Treball Final de Grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques
Universitat de Barcelona

PONCELET'S PORISM

Andrés Rojas González

Director: Dr. Joan Carles Naranjo del Val

Realitzat a: Departament d'Àlgebra i Geometria

Barcelona, Juny 2016

Introduction

The Poncelet problem

Given two non-degenerate conics C and D in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, consider the following problem: constructing a closed polygon inscribed in C and circumscribed about D.

Assuming that the polygon may have self-intersections, a first approach to build such a polygon could be the next one. Take an arbitrary point $p_0 \in C$, and choose l_0 one of the two tangent lines to D passing through p_0. If the line l_0 is not tangent to C, there exists a point $p_1 \in C \cap l_0$ different from p_0. Then, take $l_1 \neq l_0$ the tangent line to D through p_1. In a similar way, l_1 must intersect C at a point $p_2 \neq p_1$.

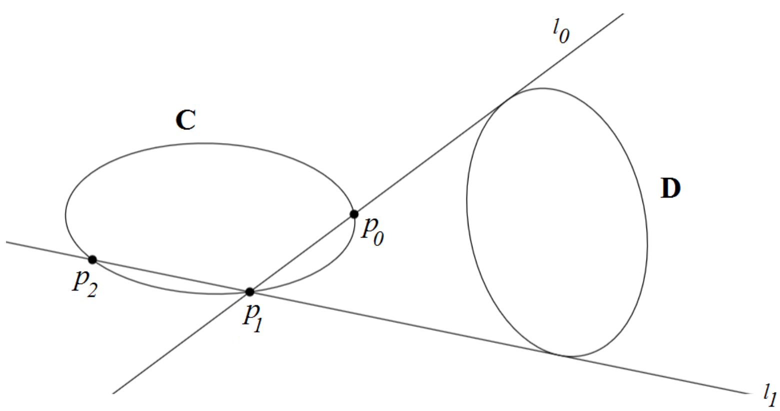

Figure 0.1

Iterating, we obtain sequences $\{p_0, p_1, p_2, \ldots\}$ and $\{l_0, l_1, l_2, \ldots\}$ of points on C and tangent lines to D, respectively.

In order to find the desired polygon, we want this process to come back to p_0, namely, $p_n = p_0$ for some $n \geq 3$. In that case, we will have an n-sided polygon inscribed in C and circumscribed about D, with vertices $p_0, \ldots, p_{n-1}$ and edges $l_0, \ldots, l_{n-1}$.

Poncelet's porism[1] is a beautiful result concerning this problem:

[1]The word *porism* has classical nature, and it's used in geometry to denote properties that are either never satisfied, or are satisfied in infinitely many cases.

Another famous result of this kind is Steiner's porism.

Theorem (Poncelet's porism). *This process closes (after n steps) for some initial point $p_0 \in C$ if, and only if, it closes (after n steps) for any initial point on C.*

In other words: if there exists an n-sided polygon inscribed in C and circumscribed about D, then any point on C is the vertex of an n-sided polygon which is also inscribed in C and circumscribed about D. In particular, there are infinitely many of these polygons.

A bit of history

This result was discovered by Jean-Victor Poncelet (1788-1867), a member of Napoleon's army, while he was a war prisoner in Saratov (Russia), in the period 1812-1814. Obviously, it was not raised in the "modern" terms we have used, and it was restricted to the case of two ellipses in the plane, with one lying inside the other.

During the captivity, Poncelet discovered many other important theorems on the development of projective geometry, which were gathered in the treatise *Traité des propriétés projectives des figures* (1822). This publication contained the first proof of Poncelet's porism, that had synthetic nature. A few years later, Jacobi gave another proof, based on the additivity of elliptic functions.

Cayley, in 1853, found explicit analytic conditions determining whether, for two given conics, there exists an n-sided polygon as the desired in the Poncelet problem. Lebesgue translated Cayley's results to the geometric language, and published these progress in *Les coniques* (1888).

Almost a hundred years later, the problem was studied again by Phillip Griffiths and Joe Harris, in terms of modern algebraic geometry.

The project

With the aim of making a first approach to the theory of complex algebraic curves, we will try to understand the point of view adopted by Griffiths and Harris in the papers [8] and [9]. All the necessary background, which involves different branches of mathematics (not only Algebraic Geometry), is exposed in the two first chapters.

Chapter 1 includes essential notions on conics and quadrics (many of them worked on the subject Projective Geometry) and introduces plane algebraic curves, which will play a fundamental role in the study of the Poncelet problem.

The second chapter is devoted to Riemann surfaces, a very particular case of topological surfaces. The concept of Riemann surface was devised in order to work in Complex Analysis (Riemann wanted to extend the domain of some analytic functions), but has become an essential tool in Geometry because of its identification with complex algebraic curves. At the end of the chapter, we specially focus on elliptic curves, a certain type of Riemann surfaces which can be endowed with a group structure.

After this previous theory, in chapter 3 we prove Poncelet's porism. The proof consists on translating the Poncelet problem to the study of the fixed points of a certain automorphism on the *Poncelet correspondence*, an elliptic curve.

In the next chapter we deal with Cayley's theorem, a criterion about the existence or not of polygons inscribed in a conic and circumscribed about another. This result is a consequence of the characterization of the torsion points on the Poncelet correspondence.

Once the Poncelet problem in the plane has been studied, chapter 5 discusses a generalization to the three-dimensional space. In particular, we analyze whether there exist polyhedra simultaneously inscribed in and circumscribed about a pair of quadric surfaces.

The project is finished by commenting, briefly, a topic linking the Poncelet problem with the area of Dynamical Systems: the mathematical billiards. Nowadays, this subject is constantly expanding and has many open problems.

Contents

Introduction **i**

1 Conics, quadrics and algebraic curves **1**

1.1 Projective conics and quadrics and their classification . . . 1

1.2 Polarity and tangency. Quadric envelopes . . . 2

1.3 Ruled quadrics in $\mathbb{P}^3_{\mathbb{C}}$ . . . 4

1.4 Plane algebraic curves . . . 7

1.5 Intersection of curves. Bézout theorem . . . 10

1.6 Conic pencils . . . 13

2 Riemann surfaces **15**

2.1 Definition and first examples . . . 15

2.2 Orientation and genus . . . 16

2.3 The complex torus . . . 17

2.4 Algebraic curves and Riemann surfaces . . . 18

2.5 Maps between Riemann surfaces . . . 20

2.6 Ramification and degree. Hurwitz formula . . . 22

2.7 Automorphisms of a complex torus . . . 24

2.8 Elliptic curves . . . 26

3 Poncelet's porism **31**

3.1 Poncelet correspondence . . . 31

3.2 $\mathfrak{M}$ as an algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ . . . 32

3.3 Structure of elliptic curve on $\mathfrak{M}$ . . . 33

3.4 Proof of Poncelet's porism . . . 36

4 Cayley's theorem **41**

4.1 A new algebraic equation for $\mathfrak{M}$ . . . 41

4.2 Cayley's theorem . . . 44

4.3 Some examples . . . 45

5 The Poncelet problem in $\mathbb{P}^3_{\mathbb{C}}$ **49**

5.1 Intersection of quadrics in $\mathbb{P}^3_{\mathbb{C}}$ . . . 49

5.2 Construction of polyhedra . . . 50

5.3 A Poncelet theorem in space . . . 52

Appendix. Mathematical billiards **55**

The billiard problem . . . 55

Elliptic billiards . . . 56

Bibliography **59**

Chapter 1

Conics, quadrics and algebraic curves

In this chapter, we explain all the geometric preliminaries that are necessary for the project.

With respect to conics and quadrics, some basic definitions and principles have been revised, in order to fix ideas and notations. In that case, proofs have been avoided. A detailed exposition can be found in [1].

Even if most of the concepts can be considered over an arbitrary field K, we will study only complex projective spaces, as they are the necessary ones to prove Poncelet's porism using Riemann surfaces.

1.1 Projective conics and quadrics and their classification

Definition. A *quadric* of $\mathbb{P}^n_{\mathbb{C}}$ is a class, modulo proportionality, of non-zero symmetric bilinear forms on $\mathbb{C}^{n+1}$.

Definition. Let $[\varphi]$ be the quadric represented by a non-zero symmetric bilinear form φ on $\mathbb{C}^{n+1}$. A point $p = [v] \in \mathbb{P}^n_{\mathbb{C}}$ is said to be a *point of the quadric* $[\varphi]$ if, and only if, $\varphi(v,v) = 0$. In that case, we will write $p \in [\varphi]$.

Let Δ a reference of $\mathbb{P}^n_{\mathbb{C}}$, and $A = \left(a_{ij}\right)_{0\leq i,j\leq n}$ the (symmetric) matrix of a non-zero symmetric bilinear form φ in a basis of $\mathbb{C}^{n+1}$ adapted to Δ. Then,

$$(x_0 : \ldots : x_n)_\Delta \in [\varphi] \iff 0 = \varphi((x_0,\ldots,x_n),(x_0,\ldots,x_n)) = \sum_{i,j=0}^{n} a_{ij}x_ix_j$$

Definition. A and $\sum\limits_{i,j=0}^{n} a_{ij}x_ix_j = 0$ are, respectively, the *matrix* and the *equation* of $[\varphi]$ relative to the reference Δ.

Definition. The *rank* of a quadric $[\varphi]$ of $\mathbb{P}^n_{\mathbb{C}}$ is the rank of the matrix of $[\varphi]$ relative to any reference of $\mathbb{P}^n_{\mathbb{C}}$. The quadric is *non-degenerate* when its rank is $n+1$.

Remarks.

1. Since the set of points of a quadric determines univoquely the quadric, the word *quadric* will also refer to it.

2. Fixed a reference of $\mathbb{P}^n_{\mathbb{C}}$, the matrix and the equation of a quadric are unique up to scalar multiplication.

3. Quadrics of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ are called *conics*.

An essential fact is that quadrics are projective notions, that is, the image $f(Q)$ of a quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$ under a projectivity $f : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ is a quadric of $\mathbb{P}^n_{\mathbb{C}}$.

In particular: if A is the matrix of Q relative to a reference Δ and f is given by M in the references Δ and Δ', then the matrix of $f(Q)$ relative to Δ' is $(M^{-1})^t A M^{-1}$.

Hence, it makes sense to classify quadrics of a projective space $\mathbb{P}^n_{\mathbb{C}}$ under the action of projectivities.

Definition. Two quadrics $Q, Q' \subset \mathbb{P}^n_{\mathbb{C}}$ are said to be *projectively equivalent* if, and only if, there exists a projectivity $f : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ such that $f(Q) = Q'$.

Theorem 1.1. *$Q, Q' \subset \mathbb{P}^n_{\mathbb{C}}$ are projectively equivalent $\Longleftrightarrow$ Q and Q' have the same rank.*

Example. Each conic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to a:

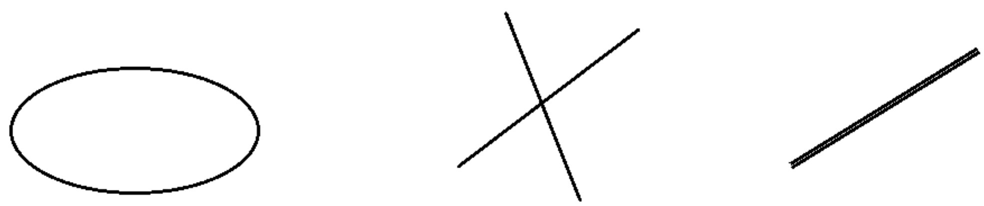

Figure 1.1

Non-degenerate conic (rank 3) Pair of lines (rank 2) Line counted twice (rank 1)

Remark. This projective classification holds when we work over any algebraically closed field K. Nevertheless, in real projective spaces, it's necessary to work with an extra projective invariant (aside from the rank): the index.

1.2 Polarity and tangency. Quadric envelopes

Let $Q = [\varphi]$ and $L = [F]$ be, respectively, a quadric and a linear variety of $\mathbb{P}^n_{\mathbb{C}}$.

- If $\varphi_{|F \times F} = 0$, we have $\varphi(v, v) = 0$ for each $v \in F$, and thus $L \subset Q$.
- If $\varphi_{|F \times F} \neq 0$, we have the quadric $Q \cap L = [\varphi_{|F \times F}]$ of L.

In particular, when we take a line l of $\mathbb{P}^n_{\mathbb{C}}$, there are three possible cases:

1. $Q \cap l$ is a single point (called *double point*)
2. $Q \cap l$ are two different points
3. $l \subset Q$

Definition. The line l is said to be *tangent* to Q in the cases 1 (*proper tangent line*) and 3. If l is tangent to Q, the points of $Q \cap l$ are called *contact points* of Q and l.

Definition. Let Q and L be, respectively, a quadric and a linear variety of $\mathbb{P}^n_{\mathbb{C}}$. Then, L is *tangent* to Q at a point $p \in \mathbb{P}^n_{\mathbb{C}}$ when the following conditions are satisfied:

- $p \in Q \cap L$
- For each point $q \in L \setminus \{p\}$, the line $p \vee q$ is tangent to Q.

Definition. Let Q be a non-degenerate quadric of $\mathbb{P}^n_{\mathbb{C}}$, with matrix A relative to a certain reference Δ. The *polarity induced by* Q is the projectivity

$$P_Q : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^{n\vee}_{\mathbb{C}}$$

that, in the references Δ of $\mathbb{P}^n_{\mathbb{C}}$ and $\Delta^\vee$ of $\mathbb{P}^{n\vee}_{\mathbb{C}}$, is given by the regular matrix A.

The image $P_Q(p)$ of a point $p \in \mathbb{P}^n_{\mathbb{C}}$ by P_Q is called the *polar hyperplane of p.*

Lemma 1.2. *If p is a point of a non-degenerate quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$, then $P_Q(p)$ is the tangent hyperplane to Q at p.*

Definition. A *quadric envelope* of $\mathbb{P}^n_{\mathbb{C}}$ is a quadric of the dual space $\mathbb{P}^{n\vee}_{\mathbb{C}}$.

Definition. The *envelope* of a non-degenerate quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$ is the image $Q^* = P_Q(Q)$ of Q by its own polarity.

Since Q^* is the image of a non-degenerate quadric by a projectivity, Q^* is a non-degenerate quadric of $\mathbb{P}^{n\vee}_{\mathbb{C}}$ (i.e., a non-degenerate quadric envelope of $\mathbb{P}^n_{\mathbb{C}}$). Conversely:

Lemma 1.3. *Each non-degenerate quadric envelope of $\mathbb{P}^n_{\mathbb{C}}$ is the envolope Q^* of a non-degenerate quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$ (univoquely determined).*

Proof. Let A be the symmetric matrix of a quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$ relative to a reference Δ. Since P_Q is given by A in the references Δ and $\Delta^\vee$, the matrix of the envelope Q^* relative to $\Delta^\vee$ is $(A^{-1})^t A A^{-1} = A^{-1}$.

Proportional regular matrices have proportional inverses and conversely. Hence, mapping each non-degenerate quadric $Q \subset \mathbb{P}^n_{\mathbb{C}}$ to its envelope Q^* is a bijection between the set of non-degenerate quadrics and the set of non-degenerate quadric envelopes of $\mathbb{P}^n_{\mathbb{C}}$. Namely:

- Each non-degenerate quadric envelope is the envelope of a non-degenerate quadric.
- Each non-degenerate quadric is determined by its envelope. □

Remark. Joining lemmas 1.2 and 1.3, we obtain that a non-degenerate quadric of $\mathbb{P}^{n\vee}_{\mathbb{C}}$ is the set of tangent hyperplanes to a non-degenerate quadric of $\mathbb{P}^n_{\mathbb{C}}$. For example, for $n = 2$, a conic envelope has the form

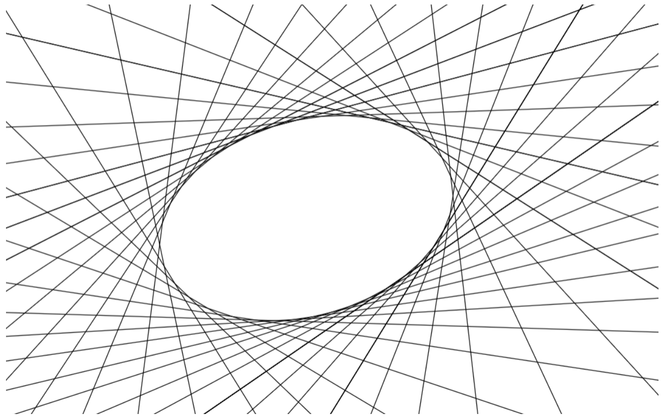

Figure 1.2

1.3 Ruled quadrics in $\mathbb{P}^3_{\mathbb{C}}$

In this section, we focus on non-degenerate quadrics of $\mathbb{P}^3_{\mathbb{C}}$. As we will see, they are ruled quadrics (there exist lines contained in them) and their tangent planes can be described easily.

This description will be very useful to generalize Poncelet's porism to the three-dimensional space.

Theorem 1.4. *If $Q \subset \mathbb{P}^3_{\mathbb{C}}$ is a non-degenerate quadric, there are two families A and B of lines lying on Q such that:*

1. *Any line contained in Q belongs to one (and only one) of the families.*
2. *Two different lines of the same family are disjoint.*
3. *Any two lines of different families meet.*
4. *For each point $p \in Q$, there is one line of each family going through p.*

Proof. Since, by theorem 1.1, all non-degenerate quadrics of $\mathbb{P}^3_{\mathbb{C}}$ are projectively equivalent, we can take a reference such that the equation for Q relative to it is $xt - yz = 0$.

Consider A the family of lines whose equations are

$$\begin{cases} \alpha_0 t - \alpha_1 y = 0 \\ \alpha_1 x - \alpha_0 z = 0 \end{cases}$$

for some $(\alpha_0, \alpha_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Similarly, let B the family of lines with equations

$$\begin{cases} \beta_0 t - \beta_1 z = 0 \\ \beta_1 x - \beta_0 y = 0 \end{cases}$$

for some $(\beta_0, \beta_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$. The lines described are trivially contained in Q. Furthermore, looking at the parameters α_i and β_i as homogeneous coordinates, A and B can be endowed with structure of $\mathbb{P}^1_{\mathbb{C}}$.

Suppose that we have two different lines $l_1, l_2 \in A$ with coordinates α_i and α_i', respectively. The coordinates of the points of $l_1 \cap l_2$ are the solutions of the system

$$\begin{cases} \alpha_0 t - \alpha_1 y = 0 \\ \alpha_1 x - \alpha_0 z = 0 \\ \alpha_0' t - \alpha_1' y = 0 \\ \alpha_1' x - \alpha_0' z = 0 \end{cases}$$

whose determinant can be easily checked to be $\begin{vmatrix} \alpha_0 & \alpha_1 \\ \alpha_0' & \alpha_1' \end{vmatrix}^2 \neq 0$ (this 2×2 determinant not being zero is equivalent to the lines l_1, l_2 being different). Hence, the system above has no non-zero solutions and $l_1 \cap l_2 = \emptyset$. A similar reasoning will give us that two different lines of B are disjoint.

On the other hand, the intersection of an A-line with a B-line is given by the system

$$\begin{cases} \alpha_0 t - \alpha_1 y = 0 \\ \alpha_1 x - \alpha_0 z = 0 \\ \beta_0 t - \beta_1 z = 0 \\ \beta_1 x - \beta_0 y = 0 \end{cases}$$

whose determinant is

$$\begin{vmatrix} 0 & -\alpha_1 & 0 & \alpha_0 \\ \alpha_1 & 0 & -\alpha_0 & 0 \\ 0 & 0 & -\beta_1 & \beta_0 \\ \beta_1 & -\beta_0 & 0 & 0 \end{vmatrix} = \alpha_1 \beta_0 \begin{vmatrix} 0 & \alpha_0 \\ -\beta_1 & \beta_0 \end{vmatrix} + \alpha_1 \beta_1 \begin{vmatrix} -\alpha_0 & 0 \\ -\beta_1 & \beta_0 \end{vmatrix} = \alpha_1 \beta_0 \alpha_0 \beta_1 - \alpha_0 \beta_0 \alpha_1 \beta_1 = 0$$

so that the system has non-trivial solutions and the intersection is non-empty. This proves 3.

Finally, let $p = (a : b : c : d)$ a point of Q. Then, there exists an A-line through p if, and only if,

$$\begin{cases} \alpha_0 d - \alpha_1 b = 0 \\ \alpha_1 a - \alpha_0 c = 0 \end{cases}$$

for some $(\alpha_0, \alpha_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$. This is equivalent to the determinant of the system (in variables α_0, α_1) being zero, i.e., $ad - bc = 0$. And this equality holds since $p \in Q$.

Hence for each point of Q there exists an A-line going through it. Note that this A-line will be unique, inasmuch as two different A-lines fail to meet.

The same argument holds for the existence of a B-line through p. $\square$

Remark. Consequently, mapping each pair of lines (l_A, l_B) (with l_A an A-line and l_B a B-line) to the point $l_A \cap l_B \in Q$ gives a bijection from $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ to Q.

Theorem 1.5. *The section of Q by a tangent plane is the pair of lines (contained in Q) through the contact point.*

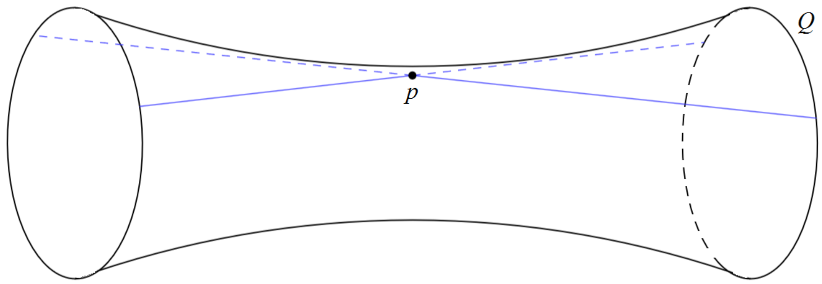

Figure 1.3

Proof. Given a point $p \in Q$, denote by T_pQ the tangent plane to Q at p. We want to study $Q \cap T_pQ$.

We know that any line meeting Q in three or more points is contained in Q. We claim that:

$$q \in Q \cap T_pQ \implies p \vee q \subset Q \cap T_pQ$$

In fact, the line $p \vee q$ lies on T_pQ (since $p, q \in T_pQ$) and meets Q in at least three points: p (a double point) and q. Thus $p \vee q \subset Q \cap T_pQ$.

Namely, we have a conic $Q \cap T_pQ$ (in the plane T_pQ) such that:

$$q \in Q \cap T_pQ \implies p \vee q \subset Q \cap T_pQ$$

So $Q \cap T_pQ$ must be a pair of lines through p. And these lines must be different (we have not a line counted twice) because Q is non-degenerate.

Since $Q \cap T_pQ$ is pair of different lines contained at Q meeting at the contact point p, we have the result. $\square$

For any line $l \subset \mathbb{P}^3_{\mathbb{C}}$, the set $l^* = \left\{H \in \mathbb{P}^{3\vee}_{\mathbb{C}} : l \subset H\right\}$ is a linear variety of $\mathbb{P}^{3\vee}_{\mathbb{C}}$ with dimension $3 - 1 - \dim l = 1$, i.e., it's a line in $\mathbb{P}^{3\vee}_{\mathbb{C}}$.

Then $l \subset Q$ if, and only if, $l^* \subset Q^*$. This claim can be easily proved assuming, by a change of coordinates, that Q has equation $xt - yz = 0$ (in that case, the equation for Q^* in $\mathbb{P}^{3\vee}_{\mathbb{C}}$ is the same one).

A consequence of this equivalence is:

Lemma 1.6.

1. *Let $l \subset \mathbb{P}^3_{\mathbb{C}}$ be a line such that $l \not\subset Q$. Then, there are exactly two tangent planes to Q containing l.*
2. *Any plane containing a line $l \subset Q$ is tangent to Q at some point of l.*

Proof. In 1, we want to study the planes tangent to Q and containing l, i.e., the elements of $l^* \cap Q^*$.

But if $l \not\subset Q$, we know that $l^* \not\subset Q^*$. So l^* is a line of $\mathbb{P}^{3\vee}_{\mathbb{C}}$ not contained in the non-degenerate quadric Q^* of $\mathbb{P}^{3\vee}_{\mathbb{C}}$, and the intersection $l^* \cap Q^*$ is a pair of points of $\mathbb{P}^{3\vee}_{\mathbb{C}}$.

In order to prove 2, suppose that a plane $\pi \subset \mathbb{P}^3_{\mathbb{C}}$ contains a line $l \subset Q$.

Then, joining $l^* \subset Q^*$ with $\pi \in l^*$ we deduce that $\pi \in Q^*$, namely, the plane π is tangent to Q. $\square$

1.4 Plane algebraic curves

Now, we introduce algebraic curves. They will be defined as the zero locus of polynomials in $\mathbb{P}^2_{\mathbb{C}}$ or the complex affine plane $\mathbb{C}^2$, and it will be a generalization of the conics we have described in section 1.1.

By the identification of $\mathbb{C}^2$ with $\{(x:y:z) \in \mathbb{P}^2_{\mathbb{C}} : z \neq 0\}$, the study of algebraic curves in both spaces will be essentialy the same. This allows us to choose the most appropiate context (affine or projective) for each case.

Definition. A *plane affine curve* is a zero locus

$$\{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$$

where $f \in \mathbb{C}[X,Y]$ is a non-zero polynomial.

Definition. A polynomial $F \in \mathbb{C}[X_1, \ldots, X_n]$ is *homogeneous of degree d* if all its monomials have degree d. Equivalently, if $F(\lambda X_1, \ldots \lambda X_n) = \lambda^d \cdot F(X_1, \ldots, X_n)$ for each $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition. A *plane projective curve* is a zero locus

$$\{(x:y:z) \in \mathbb{P}^2_{\mathbb{C}} : F(x,y,z) = 0\}$$

where $F \in \mathbb{C}[X,Y,Z]$ is a non-zero homogeneous polynomial.

Remark. If F is an homogeneous polynomial, the condition $F(x,y,z) = 0$ does not depend on the choice of coordinates for the points $(x:y:z) \in \mathbb{P}^2_{\mathbb{C}}$: for each $\lambda \in \mathbb{C} \setminus \{0\}$,

$$0 = F(\lambda x, \lambda y, \lambda z) = \lambda^d \cdot F(x,y,z) \iff 0 = F(x,y,z)$$

Therefore, the zero locus of an homogeneous polynomial in $\mathbb{P}^2_{\mathbb{C}}$ is well-defined.

Definition.

1. If γ is a projective plane curve given by an homogeneous polynomial $F(x,y,z)$, the plane affine curve $\{(x,y) \in \mathbb{C}^2 : F(x,y,1) = 0\}$ is called the *affine part of* γ.
2. Conversely, let γ' an affine plane curve given by a polynomial $f(x,y)$ of degree d. Then, $F(x,y,z) = z^d \cdot f(\frac{x}{z}, \frac{y}{z})$ is an homogeneous polynomial of degree d, such that $F(x,y,1) = f(x,y)$.

   The plane projective curve $\{(x:y:z) \in \mathbb{P}^2_{\mathbb{C}} : z^d \cdot f(\frac{x}{z}, \frac{y}{z}) = 0\}$ is called the *projective completion of* γ', and its affine part is γ'.

Definition.

1. A projective curve $\gamma = \{(x:y:z) \in \mathbb{P}^2_{\mathbb{C}} : F(x,y,z) = 0\}$ is said to be *non-singular at $a \in \gamma$* when

   $$\frac{\partial F}{\partial x}(a) \neq 0,\ \frac{\partial F}{\partial y}(a) \neq 0 \text{ or } \frac{\partial F}{\partial z}(a) \neq 0.$$

   In that case, the *tangent line to γ at a* is the line with equation $\frac{\partial F}{\partial x}(a) \cdot x + \frac{\partial F}{\partial y}(a) \cdot y + \frac{\partial F}{\partial z}(a) \cdot z = 0$.

2. An affine curve $\gamma' = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$ is said to be *non-singular at* $a = (a_1, a_2) \in \gamma'$ when

$$\frac{\partial f}{\partial x}(a) \neq 0 \text{ or } \frac{\partial f}{\partial y}(a) \neq 0.$$

In that case, the *tangent line to* γ' *at* a is the line with equation $\frac{\partial f}{\partial x}(a) \cdot (x - a_1) + \frac{\partial f}{\partial y}(a) \cdot (y - a_2) = 0$.

Example (the conic as a plane projective curve). Consider an arbitrary conic C of $\mathbb{P}^2_{\mathbb{C}}$, with matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

Its set of points is the zero locus of the homogeneous polynomial $F(x,y,z) = a_{00}x^2 + 2a_{01}xy + 2a_{02}xz + a_{11}y^2 + 2a_{12}yz + a_{22}z^2$, so we can see the conic as a plane projective curve. Since

$$\frac{\partial F}{\partial x} = 2a_{00}x + 2a_{01}y + 2a_{02}z,\ \frac{\partial F}{\partial y} = 2a_{01}x + 2a_{11}y + 2a_{12}z \text{ and } \frac{\partial F}{\partial z} = 2a_{02}x + 2a_{12}y + 2a_{22}z,$$

it's not difficult to see that:

- C is non-degenerate if, and only, it has no singular points.
- If C is a pair of lines (i.e., A has rank 2) the only singular point of C is the intersection of both lines.
- If C is a line counted twice (i.e., A has rank 1), each point of C is singular.

Furthermore, the tangent line to C at a non-singular point is the polar line.

Now, we want to see that the definition of a singular point in a projective curve is consistent when we consider its affine part. First of all, we need Euler's theorem for homogeneous polynomials:

Theorem 1.7 (Euler). *If $F(X_1, \ldots, X_n)$ is an homogeneous polynomial of degree d, then* $\sum_{i=1}^{n} X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F(X_1, \ldots, X_n)$.

Proof. By definition of homogeneous polynomial, for each $t \in \mathbb{C}$ the equality

$$F(tX_1, \ldots, tX_n) = t^d \cdot F(X_1, \ldots, X_n)$$

holds. Differenciating with respect to t, we have

$$\sum_{i=1}^{n} X_i \cdot \frac{\partial F}{\partial X_i}(tX_1, \ldots, tX_n) = dt^{d-1} \cdot F(X_1, \ldots, X_n)$$

(on the left side we use the chain rule). Taking $t = 1$, the result is obtained. □

Theorem 1.8. *Let $\gamma \subset \mathbb{P}^2_{\mathbb{C}}$ a plane projective curve with homogeneous equation $F(x,y,z) = 0$, and let $\gamma' \subset \mathbb{C}^2$ be its affine part. Then:*

$$(a_1 : a_2 : 1) \in \gamma \text{ is a non-singular point of } \gamma \iff (a_1, a_2) \in \gamma' \text{ is a non-singular point of } \gamma'$$

In such a case, the tangent line to γ' at (a_1, a_2) is the affine part of the tangent line to γ at $(a_1 : a_2 : 1)$.

Proof. Note that, if $f(x,y) = F(x,y,1)$ is the polynomial defining γ',

$\frac{\partial F}{\partial x}(x,y,1) = \frac{\partial f}{\partial x}(x,y)$, $\frac{\partial F}{\partial y}(x,y,1) = \frac{\partial f}{\partial y}(x,y)$

A combination of these equalities with Euler's theorem gives

$$0 = d \cdot F(a_1,a_2,1) = a_1 \cdot \frac{\partial F}{\partial x}(a_1,a_2,1) + a_2 \cdot \frac{\partial F}{\partial y}(a_1,a_2,1) + \frac{\partial F}{\partial z}(a_1,a_2,1) = a_1 \cdot \frac{\partial f}{\partial x}(a_1,a_2) + a_2 \cdot \frac{\partial f}{\partial y}(a_1,a_2)$$

$$+ \frac{\partial F}{\partial z}(a_1,a_2,1) \implies \frac{\partial F}{\partial z}(a_1,a_2,1) = -a_1 \cdot \frac{\partial f}{\partial x}(a_1,a_2) - a_2 \cdot \frac{\partial f}{\partial y}(a_1,a_2)$$

So that

$$(a_1,a_2) \in \gamma' \text{ is a singular point of } \gamma' \iff \frac{\partial f}{\partial x}(a_1,a_2) = 0 = \frac{\partial f}{\partial y}(a_1,a_2) \iff \frac{\partial F}{\partial x}(a_1,a_2,1) =$$

$$= \frac{\partial F}{\partial y}(a_1,a_2,1) = \frac{\partial F}{\partial z}(a_1,a_2,1) = 0 \iff (a_1 : a_2 : 1) \in \gamma \text{ is a singular point of } \gamma$$

Furthermore, the tangent line $\frac{\partial F}{\partial x}(a_1,a_2,1) \cdot x + \frac{\partial F}{\partial y}(a_1,a_2,1) \cdot y + \frac{\partial F}{\partial z}(a_1,a_2,1) \cdot z = 0$ has affine part

$$0 = \frac{\partial F}{\partial x}(a_1,a_2,1) \cdot x + \frac{\partial F}{\partial y}(a_1,a_2,1) \cdot y + \frac{\partial F}{\partial z}(a_1,a_2,1) = \frac{\partial f}{\partial x}(a_1,a_2) \cdot x + \frac{\partial f}{\partial y}(a_1,a_2) \cdot y - a_1 \cdot \frac{\partial f}{\partial x}(a_1,a_2)$$

$$-a_2 \cdot \frac{\partial f}{\partial y}(a_1,a_2) = (x - a_1) \cdot \frac{\partial f}{\partial x}(a_1,a_2) + (y - a_2) \cdot \frac{\partial f}{\partial y}(a_1,a_2)$$

that is the tangent line to γ' at the point (a_1,a_2). $\square$

The following result assures us that plane algebraic curves, as well as their singular points, are an invariant notion under the action of projectivities.

Lemma 1.9. *If $g : \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ is a projectivity and $\gamma \subset \mathbb{P}^2_{\mathbb{C}}$ is a plane projective curve, then $g(\gamma) \subset \mathbb{P}^2_{\mathbb{C}}$ is a plane projective curve. Furthermore:*

1. *If p is a singular point of γ, $g(p)$ is a singular point of the curve $g(\gamma)$.*
2. *If l is the tangent line to γ at a point $p \in \gamma$, then $g(l)$ is the tangent line to $g(\gamma)$ at $g(p)$.*

To finish this section, we define algebraic curves in the product space $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

As well as the concept of singular point of a projective plane curve can be studied "locally" restricting it to an affine plane curve, the same will hold for curves in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. In this case, we will use the cover of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ given by

$$A_1 = \{((x_0 : 1),(y_0 : 1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_0, y_0 \in \mathbb{C}\}, \qquad A_2 = \{((x_0 : 1),(1 : y_1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_0, y_1 \in \mathbb{C}\},$$

$$A_3 = \{((1 : x_1),(y_0 : 1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_1, y_0 \in \mathbb{C}\}, \qquad A_4 = \{((1 : x_1),(1 : y_1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_1, y_1 \in \mathbb{C}\}$$

(each of these subsets can be easily identified with the complex affine plane $\mathbb{C}^2$).

Definition. A polynomial $F \in \mathbb{C}[X_1,\dots,X_n,Y_1,\dots,Y_m]$ is said to be *bihomogenous of bidegree* (d,e) if

$$\forall \lambda, \mu \in \mathbb{C} \setminus \{0\} \quad F(\lambda X_1,\dots,\lambda X_n, \mu Y_1,\dots,\mu Y_m) = \lambda^d \mu^e \cdot F(X_1,\dots,X_n,Y_1,\dots,Y_m)$$

Equivalently, in each monomial of F the groups of indeterminates X_i and Y_j have degree d and e, respectively.

Definition. An *algebraic curve in* $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is a zero locus

$$\{((x_0 : x_1), (y_0 : y_1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : F(x_0, x_1, y_0, y_1) = 0\}$$

where $F \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$ is a non-zero bihomogeneous polynomial.

Definition. Let γ be an algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, with bihomogeneous equation $F(x_0, x_1, y_0, y_1) = 0$. A point $p = ((x_0 : 1), (y_0 : 1)) \in \gamma \cap A_1$ is a *singular point of* γ if, and only if, the point (x_0, y_0) is a singular point of the plane affine curve with equation $F(x, 1, y, 1) = 0$.

Similarly, we can describe all the singular points of γ in any subset $\gamma \cap A_i$.

1.5 Intersection of curves. Bézout theorem

In this part, we are interested in describing the intersection of two plane projective curves, focusing on the case of two conics. In order to understand the basic result, Bézout theorem, we will define the intersection number from the implicit function theorem for polynomials.

Nevertheless, the usual definition involves the notion of local ring at a point. For further details on this construction, as well as a proof of Bézout theorem, see [5].

Theorem 1.10 (implicit function theorem for polynomials). *Let $f(x, y)$ be a polynomial in two variables with complex coefficients, such that $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$ for some $a, b \in \mathbb{C}$.*

Then, there exist open neighbourhoods X and Y of a and b (respectively) in $\mathbb{C}$, and an holomorphic function $g : X \longrightarrow Y$ such that, for each $x \in X$, $f(x, g(x)) = 0$.

Remark. In other words, if $\frac{\partial f}{\partial y}(a, b) \neq 0$, the plane affine curve $f(x, y) = 0$ can be parameterized as $y = g(x)$ in a neighbourhood of the initial point (a, b). Similarly, we can express x as a function of y when $\frac{\partial f}{\partial x}(a, b) \neq 0$.

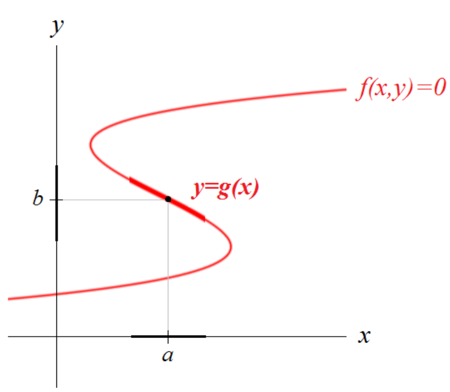

Figure 1.4

Definition. Let $p \in \mathbb{C}^2$ and C, D two plane affine curves, with respective equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$. Suppose, by changing coordinates, that $p = (0, 0)$. If C is non-singular at p, the *multiplicity of C and D at p* is

$$I(p, C \cap D) = \text{mult}_0 f_2(x, g(x))$$

where $y = g(x)$ is a parameterization of C, via the implicit function theorem, in a neighbourhood of $(0,0)$.

Remarks.

1. Likewise, if C is parameterized as $x = g(y)$ in a neighbourhood of $(0,0)$, the multiplicity is defined as
$$I(p, C \cap D) = \text{mult}_0 f_2(g(y), y).$$

2. It follows from the definition that $I(p, C \cap D) \geq 0$, with equality if, and only if, $p \notin C \cap D$.

3. If $p \in C \cap D$ is a point where C and D have different tangent lines, it must be $I(p, C \cap D) = 1$. And the multiplicity will be greater than 1 when C and D have the same tangent line at p.

This definition can be extended to the case of two plane projective curves C, D and a point $p \in \mathbb{P}^2_{\mathbb{C}}$. In fact, supposing $p = (0:0:1)$ by the action of a projectivity, we define $I(p, C \cap D)$ as the multiplicity of the affine parts of C and D at the point $(0,0)$.

Example. Consider $C : xz - y^2 = 0$, and D the conic given by the matrix

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

Let's study the multiplicity of C and D at $p = (0:0:1)$. Note that the affine parts of C and D are the respective zero locus of $f_1(x,y) = x - y^2$ and $f_2(x,y) = ax^2 + by^2 + 2dxy + 2ex + 2fy + c = 0$.

In particular, $x = y^2$ is a global parameterization of the affine part of C. Namely, $I(p, C \cap D)$ is the multiplicity of 0 with respect to

$$f_2(y^2, y) = ay^4 + 2dy^3 + (b + 2e)y^2 + 2fy + c$$

Therefore:

- If $c \neq 0$, we have $I(p, C \cap D) = 0$. It makes sense, since $c \neq 0$ if, and only if, $(0:0:1) \notin D$.
- If $c = 0$ and $f \neq 0$, it must be $I(p, C \cap D) = 1$.
- If $c = f = 0$ and $b + 2e \neq 0$, we have $I(p, C \cap D) = 2$.
- If $c = f = b + 2e = 0$ and $d \neq 0$, the multiplicity is $I(p, C \cap D) = 3$.
- Finally, if $c = f = b + 2e = d = 0$, we have $I(p, C \cap D) = 4$.

Theorem 1.11 (Bézout theorem). *Let C and D two plane projective curves given, respectively, by homogeneous polynomials $F_1(x,y,z)$ and $F_2(x,y,z)$ without a common factor. Then, $C \cap D$ is a set of* $\deg F_1 \cdot \deg F_2$ *points, counted with multiplicity.*

Example (intersection of conics). Let $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ two different conics, with homogeneous polynomials $F(x,y,z)$ and $G(x,y,z)$. If (at least) one of them is non-degenerate, its homogeneous polynomial is irreducible and F, G have no common factors. According to Bézout theorem, C and D meet at four points counting multiplicities. The possible cases are:

1. C and D meet at four different points, each of them with multiplicity 1. That is, C and D have no common tangent lines at the intersection points.

2. C and D meet at three different points, one of them being of multiplicity 2.

3. C and D meet at two different points, each of them counted twice.

4. C and D meet at two points, with multiplicities 3 and 1.

5. C and D meet at a single point, whose multiplicity is 4.

Cases 1 to 3 are too intuitive, and can be visualized in the following way:

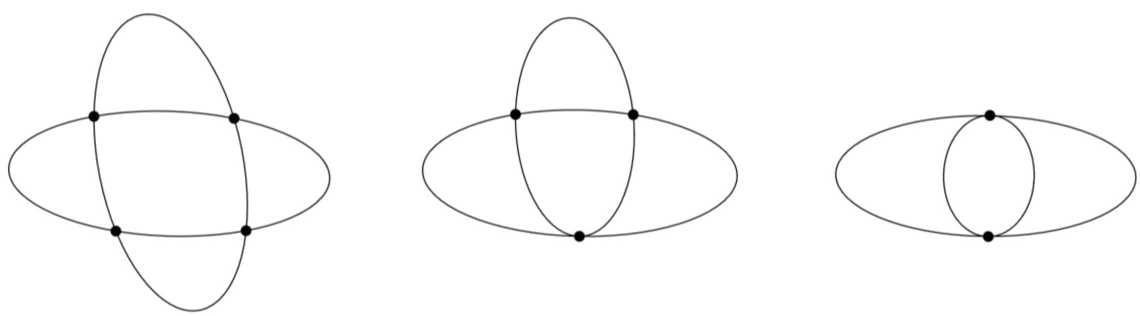

Figure 1.5

To visualize cases 4 and 5, we can use the computations made in the preceding example:

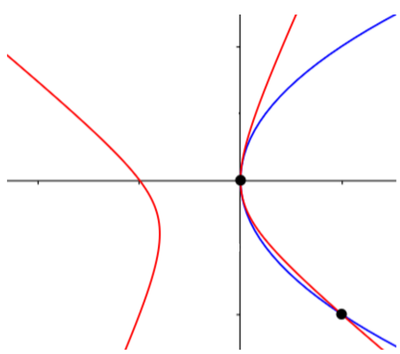

Figure 1.6

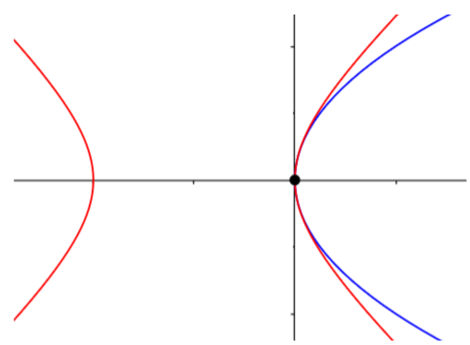

Figure 1.7

Figure 1.6 corresponds to case 4. It's a representation of the affine parts of $C: xz = y^2$ (blue) and $D: x^2 - y^2 + xy + xz = 0$ (red). Both conics meet at $(0:0:1)$ (with multiplicity 3) and $(1:-1:1)$ (with multiplicity 1).

In figure 1.7, we can see the affine parts of $C: xz = y^2$ (blue) and $D: x^2 - 2y^2 + 2xz = 0$ (red). The unique intersection point is $(0:0:1)$, with multiplicity 4.

1.6 Conic pencils

Remark. Whenever the context is clear enough, we write, by an abuse of notation, C and D to denote the conics and their respective matrices.

Definition. Let $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ two different conics, with respective matrices C and D. The *conic pencil generated by C and D*, that we will write $\{C, D\}$, is the set of conics with matrices $r_0 C + r_1 D$, for some $(r_0, r_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

Remarks.

1. Considering the parameters λ, μ as homogeneous coordinates, $\{C, D\}$ can be identified with $\mathbb{P}^1_{\mathbb{C}}$.
2. The points of $C \cap D$ are called the *base points* of $\{C, D\}$.
3. According to Bézout theorem, in a conic pencil there are from one to four different base points.

Lemma 1.12. *Let $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ two different conics, with (at least) one of them non-degenerate. Then:*

1. *Each conic of the pencil $\{C, D\}$ contains the base points.*
2. *Each point in $\mathbb{P}^2_{\mathbb{C}}$, not being a base point, is contained in an unique conic of $\{C, D\}$.*

Theorem 1.13. *Let p_1, p_2, p_3, p_4 be the points of a quadrivertex in $\mathbb{P}^2_{\mathbb{C}}$ (that is, no three of them are aligned). Then, the set of conics containing these points is a conic pencil with exactly three degenerate conics.*

Proof. Take the reference of $\mathbb{P}^2_{\mathbb{C}}$ with vertices p_1, p_2, p_3 and unit point p_4. Namely, $p_1 = (1:0:0)$, $p_2 = (0:1:0)$, $p_3 = (0:0:1)$ and $p_4 = (1:1:1)$.

It's immediate to check that any conic through p_1, p_2, p_3, p_4 has matrix

$$\begin{pmatrix} 0 & \lambda & \mu \\ \lambda & 0 & -\lambda-\mu \\ \mu & -\lambda-\mu & 0 \end{pmatrix}$$

and equation $0 = \lambda xy + \mu xz - \lambda yz - \mu yz = \lambda(xy - yz) + \mu(xz - yz)$.

Hence, the set of conics containing p_1, p_2, p_3, p_4 is the conic pencil generated by $xy - yz = 0$ and $xz - yz = 0$.

An arbitrary conic $\lambda(xy - yz) + \mu(xz - yz) = 0$ of this pencil will be degenerate if, and only if,

$$0 = \begin{vmatrix} 0 & \lambda & \mu \\ \lambda & 0 & -\lambda-\mu \\ \mu & -\lambda-\mu & 0 \end{vmatrix} = -2\lambda\mu(\lambda+\mu) \iff \lambda = 0, \mu = 0, \lambda = -\mu$$

so that this conic pencil has exactly three degenerate conics: the two generators and the conic $xz - xy = 0$. $\square$

Remark. Note that the three degenerate conics of this pencil are $(p_1 \vee p_2) \cup (p_3 \vee p_4)$, $(p_1 \vee p_3) \cup (p_2 \vee p_4)$ and $(p_1 \vee p_4) \cup (p_2 \vee p_3)$.

Corollary 1.14. *If $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ are two non-degenerate conics meeting at four different points $a, b, c, d \in \mathbb{P}^2_{\mathbb{C}}$, the degenerate conics of $\{C, D\}$ are exactly the pairs of lines*

$$(a \vee b) \cup (c \vee d),\ (a \vee c) \cup (b \vee d),\ (a \vee d) \cup (b \vee c)$$

Proof. Since C and D are non-degenerate, any line meets C and D at most two different points. Thus there are not three aligned points in the set $\{a, b, c, d\}$, and a, b, c, d form a quadrivertex.

By lemma 1.12, the pencil $\{C, D\}$ is the set of conics containing the points $\{a, b, c, d\}$. And, according to theorem 1.13, there are three degenerate conics in this set: $(a \vee b) \cup (c \vee d)$, $(a \vee c) \cup (b \vee d)$ and $(a \vee d) \cup (b \vee c)$. $\square$

Chapter 2

Riemann surfaces

2.1 Definition and first examples

Definition. Let X be a topological space. A *complex chart* on X is a pair (U,ϕ), where U is an open set in X (called *domain*) and $\phi: U \longrightarrow V$ is an homeomorphism between U and an open set $V \subset \mathbb{C}$ in the complex plane.

We say that the chart is *centered* at a point $p \in U$ if $\phi(p) = 0$.

Definition. Two charts (U_1,ϕ_1) and (U_2,ϕ_2) on X are *compatible* if either $U_1 \cap U_2 = \emptyset$, or the transition function

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$$

is holomorphic (between open sets in $\mathbb{C}$).

Lemma 2.1. *Each transition function T between two compatible charts is a conformal map.*

Proof. We only need to show that the derivative T' is never zero. Obviously, T is a bijective map, since it's the composition of two homeomorphisms. If S is its inverse function, for each $z \in \mathbb{C}$ in the domain of T we have

$$S(T(z)) = z \implies 1 = S'(T(z)) \cdot T'(z) \implies T'(z) \neq 0\,. \ \square$$

Definition. An *atlas* on X is a collection $\{(U_i,\phi_i)\}_{i\in I}$ of pairwise compatible charts such that $X = \bigcup_{i\in I} U_i$.

Definition. A *maximal atlas* on X (or *complex structure* on X) is an atlas $\{(U_i,\phi_i)\}_{i\in I}$ such that, if (V,ψ) is a chart on X compatible with each (U_i,ϕ_i), then (V,ψ) is a chart of the atlas.

Definition. A *Riemann surface* is a Hausdorff and second countable topological space, endowed with a complex structure.

Remark. By using Zorn's lemma, it can be checked that each atlas of a Hausdorff and second countable space is contained in an unique complex structure. Consequently, giving an atlas is enough to determine a Riemann surface.

Examples.

1. **The complex plane.** $\mathbb{C}$, with its usual topolgy, is a Hausdorff, connected and second countable space. Furthermore, we have an atlas given by a single chart, $(\mathbb{C}, \mathrm{Id}_{\mathbb{C}})$.

2. **The Riemann sphere.** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Alexandroff's compactification of $\mathbb{C}$. Recall that the open sets in this space are the open sets in $\mathbb{C}$ and the complementaries, in $\hat{\mathbb{C}}$, of the compact sets in $\mathbb{C}$.

   Then, $\hat{\mathbb{C}}$ is compact, connected, Hausdorff and second countable space. Let's consider the charts:

   - (U_1, ϕ_1), where $U_1 = \mathbb{C}$ and $\phi_1 = \mathrm{Id}_{\mathbb{C}}$.
   - (U_2, ϕ_2), with domain $U_2 = \hat{\mathbb{C}} \setminus \{0\}$ and $\phi_2 : \hat{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{C}$ given by $\phi_2(z) = \frac{1}{z}$ (with the rule $\frac{1}{\infty} = 0$).

   Clearly, their domains cover $\hat{\mathbb{C}}$. Let's check the compatibility condition: we have

   $$U_1 \cap U_2 = \mathbb{C} \setminus \{0\}, \quad \phi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\} = \phi_2(U_1 \cap U_2)$$

   Therefore, the transition function is the holomorphic function

   $$\begin{aligned} \phi_1 \circ \phi_2^{-1} : \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\longmapsto \frac{1}{z} \end{aligned}$$

3. **Open sets in a Riemann surface.** Let X be a Riemann surface, and $U \subset X$ a connected open set. Then, U inherites (with the subspace topology) the properties of being Hausdorff and second countable.

   Moreover, if we have a complex structure $\{(U_i, \phi_i)\}_{i \in I}$ on X, it can be easily checked that the collection defined by $\{(U_i, \phi_i) : i \in I, U_i \subset U\}$ is a complex structure on U.

2.2 Orientation and genus

Proposition 2.2. *Every Riemann surface X is an orientable topological surface.*

Proof. Taking the usual identification between $\mathbb{C}$ and $\mathbb{R}^2$, we have that X is locally homeomorphic to $\mathbb{R}^2$. Since X is also a Hausdorff and second countable space, it follows that it's a topological surface.

Furthermore, for any two charts (U, ϕ) and (V, ψ) of the atlas on X, the holomorphy condition of

$$\begin{aligned} \phi \circ \psi^{-1} : \psi(U \cap V) &\longrightarrow \phi(U \cap V) \\ x + yi &\longmapsto u(x,y) + v(x,y)i \end{aligned}$$

means that $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Considering $\phi \circ \psi^{-1}$ as a function between open sets in $\mathbb{R}^2$ (instead of open sets in $\mathbb{C}$), its Jacobian is

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 > 0$$

So the atlas on the Riemann surface X defines an oriented atlas, thinking of X as a topological surface. $\square$

By the classification theorem of topological surfaces, the following definition makes sense.

Definition. The *genus* of a compact and connected Riemann surface X is the topological genus of X, as a compact, connected and orientable topological surface. In other words, it's the number of "handles" or "holes" on X.

Example. The Riemann sphere $\hat{\mathbb{C}}$ is homeomorphic to the 2-dimensional sphere $\mathbb{S}^2$ (since it's the Alexandroff's compactification of $\mathbb{C} \cong \mathbb{R}^2$). Hence, the Riemann sphere is a compact Riemann surface with genus 0.

2.3 The complex torus

Let $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$ (i.e., $\omega_1 \neq 0$ and $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$), and

$$\Lambda = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\} = \langle \omega_1, \omega_2 \rangle$$

the lattice of the complex plane generated by ω_1 and ω_2. We have an equivalence relation on $\mathbb{C}$ given by

$$z_1 \sim z_2 \iff z_1 - z_2 \in \Lambda$$

In the quotient set (denoted by $\mathbb{C}/\Lambda$) consider the final topology with respect to the projection $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$. Namely: $U \subset \mathbb{C}/\Lambda$ is an open set $\iff \pi^{-1}(U) \subset \mathbb{C}$ is an open set.

Lemma 2.3. *The topological space $\mathbb{C}/\Lambda$ is homeomorphic to a torus.*

Proof. Consider the projection π restricted to the closed region R with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$.

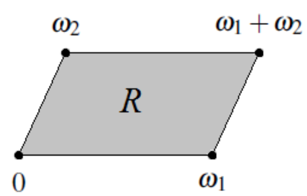

Figure 2.1

$\pi_{|R}$ is continuous and surjective (all the classes have a representative in R). And since π is an open map, so it is $\pi_{|R}$:

$$U \text{ open in } \mathbb{C} \Longrightarrow \pi^{-1}(\pi(U)) = \bigcup_{\omega \in \Lambda} (\omega + U) \text{ open in } \mathbb{C} \text{ (it's the union of open sets)} \Longrightarrow \pi(U) \text{ open in } \mathbb{C}/\Lambda$$

So $\pi_{|R}$ is an identification map. Moreover, two different points $z_1, z_2 \in R$ satisfy $\pi(z_1) = \pi(z_2)$ if, and only if, $z_1 - z_2 \in \{\omega_1, \omega_2\}$.

Hence, $\mathbb{C}/\Lambda$ is the result of gluing together the sides of R as in the following figure:

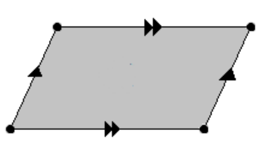

Figure 2.2

Doing these identifications, we obtain that $\mathbb{C}/\Lambda$ is homeomorphic to a torus. $\square$

Lemma 2.4. $\mathbb{C}/\Lambda$ *is a compact Riemann surface with genus 1 (that will be called complex torus).*

Proof. We only need to determine an atlas on $\mathbb{C}/\Lambda$ because, since it's homeomorphic to a torus, $\mathbb{C}/\Lambda$ is a second countable, Hausdorff, connected and compact space with topological genus 1.

Take $\varepsilon = \min\{|\omega| : \omega \in \Lambda \setminus \{0\}\}$, and for each $a \in \mathbb{C}$, define $D_a = \left\{z \in \mathbb{C} : |z-a| < \frac{\varepsilon}{4}\right\}$.

We are going to check that $\pi_{|D_a}$ is injective, for each $a \in \mathbb{C}$. Let $x, y \in D_a$ such that $\pi(x) = \pi(y)$. Then:

$$|x-y| \leq |x-a| + |y-a| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

$$\pi(x) = \pi(y) \implies x - y \in \Lambda$$

By definition of ε, it must be $x - y = 0$, i.e., $x = y$.

Since $\pi_{|D_a}$ is injective and π is an open and continuous map, we have an homeomorphism $\pi_{|D_a} : D_a \longrightarrow \pi(D_a)$. If ϕ_a denotes the inverse homeomorphism and $U_a = \pi(D_a)$, it follows that $\{(U_a, \phi_a)\}_{a\in\mathbb{C}}$ is an atlas on $\mathbb{C}/\Lambda$:

- The sets U_a cover $\mathbb{C}/\Lambda$: in fact,

  $x \in \mathbb{C}/\Lambda \implies x = \pi(a) \in U_a$, for some $a \in \mathbb{C}$

- Compatibility of the charts: suppose that $U_a \cap U_b \neq \varnothing$, and denote by w the transition function

  $$w = \phi_b \circ \phi_a^{-1} : \phi_a(U_a \cap U_b) \longrightarrow \phi_b(U_a \cap U_b)$$

  Then, for each $z \in \phi_a(U_a \cap U_b)$, we have: $\pi(w(z)) = \pi(z) \implies w(z) \sim z \implies w(z) - z \in \Lambda$.

  Let's see that $\alpha(z) = w(z) - z$ is a constant function, namely, it does not depend on z. If $z_1, z_2 \in \phi_a(U_a \cap U_b)$,

  $$z_1, z_2 \in D_a \implies |z_1 - z_2| \leq |z_1 - a| + |z_2 - a| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

  $$w(z_1), w(z_2) \in D_b \implies |w(z_1) - w(z_2)| \leq |w(z_1) - b| + |w(z_2) - b| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

  so that

  $$|\alpha(z_1) - \alpha(z_2)| = |w(z_1) - z_1 - w(z_2) + z_2| \leq |w(z_1) - w(z_2)| + |z_1 - z_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

  Since $\alpha(z_1) - \alpha(z_2) \in \Lambda$, it must be $\alpha(z_1) - \alpha(z_2) = 0$, i.e., $\alpha(z_1) = \alpha(z_2)$.

  Thus for each $z \in \phi_a(U_a \cap U_b)$ we have $\alpha(z) = C \in \Lambda$ (constant) and the transtion function $w(z) = z + C$ is a translation (in particular, is holomorphic). □

2.4 Algebraic curves and Riemann surfaces

Let's start by studying plane affine curves, with the induced topology as a subspace of $\mathbb{C}^2$. In order to determine complex structures, we will need the implicit function theorem for polynomials.

Theorem 2.5. *Every non-ninsgular algebraic curve γ of $\mathbb{C}^2$ is a Riemann surface.*

Proof. As a subspace of $\mathbb{C}^2$, γ is Hausdorff and second countable. We only need to find an atlas.

Let $f(x,y)=0$, with f a polynomial, be the equation for γ, and take an arbitrary point $(a,b)\in\gamma$. By the non-singularity of γ, we will have one of the two following cases:

- If $\dfrac{\partial f}{\partial y}(a,b)\neq 0$, we can take neighbourhoods X and Y, and a function g as in theorem 1.10. Then,

  $$U=\{(x,y)\in\gamma : x\in X, y\in Y\}=\{(x,g(x)) : x\in X\}$$

  is an open neighbourhood of (a,b) in γ, and we have an homeomorphism

  $$\begin{aligned}\phi : U &\longrightarrow X\subset\mathbb{C}\\ (x,y) &\longmapsto x\\ (x,g(x)) &\longleftarrow\!\shortmid\ x\end{aligned}$$

  Thus (U,ϕ) is a local chart on γ at the point (a,b).

- If $\dfrac{\partial f}{\partial x}(a,b)\neq 0$, we have neighbourhoods X' and Y', and a function $h : Y'\longrightarrow X'$ parameterizing the zeroes of f as $x=h(y)$. Then,

  $$V=\{(x,y)\in\gamma : x\in X', y\in Y'\}=\{(h(y),y) : y\in Y'\}$$

  is an open neighbourhood of (a,b) in γ, and we have an homeomorphism

  $$\begin{aligned}\psi : V &\longrightarrow Y'\subset\mathbb{C}\\ (x,y) &\longmapsto y\\ (h(y),y) &\longleftarrow\!\shortmid\ y\end{aligned}$$

  Thus (V,ψ) is a local chart on γ at the point (a,b).

Taking for each point of γ some of these charts, we obtain a collection of charts whose domains cover γ. In order to see that this collection is an atlas, we have to check the compatibility conditions.

Using previous notations, each chart has the form (U,ϕ) or (V,ψ). The possible transition functions will be the identity (if both charts have the same form), $(\psi\circ\phi^{-1})(z)=g(z)$ or $(\phi\circ\psi^{-1})(z)=h(z)$.

Since all these transition functions are holomorphic, the compatibility is proved. □

Remark. A plane affine curve in $\mathbb{C}^2$ is not compact, since it's not a bounded space: its projective closure meets the line at infinity (according to Bézout theorem).

Now, we work in the projective plane $\mathbb{P}^2_{\mathbb{C}}$, with the induced topology of $\mathbb{C}^3\setminus\{(0,0,0)\}$ by the projection. With this topology, $\mathbb{P}^2_{\mathbb{C}}$ is a second countable, compact and Hausdorff space that can be covered by the open sets

$$U_i=\left\{(x_0:x_1:x_2)\in\mathbb{P}^2_{\mathbb{C}} : x_i\neq 0\right\}\quad (i=0,1,2)$$

Each of these open sets is homeomorphic to the affine plane, taking

$$\varphi_0 : U_0\longrightarrow\mathbb{C}^2,\ \varphi_0((x_0:x_1:x_2))=(\tfrac{x_1}{x_0},\tfrac{x_2}{x_0}),\ \varphi_0^{-1}(a,b)=(1:a:b)$$

$$\varphi_1 : U_1\longrightarrow\mathbb{C}^2,\ \varphi_1((x_0:x_1:x_2))=(\tfrac{x_0}{x_1},\tfrac{x_2}{x_1}),\ \varphi_1^{-1}(a,b)=(a:1:b)$$

$$\varphi_2 : U_2\longrightarrow\mathbb{C}^2,\ \varphi_2((x_0:x_1:x_2))=(\tfrac{x_0}{x_2},\tfrac{x_1}{x_2}),\ \varphi_2^{-1}(a,b)=(a:b:1)$$

Theorem 2.6. *Let X be a non-singular curve of $\mathbb{P}^2_{\mathbb{C}}$, given by an homogeneous polynomial $F(x,y,z)$. Then, X is a Riemann surface.*

Proof. Again, the conditions of being Hausdorff and second countable are inherited from the ambient space $\mathbb{P}^2_{\mathbb{C}}$.

On the other hand, we can see each of the open sets $X_i = X \cap U_i$ in X as a non-singular curve of $\mathbb{C}^2$. For example,

$$X_2 = X \cap U_2 \cong \left\{(a,b) \in \mathbb{C}^2 : F(a,b,1) = 0\right\}$$

is the affine part of γ. By theorem 2.5, each X_i has, separately, a complex structure.

We want to see that they give a complex structure on X, i.e., that any two charts on different X_i are compatible.

For example, let's consider two charts (V_0, ψ_0) and (V_1, ψ_1), respectively on X_0 and X_1, such that $V_0 \cap V_1 \neq \emptyset$. By the construction we did in theorem 2.5, it's easy to check that

$$\psi_0((x_0 : x_1 : x_2)) = \frac{x_1}{x_0} \text{ or } \frac{x_2}{x_0}, \qquad \psi_1((x_0 : x_1 : x_2)) = \frac{x_0}{x_1} \text{ or } \frac{x_2}{x_1}$$

Suppose that $\psi_0((x_0 : x_1 : x_2)) = \frac{x_1}{x_0}$ and $\psi_1((x_0 : x_1 : x_2)) = \frac{x_2}{x_1}$. Then, if $z \in \psi_0(V_0 \cap V_1)$, we have

$$\psi_0^{-1}(z) = (1 : z : g(z)), \text{ with } g \text{ holomorphic } \implies (\psi_1 \circ \psi_0^{-1})(z) = \frac{g(z)}{z} \text{ is holomorphic, since } z \neq 0$$ [1]

Hence, the charts (V_0, ψ_0) and (V_1, ψ_1) are compatible. In a similar way, the remaining cases can be checked to conclude the proof. □

Remark. The curve X is compact (it's a closed set in the compact space $\mathbb{P}^2_{\mathbb{C}}$) and connected (the proof exceeds our level; it can be found in [7]). Thus it makes sense considering the genus of X. In section 2.6, we will express this genus in terms of the degree of the curve.

Theorem 2.7. *Any non-singular algebraic curve γ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is a Riemann surface.*

Proof. First of all, our curve is Hausdorff and second countable, as a subspace of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

Moreover, considering the restrictions of γ in the subsets

$$A_1 = \left\{((x_0 : 1), (y_0 : 1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_0, y_0 \in \mathbb{C}\right\} \cong \mathbb{C}^2, A_2 = \left\{((x_0 : 1), (1 : y_1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_0, y_1 \in \mathbb{C}\right\} \cong \mathbb{C}^2$$

$$A_3 = \left\{((1 : x_1), (y_0 : 1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_1, y_0 \in \mathbb{C}\right\} \cong \mathbb{C}^2, A_4 = \left\{((1 : x_1), (1 : y_1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x_1, y_1 \in \mathbb{C}\right\} \cong \mathbb{C}^2$$

we have four non-singular plane affine curves that, according to theorem 2.5, will have separate complex structures.

It can be checked, by a similar reasoning to the one used in theorem 2.6, that two charts on different restrictions $\gamma \cap A_i$ are compatible. Therefore, we have an atlas on γ and γ is a Riemann surface. □

2.5 Maps between Riemann surfaces

Definition. Let X be a Riemann surface and $p \in X$. A function $f : X \longrightarrow \mathbb{C}$ is said to be *holomorphic at p* (resp. *meromorphic at p*) if, for some chart (U, ϕ) on X satisfying $p \in U$, the composition $f \circ \phi^{-1}$ is holomorphic (resp. meromorphic) at $\phi(p)$.

[1] It must be $z \neq 0$, because $(1 : z : g(z)) = \psi_0^{-1}(z) \in V_0 \cap V_1 \subset V_1 \subset X_1$

The function f is *holomorphic* whether it is holomorphic at any point of X.

Remark. If f is meromorphic at p, the type of singularity of f at p is the type of singularity of $f \circ \phi^{-1}$ at $\phi(p)$ (pole, removable or essential singularity).

Examples.

1. If (U, ϕ) is a chart on X, the function $\phi : U \longrightarrow \mathbb{C}$ is holomorphic at U.
2. Taking $X = \mathbb{C}$, the preceding definitions agree with the usual definitions of holomorphy and meromorphy.

Definition. Let X, Y be Riemann surfaces and $p \in X$. We say that $F : X \longrightarrow Y$ is an *holomorphic map at p* (resp. *meromorphic map at p*) if, for some charts (U_1, ϕ) on X and (U_2, ψ) on Y such that $p \in U_1$ and $F(p) \in U_2$, the composition $\psi \circ F \circ \phi^{-1}$ is holomorphic (resp. meromorphic) at $\phi(p)$.

F is an *holomorphic map* whether it is holomorphic at any point of X.

Remark. Since the composition of holomorphic functions (between open sets in $\mathbb{C}$) is holomorphic, the preceding definition does not rely on the choice of local coordinates and we can change "some charts" by "all the charts".

Remark. It's not difficult to prove, using the analogous properties for holomorphic functions of $\mathbb{C}$, that:

- Any two holomorphic maps are continuous, and their composition is an holomorphic map.
- A map $F : X \longrightarrow Y$, holomorphic on $X \setminus \{p_1, \dots, p_r\}$ and continuous at $p_1, \dots, p_r$, is holomorphic on X.

Examples.

1. Taking $Y = \mathbb{C}$, an holomorphic map $F : X \longrightarrow Y$ is an holomorphic function.
2. For any Riemann surface X, $Id : X \longrightarrow X$ is an holomorphic map.
3. Every meromorphic function $f : X \longrightarrow \mathbb{C}$ induces an holomorphic map $F : X \longrightarrow \hat{\mathbb{C}}$ given by

$$F(x) = \begin{cases} \infty & \text{if } x \text{ is a pole of } f \\ f(x) & \text{otherwise} \end{cases}$$

Definition. Let X, Y two Riemann surfaces. A map $F : X \longrightarrow Y$ is an *isomorphism* if is holomorphic and bijective. In such a case, X and Y are said to be *isomorphic* Riemann surfaces.

Definition. An *automorphism* of a Riemann surface X is an isomorphism $F : X \longrightarrow X$.

Example. The automorphisms $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ are given by $\phi(z) = az + b$, with $a \neq 0$.

By the following result, we can transfer to other sets the complex structure of a Riemann surface.

Lemma 2.8. *Let X be a Riemann surface, Y a set and $f : X \longrightarrow Y$ a bijective map. We define a topology and a complex structure on Y with the rules:*

$$U \text{ open set in } Y \iff f^{-1}(U) \text{ open set in } X, \quad (U,\phi) \text{ chart on } Y \iff (f^{-1}(U), \phi \circ f) \text{ chart on } X$$

Then, $f : X \longrightarrow Y$ is an isomorphism between the Riemann surfaces X and Y.

Example. By the bijection $f : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, $f((x_0 : x_1)) = \frac{x_0}{x_1}$, the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ is a compact Riemann surface with genus 0.

For those times when we are dealing with Riemann surfaces related to projective spaces, the following lemma will be very useful, since it allows us to extend holomorphic maps.

Lemma 2.9. *Let X and $Y \subset \mathbb{P}^n_{\mathbb{C}}$ be Riemann surfaces, and $U \subset X$ an open set such that $X \setminus U$ consists of isolated points. If X is compact, every holomorphic map $f : U \longrightarrow Y$ can be extended to an holomorphic map $\widetilde{f} : X \longrightarrow Y$.*

Proof. Consider, up to scalar, the holomorphic functions $f_i : U \longrightarrow \mathbb{C}$ $(i = 0, \dots, n)$ defining the homogeneous coordinates of f. Namely, $f(x) = (f_0(x) : \dots : f_n(x))$ for each $x \in U$.

Given a point $p \in X \setminus U$, take a chart (V, ϕ) on X centered at p, and consider the Laurent series of $f_i \circ \phi^{-1}$ on a neighbourhood V_i of $z = 0$. Compactness of X ensures us that its principal part is finite:

$$\forall z \in V_i \setminus \{0\} \quad (f_i \circ \phi^{-1})(z) = \frac{a_{-m_i}}{z^{m_i}} + \dots, \text{ with } a_{-m_i} \neq 0$$

If $m = \max\{m_0, \dots, m_n\}$, for each $z \in (V_0 \cap \dots \cap V_n) \setminus \{0\}$ we can multiply projective coordinates by $z^m \neq 0$ and

$$(f \circ \phi^{-1})(z) = ((f_0 \circ \phi^{-1})(z) : \dots : (f_n \circ \phi^{-1})(z)) = ((f_0 \circ \phi^{-1})(z) \cdot z^m : \dots : (f_n \circ \phi^{-1})(z) \cdot z^m) =$$
$$= (a_{-m_0} z^{m-m_0} + \dots : \dots : a_{-m_n} z^{m-m_n} + \dots)$$

Taking limits, we can extend $f \circ \phi^{-1}$ to an holomorphic function at $z = 0$, so f can be extended to an holomorphic map at p.

Repeating this process for each point of $X \setminus U$, we obtain an holomorphic map $\widetilde{f} : X \longrightarrow \mathbb{P}^n_{\mathbb{C}}$. The condition $\widetilde{f}(X) \subset \overline{f(U)} \subset Y$ finishes the proof. □

2.6 Ramification and degree. Hurwitz formula

Theorem 2.10 (local form). *Let X, Y be Riemann surfaces and $F : X \longrightarrow Y$ a non-constant map, holomorphic at $p \in X$. Then, there exists an unique integer $m \geq 1$ with the following property: if (U_2, ϕ_2) is a chart on Y centered at $F(p)$, there exists a chart (U_1, ϕ_1) on X centered at p such that $(\phi_2 \circ F \circ \phi_1^{-1})(z) = z^m$.*

Proof. To prove the existence, fix a chart (U_2, ϕ_2) centered at $F(p)$ (i.e., $\phi_2(F(p)) = 0$), and consider (U, ψ) a chart on X centered at p. Then, the Taylor series of $T = \phi_2 \circ F \circ \phi^{-1}$ in a neighbourhood of $w = 0$ has the form

$$T(w) = \sum_{i=m}^{\infty} c_i w^i,$$

with $c_m \neq 0$ and $m \geq 1$ (since $T(0) = (\phi_2 \circ F \circ \phi^{-1})(0) = \phi_2(F(p)) = 0$). So we can write $T(w) = w^m \cdot S(w)$, with S holomorphic at $w = 0$ and $S(0) \neq 0$.

By the existence of m-th root of S, there is a function $R(w)$, holomorphic on a neighbourhood of $w = 0$, such that

$$S(w) = R(w)^m \implies T(w) = w^m \cdot R(w)^m = (w \cdot R(w))^m$$

Writing $\eta(w) = w \cdot R(w)$, we have: $\eta'(w) = w \cdot R'(w) + R(w) \implies \eta'(0) = R(0) \neq 0$.

Hence, on a neighbourhood V of 0, the funcion η is invertible (by the inverse function theorem) and holomorphic. The composition $\phi_1 = \eta \circ \psi$, considered on a neighbourhood V' of p such that $\psi(V') \subset V$, satisfies:

- It's a chart on X, since it's the composition of the chart ψ with the invertible and holomorphic function η.
- It's centered at p: $\phi_1(p) = \eta(\psi(p)) = \psi(p) \cdot R(\psi(p)) = 0 \cdot R(0) = 0$
- $\phi_2(F(\phi_1^{-1}(z))) = \phi_2(F(\psi^{-1}(\eta^{-1}(z)))) = T(\eta^{-1}(z)) = (\eta^{-1}(z) \cdot R(\eta^{-1}(z)))^m = (\eta(\eta^{-1}(z)))^m = z^m$

Now, let's see the uniqueness. In a neighbourhood of p, the points near $F(p)$ have exactly m preimages. This exponent m is determined by the topological propierties of the map in a neighbourhood of p, so it does not rely on the choice of charts. $\square$

Definition. Under the above hypothesis:

1. The integer m is called *ramification index of F at p*, and we denote it by $e_p(F)$.
2. We say that p is a *ramification point of F* when $e_p(F) > 1$.
3. A *branch point of F* is the image, for F, of a ramification point.

Remark. Ramification points of an holomorphic map form a discrete set.

Proposition 2.11. *Let $F : X \longrightarrow Y$ be a non-constant homolomorphic map between compact Riemann surfaces. For each $y \in Y$, let's consider*

$$\deg_y(F) = \sum_{p \in F^{-1}(\{y\})} e_p(F)$$

Then $\deg_y(F)$ is an integer, independent of the point $y \in Y$. It's called the degree of F, and denoted by $\deg(F)$.

Theorem 2.12 (Hurwitz formula). *If X, Y are compact Riemann surfaces and $F : X \longrightarrow Y$ is a non-constant holomorphic map, then*

$$2g(X) - 2 = \deg(F) \cdot (2g(Y) - 2) + \sum_{p \in X} (e_p(F) - 1)$$

Example. Let C a non-singular algebraic curve in $\mathbb{P}^2_{\mathbb{C}}$, given by an homogeneous polynomial $F(x, y, z)$ of degree d. We are going to see that its genus is $g(C) = \frac{(d-1)(d-2)}{2}$.

Take $p = (a : b : c) \notin C$, with $c \neq 0$, and denote by π_p the projection, from p, of the points in C over the line $z = 0$:

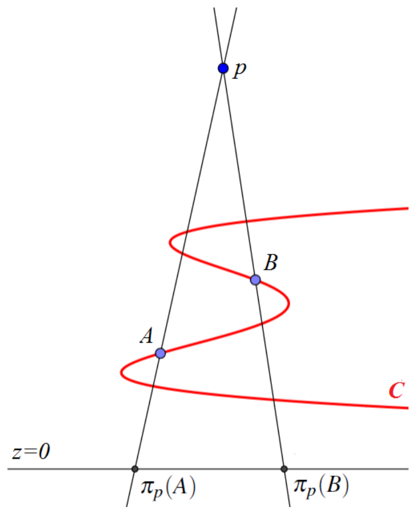

Figure 2.3

We can consider this projection as an holomorphic map $\pi_p : C \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. Then:

- The ramification points will be the points $q \in C$ such that $p \in T_C(q)$, i.e., the points $q \in C$ such that

  $$a \cdot \frac{\partial F}{\partial x}(q) + b \cdot \frac{\partial F}{\partial y}(q) + c \cdot \frac{\partial F}{\partial z}(q) = 0$$

  The points q satisfying this equation define a curve C', with degree $d-1$ (the *polar of C with respect to p*).

  So we are looking for the points of $C \cap C'$. By Bézout theorem, we have $d(d-1)$ points (counted with multiplicity). In point of fact, it can be proved that there are exactly $d(d-1)$ ramification points, all of them with index 2.

- If we take a point $\tilde{p}$ of the line $z = 0$ (not a branch point), the preimages of $\tilde{p}$ are the d different intersection points of the line $\tilde{p} \vee p$ with C, all of them with ramification index 1. By computing $\deg(\pi_p)$ with $\tilde{p}$, we obtain that $\deg(\pi_p) = d$.

According to Hurwitz formula, we have

$$2g(C) = 2 + \deg(\pi_p) \cdot (2g(\mathbb{P}^1_{\mathbb{C}}) - 2) + \sum_{p \in C} (e_p(\pi_p) - 1) = 2 - 2d + d(d-1) = d^2 - 3d + 2 = (d-1)(d-2) \Longrightarrow$$

$$\Longrightarrow g(C) = \frac{(d-1)(d-2)}{2}$$

2.7 Automorphisms of a complex torus

In this section, we want to describe the automorphisms of a complex torus. It will be a particular case of the study of isomorphisms between two arbitrary complex tori.

Let's consider Λ_1, Λ_2 two lattices of the complex plane, defining two separate tori $T_1 = \mathbb{C}/\Lambda_1$ i $T_2 = \mathbb{C}/\Lambda_2$.

Theorem 2.13. *Let $F : T_1 \longrightarrow T_2$ be an holomorphic map. Then, there exists an unique isomorphism $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ T_1 & \xrightarrow{F} & T_2 \end{array} \qquad (*)$$

is commutative, with $\pi_i : \mathbb{C} \longrightarrow T_i$ denoting the projection.

Proof. Since $g(T_1) = 1 = g(T_2)$, by Hurwitz formula F has no ramification points. Thus

$$(T_1, F) \text{ is a covering space of } T_2 \Longrightarrow (\mathbb{C}, F \circ \pi_1) \text{ is a covering space of } T_2$$

But $\mathbb{C}$ is simply connected, so $(\mathbb{C}, F \circ \pi_1)$ must be homeomorphic to the universal covering space of T_2: $(\mathbb{C}, \pi_2)$. We have an homeomorphism $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that the diagram $(*)$ commutes.

Now, we only have to check that $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ is an isomorphism of Riemann surfaces. And this is immediate, since ψ is holomorphic (the other maps in the diagram are holomorphic) and bijective (it's an homeomorfism). □

Theorem 2.14.

1. *If the tori T_1 and T_2 are isomorphic Riemann surfaces, then $\Lambda_2 = \alpha\Lambda_1$, for some $\alpha \in \mathbb{C}$.*
2. *Conversely, if $\Lambda_2 = \alpha\Lambda_1$, then T_1 and T_2 are isomorphic. The isomorphisms from T_1 to T_2 have the form*

   $\varphi([z]) = [\alpha z + \beta]$ *($z \in \mathbb{C}$), for any $\beta \in \mathbb{C}$.*

Proof. We start proving 1. Given an isomorphism $F : T_1 \longrightarrow T_2$, by theorem 2.13 there exists an automorphism $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $F(\pi_1(z)) = \pi_2(\psi(z))$, for each $z \in \mathbb{C}$.

By the characterization of automorphisms of $\mathbb{C}$, it must be $\psi(z) = \alpha z + \beta$ (with $\alpha \neq 0$, $\beta \in \mathbb{C}$). So, if $z \in \Lambda_1$,

$$\pi_2(\alpha z + \beta) = F(\pi_1(z)) = F(\pi_1(0)) = \pi_2(\beta) \Longrightarrow [\alpha z + \beta] = [\beta] \text{ in } \Lambda_2 \Longrightarrow [\alpha z] = [0] \text{ in } \Lambda_2 \Longrightarrow \alpha z \in \Lambda_2$$

This proves the inclusion $\alpha\Lambda_1 \subseteq \Lambda_2$. A similar reasoning with F^{-1} and ψ^{-1} proves the converse inclusion.

Let's see 2. Suppose that $\Lambda_2 = \alpha\Lambda_1$, and for any $\beta \in \mathbb{C}$ consider the map $\varphi : T_1 \longrightarrow T_2$, $\varphi([z]) = [\alpha z + \beta]$. Then:

- φ is injective: $\varphi([z_1]) = \varphi([z_2]) \Longrightarrow [\alpha z_1 + \beta] = [\alpha z_2 + \beta]$ in $\Lambda_2 \Longrightarrow \alpha(z_1 - z_2) = (\alpha z_1 + \beta) - (\alpha z_2 + \beta) \in \Lambda_2 = \alpha\Lambda_1 \Longrightarrow \alpha(z_1 - z_2) = \alpha x$, for a certain $x \in \Lambda_1 \Longrightarrow z_1 - z_2 = x \in \Lambda_1 \Longrightarrow [z_1] = [z_2]$ in Λ_1 .
- φ is surjective: For each $x \in T_2$, we have $x = [z_0]$, for some $z_0 \in \mathbb{C}$. Then, $x = [z_0] = \varphi(\left[\frac{z_0 - \alpha}{\beta}\right])$.
- φ is holomorphic: If $x \in T_1$, put $x = [z_0]$, for a certain $z_0 \in \mathbb{C}$. Consider an open disk D, with center z_0, such that $\pi_1 : D \longrightarrow \pi_1(D)$ is an isomorphism.

  Then, for each $p \in \pi_1(D)$, we can put $\varphi(p) = (\pi_2 \circ \psi \circ \pi_1^{-1})(p)$. Hence, φ is holomorphic on $\pi_1(D)$, particularly at $x = [z_0]$.

Therefore, φ is an isomorphism of Riemann surfaces. Moreover, every isomorphism from T_1 to T_2 has this form, according to theorem 2.13. □

Theorem 2.15 (automorphisms of a complex torus). *Let $T = \mathbb{C}/\Lambda$ be a complex torus.*

1. *The automorphisms of T are exactly $\varphi([z]) = [\alpha z + \beta]$, for any $\beta \in \mathbb{C}$ and with $\alpha \in \mathbb{C}$ such that $\alpha\Lambda = \Lambda$.*
2. *The automorphisms of T with no fixed points have the form $\varphi([z]) = [z + \beta]$, for any $\beta \in \mathbb{C} \setminus \Lambda$.*

Proof. The first part is a particular case of the preceding theorem, with $\Lambda_1 = \Lambda = \Lambda_2$.

Now, consider $\varphi([z]) = [\alpha z + \beta]$ an automorphism of T without fixed points. If it were $\alpha \neq 1$, we would have

$$\varphi(\left[\frac{\beta}{1-\alpha}\right]) = \left[\alpha\frac{\beta}{1-\alpha} + \beta\right] = \left[\frac{\alpha\beta + \beta(1-\alpha)}{1-\alpha}\right] = \left[\frac{\beta}{1-\alpha}\right]$$

and $\left[\frac{\beta}{1-\alpha}\right]$ would be a fixed point for φ, contradiction.

On the other hand, $\varphi([z]) = [z + \beta]$ has no fixed points, except when $\beta \in \Lambda$ (in that case, φ is the identity map). □

Example. A typical consequence of these results is that every complex torus is isomorphic to $\mathbb{C}/\langle 1, \tau\rangle$, with $\mathrm{Im}\tau > 0$.

Indeed, given a torus $T = \mathbb{C}/\Lambda$ (where Λ is the lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$), consider $\tau = \pm\frac{\omega_2}{\omega_1}$ (taking + o − so that $\mathrm{Im}\tau > 0$). Then, we have $\langle 1, \tau\rangle = \pm\frac{1}{\omega_1}\Lambda$ and, by theorem 2.14, the tori T and $\mathbb{C}/\langle 1, \tau\rangle$ are isomorphic.

2.8 Elliptic curves

Definition. An *elliptic curve* is a compact and connected Riemann surface with genus 1.

Example. According to the characterization of the genus of a non-singular algebraic curve in $\mathbb{P}^2_{\mathbb{C}}$, every non-singular plane projective cubic is an elliptic curve.

Theorem 2.16. *Every elliptic curve is isomorphic (as a Riemann surface) to a torus $\mathbb{C}/\Lambda$, for some lattice Λ.*

Remarks.

1. It's obvious, from the definition, that an elliptic curve is homeomorphic to any complex torus, since they are two orientable topological surfaces with the same genus.

   Nevertheless, theorem 2.16 says something quite deeper: any complex structure on an elliptic curve is the one given in a certain complex torus.

2. This result can be proved by using universal coverings and Riemann uniformization theorem. Another proof, at a higher level, requires Abel's theorem. For further reading, see [4].

Let E be an elliptic curve. Then, there exists an isomorphism of Riemann surfaces

$$\varphi : \mathbb{C}/\Lambda \longrightarrow E$$

for some lattice Λ. Observe that:

- $\mathbb{C}/\Lambda$ has an additive group structure, with the following addition inherited from $\mathbb{C}$:

  $$\forall z_1, z_2 \in \mathbb{C} \quad [z_1] + [z_2] = [z_1 + z_2]$$

  We say that $(\mathbb{C}/\Lambda, +)$ is an analytic group, that is, in terms of local charts about any two points in $\mathbb{C}/\Lambda$, the addition $+$ is an analytic function of two complex variables.

- By means of φ and the preceding addition $+$, we have an analytic addition $\oplus$ on E given by

  $$\forall p_1, p_2 \in \mathbb{C}/\Lambda \quad \varphi(p_1) \oplus \varphi(p_2) = \varphi(p_1 + p_2)$$

  The neutral element on E will be $\varphi([0])$.

Theorem 2.17. *Let E be an elliptic curve. Then, for each $\theta \in E$, E has an unique analytic group structure having θ for neutral element.*

Proof. Let's see the existence. Let $\varphi : E \longrightarrow \mathbb{C}/\Lambda$ be an isomorphism of Riemann surfaces, and consider the map

$$\begin{aligned} \alpha : \mathbb{C}/\Lambda &\longrightarrow \mathbb{C}/\Lambda \\ [z] &\longmapsto [z] - \varphi(\theta) \end{aligned}$$

According to theorem 2.15, α is an automorphism of $\mathbb{C}/\Lambda$, mapping $\varphi(\theta)$ to $[0]$. Now, if Φ denotes the isomorphism

$$\Phi = \varphi^{-1} \circ \alpha^{-1} : \mathbb{C}/\Lambda \longrightarrow E \ ,$$

we can define an addition $\oplus$ on E by the rule

$$\forall p_1, p_2 \in \mathbb{C}/\Lambda \quad \Phi(p_1) \oplus \Phi(p_2) = \Phi(p_1 + p_2)$$

Then, $(\mathbb{C}/\Lambda, \oplus)$ is a group with neutral element $\Phi([0]) = \varphi^{-1}(\alpha^{-1}([0])) = \varphi^{-1}(\varphi(\theta)) = \theta$. Furthermore, the property of being an analytic group is inherited from $(\mathbb{C}/\Lambda, +)$, since Φ is an isomorphism of Riemann surfaces.

Now, let's see the uniqueness. Let's assume, as an initial case, that $E = \mathbb{C}/\Lambda$ and $\theta = [0]$.

Suppose that, a part from $+$ (inherited from $\mathbb{C}$), there exists another analytic addition $\oplus$ on $\mathbb{C}/\Lambda$ with $[0]$ for neutral element. We want to see that

$$\forall [z_1], [z_2] \in \mathbb{C}/\Lambda \quad [z_1] + [z_2] = [z_1] \oplus [z_2]$$

Note that:

- If $[z_2] = [0]$, for each $[z_1] \in \mathbb{C}/\Lambda \quad [z_1] + [0] = [z_1] = [z_1] \oplus [0]$, since $[0]$ is the neutral element of $+$ and $\oplus$.

- If $[z_2] \neq [0]$, the map $\Phi_{[z_2]}([z]) = [z] \oplus [z_2]$ is an automorphism of $\mathbb{C}/\Lambda$: in fact, it's a bijective an holomorphic map, because $(\mathbb{C}/\Lambda, \oplus)$ is an analytic group.

  Moreover, $\Phi_{[z_2]}$ has no fixed points, so by theorem 2.15,

  $$\Phi_{[z_2]}([z]) = [z] + [\beta]$$

  for some $\beta \notin \Lambda$, i.e., $[\beta] \neq [0]$. Then, for each $[z] \in \mathbb{C}/\Lambda$,

  $$[z] \oplus [z_2] = \Phi_{[z_2]}([z]) = [z] + [\beta]$$

  In particular, taking $[z] = [0]$ we deduce $[z_2] = [\beta]$ and thus

  $$\forall [z_1] \in \mathbb{C}/\Lambda \quad [z_1] \oplus [z_2] = [z_1] + [\beta] = [z_1] + [z_2]$$

In the case of an arbitrary elliptic curve E and an arbitrary element $\theta \in E$, suppose that there exist two different analytic group structures on E, having θ for neutral element.

Defining Φ as above, Φ^{-1} transfers two different analytic additions on E (with θ for neutral element) to two different analytic additions on $\mathbb{C}/\Lambda$ with $\Phi^{-1}(\theta) = [0]$ for neutral element, which is impossible. □

Corollary 2.18. *Let E_1, E_2 be elliptic curves, with additions $+$ and $\oplus$ having for neutral elements θ_1 and θ_2, respectively. Then, any isomorphism $\varphi : E_1 \longrightarrow E_2$ of Riemann surfaces satisfying $\varphi(\theta_1) = \theta_2$ is a group isomorphism from $(E_1, +)$ to $(E_2, \oplus)$.*

Proof. Define an analytic addition $\oplus'$ on E_2 given by

$$\forall a, b \in E_1 \quad \varphi(a) \oplus' \varphi(b) = \varphi(a+b)$$

with neutral element $\varphi(\theta_1) = \theta_2$.

Then, $(E_2, \oplus)$ and $(E_2, \oplus')$ are both analytic groups with neutral element θ_2. By theorem 2.17,

$$\forall a, b \in E_1 \quad \varphi(a) \oplus \varphi(b) = \varphi(a) \oplus' \varphi(b) = \varphi(a+b)$$

This property, jointly with φ being bijective and mapping the neutral element θ_1 to the neutral element θ_2, gives us that φ is a group isomorphism. □

Example. Given a lattice $\Lambda \subset \mathbb{C}$, its Weierstrass's $\wp$-function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

This function is elliptic with period any point of Λ: that is, $\wp(z) = \wp(z+\omega)$ for each $\omega \in \Lambda$.

Furthermore, $\wp$ is meromorphic on $\mathbb{C}$ (with a double pole at each point of Λ) and satisfies an equation of degree 3 involving $\wp$ and $\wp'$.

If C is a non-singular plane projective cubic (a particular case of elliptic curve), by the properties of Weierstrass's $\wp$-functions, the isomorphism $\varphi : \mathbb{C}/\Lambda \longrightarrow C$ with a certain complex torus $\mathbb{C}/\Lambda$ has the form

$$\varphi([z]) = (1 : \wp(z) : \wp'(z))$$

The group structure on C defined through φ, with neutral element $O \in C$, is given by

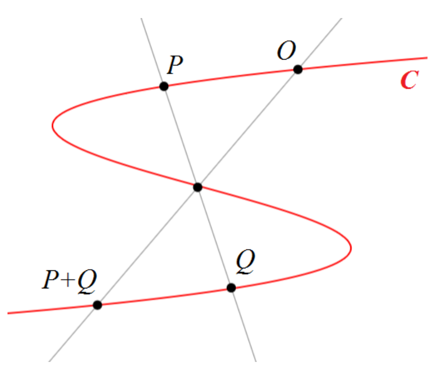

Figure 2.4

Remark. An exposition of Weierstrass's $\wp$-functions (and their relation to elliptic curves) can be found in [3].

Definition. Let E be an elliptic curve, and $+$ the analytic addition on E with neutral element $\theta \in E$. A point $p \in E$ is said to be a *torsion point of order n* (or a *n-torsion point*) if and only if

$$\theta = p + \overset{n)}{\cdots} + p = n \cdot p$$

The notion of torsion point on elliptic curves is useful in many branches of mathematics. For example, in Number Theory, one considers elliptic curves over a number field, and their torsion points are closely related to the solutions of diophantine equations.

In our case, we are studying elliptic curves over $\mathbb{C}$. In order to prove Cayley's theorem in Chapter 4, we are interested in the torsion points on a very specific elliptic curve: the plane projective cubic. The following theorem gives us an useful criterion:

Theorem 2.19. *Let E be the plane cubic with equation $y^2 z = (x - az)(x - bz)(x - cz)$ in $\mathbb{P}^2_{\mathbb{C}}$, where $a, b, c \in \mathbb{C} \setminus \{0\}$ are distinct. Consider E as elliptic curve, with neutral element $(0:1:0) \in E$.*

Suppose that $\sum_{k=0}^{\infty} A_k (x - x_0)^k$ is the Taylor expansion of $\sqrt{(x-a)(x-b)(x-c)}$ at a point $x = x_0$.

1. *If n is odd ($n = 2m+1$, for some $m \geq 1$), then:*

$$(x_0 : A_0 : 1) \text{ is a } n\text{-torsion point of } E \iff \begin{vmatrix} A_2 & \dots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \dots & A_{2m} \end{vmatrix} = 0$$

2. *If n is even ($n = 2m$, for some $m \geq 2$), then:*

$$(x_0 : A_0 : 1) \text{ is a } n\text{-torsion point of } E \iff \begin{vmatrix} A_3 & \dots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \dots & A_{2m-1} \end{vmatrix} = 0$$

Remark. Choosing the Taylor expansion of $-\sqrt{(x-a)(x-b)(x-c)}$, the same criterion holds. In fact, this choice just changes the sign of the coefficients A_k, so the determinant is multiplied either by 1 or -1.

Chapter 3

Poncelet's porism

3.1 Poncelet correspondence

Definition. Let C and D two non-degenerate conics of $\mathbb{P}^2_{\mathbb{C}}$. The *Poncelet correspondence for C and D* is

$$\mathfrak{M} = \{(p,l) \in C \times D^* : \, p \in l\},$$

where D^* denotes the conic envelope of D.

Consider two maps σ and τ on $\mathfrak{M}$, given by

$$\begin{aligned} \sigma : \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (p,l) &\longmapsto (q,l) \end{aligned} \qquad\qquad \begin{aligned} \tau : \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (q,l) &\longmapsto (q,\widetilde{l}) \end{aligned}$$

where q is the other point of the intersection $C \cap l$, and $\widetilde{l}$ is the other tangent line to D through q. It's obvious that σ and τ are involutions of $\mathfrak{M}$: $\sigma^2 = Id_{\mathfrak{M}} = \tau^2$.

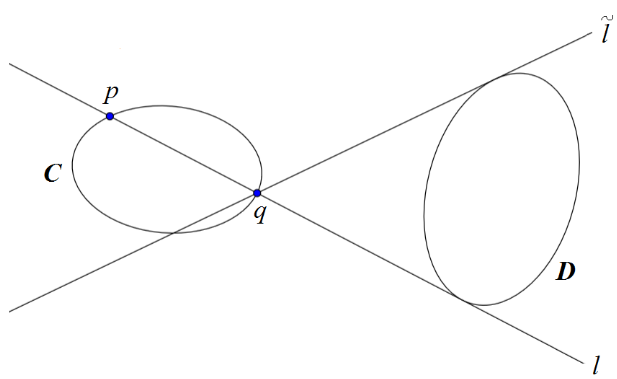

Figure 3.1

The composition $\eta = \tau \circ \sigma$ maps the pair (p,l) to $(q,\widetilde{l})$, which is equivalent, in terms of the Poncelet problem, to make a step in the construction of a polygon inscribed in C and circumscribed about D, just like we did in the Introduction.

So Poncelet's porism can be restated in the following way:

"For any integer $n \geq 3$, η^n has a fixed point if and only if $\eta^n = Id_{\mathfrak{M}}$"

To prove this new version of Poncelet's porism, we shall identify $\mathfrak{M}$ with a non-singular algebraic curve of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, so that $\mathfrak{M}$ will become a Riemann surface. Actually, we will see that $\mathfrak{M}$ is an elliptic curve.

Poncelet's porism will follow easily from the fact that η is a translation of $\mathfrak{M}$, considering $\mathfrak{M}$ with its group structure. In this point, the characterization of automorphisms on a complex torus, given in the preceding chapter, will be very useful.

3.2 $\mathfrak{M}$ as an algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

Let C and D two non-degenerate conics of $\mathbb{P}^2_{\mathbb{C}}$, and let

$$q : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}, \quad q(r) = q(r_0 : r_1) = (r_0^2 : r_0 r_1 : r_1^2)$$

a parameterization of the non-degenerate conic with equation $y^2 - xz = 0$. Since any two non-degenerate conics of $\mathbb{P}^2_{\mathbb{C}}$ are projectively equivalent, there are 3×3 regular matrices A and B such that $Aq(r) = p(r)$ and $Bq(s) = l(s)$ are, respectively, parameterizations of C and D^*.

Hence, we have a bijection

$$\begin{aligned} F : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} &\longrightarrow C \times D^* \\ (r,s) &\longmapsto (p(r), l(s)) \end{aligned}$$

Let's consider $\gamma = F^{-1}(\mathfrak{M}) = \left\{(r,s) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : \, p(r) \in l(s)\right\}$, and put $B^t A = T = (t_{ij})_{0 \leq i,j \leq 2}$. T is a regular matrix, as it's the product of two regular matrices.

In order to find an equation of γ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, observe that

$$(r,s) \in \gamma \iff p(r) \in l(s) \iff 0 = (l(s))^t \cdot p(r) \iff 0 = (Bq(s))^t \cdot Aq(r) = q(s)^t B^t A q(r) \iff$$

$$\iff 0 = \begin{pmatrix} s_0^2 & s_0 s_1 & s_1^2 \end{pmatrix} \begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} r_0^2 \\ r_0 r_1 \\ r_1^2 \end{pmatrix} = (t_{00} r_0^2 + t_{01} r_0 r_1 + t_{02} r_1^2) \cdot s_0^2 +$$

$$+ (t_{10} r_0^2 + t_{11} r_0 r_1 + t_{12} r_1^2) \cdot s_0 s_1 + (t_{20} r_0^2 + t_{21} r_0 r_1 + t_{22} r_1^2) \cdot s_1^2$$

Writing $T_i(r) = t_{i0} r_0^2 + t_{i1} r_0 r_1 + t_{i2} r_1^2$, we can describe γ as the algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with equation $H(r,s) = 0$, where $H(r,s) = T_0(r) \cdot s_0^2 + T_1(r) \cdot s_0 s_1 + T_2(r) \cdot s_1^2$ is a bihomogeneous form of bidegree $(2,2)$.

Remark. $T_0(r) = T_1(r) = T_2(r) = 0$ has no solution. In fact, since T is regular,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T_0(r) \\ T_1(r) \\ T_2(r) \end{pmatrix} = T \begin{pmatrix} r_0^2 \\ r_0 r_1 \\ r_1^2 \end{pmatrix} \implies r_0^2 = r_0 r_1 = r_1^2 = 0 \implies r_0 = 0 = r_1 \implies (r_0 : r_1) \notin \mathbb{P}^1_{\mathbb{C}}$$

Proposition 3.1. *Let $\Delta(r) = T_1(r)^2 - 4T_0(r)T_2(r)$, for each $r \in \mathbb{P}^1_{\mathbb{C}}$. Then: $\Delta(r) = 0$ if and only if $p(r) \in C \cap D$.*

Proof. For a fixed point $r \in \mathbb{P}^1_{\mathbb{C}}$, there are two distinct tangent lines to D through $p(r)$, except when $p(r) \in C \cap D$ (in that case, there is only one such line). Therefore,

$p(r) \in C \cap D \iff$ There is a single tangent line to D through $p(r) \iff H(r,s) = 0$ has a single solution $s \in \mathbb{P}^1_{\mathbb{C}}$

We are going to prove that this is equivalent to $\Delta(r) = 0$. Let's distinguish two cases:

- If $T_2(r) = 0$, we have: $0 = \Delta(r) \iff T_1(r) = 0$ (and $T_0(r) \neq 0$) $\iff$ The equation is $0 = H(r,s) = T_0(r) \cdot s_0^2$.

  And this equation has a single solution, $s = (0:1) \in \mathbb{P}^1_{\mathbb{C}}$.

- If $T_2(r) \neq 0$, it's easy to check that $H(r,s) = 0$ has no solution with $s_0 = 0$. We can suppose that $s_0 = 1$ and the equation is

  $$0 = T_0(r) + T_1(r) \cdot s_1 + T_2(r) \cdot s_1^2,$$

  that is a polynomial of degree 2 in s_1. So:

  $H(r,s) = 0$ has a single solution $s = (1 : s_1) \in \mathbb{P}^1_{\mathbb{C}} \iff 0 = T_1(r)^2 - 4T_0(r)T_2(r) = \Delta(r)$

In any case, we have the desired equivalence. □

Reversing r and s, we have $H(r,s) = \widetilde{T}_0(s) \cdot r_0^2 + \widetilde{T}_1(s) \cdot r_0 r_1 + \widetilde{T}_2(s) \cdot r_1^2$, where

$$\widetilde{T}_i(s) = t_{0i} s_0^2 + t_{1i} s_0 s_1 + t_{2i} s_1^2$$

(namely, H is given by the columns of T instead of its rows). By a similar argument, we obtain:

Proposition 3.2. *Let $\widetilde{\Delta}(s) = \widetilde{T}_1(s)^2 - 4\widetilde{T}_0(s)\widetilde{T}_2(s)$, for each $s \in \mathbb{P}^1_{\mathbb{C}}$. Then: $\widetilde{\Delta}(s) = 0$ if and only if $l(s) \in C^* \cap D^*$.*

3.3 Structure of elliptic curve on $\mathfrak{M}$

Our first goal in this section is to prove that, when C and D are in general position, $\gamma = \{(r,s) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : H(r,s) = 0\}$ is a non-singular algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Then, according to theorem 2.7, γ will become a Riemann surface, and the bijection $F_{|\gamma} : \gamma \longrightarrow \mathfrak{M}$ will endow $\mathfrak{M}$ with a complex structure, through the rules:

U is an open set in $\mathfrak{M} \iff F^{-1}(U)$ is an open set in γ,

(U, ϕ) is a chart on $\mathfrak{M} \iff (F^{-1}(U), \phi \circ F)$ is a chart on γ

Lemma 3.3. *The set of singular points of γ is $S_\gamma = \{(a,b) \in \gamma :\ p(a) \in C \cap D,\ l(b) \in C^* \cap D^*\}$.*

Proof. We are going to study γ in the cover of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ formed by the charts

$$A_1 = \{((x_0:1),(y_0:1)) : x_0, y_0 \in \mathbb{C}\}, \quad A_2 = \{((x_0:1),(1:y_1)) : x_0, y_1 \in \mathbb{C}\},$$

$$A_3 = \{((1:x_1),(y_0:1)) : x_1, y_0 \in \mathbb{C}\}, \quad A_4 = \{((1:x_1),(1:y_1)) : x_1, y_1 \in \mathbb{C}\},$$

each of them isomorphic to the complex affine plane $\mathbb{C}^2$.

For example, we can see $\gamma \cap A_1$ as the plane affine curve $\widetilde{H}(r_0,s_0) = 0$, where

$$\widetilde{H}(r_0,s_0) = H((r_0:1),(s_0:1)) = T_0((r_0:1)) \cdot s_0^2 + T_1((r_0:1)) \cdot s_0 + T_2((r_0:1)).$$

By an abuse of notation, we will write $T_i(r_0) = T_i((r_0:1))$ for $i = 0,1,2$. Then,

$$\widetilde{H}(r_0,s_0) = T_0(r_0) \cdot s_0^2 + T_1(r_0) \cdot s_0 + T_2(r_0)$$

$$\frac{\partial \widetilde{H}}{\partial s_0}(r_0,s_0) = 2T_0(r_0) \cdot s_0 + T_1(r_0)$$

and we have the identity $\left(\dfrac{\partial \widetilde{H}}{\partial s_0}(r_0,s_0)\right)^2 - 4T_0(r_0) \cdot \widetilde{H}(r_0,s_0) = T_1(r_0)^2 - 4T_0(r_0)T_2(r_0) = \Delta((r_0:1)) \quad (*)$.

On the other hand, using the expression of $H(r,s)$ in terms of $\widetilde{T}_i(s)$ $(i = 0,1,2)$, we can put

$$\widetilde{H}(r_0,s_0) = H((r_0:1),(s_0:1)) = \widetilde{T}_0((s_0:1)) \cdot r_0^2 + \widetilde{T}_1((s_0:1)) \cdot r_0 + \widetilde{T}_2((s_0:1)).$$

Abusing again of notation, we will write $\widetilde{T}_i(s_0) = \widetilde{T}_i((s_0:1))$ for $i = 0,1,2$. Then,

$$\widetilde{H}(r_0,s_0) = \widetilde{T}_0(s_0) \cdot r_0^2 + \widetilde{T}_1(s_0) \cdot r_0 + \widetilde{T}_2(s_0)$$

$$\frac{\partial \widetilde{H}}{\partial r_0}(r_0,s_0) = 2\widetilde{T}_0(s_0) \cdot r_0 + \widetilde{T}_1(s_0)$$

and we deduce $\left(\dfrac{\partial \widetilde{H}}{\partial r_0}(r_0,s_0)\right)^2 - 4\widetilde{T}_0(s_0) \cdot \widetilde{H}(r_0,s_0) = \widetilde{T}_1(s_0)^2 - 4\widetilde{T}_0(s_0)\widetilde{T}_2(s_0) = \widetilde{\Delta}((s_0:1)) \quad (**)$.

Finally, if $((a_0:1),(b_0:1))$ is an arbitrary point of $\gamma \cap A_1$ (i.e., $\widetilde{H}(a_0,b_0) = 0$),

$$((a_0:1),(b_0:1)) \text{ is a singular point of } \gamma \cap A_1 \iff \frac{\partial \widetilde{H}}{\partial r_0}(a_0,b_0) = 0 = \frac{\partial \widetilde{H}}{\partial s_0}(a_0,b_0) \iff$$

$$\iff \Delta((a_0:1)) = 0 = \widetilde{\Delta}((b_0:1)) \iff p((a_0:1)) \in C \cap D \text{ and } l((b_0:1)) \in C^* \cap D^*$$

(in the second equivalence we make use of $(*)$ and $(**)$, and in the third one propositions 3.1 and 3.2 are required).

A similar argument holds for the affine charts A_2, A_3 and A_4. □

Corollary 3.4. *If C and D meet at four different points, γ is a non-singular algebraic curve of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.*

Proof. Note that the image of S_γ under the bijection F is $F(S_\gamma) = \{(p,l) :\ p \in C \cap D,\ l \in C^* \cap D^*,\ p \in l\}$.

Under our hypothesis, this set is empty: if it were $(p,l) \in F(S_\gamma)$, C and D would meet at p with the same tangent line l. Hence, the multiplicity of the intersection point p would be greater than 1 and, according to Bézout theorem, C and D would not meet at four different points.

Since $F(S_\gamma) = \emptyset$ and F is a bijection, it follows that $S_\gamma = \emptyset$. By lemma 3.3, γ has no singular points. □

Hereinafter, we will assume that C and D meet at four different points. Once we have shown that $\mathfrak{M}$ is a Riemann surface (with a complex structure inherited from γ), we want to prove that $\mathfrak{M}$ is an elliptic curve. This fact follows from Hurwitz formula, but first we need a technical result concerning holomorphic maps and analytic manifolds.

Theorem 3.5. *Let V_1 and V_2 two connected analytic manifolds, $M \subset V_1$ a submanifold and $f : V_1 \longrightarrow V_2$ an holomorphic map. Then, $f_{|M}$ is an holomorphic map.*

Naively, we can think of analytic manifolds as a generalization of Riemann surfaces: they are topological spaces locally homeomorphic to $\mathbb{C}^n$, such that the transition maps are holomorphic functions in several complex variables.

Lemma 3.6. *$\mathfrak{M}$ is an elliptic curve.*

Proof. Consider the analytic manifolds $C \times D^*$ and C, with the complex structures determined by the parameterizations $p : \mathbb{P}^1_{\mathbb{C}} \longrightarrow C$ and $l : \mathbb{P}^1_{\mathbb{C}} \longrightarrow D^*$. Then,

$$\begin{aligned} \pi_1 : C \times D^* &\longrightarrow C \\ (p,l) &\longmapsto p \end{aligned} \qquad \begin{aligned} \pi_2 : C \times D^* &\longrightarrow D^* \\ (p,l) &\longmapsto l \end{aligned}$$

are holomorphic maps so, by theorem 3.5, their respective restrictions $\widetilde{\pi}_1, \widetilde{\pi}_2$ to the Riemann surface $\mathfrak{M} \subset C \times D^*$ are holomorphic maps too.

If we focus on $\widetilde{\pi}_1$, a general point $p \in C$ has two preimages (p,l_1) and (p,l_2), where l_1 and l_2 are the two tangent lines to D through p:

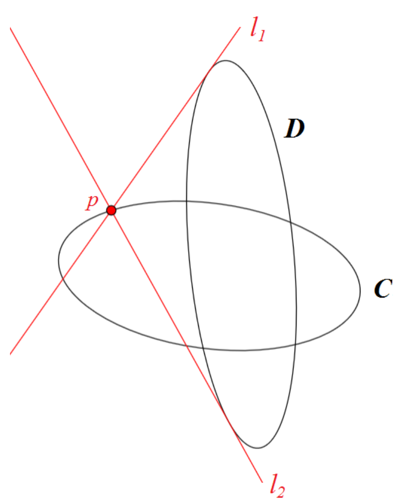

Figure 3.2

This happens except when $p \in C \cap D$. In that case, there is a single tangent line to D through the point p, so p has a single preimage.

Hence, $\widetilde{\pi}_1$ is an holomorphic map of degree 2 with four ramification points: $(p_1,l_1),(p_2,l_2),(p_3,l_3)$ and (p_4,l_4), where p_1, p_2, p_3, p_4 are the four different points of $C \cap D$ and l_i is the tangent line to D through p_i.

Furthermore, each of these ramification points has ramification index 2. By Hurwitz formula,

$$2g(\mathfrak{M})-2=\deg(\widetilde{\pi}_1)\cdot(2g(C)-2)+\sum_{p\in\mathfrak{M}}(e_p(\widetilde{\pi}_1)-1)=2\cdot(0-2)+4\cdot(2-1)=-4+4=0 \implies g(\mathfrak{M})=1$$

Therefore, to show that $\mathfrak{M}$ is an elliptic curve, it will suffice to prove that $\mathfrak{M}$ is a compact and connected space, so that Hurwitz formula can indeed be used.

Compactness of $\mathfrak{M}$ is immediately checked: the topology of $\mathfrak{M}$ is inherited from γ, which is a compact space (γ is a closed subset in the compact space $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$).

Let's prove that $\mathfrak{M}$ is a connected space. We know that $(\mathfrak{M},\widetilde{\pi}_1)$ is a double cover of C. So, if $\mathfrak{M}$ has two connected components $\mathfrak{M}_0$ and $\mathfrak{M}_1$, we have isomorphisms

$$\widetilde{\pi}_1:\mathfrak{M}_0\xrightarrow{\cong}C,\qquad \widetilde{\pi}_1:\mathfrak{M}_1\xrightarrow{\cong}C$$

and $\mathfrak{M}_0\cap\mathfrak{M}_1$ is the set of four ramification points of $\widetilde{\pi}_1$: that is, $\mathfrak{M}_0\cap\mathfrak{M}_1=\{(p_i,l_i):i=1,\dots,4\}=\mathscr{F}_1$.

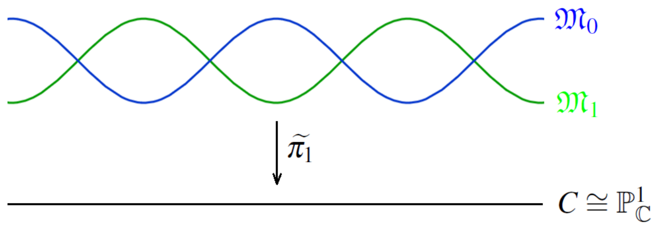

Figure 3.3

In a similar way, $(\mathfrak{M},\widetilde{\pi}_2)$ is a double cover of D^* and we will also have isomorphisms

$$\widetilde{\pi}_2:\mathfrak{M}_0\xrightarrow{\cong}D^*,\qquad \widetilde{\pi}_2:\mathfrak{M}_1\xrightarrow{\cong}D^*$$

$\mathfrak{M}_0\cap\mathfrak{M}_1$ will be the set of ramification points of $\widetilde{\pi}_2$: $\mathfrak{M}_0\cap\mathfrak{M}_1=\left\{(\widetilde{p}_i,\widetilde{l}_i):i=1,\dots,4\right\}=\mathscr{F}_2$, where $\widetilde{l}_i\in C^*\cap D^*$ and $\widetilde{p}_i=C\cap\widetilde{l}_i$.

But $\mathscr{F}_1\neq\mathscr{F}_2$ (in fact, $\mathscr{F}_1\cap\mathscr{F}_2=\emptyset$ as we saw in corollary 3.4), which is a contradiction. $\square$

3.4 Proof of Poncelet's porism

As we have said in section 3.1, we will deduce Poncelet's porism from the fact that $\eta=\tau\circ\sigma$ is a translation on $\mathfrak{M}$. The first step consists on proving that the involutions σ and τ are automorphisms of $\mathfrak{M}$, and thus so is $\eta=\tau\circ\sigma$.

Let us remember that we have an isomorphism of Riemann surfaces

$$F_{|\gamma}:\gamma\longrightarrow\mathfrak{M},\quad F(r,s)=(p(r),l(s))$$

So we can define two maps $\sigma_*,\tau_*:\gamma\longrightarrow\gamma$ given by the rules $\sigma_*=F^{-1}\circ\sigma\circ F$ and $\tau_*=F^{-1}\circ\tau\circ F$.

Remark. σ_* interchanges the points of γ with the same s-coordinate. In fact, for a point $(a,b)\in\gamma$, we have:

- If $l(b)\notin C^*\cap D^*$, the intersection $l(b)\cap C$ consists of two different points $p(a)$ and $p(a')$, for a certain $a'\in\mathbb{P}^1_{\mathbb{C}}$. Therefore, $(a,b),(a',b)$ are the two points of γ with the s-coordinate equal to b and

$$\sigma_*(a,b)=(F^{-1}\circ\sigma\circ F)(a,b)=F^{-1}(\sigma(p(a),l(b)))=F^{-1}(p(a'),l(b))=(a',b)$$

- Let's suppose that $l(b) \in C^* \cap D^*$. Since $l(b)$ is a tangent line to C, it follows that $p(a)$ is the single point of the intersection $l(b) \cap C$. In this case, (a,b) is the single point of γ with the s-coordinate equal to b and

$$\sigma_*(a,b) = (F^{-1} \circ \sigma \circ F)(a,b) = F^{-1}(\sigma(p(a),l(b))) = F^{-1}(p(a),l(b)) = (a,b)$$

In the same way, τ_* interchanges the points of γ with the same r-coordinate.

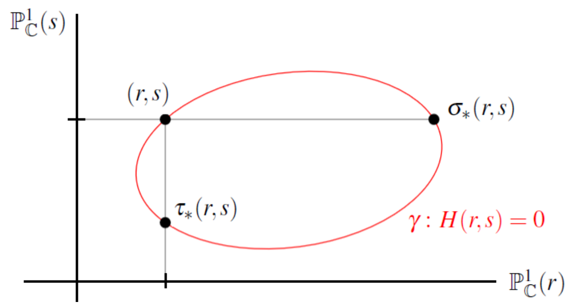

Figure 3.4

Lemma 3.7. *σ_* and τ_* are automorphisms of the Riemann surface γ.*

Proof. We will give a proof for τ_*; a similar one holds for σ_*.

Consider the set $B = \{(r,s) \in \gamma : \ p(r) \in C \cap D\}$. By the preceding remark, we can write

$$\tau_*(r,s) = \begin{cases} (r,s') \text{ (with } s' \neq s \text{ such that } p(r) \in l(s')) & \text{if } (r,s) \notin B \\ (r,s) & \text{if } (r,s) \in B \end{cases}$$

It is clearly a bijective map. In order to show that τ_* is an holomorphic map, observe that:

- τ_* is holomorphic on $\gamma \setminus B$:

  Let $p_0 = (r_0, s_0) \in \gamma \setminus B$, then $\tau_*(p_0) = (r_0, s_0') \in \gamma \setminus B$ with $s_0 \neq s_0'$. Since γ is a Hausdorff space, we can take disjoint neighbourhoods U and U' of p_0 and $\tau_*(p_0)$, respectively.

  Let's choose a neighbourhood V of r_0 in $\mathbb{P}^1_{\mathbb{C}}$, such that the projection

  $$\pi_1 : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}, \quad \pi_1(r,s) = r$$

  induces homeomorphisms $\pi_1 : U \xrightarrow{\cong} V$ and $\pi_1 : U' \xrightarrow{\cong} V$.

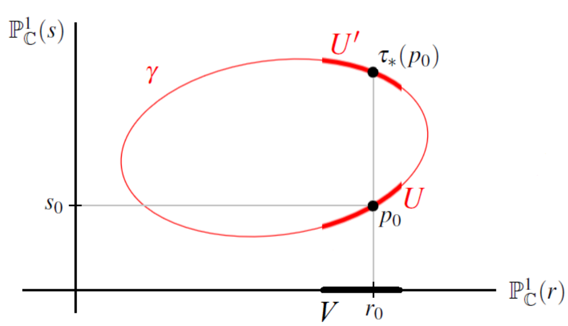

Figure 3.5

We can see (U,π_1) and (U',π_1) as local charts on γ. In terms of these charts, τ_* is given by

$$\pi_1 \circ \tau_* \circ \pi_1^{-1} : V \xrightarrow{\pi_1^{-1}} U \xrightarrow{\tau_*} U' \xrightarrow{\pi_1} V,$$

that is the identity map. Thus τ_* is holomorphic at p_0.

- τ_* is continuous on B:

  Let's suppose that τ_* is not continuous at a point $p_0 = (r_0,s_0) \in B$. Namely, we can consider a sequence $\{(r_n,s_n)\}_n \subset \gamma$ such that

  $$(r_n,s_n) \xrightarrow{n} (r_0,s_0), \quad \tau_*(r_n,s_n) = (r_n,s_n') \overset{n}{\nrightarrow} \tau_*(r_0,s_0) = (r_0,s_0)$$

  By the compactness of γ, there exists a partial sequence $\left\{(r_{n_k},s'_{n_k})\right\}_k$ converging to a point of γ. Looking at the first component, we have

  $$r_n \xrightarrow{n} r_0 \Longrightarrow (r_{n_k},s'_{n_k}) \xrightarrow{k} (r_0,s_0') \in \gamma, \text{ for some } s_0' \neq s_0$$

  So $(r_0,s_0),(r_0,s_0') \in \gamma$, with $s_0 \neq s_0'$ and $(r_0,s_0) \in B$, which is a contradiction.

Since τ_* is holomorphic on $\gamma \setminus B$ and is continuous on the finite set B, we conclude that τ_* is holomorphic on γ. □

Corollary 3.8. *σ and τ are automorphisms of the Riemann surface $\mathfrak{M}$.*

Proof. It follows from the fact that $\sigma = F \circ \sigma_* \circ F^{-1}$ and $\tau = F \circ \tau_* \circ F^{-1}$ are composition of isomorphisms. □

In the following theorem, we deal with the structure of involutional automorphisms of a complex torus.

Theorem 3.9. *Let Λ a lattice, and $\tau : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$ a nontrivial automorphism with at least one fixed point satisfying $\tau^2 = Id$. Then,*

$$\tau([z]) = [-z+\beta], \textit{ for some } \beta \in \mathbb{C}$$

Proof. Since τ is an automorphism of $\mathbb{C}/\Lambda$, by theorem 2.15 we know that $\tau([z]) = [\alpha z + \beta]$, for some $\alpha,\beta \in \mathbb{C}$ such that $\alpha\Lambda = \Lambda$. Iterating τ, we obtain

$$\tau^2([z]) = [\alpha(\alpha z+\beta)+\beta] = [\alpha^2 z + \beta(\alpha+1)]$$

By hypothesis, τ is an involution, so

$$\forall z \in \mathbb{C} \ [z] = \tau^2([z]) = [\alpha^2 z + \beta(\alpha+1)] \implies \forall z \in \mathbb{C} \ g(z) = (\alpha^2-1)z + \beta(\alpha+1) \in \Lambda$$

Note that, if $\alpha^2 \neq 1$, g is a translation and its image is not contained in Λ. Hence, $\alpha^2 = 1$.

Furthermore, if $\alpha = 1$, $\tau([z]) = [z+\beta]$ is the identity map (if $\beta \in \Lambda$) or has no fixed points (if $\beta \notin \Lambda$).

Therefore, it must be $\alpha = -1$ and $\tau([z]) = [-z+\beta]$. □

Remark. If $\{\omega_1,\omega_2\}$ is a basis of the lattice Λ, it's easy to check that τ has exactly four fixed points:

$$[\tfrac{1}{2}(\beta+\omega_1)],\ [\tfrac{1}{2}(\beta+\omega_2)],\ [\tfrac{1}{2}\beta] \text{ and } [\tfrac{1}{2}(\beta+\omega_1+\omega_2)]$$

Corollary 3.10. *Let τ_1, τ_2 automorphisms of $\mathbb{C}/\Lambda$ with at least one fixed point satisfying $\tau_1^2 = Id = \tau_2^2$. Then, $\tau_1 \circ \tau_2$ is a translation of $\mathbb{C}/\Lambda$.*

Proof. By theorem 3.9, we have $\tau_1([z]) = [-z+\beta_1]$ and $\tau_2([z]) = [-z+\beta_2]$, for some $\beta_1, \beta_2 \in \mathbb{C}$.

This gives that $(\tau_1 \circ \tau_2)([z]) = \tau_1([-z+\beta_2]) = [z-\beta_2+\beta_1]$ is a translation. □

Recall that, inasmuch as $\mathfrak{M}$ is an elliptic curve, we have an isomorphism

$$\varphi : \mathbb{C}/\Lambda \longrightarrow \mathfrak{M}$$

for some lattice Λ. This isomorphism induces an analytic group structure on $\mathfrak{M}$, with the addition

$$\varphi(x) + \varphi(y) = \varphi(x+y) \text{ for all } x, y \in \mathbb{C}/\Lambda$$

Based on the preceding corollary, we are ready to prove that η is a translation on $\mathfrak{M}$ and deduce Poncelet's porism.

Proposition 3.11. *$\eta = \tau \circ \sigma$ is a translation on $\mathfrak{M}$. Namely, there exists $m \in \mathfrak{M}$ such that*

$$\eta(p) = p + m, \text{ for all } p \in \mathfrak{M}.$$

Proof. Let's consider the maps $\widetilde{\sigma}, \widetilde{\tau}, \widetilde{\eta} : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$ given by

$$\widetilde{\sigma} = \varphi^{-1} \circ \sigma \circ \varphi, \ \widetilde{\tau} = \varphi^{-1} \circ \tau \circ \varphi, \ \widetilde{\eta} = \varphi^{-1} \circ \eta \circ \varphi \qquad (\text{note that } \widetilde{\eta} = \widetilde{\tau} \circ \widetilde{\sigma}).$$

The involutions $\widetilde{\sigma}$ and $\widetilde{\tau}$ are automorphisms of $\mathbb{C}/\Lambda$ with four fixed points (a property inherited from σ and τ). By corollary 3.10, there exists $a \in \mathbb{C}/\Lambda$ such that

$$\widetilde{\eta}(q) = q + a, \text{ for all } q \in \mathbb{C}/\Lambda.$$

Then, it's enough to take $m = \varphi(a)$. In fact, for all $p \in \mathfrak{M}$,

$$\eta(p) = (\varphi \circ \widetilde{\eta} \circ \varphi^{-1})(p) = \varphi(\varphi^{-1}(p) + a) = \varphi(\varphi^{-1}(p)) + \varphi(a) = p + m \ \square$$

Theorem 3.12 (Poncelet's porism). *For any integer $n \geq 3$, η^n has a fixed point if and only if $\eta^n = Id_{\mathfrak{M}}$.*

Proof. Suppose that η^n has a fixed point $p_0 \in \mathfrak{M}$. By proposition 3.11, there exists $m \in \mathfrak{M}$ such that

$$\forall p \in \mathfrak{M} \ \ \eta(p) = p + m \implies \forall p \in \mathfrak{M} \ \ \eta^n(p) = p + n \cdot m$$

If p_0 is a fixed point of η^n, we have

$$p_0 = \eta^n(p_0) = p_0 + n \cdot m \implies n \cdot m = 0 \implies \forall p \in \mathfrak{M} \ \eta^n(p) = p$$

and, therefore, η^n is the identity map. □

Chapter 4

Cayley's theorem

We keep on working with two non-degenerate conics C and D of $\mathbb{P}^2_{\mathbb{C}}$, meeting at four different points.

Let $n \geq 3$ an integer. By Poncelet's porism, we know that either there are no n-sided polygons simultaneously inscribed in C and circumscribed about D, or there are infinitely many of them.

Now, the problem we deal with is determining whether there exists such a polygon. Cayley's theorem provides an elegant answer to this question, by expliciting a criterion from the equations for C and D.

4.1 A new algebraic equation for $\mathfrak{M}$

Remark. By an abuse of notation, we write C and D to denote the conics and their respective matrices.

For every $r = (r_0 : r_1) \in \mathbb{P}^1_{\mathbb{C}}$, let C_r be the conic with matrix $r_0C + r_1D$. Then, $\{C_r\}_{r\in\mathbb{P}^1_{\mathbb{C}}}$ is the conic pencil with base points $p_0, p_1, p_2, p_3 \in C \cap D$.

Let's denote by l_r the tangent line to C_r through the point p_0. This line will meet again C at another point $p(r)$:

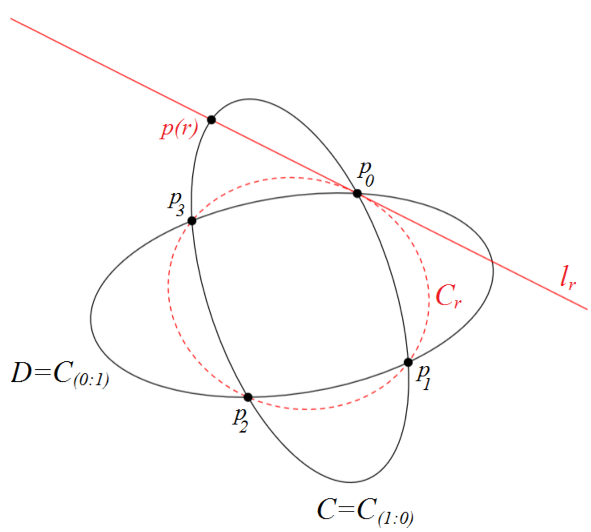

Figure 4.1

We have a bijection $p : \mathbb{P}^1_{\mathbb{C}} \longrightarrow C$, with $p((1:0)) = p_0$ since $l_{(1:0)}$ is the tangent line to $C_{(1:0)} = C$ through p_0.

This bijection p gives us a parameterization of the conic C. So there exists a 3×3 regular matrix A such that $Aq(r) = p(r)$, where

$$q : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}, \quad q(r) = q(r_0 : r_1) = (r_0^2 : r_0 r_1 : r_1^2)$$

is the parameterization of the non-degenerate conic $y^2 - xz = 0$.

Proposition 4.1. *For each $r = (r_0 : r_1) \in \mathbb{P}^1_{\mathbb{C}}$, the identity $\Delta(r) = r_1 \det(r_0 C + r_1 D)$ holds.*

Proof. We have already seen that $p_0 = p((1:0))$. Now, we want to find the preimages of the other base points p_1, p_2, p_3 of the conic pencil $\{C_r\}_{r \in \mathbb{P}^1_{\mathbb{C}}}$.

Inasmuch as C and D meet at four different points, $\{C_r\}_{r \in \mathbb{P}^1_{\mathbb{C}}}$ has exactly three degenerate conics C_{a_1}, C_{a_2} and C_{a_3}, with $a_i = (a_{i0} : a_{i1}) \in \mathbb{P}^1_{\mathbb{C}}$ satisfying $\det(a_{i0} C + a_{i1} D) = 0$.

Furthermore, we can assume that $a_i = (a_{i0} : 1)$ (C is a non-degenerate conic, so $\det C \neq 0$). Namely, a_{10}, a_{20} and a_{30} are the three complex roots of the third degree polynomial $\det(r_0 C + D)$.

According to corollary 1.14, the degenerate conics C_{a_1}, C_{a_2} and C_{a_3} are the three pairs of lines including the points p_0, p_1, p_2, p_3. If we index the a_i in a way that C_{a_i} contains the line $l_i = p_0 \vee p_i$, we have

$$l_{a_i} = l_i \implies p_i = p(a_i)$$

On the other hand, recall that $p(r) \in C \cap D = \{p_0, p_1, p_2, p_3\} \iff \Delta(r) = 0$. Now:

- Focusing on the points $(r_0 : 1)$, $r_0 \in \mathbb{C}$, we observe that $\Delta((r_0 : 1))$ and $\det(r_0 C + D)$ are polynomials in $\mathbb{C}$ with exactly the same roots (a_{10}, a_{20} and a_{30}). Consequently,

  $$\forall r_0 \in \mathbb{C} \;\; \Delta((r_0 : 1)) = \alpha \cdot \det(r_0 C + D)$$

  for some constant $\alpha \neq 0$.

- $\Delta((1:0)) = 0$ (it follows from $p((1:0)) = p_0 \in C \cap D$) and, clearly, $\alpha \cdot r_1 \det(r_0 C + r_1 D)$ vanishes at $(1:0)$.

Since $\Delta((r_0 : r_1))$ and $\alpha \cdot r_1 \det(r_0 C + r_1 D)$ are homogeneous polynomials of degree 4, it is deduced that

$$\forall (r_0 : r_1) \in \mathbb{P}^1_{\mathbb{C}} \;\; \Delta((r_0 : r_1)) = \alpha \cdot r_1 \det(r_0 C + r_1 D)$$

Changing, if necessary, the matrix A by $\frac{1}{\sqrt{\alpha}} A$ (this change does not affect the projectivity represented by A), we can assume $\alpha = 1$. $\square$

As we have seen, with this parameterization p of the conic C we have a "relatively good" expression for $\Delta(r)$. Now, we will use this expression to construct an explicit isomorphism G between a certain elliptic curve E and γ.

Lemma 4.2. *The curve γ is isomorphic to $E = \left\{(r,u) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : u_0^2 r_1^3 = u_1^2 \cdot \det(r_0 C + r_1 D)\right\}$.*

Proof. Note that E is an algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, with bidegree $(3,2)$. In order to show that E is a Riemann surface, by theorem 2.7 it will suffice to prove that it has no singular points.

Consider the affine chart $A_1 = \left\{((x:1),(y:1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x,y \in \mathbb{C}\right\}$ of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Using the notations of proposition 4.1, we can see $E \cap A_1$ as the plane affine cubic with equation

$$y^2 = \det(xC+D) = (x-a_{10})(x-a_{20})(x-a_{30})$$

where the a_{i0} are three different non-zero complex numbers.

Taking partial derivatives with respect to x and y, the conditions for singular points are

$$\begin{cases} 2y = 0 \\ (x-a_{10})(x-a_{20})+(x-a_{10})(x-a_{30})+(x-a_{20})(x-a_{20}) = 0 \\ y^2 = (x-a_{10})(x-a_{20})(x-a_{30}) \end{cases}$$

But this system has no solutions: if $y=0$, by the third equation it must be $x = a_{i0}$ for some $i \in \{1,2,3\}$ and the second equation is not satisfied.

So that E has no singular points on A_1. A similar check holds for the remaining affine charts covering $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

Now, consider the open set $E_0 = \{(r,u) \in E : T_2(r) \neq 0, r_1 \neq 0\}$ in E, and define a map

$$G : E_0 \longrightarrow \gamma, \quad G((r_0:r_1),(u_0:u_1)) = ((r_0:r_1),(2T_2(r)\cdot u_1 : -T_1(r)\cdot u_1 + u_0 \cdot r_1^2))$$

In fact, for each $(r,u) \in E_0$, $G(r,u)$ is a point of γ, because it satisfies the equation $H(r,s) = 0$:

$$\begin{aligned} &H((r_0:r_1),(2T_2(r)\cdot u_1 : -T_1(r)\cdot u_1 + u_0\cdot r_1^2)) = T_0(r)\cdot(2T_2(r)\cdot u_1)^2 + T_1(r)\cdot 2T_2(r)\cdot u_1 \cdot(-T_1(r)\cdot u_1 + u_0\cdot r_1^2) + \\ &+T_2(r)\cdot(-T_1(r)\cdot u_1 + u_0 \cdot r_1^2)^2 = 4T_0(r)T_2(r)^2u_1^2 - 2T_1(r)^2T_2(r)u_1^2 \cancel{+2T_1(r)T_2(r)u_0u_1r_1^2} + T_2(r)u_0^2r_1^4 \\ &\cancel{-2T_1(r)T_2(r)u_0u_1r_1^2} + T_1(r)^2T_2(r)u_1^2 = 4T_0(r)T_2(r)^2u_1^2 - T_1(r)^2T_2(r)u_1^2 + T_2(r)u_1^2 r_1 \cdot \det(r_0C + r_1D) = \\ &= 4T_0(r)T_2(r)^2u_1^2 - T_1(r)^2T_2(r)u_1^2 + T_2(r)u_1^2\Delta(r) = T_2(r)u_1^2\cdot(4T_0(r)T_2(r) - T_1(r)^2 + \Delta(r)) = T_2(r)u_1^2 \cdot 0 = 0 \end{aligned}$$

Moreover, G is an injective map:

$$\begin{aligned} &((r_0:r_1),(2T_2(r)\cdot u_1 : -T_1(r)\cdot u_1 + u_0\cdot r_1^2)) = ((\widetilde{r_0}:\widetilde{r_1}),(2T_2(\widetilde{r})\cdot \widetilde{u_1} : -T_1(\widetilde{r})\cdot\widetilde{u_1} + \widetilde{u_0}\cdot\widetilde{r_1}^2)) \implies \\ &\implies (r_0:r_1) = (\widetilde{r_0}:\widetilde{r_1}) \text{ and } (2T_2(r)\cdot u_1 : -T_1(r)\cdot u_1 + u_0 \cdot r_1^2) = (2T_2(r)\cdot\widetilde{u_1} : -T_1(r)\cdot\widetilde{u_1} + \widetilde{u_0}\cdot r_1^2) \implies \\ &\implies (r_0:r_1) = (\widetilde{r_0}:\widetilde{r_1}) \text{ and } 0 = \begin{vmatrix} 2T_2(r)\cdot u_1 & -T_1(r)\cdot u_1 + u_0\cdot r_1^2 \\ 2T_2(r)\cdot\widetilde{u_1} & -T_1(r)\cdot\widetilde{u_1} + \widetilde{u_0}\cdot r_1^2 \end{vmatrix} = 2T_2(r)\cdot r_1^2\cdot(\widetilde{u_0}u_1 - u_0\widetilde{u_1}) \implies \\ &\implies (r_0:r_1) = (\widetilde{r_0}:\widetilde{r_1}) \text{ and } 0 = \widetilde{u_0}u_1 - u_0\widetilde{u_1} \implies (r_0:r_1) = (\widetilde{r_0}:\widetilde{r_1}) \text{ and } (u_0:u_1) = (\widetilde{u_0}:\widetilde{u_1}) \end{aligned}$$

It can also be checked that G is an holomorphic map. So we have two holomorphic maps $G : E_0 \longrightarrow \gamma$ and $G^{-1} : G(E_0) \longrightarrow E$, both of them with degree 1 (by the injectivity of G).

Since γ and E are Riemann surfaces contained in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^3_{\mathbb{C}}$ (this last inclusion via the Segre embedding), it follows from lemma 2.9 that we have two extended holomorphic maps $\widetilde{G} : E \longrightarrow \gamma$ and $\widetilde{G^{-1}} : \gamma \longrightarrow E$.

The fact of E and γ being non-singular curves ensures us that each of these extensions has degree 1. Therefore, the map $G : E \longrightarrow \gamma$ is bijective and gives us an isomorphism from the elliptic curve E to γ. □

4.2 Cayley's theorem

Since $\mathfrak{M}$ is an elliptic curve, it can be endowed with an analytic group structure, uniquely determined by the choice of neutral element as we saw in theorem 2.17.

Now, we are going to relate the existence of our desired polygons with the torsion points of $\mathfrak{M}$.

We will use the restatement of the Poncelet problem in terms of η. Namely, there exists an n-sided polygon inscribed in C and circumscribed about D if, and only if, $\eta^n = Id_{\mathfrak{M}}$.

Lemma 4.3. *Let $n > 0$ a positive integer, and $\theta \in \mathfrak{M}$ the neutral element of the addition on $\mathfrak{M}$. Then:*

$$\eta^n = Id_{\mathfrak{M}} \iff n \cdot \eta(\theta) = \theta \text{ (i.e., } \eta(\theta) \text{ is a torsion point of order } n\text{)}$$

Proof. We know that η is a translation of $\mathfrak{M}$. Hence, there must exist $m \in \mathfrak{M}$ such that

$$\forall p \in \mathfrak{M} \;\; \eta(p) = p + m \implies \forall p \in \mathfrak{M} \;\; \eta^n(p) = p + n \cdot m$$

In particular, taking $p = \theta$, observe that $\eta(\theta) = \theta + m = m$ and

$$\eta^n = Id_{\mathfrak{M}} \iff n \cdot m = \theta \iff n \cdot \eta(\theta) = \theta \;\square$$

So we are interested in the torsion points of $\mathfrak{M}$, which provide from torsion points of E.

But note that, in the usual affine charts of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, the elliptic curve E "looks like" a plane cubic. Basing on the description of torsion points of plane cubics given in theorem 2.19, we can establish the following criterion:

Theorem 4.4 (Cayley's theorem). *Let C and D two non-degenerate conics of $\mathbb{P}^2_{\mathbb{C}}$ meeting at four different points, and let*

$$\sqrt{\det(tC + D)} = A_0 + A_1 t + A_2 t^2 + \ldots$$

be the Taylor expansion, at the point $t = 0$, of the function $\sqrt{\det(tC + D)}$. Then, there exists a n-sided polygon inscribed in C and circumscribed about D if, and only if,

$$\begin{vmatrix} A_2 & \ldots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \ldots & A_{2m} \end{vmatrix} = 0, \quad \text{when } n \text{ is odd and } n = 2m+1, \text{ for some } m \geq 1$$

$$\begin{vmatrix} A_3 & \ldots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \ldots & A_{2m-1} \end{vmatrix} = 0, \quad \text{when } n \text{ is even and } n = 2m, \text{ for some } m \geq 2$$

Proof. Recall that we have the curves $E = \left\{(r,u) : u_0^2 r_1^3 = u_1^2 \cdot \det(r_0 C + r_1 D)\right\}$ and $\gamma = \{(r,s) : H(r,s) = 0\}$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, as well as the isomorphisms

$$G : E \longrightarrow \gamma, \quad G((r_0 : r_1),(u_0 : u_1)) = ((r_0 : r_1),(2T_2(r) \cdot u_1 : -T_1(r) \cdot u_1 + u_0 \cdot r_1^2))$$

$$F : \gamma \longrightarrow \mathfrak{M}, \quad F(r,s) = (p(r), l(s))$$

Let's consider the isomorphism $\psi = G^{-1} \circ F^{-1} : \mathfrak{M} \longrightarrow E$.

If we take $\theta = (p_0, l_0) = (p((1:0)), l_0)$ as the neutral element on $\mathfrak{M}$ (where l_0 is the tangent line to D through p_0),

$$\psi(\theta) = G^{-1}(F^1(p((1:0)), l_0)) = G^{-1}((1:0), l^{-1}(l_0)) = ((1:0), (u_0 : u_1))$$

for some $(u_0 : u_1)$ such that $((1:0), (u_0 : u_1)) \in E$. That is, $\psi(\theta) = ((1:0), (1:0))$.

Thus choosing $((1:0), (1:0))$ as the neutral element on E, by corollary 2.18 ψ is also a group isomorphism.

Note that $\eta(\theta) = (\widetilde{p}, \widetilde{l})$, where $\widetilde{p}$ satisfies $C \cap l_0 = \{p_0, \widetilde{p}\}$ and $\widetilde{l}$ is the tangent line to D through $\widetilde{p}$. But $\widetilde{p} = p((0:1))$ (since l_0 is the tangent line to $D = C_{(0:1)}$ through p_0), so

$$\psi(\eta(\theta)) = G^{-1}(F^1(p((0:1)), \widetilde{l})) = G^{-1}((0:1), l^{-1}(\widetilde{l})) = ((0:1), (u_0 : u_1))$$

for some $(u_0 : u_1)$ such that $((0:1), (u_0 : u_1)) \in E$. From the equation for E, it follows that $(u_0 : u_1) = (\pm\sqrt{\det D} : 1)$.

Using that ψ is a group isomorphism and Lemma 4.3, we deduce that

$$\eta^n = Id_{\mathfrak{M}} \iff \eta(\theta) \text{ is a torsion point of } \mathfrak{M} \text{ of order } n \iff$$

$$\iff \psi(\eta(\theta)) = ((0:1), (\pm\sqrt{\det D} : 1)) \text{ is a } n\text{-torsion point of } E$$

Now, let's study the restriction of the elliptic curve E to the affine chart $A_1 = \left\{((x:1), (y:1)) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} : x, y \in \mathbb{C}\right\}$ of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Namely, consider the plane affine curve

$$E' = \left\{(x,y) \in \mathbb{C}^2 : ((x:1), (y:1)) \in E\right\} = \left\{(x,y) \in \mathbb{C}^2 : y^2 = \det(xC + D)\right\}$$

and its projective closure

$$E'' = \left\{(x:y:z) \in \mathbb{P}^2_{\mathbb{C}} : y^2 z = \det(xC + Dz)\right\}$$

Since the pair of points at infinity $((1:0), (1:0))$ is the neutral element on E, the neutral element on E'' must be on the line at infinity $z = 0$: it's the point $(0:1:0)$. Then,

$$((0:1), (\pm\sqrt{\det D} : 1)) \text{ is a } n\text{-torsion point of } E \iff (0, \pm\sqrt{\det D}) \text{ is a } n\text{-torsion point of } E' \iff$$

$$\iff (0 : \pm\sqrt{\det D} : 1) \text{ is a } n\text{-torsion point of } E''$$

and Cayley's theorem becomes a consequence of theorem 2.19. $\square$

4.3 Some examples

Let C and D two non-degenerate conics of $\mathbb{P}^2_{\mathbb{C}}$. As we have seen, simply by computing a Taylor series and a determinant, Cayley's theorem allows us to know whether there exists a n-sided polygon inscribed in C and circumscribed about D (and hence, whether there exist infinitely many).

In this section, we see two examples of this explicit criterion, checking graphically the results obtained.

Example. Consider the non-degenerate conics $C: -y^2+2xz=0$ and $D: 2xy-z^2=0$ of $\mathbb{P}^2_{\mathbb{C}}$, given by

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have $\det(tC+D) = t^3+1$ and the Taylor expansion $\sqrt{t^3+1} = 1+\frac{1}{2}t^3+\dots$.

Since C and D meet at the four different points $(\frac{1}{2}:1:1)$, $(1:0:0)$, $(\frac{-1+\sqrt{3}i}{4}:\frac{-1-\sqrt{3}i}{2}:1)$ and $(\frac{-1-\sqrt{3}i}{4}:\frac{-1+\sqrt{3}i}{2}:1)$, it follows from Cayley's theorem (taking $n=3$ and $m=1$) that there exist infinitely many triangles inscribed in C and circumscribed about D.

Let's visualize this fact at the real affine plane $\mathbb{R}^2 \cong \left\{(x:y:1) \in \mathbb{P}^2_{\mathbb{C}} : x,y \in \mathbb{R}\right\}$ with Geogebra. The affine equations for C and D are $C: -y^2+2x=0$ (a parabola) and $D: 2xy-1=0$ (an hiperbola).

Note that we will only see the intersection point $(\frac{1}{2},1)$, since $(1:0:0)$ lies on the line at infinity and the points $(\frac{-1+\sqrt{3}i}{4}:\frac{-1-\sqrt{3}i}{2}:1)$, $(\frac{-1-\sqrt{3}i}{4}:\frac{-1+\sqrt{3}i}{2}:1)$ have complex coordinates.

For any starting point on C [1], the Poncelet construction closes at the third step, and gives us a triangle simultaneously inscribed in C and circumscribed about D. The independence of the choice of starting point is easily checked with the tool «Attach / Detach Point».

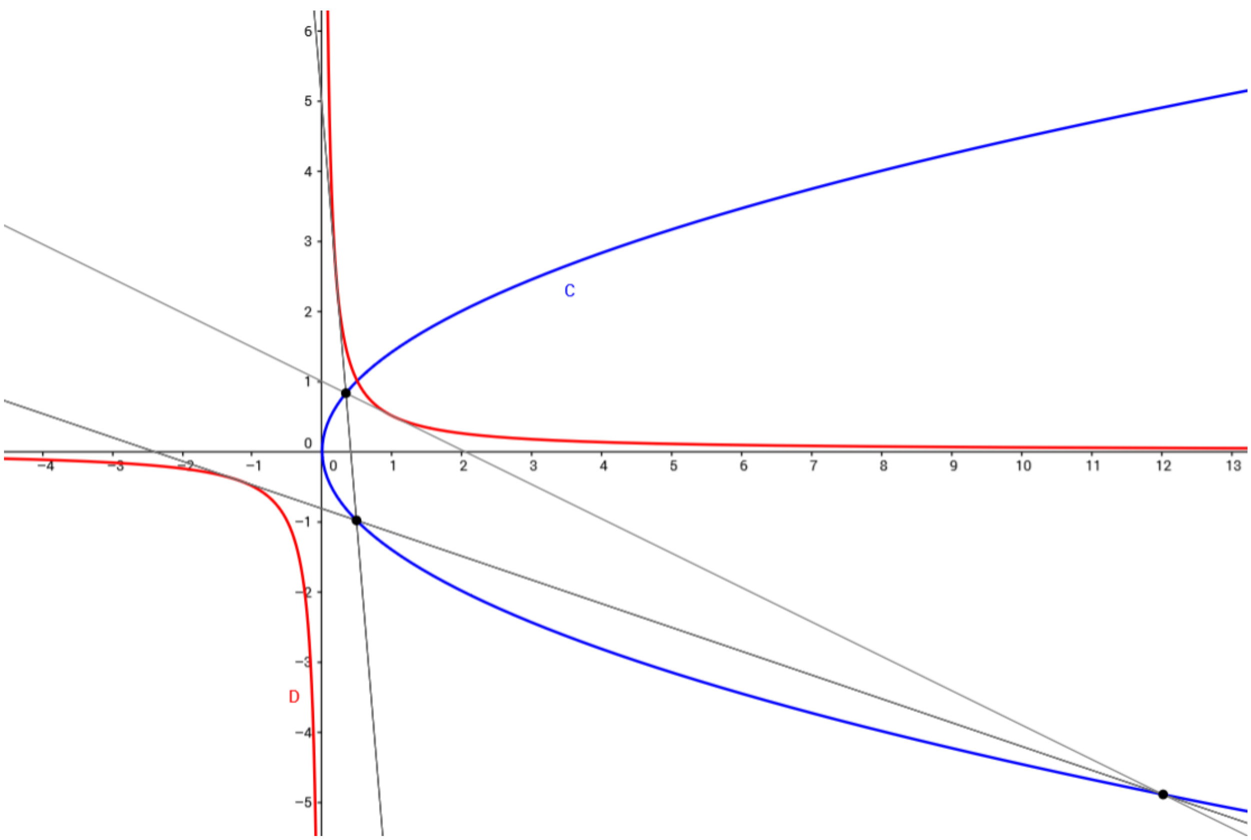

Figure 4.2

[1] The point must be good enough to ensure the existence of tangent lines to D: recall that now we are working in $\mathbb{R}^2$

Example. Let $C : -y^2+2xy+2xz=0$ and $D : -x^2+2xy+2yz=0$ be the non-degenerate conics with matrices

$$C=\begin{pmatrix}0 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}-1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

Note that C and D meet at the points $(0:0:1)$, $(-2:-2:1)$, $(-1-\frac{\sqrt{3}}{3}i:-1+\frac{\sqrt{3}}{3}i:1)$ and $(-1+\frac{\sqrt{3}}{3}i:-1-\frac{\sqrt{3}}{3}i:1)$.

It's easy to see that $\det(tC+D)=t^3+2t^2+2t+1$ and $\sqrt{t^3+2t^2+2t+1}=1+t-t^2-\frac{1}{2}t^4+\ldots$. Hence, by Cayley's theorem (with $n=4$ and $m=2$), any point on C is a vertex of a quadrilateral inscribed in C and circumscribed about D.

Proceeding as in the example above, we get a visualization at the real affine plane:

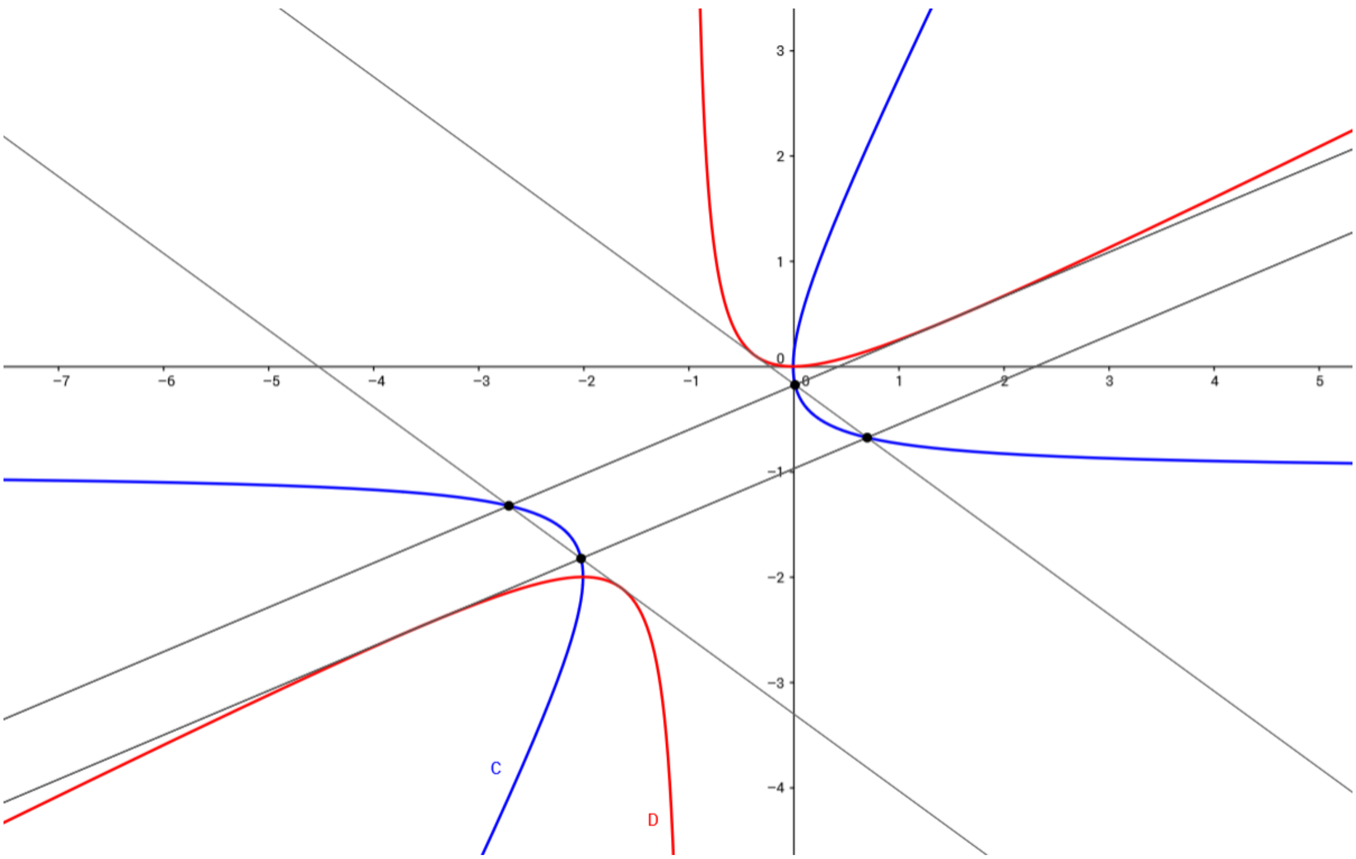

Figure 4.3

Chapter 5

The Poncelet problem in $\mathbb{P}^3_{\mathbb{C}}$

After having studied the Poncelet problem in the plane, now we want to generalize the results obtained to higher dimensions.

For example, we ask whether there exist polyhedra in $\mathbb{P}^3_{\mathbb{C}}$ inscribed in one quadric and circumscribed about another. Nevertheless, the method of construction can't be exactly the same as before: through a point in $\mathbb{P}^3_{\mathbb{C}}$ there are infinitely many tangent planes to a quadric.

We will need to construct polyhedra both inscribed in and circumscribed about a pair of quadrics. The theorem concerning their existence will be remarkably similar to Poncelet's porism.

5.1 Intersection of quadrics in $\mathbb{P}^3_{\mathbb{C}}$

During all the chapter, Q_1 and Q_2 will be two non-degenerate quadrics in $\mathbb{P}^3_{\mathbb{C}}$, with respective matrices M_1 and M_2.

In order to construct a polyhedra both inscribed in and circumscribed about Q_1 and Q_2, we must know how is the intersection of the given quadrics. We will assume that it is a *transverse intersection*.

Definition. We say that Q_1 and Q_2 are *meeting transversely* if, and only if, for each point $p \in Q_1 \cap Q_2$ the tangent planes T_pQ_1 and T_pQ_2 are different.

Remarks.

1. We can see the transverse intersection as an analogy of the intersection of two conics at four different points.

   In fact, by Bézout theorem, two conics meet at four different points if, and only if, both tangent lines to the conics at the intersection points are different.

2. If P_{Q_i} denotes the polarity induced by the quadric Q_i, the transverse intersection of Q_1 and Q_2 is equivalent to the projectivity

$$P_{Q_2}^{-1} \circ P_{Q_1} : \mathbb{P}^3_{\mathbb{C}} \xrightarrow{P_{Q_1}} \mathbb{P}^{3\vee}_{\mathbb{C}} \xrightarrow{P_{Q_2}^{-1}} \mathbb{P}^3_{\mathbb{C}}$$

not having fixed points, when restricted to $Q_1 \cap Q_2$.

Proposition 5.1. *Two quadrics $Q_1, Q_2 \subset \mathbb{P}^3_{\mathbb{C}}$ are meeting transversely if, and only, their envelopes Q_1^*, Q_2^* are meeting transversely (as non-degenerate quadrics in $\mathbb{P}^{3\vee}_{\mathbb{C}}$).*

In the case of intersecting three quadrics, we have the following result:

Lemma 5.2. *Let $Q_1, Q_2, Q_3 \subset \mathbb{P}^3_{\mathbb{C}}$ be three non-degenerate quadrics with pairwise transverse intersection. Then, $Q_1 \cap Q_2 \cap Q_3$ consists of a set with eight points.*

Remark. The idea behind lemma 5.2 is that, via the identification $Q_1 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ given in section 1.3, we can see

$$Q_1 \cap Q_2 \cap Q_3 = (Q_1 \cap Q_2) \cap (Q_1 \cap Q_3)$$

as the intersection of two non-singular curves in $Q_1 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, both of them with bidegree $(2,2)$.

According to a sort of Bézout theorem for curves in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $Q_1 \cap Q_2 \cap Q_3$ consists of $2 \cdot 2 + 2 \cdot 2 = 8$ points.

5.2 Construction of polyhedra

Hereinafter, we will assume that $Q_1, Q_2 \subset \mathbb{P}^3_{\mathbb{C}}$ are two non-degenerate quadrics meeting transversely.

We will denote by A, B the two families of lines lying on Q_1, and similarly C, D for Q_2, and we will write

$$E = Q_1^* \cap Q_2^*$$

the set of *bitangent planes*.

Consider an arbitrary bitangent plane $T \in E$. Then:

- Since $T \in Q_1^*$, it follows from theorem 1.5 that T meets Q_1 in two lines: an A-line L_A and a B-line L_B.
- Similarly, the intersection $T \cap Q_2$ is the union of a C-line L_C and a D-line L_D.

Note that the four lines must be distinct because Q_1 and Q_2 are meeting transversely.

Then, $P_1 = L_A \cap L_B$ and $P_2 = L_C \cap L_D$ are the contact points of T with Q_1 and Q_2, respectively.

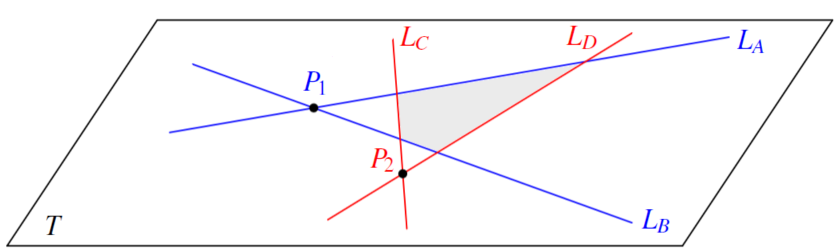

Figure 5.1

Now, since $L_A \not\subset Q_2$, according to lemma 1.6 there are exactly two tangent planes to Q_2 containing the line L_A. One of them is T. Let's write $\widetilde{T}$ the other one.

Note that $\widetilde{T}$ contains the line $L_A \subset Q_1$ so, again by lemma 1.6, $\widetilde{T}$ is tangent to Q_1 at some point of L_A.

Hence, $\widetilde{T}$ is a bitangent plane ($\widetilde{T} \in E$) and we can write

$$\begin{cases} T \cap Q_1 = L_A \cup L_B, & T \cap Q_2 = L_C \cup L_D \\ \widetilde{T} \cap Q_1 = L_A \cup \widetilde{L_B}, & \widetilde{T} \cap Q_2 = \widetilde{L_C} \cup \widetilde{L_D} \end{cases}$$

where $\widetilde{L_B}$ is a B-line, $\widetilde{L_C}$ is a C-line and $\widetilde{L_D}$ is a D-line.

Remark. We have $L_D \cap L_A = \widetilde{L_C} \cap L_A$ and $L_C \cap L_A = \widetilde{L_D} \cap L_A$.

In fact, $L_D \cap \widetilde{L_C} \neq \emptyset$, since it's the intersection of a C-line with a D-line. Then,

$$\emptyset \neq L_D \cap \widetilde{L_C} \subset T \cap \widetilde{T} = L_A$$

So the three distinct lines L_A, L_D and $\widetilde{L_C}$ are incident and it must be $L_D \cap L_A = L_D \cap \widetilde{L_C} = \widetilde{L_C} \cap L_A$. A similar reasoning holds for $L_C \cap L_A = \widetilde{L_D} \cap L_A$.

The following figure illustrates the situation:

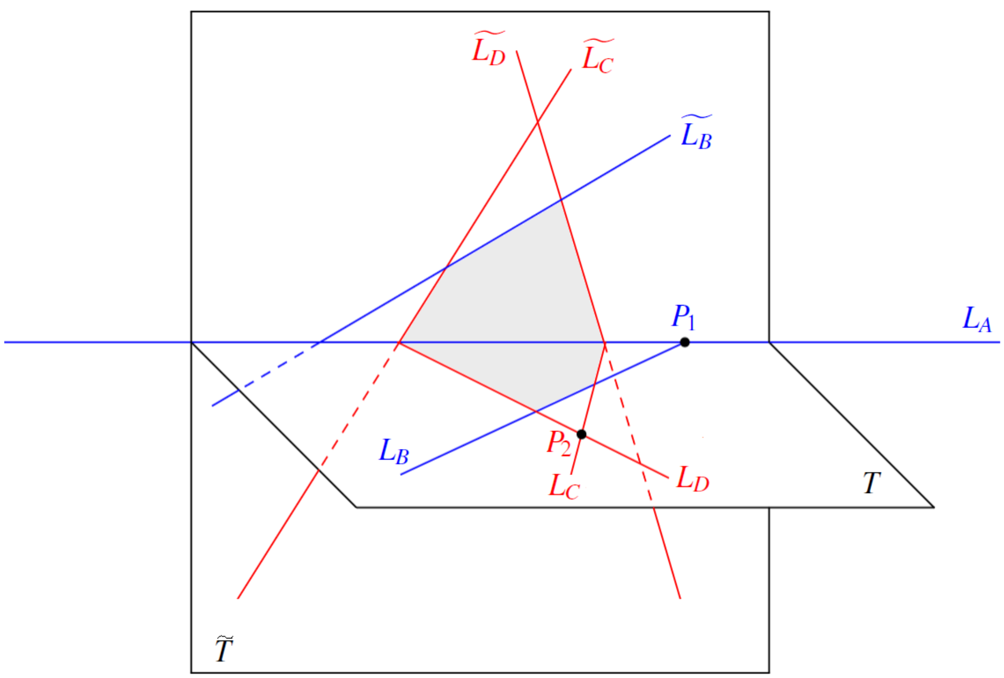

Figure 5.2

Let's denote by i_A this construction process of $\widetilde{T}$ from T: namely, $\widetilde{T} = i_A(T)$.

In a similar way, we can define maps i_B, i_C, i_D on E, by taking L_B, L_C or L_D on the plane T instead of the line L_A.

Remark. The maps i_A, i_B, i_C and i_D are involutions on E. For example, i_A interchanges the two tangent planes to Q_2 containing the line L_A.

Beginning with a fixed bitangent plane $T_0 \in E$ and succesively applying these involutions, we have a polyhedron $\Pi(T_0)$. For example, in figure 5.2, the shaded quadrilaterals are faces of the configuration $\Pi(T)$.

A polyhedron $\Pi(T_0)$ generated from a bitangent plane $T_0 \in E$ is both inscribed in and circumscribed about Q_1 and Q_2, in the following sense:

- Its planes are elements of E, that is, are tangent to both Q_1 and Q_2.
- Its vertices are points lying on $Q_1 \cap Q_2$.
- Its edges are lines alternately contained in Q_1 and Q_2.

The question we deal with is whether or not this configuration is finite, that is, the process of applying succesively the involutions comes back to the initial bitangent plane.

The answer, published by Griffiths and Harris in [8], reminds Poncelet's porism.

5.3 A Poncelet theorem in space

Lemma 5.3. *The transverse intersection $Q_1 \cap Q_2$ is a Riemann surface with genus 1.*

Proof. We can see $Q_1 \cap Q_2$ as a non-singular curve of bidegree $(2,2)$ in $Q_1 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ (see remark on page 50). Hence, when endowed with the complex structure described in theorem 2.7, $Q_1 \cap Q_2$ is a Riemann surface.

Now, let's define

$$\pi_A : Q_1 \cap Q_2 \longrightarrow \{A\text{-lines of } Q_1\}$$

mapping each point $p \in Q_1 \cap Q_2$ to the A-line through p.

We can see $\{A\text{-lines of } Q_1\}$ as a Riemann surface, by taking a bijection from $\mathbb{P}^1_{\mathbb{C}}$ to this set and defining a complex structure as in lemma 2.8. With this complex structure, π_A is an holomorphic map.

Consider an arbirary A-line L of Q_1. Note that it can't be $L \subset Q_2$, since Q_1 and Q_2 are meeting transversely. Then, generally L meets Q_2 in two points, so that the A-line L has two preimages.

But this happens except when L is a tangent line to Q_2: in such a case, L has an unique preimage.

In other words: π_A is an holomorphic map of degree 2, with branch points the A-lines of Q_1 which are tangent to Q_2. By Hurwitz formula, we deduce that

$$2 \cdot g(Q_1 \cap Q_2) = -2 + \sum_{p \in Q_1 \cap Q_2} (e_p(\pi_A) - 1)$$

So, in order to prove that $g(Q_1 \cap Q_2) = 1$, it's enough to prove that there are four branch points.

On the other hand, consider the holomorphic map

$$\pi_B : Q_1 \cap Q_2 \longrightarrow \{B\text{-lines of } Q_1\}$$

whose branch points are the B-lines of Q_1 which are tangent to Q_2.

By Hurwitz formula, π_B has the same number of branch points as π_A. Therefore, if we prove that there are exactly eight lines lying on Q_1 and tangent to Q_2, we will have four of them in each family (A or B), finishing our prove.

Let's suppose that $L \subset Q_1$ is a tangent line to Q_2 (at a point p). Then, T_pQ_2 is a plane containing the line $L \subset Q_1$ and, according to lemma 1.6, T_pQ_2 is tangent to Q_1 somewhere along L (namely, $T_pQ_2 \in Q_1^*$).

Conversely, if we have a point $p \in Q_1 \cap Q_2$ with $T_pQ_2 \in Q_1^*$ (that is, $T_pQ_2 = T_{p'}Q_1$ for some $p' \in Q_1$), then:

$$p \vee p' \subset Q_1 \cap T_{p'}Q_1 = Q_1 \cap T_pQ_2 \Longrightarrow p \vee p' \subset Q_1 \text{ and } p \vee p' \text{ is tangent to } Q_2$$

Thus: finding the lines contained in Q_1 and tangent to Q_2 is equivalent to determining the points $p \in Q_1 \cap Q_2$ such that $T_pQ_2 \in Q_1^*$.

Suppose that $p \in Q_1 \cap Q_2$ is a point with coordinates $(x_0 : x_1 : x_2 : x_3)$ in $\mathbb{P}^3_{\mathbb{C}}$. Then,

$$T_pQ_2 \in Q_1^* \Longleftrightarrow M_2 \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = M_1 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \text{ for some } (y_0 : y_1 : y_2 : y_3) \in Q_1 \Longleftrightarrow$$

$$\Longleftrightarrow M_1^{-1}M_2 \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \text{ for some } \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix} M_1 \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} M_2^t (M_1^{-1})^t M_1 M_1^{-1} M_2 \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} M_2 M_1^{-1} M_2 \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Namely, the points $p \in Q_1 \cap Q_2$ such that $T_pQ_2 \in Q_1^*$ are exactly the points $p \in Q_1 \cap Q_2 \cap Q_3$, where $Q_3 \subset \mathbb{P}^3_{\mathbb{C}}$ is the non-degenerate quadric with matrix $M_2 M_1^{-1} M_2$.

Since the quadrics Q_1, Q_2, Q_3 have pairwise transverse intersection (it can be checked from their matrices), by lemma 5.2 the intersection $Q_1 \cap Q_2 \cap Q_3$ consists of a set with eight points.

Therefore, there are exactly eight lines contained in Q_1 and tangent to Q_2, which finishes the proof. □

Corollary 5.4. *If two quadrics $Q_1, Q_2 \subset \mathbb{P}^3_{\mathbb{C}}$ have transverse intersection, the set $E = Q_1^* \cap Q_2^*$ of bitangent planes is an elliptic curve.*

Proof. According to proposition 5.1, we know that the quadrics Q_1^* and Q_2^* are meeting transversely in $\mathbb{P}^{3\vee}_{\mathbb{C}}$.

Since Q_1^* and Q_2^* are quadrics of a three-dimensional projective space meeting transversely, it follows from lemma 5.3 that $E = Q_1^* \cap Q_2^*$ is a Riemann surface with genus 1. $\square$

Theorem 5.5 (Griffiths, Harris). *Let $Q_1, Q_2 \subset \mathbb{P}^3_{\mathbb{C}}$ be two non-degenerate quadrics meeting transversely, and $E = Q_1^* \cap Q_2^*$ their set of bitangent planes.*

Then, the configuration $\Pi(T_0)$ is finite for some $T_0 \in E$ if, and only if, $\Pi(T)$ is finite for every bitangent plane $T \in E$.

Proof. By corollary 5.4, E is an elliptic curve. Hence, there exists an isomorphism $\varphi : \mathbb{C}/\Lambda \longrightarrow E$ defining a group structure on E:

$$\varphi(x) + \varphi(y) = \varphi(x+y) \text{ for all } x, y \in \mathbb{C}/\Lambda$$

Let's denote by $\theta = \varphi([0])$ the neutral element for this group law on E.

The map $\widetilde{i_A} = \varphi^{-1} \circ i_A \circ \varphi : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$ is an involutional automorphism of $\mathbb{C}/\Lambda$ with fixed points (a property inherited from i_A). So, according to theorem 3.9, there exists an element $a_1 \in \mathbb{C}/\Lambda$ such that

$$\widetilde{i_A}(q) = -q + a_1, \text{ for all } q \in \mathbb{C}/\Lambda.$$

Taking $\tau_1 = \varphi(a_1)$, we have

$$i_A(z) = (\varphi \circ \widetilde{i_A} \circ \varphi^{-1})(z) = \varphi(-\varphi^{-1}(z) + a_1) = \varphi(-\varphi^{-1}(z)) + \varphi(a_1) = -z + \tau_1$$

for all $z \in E$. Likewise, we can suppose that i_B, i_C and i_D are given by

$$i_B(z) = -z + \tau_2\,, \qquad i_C(z) = -z + \tau_3\,, \qquad i_D(z) = -z + \tau_4$$

With these notations, the condition of having a finite polyhedron $\Pi(T_0)$ for some $T_0 \in E$ becomes

$$(i_D \circ i_C \circ i_B \circ i_A)^n(T_0) = T_0, \text{ for some } n \geq 1 \Longleftrightarrow T_0 + n(\tau_4 - \tau_3 + \tau_2 - \tau_1) = T_0, \text{ for some } n \geq 1 \Longleftrightarrow$$

$$\Longleftrightarrow n(\tau_4 - \tau_3 + \tau_2 - \tau_1) = \theta, \text{ for some } n \geq 1 \Longleftrightarrow \tau_4 - \tau_3 + \tau_2 - \tau_1 \text{ is a torsion point on } E$$

And this condition does not rely on the choice of the initial bitangent plane T_0. $\square$

Appendix. Mathematical billiards

The billiard problem

The billiard problem was formulated by George D. Birkhoff (1884-1944) in his studies of certain dynamical systems concerning the three-body problem.

Let C be a simple closed convex curve in the euclidean plane $\mathbb{R}^2$ (for example, a polygon or an ellipse). We can imagine the domain bounded by C as a pool table.

Let's suppose that a point particle moves in the interior of this pool table. The motion is along a straight line, with constant velocity and the particle reflects elastically at the boundary. That is, when the particle reaches the boundary, the angle of incidence is equal to the angle of reflection.

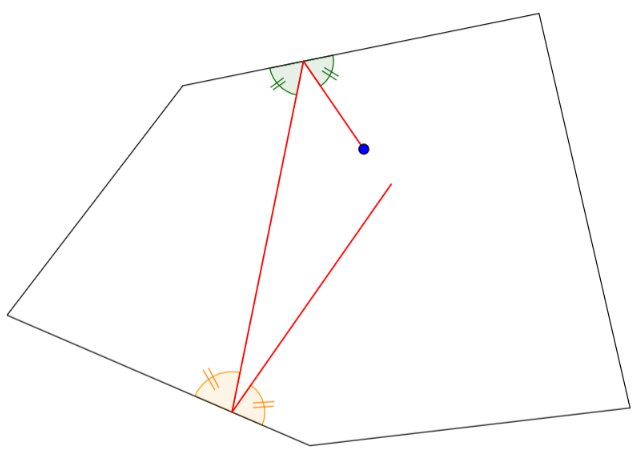

Figure A.1

The billiard problem consists on describing all the possible trajectories of the particle, and its general answer is not known.

However, if the curve C is an ellipse, the billiard problem is closely related to Poncelet's porism. We are going to see that the trajectories of the particle correspond to the construction of polygons inscribed in C and circumscribed about a conic D (an ellipse or an hyperbola), that is confocal with C.

Elliptic billiards

Hereinafter, $C \subset \mathbb{R}^2$ will denote an ellipse with foci F_1 and F_2. Namely, C is the locus of points in $\mathbb{R}^2$ the sum of whose distances from F_1 and F_2 is a fixed constant.

Lemma A.1. *Let $p_0 \in C$, and $l = T_{p_0}C$ the tangent line to C through the point p_0.*

1. *If F_2' is the symmetric point of the focus F_2 with respect to l, then the points F_1, p_0 and F_2' lie on a line.*
2. *The lines $l_1 = F_1 \vee p_0$ and $l_2 = F_2 \vee p_0$ have the same incident angle with respect to l.*

Proof. Given two points $a, b \in \mathbb{R}^2$, we write ab for the segment defined by a and b, and $|ab|$ for its length.

If $\lambda = |F_1 p_0| + |F_2 p_0|$, by definition of ellipse we have $|F_1 q| + |F_2 q| = \lambda$ for each point $q \in C$.

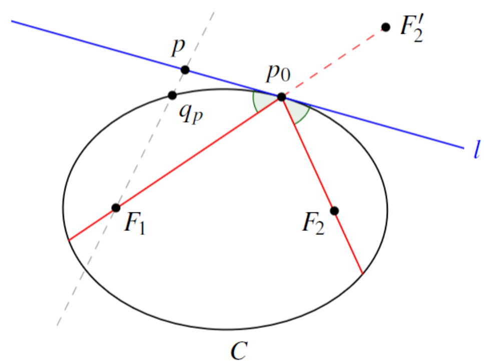

Figure A.2

We claim that, for each $p \in l \setminus \{p_0\}$, $|F_1 p| + |pF_2'| > \lambda$. Indeed, if q_p is the intersection point of C with the line spanned by F_1 and p (see figure A.2),

$$\lambda = |F_1 q_p| + |F_2 q_p| < |F_1 q_p| + |q_p p| + |pF_2| = |F_1 p| + |pF_2| = |F_1 p| + |pF_2'|$$

Since $p = p_0$ minimizes the function $|F_1 p| + |pF_2'|$ in l, it follows the result. $\square$

Definition. A *billiard trajectory* for C is a sequence $\{(p_n, l_n)\}_{n \geq 0}$ (with $p_n \in C$ and l_n a line on $\mathbb{R}^2$) such that, for each $n \geq 0$, $p_n, p_{n+1} \in l_n$ and l_{n-1}, l_n make equal angles with the tangent line $T_{p_n}C$.

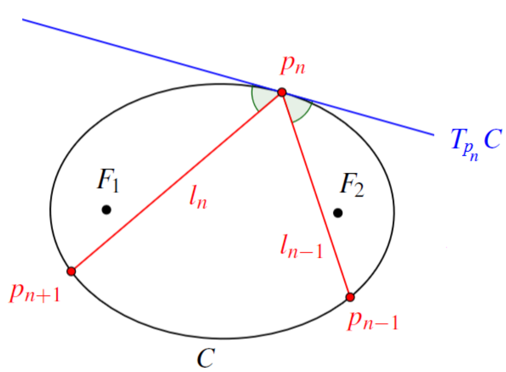

Figure A.3

Remark. By lemma A.1, if a line of a billiard trajectory for C contains one of the foci, then all lines of the trajectory contain one or the other focus, alternately.

Definition. Let D be either another ellipse or an hyperbola in $\mathbb{R}^2$. A *Poncelet trajectory* for the pair (C,D) is a sequence $\{(p_n,l_n)\}_{n\geq 0}$ such that, for each $n \geq 0$, $p_n, p_{n+1} \in C \cap l_n$ and l_n is a tangent line to D.

Theorem A.2.

1. *Let D a confocal ellipse or hyperbola with C. Then, the Poncelet trajectories for (C,D) are billiard trajectories for C.*
2. *Conversely, any billiard trajectory for C (not passing through the foci, and not along the minor axis) is a Poncelet trajectory for (C,D), for some conic D confocal with C.*

Proof. Let's prove 1, assuming that C and D are two confocal ellipses. By lemma A.1, we have a picture

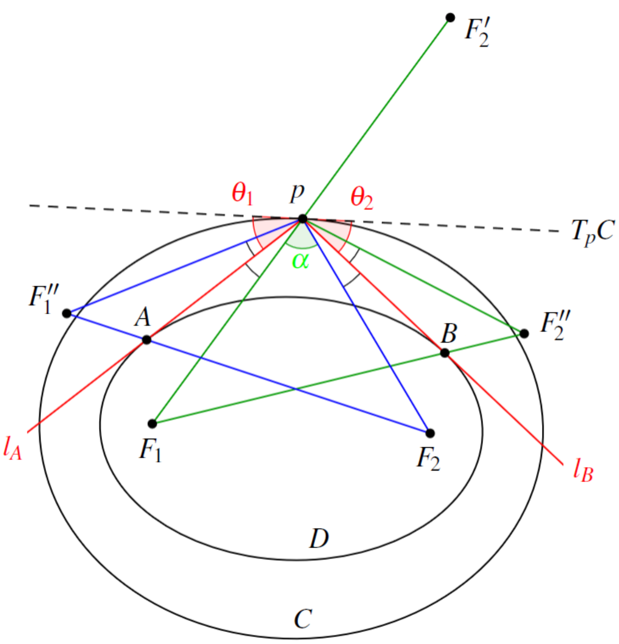

Figure A.4

where the red lines l_A and l_B, and the point p, are part of a Poncelet trajectory for (C,D). In order to prove that it's a billiard trajectory, we must show the equality of red angles $\theta_1 = \theta_2$. Consider:

- A and B the contact points of l_A and l_B with D, respectively.
- F_2' the reflexion of F_2 with respect to T_pC.
- F_1'' and F_2'' the reflexions of F_1 and F_2 with respect to l_A and l_B, respectively.
- η_1 the angle between l_A and the line $p \vee F_1''$, as well as the angle between l_A and the line $p \vee F_1$.
- η_2 the angle between l_B and the line $p \vee F_2''$, as well as the angle between l_B and the line $p \vee F_2$.

By definition of the ellipse D,

$$|F_1A| + |AF_2| = |F_1B| + |BF_2| \Longrightarrow |F_1''A| + |AF_2| = |F_1B| + |BF_2''| \Longrightarrow |F_1''F_2| = |F_1F_2''|$$

It follows that the triangles $F_1''pF_2$ and F_1pF_2'' are rotations of each other (through p), so that

$$\alpha + 2\eta_1 = \alpha + 2\eta_2 \Longrightarrow \eta_1 = \eta_2$$

Moreover, considering C and the segments pF_1 and pF_2, it must be $\theta_1 + \eta_1 = \theta_2 + \eta_2$ and thus $\theta_1 = \theta_2$.

The second statement can be proved with similar arguments. See the book [3] for further details. $\square$

Bibliography

[1] CASAS-ALVERO, EDUARDO. *Analytic Projective Geometry*. European Mathematical Society, 2014

[2] DRAGOVIC, VLADIMIR; RADNOVIC, MILENA. *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*. Frontiers in Mathematics, Springer, 2011

[3] FLATTO, LEOPOLD. *Poncelet's Theorem.* American Mathematical Society, 2009

[4] FREITAG, EBERHARD. *Complex Analysis 2: Riemann Surfaces, Several Complex Variables, Abelian Functions, Higher Modular Functions*. Springer, 2011

[5] FULTON, WILLIAM. *Algebraic Curves: An introduction to Algebraic Geometry*. Available online

[6] GRAY, JEREMY. *Worlds Out of Nothing: A Course in the History of Geometry in the* 19^{th} *Century*. Springer, 2007

[7] GRIFFITHS, PHILLIP. *Introduction to Algebraic Curves*. American Mathematical Society, 1989

[8] GRIFFITHS, PHILLIP; HARRIS, JOE. *A Poncelet theorem in space*. Commentarii Mathematici Helvetici, Vol. 52 (1977), 145-160

[9] GRIFFITHS, PHILLIP; HARRIS, JOE. *On Cayley's explicit solution to Poncelet's porism.* L'Enseigment Mathématique 24 (1978), 31-40

[10] KERR, MATT. *Lecture Notes: Algebraic Geometry III/IV*. Available online

[11] KIRWAN, FRANCES. *Complex Algebraic Curves*. Cambridge University Press, 1992

[12] MIRANDA, RICK. *Algebraic Curves and Riemann Surfaces*. American Mathematical Society, 1995