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GRAU DE MATEMÀTIQUES

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Lefschetz properties in algebra and geometry

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, January 16, 2017

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Introduction

The weak and strong Lefschetz properties on graded artinian algebras have been an object of study along the last few decades. Precisely, let be A a graded artinian algebra. We say that A has the Strong Lefschetz property (SLP) if the multiplication by a dth power of a general linear form have maximal rank (i.e. $\times L^d : A_i \to A_{i+d}$ is injective or surjective for every i). We say that A has the Weak Lefschetz property (WLP) if occurs the same with $d = 1$. These properties have connections among different areas such as algebraic geometry, commutative algebra and combinatorics. Sometimes quite surprising, these connections give new approaches and relate problems, a priori, very distant.

It is worth to mention that the study of the Lefschetz properties started with a work due to R. Stanley in 1980 [13] and then continued by J. Watanabe in 1987 [16], which connected them to the Sperner theory in combinatorics. Later there have been discovered more connections between the Lefschetz properties and vector bundles, line arrangements on the plane and the Fröberg conjecture (see [10], [11]). The study of these subjects require several tools out of reach for an undergraduate student.

The aim of this work is to study and contribute to another connection given by E. Mezzetti, R. M. Miró-Roig and G. Ottaviani in [9] which relates the failure of the weak Lefschetz property of artinian ideals to the existence of projective varieties satisfying at least one the Laplace equation. We first start introducing the notation that we will use and the two concepts that we need to state this connection via Macaulay-Matlis duality. Then, in the next chapter we define properly the Lefschetz properties and give several examples and results describing them. At the final of this second chapter, we will focus our attention on artinian ideals $I \subset k[x_0, \ldots, x_n] = R$ generated by forms of the same degree d, and the artinian graded algebra $A = R/I$. While A is *expected* to have the WLP, there are few cases in which A does not hold the WLP in a certain degree (i.e. there is $i \geq 1$ such that $\times L : A_i \to A_{i+1}$ is neither injective nor surjective for any linear form L).

Notice that, since for $i \leq d-1$ there is no difference between A_i and R_i, the first possible degree where A can fail the WLP is at degree $d-1$. Using Macaulay-Matlis duality, we can relate these artinian ideals I failing the WLP in degree $d-1$, with suitable projections of the Veronese variety $V(n, d)$ satisfying one Laplace equation of order $d-1$. The study and classification of rational varieties satisfying one Laplace equation of order s is a long-standing problem in mathematics. Hence, this result shows once more how the Lefschetz properties can be useful to study problems, a priori, far away one each other. We name the ideals generated by forms of degree d and failing the WLP in degree $d-1$ Togliatti systems (see Definition 2.1.6). The name is in honor to E. Togliatti who proved that in $k[x, y, z]$, the only smooth monomial Togliatti system of cubics is (x^3, y^3, z^3, xyz) or, equivalently, there is only one rational surface in $\mathbb{P}^5$ parametrized by cubics and satisfying a Laplace equation. It is the rational surface obtained projecting the third Veronese embedding $V(2, 3)$ in $\mathbb{P}^2$ from four points (see [14], [15]).

Once introduced what are Togliatti systems and their geometric significance, in chapter 3 we focus on the study and classification of minimal (smooth) monomial Togliatti systems (see 3.0.1. In this case, we can view the monomials generating I as integer points in the lattice $\mathbb{Z}^{n+1}$ and we can associate to them a toric variety following [4]. This approach allows us to apply combinatoric techniques and make the study much more easier. We also present a smoothness criterion due to Perkinson to see whether this toric variety is smooth by solely observing the relation with the points and the lattice.

Next we recall the classification of minimal smooth monomial Togliatti systems of quadrics and cubics, as given in [7]. This classification uses graph theory and other combinatoric tools in its proof, and cannot be easily generalized to classify all minimal smooth monomial Togliatti systems of degree $d \geq 4$. The classification of smooth minimal Togliatti systems of degree $d \geq 4$ seems out of reach. Therefore, in order to achieve new results on the problem of classifying minimal smooth monomial Togliatti systems of arbitrary degree $d \geq 4$, we have to change the strategy and find other invariants, as the number of generators of a minimal (smooth) monomial Togliatti system.

Continuing in the third chapter, we study minimal(smooth) monomial Togliatti systems I from this new perspective, following [8]. First of all give upper and lower bounds on the number of generators $\mu(I)$. Actually, we get that, if I is a minimal monomial Togliatti system in $k[x_0, \dots, x_n]$ of forms of degree $d \geq 4$, then $2n + 1 \leq \mu(I) \leq \binom{n+d-1}{n}$ where $n \geq 2$ and $d \geq 4$. The second step it to classify all smooth Togliatti systems which reach the lower bound or exceed it by one and we obtain that, except a few cases when $n = 2$, they have a very particular form. Finally, we study whether there exist minimal (smooth) monomial Togliatti systems in the range comprised between the lower and upper bound. We obtain in particular that there is no minimal smooth monomial Togliatti system in $n + 1$ variables with $2n + 3$ generators for $n \geq 3$ and $d \geq 4$, but what happens when $n = 2$?

The last chapter give new results on this topic and answer this last question. Actually, we have classified all minimal monomial Togliatti systems in $\subset k[x, y, z]$ with $\mu(I) = 7$. Finally, joining the results of third chapter and these new results, we give a complete classification of minimal smooth monomial Togliatti systems in $k[x_0, \dots, x_n]$ generated by $2n + 3$ monomials of degree $d \geq 4$. We want to point out that all results of this chapter are new and they will be published as part of the results in [12].

Acknowledgements

I would like to express my sincere gratitude to my advisor Rosa Maria Miró-Roig for her tireless weekly dedication and for introducing and guiding me to this new field of study. Also, I would thank to my whole family for their support. Especially, I would thank to L.

Chapter 1

Preliminaries

1.1 Notation

Let k be an algebraically closed field of characteristic 0. For $n \geq 2$, let R be the polynomial ring $k[x_0, \ldots, x_n]$ with its standard graduation $R = \bigoplus_{d \geq 0} R_d$. We note $n_d = \binom{n+d}{d} = \dim_k R_d$ and $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ the $n-$dimensional projective space. For every homogeneous ideal $I \subset R$, $V(I) \subset \mathbb{P}^n$ stands for the projective variety associated to I, i.e. $V(I) = \{a \in \mathbb{P}^n | \forall F \in I, \ F(a) = 0\}$. We say $I \subset R$ is an artinian ideal if $V(I) = \emptyset$. If $F_1, \ldots, F_r$ are homogeneous polynomials in R, $(F_1, \ldots, F_r) \subset R$ stands for the ideal generated by these forms, while $\langle F_1, \ldots, F_r \rangle$ stands for the correspondent $k-$vector space.

Finally, we define the Veronese map to be $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{n_d - 1}$ sending $(t_0 : \cdots : t_n)$ to $(t_0^{a_0} \cdots t_n^{a_n})_{a_0 + \cdots + a_n = d}$. The image $V(n,d) := v_{n,d}(\mathbb{P}^n) \subset \mathbb{P}^{n_d - 1}$ is a projective variety, which is called the Veronese variety.

1.2 Laplace equations

Definition 1.2.1: Let $X \subset \mathbb{P}^N$ be a rational variety of dimension n with parametrization

$$\Psi : \mathbb{P}^n \dashrightarrow X \ \text{ s.t. } \ (t_0 : \ldots : t_n) \mapsto (F_0(t_0, \ldots, t_n) : \ldots : F_N(t_0, \ldots, t_n))$$

We call s**th osculating vector space** on $x = \Psi(t_0 : \ldots : t_n)$ the vector space

$$T_x^{(s)} X := \left\langle \frac{\partial^k \Psi}{\partial_{t_0}^{k_0} \cdots \partial_{t_n}^{k_n}} (t_0 : \ldots : t_n) | k_0 + \cdots + k_n = s \right\rangle$$

Finally, we call s**th osculating projective space** on $x \in X$ the projectivization of the vector space above: $\mathbb{T}_x^{(s)} X := \mathbb{P}\left(T_x^{(s)} X\right)$

Remark 1.2.2: Since we have $n_s - 1$ vectors $(k_0, \ldots, k_n)$ which satisfy $k_0 + \cdots + k_n = s$ the dimension in a general point $x \in X$ of $T_x^{(s)} X$ is, at most, $n_s - 1$ (we call it **expected dimension**). However, if there are linear dependencies among partial derivatives of order s, this bound is not reached and we are given a linear equation involving partial derivatives of order s of Ψ.

Definition 1.2.3: Let $X \subset \mathbb{P}^N$ be a rational projective variety of dimension n. We say that X **satisfy δ Laplace equations of order** s if, and only if

(1) for all smooth point $x \in X$ we have $\dim T_x^{(s)} X < n_s$ and

(2) for general $x \in X$, $\dim T_x^{(s)} X = n_s - 1 - \delta$.

Remark 1.2.4: If $N < n_s - 1$, then $T_x^{(s)} X$ is spanned by more vectors than the ambiance space. So, X must satisfy at least one Laplace equation of order s.

1.3 Macaulay inverse system

Let $R = k[x_0, \dots, x_n]$ and $\mathcal{D} = k[y_0, \dots, x_n]$. We define an action of R over $\mathcal{D}$:

$$R_i \times \mathcal{D}_j \to \mathcal{D}_{j-i} \text{ such that } (F, G) \mapsto F \cdot G := F\left(\frac{\partial}{\partial y_0}, \dots, \frac{\partial}{\partial y_n}\right) G$$

which structures $\mathcal{D}$ as a graded $R-$module.

Definition 1.3.1: With notations above, let $I \subset R$ be a homogeneous ideal. The **Macaulay inverse system of** I is $I^{-1} := \{D \in \mathcal{D} | \forall F \in I,\ F \cdot D = 0\}$, which is a graded $R-$submodule of $\mathcal{D}$.

Remark 1.3.2: If I is a homogeneous ideal generated by monomials of degree d, then $\left[I^{-1}\right]_d$ is generated by all the monomials of degree d which are not generators of I.

Recall that if I is a homogeneous ideal of R, then R/I is a graded module where $(R/I)_d = R_d/(I \cap R_d)$.

Lemma 1.3.3: *Let $I \subset R$ be a homogeneous ideal. Then, for every $i \in \mathbb{Z}$,* $\dim_k(R/I)_i = \dim_k(I^{-1})_i$.

Finally we have the Macaulay-Matlis duality

Proposition 1.3.4: *There is a bijection*

$$\begin{array}{cccc} \varphi: & \{\textit{homogeneous ideals of } R\} & \leftrightarrow & \{\textit{graded } R-\textit{submodules of } S\} \\ & I & \mapsto & I^{-1} \\ & Ann_R(M) & \leftarrow\!\shortmid & M \end{array}$$

For more details on this subject see [9].

Chapter 2

Lefschetz properties

The main purpose of this chapter is to provide the definitions, examples and basic results on the weak Lefschetz property of artinian homogeneous ideals $I \subset R = k[x_0, \ldots, x_n]$ and see how they are codified in the Hilbert function of I.

Let us start considering the following statement, which was firstly proved by R. Stanley [13] in 1980 and which has motivated the so called Lefschetz properties:

Proposition 2.0.1: *Let $I = (x_0^{a_0}, \ldots, x_n^{a_n})$ be an artinian monomial complete intersection. Let $L \in R_1$ be a general linear form. Then, for any positive integers d and i, the homomorphism $\times L^d : [R/I]_i \to [R/I]_{i+d}$ (induced by multiplication by L^d) has maximal rank.*

Definition 2.0.2: Let $I \subset R$ be an artinian ideal and let us consider $A = R/I$ with the standard graduation $A = \bigoplus_{i=0}^{r} A_i$. Let $L \in R_1$ be a general linear form. Then:

(1) A has the Strong Lefschetz Property (SLP) if, for all positive integer d and for all $1 \leq i \leq r - d$, the homomorphism $\times L^d : [A]_i \to [A]_{i+d}$ has maximal rank.

(2) A has the Weak Lefschetz Property (WLP) if, for all $1 \leq i \leq r - 1$, the homomorphism $\times L : [A]_i \to [A]_{i+1}$ has maximal rank.

Notation 2.0.3: (1) By abuse of notation we say that the ideal I has the SLP (resp. WLP).
(2) In the case above, the linear form L is called a **Strong Lefschetz element** (resp. **Weak Lefschetz element** or simply **Lefschetz element**) of R/I.
(3) If for a general form $L \in [R/I]_1$ there is $d \geq 1$ and $1 \leq i \leq r - d$ such that the map $\times L^d$ has not maximal rank, we say that R/I fails the SLP (resp. the WLP if $d = 1$) in degrees (d, i) (resp. in degree i).

We will now establish the failure of SLP (resp. WLP) by only comparing two Hilbert functions.

Lemma 2.0.4: *Let $I \subset R$ be an artinian ideal and let $L \in R_1$ be a general linear form. Then, for each integer $d \geq 1$ and for all $1 \leq i \leq r - d$ we have the following exact sequence:*

$$[R/I]_i \to [R/I]_{i+d} \to \left[R/(I+(L^d))\right]_{i+d} \to 0.$$

Moreover, I fails the SLP in degrees (d,i) if, and only if one of these two cases holds:
(1) $\dim_k [R/I]\, i \le \dim_k [R/I]_{i+d}$ *and*

$$\dim_k \left[R/(I+(L^d))\right]_{i+d} > \dim_k [R/I]_{i+d} - \dim_k [R/I]_{i+d}.$$

In this case we can say that SLP fails in degrees (d,i) because of injectivity.
(2) $\dim_k [R/I]\, i \ge \dim_k [R/I]_{i+d}$ *and*

$$\dim_k \left[R/(I+(L^d))\right]_{i+d} > 0.$$

In this case we can say that SLP fails in degrees (d,i) because of surjectivity.

Remark 2.0.5: (1) Recall that if $I \subset R$ is an ideal, $HF(R/I,i) = \dim_k [R/I]_i$. Then, with the notations above, I fails the SLP in degrees (d,i) if, and only if,

$$HF(R/(I+(L^d)), i+d) > \max\{0, HF(R/I,i) - HF(R/I,i+d)\}.$$

(2) This Lemma gives us a tool to see whether an ideal $I \subset R$ fails the SLP (resp. the WLP) in degrees (d,i), but to use it, it is necessary to know which are the appropriate degrees (d,i) to look at. Examples below show some ideals failing the WLP. The first three have been tested with the computer implementing the Lemma 2.0.4 to Macaulay2 [3].

Remark 2.0.6: It is clear that having the SLP implies having the WLP, however, the converse is not true. For instance, using Lemma 2.0.4 one can see that $I = (x_0^2, x_1^3, x_2^5, x_0x_1, x_0x_2^2, x_1x_2^3, x_1^2x_2^2)$ has the WLP but fails the SLP in degrees $(2,1)$ and also that $I = (x_0^3, x_1^3, x_2^3, (x_0+x_1+x_2)^3)$ has the WLP but fails the SLP in degrees $(3,1)$.

Example 2.0.7: (1) $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2.
(2) $I = (x^4, y^4, z^4, t^4, xyzt)$ fails the WLP in degree 5.
(3) $I = (x^5, y^5, z^5, t^5, w^5, xyztw)$ fails the WLP in degree 8 and in degree 9.
(4) By [6] (Theorem 4.3), the ideals $I = (x_0^{n+1}, \dots, x_n^{n+1}, x_0 \dots x_n)$ fail the WLP in degree $\binom{n+1}{2} - 1$.

Remark 2.0.8: Notice that the first ideal fails the WLP in the first non trivial place while the others fail later. This particularity will be studied in the following in more detail.

2.1 The Weak Lefschetz Property

In this section, we will focus our attention on artinian ideals $I \subset R$ generated by r forms of a fixed degree d and failing the WLP in the first non trivial place. In this case, $[R/I]_i = R_i$ for all $0 \le i \le d-1$. Hence, the first non trivial place where I can fail the WLP is from degree $d-1$ to d.

If we restrict sufficiently the number of generators of I to get $\dim_k [R/I]_d > \dim_k R_{d-1}$ (for instance, $r \le \binom{n-1+d}{n-1}$) the WLP can only fail in degree $d-1$ because of injectivity.

Moreover, we have the next useful lemma:

Lemma 2.1.1: *Let $I=(F_1,\dots,F_r)\subset R$ be an artinian ideal generated by $r\le\binom{n-1+d}{n-1}$ forms of degree d. Let L be a linear form, set $\overline{R}=R/(L)$ and let $\overline{I}$ (resp. $\overline{F_i}$) be the image of I (resp. F_i) in $\overline{R}$. Then I fails the WLP in degree $d-1$ if, and only if $\overline{F_1},\dots,\overline{F_r}$ are $k-$linearly dependent.*

Proof: First notice that
(1) For $1\le i\le r$, $\deg(F_i)=d\Rightarrow [R/I]_{d-1}\cong R_d$.
(2) $\dim_k \overline{R}_d=\binom{n+d}{d}-\binom{n+d-1}{d-1}=\frac{(n+d)!-d(n+d-1)!}{n!d!}=\frac{n(n+d-1)!}{n!d!}=\binom{n+d-1}{n-1}$.
(3) $I=(F_1,\dots,F_r)$ and $\deg(F_i)=d\Rightarrow \dim_k [R/I]_d=\binom{n+d}{d}-r$.

And let us write $\varphi:[R/I]_{d-1}\to[R/I]_d$ the multiplication map in degree $d-1$. Now, using remarks above and since $r\le\binom{n+d-1}{d}$ by hypothesis, we have that

$$\binom{n+d}{d}=\binom{n+d-1}{d-1}+\binom{n+d-1}{d}\ge\binom{n+d-1}{d-1}+r\Rightarrow \dim_k [R/I]_d\ge\dim_k [R/I]_{d-1}.$$

Thus, we have proved that φ is not surjective in degree $d-1$ unless $r=\binom{n+d-1}{d}$ (in such case, φ is surjective if, and only if it is injective). Hence, φ does not have maximal rank if, and only if is not injective if, and only if $\dim_k(\operatorname{Ker}\varphi)>0$ if, and only if $\dim_k(\operatorname{Im}\varphi)=\dim_k [R/I]_{d-1}-\dim_k(\operatorname{Ker}\varphi)_{d-1}<\binom{n+d-1}{d-1}$.

On the other hand, $\operatorname{Coker}\varphi=[R/I]_d/\operatorname{Im}\varphi=[R/I]_d/L[R/I]_{d-1}\cong(R/(I,L))_d\Rightarrow \dim_k(R/(I,L))_d=\dim_k [R/I]_d-\dim_k(\operatorname{Im}\varphi)$ and replacing $\dim_k(\operatorname{Im}\varphi)=\binom{n+d}{d}-r-\dim_k(R/(I,L))_d$ we get φ is not injective $\Leftrightarrow \binom{n+d}{d}-r-\dim_k(R/(I,L))_d<\binom{n+d-1}{d-1}\Leftrightarrow \dim_k(R/(I,L))_d>\binom{n+d}{d}-\binom{n+d-1}{d-1}-r$.

Using that $\dim_k R/(I,L)=\dim_k\overline{R}-\dim_k\overline{I}$ because $R/(I,L)\cong(R/(L))/(I/(L))$ we get φ is not injective $\Leftrightarrow \dim_k\overline{R}-\dim_k\overline{I}>\binom{n+d}{d}-\binom{n+d-1}{d-1}-r\Leftrightarrow \dim_k\langle\overline{F_1},\dots,\overline{F_r}\rangle<r$.

Hence, φ is not injective if, and only if $\{\overline{F_1},\dots,\overline{F_r}\}$ are $k-$linearly dependent. □

Remark 2.1.2: Suppose that $I=(F_1,\dots,F_r)$ with $\deg(F_i)=d$ for $1\le i\le r\le\binom{n+d-1}{d}$ fails the WLP in degree $d-1$. This is equivalent that $\overline{F_1},\dots,\overline{F_r}$ are $k-$linearly dependent in $R/(L)$ for a general $L\in R_1$. Then, for any $r\le t\le\binom{n+d-1}{d}$ and also any $\{F_{r+1},\dots,F_t\}\subset R_d$ we have that the enlarged ideal $J=(F_1,\dots,F_r,F_{r+1},\dots,F_t)$ also fails the WLP.

This easy observation allows us to study only those ideals $I=(F_1,\dots,F_r)\subset R$ failing the WLP because of injectivity such that for any $1\le s\le r-1$, the ideals $I=(F_{i_1},\dots,F_{i_s})$ have the WLP in degree $d-1$.

Remark 2.1.3: As we have said above, Lemma 2.1.1 is useless if $r>\binom{n+d-1}{n-1}$. In that case, I fails the WLP in degree $d-1$ because of surjectivity. Although we will not study this case here, we will give a example: $I=(x_0^3,x_1^3,x_2^3,x_0^2x_1,x_0^2x_2)$ is generated by $5>4=\binom{2+2}{1}$ and we cannot apply Lemma 2.1.1 but by Lemma 2.0.4 we can see it fails the WLP in degree $d-1=2$ because of surjectivity.

The next step will be to relate the failure of the WLP on this type of ideals to suitable projections of the Veronese variety $V(n,d)$ satisfying at least one equation of Laplace of order $d-1$. Let us start with a definition:

Definition 2.1.4: Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r forms of degree d. Let $I^{-1} \subset \mathcal{D}$ be its inverse Macaulay system (as seen in Section 1.3). Then, we consider

$\varphi_{[I^{-1}]_d} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n_d - r - 1}$ is the rational map associated to $[I^{-1}]_d$

$\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{r-1}$ is the morphism (I is artinian) associated to I_d

We define

(1) $X_{n,[I^{-1}]_d} := \overline{\mathrm{Im}(\varphi_{[I^{-1}]_d})}$, which is the projection of $V(n,d)$ from $\langle F_1, \dots, F_r \rangle$

(2) $X_{n,I_d} := \mathrm{Im}\left(\varphi_{I_d}\right)$, which is the projection of $V(n,d)$ from $\langle [I^{-1}]_d \rangle$.

Next proposition, gives us the relation we were searching:

Proposition 2.1.5 ([9]; Theorem 3.2)**:** *Let $I \subset R$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree d. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

(1) the ideal I fails the WLP in degree $d-1$.

(2) the forms $F_1, \dots, F_r$ become $k-$linearly dependent on a general hyperplane $H \subset \mathbb{P}^n$.

(3) the n dimensional variety $X_{n,[I^{-1}]_d}$ satisfies at least one Laplace equation of order $d-1$.

The above result has motivated the following definition:

Definition 2.1.6: With the notations above, we will say that I^{-1} (or I) defines a **Togliatti system** if it satisfies the three equivalent conditions in Proposition 2.1.5.

The name is in honor to E. Togliatti who proved that the only smooth Togliatti system of cubics is $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ (see for instance, [7],[8],[9],[2],[14] or [15]).

Remark 2.1.7: Let us notice that being I not a Togliatti system does not imply that I has the WLP. As showed in example 2.0.7 (4), I can hold the WLP in the first non trivial place but fail it later. Hence it is important to remark that Togliatti systems are those ideals satisfying hypothesis of 2.1.5 such that fail the WLP in the first non trivial place because of injectivity.

The Example 2.0.7 (1) gives us a Togliatti system generated by monomials. These types of Togliatti systems will be the object of the study in the next chapters because we can associate to them a toric variety and apply combinatorial tools. However, there are also Togliatti systems generated by other forms than monomials.

Example 2.1.8: Let $n \geq 3$ and $d \geq 3$. Let $L \in R_1$ be a linear form, $F_1, \dots, F_t \in R_{d-1}$ be general forms of degree $d-1$, and $G_1, \dots, G_n \in R_d$ general forms of degree d. If $\binom{n+d-2}{n-1} + 1 \leq t \leq \binom{n+d-1}{n-1} - n$, then I is artinian and fails the WLP in degree $d-1$. Indeed: since $t \geq \binom{n+d-2}{n-1} + 1$, when we restrict $\{LF_i\}$ to a general hyperplane, they become $k-$linearly dependent. On the other hand I is generated by $r = t + n \leq \binom{n+d-1}{n-1}$ forms of degree d and, hence we can apply Proposition 2.1.5 and conclude that I is a Togliatti system.

Chapter 3

Monomial Togliatti systems

In this chapter we will restrict the study to Togliatti systems generated by monomials.

Definition 3.0.1: Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by $r \le \binom{n+d-1}{n-1}$ forms of degree d. Let us assume that I is a Togliatti system (i.e. it fails WLP in degree $d-1$). We say

(1) I is a **minimal Togliatti system** if for any $1 \le s \le r$ and $\{F_{i_1}, \dots, F_{i_s}\} \subset \{F_1, \dots, F_r\}$ we have that $I' = (F_{i_1}, \dots, F_{i_s})$ is not a Togliatti system.

(2) I is a **monomial Togliatti system** if I can be generated by monomials.

(3) I is a **smooth Togliatti system** if $X_{n,[I^{-1}]_d}$ is a smooth variety.

Remark 3.0.2: We cannot forget the assumption of I being artinian. Indeed, $I = (x_0^4, x_1^4, x_2^4, x_3^4) + x_0^3(x_1, x_2, x_3) \subset k[x_0, x_1, x_2, x_3]$ is a minimal monomial Togliatti system. If we consider $J = (x_0^4, x_1^4, x_2^4) + x_0^3(x_1, x_2)$, then we can say that $J \subset I$ and that J is also a minimal monomial Togliatti system. We have committed an abuse of notation:

When we say $J \subset I$ we are seeing J as an ideal of $k[x_0, x_1, x_2, x_3]$, but in this case J is not a Togliatti system because it is not artinian ($(0:0:0:1) \in V(J)$). Therefore, when we are saying that J is a monomial Togliatti system we are referring J as an ideal of $k[x_0, x_1, x_2]$ and, since $I \not\subseteq k[x_0, x_1, x_2]$ there is no sense in saying $J \subset I$.

In order to make the study easier, in the next section we will give some tools to test the WLP of monomial ideals and the smoothness of the associated varieties. Then we will apply these tools to classify smooth monomial Togliatti systems generated by quadrics and cubics. The classification of smooth monomial Togliatti systems of forms of degree $d \ge 4$ is still open. We will discuss it.

3.1 Preliminaries

We start with a very useful result for testing the WLP in the monomial case.

Proposition 3.1.1: *Let $I \subset R$ be an artinian monomial ideal. Then R/I has the WLP if, and only if $x_0 + \cdots + x_n$ is a Lefschetz element for R/I.*

Proof: Take $I = (m_1, \ldots, m_r)$ with $m_i = x_0^{j(i,0)} \cdots x_n^{j(i,n)} \in R$ for $i = 1, \ldots, r$. By definition, I has the WLP if, and only if there is a general linear form $L \in R_1$ such that for every $d \in \mathbb{Z}$, the map $\times L : (R/I)_d \to (R/I)_{d+1}$ either is surjective or injective, and we call L a Lefschetz element. By Lemma 2.0.4 we know that L is a Lefschetz element if $R/(I + (L))$ and $R/(I + (x_0 + \cdots + x_n))$ have the same Hilbert function.

We write $L = a_0x_0 + \cdots + a_{n-1}x_{n-1} + x_n$ with $a_0 \cdots a_{n-1} \neq 0$ by generality of L. Now, let $J := (\overline{m_1}, \ldots, \overline{m_r}) \subset S := k[x_0, \ldots, x_{n-1}]$ where $\overline{m_i} := x_0^{j(i,0)} \cdots x_{n-1}^{j(i,n-1)}(a_0x_0 + \cdots + a_{n-1}x_{n-1})^{j(i,n)}$ for $i = 1, \ldots, r$. Thus, we have $R/(I + (L)) \cong S/J$ $(\star)$. Since $a_0 \cdots a_{n-1} \neq 0$, we can replace each $\overline{m_i}$ by $\overline{n_i} = (a_0x_0)^{j(i,0)} \cdots (a_{n-1}x_{n-1})^{j(i,n-1)}(-a_0x_0 - \cdots - a_{n-1}x_{n-1})^{j(i,n)}$ for $i = 1, \ldots, r$ without changing the ideal. Furthermore, by the isomorphism $y_i \overset{\prime}{\mapsto} a_ix_i$ between S and $S' := k[y_0, \ldots, y_n]$ we have $S/J \cong S'/J'$.

Finally, we have $J' = (\overline{n_1}', \ldots, \overline{n_r}')$ with $\overline{n_i}' = y_0^{j(i,0)} \cdots y_{n-1}^{j(i,n-1)}(-y_0 - \cdots - y_{n-1})^{j(i,n)}$. So, by means of the analogous isomorphism we used in $(\star)$ we have $S'/J' \cong R/(I + (x_0 + \cdots + x_n))$. Therefore, we have seen that $R/(I + (L))$ and $R/(I + (x_0 + \cdots + x_n))$ have the same Hilbert functions, and so we have finished the proof. $\square$

Now we study one example of monomial Togliatti system of degree 3 in $\mathbb{P}^n$.

Example 3.1.2: Let $n \geq 2$ and $I^{-1} := \{x_i^2x_j\}_{0 \leq i \neq j \leq n} \subset k[x_0, \ldots, x_n]$. By simply counting there are $n(n+1)$ monomials so $\dim I^{-1} = n(n+1) - 1$.

We also let $\varphi_{[I^{-1}]_d}$ be the rational map associated with I^{-1}, which is not defined in $n+1$ points. As we saw in Definition 2.1.4, the closure of its image $X := \overline{\mathrm{Im}(\varphi_{[I^{-1}]_d})}$ is projectively equivalent to the projection of the Veronese variety $V(n,3)$ from the linear subspace $I := \langle \{x_i^3\}_{0 \leq i \leq n} \cup \{x_ix_jx_k\}_{0 \leq i < j < k \leq n} \rangle$.

We want to check that X satisfies a Laplace equation of order 2, and hence I will be a Togliatti system of cubics. Observe that $\binom{n+3}{3} - n(n+1) \leq \binom{n+3-1}{n-1}$ if, and only if $n \geq 2$. Thus, by Propositions 2.1.5 and 3.1.1 X satisfies a Laplace equation of order 2 if, and only if the cubics $\{x_i^3\}_{0 \leq i \leq n} \cup \{x_ix_jx_k\}_{0 \leq i < j < k \leq n}$ become $k-$linearly dependent when we restrict them to the hyperplane $x_0 + \cdots + x_n = 0$, which it is easy to check.

When we are dealing with monomial ideals I, I^{-1} is also a monomial ideal and the variety $X := \mathrm{Im}\,\varphi_{n,I_d^{-1}}$ is a toric variety. In the following we will define, construct and study these varieties.

3.1.1 Toric varieties

Let $\{m_0, \ldots, m_N\} \subset k[x_0, \ldots, x_n]$ a set of monomials of degree d. Each of them can be remembered as a $(n+1)-$uple of nonnegative integers (its exponents). That is:

$$m_i = x_0^{\omega_0^i} \cdots x_n^{\omega_n^i} \Rightarrow m_i \leftrightarrow \omega^{(i)} := (\omega_0^i, \ldots, \omega_n^i) \in \mathbb{Z}^{n+1}.$$

Therefore $A := \{\omega^{(0)}, \ldots, \omega^{(N)}\}$ is a finite subset of $\mathbb{Z}^{n+1}$. For any $x = (x_0, \ldots, x_n) \in (k^\star)^{n+1}$ and $\omega = (\omega_0, \ldots, \omega_n) \in \mathbb{Z}^{n+1}$ we set $x^\omega := x_0^{\omega_0} \cdots x_n^{\omega_n}$. Define

$$X_A^0 := \{(x^{\omega^{(0)}} : \cdots : x^{\omega^N}) | x = (x_0, \ldots, x_n) \in (k^\star)^{n+1}\} \subset \mathbb{P}^N.$$

Its closure defines a projective variety $X_A := \overline{X_A^0}$. We will see that X_A^0 is a Toric variety.

Definition 3.1.3: A **toric variety** is a pair (X, α) where
(1) X is a quasi-projective variety, and
(2) $\alpha : (k^\star)^n \times X \to X$ is an action with a dense orbit, i.e. there is $x \in X$ such that

$$\overline{O_\alpha(x)} = \overline{\{a \cdot x | a \in (k^\star)^n\}} = X.$$

Proposition 3.1.4: *Let $A = \{\omega^{(0)}, \ldots, \omega^{(N)}\}$ be a finite subset of $\mathbb{Z}^{n+1}$. Then X_A^0 is a toric variety.*

Proof: Let be $\alpha : (k^\star)^{n+1} \times \mathbb{P}^N \to \mathbb{P}^N$ such that $x \cdot (z_0 : \cdots : z_N) = (x^{\omega^{(0)}} z_0 : \cdots : x^{\omega^{(N)}} z_N)$. By definition $X_A^0 = \{(x^{\omega^{(0)}} : \cdots : x^{\omega^{(N)}}) | x \in (k^\star)^{n+1}\} = O_\alpha((1 : \cdots : 1))$. Then $X_A^0 = O_\alpha((1 : \cdots : 1))$ is a toric variety together with $\alpha : (k^\star)^{n+1} \times X_A^0 \to X_A^0$. □

Notation 3.1.5: By abuse of notation, when talking about toric varieties of the kind X_A we will omit the action α.

The following results and definitions will describe the properties of these varieties. We fix a finite subset $A = \{\omega^{(0)}, \ldots, \omega^{(N)}\}$ of $\mathbb{Z}^{n+1}$ and X_A the toric variety associate to it.

Definition 3.1.6: We call the **affine sublattice generated by** A the set

$$Aff_{\mathbb{Z}}(A) := \left\{ \sum_{\omega \in A} n_\omega \omega | \forall \omega \in A,\ n_\omega \in \mathbb{Z};\ \sum \omega \in A n_\omega = 1 \right\} \subset \mathbb{Z}^{n+1}.$$

We call **polytope generated by** A the set

$$P_A := \{\textstyle\sum_{\omega \in A} a_\omega \omega | \forall \omega \in A,\ a_\omega \in \mathbb{R}_+;\ \sum_{\omega \in A} a_\omega = 1\} \subset \mathbb{R}^{n+1}.$$

and the dimension of P_A to be the dimension of the smallest affine space containing P_A.

We want to give a criterion to know whether a toric variety of the type X_A is smooth. To do it we need to fix some notation:

Let S be the subsemigroup of $\mathbb{Z}^{n+1}$ generated by A and 0. Pick up a face of P_A, F, and let us define $\text{Lin}_{\mathbb{R}}(F)$ as the vector $\mathbb{R}-$subspace spanned by F in $\mathbb{R}^{n+1}$. Observe that $\dim_{\mathbb{R}} \text{Lin}_{\mathbb{R}}(F) = \dim F + 1$. Indeed, by definition of polytope dimension, F is contained in an affine subvariety of dimension $\dim F$, let us name it V. As long as F does not contain $0 \in \mathbb{R}^{n+1}$, we have $\text{Lin}_{\mathbb{R}}(F) = V \vee 0$ and $\dim_{\mathbb{R}} \text{Lin}_{\mathbb{R}}(F) = \dim(F) + 1$.

Consider now the quotient $\mathbb{Z}^{n+1}/F := \mathbb{Z}^{n+1}/(\mathbb{Z}^{n+1} \cap \text{Lin}_{\mathbb{R}}(F))$ which is a quotient of free abelian groups, and hence it is a free abelian group. Let us define S/F as the image of S in $\mathbb{Z}^{n+1}/F$. Since S is a subsemigroup of $\mathbb{Z}^{n+1}$, S/F is a subsemigroup of $\mathbb{Z}^{n+1}/F$.

Finally we define $\mathrm{Lin}_{\mathbb{Z}}(A \cap F)$ as the abelian subgroup generated by $A \cap F$ in $\mathbb{Z}^{n+1}$. Observe that $\mathrm{Lin}_{\mathbb{Z}}(A \cap F)$ is a subgroup of $\mathbb{Z}^{n+1} \cap \mathrm{Lin}_{\mathbb{R}}(F)$, in particular we have an index

$$i(F, A) := [\mathbb{Z}^{n+1} \cap \mathrm{Lin}_{\mathbb{R}}(F) : \mathrm{Lin}_{\mathbb{Z}}(A \cap F)].$$

Proposition 3.1.7: *Let $A \subset \mathbb{Z}^{n+1}$ be a finite set generating $\mathbb{Z}^{n+1}$ as an affine lattice and let P_A the polytope generated by A. Then, X_A is smooth if, and only if for each non-empty face F of P_A the following conditions hold:*

a) The semigroup S/F is free.

b) The index $i(F, A) = 1$.

To see whether these conditions hold we can understand them by means of the polytope P_A and its relation to the lattice $\mathbb{Z}^{n+1}$. In particular it can be proved

Proposition 3.1.8: *In the context of proposition 3.1.7, conditions a) and b) are respectively equivalent to*

a') For every vertex v of P_A let us consider $v_1, \dots, v_k$ the first integral points on the edges going from v. Then $\{v_i - v\}_{1 \le i \le k}$ spans $\mathbb{Z}^{n+1}$.

b') For every non empty face F of P_A, $Aff_{\mathbb{R}}(F) \cap \mathbb{Z}^{n+1} = Aff_{\mathbb{Z}}(A \cap F)$.

Also, when $n = 2$, condition b') is equivalent to b'') For every edge F of P_A, $F \cap \mathbb{Z}^3 \subset A$.

Example 3.1.9: Consider $I^{-1} = \{x_i^2 x_j\}_{0 \le i \ne j \le 3}$ and the corresponding Togliatti system $I = \{x_i^3\}_{i=0}^n \cup \{x_i x_j x_k\}_{0 \le i < j < k \le n}$ (see Example 3.1.2). The smoothness criterion tells us that $X = \overline{\mathrm{Im}(\varphi_{[I^{-1}]_d})}$ is smooth.

The next proposition will be a very important tool to see whether a toric variety X_A satisfies one Laplace equation of order two:

Proposition 3.1.10: *Let $A \subset \mathbb{Z}^{n+1}$ be a finite subset, and let $X_A \subset \mathbb{P}^n$ be the toric variety associated with A. Then $\Psi : \mathbb{P}^n \to X_A$ s.t. $(t_0 : \cdots : t_n) \mapsto (t^{m_0} : \cdots : t^{m_N})$ is a parametrization where:*

(1) $m_i = (m_{i0}, \cdots, m_{in}) \in A$

(2) $t = (t_0, \dots, t_n) \Rightarrow t^{m_i} = t_0^{m_{i0}} \cdots t_n^{m_{in}}$

If there is a hypersurface of degree d in $\mathbb{P}^n$ which contains all points of A, then X_A satisfies one Laplace equation of order d.

Proof: Let $\mathbf{l} = (l_0, \dots, l_n)$ and $\mathbf{a} = (a_0, \dots, a_n)$ be two $(n+1)-$uples of nonnegative integers. Then we consider

$$\frac{\partial t^{\mathbf{l}}}{\partial t^{\mathbf{a}}} = \frac{1}{a!} \frac{\partial^{|\mathbf{a}|} t^{\mathbf{l}}}{\partial t^{a_0} \cdots \partial t^{a_n}} = \binom{l_0}{a_0} \cdots \binom{l_n}{a_n} t^{\mathbf{l} - \mathbf{a}}$$

We can group all dth order partial derivatives of Ψ at a general point in a matrix:

$$H_d(\Psi)(\bar{t}) = \begin{pmatrix} \frac{\partial^d t^{m_0}}{\partial t_0^d}(\bar{t}) & \frac{\partial^d t^{m_1}}{\partial t_0^d}(\bar{t}) & \cdots & \frac{\partial^d t^{m_n}}{\partial t_0^d}(\bar{t}) \\ \frac{\partial^d t^{m_0}}{\partial t_0^{d-1} t_1}(\bar{t}) & \frac{\partial^d t^{m_1}}{\partial t_0^{d-1} t_1}(\bar{t}) & \cdots & \frac{\partial^d t^{m_n}}{\partial t_0^{d-1} t_1}(\bar{t}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^d t^{m_0}}{\partial t^{\mathbf{a}}}(\bar{t}) & \frac{\partial^d t^{m_1}}{\partial t^{\mathbf{a}}}(\bar{t}) & \cdots & \frac{\partial^d t^{m_n}}{\partial t^{\mathbf{a}}}(\bar{t}) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^d t^{m_0}}{\partial t_n^d}(\bar{t}) & \frac{\partial^d t^{m_1}}{\partial t_n^d}(\bar{t}) & \cdots & \frac{\partial^d t^{m_n}}{\partial t_n^d}(\bar{t}) \end{pmatrix}$$

Observe that $\dim_k \mathbb{P}\left(T^{(d)}_{\Psi(\bar{t})} X_A\right) = \operatorname{rg} H(\Psi)(\bar{t}) - 1$. Then, X_A satisfies one Laplace equation of order d if, and only if for general $\bar{t} \in \mathbb{P}^n$ there is a $k-$linear dependence among all columns of $H(\Psi)(\bar{t})$. As $\bar{t}$ is general, we can assume $\bar{t} = (1 : \cdots : 1)$, and using the partial derivative expression above, each column of $H_d(\Psi)(1 : \cdots : 1)$ can be written as

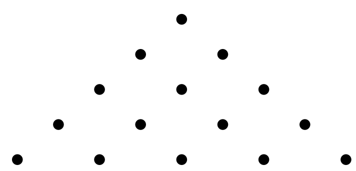
$$\left[\binom{m_{i0}}{a_0} \cdots \binom{m_{in}}{a_n}\right]_{\mathbf{a} \in \mathbb{Z}^{n+1} \quad \text{s.t.} \quad |\mathbf{a}|=d}$$

Then a null linear combination of these columns corresponds to a hypersurface of degree d going through every point of A. $\square$

Let us notice that if all monomials corresponding to the set A have the same degree d, we can see them as a subset of $d\Delta_n$ where Δ_n is the standard simplex with $n+1$ vertices. The next figure shows visually how to represent all monomials of degree 4 in 3 variables:

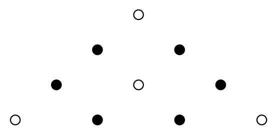
$$\begin{array}{ccccccccc} & & & & x_0^4 & & & & \\ & & & x_0^3 x_1 & & x_0^3 x_2 & & & \\ & & x_0^2 x_1^2 & & x_0^2 x_1 x_2 & & x_0^2 x_2^2 & & \\ & x_0 x_1^3 & & x_0 x_1^2 x_2 & & x_0 x_1 x_2^2 & & x_0 x_2^2 & \\ x_1^4 & & x_1^3 x_2 & & x_1^2 x_2^2 & & x_1 x_2^3 & & x_2^4 \end{array}$$

For simplicity we will consider the monomials as dots:



Therefore, when considering a monomial artinian ideal I of $k[x_0, \ldots, x_n]$ generated by forms of degree d, and its inverse system (also a monomial ideal) we can consider $A_I \subset d\Delta_n$ consisting in all integer points of $d\Delta_n$ minus the points corresponding to the minimal generators of I. Since I is artinian and monomial, A_I is $d\Delta_n$ minus the $n+1$ vertices and other points. Let us see a visual example:

If $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$, then

$A_I =$ 

Where the empty circles represent the removed monomials from $3\Delta_2$. Let us also notice that this visual representation allows us to see easily how is P_{A_I} which we will write shortly as P_I.

Finally, let us consider a corollary of Proposition 3.1.10:

Proposition 3.1.11: *Let $I \subset R$ be an artinian monomial ideal generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree d. Then I is a (monomial) Togliatti system if, and only if there exists a hypersurface of degree $d-1$ passing through all points of A_I.*

Moreover, I is a minimal monomial Togliatti system if, and only if such a hypersurface F does not contain any point of $d\Delta_n$ except, eventually, some vertex.

With these tools we address the problem of classifying all smooth monomial Togliatti systems of forms of degree d (equivalently, the classification of projections of $V(n,d)$ satisfying at least one Laplace equation of order $d-1$).

3.2 Quadratic case

In this subsection we classify all monomial smooth Togliatti systems of quadrics in $R = k[x_0, \ldots, x_n]$ following [7]. To do this, we will use combinatorial techniques and graph theory.

Notation 3.2.1: Let $I \subset R$ be an ideal. We associate a graph G with vertex set $V = \{v_0, \ldots, v_n\}$ and edges $E(G) = \{(v_i, v_j) | x_ix_j \in I\}$.

The main result we want to prove is

Proposition 3.2.2: *Let I be a minimal smooth monomial Togliatti system of quadrics in $R = k[x_0, \ldots, x_n]$ with $n \geq 3$. Then, there are $n-1 \geq a_1 \geq a_2 \geq 2$ such that $n+1 = a_1 + a_2$ and, up to permutation of the coordinates, we have*

$$I = (x_0, \ldots, x_{a_1-1})^2 + (x_{a_1}, \ldots, x_n)^2.$$

Proof: First we check that $(x_0, \ldots, x_{a_1-1})^2 + (x_{a_1}, \ldots, x_n)^2$ are minimal smooth Togliatti system for all a_1, a_2 in the hypothesis.

Now we fix a minimal smooth Togliatti system of quadrics S and let P be its inverse system. Then, we consider the graph G associated to P. We may regard P as a subset of integral points of 2Δ, where Δ is the standard simplex with $n+1$ vertices. Observe that there is a correspondence between edges of G and vertices of P.

By Proposition 2.1.5, $X_{n,P}$ satisfies at least one Laplace equation of degree one. Then, its tangent space at a general point has dimension less than the expected one (which is $n+1$). Finally, Proposition 3.1.11 implies that the set of points P lies in a hyperplane.

To continue with the proof we need some lemmas describing the graph G.

Lemma 3.2.2.1: *Each path (v_j, v_k, v_l, v_s) in G is a part of a four cycle (i.e. (v_s, v_j) is an edge).*

Proof: We can rewrite the hypothesis of the lemma as $p_1 := x_j x_k$, $p_2 := x_k x_l$ and $p_3 := x_l x_s$ are points of P. It is clear that $p_4 := x_s x_j$ is coplanar with p_1, p_2 and p_3. On the other hand, as we said before, the points of P lie in a hyperplane which contains all linear subvarieties spanned by subsets of P. Then p_4 must lie in the same hyperplane and consequently $p_4 \in P$. □

Lemma 3.2.2.2: *Each connected component of G is either a complete or a complete bipartite graph.*

Proof: Let C be a connected component of G with more than one vertex. If C does not contain odd cycles, then C is bipartite. Therefore, the minimal path joining two vertices of different parts must be of odd length, and by Lemma 3.2.2.1 must be of length one. Thus C is a complete bipartite graph.

On the other hand suppose that C contains an odd cycle. By Lemma 3.2.2.1 the shortest odd cycle of C is a triangle. Let us now consider C' be the largest complete subgraph of C, and let v be a vertex of $C - C'$. Since C is connected, v must be joined to C' with at least one edge $e = (v, v_0)$. Since C' is at least a triangle it has at least three vertices and we can choose 3−paths ending at v_0: (v_i, v_k, v_0). Then we can apply Lemma 3.2.2.1 and there must be another edge connecting v with v_i. As long as C' is complete and we can change v_i for any vertex of C', v must be joined with every vertex of C'. Thus, v is a vertex of C' and $C = C'$ is a complete graph.

Finally, if C is one isolated vertex, P is smooth if, and only if $n = 3$. But if $n = 3$ and G has an isolated vertex, P has at most 3 points which contradicts the cardinality assumption of $|S| \leq \binom{n+2-1}{n-1}$. for Togliatti systems of quadrics. □

Now we have all ingredients for the proof, and we start supposing that none of the components of G is bipartite. By the previous lemma, G is a disjoint union of $r+1$ complete graphs and then we can write

$$P = \{x_0x_1, x_0x_2, \dots, x_{j_1-2}x_{j_1-1}\} \cup \{x_{j_1}x_{j_1+1}, \dots, x_{j_2-2}x_{j_2-1}\} \cup \dots \cup \{x_{j_r}x_{j_r+1}, \dots, x_{n-1}x_n\}$$

That set of points does not fit in a single hyperplane. Indeed, the set of hyperplanes such that every hyperplane contains all points in r first sets above does not intersect with the set of hyperplane such that every hyperplane contains all points in the last set above.

On the other hand, if two components of G were bipartite graphs, then we could write $I = I_1 + I_2 + J$ were I_j would be ideals of the same type as in the text in the proposition. As we have seen these ideals are minimal smooth monomial Togliatti systems, then I would not be minimal.

Hence, we can assume that exactly one component is a bipartite graph and none of the components is an isolated vertex. If G is connected I must be as in the proposition.

Otherwise, by the minimality and the smoothness, we can assume that G has exactly two components: a triangle and a complete bipartite graph with one part consisting in a single vertex. However, this implies that $|P| = 3 + (n-3) = n < n+1$ and does not satisfy the cardinality assumption.

3.3 Cubic case

As we have mentioned above, in three variables there is only one monomial Togliatti system of cubics: $I = (x^3, y^3, z^3, xyz)$. Now, we want to classify all monomial Togliatti systems of cubics in $n+1$ variables.

3.3.1 Four variables

Next proposition classifies all monomial Togliatti systems of cubics in $k[x_0, x_1, x_2, x_3]$. It can be proved using Macaulay2.

Proposition 3.3.1 ([9]; Theorem 4.11)**:** *Let $I \subset k[x_0, x_1, x_2, x_3]$ be a monomial artinian ideal of degree 3. Let $X = X_{I^{-1}}$ the variety associated to I^{-1}. If X is a smooth threefold satisfying a Laplace equation of degree 2, then (up to a permutation of the coordinates) we have the following three options:*

(1) $I^{-1} = (x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_2^2, x_0x_3^2, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_3^2, x_2^2x_3, x_2x_3^2)$

(2) $I^{-1} = (x_0x_1x_2, x_0x_1x_3, x_0^2x_2, x_0^2x_3, x_0x_2^2, x_0x_3^2, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_3^2, x_2^2x_3, x_2x_3^2)$

(3) $I^{-1} = (x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3, x_0^2x_2, x_0x_2^2, x_0^2x_3, x_0x_3^2, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2)$

Moreover, if we delete smoothness hypothesis we have the further cases

(1) $I^{-1} = (x_0x_2x_3, x_1x_2x_3, x_0^2x_2, x_0^2x_3, x_0x_3^2, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_2^2x_3, x_2x_3^2)$

(2) *A projection of case ii removing one or both of the monomials $x_0x_1x_2, x_0x_1x_3$ or a projection of case iii removing a subset of the monomials $(x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3)$ or a projection of case iv removing one or both of the monomials $(x_0x_2x_3, x_1x_2x_3)$.*

3.3.2 General case

We will study the general case also following [7]. We begin with a family of examples of smooth Togliatti systems of cubics.

Example 3.3.2: Let us consider a partition of $n+1 = a_1 + \ldots + a_s$, with $n+1 \geq a_1 \geq \cdots \geq a_s \geq 1$ and the ideal $S = (x_0, \ldots, x_{a_1-1})^3 + \cdots + (x_{n+1-a_s}, \ldots, x_n)^3 + J$, where $J = \big(x_ix_jx_k | i < j < k;\ \forall 1 \leq \lambda \leq s,\ \#(\{i,j,k\} \cap I_\lambda) \leq 1\big)$ with $I_\lambda = \{\sum_{\alpha \leq \lambda - 1} a_\alpha, \ldots, \sum_{\alpha \leq \lambda} a_\alpha - 1\}$. Then S is a minimal monomial Togliatti system of cubics.

Proof: First of all let us observe that J is generated by $\sum_{0 \leq i < j < k \leq s} a_ia_ja_k$ monomials, and consequently S is generated by

$$\mu_{a_1,\dots,a_s} = \sum_{k=1}^{s} \binom{a_k+2}{3} + \sum_{0\le i<j<k\le s} a_i a_j a_k \text{ monomials}$$

Also we can observe that its inverse system is $P = (x_i^2 x_j, x_i x_j^2 | i < j;\ \forall 1 \le \lambda \le s,\ \#(\{i,j\} \cap I_\lambda) \le 1) + (x_i x_j x_k | i < j;\ \forall 1 \le \lambda \le s,\ \{i,j\} \subset I_\lambda,\ k \notin I_\lambda)$

The quadric $Q = 2\sum_{0\le i\le n} x_i^2 - 5\sum_{0\le i<j\le n} x_i x_j + 9\left(\sum_{0\le i<j\le a_1-1} x_i x_j + \cdots + \sum_{n+1-a_s\le i<j\le n} x_i x_j\right)$ go through every integral point of P and every other quadric Q' going through every integral point of P, Q' is projectively equivalent to Q. In fact,

Let $Q' = \sum_{i=0}^{n} \mu_i x_i^2 + \sum_{0\le i<j\le n} \mu_{ij} x_i x_j$ be a quadric going through all integral points of P. As we can see above, P is a union of two sets of points. These two sets will restrict the values of μ_i and μ_{ij} and that will give us the proof. From every point of the form $x_i^2 x_j$ and $x_i x_j^2$ such that $i < j$ and $\forall 1 \le \lambda \le s,\ \#(\{i,j\} \cap I_\lambda) \le 1$ we are given two equations:

$$\begin{cases} 4\mu_i + \mu_j + 2\mu_{ij} &= 0 \\ \mu i + 4\mu_j + 2\mu_{ij} &= 0 \end{cases} \quad \text{and we get} \quad \boxed{\mu_i = \mu_j = -\frac{2\mu_{ij}}{5}}$$

Finally, from every point of the second set, i.e. $x_i x_j x_k$ such that $i < j$, and $\forall 1 \le \lambda \le s,\ \{i,j\} \subset I_\lambda$ and $k \notin I_\lambda$ we are given the equation $\mu_i + \mu_j + \mu_k + \mu_{ij} + \mu_{ik} + \mu_{jk} = 0$. Since $\{i,k\}$ and $\{j,k\}$ satisfy the first set condition we get $\mu_i = \mu_k = \mu_j = -\frac{2\mu_{ik}}{5} = -\frac{2\mu_{jk}}{5}$. Replacing it in the above equation we have $\boxed{\mu_{ij} = 2\mu_i = -\frac{4\mu_{ik}}{5}}$. In short we have that:

$$\begin{cases} \mu_{ij} = -\frac{5}{2}\mu_i, & i < j;\ \forall 1 \le \lambda \le s,\ \#(\{i,j\} \cap I_\lambda) \le 1 \\ \mu_{ij} = 2\mu_i, & i < j;\ \{i,j\} \subset I_\lambda \end{cases}$$

Therefore, we can write

$$Q' = \sum_{0\le i\le n} \mu_i x_i^2 - \frac{5}{2} \sum_{\substack{0\le i<j\le n \,|\\ \#(\{i,j\}\cap I_\lambda)\le 1}} \mu_i x_i x_j + 2\sum_{0\le i<j\le a_1-1} \mu_i x_i x_j + \cdots + 2\sum_{n+1-a_s\le i<j\le n} \mu_i x_i x_j =$$

$$= \sum_{0\le i\le n} \mu_i x_i^2 - \frac{5}{2}\sum_{0\le i<j\le n} \mu_i x_i x_j + \frac{9}{2}\sum_{0\le i<j\le a_1-1} \mu_i x_i x_j + \cdots + \frac{9}{2}\sum_{n+1-a_s\le i<j\le n} \mu_i x_i x_j$$

which is projectively equivalent (by a change of variables) to Q. Using proposition 3.1.11, this fact proves that S is a minimal Togliatti system. ☐

Actually, it can be proved that the systems presented in this example are all the smooth Togliatti systems of cubics:

Proposition 3.3.3 ([7]; Theorem 3.4)**:** *Let P be a minimal smooth monomial Togliatti system of cubics, and let S be its inverse system. Then, up to permutation of coordinates, either P or S is one of the examples above for some partition of $n+1$.*

3.4 Number of generators of monomial Togliatti systems

In the previous sections we classified all smooth monomial Togliatti systems of cubics and quadrics. The classification of smooth monomial Togliatti systems of forms of degree $d \geq 4$ seems out of reach. In this section we study monomial Togliatti systems of forms of arbitrary degree. The results of [8], though not classify all monomial Togliatti systems, give us upper and lower bounds on their number of generators.

First of all, let us fix some notation:

Definition 3.4.1: For every $n, d \in \mathbb{N}$, we denote by $\mathcal{T}(n,d)$ the set of all minimal monomial Togliatti systems, and by $\mathcal{T}^s(n,d)$ the set of all minimal smooth monomial Togliatti systems. Furthermore, we write

(1) $\mu(n,d) = \min\{\mu(I) | I \in \mathcal{T}(n,d)\}$

(2) $\mu^s(n,d) = \min\{\mu(I) | I \in \mathcal{T}^s(n,d)\}$

(3) $\rho(n,d) = \max\{\mu(I) | I \in \mathcal{T}(n,d)\}$

(4) $\rho^s(n,d) = \max\{\mu(I) | I \in \mathcal{T}^s(n,d)\}$

where $\mu(I)$ stands for the minimal number of generators of an ideal $I \subset k[x_0, \dots, x_n]$.

Since every monomial artinian ideal $I \subset k[x_0, \dots, x_n]$ generated by forms of degree d contains the ideal $(x_0^d, \dots, x_n^d)$, and such ideal has the WLP (Proposition 2.0.1), we have

$$n + 2 \leq \mu(n,d) \leq \mu^s(n,d) \leq \rho(n,d) \leq \rho(n,d) \leq \binom{n+d-1}{n-1}$$

We can analyze the cases $d = 2,\ 3$ using the results we have shown before:

Proposition 3.4.2: *Using the above notation. For $d = 2$ it holds:*

(1) $\mathcal{T}^s(2,2) = \emptyset$

(2) For $n \geq 3$, we have $\mu^s(n,2) = \begin{cases} \lambda^2 + 2\lambda + 1 & \text{if } n = 2\lambda \\ \lambda^2 + 3\lambda + 2 & \text{if } n = 2\lambda + 1 \end{cases}$

(3) For $n \geq 3$, $\rho^s(n,2) = \binom{n}{2} + 3$.

For $d = 3$, we have:

(1) $\mu^s(2,3) = \rho^s(2,3) = 4$.

(2) $\mu^s(3,3) = \rho^s(3,3) = 8$.

(3) $13 = \mu^s(4,3) < \rho^s(4,3) = 15$.

(4) For all $n \geq 4$, we have

$$\rho^s(n,3) = \binom{n+1}{3} + n + 1 \text{ and } \mu^s(n,3) = \begin{cases} \binom{\lambda+2}{3} + 2\binom{\lambda+3}{3} & \text{if } n = 2\lambda \\ 2\binom{\lambda+3}{3} & \text{if } n = 2\lambda + 1 \end{cases}$$

The goal of this section is to find bounds for the general case when $n \geq 2$ and $d \geq 4$ and to classify monomial Togliatti systems reaching the lower bounds or close to them.

3.4.1 Lower bounds

First of all we introduce a type of Togliatti systems that we will often find along this part.

Definition 3.4.3: A Togliatti system $I \subset k[x_0, \dots, x_n]$ of forms of degree d is said to be **trivial** if there exists a form F of degree $d-1$ such that $(x_0F, \dots, x_nF) \subset I$.

Remark 3.4.4: (1) If we restrict $Fx_0, \dots, Fx_n$ to the hyperplane $x_0 + \cdots + x_n = 0$ they become (trivially) dependent.
(2) If $d = 3$, and F is a monomial, then $X_{n,[I^{-1}]_d}$ is not smooth (see Proposition 3.4.7).

In the next proposition we will give a lower bound and we will see that the Togliatti systems which reach the bound are mainly the trivial ones.

Proposition 3.4.5: *For any integers $n \geq 2$ and $d \geq 4$ we have $\mu^s(n,d) = \mu(n,d) = 2n+1$. Furthermore, all minimal monomial Togliatti systems of forms of degree $d \geq 4$ with $\mu(I) = 2n+1$ are trivial unless one of the following cases holds:*

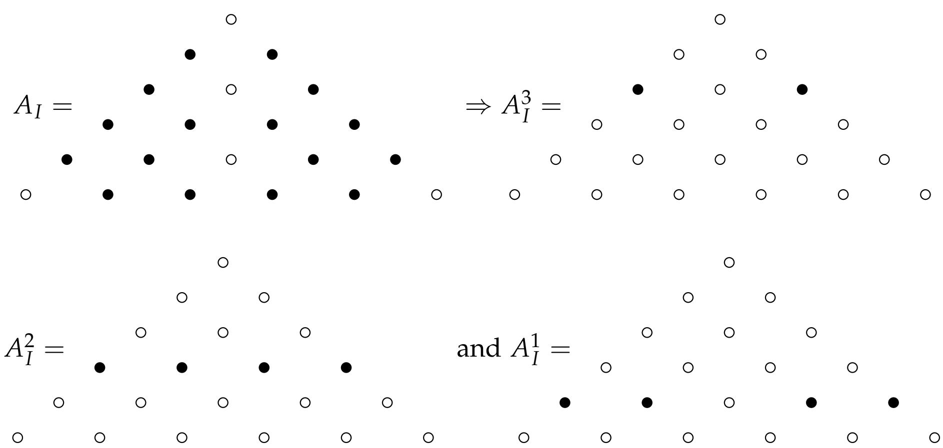
(1) $(n,d) = (2,5)$ and, up to a permutation of the coordinates, $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$.

(2) $(n,d) = (2,4)$ and, up to a permutation of the coordinates, $I = (x_0^4, x_1^4, x_2^4, x_0x_1x_2^2, x_0^2x_1^2)$.

Proof: First of all we will see that $\mu(n,d) \geq 2n+1$, which is equivalent to see that any monomial artinian ideal $I = (x_0^d, \dots, x_n^d, m_1, \dots, m_{n-1})$ has the WLP. We write $m_i = x_0^{a_0^i} \cdots x_n^{a_n^i}$ with $a_0^i + \cdots + a_n^i = d$ for $1 \leq i \leq n-1$. By Proposition 3.1.11 it is enough to see that there is no hypersurface of degree $d-1$ containing all points of A_I (where A_I is as subsection 3.1.1). To show this we will slice the set A_I into the following $d+1$ sets:

For $0 \leq i \leq d$ we define $H_i := \{(a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1} | a_0 = i\}$, and $A_I^i := A_I \cap H_i$.

We have that $A_I = \cup_{i=0}^d A_I^i$ and $A_I^d = \{x_0^d\}$. Next figures give a visualization of these sets choosing, for example $I = (x_0^5, x_1^5, x_2^5, x_0^3x_1x_2, x_0x_1^2x_2^2)$



Now we will proceed the proof by induction on n, starting with the case $n = 2$:

In this case we have $I = (x_0^d, x_1^d, x_2^d, x_0^{a_0^1} x_1^{a_1^1} x_2^{a_2^1})$ with $a_0^1 + a_1^1 + a_2^1 = d \geq 4$. Hence we can assume without loss of generality that $a_0^1 \geq 2$. Let us assume that there is a plane curve F_{d-1} of degree $d-1$ containing all points of A_I and we will get a contradiction.

Since $a_0^1 \geq 2$ there are exactly $d-1$ points in A_I^0 and d points in A_I^1. Now, since F_{d-1} is a curve of degree $d-1$ containing all d points of A_I^1 there must be a linear form L_1 such that $F_{d-1} = L_1 F_{d-2}$. Since F_{d-2} has degree $d-2$ and contains all $d-1$ points of A_I^0 there is also a linear form L_0 such that $F_{d-2} = L_0 F_{d-3}$. We have now that $F_{d-1} = L_0 L_1 F_{d-3}$ with F_{d-3} of degree $d-3$.

In case that $a_0^1 = 2$, A_I^2 has exactly $d-2$ points. Otherwise, if $a_0^1 > 2$, A_I^2 has exactly $d-1$ points. In both cases A_I^2 contains more points than the degree of F_{d-3}. Since F_{d-3} contains all points of A_I^2 we have that $F_{d-3} = L_2 F_{d-4}$, and by the minimality of I, $a_0^1 > 0$.

Repeating the argument, we get that $F_{d-1} = L_0 L_1 \cdots L_{d-2}$, so F_{d-1} does not contain the points of A_I^{d-1} which is non empty and it contradicts the existence of a plane curve of degree $d-1$ containing the integral points of A_I.

Let now $n \geq 3$ and assume, inductively that the claim is true for $n-1$. We want to prove that there is no hypersurface of degree $d-1$ containing all integral points of A_I, where

$$I = (x_0^d, \ldots, x_n^d, m_1, \ldots, m_{n-1}) \text{ and } m_i = x_0^{a_0^i} \cdots x_n^{a_n^i} \text{ with } \sum_{k=0}^{n} a_k^i = d \geq 4, \quad 1 \leq i \leq n-1.$$

By reenumerating if necessary, we can assume that $a_0^1 \geq a_1^1 \geq \cdots \geq a_n^1 \geq 0$, and also that $a_0^1 \geq a_0^2 \geq 0$. Therefore, $a_0^1 > 0$, and we can see A_I^0 as the integer points of $d\Delta_{n-1}$ minus the n vertices and, at most, minus $n-2$ vertices more. By inductive assumption, there is no hypersurface of degree $d-1$ in $\mathbb{P}^{n-1}$ containing all points of A_I^0. Hence, F_{d-1} factorizes as $F_{d-1} = L_0 F_{d-2}$ where F_{d-2} is a hypersurface of degree $d-2$ containing all points of $A_I \setminus A_I^0$.

Now, suppose that $a_0^1 = \ldots = a_0^{n-1} = 1$, then $A_I^2 = (d-2)\Delta_{n-1}, \ldots, A_I^{d-1} = \Delta_{n-1}$. Since there is no hypersurface of degree $d-j$ in $\mathbb{P}^{n-1}$ containing all integer points of $(d-j)\Delta_{n-1}$ for $j = 2, \ldots, d-1$, F_{d-2} factorizes as $L_2 \cdots L_{d-1}$. This gives a contradiction because F_{d-1} does not contain any point of A_I^1 which is non empty.

Otherwise, suppose that $a_0^1 \geq 2$. Then, A_I^1 is equal to $(d-1)\Delta_{n-1}$ minus at most $n-2$ points. By inductive assumption, there is no hypersurface of degree $d-2$ in $\mathbb{P}^{n-2}$ containing all points of A_I^1. Therefore, $F_{d-2} = F_{d-3} L_1$, and we repeat the argument until we find that $F_{d-1} = L_0 \cdots L_{d-2}$ which gives us a contradiction.

Once seen that $\mu(I) \geq 2n+1$ for every $I \in \mathcal{T}(n,d)$ we will classify all monomial Togliatti systems in $\mathcal{T}(n,d)$ such that they are generated by exactly $2n+1$ monomials.

We will start with the case $n = 2$. Let us take $I = (x_0^d, x_1^d, x_2^d, m_1, m_2) \in \mathcal{T}(2,d)$, where $m_i = x_0^{a_0^i} x_1^{a_1^i} x_2^{a_2^i}$ and $a_0^i + a_1^i + a_2^i = d$. Let us suppose that there exists $0 \leq i \leq 2$ such that $a_i^1, a_i^2 \geq 2$. We can suppose without loss of generality, that $i = 0$. Then, the plane curve F_{d-1} containing all points of A_I factorizes as $F_{d-1} = L_0 L_1 F_{d-3}$. Now,

$$\#(F_{d-3} \cap H_2) = \begin{cases} d-1, & a_0^1, a_0^2 > 2 \\ d-2, & a_0^1 > a_0^2 = 2 \\ d-3, & a_0^1 = a_0^2 = 2 \end{cases}$$

In the first case we get $F_{d-3} = L_2 F_{d-4}$. In the second case we obtain a contradiction with minimality. The last case is possible, a priori. Hence, $a_0^1 = a_0^2 = s \geq 2$, and using above argument we can see that $F_{d-1} = L_0 L_1 \cdots L_{s-1} F_{d-s-1}$. Now we can see that $\#(F_{d-s-1} \cap H_{s+1}) = d - s > d - s - 1$. Therefore we have the factorization $F_{d-s-1} = L_{s+1} F_{d-s-2}$. Applying this argument recursively we obtain that $F_{d-1} = L_0 L_1 \ldots L_{s-1} L_{s+1} \cdots L_{d-1}$. Using the minimality of I we obtain that this is only possible when $s = d-1$ and, therefore I is trivial.

Now, let us assume that for any $0 \leq i \leq 2$, there is $1 \leq j \leq 2$ such that $a_i^j \leq 1$. Since there are 3 variables and 2 monomials, by reenumerating the indices we can assume that $a_0^1, a_1^1, a_2^2 \leq 1$. Therefore it must be:

$$m_1 \in \{x_0 x_1 x_2^{d-2}, x_0 x_2^{d-1}, x_1 x_2^{d-1}\} \quad \text{and} \quad m_2 \in \{x_0^{\alpha} x_1^{d-1-\alpha}, x_0^{\alpha} x_1^{d-\alpha} | 0 \leq \alpha \leq d-1\}$$

When replacing x_0 by $x_1 + x_2$ in the above possible monomials, then $x_0^d, x_1^d, x_2^d, m_1, m_2$ remain linearly independent except when

(1) $d = 4$ and $(m_1, m_2) = (x_0^2 x_1 x_2, x_0^2 x_1^2)$

(2) $d = 5$ and $(m_1, m_2) = (x_0 x_1 x_2^3, x_0^2 x_1^2 x_2)$

Let us now suppose that $n \geq 3$ and $d \geq 4$. Let $I = (x_0^d, \ldots, x_n^d, m_1, \ldots, m_n)$ be a minimal Togliatti system where $m_i = x_0^{a_0^i} \cdots x_n^{a_n^i}$ and $\sum_{j=0}^n a_j^i = d \geq 4$. Since we have $n+1$ variables and n monomials, there must be a variable x_j appearing in more than two monomials. We can assume without loss of generality that $j = 0$ and $a_0^1, a_0^2 \geq 1$. Let us now consider the hypersurface F_{d-1} of degree $d-1$ containing all integral points of A_I given by Proposition 3.1.11. Let us consider its restriction G_{d-1} to the hyperplane H_0. G_{d-1} vanishes to all points of A_I^0 which is equal to $d\Delta_{n-1}$ minus the n vertices and at most $n-2$ other points. Let us consider now I' the ideal generated by those n vertices and the monomials not containing x_0 among $\{m_i\}$.

If G_{d-1} is not a hyperplane, we have a hypersurface in n variables containing $A_{I'}$. Therefore I' would be a minimal Togliatti system with $\mu(I') \leq 2n - 2 = 2(n-1)$. In the first part of the proof we have seen this cannot happen. Hence G_{d-1} is a hyperplane and $F_{d-1} = L_0 F_{d-2}$. Since I is minimal, we have that $a_0^1 \geq a_0^2 \geq \cdots \geq a_0^n \geq 1$.

Since I is a Togliatti system, when we restrict $x_0^d, \ldots, x_n^d, m_1, \ldots, m_n$ to $x_0 + \cdots + x_n = 0$ they become linearly dependent. In such a linear combination, the coefficient of $(x_0 + \ldots + x_{n-1})^d$ must be null because, for example, the monomial $x_1^{d-1} x_2$ cannot be canceled by any of the other monomials. Then, the coefficients of the monomials $x_1^d, \ldots, x_{n-1}^d$ must be zero in this linear combination. Therefore, we can divide the monomials m_i by x_0 and obtain new monomials m_i' of degree $d-1$. These new monomials, together with $x_0^{d-1}, \ldots, x_n^{d-1}$ must form a new Togliatti system. We can repeat this procedure until $d = 4$ and obtain that $m_i = Mm'_i$ where m'_i is a monomial of degree 4 and $I' = (x_0^4, \dots, x_n^4, m'_1, \dots, m'_n)$ is a minimal Togliatti system.

We can use the same argumentation as above to show that these monomials m'_i all contain x_j. Then, if F'_3 is the hypercubic containing $A_{I'}$, it must factorize as $F'_3 = L'_0 F'_2$ with F'_2 containing $A_{I'} \setminus A^0_{I'}$. With only n points we can remove, the only possibility is that these points are next to x_j^4, and then $(m'_1, \dots, m'_n) = x_j^3(x_0, \dots, x_n)$.

Finally, since $\mu(I) = 2n+1$, $M = x_0^{d-4}$ and $j = 0$. Hence, $I = (x_0^d, \dots, x_n^d) + x_0^{d-1}(x_1, \dots, x_n)$. □

Arguing as above we can classify all smooth minimal Togliatti systems which are generated by $2n+2$ monomials and we get

Proposition 3.4.6 ([8]; Theorem 3.17)**:** *Let $I \subset k[x_0, \dots, x_n]$ be a smooth minimal monomial Togliatti system of forms of degree $d \geq 4$ with $\mu(I) = 2n+2$. Then I is trivial unless $n = 2$ and, up to a permutations of the variables, one of the following cases hold:*
(1) *$d = 5$ and*

(a) $I = (x_0^5, x_1^5, x_2^5) + x_0x_1x_2(x_0, x_1)^2$

(b) $I = (x_0^5, x_1^5, x_2^5) + x_0x_1x_2(x_0^2, x_1^2, x_2^2)$

(c) $I = (x_0^5, x_1^5, x_2^5) + x_0x_1x_2(x_0x_1, x_0x_2, x_1x_2)$

(2) *$d = 7$ and*

(a) $I = (x_0^7, x_1^7, x_2^7) + x_0x_1x_2(x_0^2x_1^2, x_0^2x_2^2, x_1^2x_2^2)$

(b) $I = (x_0^7, x_1^7, x_2^7) + x_0x_1x_2(x_0^4, x_1^4, x_2^4)$

(c) $I = (x_0^7, x_1^7, x_2^7) + x_0x_1x_2(x_0^2x_1^2, x_0x_1x_2^2, x_2^4)$

The next proposition finishes the classification of smooth monomial Togliatti systems I with $2n+1 \leq \mu(I) \leq 2n+2$.

Proposition 3.4.7: *Let $I = (x_0^d, \dots, x_n^d) + m(x_0, \dots, x_n)$ be a trivial Togliatti system of degree d where m is a monomial of degree $d-1$. Then I is smooth if, and only if (up to permutation of variables) one of the following cases hold:*

(1) *$d = 2$ and $n = 2$ or $n = 3$.*

(2) *$d = 3$, $n = 2$ and $m = x_0^2$. In this case $5 = \mu(I) > \rho(2,3) = 4$, but we consider it to have a complete picture.*

(3) *$d \geq 4$, $n = 2$ and $m = x_0^{d-1}$ or $m = x_0^{i_0}x_1^{i_1}x_2^{i_2}$ with $i_0 \geq i_1 \geq i_2 > 0$.*

(4) *$d \geq 4$, $n \geq 3$ and $m = x_0^{d-1}$ or $m = x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}$ with $i_0 \geq i_1 \geq \cdots \geq i_n \geq 0$ and $i_2 > 0$.*

Proof: We will study cases $d = 2, 3 \geq 4$ separately:
(1) Assume $d = 2$. We can assume without loss of generality that $m = x_0$. If $n = 2$, then X is a point corresponding to x_1x_2, and it is smooth.

Suppose now that $n \geq 3$, then to get A_I we have to remove all points of $2\Delta_n$ except those corresponding to the monomials $x_i x_j$ with $0 < i < j \leq n$. Therefore, $A_I = A_I^0$ and it is equal to $2\Delta_{n-1}$ minus the n vertices. Hence, there are $2(n-2)$ edges emanating from each vertex of P_I. By the smoothness criterion 3.1.8 we have that X is smooth if, and only if $2(n-2) = n-1 \Leftrightarrow n = 3$.
(2) Assume $d = 3$. In this case we can assume m to be x_0^2 or x_0x_1. If $n = 2$, the first case is smooth because P_I corresponds to a trapezium, and the second case is singular:

> If $m = x_0x_1$, in order to obtain A_I we have to remove all vertices of $3\Delta_2$ together with $x_0^2x_1$, $x_0x_1^2$ and $x_0x_1x_2$. Hence $\overline{x_0^2x_2\ x_1^2x_2}$ is an edge of P_I which contains $x_0x_1x_2$. This contradicts the smoothness criterion 3.1.8.

If $n \geq 3$ both cases are singular:

If $m = x_0^2$, then there are more than $2(n-1)$ edges emanating from the vertex corresponding to $x_0x_1^2$. Since $n \geq 3$, then $2(n-1) > n-1$ and the smoothness criterion 3.1.8 does not hold. Otherwise, if $m = x_0x_1$ we can use the same argument than the case $n = 2$ and get that X is singular.
(3) Assume $d \geq 4$ and $n = 2$. If $m = x_0^{d-1}$ then P_I corresponds to a truncated simplex and X is smooth. If $m = x_0^{d-2}x_1$, then it is singular:

> In particular we are removing the points corresponding to x_0^d, $x_0^{d-1}x_1$ and $x_0^{d-2}x_1^2$. Also, the points corresponding to $x_0^{d-1}x_2$ and $x_0^{d-3}x_1^3$ belongs to A_I. Hence $\overline{x_0^{d-1}x_2\ x_0^{d-3}x_1^3}$ is an edge of P_I and the smoothness criterion 3.1.8 does no hold.

If $m = x_0^{d-i}x_1^{i-1}$ with $3 \leq i \leq d-2$, then $x_0^{d-1}x_1$ and $x_0x_1^{d-1}$ belong to A_I. Therefore, $\overline{x_0^{d-1}x_1\ x_0x_1^{d-1}}$ is an edge of P_I which contains $x_0^{d-i}x_1^{i-1} \notin A_I$. Hence, the smoothness criterion 3.1.8 implies X is not smooth.

Finally, if $m = x_0^{i_0}x_1^{i_1}x_2^{i_2}$ with $i_0 \geq i_1 \geq i_2 > 0$, then X is smooth because P_I is a truncated simplex and all points different from vertices removed are located in the interior of P_I.
(4) Assume $d \geq 4$ and $n \geq 3$. If $m = x_0^{d-1}$, the system is smooth because P_I is a truncated simplex with all integer points belonging to A_I. If $m = x_0^{d-i}x_1^{i-1}$ with $i \geq 3$ X is singular because P_I has a $2-$face where we can apply the same argument as in the case $n = 2$.

Finally, if m contains at least 3 different variables, then X is smooth: P_I is a truncated simplex such that all integer points on its edges belong to A_I, and on each $k-$face F ($k \geq 2$) if $p \in F \setminus A_I$ then $p \in \mathring{F}$ □

Remark 3.4.8: Without the smoothness hypothesis there are more monomial Togliatti systems I with $\mu(I) = 2n+2$. For instance, $I = (x_0^5, x_1^5, x_2^5, x_0x_1^4, x_0x_1^2x_2^2, x_0x_2^4) \in \mathcal{T}(2,5) \setminus \mathcal{T}^s(2,5)$ and $\mu(I) = 6$.

3.4.2 Study in the range

In this subsection we will study whether there exists, for each $\mu(n,d) \leq r \leq \rho(n,d)$ (resp. $\mu^s(n,d) \leq r \leq \rho^s(n,d)$), any $I \in \mathcal{T}(n,d)$ (resp. $\mathcal{T}^s(n,d)$) with $\mu(I) = r$. The case $n = 2$ is easily solved as next proposition shows:

Proposition 3.4.9: *If $n = 2$ then for any $d \geq 4$ we have:*

(1) $\mu^s(2,d) = \mu(2,d) = 5$.

(2) $\rho^s(2,d) = \rho(2,d) = d+1$.

(3) For any $5 \leq r \leq d+1$*, there exists* $I \in \mathcal{T}^s(2,d)$ *such that* $\mu(I) = r$.

Proof: (1) It follows from Proposition 3.4.5.

(2) By definition of Togliatti system we have that $\rho(2,d) \leq d+1$. The existence of $I \in \mathcal{T}^s(2,d)$ such that $\mu(I) = d+1$ will follow (3) (by choosing $r = d+1$).

(3) Let us consider the ideals

$$I_5 := (x_0^d, x_1^d, x_2^d) + x_0^{d-1}(x_1, x_2)$$

$$\text{For } 6 \leq r \leq d+1 \quad I_r := (x_0^d, x_1^d, x_2^d) + x_0^{d-r+3}x_1x_2\left[(x_0,x_1)^{r-5} + (x_2)^{r-5}\right]$$

The first ideal is a smooth monomial Togliatti system as seen in Proposition 3.4.7. For $r = 6$ $I_r = (x_0^d, x_1^d, x_2^d) + x_0^{d-3}x_1x_2(x_0,x_1,x_2)$ is also a smooth monomial Togliatti system again by Proposition 3.4.7.

Let us consider L_i, L'_i and L''_i the linear forms in $k[x_0,x_1,x_2]$ corresponding to the affine hyperplanes $\{a_0 = i\}$, $\{a_1 = i\}$ and $\{a_2 = i\}$ respectively, where $0 \leq i \leq d$ and (a_0,a_1,a_2) are the coordinates in $\mathbb{Z}^3$. With this notation, for each $7 \leq r \leq d+1$, $F_{d-1} = L_0 \cdots L_{d-r+2}L''_0L''_2 \cdots L''_{r-5}L'_0$ is a hypersurface of degree $d-1$ containing all points of A_{I_r}. Finally, this system is smooth because P_{I_r} is a truncated simplex and every integer point in its edges belongs to A_{I_r} □

However, we cannot generalize this proposition to the case $n \geq 3$. In that case we will see that there is no $I \in \mathcal{T}^s(n,d)$ with $\mu(I) = 2n+3$. We will study this separating the cases $d = 3$ and $d \geq 4$, but before we will prove a useful lemma:

Lemma 3.4.10: *Let* $I = (x_0^d, \dots, x_n^d, m_1, \dots, m_h) \in \mathcal{T}(n,d)$ *with* $h \geq n$, $d \geq 3$ *and* $m_i = x_0^{a_0^i} \cdots x_n^{a_n^i}$ *for* $i = 1, \dots, h$. *We can assume that* $a_0^1 \geq \cdots \geq a_0^h$.

If $a_0^{h-n+2} > 0$, *then* $a_0^i > 0$ *for* $i = 1, \dots, h$.

Proof: Since I is a Togliatti system there is a form F_{d-1} of degree $d-1$ in $\mathbb{P}^n$ which contains all points of A_I. By the hypothesis assumption, A_I^0 is equal to $d\Delta_{n-1}$ minus the n vertices and at most $h - (h-n+2) = n-2$ other points.

Let us consider the ideal I' generated by $x_1^d, \dots, x_n^d$ and $\{m_i | h-n+3 \leq i \leq h,\ a_0^i = 0\}$. Note that $A_{I'} \subset d\Delta_{n-1}$. If the restriction of F_{d-1} to the hyperplane H_0 is $F(0,x_1,\dots,x_n) \neq 0$, then we have a hypersurface of degree $d-1$, $G_{d-1} := F_{d-1_{|H_0}}$ in $\mathbb{P}^{n-1}$ such that it contains all points of $A_{I'}$. Then I' is a Togliatti system in n variables, generated at most by $n+n-2 = 2(n-1)$ monomials, this is a contradiction by Proposition 3.4.5. Hence $G_{d-1} = 0$ and F_{d-1} factorizes as $F_{d-1} = L_0F_{d-2}$.

Since L_0 contains all integer points of $d\Delta_n \cap H_0$ and I is a minimal Togliatti system, by Proposition 3.1.11 we conclude that $A_I^0 = \emptyset$ and then $a_0^i > 0$, $\forall 1 \leq i \leq h$. □

Lemma 3.4.11: *Assume $n \geq 4$ and let I be a minimal Togliatti system of cubics. Then $\mu(I) \geq 2n+1$ and one of the following holds: (1) $\mu(I) = 2n+1$ if, and only if (up to permutation of coordinates) $I = (x_0^3, \ldots, x_n^3) + x_0^2(x_1, \ldots, x_n)$.*
(2) $\mu(I) = 2n+2$ if, and only if (up to permutation of coordinates) $I = (x_0^3, \ldots, x_n^3) + x_i x_j(x_0, \ldots, x_n)$, with $i \neq j$.
(3) $\mu(I) \neq 2n+3$.

Proof: Let us notice that we cannot use Proposition 3.4.5 because $d < 4$.

We will start proving that there is no monomial ideal I generated by $2n$ cubics failing the WLP from degree 2 to 3. We proceed by induction and with Macaulay2 we see that $\mu(I) \geq 9$ for any $I \in \mathcal{T}(4,3)$. Assume now that $n \geq 5$ and the result is true for $n-1$. Let us consider $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_{n-1})$ with $m_i = x_0^{a_0^i} \cdots x_n^{a_n^i}$ and $a_0^i + \cdots + a_n^i = 3$. We will suppose that there is a hyperquadric F_2 containing all points of A_I. Without loss of generality we can suppose that $a_0^1 > 0$, and then A_I^0 is equal to $3\Delta_{n-1}$ minus the n vertices and at most $n-2$ other points. By induction hypothesis, the restriction of F_2 to the hyperplane H_0 must be of degree one, and therefore $F_2 = L_0 F_1$ with F_1 a hyperplane containing all points of $A_I \setminus A_I^0$. This is impossible, since at least one point of $3\Delta_n \cap H_2$ belongs to A_I.

Suppose now that $\mu(I) = 2n+1$. With Macaulay2 we can see that for $n = 4$ I as (1). Let us assume that $n \geq 5$ and that the proposition is true for $n-1$. We can write $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_n)$ with the same notation as above and let F_2 be a hyperquadric containing all points of A_I. Since at least one variable appears in to different monomials m_i and mj, we can assume without loss of generality that $a_0^1, a_0^2 \geq 1$. Therefore, argumenting as before it must be $F_2 = L_0 F_1$ with F_1 a hyperplane containing all points in $A_I \setminus A_I^0$. The only possibility is that $A_I^2 = \emptyset$ and $m_i = x_0^2 x_i$, and hence I is as (1).

Let us see what happens when $\mu(I) = 2n+2$. With Macaulay2 we can see that for $n = 4$ (2) is verified. Let us assume that $n \geq 5$ and that the proposition is true for $n-1$. We can write $I = (x_0^3, \ldots, x_n^3, m_1, \ldots, m_{n+1})$ with the same notation as above and reenumerating the indices if necessary, we can suppose that $a_0^1 \geq a_0^2 \geq \cdots a_0^{n+1} \geq 0$ and $a_0^1 > 0$. If $a_0^3 > 0$, then by Lemma 3.4.10 $a_0^{n+1} > 0$. Therefore, $F_2 = L_0 F_1$ with F_1 a hyperplane containing all points of $A_I \setminus A_I^0$. The only way to remove $n+1$ points from $3\Delta_n \setminus H_0$ different from the vertices is as follows:

> First choose an edge $\overline{x_0^3\, x_i^3}$ and consider every 2−face of $3\Delta_n$ emanating from this edge. There are in total $n-1$ of these 2−faces.
>
> Now, the inner integer point of each face, together with the two inner integer points of the edge we have chosen, they correspond to $n+1$ monomials of degree 3.
>
> Precisely they are: $\{x_0^2 x_i,\ x_0 x_i^2\} \cup \{x_0 x_i x_j\}_{j \notin 0,i}$.

This allows us to choose F_1 corresponding to the affine hyperplane $\{a_i = 0\}$.

Now suppose that $a_0^3 = 0$. Then, the restriction of I to the hyperplane $\{x_0 = 0\}$ is a Togliatti system $\overline{I}$ in n variables with $\mu(I) = 2n-1 = 2(n-1)+1$ or $\mu(I) = 2n =$

$2(n-1)+2$. Using the induction hypothesis we get that $\overline{I}$ is trivial and can be write as $\overline{I} = (x_1^3, \dots, x_n^3) + x_1^2(x_2, \dots, x_n)$ or $\overline{I} = (x_1^3, \dots, x_n^3) + x_i x_j(x_1, \dots, x_n)$ with $1 \leq i < j \leq n$.

In the first case, applying the smoothness criterion 3.1.8 A_I must contain the point $x_1^2 x_0$ and therefore I contains the trivial Togliatti system $(x_0^3, \dots, x_n^3) + x_1^2(x_0, x_2, \dots, x_n)$ which contradicts the minimality.

On the other hand, in the second case $I = (x_0^3, \dots, x_n^3) + x_i x_j(x_1, \dots, x_n) + (m_1)$. Since m_1 contains x_0, there is $1 \leq k \leq n$ with $k \neq i, j$ such that $x_i^2 x_k$ and $x_j^2 x_k$ are vertices of P_I such that $\overline{x_i^2 x_k\ x_j^2 x_k}$ is an edge of P_I. Since the point $x_i x_j x_k$ lies over the edge and does not belong to A_I, we get a contradiction with the smoothness criterion 3.1.8.

Finally, let us suppose that $\mu(I) = 2n+3$. With Macaulay2 we can see it is not possible for $n = 4$. Suppose now that $n \geq 5$ and let $I = (x_0^3, \dots, x_n^3, m_1, \dots, m_{n+2})$ with the same notation as above. As we did before, we can suppose without loss of generality that $a_0^1 \geq \cdots \geq a_0^{n+1} \geq 0$, and $a_0^1 > 0$. If $a_0^4 > 0$, then we can apply Lemma 3.4.10 and get that $a_0^{n+2} > 0$. Therefore $F_1 = L_0 F_1$ with F_1 a hyperplane passing through all points of $A_I \setminus A_I^0$ and no other point of $3\Delta_n$ apart from the vertices. Since we have to remove $n+2$ points this is impossible.

Otherwise, if $a_0^4 = 0$, we can restrict $x_0^3, \dots, x_n^3, m_1, \dots, m_{n+2}$ to the hyperplane $x_0 = 0$ and get a new ideal $\overline{I} \subset k[x_1, \dots, x_n]$ generated by $x_1^3, \dots, x_n^3$ and those m_i such that do not contain x_0. Since $a_0^1 > 0$ we have that $\mu(\overline{I}) \leq 2(n-1)+2$. Hence we can apply induction hypothesis and get that $\overline{I}$ is one of the following cases:

(1) $\overline{I} = (x_1^3, \dots, x_n^3) + x_1^2(x_2, \dots, x_n)$

(2) $\overline{I} = (x_1^3, \dots, x_n^3) + x_i x_j(x_1, \dots, x_n)$ with $1 \leq i < j \leq n$.

We can apply the argumentation we used just before and see that in any of these cases we cannot get a smooth Togliatti system I. □

Proposition 3.4.12: *Let $n \geq 3$ and $d \geq 4$. Then, there is no $I \in \mathcal{T}^s(n,d)$ with $\mu(I) = 2n+3$.*

Proof: We can write $I = (x_0^d, \dots, x_n^d, m_1, \dots, m_{n+1})$ with the usual notation. Now, we distinguish two cases:

(1) For all $0 \leq j \leq n$, $\#\{i | a_j^i \geq 1\} \leq 3$, which is equivalent to say that each variable appears in at most three monomials.

Any monomial contains all the variables. Indeed, if one monomial contained all the variables, then the other $n+1$ monomials would contain two variables each. This cannot occur as we have seen in the proof of Proposition 3.4.6.

Furthermore, at least two variables appear in three monomials. We can assume without loss of generality that x_0 appears in three monomials. Therefore, A_I^0 is equal to $d\Delta_{n-1}$ minus the vertices and $n-1$ other points. Let us consider the restriction of the hypersurface of degree $d-1$, given by Proposition 3.1.11, to the hyperplane H_0. Then the removed $n-1$ points and the n vertices of $d\Delta_{n-1}$ form a Togliatti system $\overline{I}$ in n variables with $\mu(\overline{I}) = 2(n-1)+1$. By Proposition 3.4.5 there are two possibilities:

(a) $n = 3$ and $\overline{I}$ is one of the two special Togliatti systems of degree 4 or 5 of Proposition 3.4.5. Any of them gives us a Togliatti system I.

(b) $\overline{I} = (x_1^d, \ldots, x_n^d) + x_1^{d-1}(x_2, \ldots, x_n)$. Since x_1 appears in $n-1$ monomials, it can only be $n = 3$ or $n = 4$.

In both cases, we can restrict out attention to the $2-$face $F := \langle x_0^3, x_1^3, x_2^3 \rangle$. If $x_0 x_1^{d-1} \in A_I$, then it is a vertex of P_I. On $F \cap P_I$, there are two edges adjacent to this point. One of them has to be $\overline{x_0 x_1^{d-1}\ x_1^{d-2} x_2^2}$, and this contradicts the smoothness criterion 3.1.8.

(2) There exists $0 \leq j \leq n$ such that $\#\{i | a_j^i \geq 1\} \geq 4$. We can assume $j = 0$ and $a_0^1 \geq a_0^2 \geq a_0^3 \geq a_0^4 > 0$. Applying Lemma 3.4.10 we get that $a_0^{n+1} > 0$. Then we can consider $m_i' := m_i / x_0$ for $1 \leq i \leq n+2$ and as in the proof of Proposition 3.4.6 we get that $m_1', \ldots, m_{n+2}'$ together with $x_0^{d-1}, \ldots, x_n^{d-1}$ form a Togliatti system I_1 of degree $d-1$. There are two possibilities:

(a) at least one of the monomials m_i' is the $(d-1)$th power of a variable, and then $\mu(I_1) < 2n+3$.

(b) $\mu(I_1) = \mu(I) = 2n+3$.

In the first case, if $d > 4$, then I_1 must be trivial. Hence, I would contain a Togliatti system and this would contradict the minimality of I. If $d = 4$ then $\mu(I_1) \leq 2n+2$.

In the second case, we can apply the above argument. We can do this procedure until arrive to $d = 4$. Now we have a Togliatti system I_1 of degree $d-1 = 3$ with $\mu(I_1) \leq 2n+3$. If $n = 3$ Macaulay2 gives us the result. Otherwise, we can apply Lemma 3.4.11 and get that I_1 is trivial and therefore I is not minimal. □

To finish this section we will delete the smoothness condition and we will generalize Proposition 3.4.12

Proposition 3.4.13: *For any ≥ 4, we have:* (1) *For any $n \geq 3$, $\mu(n,d) = 2n+1$.*
(2) *For any $n \geq 3$, $\rho(n,d) = \binom{n+d-1}{n-1}$*
(3) *For any integer r with $\mu(3,d) = 7 \leq r \leq \binom{d+2}{2} = \rho(3,d)$, there exists $I \in \mathcal{T}(3,d)$ with $\mu(I) = r$.*

Proof: (1) It follows from Proposition 3.4.5.

(2) By definition we have that $\rho(n,d) \leq \binom{n+d-1}{n-1}$. Let us consider $I = (x_0^d, \ldots, x_n^d) + x_1(x_1, \ldots, x_n)^{d-1} + x_2(x_2, \ldots, x_n)^{d-1} + \cdots + x_{n-2}(x_{n-2}, x_{n-1}, x_n)^{d-1} + x_0^3(x_{n-1}, x_n)^{d-3}$.

$\mu(I) = \binom{n+d-1}{n-1}$ and by restricting to the hyperplane $x_0 + \cdots + x_n = 0$ we get that $I \in \mathcal{T}(n,d)$.

(3) Now we assume $n = 3$. We have to prove that for any $7 \leq r \leq \binom{d+2}{2}$ there exists $I \in \mathcal{T}(3,d)$ with $\mu(I) = r$.

$r = 7$: $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-1}(x_1, x_2, x_3)$.

$r = 8$: $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-2} x_1 (x_0, x_1, x_2, x_3)$.

$r = 9$: $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0^{d-2}(x_1^2, x_1 x_2, x_2^2, x_2 x_3, x_3^2)$.

It only remains to prove the result for $10 \leq r \leq \binom{d+2}{2}$. We will proceed by induction over $d \geq 4$. In the case $d = 4$ we can give an explicit example for $10 \leq r \leq 13$ thanks to Macaulay2:

$r = 10$: $I = (x_0, x_1)^4 + (x_2, x_3)^4$.

$r = 11$: $I = (x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2^2x_3^2, x_3^4, x_0x_2x_3^2, x_1x_2x_3^2)$.

$r = 12$: $I = (x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2x_3^3, x_3^4, x_0^2x_3^2, x_0x_1x_3^2, x_1^2x_3^2)$.

$r = 13$: $I = (x_0, x_1)^4 + (x_2^4, x_2^3x_3, x_2x_3^3, x_3^4, x_0^3x_3, x_0^2x_1x_3, x_0x_1^2x_3, x_1^3x_3)$.

For $r = 14$ we consider $I = (x_0^4, x_1^4, x_2^4, x_3^4, x_0^2x_1x_2, x_0^2x_1x_3, x_0^2x_2x_3, x_0x_1^2x_2, x_0x_1^2x_3, x_0x_1x_2^2, x_0x_2x_3^2, x_1^2x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2)$
and the case $r = 15$ is covered by (2).

Let us suppose now that $d \geq 4$ and that the result is true for $d-1$.

For any $7 \leq s \leq \binom{d+1}{2}$ let us consider, thanks to induction hypothesis, $J \in \mathcal{T}(3, d-1)$ such that $\mu(J) = s$. Now, we define $I = (x_0^d, x_1^d, x_2^d) + x_3J$. Let us notice that $\mu(I) = s+3$, and also that $I \in \mathcal{T}(3, d)$. Hence, we have proved the result for $10 \leq r \leq \binom{d+1}{2} + 3$.

Observe now that $I = (x_0^d, x_1^d, x_2^d, x_3^d) + x_0(x_1, x_2, x_3)^{d-1} \in \mathcal{T}(3, d)$ and $\mu(I) = \binom{d+1}{2} + 4$.

Finally let us consider, for any $3 \leq i \leq d-1$, the ideal

$$I = (x_0^d, x_1^d, x_2^d, x_3^d) + (x_1^{i_1}x_2^{i_2}x_3^{i_3} | i_1 + i_2 + i_3 = d,\ 1 \leq i_1 \leq d-1) + x_0^i(x_2, x_3)^{d-i}.$$

First of all let us observe that $\mu(I_i) = 4 + \left(\binom{d+2}{2} - (d-1) - 3\right) + (d-i+1) = \binom{d+2}{2} + 3 - i$. Therefore, when i ranges from $i = 3$ to $i = d-1$ we have $\mu(I_i)$ ranges from $\binom{d+1}{2} + 5$ to $\binom{d+2}{2}$. It only remains to prove that $I_i \in \mathcal{T}(3, d)$. By Proposition 3.1.11 it is enough to prove that there is a surface F_{d-1} passing through all points of A_{I_i}. As usual, since $A_{I_i}^1 = (d-1)\Delta_2, \ldots, A_{I_i}^{i-1} = (d-i+1)\Delta_2$, we have the factorization $F_{d-1} = L_1 \cdots L_{i-1}F_{d-i}$ where F_{d-i} is a surface of degree $d-i$ containing all points of $A_{I_i} \setminus \left(\cup_{j=1}^{i-1} A_{I_i}^j\right)$.

On one hand, forms of degree $d-i$ in 4 variables are parametrized by a $k-$vector space of dimension $\binom{d-i+3}{3}$. On the other hand, we are imposing some restrictions to F_{d-i}:

Let us notice that $A_{I_i}^0$ is a set of $d-1$ aligned points. Precisely $A_{I_i}^0 = \{(0, 0, k, d-k)\}_{k=1}^{d-1}$. Containing such a set of points imposes $d-i+1$ conditions to these forms. In addition, containing the points of $A_{I_i}^{i+1}, \ldots, A_{I_i}^{d-1}$ imposes $\binom{d-i+1}{2}, \ldots, 3$ conditions, respectively. Finally, to say that F_{d-i} contains $A_{I_i}^i$ imposes $\binom{d-i+2}{2} - (d-i+1)$ more conditions. If we sum up, we get in total $\binom{d-i+3}{3} - 1$ conditions we are imposing. Then there exists at least one surface of degree $d-i$ satisfying all these restrictions and hence we have found the surface F_{d-1} we were looking for. □

Chapter 4

New results on the classification of Togliatti systems

All results of this chapter are new and they will appear in [12].

In this chapter we will use the combinatoric machinery that we have shown in the previous chapter to classify all smooth monomial minimal Togliatti systems $I \subset k[x_0, \dots, x_n]$ of forms of degree $d \geq 4$ with $\mu(I) = 2n + 3$ and $n \geq 2$, as well as all monomial minimal Togliatti systems in three variables generated by 7 monomials. Therefore, our contributions are a natural continuation of the results in [8] and classify the first case left open there.

In all this chapter we fix the following notation:

Notation 4.0.1: The ideal $T = (x^3, y^3, z^3, xyz)$ and the following sets of ideals:

$A = \{(y^3, y^2z, yz^2, z^3), (xy^2, xz^2, y^3, z^3), (x^2y, y^3, y^2z, z^3), (x^2z, y^3, y^2z, z^3), (xz^2, y^3, y^2z, z^3),$
$(xz^2, y^3, y^2z, yz^2), (x^2z, y^3, y^2z, yz^2), (xyz, xz^2, y^3, yz^2), (xy^2, xz^2, y^3, yz^2), (xyz, xz^2, y^3, y^2z),$
$(xy^2, xz^2, y^2z, yz^2), (x^2z, xy^2, y^2z, yz^2), (x^2z, xz^2, y^3, y^2z), (x^2z, xz^2, y^3, yz^2), (x^2y, xy^2, y^3, z^3),$
$(x^2z, xy^2, y^3, z^3), (x^2z, xyz, y^3, y^2z), (x^2z, xyz, y^3, yz^2), (x^2y, xz^2, y^3, y^2z), (x^2y, xz^2, y^3, yz^2),$
$(x^2z, xy^2, y^3, yz^2)\}$,

$B = \{(x^3z, xy^2z, y^4, yz^3), (x^2yz, xz^3, y^4, y^3z), (x^2z^2, xy^2z, y^4, z^4), (x^2yz, y^4, y^2z^2, z^4)\}$, and

$C = \{(x^3yz, xy^2z^2, y^5, z^5), (x^2yz^2, xy^3z, y^5, z^5)\}$.

Finally, for any $d \geq 1$ integer, let be $M(d) := \left\{ x^a y^b z^c \;\middle|\; \begin{array}{l} d-1 \geq a, b, c \geq 0 \\ a+b+c = d \end{array} \right\}$.

Now we can state the main theorem:

Theorem 4.0.2: *Let $I \subset k[x, y, z]$ be a minimal Togliatti system generated by 7 monomials of degree $d \geq 10$. Then, (up to a permutation of the variables) one of the following cases hold*

(1) There is $m \in M(d-2)$ such that

either (a) $I = (x^d, y^d, z^d) + m(x^2, y^2, xz, yz)$ *or* *(b)* $I = (x^d, y^d, z^d) + m(x^2, y^2, xy, z^2)$.

(2) *There is* $J \in A$ *such that* $I = (x^d, y^d, z^d) + x^{d-3}J$.

(3) *There is* $m \in M(d-3)$ *such that* $I = (x^d, y^d, z^d) + mT$.

(4) *There is* $J \in B$ *such that* $I = (x^d, y^d, z^d) + x^{d-4}J$.

(5) *There is* $J \in C$ *such that* $I = (x^d, y^d, z^d) + x^{d-5}J$.

Proof: It is easy to check that all of these ideals are minimal Togliatti systems. Let us prove the reciprocal. As usual, let us write $I = (x^d, y^d, z^d, m_1, m_2, m_3, m_4)$ where for $1 \leq i \leq 4$, $m_i = x^{a_i}y^{b_i}z^{c_i}$ with $a_i + b_i + c_i = d$. We consider $A_I \subset d\Delta_2 \cap \mathbb{Z}^3$ and we slice A_I with planes in three possible manners:

For $0 \leq j \leq 2$ and $0 \leq i \leq d$, we define $H_i^j := \{(t_0, t_1, t_2) | t_j = i\}$ and $A_I^{(i,j)} := A_I \cap H_i^j$. We will divide the proof in two cases:

Case 1: There exist $1 \leq i_a,\ i_b,\ i_c \leq 4$ such that $a_{i_a},\ b_{i_b},\ c_{i_c} \leq 1$.
Case 2: There exists one variable whose square divides all monomials m_i.

Case 1: None of the squares of the variables divide the four monomials m_1, m_2, m_3 and m_4. Up to permutation of the variables, we have two possibilities:

Case 1A: $I = (x^d, y^d, z^d, x^{e_1}y^a z^{d-a-e_1}, x^b y^{e_2} z^{d-b-e_2}, x^c y^{d-c-e_3} z^{e_3}, x^\alpha y^\beta z^{d-\alpha-\beta})$ with $0 \leq e_1, e_2, e_3 \leq 1$, $d-2-e_1 \geq a \geq 2$, $d-2-e_1 \geq a \geq 2$, $d-2-e_1 \geq a \geq 2$ and only one of the exponents $\alpha, \beta, d-\alpha-\beta$ is ≤ 1.

Case 1B: $I = (x^d, y^d, z^d, x^{e_1}y^{e_2}z^{d-e_1-e_2}, x^a y^{d-a-e_3} z^{e_3}, x^\alpha y^\beta z^{d-\alpha-\beta}, x^\gamma y^\delta z^{d-\gamma-\delta})$ with $0 \leq e_1, e_2, e_3 \leq 1$.

In both cases, a straightforward computation shows that when we restrict to $x+y+z$ the 7 monomials remain $k-$linearly independents. Therefore, I is not a Togliatti system.

Case 2: Without loss of generality we can suppose that x^2 divides each monomial m_i. We can also assume that $a_1 \geq a_2 \geq a_3 \geq a_4 = s \geq 2$.

First of all we will see that $a_3 = a_4 = s \geq 2$. In fact, since $s \geq 2$, F_{d-1} is a plane curve of degree $d-1$ containing all d points in $A_I^{(1,0)}$. So it factorizes as $F_{d-1} = L_1^0 F_{d-2}$. Since F_{d-2} contains all $d-1$ points of $A_I^{(0,0)}$ it factorizes as $F_{d-1} = L_0^0 L_1^0 F_{d-3}$. Repeating the process we arrive to $F_{d-1} = L_0^0 L_1^0 \cdots L_{s-1}^0 F_{d-s-1}$. Suppose $a_3 > a_4 = s$. Therefore, $A_I^{(s,0)}$ consist in $d-s$ points, and F_{d-s-1} must contain them. Hence we have $F_{d-s-1} = L_s^0 F_{d-s-2}$ which contradicts the minimality of I (Proposition 3.1.11).

Once seen this, we have that $F_{d-1} = L_0^0 \cdots L_{s-1}^0 F_{d-s-1}$ where F_{d-s-1} is a plane curve of degree $d-s-1$ which contains all integer points of $\tilde{A}_I := A_I \setminus \left(\cup_{k=0}^{s-1} A_I^{(k,0)}\right)$. Set $\tilde{A}_I^{(i,j)} = \tilde{A}_I \cap H_i^j$. We will distinguish four subcases:

Case 2A: $a_1 = a_2 = a_3 = a_4 =: s \geq 2$.
Case 2B: $u := a_1 > a_2 = a_3 = a_4 =: s \geq 2$.
Case 2C: $u := a_1 = a_2 > a_3 = a_4 =: s \geq 2$.

Case 2D: $u := a_1 > v := a_2 > a_3 = a_4 =: s \geq 2$.

Case 2A: We assume that $a_1 = a_2 = a_3 = a_4 =: s \geq 2$. In this case, it must be $F_{d-1} = L_0^0 \cdots L_{s-1}^0 L_{s+1}^0 \cdots L_{d-1}^0$. Therefore, $s = d-3$ and $I = (x^d, y^d, z^d) + x^{d-3}(y^3, y^2z, yz^2, z^3)$, which is of type (3).

Case 2B: We assume that $u := a_1 > a_2 = a_3 = a_4 =: s \geq 2$. In this case, either $u = s+1$ or $u = s+2$. Indeed, if $u > s+2$ we have, $F_{d-s-1} = L_{s+1} \cdots L_{u-1} F_{d-u}$ with F_{d-u} a plane curve of degree $d-u$ which contains in particular $A_I^{(s,0)}$. By minimality, $\#(F_{d-u} \cap A_I^{(s,0)}) = d-s-2 > d-u$ (Proposition 3.1.11) and we have $F_{d-u} = L_0^s F_{d-u-1}$, which is a contradiction. Then, up to permutation of variables, I is as one of the following cases:

Case b1: $u = s+1$ and $I = (x^d, y^d, z^d) + x^s(xy^a z^{d-a-s-1}, y^b z^{d-b-s}, y^c z^{d-c-s}, y^e z^{d-e-s})$.
Case b2: $u = s+2$ and $I = (x^d, y^d, z^d) + x^s(x^2 y^a z^{d-a-s-2}, y^b z^{d-b-s}, y^c z^{d-c-s}, y^e z^{d-e-s})$.

Case b1: In this case we are removing three points from H_s^0 and one from H_{s+1}^0. Up to permutation of the variables y and z, we can suppose that $d-s-1 \geq a \geq \lfloor \frac{d-s-1}{2} \rfloor$, and also we can assume that $d-s \geq b > c > e \geq 0$. Let us suppose first that $\lfloor \frac{d-s-1}{2} \rfloor > e \geq 0$.

In this case $\#(F_{d-s-1} \cap \tilde{A}_I^{(0,1)}) = \begin{cases} d-s & e \geq 1 \\ d-s-1 & e = 0 \end{cases}$

If $e \geq 1$, then $\#\tilde{A}_I^{(0,1)} = d-s$ and we have the factorization $F_{d-s-1} = L_0^1 F_{d-s-2}$. Inductively, we have $F_{d-s-1} = L_0^1 \cdots L_{e-1}^1 F_{d-s-e-1}$ and $F_{d-s-e-1}$ contains, in particular, the integer points of $\tilde{A}_I^{(e,1)}$. Since $a > e$ and $b > c > e$, we have that $\#\tilde{A}_I^{(e,1)} = d-s-e$ and it must be $F_{d-s-e-1} = L_e^1 F_{d-s-e-2}$ contradicting the minimality of I. Therefore it must be $e = 0$, and $m_4 = x^s z^{d-s}$ with $d-s-1 \geq c \geq 1$. Let us consider

$$\#(F_{d-s-1} \cap \tilde{A}_I^{(1,1)}) = \begin{cases} d-s & a, c \geq 2 \\ d-s-1 & a = 1,\ c \geq 2 \\ d-s-1 & a \geq 2,\ c = 1 \\ d-s-2 & a = c = 1 \end{cases}$$ and we will study the four possibilities.

If $a, c \geq 2$ then we have the factorization $F_{d-s-1} = L_1^1 F_{d-s-2}$. In particular F_{d-s-2} is a plane curve of degree $d-s-2$ containing the $d-s-1$ points of $\tilde{A}_I^{(0,1)}$, then we have the factorization $F_{d-s-2} = L_0^1 F_{d-s-3}$ which contradicts again the minimality of I. Therefore, if $a \geq 2$, then $c = 1$.

If $a = 1$, we have $d-2 \geq s \geq d-3$. If $s = d-2$, then $c = 1$ and $I = (x^d, y^d, z^d, x^{d-1}y, x^{d-2}y^2, x^{d-2}yz, x^{d-2}z^2)$ which is not a Togliatti system. Otherwise, $s = d-3$, then we have several possibilities:
(i) $c \geq 2$ and $I = (x^d, y^d, z^d) + x^{d-3}(y^3, z^3, xyz) + (x^{d-3}y^2z)$ which is not minimal.
(ii) $c = 1$ and either $I = (x^d, y^d, z^d) + x^{d-3}(xyz, y^3, z^3) + (x^{d-3}yz^2)$ or $I = (x^d, y^d, z^d) + x^{d-3}yz(x, y, z) + (x^{d-3}z^3)$. Both of them are not minimal.

So we can suppose $d-s-1 \geq a \geq 2$ and $s \leq d-3$. We have seen that $e = 0$ and $c = 1$. We have $(m_1, m_2, m_3, m_4) = (x^{s+1}y^a z^{d-a-s-1}, x^s y^b z^{d-b-s}, x^s yz^{d-s-1}, x^s z^{d-s})$, with $d-s \geq b \geq 2$. Let us consider

$$\#(F_{d-s-1}\cap \tilde{A}_I^{(0,2)}) = \begin{cases} d-s & d-s-1\geq b,\ d-s-2\geq a\\ d-s-1 & a=d-s-1,\ d-s-1\geq b\\ d-s-1 & b=d-s,\ d-s-2\geq a\\ d-s-2 & b=d-s,\ a=d-s-1\end{cases}$$

In the first case, we have the factorization $F_{d-s-1}=L_0^2F_{d-s-2}$. As we have seen before, this implies that $F_{d-s-2}=L_1^0F_{d-s-3}$ and once more it contradicts the minimality of I.

Now, if $a=d-s-1$ and $d-s-1\geq b$, then it must be $b=d-s-1$. Otherwise, we would have the factorization $F_{d-s-1}=L_1^2F_{d-s-2}$ and it would contradict the minimality of I. Therefore we have $I=(x^d,y^d,z^d)+x^s(xy^{d-s-1},y^{d-s-1}z,yz^{d-s-1},z^{d-s})$ with $s\leq d-3$. For $s=d-3$ it corresponds to a Togliatti system of type (2), while for $s\leq d-4$ is not Togliatti because when we restrict to $x+y+z=0$ the generators, they remain $k-$linearly independent.

In the third case, using the same argument, if $d-s-2\geq a$ and $b=d-s$, then it must be $a=d-s-2$. Hence we have $s\leq d-4$ and $I=(x^d,y^d,z^d)+x^s(xy^{d-s-2}z,y^{d-s},yz^{d-s-1},z^{d-s})$ which is never a Togliatti system.

Finally, in the last case we assume $b=d-s$ and $a=d-s-1$. Then $s\leq d-3$ and $I=(x^d,y^d,z^d)+x^s(xy^{d-s-1},y^{d-s},yz^{d-s-1},z^{d-s})$, which is never a Togliatti system.

To finish with the case b1, we have to see what happens when $d-s-2\geq e\geq\lfloor\frac{d-s-1}{2}\rfloor$. In this case $s\leq d-3$. Let us see that $a=e$. Otherwise, we can suppose $a>e$ (the other case is symmetric) and we have the factorization $F_{d-s-1}=L_0^1\cdots L_{e-1}^1F_{d-s-e-1}$. Since $a>e$ and $b>c>e$, $\tilde{A}_I^{(e,1)}$ has $d-s-e$ points and we have the factorization $F_{d-s-e-1}=L_e^1F_{d-s-e-2}$ which contradicts the minimality of I. Hence $a=e$ and in particular $d-s-1>a$ and $d-s\geq b>c>a$.

Let us consider $\tilde{\tilde{A}}_I:=\tilde{A}_I\setminus\left(\cup_{k=0}^{a-1}\right)$ in the same spirit as A_I and $\tilde{A}_I$. If $b=d-s$, then $\tilde{\tilde{A}}_I^{(0,2)}$ consists in $d-s-e$ different points. Otherwise, $d-s-1\geq b$, then $\#\tilde{\tilde{A}}_I^{(0,2)}=d-s+1-e$. In both cases $\tilde{\tilde{A}}_I^{(0,2)}$ have more points than the degree of $F_{d-s-e-1}$ which passes through them. Therefore we have the factorization $F_{d-s-e-1}=L_0^2F_{d-s-e-2}$ and, since $F_{d-s-e-2}$ contains all $d-s-e-1$ points of $\tilde{\tilde{A}}_I^{(e,1)}$ it factorizes as $F_{d-s-e-2}=L_e^1F_{d-s-e-3}$ contradicting the minimality of I.

CASE b2: We are removing from $d\Delta_2$ to get $\tilde{A}_I$: three points of H_s^0 and one from H_{s+2}^0. Up to permutation of the variables y and z, we can suppose that $d-s-2\geq a\geq\lfloor\frac{d-s-2}{2}\rfloor$, and also we can assume that $d-s\geq b>c>e\geq 0$.

Let us suppose first that $\lfloor\frac{d-s-2}{2}\rfloor>e\geq 0$. In this case we argue as in the case $u=s+1$ to prove that $e=0$. As we did above, let us consider $\#(F_{d-s-1}\cap\tilde{A}_I^{(1,1)})$. Using the same argumentation we can prove that if $a,c\geq 2$ we get a contradiction. If $a=1$, then either $s=d-3$ or $s=d-4$ and we have the following cases:
(i) If $s=d-3$, then (m_1,m_2,m_3,m_4) can be $x^{d-3}(x^2y,y^3,y^2z,z^3)$, $x^{d-3}(x^2y,y^3,yz^2,z^3)$ or $x^{d-3}(x^2y,y^2z,yz^2,z^3)$. And all of them become Togliatti systems of type (2).
(ii) If $s=d-4$, then (m_1,m_2,m_3,m_4) can be $x^{d-4}(x^2yz,y^4,y^3z,z^4)$, $x^{d-4}(x^2yz,y^4,y^2z^2,z^4)$, $x^{d-4}(x^2yz,y^4,yz^3,z^4)$, $x^{d-4}(x^2yz,y^3z,y^2z^2,z^4)$, $x^{d-4}(x^2yz,y^3z,yz^3,z^4)$ or $x^{d-4}(x^2yz,y^2z^2,yz^3,z^4)$. The only one of them which becomes a minimal Togliatti system is the second one and it corresponds to a Togliatti system of type (4).

Now, we can suppose that $e = 0$, $c = 1$ and $d - s - 2 \geq a \geq 2$. In particular, $s \leq d - 4$. As we did before, we can consider now $\#(F_{d-s-1} \cap \tilde{A}_I^{(0,2)})$, and see that if $d - s - 1 \geq b \geq 2$ and $d - s - 3 \geq a \geq 2$, there is a contradiction with the minimality of I. Also as we did before, if $b = d - s$ (resp. $a = d - s - 2$) then it must be $a = d - s - 3$ (resp. $b = d - s - 1$). Otherwise we would incur again to a contradiction with the minimality of I. So, we have three possibilities.

(i) $a = d - s - 3 \geq 2$, $b = d - s$, $s \leq d - 5$ and $I = (x^d, y^d, z^d) + x^s(x^2y^{d-s-3}z, y^{d-s}, yz^{d-s-1}, z^{d-s})$.

(ii) $a = d - s - 2$, $b = d - s - 1$, $s \leq d - 4$ and $I = (x^d, y^d, z^d) + x^s(x^2y^{d-s-2}, y^{d-s-1}z, yz^{d-s-1}, z^{d-s})$.

(iii) $a = d - s - 2$, $b = d - s$, $s \leq d - 4$ and $I = (x^d, y^d, z^d) + x^s(x^2y^{d-s-2}, y^{d-s}, yz^{d-s-1}, z^{d-s})$.

After restricting to $x + y + z = 0$, we see that none of them corresponds to a Togliatti system.

To finish with the case b2, we see what happens when $d - s - 2 \geq e \geq \lfloor \frac{d-s-2}{2} \rfloor$. With the same argument that we use before, we can see that $a = e$. The difference with the case $u = s + 1$ is that in this case we can have m_1 and m_2 aligned vertically. This condition can be translated as the case when $d - b - s = d - a - s - 2 \Leftrightarrow b = a + 2$. If this does not happen (i.e. $b > a + 2$), then it will contradict the minimality of I. Indeed: let us suppose that $0 \leq k := d - b - s < d - a - s - 2$. Inductively we have the factorization $F_{d-s-e-1} = L_0^2 \cdots L_{k-1}^2 F_{d-s-e-k-1}$. $F_{d-s-e-k-1}$ is a plane curve of degree $d - s - e - k - 1$ which passes through all $d - s - e - k$ points of $\tilde{\tilde{A}}_I^k$. Hence, we have the factorization $F_{d-s-e-k-1} = L_k^2 F_{d-s-e-k-2}$, contradicting the minimality assumption.

Therefore it must be $b = a + 2$ and, since $b > c > a$ we have $c = a + 1$. Finally we get: $I = (x^d, y^d, z^d) + x^s(x^2y^az^{d-a-s-2}, y^{a+2}z^{d-a-2-s}, y^{a+1}z^{d-a-1-s}, y^az^{d-a-s}) = (x^d, y^d, z^d) + x^sy^az^{d-a-s-2}(x^2, y^2, yz, z^2)$ which is of type (1).

Case 2C: We assume that $u := a_1 = a_2 > a_3 = a_4 =: s \geq 2$. Arguing as in case 2B we get $u = s + 1$ and $I = (x^d, y^d, z^d) + x^s(xy^az^{d-a-s-1}, xy^bz^{d-b-s-1}, y^cz^{d-c-s}, y^ez^{d-e-s})$. We can assume $d - s - 1 \geq a > b \geq \lfloor \frac{d-s-1}{2} \rfloor$ and $d - s \geq c > e \geq 0$. Let us suppose first that $\lfloor \frac{d-s-1}{2} \rfloor > e \geq 0$. Argumenting as in case 2B, it must be $e = 0$. Let us consider

$$\#(F_{d-s-1} \cap \tilde{A}_I^{(1,1)}) = \begin{cases} d-s & b, c \geq 2 & c1 \\ d-s-1 & c = 1, b \geq 2 & c2 \\ d-s-1 & c \geq 2, b = 1 & c3 \\ d-s-2 & c = b = 1 & c4 \end{cases}$$

Case c1: We have the factorization $F_{d-s-1} = L_1^1 F_{d-s-2}$ and then $F_{d-s-1} = L_0^1 L_1^1 F_{d-s-3}$ which, as usual, contradicts the minimality of I.

Case c3 and c4: Assume $b = 1$. Since $d - s - 2 \geq b \geq \lfloor \frac{d-s-1}{2} \rfloor$ it must be $s = d - 3$. Hence, $a = 2$, $b = 1$, $3 \geq c \geq 1$ and I have three possibilities: $(x^d, y^d, z^d) + x^{d-3}(xy^2, xyz, y^3, z^3)$, $(x^d, y^d, z^d) + x^{d-3}(xy^2, xyz, y^2z, z^3)$ and $(x^d, y^d, z^d) + x^{d-3}(xy^2, xyz, yz^2, z^3)$. The first one is not minimal while the remaining two are of type (2).

Case c2: Assume $c = 1$ and $s \leq d - 4$. Let us consider

$$\#(F_{d-s-1}\cap \tilde{A}_I^{(0,2)}) = \begin{cases} d-s & d-s-2\geq a\\ d-s-1 & a=d-s-1\end{cases}$$

In the first possibility, we have the factorization $F_{d-s-1} = L_0^2 F_{d-s-2}$ and then the factorization $F_{d-s-1} = L_0^2 L_0^1 F_{d-s-3}$ which contradicts the minimality of I. On the other hand, $a = d-s-1$ and, arguing in the same manner we can see that $b = d-s-2$. Therefore $I = (x^d, y^d, z^d) + x^s(xy^{d-s-1}, xy^{d-s-2}z, yz^{d-s-1}, z^{d-s})$ which is not Togliatti for any $s \leq d-4$.

To finish case 2C we see what happens when $e \geq \lfloor \frac{d-s-1}{2} \rfloor$. Arguing as in case 2B we can see that $b = e$. Now we have the factorization $F_{d-s-1} = L_0^0 \cdots L_{e-1}^0 F_{d-e-s-1}$ and we can consider $\tilde{\tilde{A}}_I$ as we did in case 2B. Let us consider

$$\#(F_{d-e-s-1}\cap \tilde{\tilde{A}}_I^{(0,2)}) = \begin{cases} d-e-s+1 & d-s-2\geq a, d-s-1\geq c\\ d-e-s & a=d-s-1, d-s-1\geq c\\ d-e-s & d-s-2\geq a, c=d-s\\ d-e-s-1 & a=d-s-1, c=d-s\end{cases}$$

To second and third cases means to a contradiction with the minimality of I directly. The last case means to a contradiction as we have seen earlier. Hence, the only viable possibility is the first one and we can apply recursively this argument until get $a = b+1$ and $c = e+2$. Therefore $I = (x^d, y^d, z^d) + x^s(xy^{b+1}z^{d-b-s-2}, xy^b z^{d-b-s-1}, y^{b+2}z^{d-s-b-2}, y^b z^{d-s-b})$ which is of type (1).

Case 2D: We assume that $u := a_1 > v := a_2 > a_3 = a_4 =: s \geq 2$. Recall that we have the factorization $F_{d-1} = L_0^0 L_1^0 \cdots L_{s-1}^0 F_{d-s-1}$. If $v > s+1$, then $\#\tilde{A}_I^{(s+1,0)} = d-s$. Since F_{d-s-1} is a plane curve containing $\tilde{A}_I^{(s+1,0)}$, it factorizes as $F_{d-s-1} = L_{s+1}^0 F_{d-s-2} = L_{s+1}^0 L_s^0 F_{d-s-3}$, which contradicts the minimality of I. Therefore $v = s+1$ and we can write $I = (x^d, y^d, z^d) + x^s(x^r y^a z^{d-a-s-r}, xy^b z^{d-b-s}, y^c z^{d-c-s}, y^e z^{d-e-s})$ where $u = s+r$ with $d-s-1 \geq r \geq 2$. As before, we can assume also that $d-s-1 \geq b \geq \lfloor \frac{d-s-1}{2} \rfloor$ and $d-s \geq c > e \geq 0$, and we have that $d-s-r \geq a \geq 0$ and $s \leq d-3$ Let us suppose first that $\lfloor \frac{d-s-1}{2} \rfloor > e \geq 0$, and we can consider

$$\#(F_{d-s-1}\cap A_I^{(0,1)}) = \begin{cases} d-s & e\geq 1, a\geq 1 & (d1)\\ d-s-1 & e=0, a\geq 1 & (d2)\\ d-s-1 & e\geq 1, a=0 & (d3)\\ d-s-2 & e=a=0 & (d4)\end{cases}$$

Case d1: In this case it must be $a = e$. Indeed, if we suppose $a > e \geq 1$ (the other case is symmetric) we have the factorization $F_{d-s-1} = L_0^1 \ldots L_{e-1}^1 F_{d-s-e-1}$, and $\#(F_{d-s-e-1} \cap \tilde{A}_I^{(e,1)}) = d-s-e$. Then, $F_{d-s-e-1} = L_e^1 F_{d-s-e-2}$ which contradicts the minimality of I. Let us now consider

$$\#(F_{d-s-e-1}\cap \tilde{A}_I^{(e+1,1)}) = \begin{cases} d-s-e & b\geq e+2, c\geq e+2 & (i)\\ d-s-e-1 & b=e+1, c\geq e+2 & (ii)\\ d-s-e-1 & b\geq e+2, c=e+1 & (iii)\\ d-s-e-2 & b=c=e+1 & (iv)\end{cases}$$

Case (i). We have the factorization $F_{d-s-e-1} = L_{e+1}^1 F_{d-s-e-2}$, and since $F_{d-s-e-2}$ passes through all $d-s-e-1$ points of $\tilde{A}_I^{(e,1)}$ we get a contradiction with the minimality of I.

Case (ii). We assume $b = e+1$. Let us consider $\tilde{\tilde{A}}_I$ as we did before and we can examine $\#(F_{d-s-e-1} \cap \tilde{\tilde{A}}_I^{(0,2)}) = \begin{cases} d-s-e+1 & d-s-1 \geq c, 1 \leq d-s-e-r \\ d-s-e & c = d-s, 1 \leq d-s-e-r \\ d-s-e & d-s-1 \geq c, d-s-e-r = 0 \\ d-s-e-1 & c = d-s, d-s-e-r = 0 \end{cases}$

In the second and third possibilities we obtain directly a contradiction with the minimality of I. In the last possibility we also obtain a contradiction. In fact, if $c = d-s$ and $s+r = d-e$, we do not remove any point of H_1^2 and we have $\#(F_{d-s-e-1} \cap \tilde{\tilde{A}}_I^{(1,2)}) = d-s-e$. Then $F_{d-s-e-1} = L_1^2 F_{d-s-e-2}$, and this implies $F_{d-s-e-2} = L_0^2 F_{d-s-e-3}$, which contradicts the minimality of I.

Therefore if $b = e+1$, it must be $d-s-1 \geq c \geq e+2$ and $1 \leq d-s-e-r$. Iterating this argument we can conclude that either $c = e+2$ and $r = 2$ or $c = e+3$ and $r = 3$. Therefore, either $I = (x^d, y^d, z^d) + x^s(x^2y^ez^{d-s-e-2}, xy^{e+1}z^{d-s-e-2}, y^{e+2}z^{d-s-e-2}, y^ez^{d-s-e}) = (x^d, y^d, z^d) + x^sy^ez^{d-s-e-2}(x^2, xy, y^2, z^2)$ which is of type (1); or $I = (x^d, y^d, z^d) + x^s(x^3y^ez^{d-s-e-3}, xy^{e+1}z^{d-s-e-2}, y^{e+3}z^{d-s-e-3}, y^ez^{d-s-e}) = (x^d, y^d, z^d) + x^sy^ez^{d-s-e-3}(x^3, xyz, y^3, z^3)$ which is of type (3).

Case (iii). Arguing as in case (ii) we get that $b = e+2$ and $r = 2$. Therefore, $I = (x^d, y^d, z^d) + x^s(x^2y^ez^{d-s-e-2}, xy^{e+2}z^{d-s-e-2}, y^{e+1}z^{d-s-e-1}, y^ez^{d-s-e}) = (x^d, y^d, z^d) + x^sy^ez^{d-s-e-2}(x^2, yx, y^2, z^2)$, which is of type (1).

Case (iv). Arguing as in case (ii) we get that $r = 2$ and I is of type (1).

Case d2: In this case we assume $e = 0$ and $a \geq 1$. We will separate the case $b = 1$ from the case $b \geq 2$.

Let us assume $b = 1 \geq \lfloor \frac{d-s-1}{2} \rfloor$ which implies $s = d-3$ and $r = 2$. Hence, I can be $(x^d, y^d, z^d) + x^{d-3}(xyz, y^3, z^3) + (x^{d-3}x^2y)$, $(x^d, y^d, z^d) + x^{d-3}(x^2y, xyz, y^2z, z^3)$ or $(x^d, y^d, z^d) + x^{d-3}(x^2y, xyz, yz^2, z^3)$. The first one is not minimal and the remaining two are of type (2).

Now we can assume $b \geq 2$. Let us suppose first $d-s-r-1 \geq 0$ (i.e. $m_1 \notin H_0^2$) and we consider

$$\#(F_{d-s-1} \cap \tilde{A}_I^{(1,1)}) = \begin{cases} d-s & a \geq 2, c \geq 2 & (i) \\ d-s-1 & a = 1, c \geq 2 & (ii) \\ d-s-1 & a \geq 2, c = 1 & (iii) \\ d-s-2 & a = c = 1 & (iv) \end{cases}$$

Case (i). We get the factorization $F_{d-s-1} = L_1^1 F_{d-s-2}$, and hence the factorization $F_{d-s-1} = L_0^1 L_1^1 F_{d-s-3}$ which contradicts the minimality of I.

Case (ii). Assume that $a = 1$ and $c \geq 2$. Suppose that $d-s-r-1 > 0$ and let us consider

$$\#(F_{d-s-1} \cap \tilde{A}_I^{(0,2)}) = \begin{cases} d-s & d-s-2 \geq b, d-s-1 \geq c \\ d-s-1 & b = d-s-1, d-s-1 \geq c \\ d-s-1 & d-s-2 \geq b, c = d-s \\ d-s-2 & b = d-s-1, c = d-s \end{cases}$$

The first possibility means to a contradiction with the minimality of I.

Now let us suppose that $d-s-r-1 > 1$. In this case, the second (resp. the third) possibility can occur if, and only if $b = c = d-s-1$ (resp. $b = d-s-2$ and $c = d-s$). Therefore I can only be as one of the next types:

$I = (x^d, y^d, z^d) + x^s(x^ryz^{d-s-r-1}, xy^{d-s-1}, y^{d-s-1}z, z^{d-s})$, which does not correspond to a Togliatti system.

$I = (x^d, y^d, z^d) + x^s(x^r yz^{d-s-r-1}, xy^{d-s-2}z, y^{d-s}, z^{d-s})$, which is a Togliatti system if, and only if $r = 2$ and $s = d-5$ (resp. of type (5)).

If $d-s-r-1 = 1$, then there are no special restrictions for the second and third case. Therefore I is one of the next types:

$I = (x^d, y^d, z^d) + x^s(x^{d-s-2}yz, xy^{d-s-1}, y^c z^{d-s-c}, z^{d-s})$, which is a Togliatti system if, and only if $s = d-4$ and $c = 3$ (of type (4)).

$I = (x^d, y^d, z^d) + x^s(x^{d-s-2}yz, xy^b z^{d-s-b-1}, y^{d-s}, z^{d-s})$, which is a Togliatti system if, and only if $s = d-5$ and $b = 2$ (of type (5)).

Finally, the last case gives us $I = (x^d, y^d, z^d) + x^s(x^r yz^{d-s-r-1}, xy^{d-s-1}, y^{d-s}, z^{d-s})$, which is a Togliatti system if, and only if $r = 2$ and $s = d-3$.

Now, let us suppose that $d-s-r-1 = 0$. Arguing as usual, we can see it cannot be $d-s-2 \geq c$ and $d-s-3 \geq b$. Therefore $d-s \geq c \geq d-s-1$ or $d-s-1 \geq b \geq d-s-2$, and we have the following possibilities:

$b = d-s-2$, $d-s-2 \geq c$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^{d-s-2}z, y^c z^{d-s-c}, z^{d-s})$. Which is a Togliatti system if, and only if $s = d-3$ and $c = d-s-2$ (of type (3)).

$d-s-3 \geq b$, $c = d-s-1$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^b z^{d-s-b}, y^{d-s-1}z, z^{d-s})$. Which is a Togliatti system if, and only if $d-3 \geq s \geq d-4$ and $b = d-s-3$ (resp. of type (3) and (4)).

$b = d-s-2$, $c = d-s-1$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^{d-s-2}z, y^{d-s-1}z, z^{d-s})$. Which is a Togliatti system if, and only if $s = d-3$ (of type (3)).

$b = d-s-1$, $d-s-1 \geq c$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^{d-s-1}, y^c z^{d-s-c}, z^{d-s})$. Which is a Togliatti system if, and only if $s = d-3$ and $d-s-1 \geq c \geq d-s-2$ (of type (3)).

$d-s-2 \geq b$, $c = d-s$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^b z^{d-s-b}, y^{d-s}, z^{d-s})$. Which is a Togliatti system if, and only if $s = d-3$ and $d-s-2 \geq b \geq d-s-3$ (of type (3)).

$b = d-s-1$, $c = d-s$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}y, xy^{d-s-1}, y^{d-s}, z^{d-s})$. Which is a Togliatti system if, and only if $s = d-3$.

Case (iii). Assume $c = 1$ and $a \geq 2$, and let us consider as before $\#(F_{d-s-1} \cap \tilde{A}_I^{(0,2)})$. In this case we obtain that I can only be one of the next types:

$I = (x^d, y^d, z^d) + x^s(x^r y^{d-s-r-1}z, xy^{d-s-1}, yz^{d-s-1}, z^{d-s})$, which is a Togliatti system if, and only if $r = 2$ and $d-3 \geq s \geq d-4$ (of type (3)).

$I = (x^d, y^d, z^d) + x^s(x^r y^{d-s-r}, xy^{d-s-2}z, yz^{d-s-1}, z^{d-s})$, which is a Togliatti system if, and only if $r = 2$ and $s = d-3$ (of type (3)).

$I = (x^d, y^d, z^d) + x^s(x^r y^{d-s-r}, xy^{d-s-1}, yz^{d-s-1}, z^{d-s})$. In this case, let us consider $\#(F_{d-s-1} \cap \tilde{A}_I^{(1,2)}) = d-s$ and inductively we obtain $F_{d-s-1} = L_1^2 \cdots L_{d-s-2}^1 F_1$. Therefore it must be $s = d-3$ and I is of type (3).

Case (iv). Assume $a = c = 1$. Let us suppose first that $d-s-r-1 > 0$. If $d-s-2 \geq b$ we can factorize F_{d-s-1} as $F_{d-s-1} = L_0^2 F_{d-s-2}$. As we have seen before, this contradicts the minimality of I. Therefore, $b = d-s-1$ and we can factorize $F_{d-s-1} = L_1^2 \cdots L_{d-s-r-2} F_{r+1}$. Since $\#(F_r \cap \tilde{A}_I^{(0,2)}) = d-s-1$ we have $r+1 \geq d-s-1$ and then $d-s-r-1 \leq 1$. Therefore, $d-s-r-1 = 1$ and $I = (x^d, y^d, z^d) + x^s(x^{d-s-2}yz, xy^{d-s-1}, yz^{d-s-1}, z^{d-s})$. Which is a Togliatti system if, and only if $d-3 \geq s \geq d-4$.

Now, if $d-s-r-1=0$, then we can use the same argumentation to prove that $d-s-1 \geq b \geq d-s-2$ and then we have two possibilities:
$b=d-s-1$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}y,xy^{d-s-1},yz^{d-s-1},z^{d-s})$
$b=d-s-2$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}y,xy^{d-s-2}z,yz^{d-s-1},z^{d-s})$
Which are Togliatti systems if, and only if $s=d-3$ (of type (3)).

Case d3: Let us assume $e \geq 1$ and $a=0$. Actually, it must be $e=1$. Otherwise, $e>1$ and $\#(F_{d-s-1} \cap \tilde{A}_I^{(1,1)})=d-s$, and we have seen that this cannot happen.

Now, let us suppose $d-s-r>1$. Arguing as before we can see that there are three possibilities:
$b=d-s-1$, $c=d-s-1$ and $I=(x^d,y^d,z^d)+x^s(x^rz^{d-s-r},xy^{d-s-1},\ y^{d-s-1}z,yz^{d-s-1})$.
$b=d-s-2$, $c=d-s$ and $I=(x^d,y^d,z^d)+x^s(x^rz^{d-s-r},xy^{d-s-2}z,\ y^{d-s},yz^{d-s-1})$.
Which do not correspond to a Togliatti system.
$b=d-s-1$, $c=d-s$. If $d-2>s+r$, then we have the factorization $F_{d-s-1}=L_1^2\cdots L_{d-s-r-1}F_r$ and $\#(F_r \cap \tilde{A}_I^{(0,2)})=d-s-2>r$, which contradicts the minimality of I. Hence we have $s+r=d-2$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-2}z^2,xy^{d-s-1},y^{d-s},yz^{d-s-1})$.
Which are Togliatti system if, and only if $s=d-3$ (of type (3)).

To finish, assume $d-s-r=1$. Arguing in the same manner, we can see that it cannot occur $d-s-3 \geq b$ and $d-s-2 \geq c$. Therefore we have the following possibilities:
$b=d-s-2$, $d-s-2 \geq c$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^{d-s-2}z,y^cz^{d-s-c},yz^{d-s-1})$.
Which is a Togliatti system if, and only if $s=d-3$ and $c=d-s-3$ (of type (3)).
$d-s-3 \geq b$, $c=d-s-1$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^bz^{d-s-b-1},y^{d-s-1}z,yz^{d-s-1})$.
Which is a Togliatti system if, and only if $s=d-3$ and $b=d-s-3$ (of type (3)).
$b=d-s-2$, $c=d-s-1$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^{d-s-2}z,y^{d-s-1}z,yz^{d-s-1})$.
Which is a Togliatti system if, and only if $s=d-3$ (of type (3)).
$b=d-s-1$, $d-s-1 \geq c$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^{d-s-1},y^cz^{d-s-c},yz^{d-s-1})$.
Which is a Togliatti system if, and only if $s=d-3$ and $c=d-s-1$ (of type (3)).
$d-s-2 \geq b$, $c=d-s$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^bz^{d-s-b-1},y^{d-s},yz^{d-s-1})$.
Which is a Togliatti system if, and only if $s=d-3$ and $d-s-2 \geq b \geq d-s-3$ (of type (3)), or $s=d-4$ and $b=d-s-2$.
$b=d-s-1$, $c=d-s$ and $I=(x^d,y^d,z^d)+x^s(x^{d-s-1}z,xy^{d-s-2}z,y^{d-s},yz^{d-s-1})$. Which is a Togliatti system if, and only if $s=d-3$.

Case d4: Let us assume $e=a=0$. If $b \geq 3$ and $c \geq 3$, then we have the factorization $F_{d-s-1}=L_1^1L_2^1F_{d-s-3}$ and $\#(F_{d-s-3} \cap \tilde{A}_I^{(0,1)})=d-s-2$, which means to a contradiction with the minimality of I. Hence we distinguish three cases: $b=1$, $b=2$ and $b \geq 3$.

Case (i). We assume $b=1$. Since $b \geq \lfloor \frac{d-s-1}{2} \rfloor$ it must be $s=d-3$. Therefore I has three possibilities: $(x^d,y^d,z^d)+x^{d-3}(xyz,y^3,z^3)+(x^{d-1}z)$, $(x^d,y^d,z^d)+x^{d-3}(x^2z,xyz,y^2z,z^3)$ and $(x^d,y^d,z^d)+x^{d-3}(x^2z,xyz,yz^2,z^3)$. The first one is not minimal while the remaining two are of type (1).

Case (ii). We assume $b=2$. Since $b \geq \lfloor \frac{d-s-1}{2} \rfloor$ it must be $d-3 \geq s \geq d-5$.

If $s=d-3$, I has three possibilities: $(x^d,y^d,z^d)+x^{d-3}(x^2z,xy^2,y^3,z^3)$, $(x^d,y^d,z^d)+x^{d-3}(x^2z,xy^2,y^2z,z^3)$ and $(x^d,y^d,z^d)+x^{d-3}(x^2z,xy^2,yz^2,z^3)$. All of them are of type (2).

If $s=d-4$, I has eight possibilities: $(x^d,y^d,z^d)+x^{d-4}(x^3z,xy^2z,y^4,z^4)$, $(x^d,y^d,z^d)+x^{d-4}(x^3z,xy^2z,y^3z,z^4)$, $(x^d,y^d,z^d)+x^{d-4}(x^3z,xy^2z,y^2z^2,z^4)$, $(x^d,y^d,z^d)+x^{d-4}(x^3z,xy^2z,$

$yz^3, z^4)$, $(x^d, y^d, z^d) + x^{d-4}(x^2z^2, xy^2z, y^4, z^4)$, $(x^d, y^d, z^d) + x^{d-4}(x^2z^2, xy^2z, y^3z, z^4)$, $(x^d, y^d, z^d) + x^{d-4}(x^2z^2, xy^2z, y^2z^2, z^4)$, $(x^d, y^d, z^d) + x^{d-4}(x^2z^2, xy^2z, yz^3, z^4)$. And only the fifth one is a minimal Togliatti system, and it is of type (4).

Finally, if $s = d-5$, I has 15 possibilities, but any of them is a minimal Togliatti system.

Case (iii). We assume $b \geq 3$. Then, either $c = 1$ or $c = 2$.

Case $c = 1$. We will see that it must be $b = d-s-1$. Suppose that $d-s-2 \geq b \geq 3$, then $\#(F_{d-s-1} \cap \tilde{A}_I^{(0,2)}) = d-s$ and we have the factorization $F_{d-s-1} = L_0^2 F_{d-s-2}$. First we will see that this implies that m_1 and m_2 are aligned vertically (i.e. $d-s-b-1 = d-s-r \Leftrightarrow b = r-1$). We can suppose that $d-s-b-1 \leq d-s-r$, and then $b \geq r-1$ (the other case is symmetric). Inductively we obtain that F_{d-s-1} factorizes as $F_{d-s-1} = L_0^2 L_1^2 \cdots L_{d-s-b-2}^2 F_b$. If $b > r-1$, then $\#\tilde{A}_I^{(d-s-b-1,2)} = b+1$ and it would mean to a contradiction with the minimality of I. Hence, $b = r-1$ and we can obtain the factorization $F_{d-s-1} = L_0^2 L_1^2 \cdots L_{d-s-b-2}^2 L_{d-s-b}^2 \cdots L_{d-s-2} F_1$. Since $\#\tilde{A}_I^{(d-s-b-1,2)} = b \geq 3$ we have a contradiction with the minimality again.

Once we have seen that $b = d-s-1$, using the usual argumentation we see that $d-s-r = 1 \Leftrightarrow r = d-s-1$. Therefore $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}z, xy^{d-s-1}, yz^{d-s-1}, z^{d-s})$ with $s \leq d-3$, which is Togliatti if, and only if $s = d-3$ and it is of type (3).

Case $c = 2$. Since $\#\tilde{A}_I^{(1,1)} = d-s$ we have the factorization $F_{d-s-1} = L_1^1 F_{d-s-2}$. If $d-s-2 \geq b \geq 3$, then $\#\tilde{\tilde{A}}_I^{(0,2)} = d-s-1$ and F_{d-s-1} would factorize as $F_{d-s-1} = L_1^1 L_0^2 F_{d-s-3}$. This contradicts the minimality of I because $\#\tilde{\tilde{A}}_I^{(0,1)} = d-s-2$ which would force the factorization $F_{d-s-1} = L_1^1 L_0^2 L_0^1 F_{d-s-4}$. Therefore $b = d-s-1$ and by minimality again we can see that $d-s-r = 1$. Hence, $I = (x^d, y^d, z^d) + x^s(x^{d-s-1}z, xy^{d-s-1}, y^2z^{d-s-2}, z^{d-s})$, which is Togliatti if, and only if $s = d-3$ and in this case it is of type (3)

To finish case 2D we see what happens when $d-s \geq c > e \geq \lfloor \frac{d-s-1}{2} \rfloor$. We can see using the minimality that either $a \geq b = e$, $b \geq a = e$ or $e \geq a = b$.

Arguing as before we can see that in the first possibility m_1 and m_3 must be vertically aligned and in particular $c = e+2$, $a = e$ and $r = 2$. Therefore $I = (x^d, y^d, z^d) + x^s(x^2y^ez^{d-s-e-2}, xy^ez^{d-s-e-1}, y^{e+2}z^{d-s-e-2}, y^ez^{d-s-e}) = (x^d, y^d, z^d) + x^sy^ez^{d-s-e-2}(x^2, xy, y^2, z^2)$ which is of type (1).

In the second possibility, we assume $b \geq a = e$. If $b, c \geq e+1$, then we have the factorization $F_{d-s-e-1} = L_{e+1}^1 F_{d-s-e-2}$ and, since $\#\tilde{\tilde{A}}_I^{(e,1)} = d-s-e-1$ we get $F_{d-s-e-1} = L_e^1 L_{e+1}^1 F_{d-s-e-3}$ which contradicts the minimality. Now, suppose $b = e+1$ and $c \geq e+2$ (resp. $b \geq e+2$ and $c = e+1$). As we have seen earlier, m_1 and m_3 (resp. m_2) must be aligned. Therefore, we can see using the minimality assumption that $r = 2$ and $c = e+2$ (resp. $r = 2$ and $b = e+2$). In both cases I is of type (1).

Finally, let us assume that $e \geq a = b$. If $e \geq a+2$, then we will get a contradiction with the minimality of I. Hence either $e = a$ or $e = a+1$. If $e = a$ we can see that it has to be $c = a+2$ and $r = 2$. Therefore I is of type (1). Otherwise $e = a+1$ and we can see that $c = a+2$ and $r = 2$. □

For any integer $d \geq 3$ set $M^0(d) = \{x_0^a x_1^b x_2^c | a+b+c = d \text{ and } a,b,c \geq 1\}$.

Theorem 4.0.3: *Let $I \subset k[x_0, \ldots, x_n]$ be a smooth minimal monomial Togliatti system of forms of degree $d \geq 10$. Assume that $\mu(I) = 2n+3$. Then $n = 2$ and, up to permutation of the coordinates, one of the following cases holds:*

(1) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^2, x_1^2, x_0x_2, x_1x_2)$ with $m \in M^0(d-2)$.

(2) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^2, x_1^2, x_0x_1, x_2^2)$ with $m \in M^0(d-2)$.

(3) $I = (x_0^d, x_1^d, x_2^d) + m(x_0^3, x_1^3, x_2^3, x_0x_1, x_2)$ with $m \in M^0(d-3)$.

Proof: By [8]; Proposition 4.1, for $n \geq 3$ and $d \geq 4$ there are no *smooth* minimal monomials Togliatti systems $I \subset k[x_0, \ldots, x_n]$ of forms of degree d with $\mu(I) = 2n+3$. So, $n = 2$. For $n = 2$, the results follows from Theorem 4.0.2 together with the smoothness criterion (Proposition 3.1.8). $\square$

Remark 4.0.4: For $6 \leq d \leq 9$ there are other examples of minimal monomial Togliatti systems $I = (x^d, y^d, z^d) + J \subset k[x,y,z]$ than those given in Theorem 4.0.2. This additional J's were computed with the help of Macaulay2 and we give the full list of these extra possible $J's$:

$\underline{d=6:}$ $(x^5y, x^3z^3, x^2y^3z, y^5z), (x^5z, x^3y^3, x^2y^2z^2, y^5z), (x^3z^3, x^2y^4, x^2y^2z^2, y^5z), (x^5z, x^3y^3, xyz^4, y^5z), (x^4z^2, x^3y^3, x^2y^2z^2, y^4z^2), (x^3z^3, x^2y^4, x^2y^2z^2, y^4z^2), (x^4z^2, x^3y^3, xyz^4, y^4z^2), (x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3), xy(x^4, x^2y^2, xyz^2, y^4), xy(x^3z, x^2y^2, xyz^2, y^3z), xy(x^2y^2, x^2z^2, xyz^2, y^2z^2), xy(x^2y^2, x^2z^2, xz^3, y^2z^2), xy(x^4, xz^3, y^4, y^2z^2), xy(x^4, x^2y^2, y^4, z^4), xy(x^4, xyz^2, y^4, z^4), xy(x^3z, x^2y^2, y^3z, z^4), xz(x^3z, x^2z^2, xyz^2, y^4)xz(x^2yz, x^2z^2, xyz^2, y^4), xz(x^3z, xy^2z, xyz^2, y^4), xz(x^3z, x^2yz, xz^3, y^4), xz(x^3z, x^2z^2, xz^3, y^4), xz(x^3z, xy^2z, xz^3, y^4), xz(x^2y^2, x^2z^2, xy^3, y^3z), xz(x^2y^2, x^2z^2, xy^2z, y^3z), xz(x^2z^2, xy^3, xy^2z, y^3z), xz(x^2y^2, x^2z^2, xy^3, y^4), xz(x^2y^2, x^2z^2, y^4, y^3z), xz(x^2y^2, x^2z^2, y^4, y^2z^2), xz(x^3z, x^2yz, xy^2z, y^4), xz(x^3z, x^2yz, xyz^2, y^4), x(xy^4, xyz^3, xz^4, y^3z^2), x(x^4z, x^2y^3, xy^2z^2, y^5), x(x^4z, xyz^3, y^5, y^3z^2), x(x^2z^3, xy^4, xy^2z^2, y^5), x(x^4z, x^2yz^2, y^5, y^2z^3), x(x^2z^3, xy^4, xyz^3, y^3z^2), x(x^4z, x^2z^3, xy^3z, y^5), x(x^4z, x^2y^3, y^5, yz^4), x(x^3z^2, x^2y^3, xz^4, y^3z^2), x(x^4z, xy^2z^2, y^5, yz^4), x(x^2z^3, xy^4, xz^4, y^3z^2), x(x^2y^3, x^2z^3, y^4z, yz^4), x(x^4y, x^2z^3, xy^3z, y^5), x(x^2yz^2, xy^3z, y^5, z^5)$

$\underline{d=7:}$ $xy(x^2z^3, xy^4, xy^2z^2, y^5), xy(x^5, x^2y^2z, xyz^3, y^5), xy(x^4y, x^3y^2, xz^4, y^3z^2), xy(x^3y^2, x^2y^3, x^2z^3, y^2z^3), xy(x^5, x^2y^2z, y^5, z^5), xy(x^4z, xy^4, y^5, z^5), xy(x^5, xyz^3, y^5, z^5), xz(x^3z^2, x^2z^3, xy^3z, y^5), xz(x^4z, x^2yz^2, xz^4, y^5), xz(x^4z, xy^3z, xz^4, y^5), x(x^5z, x^2y^3z, xy^2z^3, y^6), x(xy^5, xy^2z^3, xz^5, y^4z^2), x(x^5z, x^4y^2, x^2y^2z^2, y^3z^3), x(x^4y^2, x^4z^2, x^2y^2z^2, y^3z^3), x(x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3), x(x^4yz, x^2y^4, x^2z^4, y^3z^3), x(x^2y^4, x^2y^2z^2, x^2z^4, y^3z^3), x(x^4yz, xy^5, xz^5, y^3z^3), x(x^2y^2z^2, xy^5, xz^5, y^3z^3), x(x^5z, x^2y^3z, y^6, yz^5), x(x^5z, xy^2z^3, y^6, yz^5), x(x^5z, x^4y^2, y^5z, yz^5), x(x^4y^2, x^4z^2, y^5z, yz^5), x(x^3y^3, x^3z^3, y^5z, yz^5), x(x^4yz, x^2y^2z^2, y^6, z^6), x(x^4yz, y^6, y^3z^3, z^6), x(x^2y^2z^2, y^6, y^3z^3, z^6), xyz(x^2y^2, x^2z^2, xy^3, y^4), xyz(x^3z, x^2yz, xy^2z, y^4), xyz(x^4, x^2y^2, xyz^2, y^4), xyz(x^3z, x^2yz, xyz^2, y^4), xyz(x^3z, x^2z^2, xyz^2, y^4), xyz(x^2yz, x^2z^2, xyz^2, y^4), xyz(x^3z, xy^2z, xyz^2, y^4), xyz(x^3z, x^2yz, xz^3, y^4), xyz(x^3z, x^2z^2, xz^3, y^4), xyz(x^3y, xy^3, xz^3, y^4), xyz(x^3z, xy^2z, xz^3, y^4), xyz(x^2y^2, x^2z^2, xy^3, y^3z), xyz(x^2y^2, x^2z^2, xy^2z, y^3z), xyz(x^2z^2, xy^3, xy^2z, y^3z), xyz(x^3z, x^2y^2, xyz^2, y^3z), xyz(x^3y, x^2z^2, xyz^2, y^3z), xyz(x^2y^2, x^2z^2, y^4, y^3z), xyz(x^4, xy^3, xz^3, y^2z^2), xyz(x^4, xz^3, y^4, yz^3)$.

$\underline{d = 8:}\ xy(x^4z^2, x^3y^3, xyz^4, y^4z^2), xz(x^3z^3, x^2y^2z^2, xy^4z, y^6)$

$\underline{d = 9:}\ xyz(x^3z^3, x^2y^2z^2, xy^4z, y^6), xyz(x^3y^3, x^3z^3, x^2y^2z^2, y^3z^3), xyz(x^6, x^2y^2z^2, y^6, z^6).$

Appendix A

Scripts of Macaulay2

The first script is the implementation of Lemma 2.0.4:

```
 --Does I fail the SLP because of multiplication of l^d
 --from degree m to degree m-1?
 DoesFailLefschetzDegrees=(I,l,d,m)->(
     M=max(hilbertFunction(m+d,I)-hilbertFunction(m,I),0);
     if hilbertFunction(m+d,I+ideal(l^d))>M then 1 else 0
)
```

The next script is the set of functions that can be used to find all Monomial Togliatti systems of degree d in n variables up to permutation.

```
NotInPermIdeal=(J,I,n)->(
     Perm=permutations(toList(x_0..x_n));
     NumPerm=length(Perm);
     LengthJ=length J;
     if LengthJ==0 then 1 else(
          Ban=1;
          k0=LengthJ-1;
          while ((k0>-1)and(Ban==1)) do(
               l0=0;
               Ban1=1;
               while((l0<NumPerm)and(Ban1==1)) do(
                    RepList={};
                    for j0 from 0 to n do(
                         RepList = append(RepList,x_j0=>Perm_l0_j0);
                    );
                    Aux=sub(I,RepList);
                    if (isSubset(J_k0, Aux)) then (Ban1=0);
                    l0=l0+1;
                    if Ban1==0 then Ban=0;
               );
```

```
            k0=k0-1;
        );
        if Ban==1 then 1 else 0;
    );
)

MonomialList=(n,deg)->(
    rrd={};
    MM=basis (deg, R);
    N=numcols MM-1;
    for k from 0 to N do(rrd=append(rrd,MM_k_0));
    for k from 0 to n do(rrd=delete(x_k^deg,rrd));
    rrd
)

MonTogl=(n,deg)->(
    R=kk[x_0..x_n];
    rrdeg=MonomialList(n,deg);
    poli=x_1;
    for k from 2 to n do(poli=poli+x_k);
    N=binomial(n+deg-1,n-1)-n;
    A=ideal(x_0^deg);
    for k from 1 to n do(A=A+ideal(x_k^deg));
    j=n;
    --;
    --Optionally add next line:;
    --JJ={};
    --(See comment below);
    --;
    J={};
    --;
    --We can change j<N by j<n+r if we;
    --want to restrict $\mu(I)\leq 2n+r$ and add;
    --to JJ these $I$ with $\mu(I)=2n+r$;
    --(See comment below);
    --;
    while(j<N) do(
        s4=subsets(rrdeg,j);
        Lengths4=(length s4);
        i=0;
        while(i<Lengths4) do(
            Ii=ideal(s4_i);
            H=A+Ii;
            if(numcols (mingens(sub(H,x_0=>poli)))<n+j+1) then (
                if NotInPermIdeal(J,H,n)==1 then (
```

```
                    J=append(J,H);
                    --;
                    --To contruct JJ as we said above,;
                    --introduce the next line:;
                    -- if(j==n+r) then JJ=append(JJ,i);
                    --;
                );
            );
            i=i+1;
        );
        j=j+1;
    );
)

--;
--This final function substract the gcd of each $(m_1,...,m_{n+r})$ in JJ;
--;
H={};
i=0;
while(i<length(JJ)) do(
    k=JJ_i;
    Aux=gcd(s4_k);
    Laux={};
    j=0;
    while(j<length(s4_k)) do(
        Laux=append(Laux,s4_k_j/Aux);
        j=j+1;
    );
    H=append(H,Aux);
    H=append(H,Laux);
    i=i+1;
)
toString(H)
```

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