} d^{(n)}_{q_n}\right) x^q,
\end{aligned}$$

where the coefficients $d^{(j)}$ and $e^{(j)}$ are computed according to (2.4). In the left hand side of the systems there are the derivatives (2.11). Equalizing each side of the systems gives an equality of series. Considering that the terms of the same order q in x in either side have to be equal, the following recursive formula for the coefficients is obtained

$$\begin{aligned}
\boldsymbol{\alpha}_{q+1} &= \frac{1}{q+1} \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} c^{(1)}_{q_1}\right) \ldots \left(\sum_{q_n} c^{(n)}_{q_n}\right), \\
\boldsymbol{\beta}_{q+1} &= \frac{1}{q+1} \sum_{\boldsymbol{l}} \boldsymbol{b}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} d^{(1)}_{q_1}\right) \ldots \left(\sum_{q_n} d^{(n)}_{q_n}\right),
\end{aligned} \tag{2.11}$$

where the first terms have to be $\boldsymbol{\alpha}_0 = 0$ and $\boldsymbol{\beta}_0 = 0$, so that the formal solutions satisfy the initial condition. This assembly to find the formal solutions and recursive the formula it leads to, is of utmost importance and basic in the proof of the main theorem.

(*) A consequence of (2.11) for the second expression where $\boldsymbol{b}_{k,\boldsymbol{l}}$ were real positive, is that each coefficient $\boldsymbol{\beta}_{k,q}$ is real positive as well. Using induction it is easy to check. For the initial case, $\boldsymbol{\beta}_0 = \boldsymbol{0}$, hence real positive. Then as from (2.10), $\boldsymbol{\beta}_{q+1}$ is computed as sums and products of the previous coefficients, which are all real positive by induction hypothesis, and the terms $\boldsymbol{b}_{\boldsymbol{l}}$, which are also real positive by definition, it implies that the coefficient is real positive.

Now, applying induction in q it will be proven that $|\boldsymbol{\alpha}_{k,q}| < \boldsymbol{\beta}_{k,q}$. The first case $\boldsymbol{\alpha}_0 = \boldsymbol{\beta}_0 = \boldsymbol{0}$ is trivial. Taking modulo on the first equation in (2.11), using the triangular inequality (2), the induction hypothesis and the definition of majorat $|\boldsymbol{a}_{k,\boldsymbol{l}}| < \boldsymbol{b}_{k,\boldsymbol{l}}$ (3),

$$
\begin{aligned}
|\boldsymbol{\alpha}_{k,q}| &= \left| \frac{1}{q+1} \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} c_{q_1}^{(1)} \right) \ldots \left(\sum_{q_n} c_{q_n}^{(n)} \right) \right| \\
&\overset{(2)}{\leq} \frac{1}{q+1} \sum_{\boldsymbol{l}} |\boldsymbol{a}_{\boldsymbol{l}}| \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} |c_{q_1}^{(1)}| \right) \ldots \left(\sum_{q_n} |c_{q_n}^{(n)}| \right) \\
&\overset{(3)}{\leq} \frac{1}{q+1} \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} |e_{q_1}^{(1)}| \right) \ldots \left(\sum_{q_n} |e_{q_n}^{(n)}| \right) \\
&\overset{(1)}{=} \frac{1}{q+1} \sum_{\boldsymbol{l}} \boldsymbol{a}_{\boldsymbol{l}} \sum_{q_1+\ldots+q_n=q} \left(\sum_{q_1} e_{q_1}^{(1)} \right) \ldots \left(\sum_{q_n} e_{q_n}^{(n)} \right) = \boldsymbol{\beta}_{k,q}.
\end{aligned}
$$

Thus, we have actually proven that the definition of majorant holds, $\boldsymbol{y}_f \prec \boldsymbol{y}_g$. Finally, suppose $\boldsymbol{y}_g$ converges in $\mathcal{V}$, then applying the comparison test for convergence of series it follows that $\boldsymbol{y}_f$ also converges in $\mathcal{V}$, which concludes the last part. ■

▶ 2.1.6. Local Cauchy Theorem. Let $\mathcal{U} \subseteq \mathbb{C}^n$ be an open subset and $\boldsymbol{f} : \mathcal{U} \to \mathbb{C}^n$ be a function. Consider the first order autonomous ODE system in the variable $\boldsymbol{y} \in \mathcal{U}$, given by

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} = \boldsymbol{f}(\boldsymbol{y}), \qquad x \in \mathbb{C}.$$

Given a poly-disk $\boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{y}_{\boldsymbol{0}}) \subseteq \mathcal{U}$, suppose the vector field is analytic, $\boldsymbol{f}(\boldsymbol{y}) \in C^{\omega}(\boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{\xi}))$, and is also bounded*, $\sup_{\boldsymbol{y} \in \boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{y}_{\boldsymbol{0}})} |\boldsymbol{f}(\boldsymbol{y})| < M$, in that poly-disk. Then, given initial conditions $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$, there exists a unique solution to the initial value problem and such solution is analytical on a disk $D_{\rho}(x_0)$, where the radius is

$$\rho = \frac{\min(r_i)}{(n+1)} M.$$

Proof: Applying the following changes of variables

$$y_k \longrightarrow y_{0,k} + r_k y_k,$$

$$\boldsymbol{f} \longrightarrow M\boldsymbol{f},$$

$$x \longrightarrow x_0 + M^{-1}(\min r_i)x,$$

the parameters in the theorem are set to $\boldsymbol{y}_{\boldsymbol{0}} = 0$, $\boldsymbol{r} = \boldsymbol{1}$, $M = 1$ and $x_0 = 0$. Therefore, the proof is completed by finding an ODE system whose vector field majorates $\boldsymbol{f}(\boldsymbol{x})$ and has analytic solutions that can be solved explicitly, for these particular parameters, $\boldsymbol{D}_{\boldsymbol{1}}(\boldsymbol{0})$ and $\boldsymbol{y}(0) = \boldsymbol{0}$. Then Lemma 2.1.5 concludes the demonstration.

* This assumption is made for commodity but is actually superfluous. If $\boldsymbol{f}$ is globally analytic in larger domain containing the poly-disk then $\boldsymbol{f}$ is bounded in $\mathrm{cl}(\boldsymbol{D}_{\boldsymbol{r}}(\boldsymbol{\xi}))$ which is a compact. In case such larger domain does not exist we would just take a slightly smaller poly-disk and apply the same reasoning.

Consider the ODE system $d\boldsymbol{y}/dx = \boldsymbol{g}(\boldsymbol{y})$ with vector field $\boldsymbol{g}(\boldsymbol{y})$ defined on $\boldsymbol{D_1}(\boldsymbol{0})$, which consists of n copies of (2.2) from formula 2.1.2, and initial conditions $\boldsymbol{y}(0) = \boldsymbol{0}$, namely

$$y_k' = \prod_k^n (1 - y_k)^{-1}, \qquad y_k(0) = 0, \qquad 1 \leq k \leq n. \tag{2.12}$$

In formula 2.1.2 it was shown that this vector field is analytic in $\boldsymbol{D_1}(\boldsymbol{0})$, actually its Taylor series are n copies of (2.1). According to Lemma 2.1.4, $\boldsymbol{a_l}$, the coefficients of the Taylor series of $\boldsymbol{f}(\boldsymbol{y})$, defined on $\boldsymbol{D_1}(\boldsymbol{0})$, are bounded by

$$|\boldsymbol{a}_{k,\boldsymbol{l}}| \leq 1, \qquad 1 \leq k \leq n.$$

Since coefficients of the Taylor series of $\boldsymbol{g}(\boldsymbol{y})$ are all 1, it follows that $\boldsymbol{f} \prec \boldsymbol{g}$. Now, it only remains to solve explicitly the initial value problem for (2.12) and find the radius of convergence for the solution.

All equations of (2.12) are decoupled and the same, which means that all component of the solutions y_k are the same, hence the initial value problem is analogous to

$$y_k' = (1 - y_k)^{-n}, \qquad y_k(0) = 0, \qquad 1 \leq k \leq n.$$

The previous initial value problem is solvable explicitly by separation of variables, and has solution

$$\int_0^{y_k} (1 - y_k)^n \mathrm{d}y_k = \int_0^x \mathrm{d}x \qquad \Rightarrow \qquad y_k = 1 - [1 - (n+1)x]^{\frac{1}{n+1}}. \tag{2.13}$$

At $x_0 = 0$, the derivatives of each component of the (2.13) are

$$\begin{aligned}
y_k(0) &= 1 - [1 - (n+1)x]^{\frac{1}{n+1}}\Big|_{x=0} = 0, \\
y_k'(0) &= [1 - (n+1)x]^{\frac{1}{n+1}-1}\Big|_{x=0} = 1 \\
y_k''(0) &= [(n+1) - 1] \cdot [1 - (n+1)x]^{\frac{1}{n+1}-2}\Big|_{x=0} = [(n+1) - 1] \\
y_k'''(0) &= [(n+1) - 1][2(n+1) - 1] \cdot [1 - (n+1)x]^{\frac{1}{n+1}-3}\Big|_{x=0} = [(n+1) - 1][2(n+1) - 1] \\
&\ldots \\
y_k^{(p)}(0) &= \prod_{j=1}^{p-1} (j[n+1] - 1) \\
&\ldots
\end{aligned}$$

Then, the coefficients of the Taylor series of (2.13), in a neighbourhood of $x_0 = 0$, are n copies of the previous derivatives divided by the corresponding factorial. Hence, the radius of convergence of each each component of the Taylor series of (2.13) is $\rho = 1/(n+1)$.

$$\rho^{-1} = \overline{\lim}_{p\to\infty} \left(\frac{1}{p!} \prod_{j=1}^{p-1} (j[n+1] - 1) \right)^{\frac{1}{p}} = \overline{\lim}_{p\to\infty} \left(\frac{1}{p!} \prod_{j=1}^{p-1} j[n+1] \right)^{\frac{1}{p}} = \overline{\lim}_{p\to\infty} \left(\frac{1}{p!} p![n+1]^p) \right)^{\frac{1}{p}}.$$

Calling upon Lemma 2.1.6 the formal solution (as a formal series) of the original initial value problem converges to a solution on $D_{\rho}(0)$. This proves the existence of an analytic solution. By undoing the change or variables the ρ in the theorem is obtained.

Uniqueness of this solution is a consequence of the recursive formula (2.10), which give a unique way of computing the coefficients, and the Taylor theorem, which states that an analytic function is univocally determined by its Taylor series. ■

► 2.1.7. Definition (Maximal Solutions). It has already been proven that there exist a unique analytical solution to an initial value problem of a system with analytical vector field. At this point we consider our variables to be in $\mathbb{R}$. In order to extend these results the following definition is required.

A solution of an initial value problem $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a *maximal solution* if for any other solution $\psi : J \to \mathbb{R}^n$ such that $I \subseteq J$ and $\psi|_I = \varphi$, then $I = J$ and consequently $\varphi = \psi$.

► 2.1.8. Global Cauchy Theorem. Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ analytic, $\boldsymbol{f} \in C^{\omega}(\Omega)$. Consider the first order autonomous ODE system in the variable $\boldsymbol{y} \in \mathbb{C}^n$, given by

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} = \boldsymbol{f}(\boldsymbol{y}), \qquad x \in \mathbb{R}.$$

Then for a given initial value problem $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$, there exists a unique maximal solution which is analytic in Ω.

Proof: The proof of this theorem is threefold. To prove the existence of a maximal solution consider the union of all the solutions for that same initial value problem. Since locally there exist local solutions this union is not the empty set and the for resulting curve Definition 2.1.7 holds.

Uniqueness and analyticity is proven in the same way. Suppose there is a time x_1 for which the solution splits into other solutions (or in which the solution is no longer analytic), then applying the local theorem at that point we reach a contradiction. ■

2.2. Volume Preserving Flow. Roughly speaking a system is volume preserving if any volume under the action of the flow remains constant. Note in many articles, such as [7], the flow is treated as family of operators $\{\varphi_x : \Omega \to \Omega \mid x \in \mathbb{R}\}$ mapping points of the phase space into itself. In this section we are not taking this approach. However, this point of view of the flow is quite convenient since most of the theory regarding measure preserving applications is focused on maps. The subject that study these measure preserving applications is ergodic theory.

Volume preserving flows are of significant important in physics mainly in modelling of incompressible fluid dynamics.

Most of the discussion in this section is aimed to prove the Liouville theorem which establishes the relation between measure preserving flows and divergence free vector fields. It is this theorem which allow us to assert that the Michelson system, which has $\boldsymbol{div f} = 0$, preserves volume.

Regarding applications we will only be commenting on the following simple result of systems bearing this property. The result is that the sum of the eigenvalues of the linearised system is zero, which is quite trivial (see the last part of Definition 2.2.1). A consequence of this fact the fixed points, whose stability is studies linearising the system, can not be neither completely attracting nor completely repelling and the attracting and repelling directions must compensate. This also imply that, since the stability of periodic orbits is given by the stability of their corresponding fixed points in the Poincaré map, and the Poincaré map of a volume preserving flow is area preserving, then there are no completely attracting or repelling periodic orbits either.

► 2.2.1. Definition. Consider an autonomous ODE system with vector field $\boldsymbol{f} : \Omega \to \mathbb{R}^n$, in which $\Omega \subseteq \mathbb{R}^n$ is the phase space. The flow is *measure preserving* if for any fixed time x, $\varphi_x : \Omega \to \Omega$ where $\varphi_x(\boldsymbol{y}) = \varphi(x, \boldsymbol{y})$ and any given D relative compact in Ω, then $V(D) = V(\varphi_x(D))$, where V is a defined as

$$V(D) = \int_V \mathrm{d}\boldsymbol{y}$$

► 2.2.2. Definition. An autonomous ODE system is said to be *divergence free* if the vector field satisfies $\boldsymbol{div f} = 0$. For any function the definition of gradient implies that $\boldsymbol{div f} = tr(\boldsymbol{D_y f})$, where $tr(\cdot)$ is the trace of a matrix. Consequently since the trace of a matrix is the sum of its eigenvalues, the sum of eigenvalues of the linearisation of a divergence free system is zero.

► 2.2.3. Definition. Given a linear ODE system $\boldsymbol{y}' = A_x \boldsymbol{y}$ in $\mathbb{R}^n$ and a base $\{\boldsymbol{e}_i\}_{1 \leq i \leq n}$ of $\mathbb{R}^n$, the solutions of the system form a vector space and the solutions $\varphi_i(x)$ for initial conditions $\boldsymbol{y}(0) = (\boldsymbol{e}_i)$ for a base of such vector space. (For a proof of this fact see [13]). The *Wronskian* of the linear system is defined as

$$W(x) = \det(\varphi_1(x)|..., |\varphi_n(x)),$$

the determinant of the matrix whose columns a base of solutions of the linear system. [1]

► 2.2.4. Abel's Identity or Liouville's Formula. The Wronskian of linear ODE system with vector field $A(x)$, is a solution of the differential equation

$$\dot{W} = tr(A(x))W, \tag{2.14}$$

where $tr(A(x))$ stands for the trace. Integrating (2.13), $W(x) = \exp\left(\int_{x_0}^{x} tr(A(\hat{x})\mathrm{d}\hat{x}\right) W(x_0)$. [13]

Proof: The proof is obtained deriving with respect to x the definition of Wronskian

$$W(x)' = \frac{\mathrm{d}}{\mathrm{d}x} \det(\varphi_1(x)|...|\varphi_n(x)).$$

Using Leibniz formula for the derivative of a determinant gives

$$W(x)' = \sum_i^n \det(\varphi_1(x)|...|\varphi_i'(x)|...|\varphi_n(x)).$$

Since each $\varphi_i(x)$ are solutions they must satisfy $\varphi_i'(x) = \sum_j^n a_{i,j}(x)\varphi_i(x)$, where $a_{i,j}(x)$ are the coefficients of $A(x)$. Hence

$$W(x)' = \sum_i^n \det(\varphi_1(x)|...|\sum_j^n a_{i,j}(x)\varphi_i(x)|...|\varphi_n(x)) = \sum_i^n a_{i,i}(x)\det(\varphi_1(x)|...|\varphi_i(x)|...|\varphi_n(x)),$$

where this is already (2.14). ■

► 2.2.5. Liouville Theorem. Let $\boldsymbol{f} : \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^n$ and $\boldsymbol{f} \in C^1$, be a vector field of an autonomous ODE system. Say $D \in \Omega$ is a bounded subset, and $V(D)$ denotes its volume. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}V(D_x) = \int_{D_x} \boldsymbol{div}\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{2.15}$$

where, $D_x = \varphi_x(D) = \{\varphi_x(\hat{\boldsymbol{y}}) \mid \hat{\boldsymbol{y}} \in D\}$ is the image of D under the action of a given flow.

Proof. Considering the definition of the the volume and changing variables we obtain

$$V(D_x) = \int_{D_x} \mathrm{d}\boldsymbol{y} = \int_{\varphi_x(D)} \mathrm{d}\boldsymbol{y} = \int_D \left|\det(D_{\boldsymbol{y}}\varphi_x(\hat{\boldsymbol{y}}))\right| \ \mathrm{d}\hat{\boldsymbol{y}}.$$

Note that the term $det(D_{\boldsymbol{y}}\varphi_x(\hat{\boldsymbol{y}}))$ which appear due to the change of variables in nothing else but the Wronskian W of the linearised system $\mathrm{d}\boldsymbol{v}/\mathrm{d}x = D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y})\ \boldsymbol{v}$. Differentiating the previous expression with resp. to x, using Liouville formula (*) and undoing the change of variables (**),

$$\frac{\mathrm{d}}{\mathrm{d}x}V(D_x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_D |W| \ \mathrm{d}\hat{\boldsymbol{y}} = \int_D |\dot{W}| \ \mathrm{d}\hat{\boldsymbol{y}} \overset{*}{=} \int_D |tr(D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y}))| \ |W| \ \mathrm{d}\hat{\boldsymbol{y}} \overset{**}{=} \int_{D_x} |tr(D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y}))| \ \mathrm{d}\boldsymbol{y}. \ ■$$

► 2.2.6. Corollary. A flow is volume preserving if and only if it is divergence free ($\boldsymbol{\nabla}\boldsymbol{f} = 0$). ■

2.3. Time Reversible Flow. First of all, we give the following simple example on what a system with a time reversing symmetry would be. Imagine a film of pendulum swinging and then play the film backward in time. In doing so the motion observed is another plausible movement (another orbit in the phase space) with the only difference that the initial velocity is of opposite sign. Hence the transformation of the momentum $p \to -p$ is a time reversing symmetry for the system. Needless to say the importance of this property in physics where these symmetries arise in many contexts such as classical mechanics, quantum mechanics or thermodynamics. [7]

The following discussion is aimed to introduce the formal definition of time reversing symmetries and symmetric orbits, and some results concerning them. Bear in mind that the development of this theory is taken from the prespective of autonomous systems. Still, these definitions and results also exist for maps and non autonomous systems. For such cases one should refer to [7].

In point 2.3.7 a time reversing symmetry for the Michelson system is given. The fact that the Michelson system has such symmetry is not surprising in itself. It is well known that in many wave propagation PDEs (as our in case), the equations governing the steady state solutions have time reversible symmetries.

For a complete discussion on time reversing symmetries and the definitions for maps and non autonomous systems refert to [7].

► 2.3.1. Definition (Reversing Symmetry). Let $\mathrm{d}\boldsymbol{y}/\mathrm{d}x = \boldsymbol{f}(\boldsymbol{y})$ be an autonomous ODE system, where $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ is the vector field and Ω its phase space. An invertible C^1 map, $R : \Omega \to \Omega$ is a *reversing symmetry* if for any solution of the system $\varphi(x; \boldsymbol{y}_0)$, then $R \circ \varphi(-x; \boldsymbol{y}_0)$ is also a solution. In the particular case $R^2 = Id$ the reversing symmetry is called an *involution*.

Notation: The set of all the fixed points of R is denoted as $Fix(R) = \{\boldsymbol{y} \in \Omega \mid R(\boldsymbol{y}) = \boldsymbol{y}\}$. [7]

► 2.3.2. Definition (R-Symmetric Orbits). Let $o(\boldsymbol{y}) = \{\varphi(x; \boldsymbol{y}) \mid x \in I\}$ be an orbit a certain autonomous ODE system with a reversing symmetry R, where I is the maximal domain in which the solution is defined. Then, the orbit is *R-symmetric* if it is invariant under the action of R, i.e. $R(o(\boldsymbol{y})) = o(\boldsymbol{y})$. [7]

Observations: Note that given an a non R-symmetric orbit, by applying the reversing symmetry we obtain a second orbit with the same behaviour for negative time. This means that finding a non R-symmetric equilibrium point yield a second equilibrium point, finding a non R-symmetric periodic orbit yield a second periodic orbit of the same period, and so on.

An interesting case is when the the R-symmetric orbits are homoclinic or heteroclinic. Recall that these orbits are those which lay in the intersections of the stable and unstable manifolds of hyperbolic fixed points, being homoclinic if the fixed point is the same and heteroclinic otherwise. Formally, given fixed points $\hat{\boldsymbol{y}}_1$ and $\hat{\boldsymbol{y}}_2$, the solutions $\varphi(x; \boldsymbol{y}_0)$ such that satisfy

$$\lim_{x\to\infty} \varphi(x; \boldsymbol{y}_0) = \hat{\boldsymbol{y}}_1, \qquad \lim_{x\to-\infty} \varphi(x; \boldsymbol{y}_0) = \hat{\boldsymbol{y}}_2,$$

are called homoclinic if $\hat{\boldsymbol{y}}_1 = \hat{\boldsymbol{y}}_2$ and heteroclinic if $\hat{\boldsymbol{y}}_1 \neq \hat{\boldsymbol{y}}_2$.

Intuitively, we can assert that the presence of a R-symmetric fixed point imply the possibility of a R-symmetric homoclinic orbit but not a R-symmetric heteroclinic orbit connecting that point to another fixed point. Moreover, the presence of a pair of R-symmetric fixed points imply the possibility of a R-symmetric heteroclinic orbit connecting them but not of R-symmetric homoclinic orbits.

Under some conditions, the presence of R-symmetric homoclinic and heteroclinic orbits imply the existence of certain types of global bifurcations. In fact, this occurs in the case of the Michelson system which actually has R-symmetric heteroclinic orbits for all $c > 0$.

► 2.3.3. Lemma (Equivalent Definition of Revering Symmetry). The previous definition is not practical to check given a system. Under the same assumptions as in definition 2.3.1, an invertible C^1 map $R : \Omega \to \Omega$ is a reversing symmetry if and only if,

$$(D_{\boldsymbol{y}} R)(\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}) = -\boldsymbol{f} \circ R(\boldsymbol{y}).$$

Beware that the left hand side is an inner product while the right-hand side is the composition.

Proof: Assume $\varphi(x; \boldsymbol{y}_0)$ is a solution, then in order to be $R \circ \varphi(-x; \boldsymbol{y}_0)$ a solution it must satisfy the system, namely,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(R \circ \varphi(-x; \boldsymbol{y}_0)\right) = \boldsymbol{f}\left(R \circ \varphi(-x; \boldsymbol{y}_0)\right).$$

Using the chain rule

$$D_{\boldsymbol{y}} R\left(\varphi(-x; \boldsymbol{y}_0)\right) \cdot \left(\frac{\mathrm{d}\varphi(-x; \boldsymbol{y}_0)}{\mathrm{d}(-x)}\right)\left(\frac{\mathrm{d}(-x)}{\mathrm{d}x}\right) = \boldsymbol{f} \circ R(\boldsymbol{\varphi}(-x; \boldsymbol{y}_0)),$$

$$-D_{\boldsymbol{y}} R\left(\varphi(-x; \boldsymbol{y}_0)\right) \cdot \boldsymbol{f}\left(\varphi(-x; \boldsymbol{y}_0)\right) = \boldsymbol{f} \circ R(\boldsymbol{\varphi}(-x; \boldsymbol{y}_0)).$$

Thus we have proven that $(D_{\boldsymbol{y}} R)(\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}) = -\boldsymbol{f} \circ R(\boldsymbol{y})$. ■

► 2.3.4. Theorem (Characterization of R-symmetric orbits). Given an orbit $o(\boldsymbol{y})$ of the flow of an autonomous ODE system with reversing symmetry R. Then, if the orbit is not a fixed point

- The orbit is R-symmetric if and only if $o(\boldsymbol{y})$ intersects Fix(R), in which case the orbit intersects $Fix(R)$ in no more than two points.
- The orbit is an R-symmetric periodic orbit if and only if it intersects $Fix(R)$ in precisely two points.

Observation: This theorem gives a way to find R-symmetric periodic orbits and in particular it will be used to design methods to track R-symmetric periodic orbits.

Proof: First it will be proven that an orbit is R-symmetric if an only if it intersects $Fix(R)$.

Assume that $o(\boldsymbol{y}_0)$ intersects $Fix(R)$ in $\boldsymbol{y}_0$ (since the system is autonomous we can move the initial condition so that it coincides with the intersection), then the orbit is R-symmetric. This comes from the fact that $R(o(\boldsymbol{y}_0)) = \{\varphi(x; R(\boldsymbol{y}_0)) \mid x \in I \subseteq \mathbb{R}\}$, and since $R(\boldsymbol{y}_0) = \boldsymbol{y}_0$ as it is a point of $Fix(R)$, the definition of R-symmetry holds.

Conversely suppose the orbit $o(\boldsymbol{y}_0)$ is R-symmetric, then $o(\boldsymbol{y}_0) \cap Fix(R) \neq \emptyset$. This is because the solutions are the same under the symmetry, thus $\varphi(x; \boldsymbol{y}_0) = \varphi(\tau - x; R(\boldsymbol{y}_0))$ this solutions are the same except for the different parametrizations of time x. Then in the time $\tau/2$ lays a point invariant under R.

For the next part we will use that the solutions of autonomous ODE systems, are either fixed points, close orbits, or have no self intersections. [1]

To prove that the R-symmetric orbits only intersect at maximum twice, suppose that a given R-symmetric orbit, $\varphi(x; \boldsymbol{y}_0)$ with $x \in I$, intersect $Fix(R)$ in minimum three points $\boldsymbol{y}_0$, $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ in times 0, x_1, x_2 respectively, with $x_1, x_2 > 0$. Applying the symmetry R to the solution φ and the fact that it is R-symmetric gives that the solution also intersect $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ in $-x_1$ and $-x_2$. This imply that the solution that has self intersections, which is a not valid in autonomous systems, thus yields contradiction.

Proving, that if the R-symmetric orbit intersects twice $Fix(R)$ then the orbit is periodic, is done in a similar way. We should assume that the orbit intersects in $Fix(R)$ in time 0 and x_1, and use the symmetry to see that it also intersects in time $-x_1$. Therefore the only acceptable behaviour for an autonomous system is if the orbit is periodic. ■

► 2.3.5. A Reversing Symmetry for the Michelson System. The following C^1 an invertible linear map $R : \mathbb{R}^3 \to \mathbb{R}^3$, defined as

$$(y_1, y_2, y_3) \longrightarrow (-y_1, y_2, -y_3),$$

is a time reversing symmetry for the Michelson system. In particular, R is an involution and its invariant set $Fix(R)$ is the y_2 axis.

Verification: The analogous definition in lemma 2.3.3. holds, hence it is a reversing symmetry.

$$(D_{\boldsymbol{y}} R) \cdot \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \\ c^2 - y_2 - \dfrac{y_1^2}{2} \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_3 \\ -c^2 + y_2 + \dfrac{y_1^2}{2} \end{pmatrix},$$

$$-\boldsymbol{f} \circ R(\boldsymbol{y}) = -\boldsymbol{f} \begin{pmatrix} -y_1 \\ y_2 \\ -y_3 \end{pmatrix} = - \begin{pmatrix} y_2 \\ -y_3 \\ c^2 - y_2 - \dfrac{(-y_1)^2}{2} \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_3 \\ -c^2 + y_2 + \dfrac{y_1^2}{2} \end{pmatrix}. \qquad \blacksquare$$

3. DYNAMICS OF THE MICHELSON SYSTEM

In this section we will put aside the general properties and focus on giving proofs concerning only the dynamics of the Michelson system. In general, dynamical system as a subject makes a high level use of other branches of mathematics such as algebra, topology, number theory or probability to give insight of the systems in question. The approach taken in this work is from an undergraduate standpoint and perhaps the results lacks the relevance of other more cutting edge results. Nonetheless the analysis performed give a good small sample of the kind of demonstrations preformed in dynamical systems.

There are three distinct topics which comprise section. The first one is the local analysis of equilibrium points, in which we include the study of the stability, local bifurcations with respect to the parameter c^2, and the invariant manifolds. In the second topic, we will be proving by means of index theory that the fixed points are not isolated for $c^2 > 0$, which imply that there is always at least one heteroclinic or homoclinic orbit. Finally, there is a third topic where employing a perturbation theorem it is shown that for small values of c^2 there exists a 2π-periodic orbit.

Recall that the Michelson system is given by

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = c^2 - y_2 - \dfrac{y_1^2}{2} \end{cases}$$

where $y_i' = \mathrm{d}y_i/\mathrm{d}x$ and the parameter $c \geq 0$.

3.1. Equilibrium Points Analysis. As stated before the point of this subsection is to study the equilibrium points of the Michelson system. Equilibrium points (often called critical or fixed points) are points of the phase space which are invariant under the flow. This imply that the phase velocity at such points is zero and therefore equilibrium points are those for which $\boldsymbol{f}(\boldsymbol{y}) = \mathbf{0}$. Hence, in the case of the Michelson system the equilibrium points are the solutions of

$$\begin{cases} y_2 = 0, \\ y_3 = 0, \\ c^2 - y_2 - \dfrac{y_1^2}{2} = 0. \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 &= 0, \\ y_2 &= 0, \\ y_0 &= \pm\sqrt{2}c. \end{cases}$$

Depending on the value of the parameter c we may have one or two equilibrium points. For $c \neq 0$ there are two such points which we shall denote by $P_+ = (\sqrt{2}c, 0, 0)$ and $P_+ = (-\sqrt{2}c, 0, 0)$. On the other hand, when $c = 0$ (the degenerate case) there is only one equilibrium point at the origin which we shall denote by $P_0 = (0, 0, 0)$.

See that both P_+ and P_- are related by the symmetry of the Michelson system mentioned in point 2.3.5. and the degenerate case P_0 is an R-symmetric fixed point.

Note as well that for $c < 0$, the dynamics and the equilibrium points are essentially the same.

► 3.1.1. Stability of Equilibrium Points. The first approach to study the nature of an equilibrium point can be determined by considering the linearised system at such point. The linearisation may give a local approximation of the dynamics of the system in a neighbourhood of the equilibrium point. This comes from the fact that expanding the vectorfield in a neighbourhood of an equilibrium point $\hat{\boldsymbol{y}}$ gives

$$\dot{\boldsymbol{y}} = \cancel{\boldsymbol{f}(\hat{\boldsymbol{y}})} + \boldsymbol{D}_{\boldsymbol{y}}\boldsymbol{f}(\hat{\boldsymbol{y}})(\boldsymbol{y} - \hat{\boldsymbol{y}}) + \cancel{O(2)}.$$

To keep for the moment things as general as possible we will denote by $\hat{\boldsymbol{y}} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ any equilibrium point of the Michelson system. Thus, to obtain the linearised system we are required to compute the differential $\boldsymbol{D}_{\boldsymbol{y}}\boldsymbol{f}(\hat{\boldsymbol{y}})$ which is the Jacobian matrix of the vector field, hence from now on $J(\hat{\boldsymbol{y}})$ for simplicity,

$$J(\boldsymbol{y}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\hat{y}_1 & -1 & 0 \end{pmatrix}.$$

The stability of the linearised system is found by studying the eigenvalues of the matrix vector field, i.e. characteristic equation of the Jacobian system at $\hat{\boldsymbol{y}}$,

$$det\,(J(\hat{\boldsymbol{y}}) - zId) = \begin{vmatrix} -z & 1 & 0 \\ 0 & -z & 1 \\ -\hat{y}_1 & -1 & -z \end{vmatrix} = 0 \qquad \Leftrightarrow \qquad z^3 + z + \hat{y}_1 = 0. \tag{3.1}$$

To discuss the form of the roots of the characteristic equation we will use the concepts explained in Appendix B.

Since the discriminant of (3.1) is $\Delta = -4 - 27\hat{y}_1^2$, it follows that Δ is always strictly negative and (3.1) always has a real root, and two complex conjugated roots for any equilibrium point. Say for a given equilibrium point $\hat{\boldsymbol{y}}$ these roots are

$$\lambda, \qquad \mu = a + ib, \qquad \nu = a - ib, \qquad \text{with } \lambda, a, b \in \mathbb{R}.$$

Using Vieta's formula and imposing that the roots have the previous form, it is possible to express the two complex roots in terms of the real one,

$$\begin{cases} \lambda + \mu + \nu = 0 & \Rightarrow \quad \lambda + 2a = 0 & \Rightarrow \quad a = -\dfrac{\lambda}{2}, \\ \lambda\mu + \mu\nu + \nu\lambda = 1 & \Rightarrow \quad 2\lambda a + a^2 + b^2 = 1 & \Rightarrow \quad b = \sqrt{1 + \dfrac{3\lambda^2}{4}}, \\ \lambda\mu\nu = -\hat{y}_1 & \Rightarrow \quad \lambda(a^2 + b^2) = \hat{y}_1 & \Rightarrow \quad \lambda^3 + \lambda + \hat{y}_1 = 0. \end{cases}$$

Obviously the third equation is superfluous since λ is a root of (3.1). Nevertheless, from the other two we may write the roots of (3.1) as

$$\lambda, \qquad \mu = -\frac{\lambda}{2} + i\sqrt{1 + \frac{3\lambda^2}{4}}, \qquad \nu = -\frac{\lambda}{2} - i\sqrt{1 + \frac{3\lambda^2}{4}}. \tag{3.2}$$

Recall that an equilibrium point is hyperbolic if non of the eigenvalues of the linearised system have real part zero, and elliptic or parabolic otherwise. From (3.2), it follows the only non hyperbolic equilibrium point of the Michelson system is P_0 when $c = 0$ and in such case the eigenvalues are 0 which yield a parabolic part and , $+i, -i$ which yield an elliptic part. Conversely, for $c \neq 0$ the equilibrium points P_+ and P_- are hyperbolic. At this point we will study both cases separately.

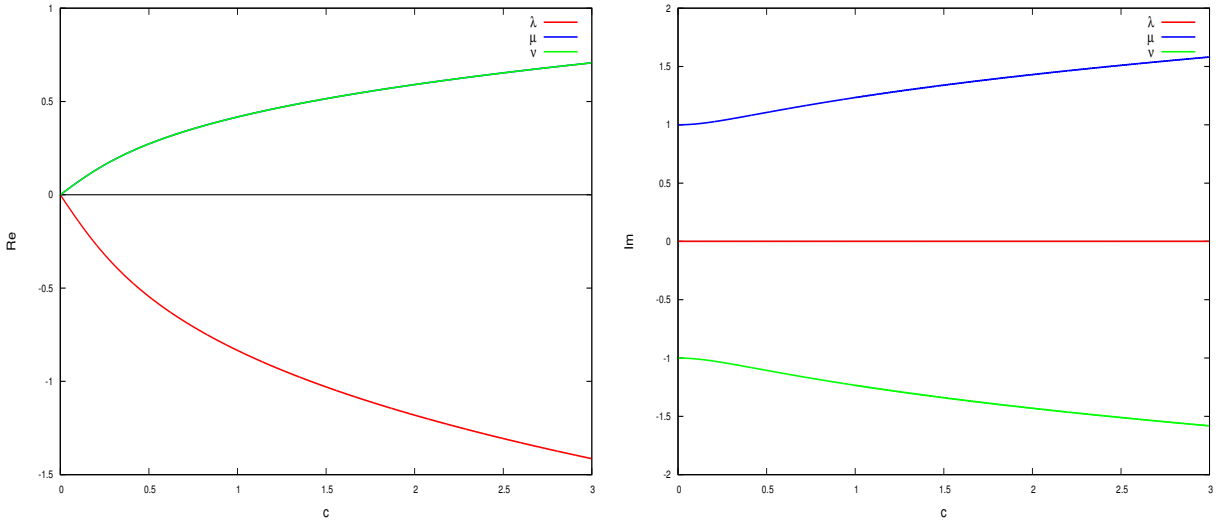


Figure 1: Diagram showing the possible real an imaginary parts of the eigenvalues with respect to the parameter c computed using a Newto method in (3.1) and using (3.2).

In the case of the hyperbolic equilibrium points, the Harman-Grobman theorem states that the linearised system give an really good approximation of the stability of such points. This means that the stability of the linearised system is the same as the original in a neighbourhood of the equilibrium point. More precisely, what the theorem says is that in a neighbourhood of an hyperbolic equilibrium point the flow can be seen as an slightly deformed version of flow for the linearised system.

Next we state the theorem which will not be proved. This theorem and its proof can be found in [10] in section 9.3.

► 3.1.2. Hartman-Grobman Theorem. Suppose $\boldsymbol{f} \in C^1$ is a vector field of a continuous ODE system that has an equilibrium point in $\mathbf{0}$. Denote by $\varphi(x, \boldsymbol{y})$ the corresponding flow and J the Jacobian of the matrix of the vector field at $\mathbf{0}$. Then there is an homeomorphism $h(\boldsymbol{y}) = \boldsymbol{y} + g(\boldsymbol{y})$ with g bounded such that the flow is topologically conjugated to the linearised system

$$h \circ e^{xJ} = \varphi \circ h,$$

in a sufficiently small neighbourhood of $\mathbf{0}$.

► 3.1.3. Stability of Hyperbolic Equilibrium Points: This case correspond to values of $c > 0$. We have already seen that in this case there are two equilibrium points given by $P_+ = (\sqrt{2}c, 0, 0)$ and $P_- = (-\sqrt{2}c, 0, 0)$. Using the Hartman-Grobman we will determine the stability of the equilibrium points by determining the stability of the linearised system.

From now on we shall denote by λ_+, μ_+, ν_+ and λ_-, μ_-, ν_- the set of eigenvalue of P_+ and P_- respectively.

For P_+, replacing the real eigenvalue λ_+ into (3.1) gives $\lambda_+^3 + \lambda_+ = -\sqrt{2}c$, thus since $c > 0$ yields $\lambda_+ < 0$. Similarly, for P_- we have $\lambda_-^3 + \lambda_- = \sqrt{2}c$ hence since $c > 0$ yields $\lambda_- > 0$. From (3.2) we can summarize

- P_+ has $\lambda_+ < 0$, $Re(\mu_+) > 0$, $Re(\nu_+) > 0$, $Im(\mu_+) > 0$, $Im(\nu_+) < 0$.
- P_- has $\lambda_- > 0$, $Re(\mu_-) < 0$, $Re(\nu_-) < 0$, $Im(\mu_-) > 0$, $Im(\nu_-) < 0$.

Then according to the classification of hyperbolic linear systems,

- The two equilibrium points are saddle-focus type. Furthermore, the orbits close to P_+ spiral outwards in positive time and the orbits close to P_- spirals inwards.

Elliptic Equilibrium Points: This case correspond to $c = 0$. In general, the center manifold theorem asserts that in equilibrium points the eigenvalues with positive real part spawn a unique unstable manifold, the eigenvalues with negative real part spawn a unique stable manifold and the eigenvalues with zero real part spawn center manifold which may not be unique.

Since for P_0, there are three real part eigenvalues 0, $+i$, $-i$, there are many possibilities for this invariant manifolds. In the following part we will try to shed light on which type on manifold is.

One should expect the center manifolds, at least locally, to be tangent to the eigenvectors which are $(1, 0, 0)$, $(1, 0, -1)$ and $(0, 1, 0)$. Thus we have performed the following numerical simulation which is displayed in figure 2. Some points were taken in the direction of the eigenvectors at a distance close enough to the P_0 (of 10^{-7}) and then they were integrated numerically with the Taylor method developed in the next section. (See Figure 2.)

It seems that near P_0 there is a family of periodic orbits laying in the plane spawned by the eigenvectors $(1, 0, -1)$ and $(0, 1, 0)$ which correspond eigenvalues i, $-i$ and there is a semi-stable 1D manifold spawned by the eigenvector $(1, 0, 0)$ which correspond to 0. Nevertheless, this is no rigorous proof of behaviour of the system, but it will motivate the following analysis.

The idea is to apply a cylindrical-type change of variables to render the clear the existence of periodic orbits.

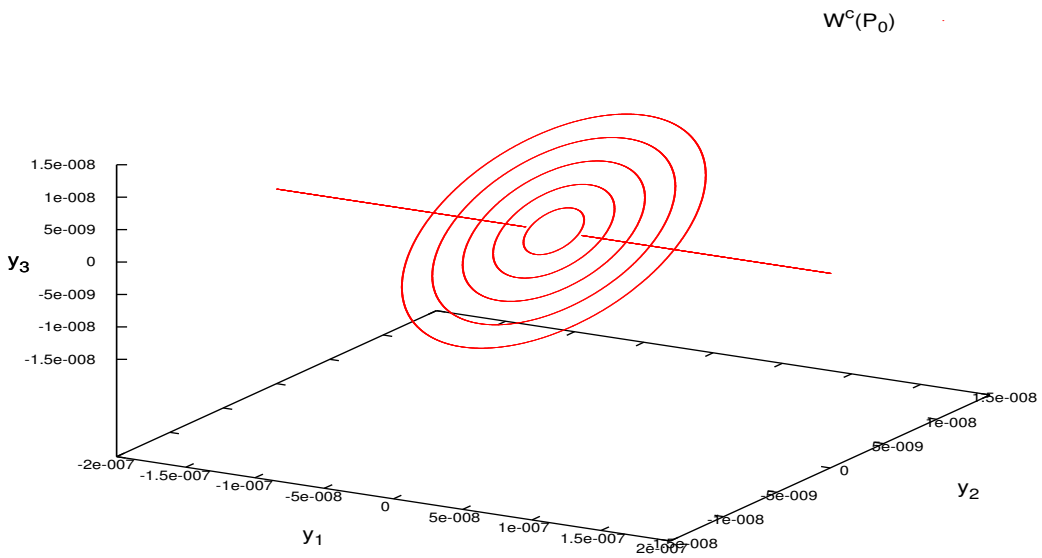


Figure 2: Orbits integrated numerically from points at a distance 10^{-7} from P_0 in the directions of the eigenvectors, for $c = 0$.

This change of coordinates will be carried out in two steps. The first is the following change of variables is $(y_1, y_2, y_3) \to (y_1, y_2, z = y_1 + y_3)$. In the new coordinates the ODE system is

$$\begin{cases} \dot{y_1} = y_2, \\ \dot{y_2} = z - y_1, \\ \dot{z} = -\dfrac{1}{2} y_1^2. \end{cases}$$

What this linear change of variables, does is to fix the Lyapunov family in the plane $z = 0$, for the new coordinates. Obviously the transformed system still has an equilibrium point in 0, with the same eigenvalues 0,+i,-i, but the respective eigenvectors now are (1,0,1), (1,0,0) and (0,1,0).

Finally make the change of variables to the cylindrical variables $(y_1, y_2, z), \longrightarrow (r\cos\theta, r\sin\theta, z)$

$$\begin{cases} \dot{r}\cos\theta - r\sin\theta\ \dot{\theta} = r\sin\theta, \\ \dot{r}\sin\theta + r\cos\theta\ \dot{\theta} = z - r\cos\theta, \\ \dot{z} = -\dfrac{1}{2} r^2\cos^2\theta. \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} r\sin\theta \\ z - r\cos\theta \\ -\dfrac{1}{2} r^2\cos^2\theta \end{pmatrix}$$

Then multiplying by the inverse of the left matrix, the system in the new coordinates is

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r^{-1}\sin\theta & r^{-1}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\sin\theta \\ z - r\cos\theta \\ -\dfrac{1}{2} r^2\cos^2\theta \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} \dot{r} = z\sin\theta, \\ \dot{\theta} = r^{-1} z\cos\theta - 1, \\ \dot{z} = -\dfrac{1}{2} r^2\cos^2\theta. \end{cases}$$

Hence, in a neighbourhood of P_0 which imply $r << 1$, for the plane $z = 0$, the system is then $(\dot{r}, \dot{\theta}, \dot{z}) = (0, -1, 0)$ for which all solutions are periodic. Thus we locally it is likely that there may be family of periodic orbits. A complete proof should use the Lyapunov center theorem.

Next we will verify the stability of the 1D semi stable manifold which is tangent to the (1,0,0) eigenvector. This will be done by means of the graph transform method. Assume that the invariant manifold W can be written as graph G of y_1 and it is analytic so that

$$W : \begin{cases} y_2 = g_2(y_1) = g_{2,2}y_1^2 + g_{2,3}y_1^3 + g_{2,4}y_1^4 + ... \\ y_3 = g_3(y_1) = g_{2,2}y_1^2 + g_{2,3}y_1^3 + g_{2,4}y_1^4 + ... \end{cases}$$

where the expansions starts at the terms of order y_1^2, because it must contain $(0, 0, 0)$ and be tangent to (1,0,0). Since W is an invariant manifold it must satisfy an invariance system, which is obtained by inserting G into the system of equations for $c = 0$.

$$\begin{cases} \dot{y}_1 = g_2(y_1), \\ \dot{g}_2(y_1) = g_3(y_1), \\ \dot{g}_3(y_1) = -g_2(y_1) - \dfrac{1}{2}y_1^2. \end{cases} \Rightarrow \begin{cases} \dot{y}_1 = g_2(y_1), \\ g_2'(y_1)\dot{y}_1 = g_3(y_1), \\ g_3'(y_1)\dot{y}_1 = -g_2(y_1) - \dfrac{1}{2}y_1^2. \end{cases} \Rightarrow \begin{cases} g_2'(y_1)g_2(y_1) = g_3(y_1), \\ g_3'(y_1)g_2(y_1) = -g_2(y_1) - \frac{1}{2}y_1^2, \end{cases}$$

where $g_i'(y_0)$ is the derivative with respect to y_0.

At this point the invariance system should be solved at each order in y_1 to obtain the solutions of the graph. However, in this particular case, by deriving the first equation an replacing $g_3'(y_1)$ into the second equation, the system can be reduced to the one equation

$$\left((g_2'(y_1))^2 + g_2(y_1)g_2''(y_1)\right) g_2(y_1) = -g_2(y_1) - \frac{1}{2}y_1^2$$

It is enough to compute the coefficients for the lower order which is order of y_1^2. Then

$$g_2(y_1) = -\frac{1}{2}y_1^2 + o(3)$$

$$g_3(y_1) = +\frac{1}{2}y_1^3 + o(4)$$

Recall that $\dot{y}_1 = g_2(y_1)$, so consequently $\dot{y}_1 = -\frac{1}{2}y_1^2 < 0$ is always negative. This implies that on the invariant manifold, which is tangent to $(1, 0, 0)$, the coordinate y_1 always point towards the same direction and the manifold is semi-stable.

► 3.1.4. Invariant Manifolds for the Hyperbolic Case. The invariant manifolds unfolding from the equilibrium point P_0 have already been fairly discussed. From here on we will be concerned with the invariant manifolds spawned by the hyperbolic equilibrium points.

The stable manifold theorem states that given an hyperbolic equilibrium $\hat{\boldsymbol{y}}$ point with k negative real part eigenvalues for $J(\hat{\boldsymbol{y}})$, then there exists a k-dimensional stable manifold W^s tangent to the stable subspace E^s of the linearised system at $\hat{\boldsymbol{y}}$. Analogously, the unstable manifold theorem

states that given an hyperbolic equilibrium $\hat{\boldsymbol{y}}$ point with $n-k$ positive real part eigenvalues for $J(\hat{\boldsymbol{y}})$, then there exists a $(n-k)$-dimensional unstable manifold W^u tangent to the unstable subspace E^u of the linearised system at $\hat{\boldsymbol{y}}$.

Recall that a manifold W is invariant if $\varphi(x, W) \subseteq W$, $\forall x \in \mathbb{R}$. In particular, an invariant manifold is stable if $\forall \boldsymbol{y} \in W$ then $\varphi(x;\boldsymbol{y}) \to \hat{\boldsymbol{y}}$ when $x \to +\infty$ and conversely it is unstable if $\forall \boldsymbol{y} \in W$ then $\varphi(x;\boldsymbol{y}) \to \hat{\boldsymbol{y}}$ when $x \to -\infty$.

The previous theorems established a way to obtain a linear approximation of the stable and unstable manifolds at first order in a neighbourhood of the equilibrium point, as E^s and E^u. (From now on, since E^s and E^u are local approximations, they will be denoted by $W^{s,loc}$ and $W^{u,loc}$.) Therefore, to obtain a first order estimate of the invariant manifolds for a given $c \neq 0$ we need the eigenvectors of the linearisation at P_+ and P_-. Similarly to how the eigenvectors were expressed in terms of the real one, using (3.2), it follows that

$$
\begin{aligned}
&ker\left(J_{\pm}(P_{\pm}) - \lambda_{\pm}\mathbb{I}\right) = \left\langle \left(1, \lambda_{\pm}, \lambda_{\pm}^2\right) \right\rangle, \\
&ker\left(J_{\pm}(P_{\pm}) - \mu_{\pm}\mathbb{I}\right) = \left\langle (1, \mu_{\pm}, \mu_{\pm}^2) \right\rangle \overset{(3.2)}{=} \left\langle \left(1, -\frac{\lambda_{\pm}}{2}, -1 - \frac{\lambda_{\pm}^2}{2}\right) + i\left(0, \sqrt{1 + \frac{3\lambda_{\pm}^2}{4}}, -\lambda_{\pm}\sqrt{1 + \frac{3\lambda_{\pm}^2}{4}}\right) \right\rangle, \\
&ker\left(J_{\pm}(P_{\pm}) - \nu_{\pm}\mathbb{I}\right) = \left\langle (1, \nu_{\pm}, \nu_{\pm}^2) \right\rangle \overset{(3.2)}{=} \left\langle \left(1, -\frac{\lambda_{\pm}}{2}, -1 - \frac{\lambda_{\pm}^2}{2}\right) + i\left(0, -\sqrt{1 + \frac{3\lambda_{\pm}^2}{4}}, \lambda_{\pm}\sqrt{1 + \frac{3\lambda_{\pm}^2}{4}}\right) \right\rangle.
\end{aligned}
$$

Hence, $(1, \lambda_{\pm}, \lambda_{\pm}^2)$ is an eigenvector for $\lambda_{\pm}$ and $(1, -\lambda_{\pm}, -1 - \lambda_{\pm}^2/2)$, $(0, 1, -\lambda_{\pm})$ are a pair of eigenvectors for $\mu_{\pm}$ and $\nu_{\pm}$. At this point, we should point out that expressing the eigenvectors and eigenvectors in terms of real roots has some utility besides give a somewhat close expression. Following this part and in order to check these results the invariant manifolds will be computed numerically. To this end the computation of the eigenvectors is required which in principle would imply solving three linear systems, two of them with complex entries. Using this approach the computations are simpler since using a Newton-Raphson method in (3.1) to obtain λ is enough to obtain the eigenvectors.

Summarizing the local invariant manifolds for P_+ and P_- and $c > 0$ which are generated by the eigenvectors of the linearised system, are

$$
\begin{aligned}
W^{u,loc}_{P_+}: &\quad (y_1, y_2, y_3) = (\sqrt{2}c, 0, 0) + \left\langle (1, 0, -1 - \lambda_+^2), (0, 1, -\lambda_+) \right\rangle, \\
W^{s,loc}_{P_+}: &\quad (y_1, y_2, y_3) = (\sqrt{2}c, 0, 0) + \left\langle (1, \lambda_+, \lambda_+^2) \right\rangle, \\
W^{u,loc}_{P_-}: &\quad (y_1, y_2, y_3) = (-\sqrt{2}c, 0, 0) + \left\langle (1, \lambda_-, \lambda_-^2) \right\rangle, \\
W^{s,loc}_{P_-}: &\quad (y_1, y_2, y_3) = (-\sqrt{2}c, 0, 0) + \left\langle (1, 0, -1 - \lambda_-^2), (0, 1, -\lambda_-) \right\rangle.
\end{aligned}
\tag{3.3}
$$

Since $W^{u,loc}_{P_+}$ and $W^{s,loc}_{P_-}$ are planes it is useful to express them in implicit form

$$
\begin{vmatrix} (y_1 \mp \sqrt{2}c) & y_2 & y_3 \\ 1 & 0 & -(1 + \lambda_{\pm}^2) \\ 0 & 1 & -\lambda_{\pm} \end{vmatrix} = 0 \qquad \Rightarrow \qquad \begin{aligned} W^{u,loc}_{P_+}: &\quad \left(1 + \lambda_+^2\right)(y_1 - \sqrt{2}c) + \lambda_+ y_2 + y_3 = 0, \\ W^{s,loc}_{P_-}: &\quad \left(1 + \lambda_-^2\right)(y_1 + \sqrt{2}c) + \lambda_- y_2 + y_3 = 0. \end{aligned}
$$

A Higher Order Approximation of Invariant Manifolds: The point of this part is to illustrate the graph transformation method which allow us to compute approximations of invariant manifolds up to any order. This method does not come new since we have already used it to determine the parabolic manifold in the hyperbolic case. Since the stable and the unstable manifold theorem states that the manifolds in question are as many times differentiable as the vector field, in this case it is safe to assume that the manifold can approximated as an analytic graph of the some of variables. Then the coefficients are determined by imposing invariance in the system.

This method will be illustrated with the manifold $W^u_{P_+}$ and we will see that at first order the previous local approximation is recovered. First, the following translation $y_1 \to y_1 - \sqrt{2}c$ is made so that the fixed point P_+ relocates to the origin. (Note that the invariant manifold found is also translated with respect to the original system). After the translation the system renders

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = -y_2 + \sqrt{2}c\ y_1 - \dfrac{y_1^2}{2}. \end{cases}$$

Lets assume that $W^u_{P_+}$ can be expressed as a graph of y_1 and y_2, namely

$$W^u_{P_+}: \quad y_3 = g(y_1, y_2) = a_{1,0}y_1 + a_{0,1}y_2 + a_{2,0}y_1^2 + a_{1,1}y_1y_2 + a_{0,2}y_2^2 + o(3),$$

where the term $a_{0,0} = 0$ since the P_+ which is now in the origin is in the manifold. The invariance system is obtained replacing the graph into the ODE system because the manifold is invariant,

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = g(y_1, y_2), \\ \dot{g}(y_1, y_2) = -y_2 + \sqrt{2}c\ y_1 - \dfrac{y_1^2}{2}. \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = g(y_1, y_2), \\ \dfrac{\partial g(y_1,y_2)}{\partial y_1} y_2 + \dfrac{\partial g(y_1,y_2)}{\partial y_2} g(y_1, y_2) = -y_2 + \sqrt{2}c\ y_1 - \dfrac{y_1^2}{2}. \end{cases}$$

Then the system is solved for each order to obtain the coefficients. This means equalizing the terms with same power of y_1 and y_2 at each side of the last equation.

At first order,

$$\begin{aligned} o(y_1): &\qquad a_{1,0}a_{0,1} = \sqrt{2}c, \\ o(y_2): &\qquad a_{1,0} + a_{0,1}^2 = 1. \end{aligned}$$

It is easy to see that multiplying the second equation by $a_{0,1}$ and replacing the first one into it gives $a_{0,1}^3 + a_{0,1} - \sqrt{2}c = 0$, which is the characteristic equation (3.1). Then since the coefficients are real $a_{0,1} = -\lambda_+$, and from the original second equation $a_{1,0} = -1 - \lambda_+^2$. Hence

$$g(y_1, y_2) = -\lambda y_1 - (1 + \lambda_+^2)y_1 + a_{2,0}y_1^2 + a_{1,1}y_1y_2 + a_{0,2}y_2^2 + o(3).$$

The tangent vectors at the origin for the graph $(y_1, y_2) \to (y_1, y_2, g(y_1, y_2))$ are

$$\left(\frac{\partial y_1}{\partial y_1}, \frac{\partial y_2}{\partial y_1}, \frac{\partial g(0,0)}{\partial y_1}\right) = \left(1, 0, -1 - \lambda_+^2\right), \qquad \left(\frac{\partial y_1}{\partial y_2}, \frac{\partial y_2}{\partial y_2}, \frac{\partial g(0,0)}{\partial y_2}\right) = \left(0, 1, -\lambda_+\right),$$

recovering the previous results. At second order,

$$\begin{array}{ll} o(y_1^2): & a_{0,1}a_{2,0} + a_{1,1}a_{1,0} = -\dfrac{1}{2}, \\ o(y_1y_2): & 2a_{2,0} + 2a_{0,2}a_{1,0} + a_{0,1}a_{1,1} = 0, \\ o(y_2^2): & a_{1,1} + 3a_{0,2}a_{0,1} = 0. \end{array} \quad \Rightarrow \quad \begin{pmatrix} -\lambda_+ & -(1+\lambda_+^2) & 0 \\ 2 & -\lambda_+ & -(1+\lambda_+^2) \\ 0 & 1 & -3\lambda_+ \end{pmatrix} \begin{pmatrix} a_{2,0} \\ a_{1,1} \\ a_{0,2} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Solving the system gives the coefficients up to order two. In general, for order n the coefficients are found by solving a $n+1$ linear system of equations. This way it is possible to obtain an approximation up to any arbitrary order.

Numerical Computations of Manifolds: After all this discussion on invariant manifolds, some numerical experimentations have to be made to confirm the theory. With the numerical integrator that we developed (explained in the next section) some orbits on the local approximations of the invariant manifolds were computed (for $c \neq 0$).

The criterion followed is the following. Since we have been working with a double floating point precision (10^{-14}) the error should be of that order of magnitude. Then as the approximations are linear the error on the manifolds is of order $o(|\boldsymbol{y} - P_\pm|^2)$ which imply that the starting point should not be at a distance larger than 10^{-7} from the equilibrium point.

As suggested before, for both P_+ and P_-, the real eigenvalues λ_+ and λ_- were computed using Newton-Raphson method. Then from (3.3) the tangent vectors were found and normalized .At this point there is some difference in the process for the 1D and 2D invariant manifold.

For the 1D manifolds the two branches were computed integrating for the two starting points given by $P_\pm \pm 10^{-7}\hat{v}_\pm$, where $\hat{v}_\pm$ are the normalized tangent vectors for P_+ and P_-.

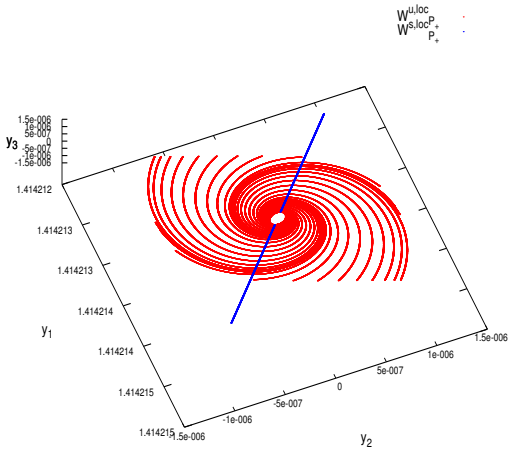
For the 2D manifolds there were many possible points to integrate. We have used as starting points to integrate, the points of a ring on the invariant planes with center the equilibrium point and the distance stated before. The ring was generated by a rotation of the normalized eigenvectors of the plane $\hat{v}_\pm$ and $\hat{u}_\pm$, namely

$$(y_1, y_2, y_3) = P_\pm + 10^{-7}\hat{v}\ \cos\theta + 10^{-7}\hat{v}\ \sin\theta.$$

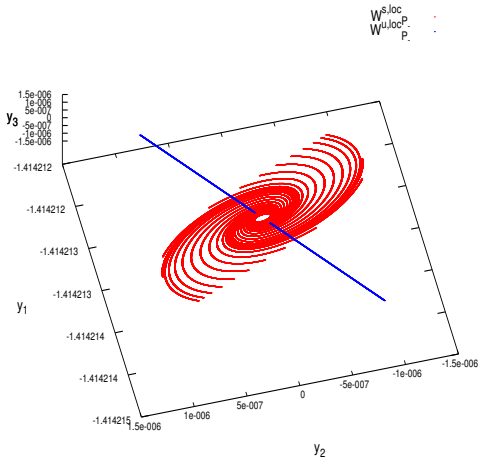
The points taken were equispaced within the angle theta angle θ for 30 divisions, i.e $\theta = 2\pi j/30$, for $j = 0 \div 30$.

Note that all the analysis on the dynamics of the equilibrium points performed corroborate the numerical results found.

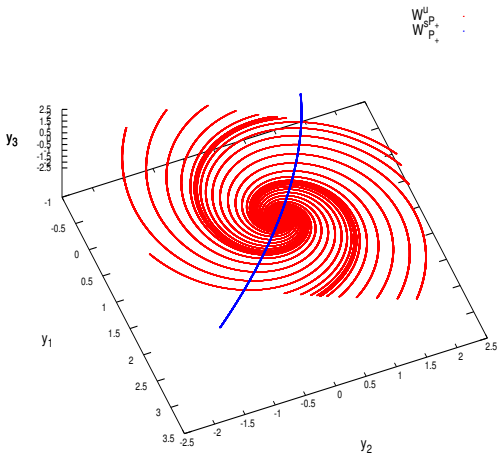
For large values of c such as those in figure 3, the invariant manifolds do not interact but for a heteroclinic orbit. As c starts to decrease the equilibrium points get closer and the invariant manifolds interact more, see figure 4. For small values of c there is a Hopf-Zero bifurcation which is responsible of the creation and annihilation of homoclinic, heteroclinic and periodic orbits and has chaotic behaviour, see figure 5.



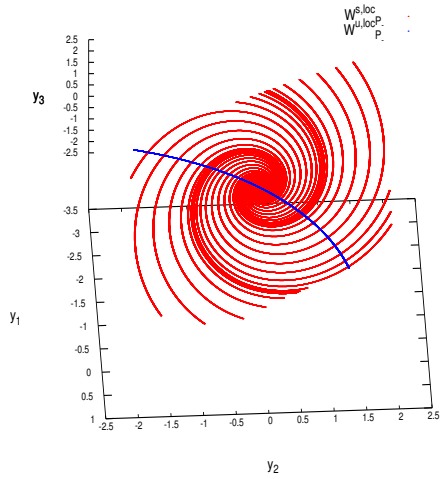
(a) Local stable and unstable manifolds of P_+.



(b) Local stable and unstable manifolds of P_-.



(c) Stable and unstable manifolds of P_+.



(d) Stable and unstable manifolds of P_-.

Figure 3: Invariant manifolds for c=1.

Observations: Note that as a volume preserving flow the equilibrium points had both contacting and expanding directions all of which were compensated. Furthermore, the equilibrium points appeared in a couple related by the time reversing symmetry except for the case $c = 0$ which was actually R-symmetric. The invariant manifolds also were related by this symmetry in the same way as the equilibrium points, being $R(W^u)_{P_+} = W^s_{P_-}$ and $R(W^s)_{P_+} = W^u_{P_-}$. Intuitively, the symmetry of the Michelson system is a rotation over the y_2 axis, so it is consistent that the behaviour on one side of the axis is the same as in the other for negative time.

Finally, for $c = 0$ the same linear reversing symmetry of the Michelson system, is a reversing symmetry of the linearised system. Thus applying 2.3.7 and since it is dimension 3, there had to be a singlet and a a doublet which actually did.

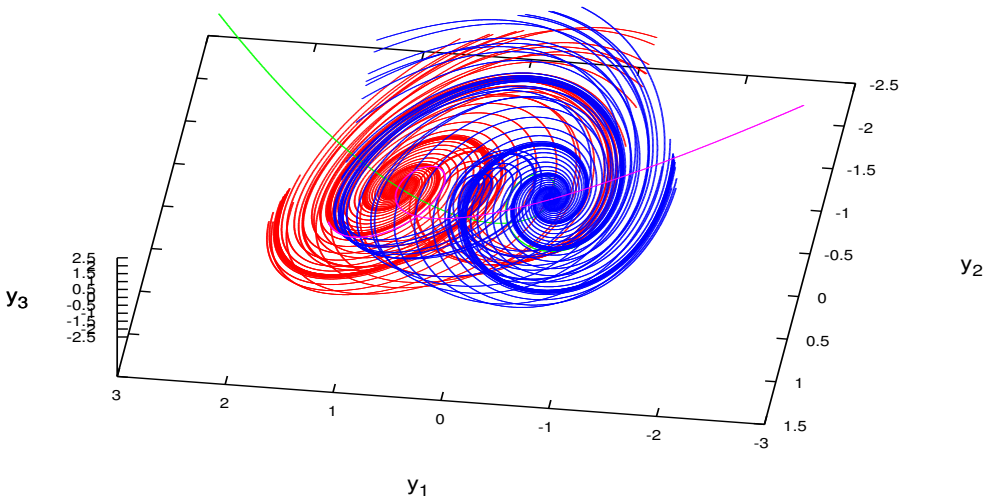


Figure 4: Invariant manifolds for c=0.5.

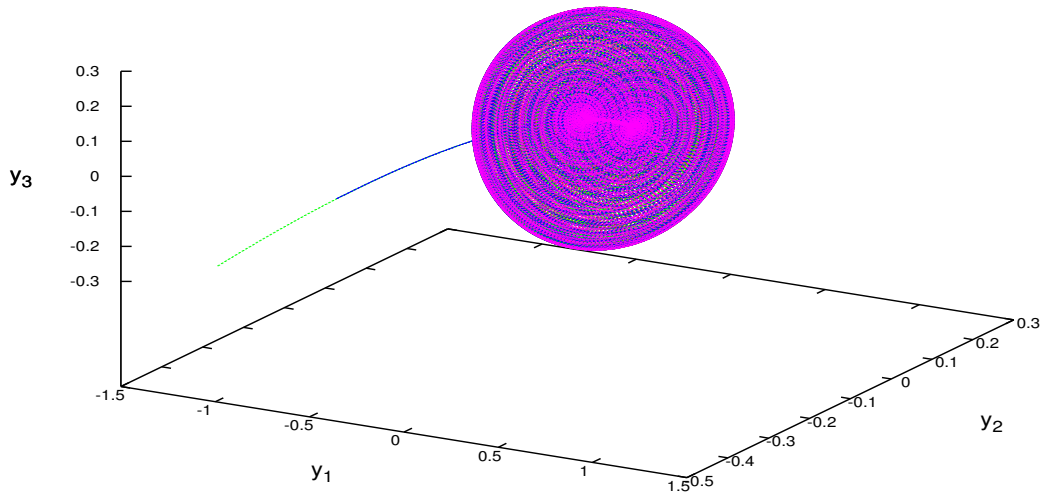


Figure 5: Invariant manifolds for c=0.1. The solutions on the invariant manifolds cover densely a Torus.

3.2. Existence of Orbits Connecting the Equilibrium points. In this section it will be proven that for all $c > 0$, the two equilibrium points have at least one homoclinic or one heteroclinic connexion. The proof has two steps, showing that the bounded solutions are uniformly bounded and using the Conley index to prove the existence. This result is a partial prove since it has been proven for the Michelson system that it is a R-symmetric heteroclinic orbit that persists for $c > 0$. Nonetheless, the point of this section is to apply index theory to show the existence of some type of orbits. Index theory is useful in these kind of existence demonstrations but usually fail in giving constructive ways to find such orbits.

▶ 3.2.1. Lemma. Consider an autonomous ODE system, depending on a parameter $\boldsymbol{c} \in \mathbb{R}^m$,

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}x} = \boldsymbol{f}_{\boldsymbol{c}}(\boldsymbol{y}), \qquad x \in (-\infty, +\infty),$$

where $\boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{f}(\boldsymbol{y}) \in C^1(\mathbb{R}^n)$. Assume we can separate the vector field $\boldsymbol{f}_{\boldsymbol{c}}(\boldsymbol{y})$ into two parts, namely $\boldsymbol{f}_{\boldsymbol{c}}(\boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{y}) + \boldsymbol{g}_{\boldsymbol{c}}(\boldsymbol{y})$ satisfying

$$\rho^r \rho^{-\boldsymbol{s}} \boldsymbol{h}(\rho^{\boldsymbol{s}} \boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{y}),$$

$$\rho^r \rho^{-\boldsymbol{s}} \boldsymbol{g}_{\boldsymbol{c}}(\rho^{\boldsymbol{s}} \boldsymbol{y}) \longrightarrow \boldsymbol{0}, \qquad \text{as } \rho \to \boldsymbol{0}$$

uniformly in $|\boldsymbol{y}| \leq 1$ and a compact domain of c. Then, under the above assumptions and having $\dot{\boldsymbol{y}} = \boldsymbol{h}(\boldsymbol{y})$ no bounded solutions but $\boldsymbol{y} = \boldsymbol{0}$, if the set of bounded solutions is not empty, it is uniformly bounded in the maximum norm in the compact domain of the parameter c.

Proof: Suppose the contrary holds, thus it exists a sequence of bounded solutions $\boldsymbol{y}_m(x)$ such that, $\sup_{x\in\mathbb{R}} |\boldsymbol{y}_m(x)| \to \infty$ as $m \to \infty$. Then consider the norm $|\boldsymbol{y}|_{\boldsymbol{s}} = \sum_{i=1}^n |y_i|^{1/s_i}$. Without loss of generality, since the system is autonomous and we can shift $x = 0$, we may assume that

$$|\boldsymbol{y}_m(0)|_{\boldsymbol{s}} = \rho_m \to \infty, \qquad \sup_{x\in\mathbb{R}} |\boldsymbol{y}_m(x)|_{\boldsymbol{s}}/\rho_m \leq 1.$$

Apply the linear changes of variables $\boldsymbol{z}_m = \rho_m^{-\boldsymbol{s}} \boldsymbol{y}_n$, $\xi = \rho_m^r x$ to each $\boldsymbol{y}_m(x)$ in the sequence of bounded solutions. Then the sequence in the new variables $\boldsymbol{z}_m(\xi)$, now satisfy the equations

$$\frac{\mathrm{d}\boldsymbol{z}_m}{\mathrm{d}\xi} = \boldsymbol{h}(\boldsymbol{z}_m) + \rho_m^r \rho_m^{-\boldsymbol{s}} \boldsymbol{g}_{\boldsymbol{c}}(\rho_m^{\boldsymbol{s}} \boldsymbol{z}_m). \tag{3.4}$$

and

$$|\boldsymbol{z}_m(0)|_{\boldsymbol{s}} = 1, \qquad \sup_{\xi\in\mathbb{R}} |\boldsymbol{z}_m(\xi)|_{\boldsymbol{s}} \leq 1. \tag{3.5}$$

Finally, in $|\boldsymbol{z}_m(0)|_{\boldsymbol{s}} \leq 1$, the term $\rho_m^r \rho_m^{-\boldsymbol{s}} \boldsymbol{g}_{\boldsymbol{c}}(\rho_m^{\boldsymbol{s}} \boldsymbol{z}_m)$ of (3.4) tends uniformly to 0 as $\rho \to 0$, since it was an hypothesis of the lemma. Therefore the set of bounded solutions $\boldsymbol{z}_m(\xi)$ tend uniformly to a $\hat{\boldsymbol{z}}(\xi)$ which is a bounded solution of the equation $\mathrm{d}\boldsymbol{z}/\mathrm{d}\xi = h(\boldsymbol{z})$. This is a contradiction since it is an hypothesis of the lemma that the only bounded solutions of the previous equation is $\hat{\boldsymbol{z}}(\xi) = \boldsymbol{0}$ and on the other hand from (3.5), $|\hat{\boldsymbol{z}}(0)|_{\boldsymbol{s}} = 1$. ■

▶ 3.2.2. Corollary. The set of bounded solutions of the Michelson system is uniformly bounded in any compact domain of $c \in [0, c_*^2]$.

Proof: For the Michelson system the hypotheses of the previous lemma apply for $\boldsymbol{s} = (1, 4/3, 5/3)$, $r = -1/3$ and the following decomposition,

$$\underbrace{\left(y_2, y_3, c^2 - y_2 - \frac{y_1^2}{2}\right)}_{\boldsymbol{f}(\boldsymbol{y})} = \underbrace{\left(y_2, y_3, -\frac{y_1^2}{2}\right)}_{\boldsymbol{h}(\boldsymbol{y})} + \underbrace{(0, 0, c^2 - y_2)}_{\boldsymbol{g}_c(\boldsymbol{y})},$$

We briefly check that this is true

$$\begin{aligned}
\rho^r \rho^{-\boldsymbol{s}} \boldsymbol{h}(\rho^{\boldsymbol{s}} \boldsymbol{y}) &= \rho^r \rho^{-\boldsymbol{s}} \boldsymbol{h}(\rho y_1, \rho^{4/3} y_2, \rho^{5/3} y_3) = \rho^r \rho^{-\boldsymbol{s}} \left(\rho^{4/3} y_2, \rho^{5/3} y_3, -\frac{1}{2}(\rho y_1)^2\right) = \\
&= \rho^r \left(\rho^{-1} \rho^{4/3} y_2, \rho^{-4/3} \rho^{5/3} y_3, -\frac{1}{2} \rho^{-5/3} \rho^2 y_1^2\right) = \\
&= \rho^{-1/3} \left(\rho^{1/3} y_2, \rho^{1/3} y_3, -\frac{1}{2} \rho^{1/3} y_1^2\right) = \left(y_2, y_3, -\frac{1}{2} y_1^2\right) = \boldsymbol{h}(\boldsymbol{y}).
\end{aligned} \quad ,$$

$$\begin{aligned}
\rho^r \rho^{-\boldsymbol{s}} \boldsymbol{g}_c(\rho^{\boldsymbol{s}} \boldsymbol{y}) &= \rho^r \rho^{-\boldsymbol{s}} \boldsymbol{g}_c(\rho y_1, \rho^{4/3} y_2, \rho^{5/3} y_3) = \rho^r \rho^{-\boldsymbol{s}} (0, 0, c^2 - \rho^{4/3} y_1) = \\
&= \rho^r (0, 0, \rho^{-5/3} c^2 - \rho^{-5/3} \rho^{4/3} y_2) = \rho^{-1/3} (0, 0, \rho^{-5/3} c^2 - \rho^{-1/3} y_2) = \\
&= (0, 0, \rho^{-2} c^2 - \rho^{-2/3} y_2) \to (0, 0, 0), \qquad \text{as } \rho \to 0, \text{ uniformly in } |\boldsymbol{y}| < 1,
\end{aligned}$$

and the system $\dot{\boldsymbol{y}} = \boldsymbol{h}(\boldsymbol{y})$ has no bounded solutions but the equilibrium point at the origin. This last part is simple to check since $\dot{y}_3 = -1/2\ y_1^2$, thus all orbits eventually scape to infinity for all $y_1 \neq 0$ as $\dot{y}_3 < 0$. In the plane $y_1 = 0$ since $\dot{y}_1 = y_2$, the orbits leave the plane for $y_2 \neq 0$, eventually escaping to infinity. Finally it only remains that the bounded solutions are in the line $y_1 = 0$, $y_2 = 0$, and since $\dot{y}_2 = y_3$ the only point which does not exit the line is the origin which remains fixed.

▶ 3.2.3. Gradient-like Systems. A gradient-like function is somewhat a generalization concept of the Lyapounov functions. Given an autonomous ODE system $\dot{\boldsymbol{y}} = f(\boldsymbol{y})$, it is said to be a *gradient-like system* if there exists a function $L(\boldsymbol{y})$ which decrease on solutions for $x > 0$. The function $L(\boldsymbol{y})$ is called *gradient-like function.*

▶ 3.2.4. Lemma. The system

$$\frac{dy}{dx} = \left(y_2, y_3, \mu - y_2 - \frac{1}{2} y_1^2\right)$$

has no bounded solutions for $\mu < 0$.

Proof: The system has the following gradient-like function $L = y_1 + y_3$. The function L decreases along with solutions since $\dot{L} = \mu - \frac{1}{2}y_1^2 < 0$ for $x > 0$. Hence, as the gradient-like function is unbounded, the solutions of the system are unbounded as well.

► 3.2.5. Theorem. For all $c > 0$ there is a homoclinic or heteroclinic connexion in P_+ and P_-.

Proof: The key to this demonstration is to use that the Conley index of a system is invariant under homotopy transformations of the flow as long as the invariant sets remain invariant (See Appendix C for definitions and properties).

As of corollary 3.2.2, the set of bounded solutions of the Michelson system is uniformly bounded in $[0, c_*^2]$. Thus, there is a compact subset of the phase space $\mathcal{U} \in \Omega$ such that $\forall c^2 \in [0, c_*^2]$ all bounded solutions are contained inside independent of the parameter c^2. Now consider the family of systems related by an homotopy of μ,

$$\frac{dy}{dx} = \left(y_2, y_3, \mu - y_2 - \frac{1}{2}y_1^2\right), \qquad \mu \in [-c_*^2, c_*^2]. \tag{3.6}$$

and let $S_{c_*^2}$ be the set of bounded solutions for the parameter c_*^2, and be $S_{-c_*^2}$ the set of bounded solutions for the parameter $-c_*^2$. According to lemma 3.2.4, there are no bounded solutions in (3.6) for $\mu < 0$, hence we have $S_{-c_*^2} = \varnothing$. At this point, note that $\mathcal{U}$ remains an isolating neighbourhood for all $\mu \in [-c_*^2, c_*^2]$, therefore the systems are related by continuation and $h(S_{c_*^2}) = h(S_{-c_*^2})$, the Conley index are the same by one of its properties. Since the Conley index of the empty set is $\bar{0}$, the pointed point, we have proven that $h(S_{c_*^2}) = \bar{0}$.

Finally we will show how this last fact contradicts to the equilibrium points P_+ and P_- not being connected. Suppose there are no bounded solutions which connect the equilibrium points for the parameter c_*^2. Then it is possible to decompose $S_{c_*^2}$ as $P_+ \cup P_- \cup S_{c_*^2} \setminus (P_+ \cup P_-)$ which is disjoint union as the equilibrium points are isolated from all other bounded solutions.

The Conley index of a union is the sum of indices, recall appendix C, in the case of disjoint unions we know how to compute it, we have to collapse the pointed points of the homotopy types of each index. The hyperbolic fixed points have Conley index a one pointed sphere and a two pointed sphere (as they have one and two positive real part eigenvalues) and the remaining set is unknown. However, it is impossible to deform with a pointed homotopy a pointed topological space which consist of a one dimension sphere a two dimension sphere and an unknown shape connected by a point into a pointed point. This imply that $h(S_{c_*^2}) \neq \bar{0}$ which is the pointed point, but this contradicts that $h(S_{c_*^2}) = \bar{0}$. Hence the assumption of the equilibrium points being isolated in the set of bounded solutions is wrong and the argument is valid $\forall c_*^2$. ■

3.3. Existence of periodic solutions for small c. Recall that the Michelson system had a family of periodic orbits for $c = 0$. In this part we will be proving that at least one of this periodic orbits persists for small parameters of c.

This demonstration is conducted by directly applying the perturbation theorem 3.3.1. The proof of this theorem is not given here since it requires ideas such as Lyapunov-Schmidt reduction and averaging methods which are overhead of this level. This demonstration and others can be found in [2]. Note that the name of the theorem "Perturbations of an isochronus set" means that the unperturbed system has to have a set of periodic orbits of the same period (isochronus means same period).

Observe that this theorem may have many direct applications, it can be used with second order perturbations of the ideal pendulum, for instance. In our case, we will be applied to the orbital equations of the Michelson system.

► 3.3.1. Theorem (Perturbations of an isochronus set). Consider the following ODE system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_0(x, \boldsymbol{y}) + \varepsilon f_1(x, \boldsymbol{y}) + \varepsilon^2 \boldsymbol{f}_2(x, \boldsymbol{y}, \varepsilon), \tag{3.7}$$

with $(x, \boldsymbol{y}, \varepsilon) \in \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0)$ and $\Omega \subseteq \mathbb{R}^n$ an open subset. Assume that $\boldsymbol{f}_0$, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are C^2 and periodic with respect to x. For any $\boldsymbol{y_0} \in \Omega$, denote by $M_{\boldsymbol{y}_0}(x)$ the principal fundamental matrix of the variational system

$$\dot{\boldsymbol{v}} = D_{\boldsymbol{y}} \boldsymbol{f}_0(x, \boldsymbol{y}_0)\ \boldsymbol{v}.$$

Suppose that for $\varepsilon = 0$ there is an open bounded subset V, such that $\forall \boldsymbol{y}_0 \in cl(V)$, the solution of the system (3.7) at that point $\varphi_{\varepsilon=0}(x; \boldsymbol{y}_0)$ is T-periodic in x. If $\hat{\boldsymbol{y}} \in cl(V)$ is a zero of the following map $\boldsymbol{F} :\ cl(V) \to \mathbb{R}^n$ defined as

$$\boldsymbol{F}(\boldsymbol{y}_0) = \int_0^T M_{\boldsymbol{y}_0}^{-1}(x) \boldsymbol{f}_1(x, \varphi(x; \boldsymbol{y}_0, 0)) dx,$$

and $\det(\boldsymbol{D}_{\boldsymbol{y}_0} \boldsymbol{F}(\hat{\boldsymbol{y}})) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there is a T-periodic solution $\varphi_\varepsilon(x; \boldsymbol{y}_0)$ of the system (3.7) such that $\varphi_\varepsilon(x; \boldsymbol{y}_0) \to \hat{\boldsymbol{y}}$ as $\varepsilon \to 0$.

► 3.3.2. Corollary. For $c > 0$ sufficiently small the Michelson system has at least one periodic orbit.

Proof. First we will give the general outlines of this demonstration. As we have seen in the analysis of the elliptic equilibrium point, there was a family of periodic solutions in a neighbourhood of the only equilibrium point, this is our set V mentioned in the previous theorem. The point is to make again a change to cylindrical and find the orbital system of equations with respect to the angle θ. Hence, with respect to orbital system theorem can be applied.

Now we shall begin the proof. Start by considering a rescaling of variables in the Michelson system, $y_1 \to \varepsilon y_1$, $y_2 \to \varepsilon y_2$, $y_3 \to \varepsilon y_3$ and $c \to \varepsilon c$, with the parameter $\varepsilon \neq 0$. The system renders

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = \varepsilon c^2 - y_2 - \dfrac{\varepsilon}{2} y_1^2. \end{cases}$$

Then recall that the first step is to apply a change of variables to cylindrical coordinates given by $y_1 \to z$, $y_2 \to r\sin\theta$, $y_3 \to r\cos\theta$,

$$\begin{cases} \dot{z} = r\sin\theta, \\ \dot{r}\sin\theta + r\cos\theta\ \dot{\theta} = r\cos\theta, \\ \dot{r}\cos\theta - r\sin\theta\ \dot{\theta} = \varepsilon c^2 - r\sin\theta - \dfrac{\varepsilon}{2} z^2, \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & r\cos\theta \\ 0 & \cos\theta & -r\sin\theta \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r\sin\theta \\ r\cos\theta \\ \varepsilon c^2 - r\sin\theta - \dfrac{\varepsilon}{2} z^2 \end{pmatrix}.$$

Inverting the left hand side matrix yield

$$\begin{pmatrix} \dot{z} \\ \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & r^{-1}\cos\theta & -r^{-1}\sin\theta \end{pmatrix} \begin{pmatrix} r\sin\theta \\ r\cos\theta \\ \varepsilon c^2 - r\sin\theta - \dfrac{\varepsilon}{2} z^2 \end{pmatrix}. \quad \Leftrightarrow \quad \begin{cases} \dot{z} = r\sin\theta, \\ \dot{r} = \dfrac{\varepsilon}{2}(2c^2 - z^2)\cos\theta, \\ \dot{\theta} = 1 - \dfrac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta. \end{cases}$$

Once this has been done the second step is to obtain the orbital system with respect to θ. This is done by using the chain rule on the previous system and replacing one of the equations into the others,

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = r\sin\theta, \\ \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{\varepsilon}{2}(2c^2 - z^2)\cos\theta, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 1 - \dfrac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta, \end{cases} \quad \Rightarrow \quad \begin{cases} \left(1 - \dfrac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta\right)\dfrac{\mathrm{d}z}{\mathrm{d}\theta} = r\sin\theta, \\ \left(1 - \dfrac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta\right)\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{\varepsilon}{2}(2c^2 - z^2)\cos\theta, \end{cases}$$

This is already the orbital system. However, we would like to give system the aspect of a regular ODE system, thus invert and send to the other side the nasty term accompanying the derivatives. To do so we use the following trick. Since the expression is analytic with respect to all its variables, the the inverse is too and we can consider the inverse as power series in ε, the coefficients of which must satisfy

$$\left(1 - \frac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta\right) \cdot \sum_{n=0} a_n(z, r, \theta)\ \varepsilon^n = 1$$

Equalizing terms at each side of the equation with same power in ε, the first two coefficients are

$$a_0(z, r, \theta) = 1,$$

$$a_1(z, r, \theta) = \frac{1}{2r}(2c^2 - z^2)\sin\theta.$$

Hence

$$\begin{aligned} \left(1 - \frac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta\right)^{-1} &= 1 + \frac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta + \varepsilon^2 \sum_{n=3} a_n(z, r, \theta)\ \varepsilon^{n-2} \\ &= 1 + \frac{\varepsilon}{2r}(2c^2 - z^2)\sin\theta + \varepsilon^2 g(z, r, \theta, \varepsilon), \end{aligned}$$

where $g(z, r, \theta, \varepsilon)$ is analytic on all its variables. The orbital equation then is

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\theta} = r\sin\theta + \dfrac{\varepsilon}{2}(2c^2 - z^2)\sin^2\theta + \varepsilon^2 f_{2,1}(z, r, \theta, \varepsilon), \\ \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{\varepsilon}{2}(2c^2 - z^2)\cos\theta + \varepsilon^2 f_{2,2}(z, r, \theta, \varepsilon), \end{cases} \tag{3.8}$$

where all the terms are analytic. Note that from now on it is convenient to use a prime to represent the derivatives with respect to θ. This system has the form of (3.7) as in the theorem and for $\varepsilon = 0$ the system is $(z', \theta') = (r\sin\theta, 0)$ for which given for all initial conditions (z_0, r_0) all solutions are 2π-periodic, in particular

$$\varphi_{\varepsilon=0}(\theta; z_0, r_0) = (r_0 + z_0 - r_0\cos\theta, r_0).$$

Therefore we are under the hypothesis of the theorem and it only remains to find the map $\boldsymbol{F}$ and see if the conditions apply. To do so, it is required the principal fundamental matrix for $\varepsilon = 0$ whose columns the solution of the linearised system

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

for initial conditions $\boldsymbol{v} = \boldsymbol{e}_1$ and $\boldsymbol{v} = \boldsymbol{e}_2$. Note that this is the linearised system for all initial conditions (z_0, x_0) since the Jacobian does not depend on z nor r. For the previous linear system, as the matrix commutes with itself the solution is

$$M(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + exp\left(\int_0^\theta \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix} \mathrm{d}\theta\right).$$

thus

$$M(\theta) = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}$$

which is independent of the initial conditions (z_0, x_0) as stated before. Finally, we build the map from the theorem $\boldsymbol{F}(z_0, x_0)$,

$$\begin{aligned} \boldsymbol{F}(z_0, x_0) &= \int_0^{2\pi} \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}^{-1} \frac{1}{2}(2c^2 - z^2) \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta \,\Bigg|_{\varphi_{\varepsilon=0}(\theta; z_0, r_0)} = \\ &= \frac{1}{2}\int_0^{2\pi} \left(2c^2 - (r_0 + z_0 - r_0\cos\theta)^2\right) \begin{pmatrix} 1 & \cos\theta - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta = \\ &= \frac{1}{2}\int_0^{2\pi} \left[\begin{pmatrix} 2c^2 - (r_0 + z_0)^2 \\ 0 \end{pmatrix} + \begin{pmatrix} -r_0^2 - 2r_0(r_0 - z_0) \\ 2r_0(r_0 + z_0) \end{pmatrix} \cos^2\theta\right] \mathrm{d}\theta = \\ &= \frac{\pi}{2} \begin{pmatrix} 4c^2 - 5r_0^2 - 6r_0 z_0 - 2z_0^2 \\ 2r_0(r_0 + z_0) \end{pmatrix}. \end{aligned}$$

To conclude the demonstration we must find the zeroes of the map for which the determinants of the differentials are not zero. The last equation imply that, since $r_0 \neq 0$, the only zeros of $\boldsymbol{F}$ must be of the form $r_0 = -z_0$ and from the second equation it follows that $(z_0, r_0) = (-2c, 2c)$ is the only zero. At this point the differential is

$$\boldsymbol{D}_{(z_0,r_0)}\boldsymbol{F}(z_0, r_0) = \frac{\pi}{2}\begin{pmatrix} -6z_0 - 4z_0 & -10r_0 - 6z_0 \\ 2r_0 & 4r_0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{D}_{(z_0,r_0)}\boldsymbol{F}(-2c, 2c) = \frac{\pi}{2}\begin{pmatrix} 4c & -8c \\ 4c & 8c \end{pmatrix},$$

for which $det(\boldsymbol{D}_{(z_0,r_0)}\boldsymbol{F}(-2c, 2c)) = 8\pi^2c^2$. Therefore, all the conditions of the theorem are met and for $|\varepsilon| > 0$ small the orbital equation (3.8) has a 2π periodic solution $\varphi_{per,\varepsilon}(\theta; z_0, r_0)$ such that (z_0, r_0) tend to $(-2c, 2c)$ as $\varepsilon \to 0$. ■

4. NUMERICAL EXPERIMENTS

Numerical experiments are a fundamental part of dynamical systems. It is thanks to them that it is possible to shed light on dynamics a system and it is them who show what is to be expected from its solutions. With numerical computations it is possible to visualize the skeleton of the system, i.e. equilibrium points, periodic orbits, invariant tori, attractors and many more.

It has already been stated that the numerical analysis is of utmost importance to gain insight of a system. Nonetheless, this experiments do not replace the formal proofs of the presence of some property or dynamics of a system. Still, in practice it is pointless to try and prove some behaviour that has not been observed, and besides a numerical experiment can give an idea into how to approach a demonstration. Finally we would like to comment that nowadays numerical techniques are becoming more relevant due to growing popularity of the so called computer assisted proofs, and in particular the Michelson system is no exception in this regard.

This section is divided into three subsections. The first part is the most important and is the basic tool for all the other experiments. We are referring to the Taylor method, which allow us to integrate to integrate the solutions of the system. Then a second part gives a method using a first hit map to find 2D-heteroclinic orbits (those laying in the intersections of 2D invariant manifolds). Finally in the third part it is explained how to locate periodic orbits using a Poncaré maps. Both the first and second part have been programmed using C language and the code can be found in the annex.

4.1. Taylor Method: This method is used to integrate, given initial conditions, the solutions of an ODE systems with sufficient smoothness. There are many already created generic programmes using this method which use automatic differentiation. However, in this work we are developing our own Taylor method ad-hoc for the Michelson system, since this way we illustrate the method and the resulting program would be faster. In this section, it is also discussed how some of the technical issues were overcome such us how to truncate the series or which step-size is chosen to control the error.

The general idea behind this method idea is to apply the Taylor theorem. Starting at a given point x_0, is approximated locally by a power series expansion, then the series is truncated at some point to effectively be able to do computations. Finally a new point of the solution $x_0 + h$ is found by evaluating the truncated series in our case using a Horner method.

Note that in section 2.1 it was proven that the solutions of the Michelson system exist, are unique and, the most important part, that they are analytic. Therefore, in our case we have as much smoothness as possible to apply this method. In fact, originally, the main motivation of section 2.1 was to prove that there was enough differentiability to be able to develop this method.

Before starting, recall that this same Taylor created here has already been used when studying the elliptic equilibrium point and in plotting the invariant manifolds in section 3.1.

▶ 4.1.1. Taylor Method Algorithm. As stated before, for each step we consider a generic formal series expansion and its derivative in a neighbourhood of x_0, which is the initial point for a given step. It is convenient to define the step-size as $h = (x - x_0)$ and relabelled the indexes m in the case of the derivative for bookkeeping the degree of h. Thus

$$\boldsymbol{y} = \sum_{m=0}^{\infty} \boldsymbol{a}_m h^m \longrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} a_{1,m} \\ a_{2,m} \\ a_{3,m} \end{pmatrix} h^m. \tag{4.1}$$

$$\boldsymbol{y}' = \sum_{m=0}^{\infty} (m+1)\boldsymbol{a}_{m+1} h^m \longrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \sum_{m=0}^{\infty} (m+1) \begin{pmatrix} a_{1,m+1} \\ a_{2,m+1} \\ a_{3,m+1} \end{pmatrix} h^m. \tag{4.2}$$

As in Lemma 2.1.5, for the series to be a solution, it must satisfy the ODE system and as a matter of fact the coefficients are determined by replacing the series into the system. Hence we insert (4.1) and (4.2) into the Michelson system, obtaining the following

$$\sum_{m=0}^{\infty} (m+1) \begin{pmatrix} a_{1,m+1} \\ a_{2,m+1} \\ a_{3,m+1} \end{pmatrix} h^m = \begin{pmatrix} 0 \\ 0 \\ c^2 \end{pmatrix} + \sum_{m=0}^{\infty} \begin{pmatrix} a_{2,m} \\ a_{3,m} \\ -a_{1,m} - \dfrac{1}{2} \displaystyle\sum_{k=0}^{m} a_{1,k} a_{1,m-k} \end{pmatrix} h^m. \tag{4.3}$$

Recall that the third equation of the Michelson system has a quadratic term y_1^2, so when substituting the series we had to consider the Cauchy product of formal series. In general, the product of two formal series may not be convergent so the previous relation may not be well defined. For the Taylor method though, since the series are truncated (in the end we consider polynomials) the product is well defined and convergence is unnecessary. However, in this particular case we can argue that this product is convergent since the solutions are analytic, therefore y_1 is analytic and the product of analytic functions is analytic.

Finally, the recurrence formula that allow to determine the coefficients is obtained by equalizing the terms of equal power in h in (4.3). Doing so gives

$$\begin{pmatrix} a_{1,m+1} \\ a_{2,m+1} \\ a_{3,m+1} \end{pmatrix} = \frac{1}{m+1} \left\{ \begin{pmatrix} 0 \\ 0 \\ c^2 \end{pmatrix} \delta_{m,0} + \begin{pmatrix} a_{2,m} \\ a_{3,m} \\ -a_{2,m} - \dfrac{1}{2} \displaystyle\sum_{k=0}^{m} a_{1,k} a_{1,m-k} \end{pmatrix} \right\}, \tag{4.4}$$

for $m \geq 0$. Note that in the case $m = 0$ there is an extra term, so in order to make the expression compact, we have used the Kronecker delta, $\delta_{m,0}$.

At this point we have an expression that allow us to compute the coefficients of the solutions as a series, using the previous ones. Nonetheless, for $m = 0$, the coefficient $\boldsymbol{a_1}$ requires $\boldsymbol{a_0}$. Hence we need initialize the recurrence by setting $\boldsymbol{a_0}$. This is done by using the previous computed point, this is replacing $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ in (4.1). Therefore the coefficient $\boldsymbol{a_0}$ is set with the value $\boldsymbol{y}_0$, the point in the previous step (or the initial conditions in case it is the first step).

The speed of the process depend on the number of computations, the less computations the faster the process. Thus, for the sake of reducing the number of computations, in this particular case, we can consider the symmetry of the Cauchy product in expression (4.4) to pair up the terms which are the same, $a_{1,k}a_{1,m-k}$ and $a_{1,m-k}a_{1,k}$. Thus, rewrite (4.4) as follows

$$\begin{pmatrix} a_{1,m+1} \\ a_{2,m+1} \\ a_{3,m+1} \end{pmatrix} = \frac{1}{m+1}\left\{ \begin{pmatrix} 0 \\ 0 \\ c^2 \end{pmatrix} \delta_{m,0} + \begin{pmatrix} a_{2,m} \\ a_{3,m} \\ -a_{2,m} - \sum_{k=0}^{[(m+1)/2]} a_{1,k}a_{1,m-k} + \frac{1}{2}a_{1,m/2}^2\Theta \end{pmatrix} \right\}. \tag{4.5}$$

where $[\,\cdot\,]$ stands for the integer part and Θ is an operator which is 1 for m even and 0 for m odd. Note that with respect to (4.4), the number operations in summation has been cut in half.

Using (4.5) and that $\boldsymbol{a_0}$ is the value of $\boldsymbol{y}$ for the previous computed point, the coefficients of the solution series (4.1) can be computed. In practice it is impossible for a computer to compute the infinite coefficients of the series. Thus it is necessary to truncate the series at some order p which will give an error. According to Taylor theorem, the reminder error (error made by truncating the series), ε, can be approximated by,

$$\mathcal{E} \approx \frac{1}{(p+1)!}\left|\frac{\mathrm{d}^{(p+1)}\boldsymbol{y}(x)}{\mathrm{d}x^{p+1}}\right|_{\xi} h^{p+1} = M\ h^{p+1}, \qquad \text{where } M \text{ is a constant.}$$

Looking at the previous, we realize that $\varepsilon = \varepsilon(p,h)$, thus there are many suitable step-sizes h and truncation orders p that suit a fixed error. At this point we would like the error to be $\varepsilon = 10^{-14}$, as we will be programming in double-precision floating-point variables and this is the maximum precision possible. The step-size and truncation order are chosen to minimize the number of operations (maximize the speed), according to the following two lemmas, the second of which is attributed to [4].

► 4.1.2. Lemma (Number of operations). The total number of elemental arithmetic operations (sums, differences, products and divisions) to compute the all of coefficients $\boldsymbol{a}_m$ of (4.5), up to an order p is of order $o(p^2)$, for p sufficiently large.

Proof: Counting in (4.5) the number of elemental arithmetic operations to compute a given coefficient $\boldsymbol{a}_m$, we obtain utmost $\#op(m) = (m/2) + 5$. Hence, the total number of operations to compute up to the p-th coefficient is the summation

$$\#op(total) = \sum_{m=0}^{p} \#op(m) = \sum_{m=0}^{p} (m/2) + 5 = (p(p+1)/4) + 5p.$$

Finally for p large yields $o(p^2)$. ■

► 4.1.3. Lemma (Optimal Step-size and Truncation). Assume that the Taylor series of a given analytic function has radius of convergence ρ, that there exist constants $M_1, M_2 > 0$ such that the derivatives satisfy

$$\frac{M_1}{\rho^m} \leq \left| \frac{\mathrm{d}^m \boldsymbol{y}(x)}{\mathrm{d}x^m} \right| \leq \frac{M_2}{\rho^m}, \qquad \forall m \in \mathbb{N}, \tag{4.6}$$

and that the number of elementary operations to compute the coefficients of the series up to order p is of order $o(p^2)$. Then, if the required accuracy $\varepsilon \to 0$ (i.e. is sufficiently small), the optimal truncation order and step-size that minimizes the number of operations tend to

$$h_{op} \to \pm \frac{\rho}{e^2}. \tag{4.7}$$

$$p \to -\frac{1}{2} \ln \varepsilon, \tag{4.8}$$

where the positive sign is for positive time (forward) integration and the negative sign for negative (backward) time integration.

Proof: From the Taylor theorem reminder and condition (4.6),

$$\varepsilon \approx o(h^{p+1}) \approx \frac{1}{(p+1)!} \left| \frac{\mathrm{d}^{(p+1)} \boldsymbol{y}(x)}{\mathrm{d}x^{p+1}} \right|_{\xi} \approx M \left(\frac{h}{\rho} \right)^{p+1}.$$

Isolating the step-size,

$$h \approx \rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{p+1}}.$$

Now we compute the expression for the speed ϕ as the total number of computations of a step (which is of order $o(p^2)$) divided by how much we advance in that step which is given h.

$$\phi(p) \approx \frac{K(p+1)^2}{\rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{p+1}}}.$$

Then to find the minimum for this speed, we must solve $\phi'(p) = 0$,

$$\phi'(p) = \frac{2K(p+1)}{\rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{p+1}}} + \frac{K(p+1)^2}{\rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{p+1}}} \frac{1}{(p+1)^2} \ln \left(\frac{\varepsilon}{M} \right) = 0 \qquad \Rightarrow \qquad p = -\frac{1}{2} \ln \left(\frac{\varepsilon}{M} \right) - 1.$$

From this last equation, taking $\varepsilon \to 0$ we obtain (4.8), and by replacing it in the expression for h and taking the limit case $\varepsilon \to 0$ we obtain (4.9),

$$h_{op} \approx \rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{-\frac{1}{2} \ln \varepsilon + 1}} = \rho \left(\frac{\varepsilon}{M} \right)^{\frac{1}{1 - \ln \varepsilon^{\frac{1}{2}}}} \quad \longrightarrow \quad \frac{\rho}{e^2} \qquad \blacksquare.$$

► 4.1.4. Corollary For a double-precision error of $\varepsilon = 10^{-14}$, the optimal truncation order to minimize the number of operations is $p = 24$. ■

In our Taylor method we have used as seen in Corollary 4.1.4 a truncation at order 24 for all steps. On the contrary the optimal step-size depends on the step in which we are computing since from (4.7) it is clear that depends on the radius of convergence. This poses the problem of compting the radius of convergence. Recall the formal definition of the radius of convergence

$$\rho = \frac{1}{\overline{\lim}_{m\to\infty} \sqrt[m]{|\boldsymbol{a}_m|}}.$$

It is not possible to compute analytically this limit with the recurrence formulas obtained. Thus, we are forced to work with an estimate $\overline{\rho}$ which we compute as

$$\overline{\rho} = \min\{\rho_{24}, \rho_{25}, \rho_{26}\}, \qquad \rho_j^{-1} = \sqrt[j]{|\boldsymbol{a}_j|}. \tag{4.9}$$

Note that we take the minimum of the three values with larger j for safety in case some coefficient is zero. This idea is also borrowed from [4]. Summarizing the optimal step-size is computed with (4.7) and the estimate (4.9).

► 4.1.5. Step-size Control. We have already seen how to compute each step and what step-size and truncation order use to control the error maximizing the speed. However, note that the step-size computed using Lemma 4.1.3 may be "too good" for plotting the phase space.

Now we shall explain what "too good" means. Note that the optimal step-size computed with (4.7) depend on the radius of convergence. Thus if the radius of convergence is too large when representing the points in the phase space they may be too separated from each other. In fact, this happens in the solutions of Michelson system, where the radius of convergence of most of the series is quite large.

The idea is for the newly computed point to be at approximately the same distance d from the previous one in the phase space. Therefore, for each step, we compute the point with the optimal step-size given in (4.7) and we do the following depending on the distance in the phase space with the previous point:

- If the newly computed point is at a distance smaller than d, then we do not save it in the file containing the points for latter representation. Instead we compute another step since we cannot increase the step-size as it would increase the error.
- If the newly computed point is at a distance larger than d from the original point $\boldsymbol{y_0}$ (note that the computed point could be computed in more that one step given the previous case), then we adjust heuristically the step-size as follows.

  Suppose after n steps, finally $|\boldsymbol{y}_n(h_{op}) - \boldsymbol{y}_0| > d$. Then the correct step-size is given by the solution of $|\boldsymbol{y}_n(h) - \boldsymbol{y}_0| = d$ where $h < h_{op}$ and $\boldsymbol{y}_n(h)$ is the truncated Taylor series computed for this last step. Note that the truncated Taylor series $\boldsymbol{y}_n(h)$ is a polynomial of degree 24 in h. However, heuristically, if d is small we can suppose that h is small too, thus we can approximate the polynomial up to the linear terms

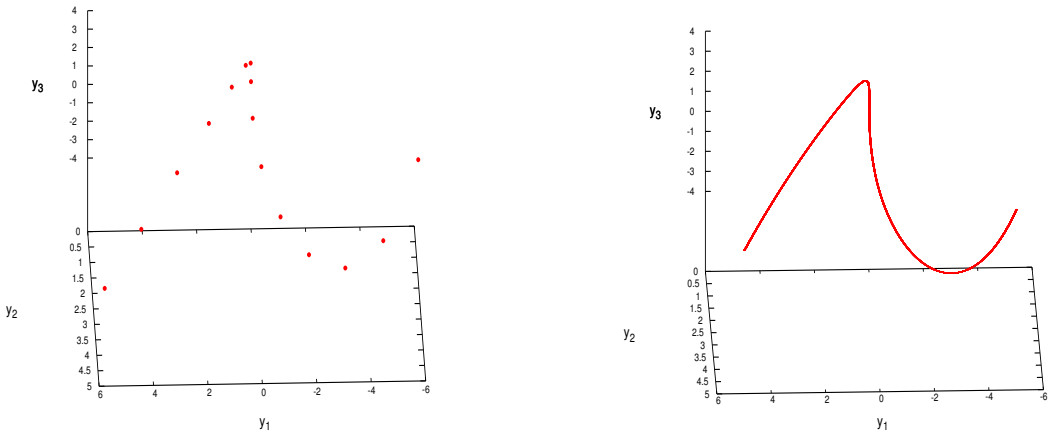
  $$d = |\boldsymbol{y}_n(h) - \boldsymbol{y}_0| \approx |\boldsymbol{a}_0 + \boldsymbol{a}_1 h - \boldsymbol{y}_0| = |(\boldsymbol{y}_{n-1} - \boldsymbol{y}_0) + h\boldsymbol{f}(\boldsymbol{y}_{n-1})| \approx |(\boldsymbol{y}_{n-1} - \boldsymbol{y}_0)| + |h|\,|\boldsymbol{f}(\boldsymbol{y}_{n-1})|,$$

where we have used that the first two coefficients of the Taylor in the last step are $\boldsymbol{a}_0 = \boldsymbol{y}_{n-1}$ and $\boldsymbol{a}_1 = \boldsymbol{f}(\boldsymbol{y}_{n-1})$. Therefore, isolating h, the step-size that we use is

$$h = \pm \left| \frac{d - |(\boldsymbol{y}_{n-1} - \boldsymbol{y}_0)|}{|\boldsymbol{f}(\boldsymbol{y}_{n-1})|} \right|,$$

where the sign depends on whether we integrate the orbits forward or backwards in time.

The following figure show an example of the difference when representing orbits in the phase space with this protocol and without it.



(a) Orbit computed using the optimal step-size given in Lemma 4.1.3.

(b) Orbit computed by using the Step-size Control 4.1.5.

Figure 6: The same orbits for c=2 and $y_1(0) = y_2(0) = y_3(0) = 0$.

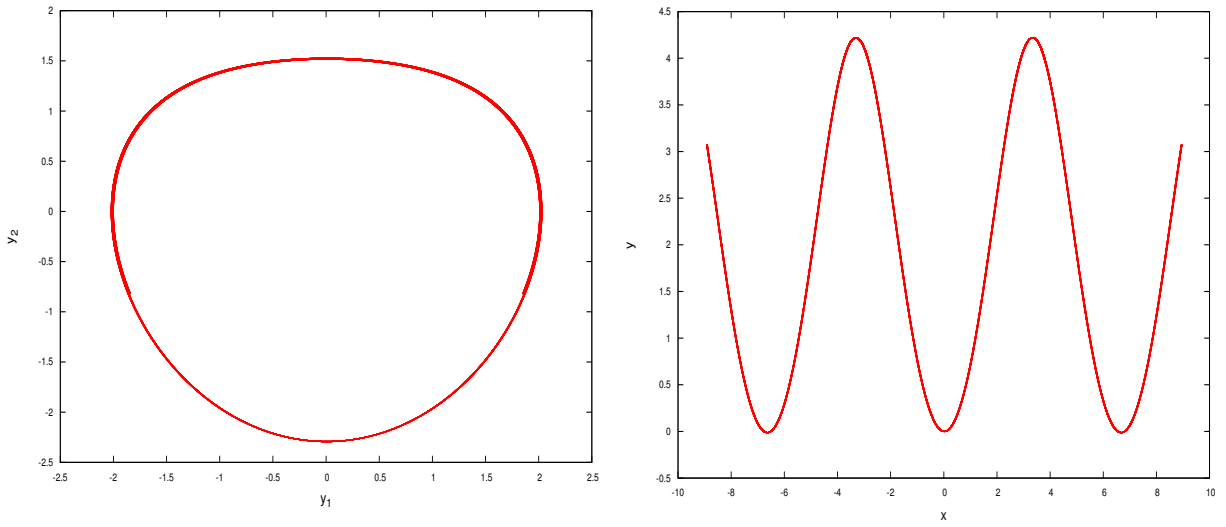
Note that this readjustment of the step-size will only be done when representing orbits in the phase space. For all other purposes we will be using the optimal step-size (4.7) which gives the fastest advance with the maximum speed.

▶ 4.1.6. The Kuramoto-Sivashinsky Waves. Recall that the Michelson system arise from the the problem of finding the steady solutions of the Kuramoto-Sivashinsky equation. As it stated in the introduction the travelling-waves of the Kuramoto-Sivashinsky equation are given by

$$y_0(x) = \int_0^x y_1(\hat{x}; 0) \mathrm{d}\hat{x}$$

Therefore, given a solution of the Michelson system we have designed a programme that compute the travelling-waves evaluating the previous integral. To solve the integral we have resorted to the trapezoidal rule. Since the solutions of the Michelson system are saved in a file containing the coordinates (x, y_1, y_2, y_3), the point of the travelling-waves are computed summing trapezoids

$$y_0(x_{n+1}) = \sum_{i=0}^{n} \frac{y_1(x_{i+1}) + y_1(x_i)}{2(x_{i+1} - x_i)}.$$



(a) Periodic orbit of the Michelson system projected on the (y_1, y_2) plane.

(b) The travelling-wave of KS system corresponding to the previous orbit of the Michelson system.

Figure 7: Periodic orbit found by Troy in [11] with $y_1 = y_3 = 0$, $y_2 \approx 1.5265$ and $c = 1$, using our integrator for the Michelson and KS systems.

In general, when considering the steady state equations of a certain wave modelling EDP, there are two types of solutions which have special interest. The first are periodic solutions which correspond to periodic travelling waves in the original system, as illustrated by Figure 7. The second are homoclinic orbits which correspond to solitary waves (called "solitions" in [6]), which are waves that start constant, at some point they oscillate and in the end they return to the same constant state. This two types of waves have many applications, for instance in optical communitarian systems.

4.2. Heteroclinic Orbits. In this section we will design an algorithm to find 2D-heteroclinic orbits. By 2D-heteroclinic orbits we mean the heteroclinic orbits laying in the intersection of the two dimensional stable and unstable manifolds of the two equilibrium points of the Michelson system.

The idea is to take section (a plane) Σ between the two equilibrium points and define two maps (the first hit maps) which send points of the linear approximation of the 2D invariant manifolds along the flow to the section. Since the heteroclinic orbits connect both equilibrium points, it is obvious that they will intersect the section a some point. Then the heteroclinic orbits are those point in which the two maps coincide and they can be computed using a Newton method. The concept will become more clear as the method is described.

► 4.2.1. Starting Points. The first step is to recall the linear approximations (planes) of the 2D invariant manifolds for both equilibrium points which were computed in 3.1.2.

$$E^u_{P_+} : (1+\lambda_+^2)(y_1 - \sqrt{2}c) + \lambda_+ y_2 + y_3 = 0, \qquad E^s_{P_-} : (1+\lambda_-^2)(y_1 + \sqrt{2}c) + \lambda_- y_2 + y_3 = 0,$$

where λ_+ (resp. λ_-) was the only real eigenvalue for the fixed point P_+ (resp. P_-).

Now we would like to inscribe a small ring on each plane centred in their respective equilibrium points. The purpose of this rings is to use its points as initial conditions to integrate the flow. This way we make sure all possible orbits on the manifold are represented with one variable which is the angle. This part can be done in two steps.

Take both linear approximation of the invariant manifolds and write them as a graph of y_3,

$$E^u_{P_+}: \qquad \boldsymbol{G}_{P_+}(y_1, y_2, y_3(y_1, y_2)) = (y_1, y_2, -(1+\lambda_+^2)(y_1 - \sqrt{2}c) - \lambda_+ y_2),$$
$$E^s_{P_-}: \qquad \boldsymbol{G}_{P_+}(y_1, y_2, y_3(y_1, y_2)) = (y_1, y_2, -(1+\lambda_-^2)(y_1 + \sqrt{2}c) - \lambda_- y_2).$$

Then for each invariant plane consider two rings $(y_1(\theta_1), y_2(\theta_1)) = (\sqrt{2}c + \varepsilon\cos\theta_1, \varepsilon\sin\theta_1)$ and $(y_1(\theta_2), y_2(\theta_2)) = (-\sqrt{2} - \varepsilon\cos\theta_2, \varepsilon\sin\theta_2)$ with center in the equilibrium points and radius $\varepsilon = 10^{-7}$, as the error in the linear approximation is of order $o(\varepsilon^2)$ and we want an the error of 10^{-14}. Then we can inscribe the rings into the planes by composing them with the graphs

$$\boldsymbol{R}_{P_+}(\theta_1) = (y_1, y_2, y_3)(\theta_1) = \left(\sqrt{2}c + \varepsilon\cos\theta_1, \varepsilon\sin\theta_1, -(1+\lambda_+^2)\varepsilon\cos\theta_1 - \lambda_+\varepsilon\sin\theta_1\right),$$
$$\boldsymbol{R}_{P_-}(\theta_2) = (y_1, y_2, y_3)(\theta_2) = \left(-\sqrt{2}c + \varepsilon\cos\theta_2, \varepsilon\sin\theta_2, -(1+\lambda_-^2)\varepsilon\cos\theta_2 - \lambda_-\varepsilon\sin\theta_2\right),$$

► 4.2.2. First Hit Maps. At this point, we may build the first hit maps. This map is built by taking points in the ring, which give a good representation of the orbits of the manifold, and integrate them until they intersect a section Σ. Therefore it is the composition of

$$\boldsymbol{P}_1(\theta_1) = \hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right),$$
$$\boldsymbol{P}_2(\theta_2) = \hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_-\left(\boldsymbol{R}_{P_-}(\theta_2)\right); \boldsymbol{R}_{P_-}(\theta_2)\right).$$

where $\hat{\pi} : \mathbb{R}^3 \to \Sigma$ is the projection from the phase space to the section Σ, and $\tau_\pm : \mathbb{R}^3 \to \mathbb{R}$ which is the characteristic time function that for any point on the ring gives the time required to reach Σ. Note that $\boldsymbol{P}_1 : S^1 \to \Sigma$ and $\boldsymbol{P}_2 : S^1 \to \Sigma$.

For the section Σ we have chosen the plane $y_1 + y_2 + y_3 = 0$ since the only condition required was for the section to be between the two equilibrium points. The local coordinates of the plane chosen are the two first coordinates (y_1, y_2), thus the projection is $\hat{\pi}(y_1, y_2, y_3) = (y_1, y_2)$.

In practice it is easy to compute numerically the points of the first hit maps with some easy modifications of the Taylor method designed in the previous section. We just have to evaluate the function $g(y_1, y_2, y_3) = y_1 + y_2 + y_3$ at the same time we integrate the solutions. When the function g changes sign, it means that in that step we have crossed the section Σ. Then using a Newton method we can refine the point of the intersection. If $\boldsymbol{y}_n(h)$ is the Taylor series of the last step we have to find the step-size h such that $g(\boldsymbol{y}_n(h)) = 0$, therefore the Newton method in question is

$$h_{new} = h_{old} - \frac{\boldsymbol{y}_{n,1}(h_{old}) + \boldsymbol{y}_{n,2}(h_{old}) + \boldsymbol{y}_{n,3}(h_{old})}{\boldsymbol{y}'_{n,1}(h_{old}) + \boldsymbol{y}'_{n,2}(h_{old}) + \boldsymbol{y}'_{n,3}(h_{old})},$$

where $\boldsymbol{y}'_n(h)$ and $\boldsymbol{y}'_n(h)$ is evaluated using a Horner method and a modified Horner method for the derivatives of a polynomial.

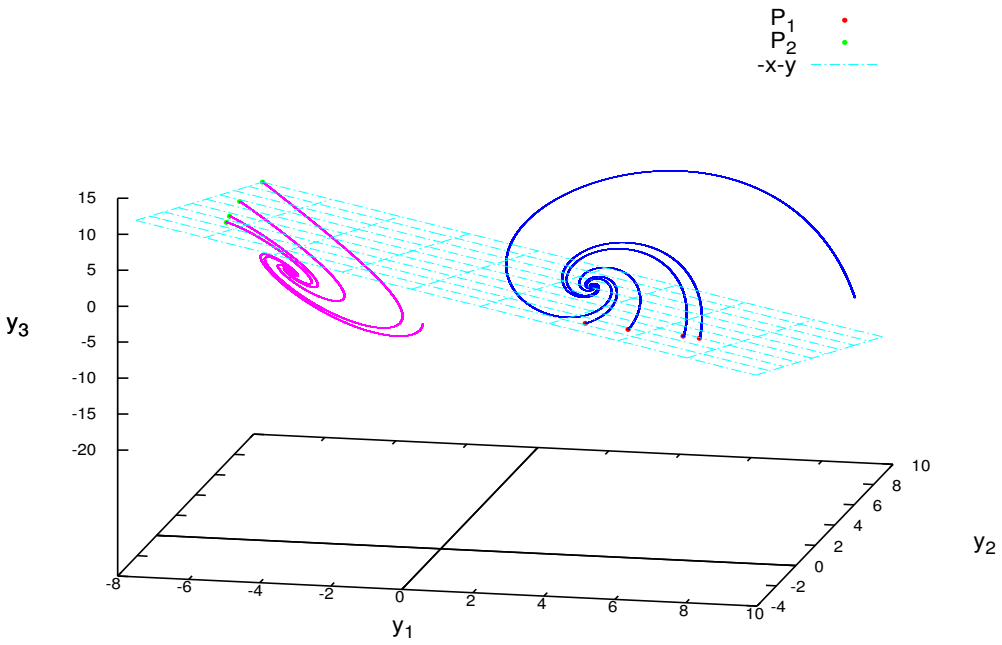


Figure 8: Illustration on how the first hit maps are computed numerically for c=3 and 5 different starting angles on each ring.

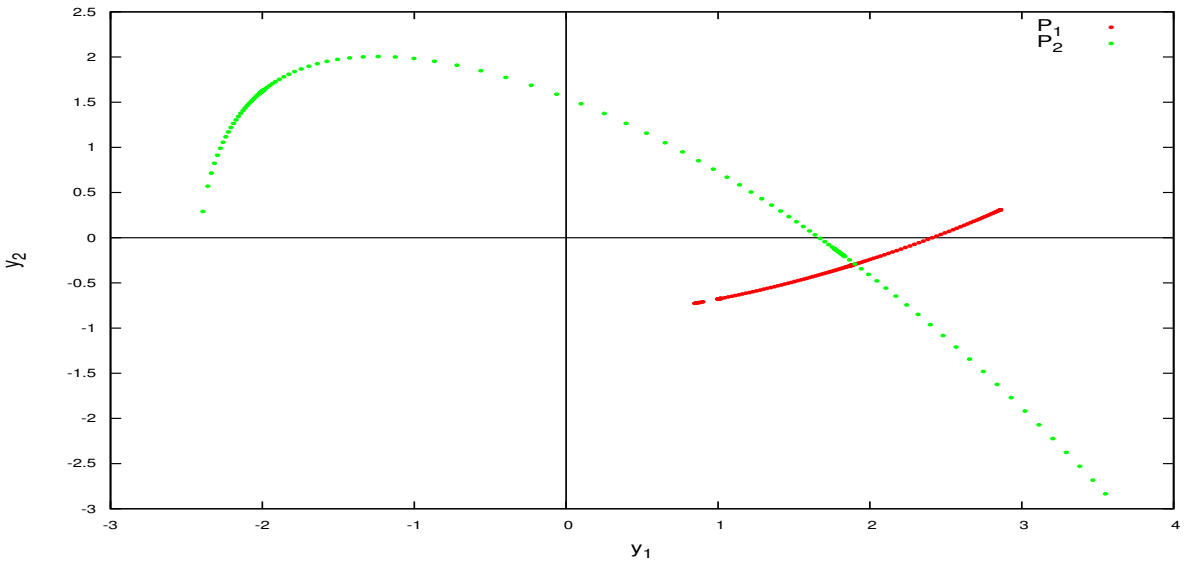


Figure 9: First hit maps for c=0.8 and 150 equispaced angles for each ring, in the coordinates of the section Σ.

▶ 4.2.3. The Newton Method. The heteroclinic connexions occur when two orbits on each of invariant manifold coincide. Therefore on Σ it correspond to the situation of two points of the maps coinciding $\boldsymbol{P}_1(\theta_1) = \boldsymbol{P}_2(\theta_2)$. For example, see from Figure 9 that for $c = 0.8$, the maps intersect so this means that orbit passing through that point is an heteroclinic orbit.

Still, we do not want to find this connexions graphically so we should design some method to do this numerically. From the argument stated before the problem of finding heteroclinic orbits can be reduced to the problem of finding the zeroes of the map $\boldsymbol{\Gamma}(\theta_1, \theta_2) = \boldsymbol{P}_1(\theta_1) - \boldsymbol{P}_2(\theta_2)$ which can be found by a two dimensional Newton method.

To compute the Newton method it is required the differential of the map $\boldsymbol{\Gamma}(\theta_1, \theta_2) : S^1 \times S^1 \to \Sigma$, thus differentiating

$$\boldsymbol{D}_{(\theta_1,\theta_2)}\boldsymbol{\Gamma}(\theta_1, \theta_2) = \begin{pmatrix} \dfrac{\partial \boldsymbol{P}_1(\theta_1)}{\partial \theta_1} & \dfrac{\partial \boldsymbol{P}_2(\theta_2)}{\partial \theta_2} \end{pmatrix}.$$

Next we will compute the first of this derivatives, for the one with $\boldsymbol{P}_2$ it is exactly the same.

$$\begin{aligned}
\frac{\partial \boldsymbol{P}_1(\theta_1)}{\partial \theta_1} &= \frac{\partial}{\partial \theta_1}\left(\hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right)\right) \\
&= \boldsymbol{D}_{(y_1,y_2,y_3)}\hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right) \cdot \frac{\partial}{\partial \theta_1}\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right)\right) \\
&= \boldsymbol{D}_{(y_1,y_2,y_3)}\hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right) \cdot \left[\frac{\partial}{\partial x}\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1)\right)\right) \cdot \right. \\
&\quad \left. \cdot \frac{\partial}{\partial \theta_1}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right) + \frac{\partial}{\partial \theta_1}\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right]
\end{aligned}$$

Each of these derivatives are

$$\boldsymbol{D}_{(y_1,y_2,y_3)}\hat{\pi} \circ \boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right) = \left.\begin{pmatrix} \dfrac{\partial \hat{\pi}_1}{\partial y_1} & \dfrac{\partial \hat{\pi}_1}{\partial y_2} & \dfrac{\partial \hat{\pi}_1}{\partial y_3} \\ \dfrac{\partial \hat{\pi}_2}{\partial y_1} & \dfrac{\partial \hat{\pi}_2}{\partial y_2} & \dfrac{\partial \hat{\pi}_2}{\partial y_3} \end{pmatrix}\right|_{\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\frac{\partial}{\partial x}\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1), 0\right)\right) = \boldsymbol{f}\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right); \boldsymbol{R}_{P_+}(\theta_1), 0\right)\right),$$

$$\frac{\partial}{\partial \theta_1}\left(\boldsymbol{R}_{P_+}\right)(\theta_1) = \begin{pmatrix} \dfrac{\partial y_1(\theta_1)}{\partial \theta_1} & \dfrac{\partial y_2(\theta_1)}{\partial \theta_1} & \dfrac{\partial y_3(\theta_1)}{\partial \theta_1} \end{pmatrix}^T = \begin{pmatrix} -\varepsilon \sin\theta_1 & \varepsilon\cos\theta_1 & +(1+\lambda_+^2)\varepsilon\sin\theta_1 - \lambda_+\varepsilon\cos\theta_1 \end{pmatrix}^T.$$

Note that the second derivative is values of the vector field at the intersection which we compute numerically along the solutions.

There is still a remaining term which apparently we do not know, which is the derivative of the characteristic time function which we do not know implicitly. We shall denote it N and L the following terms

$$L = \frac{\partial}{\partial x}\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right);\boldsymbol{R}_{P_+}(\theta_1),0\right)\right).$$

$$N = \frac{\partial}{\partial \theta_1}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right).$$

Note that L was the value of the vector field at the intersection and has dimension 3×1 and N is a scalar since $\tau_+ : \mathbb{R}^3 \to \mathbb{R}$ and $\boldsymbol{R}_{P_+} : S^1 \to \mathbb{R}^3$. This term N which is unknown, is obtained by the differentiating implicit equation $g(y_1, y_2, y_3) = y_1 + y_2 + y_3 = 0$ of Σ.

$$g\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right);\boldsymbol{R}_{P_+}(\theta_1)\right)\right) = 0,$$

differentiating

$$\frac{\partial}{\partial \theta_1} g\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right);\boldsymbol{R}_{P_+}(\theta_1)\right)\right) = 0,$$

$$\boldsymbol{D}_{(y_1,y_2,y_3)} g\left(\boldsymbol{\varphi}\left(\tau_+\left(\boldsymbol{R}_{P_+}(\theta_1)\right);\boldsymbol{R}_{P_+}(\theta_1)\right)\right)\cdot\left[NL + \frac{\partial}{\partial \theta_1}\left(\boldsymbol{R}_{P_+}(\theta_1)\right)\right] = 0,$$

therefore

$$(1,1,1)\cdot\left[N\begin{pmatrix}L_1\\L_2\\L_3\end{pmatrix} + \begin{pmatrix}-\varepsilon\sin\theta_1\\ \varepsilon\cos\theta_1\\ +(1+\lambda_+^2)\varepsilon\sin\theta_1 - \lambda_+\varepsilon\cos\theta_1\end{pmatrix}\right] = 0,$$

and

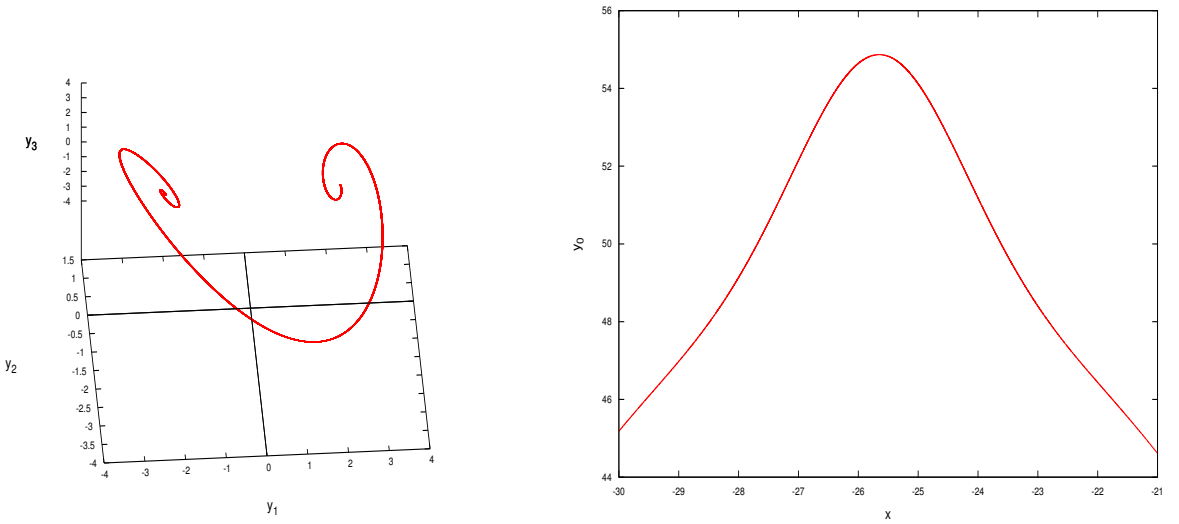
$$N = \frac{\lambda_+^2\varepsilon\sin\theta_1 + (1-\lambda_+)\varepsilon\cos\theta_1}{L_1+L_2+L_3}.$$

Once the differential of $\boldsymbol{\Gamma}(\theta_1, \theta_2)$ is

$$\begin{pmatrix}\theta_{1,new}\\ \theta_{2,new}\end{pmatrix} = \begin{pmatrix}\theta_{1,old}\\ \theta_{2,old}\end{pmatrix} - \boldsymbol{D}_{(\theta_1,\theta_2)}\boldsymbol{\Gamma}(\theta_{1,old},\theta_{2,old})^{-1}\boldsymbol{\Gamma}(\theta_{1,old},\theta_{2,old})^T.$$

► 4.2.4. Heteroclinic orbits and the KS. Given an heteroclinic orbit of the Michelson system, the corresponding travelling wave in KS equation is what in [6] is called a conical flame front. It is a kind of travelling wave that has a single maximum and its slopes tends to $\pm\sqrt{2}$ as $x \to \mp\infty$.

In the following figure it is shown a heteroclinic orbit computed with the Newton method seen above and the corresponding travelling wave. Note from the subfigure (b) why these kind of flames are called conical fronts.



(a) Heteroclinic orbit of the Michelson system.

(b) Corresponding travelling-wave of KS system of the heteroclinic orbit.

Figure 10: Heteroclinic orbit found with the Newton method of this section for c=1.5.

4.3. Poincaré Map: This part covers the concept of Poincaré maps and their use in tracking periodic orbits. Moreover, an outline of an algorithm to find periodic orbits using a Poincaré map is given for the Michelson system. However, unlike the two previous sections, in this part the method has not been programmed.

▶ 4.3.1. Poincaré Section and Map. Let $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$ be any autonomuous ODE system with vector field $\boldsymbol{f} : \Omega \to \mathbb{R}^n$. A codimension one surface $\Sigma = \{\boldsymbol{y} \in \Omega \mid g(\boldsymbol{y}) = 0\} \subseteq \Omega$ is said to be *transversal* to the vector field if $\forall \boldsymbol{y} \in \Sigma$, $\left\langle \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{D}_{\boldsymbol{y}} g(\boldsymbol{y}) \right\rangle \neq 0$, where $\langle \cdot, \cdot \rangle$ stands for the scalar product and the term $\boldsymbol{D}_{\boldsymbol{y}} g(\boldsymbol{y})$ is the normal vector of the surface. Note that what this transversality condition imply is that the solutions of the system cannot be tangent to the section. A codimension one surface with the previous transversality condition is called a *Poincaré section*.

A *Poncaré map* or a *first return map*, $\mathcal{P} : \Sigma \to \Sigma$, is an application from a Poincaré section Σ into itself, defined as

$$\mathcal{P}(\hat{\boldsymbol{y}}) = \hat{\pi} \circ \varphi(\tau(\nu(\hat{\boldsymbol{y}})), \nu(\hat{\boldsymbol{y}})),$$

where $\hat{\pi} : \Omega \to \Sigma$ is a projection , $\varphi : I \times \Omega \to \Omega$ is the flow of the system, $\nu : \Sigma \to \Omega$ is an inclusion and $\tau : \Sigma \to \mathbb{R}$ is the return time function for which given a point in the section gives the time x it takes for the point to return to the section.

From a practical standpoint, what the Poincaré map does is take points in a section and integrates them until they return too the section. The transversality condition imply that when integrating the points in the section the points immediately leave the section, hence the map is well defined as a discreet application.

► 4.3.2. Periodic Orbits and Poincaré Maps. This kind of maps are of great use since they allow us to study some dynamics of a system with a map which has one dimension less than the original system. This idea is the same than the one used in the previous section where heteroclinic orbits of the Michelson system were found using the map $\mathbf{\Gamma}(\theta_1, \theta_2)$, reducing an apparent three dimensional problem into a problem of finding fixed points in two dimensions.

The main use of the Poincaré maps is the location of periodic orbits. From the way the Poincaré maps are defined, the fixed points of such maps correspond to periodic orbits for the original system. Therefore the problem of finding periodic orbits is reduced to the problem of finding the fixed points of the Poncaré maps, $\mathcal{P}(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}$, or equivalently finding the zeroes of $\mathcal{P}(\hat{\boldsymbol{y}}) - \hat{\boldsymbol{y}}$. In order to find the zeroes in this last expression one can use a Newton method.

Observation: For the the periodic orbits to be fixed points of the Poincaré map we shall define the return time τ as the time it takes for a point $\boldsymbol{y} = \nu(\hat{\boldsymbol{y}})$ to return to the section with the same orientation that originally had, i.e. we have to additionally ask that

$$sign\left(\left\langle \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{D}_{\boldsymbol{y}} g(\boldsymbol{y})\right\rangle\right) = sign\left(\left\langle \boldsymbol{f}(\varphi(\tau(\boldsymbol{y}), \boldsymbol{y})), \boldsymbol{D}_{\boldsymbol{y}} g(\varphi(\tau(\boldsymbol{y}), \boldsymbol{y}))\right\rangle\right).$$

► 4.3.3. Poincaré Maps for the Michelson System. Up until now it looks as if finding periodic orbits with a Poincaré map is trivial. Nonetheless, it is not the case and we will illustrate the problems that arise using the Michelson system as an example.

The main problem is the choice of Poincaré section. It may occur that a such section simply do not exist or it is impossible to find. What's more, even though such a section is found, there is no guarantee of any periodic orbits crossing it thus it rendering useless.

Now, we will try to compute a Poincaré section for the Michelson system. Consider a general surface Σ which has implicit equation $g(\boldsymbol{y}) = 0$. Then for it to be a Poincaré section it must satisfy the transverslity condition, therefore

$$\left\langle \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{D}_{\boldsymbol{y}} g(\boldsymbol{y})\right\rangle = \frac{\partial g(\boldsymbol{y})}{\partial y_1} y_2 + \frac{\partial g(\boldsymbol{y})}{\partial y_2} y_3 + \frac{\partial g(\boldsymbol{y})}{\partial y_3}\left(c^2 - y_2 - \frac{y_1^2}{2}\right) \neq 0,$$

for all $\boldsymbol{y} \in \Sigma$. Note that for the previous inequality there is no unique solution.

Our suggestion for a Poincaré section would be the surface $\Sigma = \{\boldsymbol{y} \in \mathbb{R}^3 \mid y_1 = 0, y_2 \neq 0\}$. This choice is based on the assumption that the interesting dynamics occur amidst the two equilibrium points, so the idea was to choose the $y_1 = 0$ plane. However, the points with $y_2 = 0$ had to be subtracted to make the section satisfy the transversality condition.

► 4.3.4. Periodic Orbit Algorithm for the Michelson system. Recall that periodic orbits were the zeroes of $\mathcal{P}(\hat{\boldsymbol{y}}) - \hat{\boldsymbol{y}}$, and thus the differential of the Poincaré section is needed to implement a Newton method.

Differentiating and using the chain rule,

$$\begin{aligned}
\boldsymbol{D}_{\hat{\boldsymbol{y}}}\mathcal{P}(\hat{\boldsymbol{y}}) &= \boldsymbol{D}_{\hat{\boldsymbol{y}}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\\
&= \boldsymbol{D}_{\boldsymbol{y}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}\left(\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\\
&= \boldsymbol{D}_{\boldsymbol{y}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\cdot\Big[\frac{\partial}{\partial x}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\tau(\nu(\hat{\boldsymbol{y}})))\\
&\quad+\boldsymbol{D}_{\boldsymbol{y}}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\nu(\hat{\boldsymbol{y}}))\Big]\\
&= \boldsymbol{D}_{\boldsymbol{y}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\cdot\Big[\frac{\partial}{\partial x}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\boldsymbol{y}}(\tau(\nu(\hat{\boldsymbol{y}})))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\nu(\hat{\boldsymbol{y}}))+\\
&\quad+\boldsymbol{D}_{\boldsymbol{y}}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\nu(\hat{\boldsymbol{y}}))\Big].
\end{aligned}$$

For the section $\Sigma=\{\boldsymbol{y}\in\mathbb{R}^3\mid y_1=0, y_2\neq 0\}$, we have that $\hat{\pi}(y_1,y_2,y_3)=(y_2,y_3)$ and $\nu(\hat{y}_2,\hat{y}_3)=(0,\hat{y}_2,\hat{y}_3)$. Then, each derivative is

- The differential of the projection at the point at which returns is

$$\boldsymbol{D}_{\boldsymbol{y}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)=\begin{pmatrix}\frac{\partial\hat{\pi}_1}{\partial y_1} & \frac{\partial\hat{\pi}_1}{\partial y_2} & \frac{\partial\hat{\pi}_1}{\partial y_3}\\ \frac{\partial\hat{\pi}_2}{\partial y_1} & \frac{\partial\hat{\pi}_2}{\partial y_2} & \frac{\partial\hat{\pi}_2}{\partial y_3}\end{pmatrix}\Bigg|_{\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))}=\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}.$$

- The differential of the inclusion at the initial point is

$$\boldsymbol{D}_{\hat{\boldsymbol{y}}}\left(\nu(\hat{\boldsymbol{y}})\right)=\begin{pmatrix}\frac{\partial\nu_1}{\partial\hat{y}_1} & \frac{\partial\nu_1}{\partial\hat{y}_2}\\ \frac{\partial\nu_2}{\partial\hat{y}_1} & \frac{\partial\nu_2}{\partial\hat{y}_2}\\ \frac{\partial\nu_3}{\partial\hat{y}_1} & \frac{\partial\nu_3}{\partial\hat{y}_2}\end{pmatrix}=\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}.$$

- $\boldsymbol{D}_{\boldsymbol{y}}\left(\hat{\pi}\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)$ is the fundamental matrix found by solving the variational equations at the point of the section at which returns.
- $\frac{\partial}{\partial x}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))=\boldsymbol{f}(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))$ is the value of the vector field at the point of the section at which returns.

Finally the term $\boldsymbol{D}_{\boldsymbol{y}}(\tau(\nu(\hat{\boldsymbol{y}})))$ which has dimension 2×1 is obtained differentiating the implicit equation of the Poincaré section Σ, which is $g(\boldsymbol{y})=y_1=0$.

$$g\left(\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)=0$$

Differentiating with respect to $\hat{\boldsymbol{y}}$,

$$\begin{aligned}
\boldsymbol{D}_{\hat{\boldsymbol{y}}}\left(g\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right) &= \boldsymbol{D}_{\boldsymbol{y}}\left(g\circ\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\right)\cdot\Big[\frac{\partial}{\partial x}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\boldsymbol{y}}(\tau(\nu(\hat{\boldsymbol{y}})))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\nu(\hat{\boldsymbol{y}}))+\\
&\quad+\boldsymbol{D}_{\boldsymbol{y}}\varphi(\tau(\nu(\hat{\boldsymbol{y}})),\nu(\hat{\boldsymbol{y}}))\cdot\boldsymbol{D}_{\hat{\boldsymbol{y}}}(\nu(\hat{\boldsymbol{y}}))\Big]=\boldsymbol{0}.
\end{aligned}$$

Isolating $\boldsymbol{D}_{\boldsymbol{y}}(\tau(\nu(\hat{\boldsymbol{y}})))$ in this system of two equations, where all the derivatives are known, give the two components of the term.

Finally the fixed points are computes using a Newton method on $\mathcal{P}(\hat{\boldsymbol{y}}) - \hat{\boldsymbol{y}} = \boldsymbol{0}$, where the steps are given by

$$\begin{pmatrix} \hat{y}_{2,new} \\ \hat{y}_{3,new} \end{pmatrix} = \left(\boldsymbol{D}_{(\hat{y}_2,\hat{y}_3)} \mathcal{P}(\hat{\boldsymbol{y}}) - Id \right)^{-1} \left[\mathcal{P}(\hat{y}_{2,old}, \hat{y}_{3,old}) - \begin{pmatrix} \hat{y}_{2,old} \\ \hat{y}_{3,old} \end{pmatrix} \right]$$

► 4.3.5. Variational System. For this part we will focus on how to compute the fundamental matrix, by solving the variational equations. Recall that this matrix was needed to find the differential of the Poincaré section. In the end it is explained how to solve find the matrix numerically for the Michelson system.

First we will explain what the variational system is and how it is computed in the more general case. Consider the initial value problem of an ODE system with $\boldsymbol{f} : \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$,

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}). \tag{4.10}$$

Then the variational system with respect to the initial conditions of a given initial value problem $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ are given by the following system

$$\boldsymbol{v}' = D\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{y}_0, x)\boldsymbol{v}. \tag{4.11}$$

What this variational system does is, given a vector $\boldsymbol{v}_0 \in \mathbb{R}^n$, the solution of (4.11) with initial conditions $\boldsymbol{v}(x_0) = \boldsymbol{v}_0$ give a first order approximation of the solutions of (4.10) with perturbed initial conditions $\boldsymbol{y}(x_0) = \boldsymbol{y_0} + \boldsymbol{v}_0$.

In particular we are interested in the fundamental matrix of (4.10). Fixing an orthonormal base of $\mathbb{R}^n$, for simplicity take the usual base $\{\boldsymbol{e}_i\}_{1 \le i \le n}$, we shall solve the n variational systems with initial conditions the elements of the base

$$\begin{cases} \boldsymbol{v}_i' = D\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{y}, x)\boldsymbol{v}_i, \\ \boldsymbol{v}_i(x_0) = \boldsymbol{e}_i, \end{cases} \qquad 1 \le i \le n.$$

The matrix $M(x) = (\boldsymbol{v}_1(x)|...|\boldsymbol{v}_n(x))$, whose columns are the solutions of the previous systems, is known as the *principal fundamental matrix* of the system (4.10) and the point $\boldsymbol{y}_0$. Recall that since the solutions of a linear system form a base, therefore the solutions for (4.11) with any initial condition $\boldsymbol{v}(x_0) = \boldsymbol{v}_0$ is given by $M(x)\boldsymbol{v}_0$.

In the case of the Michelson, to obtain the principal fundamental matrix, we have to solve the systems

$$\begin{pmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_1 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \end{pmatrix} \qquad \Rightarrow \qquad \begin{cases} v_{i,1}' = v_2, \\ v_{i,2}' = v_3, \\ v_{i,3}' = -y_0 v_1 - v_2, \end{cases}$$

for $1 \leq i \leq 3$ and initial conditions $\boldsymbol{v}_i = \boldsymbol{e}_i$. Note that the principal fundamental matrix varies along the solutions of the Michelson system. Thus, in practice we integrate the previous initial value problem yielding the principal fundamental matrix at the same time we integrate the solutions of the original system. This is done numerically using again a Taylor method with the same criterion as the previous section with the change in the recurrence formula

$$\begin{pmatrix} b_{1,m+1} \\ b_{2,m+1} \\ b_{3,m+1} \end{pmatrix} = \frac{1}{m+1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_1 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_{1,m} \\ b_{2,m} \\ b_{3,m} \end{pmatrix}.$$

A. NOTATION

A.1. Multi-Index Notation. This notation is a powerful tool when writing "objects" in several variables with many indices in a compact way .A *multi-index* is no more than a positive integer vector, $\boldsymbol{l} = (l_1, ..., l_n) \in (\mathbb{N} \cup \{0\})^n$. The following points specify their use in different contexts.

► A.1.1. Multi-Index in Partial Derivatives. Let $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{C}^n$ be a variable of many variables, and $f(\boldsymbol{x}) \in C^\omega(\mathcal{U})$ (so that the derivatives commute) a function on $\boldsymbol{x}$, where $\mathcal{U} \in \mathbb{C}^n$. Then its partial derivatives in multi-index notation are denoted by

$$\frac{\partial^{|\boldsymbol{l}|_1} f(x_1, ..., x_n)}{\partial^{l_1} x_1 ... \partial^{l_n} x_n} \overset{\text{not.}}{\equiv} \partial^{\boldsymbol{l}}_{\boldsymbol{x}} f(\boldsymbol{x}), \tag{A.1}$$

where from now $|\boldsymbol{l}|_1 = l_1 + ... + l_n$. Note that the component l_i of the multi-index denote the order of the derivative for the variable x_i. For EDPs with several variables and high orders this notation is particularly useful.

► A.1.2. Multi-Index in Polynomials. Let $\boldsymbol{x} = (x_1, ..., x_n)$ be several indeterminates grouped in a vector for commodity, and $P \in \mathbb{C}^n[\boldsymbol{x}]$ be a polynomial. Then, the multi-indices denote the powers at which each of the indeterminates are raised and help in book-keeping the coefficients, as follows

$$a_{(l_1,...,l_n)} x^{l_1} ... \, x^{l_n} \overset{\text{not.}}{\equiv} a_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}} \qquad \Longrightarrow \qquad P(\boldsymbol{x}) \overset{\text{not.}}{\equiv} \sum_{\boldsymbol{l} \in L} a_{\boldsymbol{l}} x^{\boldsymbol{l}}, \tag{A.2}$$

where L is a set of multi-indeces.

► A.1.3. Multi-Index in Formal Series. The case of series of several indeterminates is kind of a generalization of the previous case adding for all possible values of $\boldsymbol{l}$. However, the notation is more practical, since the resulting expression resembles the one indeterminate case. Let $S \in \mathbb{C}^n[[\boldsymbol{x}]]$ be a formal series, then it is expressed in multi-index notation as

$$S(\boldsymbol{x}) = \sum_{l_1} \cdots \sum_{l_n} a_{(l_1,...,l_n)} x^{l_1} ... \, x^{l_n} \overset{\text{not.}}{\equiv} \sum_{\boldsymbol{l}} a_{\boldsymbol{l}} \boldsymbol{x}^{\boldsymbol{l}}, \tag{A.3}$$

where the $\boldsymbol{l}$ under the right-hand side summation means the sum for all possible multi-index.

An interesting case comes from combining the this case with the first one. Thus, the Taylor series expansion of a given function $f(\boldsymbol{x}) \in C^\omega(\mathcal{U})$, in several variables for a neighbourhood of $\boldsymbol{\xi} = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, can be written as

$$f(\boldsymbol{x}) \overset{\text{not.}}{\equiv} \sum_{\boldsymbol{l}} \frac{1}{l_1! \, ... \, l_n!} \, \partial^{\boldsymbol{l}}_{\boldsymbol{x}} f(\boldsymbol{\xi}) \, (\boldsymbol{x} - \boldsymbol{\xi})^{\boldsymbol{l}}. \tag{A.4}$$

B. REVIEW ON SOME ALGEBRAIC TOPICS.

B.1. Discriminant: Let $P(x)$ be a polynomial with complex coefficients, in one indeterminate, and of degree n.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

According to the fundamental theorem of algebra, the polynomial $P(x)$ has n roots (not required to be different). Say λ_1,...,λ_n are such roots, then the discriminant is defined as

$$\Delta := a_n^{2n-2} \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

Depending on the value of the discriminant, the following can be said about the roots,

- $\Delta > 0$, imply that for some integer k, such that $0 \leq k \leq n/4$, there are $2k$ pairs of complex conjugated roots and $n - 4k$ real roots (all different).
- $\Delta = 0$, imply that at least two roots coincide which may be real or complex.
- $\Delta < 0$, imply that for some integer k, such that $0 \leq k \leq (n-2)/4$, there are $2k + 1$ pairs of complex conjugated roots and $n - 4k - 2$ real roots (all different).

Particular case: In the case of a third degree polynomial,

$$P(x) = x^3 + ax^2 + bx + c = 0,$$

the discriminant has the form $\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^2$. Furthermore if $a = 0$, then it follows $\Delta = -4b^3 - 27c^2$. Depending on the value of the discriminant the nature of the roots are

- $\Delta > 0$, imply that all roots are real and different.
- $\Delta = 0$, imply that all roots are real and one has multiplicity larger than one (is repeated).
- $\Delta < 0$, imply that one root is real and the other two are non-real complex conjugated roots.

B.2. Vieta's formulas: Let $P(x)$ be a polynomial with complex coefficients, in one indeterminate, and of degree n.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

According to the fundamental theorem of algebra, the polynomial $P(x)$ has n roots (not required to be different). Say λ_1,...,λ_n are such roots, then Vieta's formulas are

$$\sum_i \lambda_i = -\frac{a_{n-1}}{a_n}, \qquad \sum_{i \neq j} \lambda_i \lambda_j = \frac{a_{n-2}}{a_n}, \qquad \sum_{i \neq j \neq k \neq i} \lambda_i \lambda_j \lambda_k = -\frac{a_{n-3}}{a_n}, \qquad \lambda_1 \cdot ... \cdot \lambda_n = (-1)^n \frac{a_0}{a_n}.$$

Particular case: For a third degree polynomial of the type

$$P(x) = x^3 + ax^2 + bx + c = 0,$$

which has roots $\lambda_1, \lambda_2, \lambda_3$, Vieta's formulas are

$$\lambda_1 + \lambda_2 + \lambda_3 = -a, \qquad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = b, \qquad \lambda_1\lambda_2\lambda_3 = -c.$$

C. A QUALITATIVE INTRODUCTION TO INDEX THEORY

This appendix purpose is to give some qualitative definitions and basic results on the Coley index theory for flows. Note that this a complete theory in itselt and it is not our purpose to deepen into it. Most of these definitions and results require further concepts that are not explained here. For complete discussion one should refer to [3] and a brief yet more detailed introduction than this refer to [8].

C.1. Conley Index. Before introducing the index some previous definitions are required. A set S of the phase space Ω is called an *invariant set* if it is the union of solution curves of the system,

$$S = \bigcup_{x \in \mathbb{R}} \varphi(x, S)$$

where $\varphi : \mathbb{R} \times \Omega \to \Omega$ is the flow. In particular, an invariant set is said to be *isolated* (an isolated invariant set) if it is the maximal invariant set in some neighbourhood of itself. For example, an hyperbolic equilibrium point is isolated invariant set since in a small neighbourhood all solutions leave in positive or negative time.

It is isolated invariant set that we want to study. However, since invariant sets themselves are difficult to study, as they can have fractal structure, be chaotic or be structurally unstable (sensible to perturbations), the approach to avoid these complications is to consider isolating neighbourhoods instead. A compact set $N \subseteq \Omega$ is an *isolating neighbourhood* of an isolated invariant set S, if $S \subseteq int(N)$. Unlike the isolated invariant sets, the isolating neighbourhoods are relatively easy to study and robust under perturbations (formally speaking, robust under continuation).

At this point we need one last definition which is that of an exit set. Given an isolating neighbourhood N, its *exit set*, $L \subseteq N$ is the maximal set such that $\forall \boldsymbol{y} \in L$, $\exists x > 0$ such that $\varphi(x, \boldsymbol{y}) \notin N$. In other words, the exit set L of N is the larger subset such that all its points eventually end up leaving N in positive time under the action of the flow. Recall that in this work time is regarded as x.

Finally, the *homotopy Conley index* or (simply Conley index) of an isolated invariant set S, is the homotopy type of the pointed topological space obtained by collapsing the exit set L of any isolating neighbourhood N, to a point. This is formally

$$h(S) \sim (N/L, [L]),$$

for any isolating neighbourhood N. Now we will briefly see some properties of the Conley index which are inherited by the properties of isolating neighbourhood and exit sets.

- Let N and N' be two isolating neighbourhood for a same isolated invariant set S, then

  $$h(S) \sim (N/L, [L]) \sim (N'/L', [L']).$$

  This is, the Conley index does not depend on the isolating neighbourhood chosen.

- If the Conley index is non trivial $h(S) \neq \bar{0}$ (we will see how $\bar{0}$ is defined in the next section), then $S \neq \varnothing$.
- If φ_λ is a family of flows related by continuation (i.e. the flows are related by an homotopy depending on the parameter λ) and S_{λ_1}, S_{λ_2} isolated invariant sets related by continuation, then $h(S_{\lambda_1}) = h(S_{\lambda_2})$. This is the Conley index being invariant under continuation.

C.2. Conley Index Results. The point of this work is not to discuss index theory. Still some two more results should be give away for completeness since they appear in section 3.2.

► C.2.1. Sum of Indices. Two operations can be done developed for the Conley index, namely the sums and products. We are interested in the sum which is referred as wedge product in the context of pointed topological spaces. In particular the sum of isolated invariant sets.

Given two isolated invariant sets S_1 and S_2, the sum (or wedge product) is achieved by considering two the pointed homotopy types of the Conley index and collapsing the two points. The sum properly speaking the Conley is index of $S_1 \cup S_2$, but in the case of isolated invariant sets since they are disjoint and there are disjoint isolating neighbourhood, it can be computed as stated previously.

The sum is well defined and obviously has neutral element it is referred to as $\bar{0}$ which correspond to the homotopy type of a pointed point. This element obviously has to be $h(\varnothing) = \bar{0}$.

► C.2.2. Lemma. The Conley index of an hyperbolic fixed point is an n dimensional pointed sphere (n-sphere), where n is the number of positive real part eigenvalues of the linearised system.

D. PROGRAMS SOURCE CODE

D.1. Header File. File containing all subroutines.

```
/* File Calling Functions. */

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

/* Makes a copy a vector of dimension 4. */
void copyvector (double y0[4], double y1[4]) {
        int i;
        for(i=0; i<4; i++){
                y1[i]=y0[i];
        }
        return;
}

/* Normalizes vectors of dimension 3. */
void normalize (double vec[3]){
        int i;
        double mod;
        mod=sqrt(vec[0]*vec[0]+vec[1]*vec[1]+vec[2]*vec[2]);
        for(i=0; i<3; i++){
                vec[i]/=mod;
        }
        return;
}

/* Evaluates three polynomials of degree 24, using Horner algorithm and saves
    them in the first three components of y0. */
void horner (double a[3][27], double y0[4], double h){
        int i, k;
        double b[3][25];

        b[0][24]=a[0][24];
        b[1][24]=a[1][24];
        b[2][24]=a[2][24];

        for(k=0; k<3; k++){
                for(i=23; i>=0; i--){
                        b[k][i]=a[k][i]+b[k][i+1]*h;
                }
        }

        y0[0]=b[0][0];
        y0[1]=b[1][0];
        y0[2]=b[2][0];
        return;
}

/* Evaluates the derivative of three polynomials of degree 24, using Horner
    algorithm and saves them in the first three components of y0der. */
void hornerder (double a[3][27], double y0der[4], double h){
        int i, k;
        double b[3][24];
```

```
        b[0][23]=24*a[0][24];
        b[1][23]=24*a[1][24];
        b[2][23]=24*a[2][24];

        for(k=0; k<3; k++){
                for(i=22; i>=0; i--){
                        b[k][i]=a[k][i+1]*(i+1)+b[k][i+1]*h;
                }
        }

        y0der[0]=b[0][0];
        y0der[1]=b[1][0];
        y0der[2]=b[2][0];

        return;
}

/* Computes the Taylor series up to order 26 of the ODE at the point y0[4]. */
void taylorseries (double c, double y0[4], double a[3][27]){
        int m, m2, k;
        double sum;

    /* The first coefficient of the series, a0 is the initial condition. */
    a[0][0]=y0[0];
    a[1][0]=y0[1];
    a[2][0]=y0[2];

    /* For the next coefficients of the series, iterate the recurrence formulas.
       (The first case is done separately). */
    a[0][1]=a[1][0];
    a[1][1]=a[2][0];
    a[2][1]=c*c-a[1][0]-(1./2)*a[0][0]*a[0][0];

    for(m=1; m<=25; m++){
        a[0][m+1]=a[1][m]/(m+1);
        a[1][m+1]=a[2][m]/(m+1);
        sum=0;
        m2=(m+1)/2;
        for(k=0; k<m2; k++){
                sum+=a[0][k]*a[0][m-k];
        }
        if(m%2==0){
                sum+=(1./2)*a[0][m2]*a[0][m2];
        }
        a[2][m+1]=-((a[1][m]+sum)/(m+1));
    }
    return;
}

/* Computes the optimal step-size (largest) for which the series has a fixed
   error and minimum n° operations. */
double optimalstep (double a[3][27], double tol){
        int i;
        double h=0., hprov, hopt, bound, modulo;
        double mod[3], rho[3];

        /* Makes three estimates of the radius of convergence. */
```

```
        mod[0]=sqrt(a[0][24]*a[0][24]+a[1][24]*a[1][24]+a[2][24]*a[2][24]);
        mod[1]=sqrt(a[0][25]*a[0][25]+a[1][25]*a[1][25]+a[2][25]*a[2][25]);
        mod[2]=sqrt(a[0][26]*a[0][26]+a[1][26]*a[1][26]+a[2][26]*a[2][26]);

        rho[0]=1./pow(mod[0],1./24);
        rho[1]=1./pow(mod[1],1./25);
        rho[2]=1./pow(mod[2],1./26);

        /* Takes the minimum of the estimates of the radius of convergence. */
        if(rho[0]<rho[1]){
                if(rho[0]<rho[2]){
                        hopt=rho[0]/exp(2);
                }else{
                        hopt=rho[2]/exp(2);
                }
        }else{
                if(rho[1]<rho[2]){
                        hopt=rho[1]/exp(2);
                }else{
                        hopt=rho[2]/exp(2);
                }
        }

        /* Control possible cancellations and give if necessary a smaller step-
            size. */
        bound=sqrt(a[0][0]*a[0][0]+a[1][0]*a[1][0]+a[2][0]*a[2][0])+hopt*sqrt(a
            [0][1]*a[0][1]+a[1][1]*a[1][1]+a[2][1]*a[2][1]);
        for(i=2; i<=25; i++){
                modulo=sqrt(a[0][i]*a[0][i]+a[1][i]*a[1][i]+a[2][i]*a[2][i]);
                hprov=pow(bound/modulo,1./i);
                if(hopt>hprov && hprov>h){
                        h=hprov;
                }
        }
        if(fabs(h)<tol){
                h=hopt;
        }
        return h;
}

/* Estimates the step-size for which the distance in the phase space is a defined
     value (heuristically). */
double adjuststep (double c, double dist, double tol, double y0safe[4], double y0
    [4], int sign){
        double h;
        double f[3], dify0[3], p[3];

        /* Computes the vector field. */
        f[0]=y0[1];
        f[1]=y0[2];
        f[2]=c*c-y0[1]-y0[0]*y0[0]/2;

        /* Computes the difference between y_{n-1} and y_{0}. */
        dify0[0]=y0[0]-y0safe[0];
        dify0[1]=y0[1]-y0safe[1];
        dify0[2]=y0[2]-y0safe[2];
```

```
        /* Computes the coefficients of the second degree polynomial the solution
             of which is the approximated h. */
        p[0]=f[0]*f[0]+f[1]*f[1]+f[2]*f[2];
        p[1]=f[0]*dify0[0]+f[1]*dify0[1]+f[2]*dify0[2];
        p[2]=dify0[0]*dify0[0]+dify0[1]*dify0[1]+dify0[2]*dify0[2];

        /* Evaluates the polynomial and chooses the right sign. */
        if(fabs(p[0])>tol){
                if(sign==1){
                        h=((-p[1])+sqrt(p[1]*p[1]+p[0]*(dist*dist-p[2])))/p[0];
                }else{
                        h=((-p[1])-sqrt(p[1]*p[1]+p[0]*(dist*dist-p[2])))/p[0];
                }
        }else{
                h=0.;
        }
        return h;
}

/* Uses a Newton method on the implicit equation of a section to find the
    intersection with the orbit. */
void newtonpoincare (double c, double y0[4], double a[3][27], double tol){
        double h=0.1, g, gder;
        double y0der[4];

        do{
                horner(a,y0,h);
                hornerder(a,y0der,h);
                g=y0[0]+y0[1]+y0[2];
                gder=y0der[0]+y0der[1]+y0der[2];
                h-=g/gder;
        }while(fabs(g)>tol);
        horner(a,y0,h);
        y0[3]+=h;
        return;
}

/* Given an initial y0 point computes the next point in the orbit forward or back
    in time (depending on the sign variable introduced) and saves it in y0.
 * Integrates at maximum speed with no restrictions.
 *
 * integrator Variables:
 *              h -> The step-size for each iteration.
 *              y0[4] -> Vector containing the initial data (y0,y1,y2,x), and in
    the end the output.
 *              a[3][27] -> Coefficients of the truncated Taylor series.
 *              sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *              event -> 0 the process can continue. 1 the process is to slow and
     stops. */

void integrator (double c, double y0[4], int sign, int *event, double tol){
        double mod, h;
        double y1[4], a[3][27];

        *event=0;
        taylorseries(c,y0,a);
        h=((double)sign)*optimalstep(a,tol);
```

```
        horner(a,y1,h);
        mod=sqrt((y1[0]-y0[0])*(y1[0]-y0[0])+(y1[1]-y0[1])*(y1[1]-y0[1])+(y1[2]-
            y0[2])*(y1[2]-y0[2]));
        copyvector(y1, y0);

        /* If the ''advance'' in the phase space is close to zero, stop iterating
            . */
        if(mod<tol){
                *event=1;
                printf("The integration found a problem.\n");
        }
        y0[3]+=h;
        return;
}

/* Given an initial y0 point computes the next point in the orbit forward or back
    in time (depending on the sign variable introduced) and saves it in y0.
 * Additionally, controls that the distance of the point returned is constant in
    the phase space.
 *
 * integratorcont Variables:
 *              h -> The step-size for each iteration.
 *              hfinal -> The total h, as the sum of the step-size of all
    iterations. (May iterate more than once if h too small).
 *              itermax -> Sets a maximum number of iterates.
 *              y0[4] -> Vector containing the initial data (y0,y1,y2,x), and in
    the end the output.
 *              a[3][27] -> Coefficients of the truncated Taylor series.
 *              sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *              event -> 0 the process can continue. 1 the process is to slow and
     stops.
 *              dist -> distance between points in the phase space. */

void integratorcont (double c, double y0[4], int sign, int *event, double dist,
    double tol){
        int iter=0, itermax=1.e2;
        double h, hfinal=0., mod;
        double y0safe[4], y1[4], a[3][27];

        copyvector(y0,y0safe);
        copyvector(y0,y1);
        *event=0;
        do{
                taylorseries(c,y0,a);
                h=((double)sign)*optimalstep(a,tol);
                horner(a,y1,h);
                mod=sqrt((y1[0]-y0safe[0])*(y1[0]-y0safe[0])+(y1[1]-y0safe[1])*(
                    y1[1]-y0safe[1])+(y1[2]-y0safe[2])*(y1[2]-y0safe[2]));
                /* If the distance in the phase space is too large, finds the
                    step-size that advances the preset distance. */
                if(mod>dist){
                        /* Estimates the approximated step and uses Horner to
                            compute the final solution. */
                        h=adjuststep(c,dist,tol,y0safe,y0,sign);
                        horner(a,y1,h);
                }
                copyvector(y1, y0);
```

```
                hfinal+=h;
                iter++;
                /* If the step is too small do another iterate since it uses the
                    optimal step-size which is the fastest in n° of operations.
                    */
                /* If the ''advance'' in the phase space is close to zero or the
                    iterates exceed a certain number, stop iterating. */
        }while(mod<dist && mod>tol && iter<=itermax);

        if(mod<tol || iter>itermax){
                *event=1;
                printf("The integration found a problem.\n");
        }
        y0[3]+=hfinal;
        return;
}

/* Given an initial y0 point computes the next point in the orbit forward or back
    in time (depending on the sign variable introduced) and saves it in y0.
 * Additionally, controls that the distance of the point returned is constant in
    the phase space.
 * Additionally, controls if the orbit crosses a given section, and if it does
    stops.
 *
 * integratorcont Variables:
 *              h -> The step-size for each iteration.
 *              hfinal -> The total h, as the sum of the step-size of all
    iterations. (May iterate more than once if h too small).
 *              itermax -> Sets a maximum number of iterates.
 *              g, gold -> Value of the implicit equation of at the integrated
    point and the previous one rep.
 *              y0[4] -> Vector containing the initial data (y0,y1,y2,x), and in
    the end the output.
 *              a[3][27] -> Coefficients of the truncated Taylor series.
 *              sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *              event -> 0 the process can continue. 1 the process is to slow and
     stops. 2 if the orbit crosses the section.
 *              dist -> distance between points in the phase space. */

void integratorcontsect (double c, double y0[4], int sign, int *event, double
    dist, double tol){
        int iter=0, itermax=1.e2;
        double h, hfinal=0., mod, g, gold;
        double y0safe[4], y1[4], a[3][27];

        copyvector(y0, y0safe);
        copyvector(y0, y1);
        *event=0;
        g=y0[0]+y0[1]+y0[2];
        do{
                taylorseries(c,y0,a);
                h=((double)sign)*optimalstep(a,tol);
                horner(a,y1,h);
                mod=sqrt((y1[0]-y0safe[0])*(y1[0]-y0safe[0])+(y1[1]-y0safe[1])*(
                    y1[1]-y0safe[1])+(y1[2]-y0safe[2])*(y1[2]-y0safe[2]));
                /* If the distance in the phase space is too large, finds the
                    step-size that advances the fixed distance. */
```

```
                if(mod>dist){
                        /* Estimates the appropriate h and uses Horner to compute
                             the final solution. */
                        h=adjuststep(c,dist,tol,y0safe,y0,sign);
                        horner(a,y1,h);
                }
                /* Checks if it crosses the section and refines the intersection
                    with a Newton method. */
                gold=g;
                g=y1[0]+y1[1]+y1[2];
                if(g*gold<=0){
                        newtonpoincare(c,y0,a,tol);
                        *event=2;
                        break;
                }
                /* Prepares the next step. */
                copyvector(y1, y0);
                hfinal+=h;
                iter++;
                /* If the step is too small do another iterate since it uses the
                    optimal step-size which is the fastest in n° of operations.
                    */
                /* If the ''advance'' in the phase space is close to zero stop
                    iterating. (Close to a fixed point). */
        }while(mod<dist && mod>tol && iter<=itermax);

        if(mod<tol || iter>itermax){
                *event=1;
                printf("The integration found a problem.\n");
        }
        y0[3]+=hfinal;
        return;
}

/* Entering a file with an orbit of the Michelson system, computes the integral
    to obtain a solution of the KS and writes it in another file.
 * Since the points are given numerically, the method is the trapezoidal rule. */
void solutionKS (FILE *output1, FILE *output2){
        double sum=0.;
        double y0[4], y1[4], y0safe[4];

        rewind(output1);
        fscanf(output1, " %le %le %le %le", &y0safe[0], &y0safe[1], &y0safe[2], &
            y0safe[3]);
        copyvector(y0safe,y0);
        while(fscanf(output1, " %le %le %le %le", &y1[0], &y1[1], &y1[2], &y1[3])
             == 4){
                /* Controls resets the initial condition when it detects the
                    change between semiorbits forward and back in time. */
                if((y0[0]-y0safe[0])*(y1[0]-y0safe[0])<0){
                        copyvector(y0safe,y0);
                        sum=0.;
                }
                sum+=(y1[1]+y0[1])*(y1[0]-y0[0])/2;

                fprintf(output2, "%.14le %.14le\n", y1[0], sum);
                copyvector(y1,y0);
        }
```

```
        return;
}

/* Return the real eigenvalue, the eigenvectors, the sign to integrate the node
    manifold, and the implicit equation of the focus invariant plane,
 * for the hyperbolic fixed points. (For c different from 0). */
double eigen (double c, double evec1[3], double evec2[3], double evec3[3], double
     invplane[4], int *sign, double tol){
        double eval=c;
        double x0;

        /* Finds the real root of the characteristic polynomial by a Newton
            method. */
        do{
                x0=eval;
                eval=x0-(x0*x0*x0+x0+sqrt(2)*c)/(3*x0*x0+1);
        }while(fabs(eval-x0)>tol);

        /* Computes the eigenvector of the node eigenvalue and normalizes it. */
        evec1[0]=1.;
        evec1[1]=eval;
        evec1[2]=eval*eval;
        normalize(evec1);

        /* Computes the eigenvectors of the focus eigenvalues and normalizes them
            . */
        evec2[0]=1.;
        evec2[1]=0.;
        evec2[2]=-(1.+eval*eval);
        normalize(evec2);
        evec3[0]=0.;
        evec3[1]=1.;
        evec3[2]=-eval;
        normalize(evec3);

        /* Sets the sign (to integrate forward or backward) for the node's
            manifold. */
        if(eval>=0){
                *sign=1;
        }else{
                *sign=-1;
        }

        /* Computes the coefficients of the implicit equation of the eigenplane
            of the focus manifold. */
        invplane[0]=1+eval*eval;
        invplane[1]=eval;
        invplane[2]=1.;
        invplane[3]=-(1+eval*eval)*sqrt(2)*c;

        return eval;
}
```

D.2. Taylor Method. Program that integrates orbits using the Taylor method.

```
/*
 * Integrator-v5.c       (Taylor Method)
 * Author: Alberto García Molina
 */

#include "Auxiliary.h"

void writefile (double, double[4], double[4], int, double, double, FILE *);

/* main Variables:
 *               c -> Parameter of the vector field.
 *               y0[4] -> Vector containing the initial data (y0,y1,y2,x).
 *               sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *               dist -> distance between points in the phase space.
 * Files:
 *               Solution.dat -> File contains (x0,y0,y1,y2) for the orbit. */


int main (void) {
        /* Initialize variables. */
        int sign;
        double c=1., dist=1.e-3, tol=1.e-12;
        double y0[4]={0.,0.,0.,0.}, y0safe[4];

    /* Opens the file. */
    FILE *output1, *output2;
    output1=fopen("SolutionMichelson.dat", "w+");
    output2=fopen("SolutionKS.dat", "w");
    if (output1 == NULL || output2 == NULL){
        printf("Error with output.\n");
        exit(1);
    }
    /* Copies the initial conditions on a safety vector. */
    copyvector(y0,y0safe);
    fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0], y0[1], y0[2])
        ;
    /* Integrate forward in time for y0[4] and writes in file. */
    sign=1;
    writefile(c,y0,y0safe,sign,dist,tol,output1);
    /* Resets the initial condition. */
    copyvector(y0safe,y0);
    /* Integrates back in time for y0[4] and writes in file. */
    sign=-1;
    writefile(c,y0,y0safe,sign,dist,tol,output1);

    /* Integrates to find the solution to the KS. */
    solutionKS(output1,output2);

    fclose(output1);
    fclose(output2);
    printf("Process Completed.");
    return 0;
}

/* Writes the points of a semiorbit in a file.
```

```
 *
 * writefile Variables:
 *               itermax -> Maximum number of iterates for control. (For instance,
     to avoid infinite computations in periodic orbits).
 *               plotlim -> maximum distance of the computations wrt the initial
    condition. (Plot restricted in a domain).
 *               event -> 0 the process can continue one more step. 1 the process
    has some cannot continue. */

void writefile (double c, double y0[4], double y0safe[4], int sign, double dist,
    double tol, FILE *output1){
        int iter, itermax=1.e6, event;
        double plotlim=10;

        iter=0;
        do{
                integratorcont(c,y0,sign,&event,dist,tol);
            iter++;
            fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0], y0
                [1], y0[2]);
        }while(iter<itermax && fabs(y0[0]-y0safe[0])<plotlim && fabs(y0[1]-y0safe
            [1])<plotlim && fabs(y0[2]-y0safe[2])<plotlim && event==0);
        return;
}
```

D.3. Invariant Manifolds. Program that integrates orbits using the Taylor method on the linear approximations of the invariant manifolds.

```
/*
 * Integrator-v5.c       (Taylor Method)
 * Author: Alberto García Molina
 */

#include "Auxiliary.h"

void writefile (double, double[4], double[4], int, double, double, FILE *);

/* main Variables:
 *               c -> Parameter of the vector field.
 *               y0[4] -> Vector containing the initial data (y0,y1,y2,x).
 *               sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *               dist -> distance between points in the phase space.
 *               eval -> node eigenvalue.
 *               evec1[2] -> node eigenvector
 *               evec2[3], evec3[3] -> focus eigenvectors.
 *               invplane[4] -> coefficients of the invariant plane of the focus A
    (y0)+B(y1)+C(y2)+D)=0.
 *               radius -> distance from the fixed point to the invariant manifold
     starting computation point.
 *               theta -> angle to compute orbits on a ring in the focus manifold.
 *               divisions -> number subdivisions of the ring where the focus
    manifold orbits are computed.
 * Files:
 *               NodeManifold.dat -> File contains (x0,y0,y1,y2) for orbits in the
     node manifold.
```

```
 *              FocusManifold.dat -> File contains (x0,y0,y1,y2) for orbits in
    the focus manifold.
 *              ManifoldInfo -> Self explicatory, contains information of the
    linearized fixed point. */


int main (void) {
        /* Initialize variables. */
        int i, divisions=10, sign;
        double c=1., dist=1.e-9, tol=1.e-20, eval, radius=1.e-7, theta;
        double y0[4], y0safe[4], evec1[3], evec2[3], evec3[3], invplane[4];

    /* Opens the file. */
    FILE *output1, *output2, *output3;
    output1=fopen("NodeManifold.dat", "w");
    output2=fopen("FocusManifold.dat", "w");
    output3=fopen("ManifoldInfo.dat", "w");
    if (output1 == NULL || output2 == NULL || output3 == NULL){
        printf("Error with output.\n");
        exit(1);
    }

    eval=eigen(c,evec1,evec2,evec3,invplane,&sign,tol);
        /* Computes one branch of the node manifold. */
        y0[0]=sqrt(2)*c+radius*evec1[0];
        y0[1]=radius*evec1[1];
        y0[2]=radius*evec1[2];
        y0[3]=0.;
        copyvector(y0,y0safe);
        fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0], y0[1], y0
            [2]);
    writefile(c,y0,y0safe,sign,dist,tol,output1);
    /* Computes the other branch of the node manifold. */
        y0[0]=sqrt(2)*c-radius*evec1[0];
        y0[1]=-radius*evec1[1];
        y0[2]=-radius*evec1[2];
        y0[3]=0.;
        copyvector(y0,y0safe);
        fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0], y0[1], y0
            [2]);
        writefile(c,y0,y0safe,sign,dist,tol,output1);

        /* Computes the orbits of a ring in the focus manifold. */
        sign=-sign;
        for(i=0; i<divisions; i++){
                theta=i*2*M_PI/divisions;

                y0[0]=sqrt(2)*c+radius*cos(theta)*evec2[0]+radius*sin(theta)*
                    evec3[0];
                y0[1]=radius*cos(theta)*evec2[1]+radius*sin(theta)*evec3[1];
                y0[2]=radius*cos(theta)*evec2[2]+radius*sin(theta)*evec3[2];
                y0[3]=0.;
                copyvector(y0,y0safe);
                fprintf(output2, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0],
                    y0[1], y0[2]);
                writefile(c,y0,y0safe,sign,dist,tol,output2);
        }
```

```
        /* Saves information of the linearized system. */
        fprintf(output3, "Node eigenvalue: %.14le\n", eval);
        fprintf(output3, "Focus eigenvalue (1): %.14le+i%.14le\n", -eval/2, sqrt
            (1+(3./4)*eval*eval));
        fprintf(output3, "Focus eigenvalue (2): %.14le-i%.14le\n\n", -eval/2,
            sqrt(1+(3./4)*eval*eval));
        fprintf(output3, "Node eigenvector: (%.14le,%.14le,%.14le)\n", evec1[0],
            evec1[1], evec1[2]);
        fprintf(output3, "Focus eigenvector (1): (%.14le,%.14le,%.14le)\n", evec2
            [0], evec2[1], evec2[2]);
        fprintf(output3, "Focus eigenvector (2): (%.14le,%.14le,%.14le)\n\n",
            evec3[0], evec3[1], evec3[2]);
        fprintf(output3, "Implicit equation focus invariant plane: %+.14le(y0)
            %+.14le(y1)%+.14le(y2)%+.14le=0\n", invplane[0], invplane[1],
            invplane[2], invplane[3]);

    fclose(output1);
    fclose(output2);
    fclose(output3);
    printf("Process Completed.");
    return 0;
}

/* Writes the points of a semiorbit in a file.
 *
 * writefile Variables:
 *              itermax -> Maximum number of iterates for control. (For instance,
     to avoid infinite computations in periodic orbits).
 *              plotlim -> maximum distance of the computations wrt the initial
    condition. (Plot restricted in a domain).
 *              event -> 0 the process can continue one more step. 1 the process
    has some cannot continue. */

void writefile (double c, double y0[4], double y0safe[4], int sign, double dist,
    double tol, FILE *output1){
        int iter, itermax=1.e5, event;
        double plotlim=1.e-6;

        iter=0;
        do{
                integratorcont(c,y0,sign,&event,dist,tol);
            iter++;
            fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3], y0[0], y0
                [1], y0[2]);
        }while(iter<itermax && fabs(y0[0]-y0safe[0])<plotlim && fabs(y0[1]-y0safe
            [1])<plotlim && fabs(y0[2]-y0safe[2])<plotlim && event==0);
        return;
}
```

D.4. 2D-Heteroclinic Orbits. Program that find the 2D-heteroclinic orbits.

```
/*
 * Integrator-v5.c      (Taylor Method)
 * Author: Alberto García Molina
 */

#include "Auxiliary.h"

void writefile (double, double, double, FILE *);

/* main Variables:
 *              c -> Parameter of the vector field.
 *              y0[4] -> Vector containing the initial data (y0,y1,y2,x).
 *              sign -> +1 if iterates are forward in time. -1 if iterates are
    back in time.
 *              dist -> distance between points in the phase space.
 * Files:
 *              Solution.dat -> File contains (x0,y0,y1,y2) for the orbit. */


int main (void) {
        /* Initialize variables. */

        double c=0.8, dist=1.e-3, tol=1.e-12;

    /* Opens the file. */
    FILE *output1, *output2;
    output1=fopen("SectionFocus1.dat", "w+");
    output2=fopen("SectionFocus2.dat", "w");
    if (output1 == NULL || output2 == NULL){
        printf("Error with output.\n");
        exit(1);
    }

    writefile (c,dist,tol,output1);
    writefile (-c,dist,tol,output2);

    fclose(output1);
    fclose(output2);
    printf("Process Completed.");
    return 0;
}

/* Writes the points of a semiorbit in a file.
 *
 * writefile Variables:
 *              itermax -> Maximum number of iterates for control. (For instance,
     to avoid infinite computations in periodic orbits).
 *              plotlim -> maximum distance of the computations wrt the initial
    condition. (Plot restricted in a domain).
 *              event -> 0 the process can continue one more step. 1 the process
    has some cannot continue. */

void writefile (double c, double dist, double tol, FILE *output1){
        int iter, itermax=1.e5, event, i, divisions=150, sign;
        double plotlim=10, radius=1.e-7, theta;
        double y0[4], y0safe[4], evec1[3], evec2[3], evec3[3], invplane[4];
```

```
        eigen(c,evec1,evec2,evec3,invplane,&sign,tol);
        /* Computes the orbits of a ring in the focus manifold. */
        sign=-sign;
        for(i=0; i<divisions; i++){
                theta=i*2*M_PI/divisions;

                y0[0]=sqrt(2)*c+radius*cos(theta)*evec2[0]+radius*sin(theta)*
                    evec3[0];
                y0[1]=radius*cos(theta)*evec2[1]+radius*sin(theta)*evec3[1];
                y0[2]=radius*cos(theta)*evec2[2]+radius*sin(theta)*evec3[2];
                y0[3]=0.;
                copyvector(y0,y0safe);

                iter=0;
                do{
                        integratorcontsect(c,y0,sign,&event,dist,tol);
                    iter++;
                }while(iter<itermax && fabs(y0[0]-y0safe[0])<plotlim && fabs(y0
                    [1]-y0safe[1])<plotlim && fabs(y0[2]-y0safe[2])<plotlim &&
                    event==0);

                if(event==2){
                        fprintf(output1, "%.14le %.14le %.14le %.14le\n", y0[3],
                            y0[0], y0[1], y0[2]);
                }
        }
        return;
}
```

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