Dynamic Games in Transboundary Pollution Problems: An Air Quality Approach

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Abstract

We address a problem of transboundary pollution from the perspective of air quality, which enables us to modelize the welfare function of the countries as a product of the air quality stock and their production. In this setting we analyse and compare Markovian optimal policies under cooperation and under Stackelberg and Nash competition, studying also the cases where one of the countries does not value the stock of air quality as an important component to increase the welfare of its population. We show that in the Markovian Stackelberg equilibria the evolution of the air quality is better and that, regardless of their roles, all the countries achieve a higher value of discounted welfare than in the Markov Perfect Nash equilibria. The possibility of the leader of select the optimal strategy knowing how the opponent will react works against him, leading him to obtain levels of welfare lower than those obtained by the follower.

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1 Introduction

In Europe, emissions of many air pollutants have decreased substantially over the past decades, resulting in improved air quality across the region. However, air pollutant concentrations are still too high, and air quality problems persist. A significant proportion of European population lives in areas where air quality standards are not reached: ozone, nitrogen dioxide and particulate matter (PM) pollution pose serious health risks. Air quality enters into the social utility function with a positive marginal utility. The important thing about air quality as a social problem is that we frequently are not capable of exercising direct control over the desired level. Planning becomes imperative. By ignoring the problem or by overreacting to it, enormous quantities of resources are wasted.

From a pollution control point of view, air quality is known for being a common resource pool that may suffer from *the tragedy of the commons*, i.e., is a resource threatened by excessive exploitation, partly because of lack of cooperation among the agents that have common access to it, because pollutants can move across countries. Consequently, the private exploitation of the air quality induces inter-temporal production spillovers that prevent the maximization of value derived from a high air quality level.

Within the framework of differential games, Nash and Stackelberg equilibria can be defined in many different ways, depending on the assumptions imposed on the information available to the players. By computing a nondegenerate Markovian equilibirum one makes the assumption that the state variable can be observed and that the players condition their actions on these observations. On the other hand, if a player uses an open-loop strategy, he either cannot observe the state variable or he chooses to commit to a fixed time function. Regrading time-consistency and subgame perfectness, different informational assumptions lead to different properties and implications.

To interpret the property of time-consistency assume that, in equilibrium, each player determines his action (his control) at every time following his optimal strategy, which depends on the state variable he observes at that time. Although the control value may (and usually does) change at time evolves, the rule he follows does not change. Now assume that at some time, the players are allowed to reconsider their strategy choices. At that time the players find themselves at the beginning of a subgame where the state has changed due to the equilibrium strategies. If the initial strategies constitutes an equilibrium for the subgame, then the original equilibrium is time-consistent. Every Markovian Nash equilibrium of a differential game is time-consistent. Time consistency could be seen as a minimal requirement for the credibility of an equilibrium strategy. If one player had an incentive to deviate from his strategy, then, the other players would not believe his announcement in first place. Consequently, they would compute their own strategies by taking into account the expected future deviation of that player, which, in general, would lead to different strategies. In contrast to time consistency, subgame perfectness not only requires that the initial strategies constitute an equilibrium for the subgame where the state trajectory is generated by the equilibrium strategies, but that it is an equilibrium for all subgames. By construction, subgame perfectness of an equilibrium implies its time-consistency.

In the previous literature, the studies that characterize and contrast noncooperative and cooperative strategies in transboundary pollution problems (TPP) usually follow the model introduced by Van der Ploeg and de Zeeuw (1992) where the world is made by N identical countries engaged in production activities that generate pollution $S(t)$ which accumulates over time according to the dynamics

$$\dot{S}(t) = \frac{\alpha}{N}\sum_{i=1}^{n} y_i - \delta S(t), \qquad S(0) = 0,$$

where $y_i(t)$ is the production rate of country i, $i = 1, ..., N$, at time t, $S(t)$ is the stock of pollution, $\delta \geq 0$ is the constant decay rate of the stock of pollution and $\alpha > 0$ denotes the emission-output ratio. Country i obtains profits from production, measured by a concave function $B(y_i)$, and incurs a damage cost $D(S)$ due to pollution, $D'(S) > 0, \quad D''(S) \geq 0$. Each player i (i.e., the government of each country i) maximizes the welfare function

$$W_i = \int_0^\infty e^{-r_i t}(B(y_i(t)) - D(S(t)))dt,$$

subject to the pollution stock dynamics,where $r > 0$ is the social rate of discount. With this setting, the authors analyse and compare the strategies under international coordination with open-loop and feedback Nash equilibrium, finding that production and emission levels are lower in the cooperative case, and open-loop equilibrium leads to lower pollution than in feedback strategies. Other authors as Long (1992), also analyse an open-loop Stackelberg equilibrium with the drawback that the solution is in general time inconsistent, and only make sense if the leader can make a binding commitment. In the other hand, Dockner and Long (1993) analyse feedback Nash equilibria along with the cooperative solution but assuming a linear-quadratic structure of the differential game.

The open-loop Stackelberg equilibrium is in general time-inconsistent, thus it is not a plausible equilibrium in situations where the agents cannot credibly commit to a fixed control path. One of our main contributions is to obtain a nondegenerate Markovian Stackelberg equilibrium, which is time-consistent. The analysis of such equilibrium in a differential game usually leads to considerable technical difficulties and this is the reason why this type of equilibrium is seldom analysed. The other main contribution is to provide a different point of view to analyse transboundary pollution problems in differential games that can lead to a new type of characterizations.

The paper is organized as follows. In Section 2, we present our model and we derive the nondegenerate Markovian Stackelberg Equilibrium, the Markov Perfect Nash Equilibrium and the Social optimum. Section 3 is devoted to analyse and compare the different equilibria from the perspective of the countries. Finally, in Section 4, we conclude. All the equilibria derivations are relegated to an appendix.

2 The Model

In contrast with the previous authors, we analyse the problem of pollution by the point of view of the quality of the environment, specifically the air quality. As in the previous literature, the production activities damage the air quality. Air quality have a positive effect on population welfare while air pollutants released in one country may be transported in the atmosphere, contributing to poor air quality elsewhere. Thus, we have a welfare function for each country that depends on production and air quality $F_i(y_i, Q)$ where $Q \geq 0$ denote the air quality level and $y_i > 0$ represents each countries i production[1]. We also assume that there exist a non-

[1]We do not consider explicit bounds in the controls or state. While this setting is more realistic, because of technical difficulties we leave it for future work.

linear decreasing relation between the pollution stock and the stock of air quality. Each country maximizes the welfare in an infinite time horizon

$$W_i = \int_0^\infty e^{-r_i t} F_i(y_i, Q) dt$$

where the discount rate $r_i > 0$ for all country i, the use of air quality as the state variable instead of pollution enables us to introduce the product between air quality and production in the welfare function. Then, we use a function similar to that used by Cornes et al.(2001), $F_i(y_i, Q) = n_i(y_i(t)\beta_i Q)^\alpha$ where $n_i > 0$ and $\beta_i > 0$ (for all countries i) are parameters that represent the size and the different relation between the production and the air quality level in the welfare function respectively for each country and with $0 < \alpha < 1/2$, this assumption implies that the marginal social welfare of the production, weighted with the air quality level, is always positive. It allows us to derive the solutions for our Markov equilibria. Indeed Long and Shinomura (1998) showed that objective functions that are homogenneous are relevant when looking for Markov equilibria. The air quality dynamics is negatively affected by the production of each country and have a positive recuperation rate which is greater for high levels of air quality and lower for small stock levels, fact that reflects the warnings made by ecologists that the regeneration of high polluted environments is less efficient,

$$\dot{Q}(t) = \sum_{i=1}^{N} -n_i y_i + kQ(t),$$

where $k < 0$ is the recuperation rate.

Model 1: When Air Quality Always Matters

In this paper we focus on the case with two players $N = 2$, and we compare the different results in nondegenerate Markovian Stackelberg equilibrium (MSE), Markov perfect Nash equilibrium (MPNE) and the Social optimum (or cooperative equilibrium), hence the game we will analyse is

$$\max_{y_1} W_1 = \int_0^\infty e^{-r_1 t} n_1 (y_1(t)\beta_1 Q)^\alpha dt, \tag{1}$$

$$\max_{y_2} W_2 = \int_0^\infty e^{-r_2 t} n_2 (y_2(t)\beta_2 Q)^\gamma dt, \tag{2}$$

$$s.t. \quad \dot{Q}(t) = -n_1 y_1 - n_2 y_2 + kQ(t), \qquad Q(0) = Q_0. \tag{3}$$

2.1 Nondegenerate Markovian Stackelberg Equilibrium

We consider the case where player 2 is the leader and can announce to the follower the policy rule she (player 2) will use throughout the game. The follower, taking this rule as given, seeks to maximize his own welfare and finds an optimal policy that will depend on the leader's policy. Then the leader, knowing the follower's reaction function chooses, among all possible decisions, the one that maximizes her welfare. One way to solve this problem is to restrict the functional form of the strategies among which she can choose. In our case, we restrict the possible leader's optimal strategy to be a linear function [2] of the state variable of the form $\phi_2(Q) = bQ$ with

[2] Although the restriction to linear strategies could be a severe assumption, it can be justified from an economic point of view because of being an easy rule to implement by policymakers.

$b > 0$. Hence, the follower's problem (player 1) is

$$\max_{y_1} W_1 = \int_0^\infty e^{-r_1 t} n_1 (y_1(t)\beta_1 Q)^\alpha dt,$$

$$s.t. \quad \dot{Q}(t) = -n_1 y_1 - n_2\phi_2(Q) + kQ(t), \qquad Q(0) = Q_0.$$

For his problem the Hamilton-Jacobi-Bellman equation is

$$r_1 V_1(Q) = \max_{y_1} \left\{ n_1 (y_1 \beta_1 Q)^\alpha + V_1'(Q)(-n_1 y_1 - n_2 \phi_2(Q) + kQ) \right\}. \tag{4}$$

This yields the optimal follower's production function depending on the air quality stock and on the policy announced by the leader, i.e., the reaction function of the follower

$$y_1(b) = \frac{r_1 + 2\alpha\,(n_2 b - k)}{2\,(1-\alpha)\,n_1} \cdot Q \tag{5}$$

We can undo the announce of the leader $\phi_2(Q) = bQ$ and rewrite that reaction function as a function of the leader's optimal production

$$y_1(b) = \frac{r_1 - 2\alpha k}{2\,(1-\alpha)\,n_1} \cdot Q + \frac{2\alpha n_2}{2(1-\alpha)n_1} \cdot bQ = \frac{r_1 - 2\alpha k}{2\,(1-\alpha)\,n_1} \cdot Q + \frac{\alpha n_2}{(1-\alpha)n_1} \cdot \phi_2(Q)$$

The leader adopts the information about the follower's reaction function and maximize her welfare by choosing her optimal production $y_2(t)$, therefore her problem becomes

$$\max_{y_2} W_2 = \int_0^\infty e^{-r_2 t} n_1 \left(y_2(t)\beta_2 Q\right)^\gamma dt$$

$$s.t. \qquad \dot{Q}(t) = -n_1 \cdot \left(\frac{r_1 - 2\alpha k}{2\,(1-\alpha)\,n_1} \cdot Q(t) + \frac{\alpha n_2}{(1-\alpha)n_1} \cdot y_2 \right) - n_2 y_2 + kQ(t), \qquad Q(0) = Q_0.$$

Solving it by the HJB equation

$$r_2 V_2(Q) = \max_{y_2} \left\{ n_2 (y_2 \beta_2 Q)^\gamma + V_2'(Q) \left[-\frac{r_1 - 2k}{2(1-\alpha)} \cdot Q - \frac{n_2}{(1-\alpha)} \cdot y_2 \right] \right\}, \tag{6}$$

yields

$$y_2(Q) = \frac{r_2(1-\alpha) + r_1\gamma - k2\gamma}{2(1-\gamma)n_2} \cdot Q \equiv \phi_2^*(Q) \tag{7}$$

and then, since $y_2(t) = bQ$, this implies that

$$b = \frac{r_2(1-\alpha) + r_1\gamma - 2\gamma k}{2(1-\gamma)n_2}.$$

Hence, substituting it in the reaction function we obtain the follower's optimal strategy as a linear function of the stock of air quality

$$y_1(Q) = \frac{r_1(1-\gamma+\gamma\alpha) + r_2\alpha(1-\alpha) - k2\alpha}{2(1-\gamma)(1-\alpha)n_1} \cdot Q \tag{8}$$

We can also obtain the value function for each player

$$V_1^*(Q) = \left(\frac{\beta_1}{2}\right)^\alpha \left(\frac{r_1(1-\gamma+\gamma\alpha) + r_2\alpha(1-\alpha) - 2\alpha k}{(1-\gamma)(1-\alpha)n_1} \right)^{\alpha-1} \cdot Q^{2\alpha} \tag{9}$$

$$V_2^*(Q) = \left(\frac{\beta_2(1-\alpha)}{2}\right)^{\gamma} \left(\frac{r_2(1-\alpha)+\gamma r_1 - 2\gamma k}{(1-\alpha)(1-\gamma)n_2}\right)^{\gamma-1} \cdot Q^{2\gamma} \tag{10}$$

and the resulting time path of the state variable is

$$Q(t) = Q_0 \cdot \exp\left[\theta_1^S \cdot t\right]$$

where

$$\theta_1^S = \frac{2k - r_1 - r_2(1-\alpha)}{2(1-\gamma)(1-\alpha)} \tag{11}$$

so that, depending on the recuperation rate and on the discount rates, we can have three different scenarios.

- $k > \frac{r_1+r_2(1-\alpha)}{2}$: Air quality grows continuously.
- $k = \frac{r_1+r_2(1-\alpha)}{2}$: Air quality remains constant at the initial level Q_0.
- $k < \frac{r_1+r_2(1-\alpha)}{2}$: Air quality decreases to along time.

2.2 Markov Perfect Nash Equilibrium

Now we analyse the case in which none of the players has an advantage, thus both countries decide their optimal strategies $\phi_1^*(Q)$ and $\phi_2^*(Q)$ at the same time taking the other player's strategy as given to maximize (1) and (2) with the restriction (3). In order to obtain the optimal strategies we solve the system of HJB equations for each player

$$r_1 V_1(Q) = \max_{y_1} \left\{ n_1 (y_1 \beta_1 Q)^{\alpha} + V_1'(Q)\left[-n_1 y_1 - n_2 \phi_2(Q) + kQ\right]\right\}, \tag{12}$$

$$r_2 V_2(Q) = \max_{y_2} \left\{ n_2 (y_2 \beta_2 Q)^{\gamma} + V_2'(Q)\left[-n_1 \phi_1(Q) - n_2 y_2 + kQ\right]\right\} \tag{13}$$

that yields

$$y_1(Q) = \frac{r_1\,(1-\gamma) + r_2\alpha - k2\alpha}{2\,(1-\gamma-\alpha)\,n_1} \cdot Q \equiv \phi_1^*(Q), \tag{14}$$

$$y_2(Q) = \frac{r_2\,(1-\alpha) + r_1\gamma - k2\gamma}{2\,(1-\gamma-\alpha)\,n_2} \cdot Q \equiv \phi_2^*(Q), \tag{15}$$

the value functions for the players

$$V_1^*(Q) = \left(\frac{\beta_1}{2}\right)^{\alpha} \left(\frac{r_1(1-\gamma)+r_2\alpha - 2\alpha k}{(1-\gamma-\alpha)n_1}\right)^{\alpha-1} \cdot Q^{2\alpha}, \tag{16}$$

$$V_2^*(Q) = \left(\frac{\beta_2}{2}\right)^{\gamma} \left(\frac{r_2(1-\alpha)+\gamma r_1 - 2\gamma k}{(1-\gamma-\alpha)n_2}\right)^{\gamma-1} \cdot Q^{2\gamma}, \tag{17}$$

and the optimal path of Q is

$$Q(t) = Q_0 \cdot \exp\left[\theta_1^N \cdot t\right],$$

where

$$\theta_1^N = \frac{2k - r_1 - r_2}{2\,(1-\gamma-\alpha)}. \tag{18}$$

As in the previous case, the air quality depends on the parameters α, γ, k, r_1 and r_2, but notice that for all possible values of the parameters, $\theta_1^S > \theta_1^N$, i.e., air quality is higher under MPNE competition than in the MSE equilibrium. In some way, the fact that the leader can choose their strategy anticipating what the other country will do, make the leader to internalize through the follower's reaction function the indirect negative effect of her production on the air quality ,i.e., as we can see in the dynamics of the air quality in the MSE, the leader's production effect on air quality is higher than when countries only care about their direct pollution. Then, instead of the leader reduce her production knowing that in this way, the follower will do the same, they compete aggressively increasing the production trying to benefit from the good air quality as long as it remains unpolluted.

2.3 Social Optimum

We assume in this section that there is a social planner who wants to maximize the discounted sum of the social welfare of the two countries. This approach is the same than assuming some type of agreement to cooperate and choose their optimal strategies jointly. Thus, the planner chooses $\phi_1^*(Q)$ and $\phi_2^*(Q)$ to maximize

$$W = W_1 + W_2.$$

In order to solve analytically this problem we need to reduce the degree of asymmetry assuming that $\gamma = \alpha$ and that both players have the same discount rate $r_1 = r_2 = r$, but the rest of countries' asymmetry still exist. In the case where $r_1 \neq r_2$, we would have a problem of time inconsistency as showed by De-Paz, Marín-Solano and Navas (2013). Then the problem to solve is to maximize

$$W = \int_0^\infty e^{-rt} \left[n_1 (y_1(t)\beta_1 Q)^\alpha) + n_2 (y_2(t)\beta_2 Q)^\gamma \right] dt,$$

subject to the equation (3). We use the HJB equation for the joint problem

$$rV(E) = \max_{y_1(t),y_2(t)} \{ n_1 (y_1(t)\beta_1 Q)^\alpha + n_2 (y_2(t)\beta_2 Q)^\gamma + \\ + V'(E) \left[-n_1 y_1(t) - n_2 y_2(t) + kQ \right] \}, \tag{19}$$

that leads to the equilibrium where the optimal strategies as a linear function of the stock are

$$y_1(Q) = \frac{\beta_1^{\frac{\alpha}{\alpha-1}}}{2} \cdot \frac{r - 2\alpha k}{(1-\alpha)\left(\beta_1^{\frac{\alpha}{\alpha-1}} n_1 + \beta_2^{\frac{\alpha}{\alpha-1}} n_2\right)} \cdot Q, \tag{20}$$

$$y_2(Q) = \frac{\beta_2^{\frac{\alpha}{\alpha-1}}}{2} \cdot \frac{r - 2\alpha k}{(1-\alpha)\left(\beta_1^{\frac{\alpha}{\alpha-1}} n_1 + \beta_2^{\frac{\alpha}{\alpha-1}} n_2\right)} \cdot Q, \tag{21}$$

the value function of the coalition

$$V^*(Q) = \left(\frac{\beta_1 \beta_2}{2}\right)^\alpha \left(\frac{r - 2\alpha k}{(1-\alpha)\left(\beta_1^{\frac{\alpha}{\alpha-1}} n_1 + \beta_2^{\frac{\alpha}{\alpha-1}} n_2\right)} \right)^{\alpha-1} \cdot Q^{2\alpha}, \tag{22}$$

and the time path of the state variable is

$$Q(t) = Q_0 \exp\left[\theta_1^C \cdot t\right],$$

where

$$\theta_1^C = \frac{2k - r}{2(1-\alpha)}. \tag{23}$$

We can only compare this result with the ones obtained previously by doing the same assumptions we made at the beginning of this analysis, i.e., $r_1 = r_2 = r$ and $\gamma = \alpha$, by doing so, we see that $\theta_1^C > \theta_1^S > \theta_1^N$, hence under the same conditions (same parameter's values) air quality evolves better under cooperation than in competition in which case the MSE is better than the MPNE. In cooperation countries internalize their production externalities and contaminate less, benefiting both by the increase in air quality that increase their social welfare.

Model 2: When A Country Neglects The Air Quality Problem

In this section we go a step further, now we want to analyse the case in which one of the countries does not value the air quality as an important component of their population's welfare, nonetheless his production function is affected by the stock of air quality because even in the case that the government does not value the pollution adverse effects on population health, for low air quality levels these health problems reduce the production. Then in the model we will use in this section the players wish to maximize their respective social welfare

$$\max_{y_1} W_1 = \int_0^\infty e^{-r_1 t} n_1 (y_1 \beta_1 Q)^\alpha dt, \tag{24}$$

$$\max_{y_2} W_2 = \int_0^\infty e^{-r_2 t} n_2 (y_2)^\gamma dt, \tag{25}$$

subject to the same equation (3) we used in the previous model. In the following sections we begin with the MSE, notice that due to the asymmetry in the valuation or not of the stock of air quality in the social welfare, we have to calculate separately the two cases of MSE depending on which country is the leader and which one is the follower and we finish with the MPNE. Because of the asymmetry introduced in this model we cannot obtain analytically the social optimum strategies, but we can still use the social optimum solution of model 1 as a benchmark to compare the different equilibriums.

2.4 Nondegenerate Markovian Stackelberg Equilibrium: The Leader Neglects

The first MSE we analyse is the case in which the country that values the air quality (player 1) is the follower. As we commented before, although player 2 does not value directly the air quality stock, it affects her production and therefore when she has to choose among all possible strategies, the optimal one will necessarily depend on the air quality stock. In this sense, as in the previous model, we assume that the leader (player 2) announces her strategy as a linear function of the state variable $\phi_2(Q) = bQ$ with $b > 0$. Since the announce made by the leader and the problem of the follower is the same than in model 1, the follower's reaction function is equal to (5), the reaction function calculated in the MSE of the previous model. And then the leader's problem becomes

$$\max_{y_2} W_2 = \int_0^\infty e^{-r_2 t} n_2 (y_2(t))^\gamma dt,$$

$$s.t. \qquad \dot{Q(t)} = -n_1 \cdot \left(\frac{r_1 - 2\alpha k}{2\,(1-\alpha)\,n_1} \cdot Q(t) + \frac{\alpha n_2}{(1-\alpha)n_1} \cdot y_2 \right) - n_2 y_2 + kQ(t), \qquad Q(0) = Q_0.$$

This problem is similar to the one solved for model 1, the only difference is that now the objective function does not depend explicitly on the air quality. We solve this problem following the same approach used in the model 1, i.e., by the HJB equation

$$r_2 V_2(Q) = \max_{y_2} \{ n_2 (y_2)^\gamma + V_2'(Q) \left[- \frac{r_1 - 2k}{2(1-\alpha)} \cdot Q - \frac{n_2}{(1-\alpha)} \cdot y_2 \right] \}, \tag{26}$$

that yields

$$y_2(Q) = \frac{2r_2(1-\alpha) + r_1\gamma - k2\gamma}{2(1-\gamma)n_2} \cdot Q, \tag{27}$$

and it implies

$$b = \frac{2r_2(1-\alpha) + r_1\gamma - 2\gamma k}{2(1-\gamma)n_2},$$

therefore the optimal strategy for the follower is

$$y_1(Q) = \frac{r_1(1-\gamma+\gamma\alpha) + 2r_2\alpha(1-\alpha) - 2k\alpha}{2(1-\gamma)(1-\alpha)n_1} \cdot Q, \tag{28}$$

the value functions are

$$V_1^*(Q) = \left(\frac{\beta_1}{2} \right)^\alpha \left(\frac{r_1(1-\gamma+\gamma\alpha) + 2r_2\alpha(1-\alpha) - 2\alpha k}{(1-\gamma)(1-\alpha)n_1} \right)^{\alpha-1} \cdot Q^{2\alpha}, \tag{29}$$

$$V_2^*(Q) = (1-\alpha)^\gamma \left(\frac{2r_2(1-\alpha) + \gamma r_1 - 2\gamma k}{2(1-\alpha)(1-\gamma)n_2} \right)^{\gamma-1} \cdot Q^\gamma, \tag{30}$$

and the resulting time path of the air quality stock is

$$Q(t) = Q_0 \cdot \exp\left[\theta_2^{S_1} \cdot t \right],$$

where

$$\theta_2^{S_1} = \frac{2k - r_1 - 2r_2(1-\alpha)}{2(1-\gamma)(1-\alpha)}. \tag{31}$$

Comparing that result with (11),(18) and (23) we see that the evolution of air quality is always worse than in the different scenarios analysed for model 1. The follower react in the same way to the decision of the leader but in this case although the leader knows that her production decision has a higher impact in the air quality evolution than in the previous Nash model, she does not value directly the effect of the air quality on the population's welfare and therefore initially produce more than in the previous equilibriums, but this implies that the pollution is also higher and the air quality evolution is worse.

2.5 Nondegenerate Markovian Stackelberg Equilibrium: The Leader Worries

In this section we analyse how the previous equilibrium is modified when the roles are changed, now player 1 is the leader while player 2 is the follower. As in previous MSE analysis we assume that the announce that the leader can do is restricted to linear functions of the state variable $\phi_1(Q) = bQ$ with $b > 0$. Hence the follower problem is to maximize (25) subject to the restriction

$$\dot{Q}(t) = -n_1\phi_1(Q) - n_2 y_2 + kQ(t),$$

whose HJB equation is

$$r_2V_2(Q) = \max_{y_2(t)} \left\{n_2(y_2(t))^\gamma + V_2'(Q)\left[-n_1\phi_1(Q) - n_2y_2(t) + kQ\right]\right\}, \tag{32}$$

that yields the optimal strategy for player 2 as a function of the announce made by the leader, i.e., the reaction function of the follower is

$$y_2(b) = \frac{r_2 + \gamma(n_1b - k)}{(1-\gamma)n_2} \cdot Q. \tag{33}$$

As in the previous MSE, we can transform the follower's reaction function into a function that depends on the leader's production

$$y_2(b) = \frac{r_2 - \gamma k}{(1-\gamma)n_2} \cdot Q + \frac{\gamma n_1}{(1-\gamma)n_2} \cdot bQ = \frac{r_2 - \gamma k}{(1-\gamma)n_2} \cdot Q + \frac{\gamma n_1}{(1-\gamma)n_2} \cdot \phi_1(Q).$$

Hence, the Leader's problem (player 1) becomes

$$\max_{y_1} W_1 = \int_0^\infty e^{-r_1t}\left(y_1(t)\beta_1Q\right)^\alpha dt,$$

$$s.t. \quad \dot{Q}(t) = -n_1y_1 - n_2\left[\frac{r_2 - \gamma k}{(1-\gamma)n_2} \cdot Q(t) + \frac{\gamma n_1}{(1-\gamma)n_2} \cdot y_1\right] + kQ(t).$$

The HJB for this problem is

$$r_1V_1(Q) = \max_{y_1} \left\{n_1(y_1\beta_1Q)^\alpha + V_1'(Q)\left[-\frac{r_2 - k}{1-\gamma} - \frac{n_1}{1-\gamma} \cdot y_1\right]\right\}, \tag{34}$$

that yields

$$y_1(Q) = \frac{r_1(1-\gamma) + 2r_2\alpha - 2\alpha k}{2(1-\alpha)n_1} \cdot Q, \tag{35}$$

and it implies

$$b = \frac{r_1(1-\gamma) + 2r_2\alpha - 2\alpha k}{2(1-\alpha)n_1},$$

which, introduced in the reaction function of the follower gives us his optimal strategy

$$y_2(Q) = \frac{2r_2(1-\alpha+\alpha\gamma) + r_1\gamma(1-\gamma) - 2\gamma k}{2(1-\alpha)(1-\gamma)n_2} \cdot Q, \tag{36}$$

the value functions

$$V_1^*(Q) = \left(\frac{\beta_1(1-\alpha)}{2}\right)^\alpha \left(\frac{r_1(1-\gamma) + 2r_2\alpha - 2\alpha k}{(1-\gamma)(1-\alpha)n_1}\right)^{\alpha-1} \cdot Q^{2\alpha}, \tag{37}$$

$$V_2^*(Q) = \left(\frac{2r_2(1-\alpha+\alpha\gamma) + r_1\gamma(1-\gamma) - 2\gamma k}{2(1-\alpha)(1-\gamma)n_2}\right)^{\gamma-1} \cdot Q^\gamma, \tag{38}$$

and the optimal time path of the air quality stock

$$Q(t) = Q_0 \cdot \exp\left[\theta_2^{S_1} \cdot t\right],$$

where

$$\theta_2^{S_2} = \frac{2k - r_1(1-\gamma) - 2r_2}{2(1-\alpha)(1-\gamma)}. \tag{39}$$

In this case the roles from the preceding equilibrium have changed, and the evolution of the air quality is even worse, $\theta_2^{S_2} < \theta_2^{S_1}$, as long as $\gamma < 2\alpha$, which we assume is satisfied, otherwise the asymmetry between the countries would be very strong and it is a case that we will not analyse. In this Equilibrium, since player 2 does not value directly the stock of air quality, her reaction function is higher than when player 1 was the follower. Then the leader, that values the air quality, reduce his production knowing that the follower will do the same, but he is no able to offset the effect produced by the fact that the follower does not value directly the air quality and the result is that the initial joint production is higher and the evolution of the state variable is worse.

2.6 Markov Perfect Nash Equilibrium

We consider now the case in which both countries are in equal footing. Then to obtain the MPNE we have to solve the system of HJB equations

$$r_1 V_1(Q) = \max_{y_1} \left\{ n_1 (y_1 \beta_1 Q)^\alpha + V_1'(Q) \left[-n_1 y_1 - n_2 \phi_2(Q) + kQ \right] \right\}, \tag{40}$$

$$r_2 V(Q) = \max_{y_2} \left\{ n_2 (y_2)^\gamma + V_2'(Q) \left[-n_1 \phi_1(Q) - n_2 y_2 + kQ \right] \right\}, \tag{41}$$

that yields

$$y_1(Q) = \frac{r_1(1-\gamma) + 2r_2\alpha - 2\alpha k}{2(1-\alpha-\gamma)n_1} \cdot Q \equiv \phi_1^*(Q), \tag{42}$$

$$y_2(Q) = \frac{r_2(1-\alpha) + \frac{\gamma}{2} r_1 - \gamma k}{(1-\alpha-\gamma)n_2} \cdot Q \equiv \phi_2^*(Q), \tag{43}$$

the value functions

$$V_1^*(Q) = \left(\frac{\beta_1}{2}\right)^\alpha \left(\frac{r_1(1-\gamma) + 2r_2\alpha - 2\alpha k}{(1-\gamma-\alpha)n_1}\right)^{\alpha-1} \cdot Q^{2\alpha}, \tag{44}$$

$$V_2^*(Q) = \left(\frac{2r_2(1-\alpha) + \gamma r_1 - 2\gamma k}{2(1-\gamma-\alpha)n_2}\right)^{\gamma-1} \cdot Q^\gamma, \tag{45}$$

and the dynamic of the state variable

$$Q(t) = Q_0 \cdot \exp\left[\theta_2^N \cdot t\right],$$

where

$$\theta_2^N = \frac{2k - r_1 - 2r_2}{2(1-\alpha-\gamma)}. \tag{46}$$

In this last equilibrium the evolution of the state variable is the worst of all the equilibriums analysed, as in the first MPNE, both countries compete aggressively by increasing their initial productions and in addition, as player 2 does not value directly the air quality stock, her initial production is even higher and then induces a faster pollution, and therefore a worse quality evolution.

3 Results

Until now we have focused in how the stock of air quality evolves depending on the type of competition, and the concern or not about the air quality of the countries. Now we will analyse and compare the different components of players' behaviour with respect to stock of air quality, their production and their value functions, i.e., the integral of their discounted utility in equilibrium. Recall that the air quality changes along time in different ways depending on the strategies adopted by the countries, therefore in this section we analyse and compare the different reactions of the players to a given stock of air quality. For simplicity we assume a high degree of symmetry between the players, and to obtain the following graphical representations we set $r_1 = r_2 = r = 0.05$, $k = 0.05$, $n_1 = n_2 = 1$, $\beta_1 = \beta_2 = 1$ and $\gamma = \alpha = 0.25$.

3.1 Model 1: When Air Quality Always Matters

In the first model all countries value the air quality stock as an important component of their population's welfare. In the initial analysis we saw that, from the air quality perspective, the cooperative equilibrium Pareto dominates the other non-cooperative equilibria analysed among which the MSE dominates the MPNE. Now we compare the results of these equilibriums from the point of view of the countries.

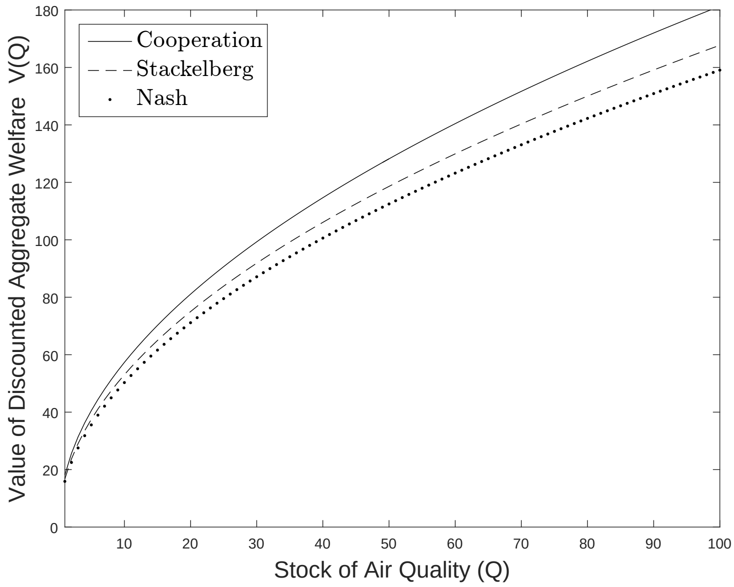

Figure 1: Model 1: Aggregate Value

Figure 1 shows that for larger values of the air quality stock, the aggregate welfare is higher. It also shows that, for whatever level of air quality, the value of discounted aggregate welfare is always higher under cooperation than in the MSE, which Pareto dominates the outcome of the MPNE. Considering that, in cooperation, the two countries decide as if they were one, and they internalize completely the externalities of their productions in terms of air quality loss. In the case of the MSE, the leader internalize part of her externality by taking into account the reaction function of the follower. And in the case of the MPNE, each country only cares about the negative externality of his production on his own welfare. We reach the conclusion that the more the agents care about the negative effect that their decisions have on the other country, the higher is the aggregate welfare. Jointly the two countries are better off when one of them is the leader and the other is the follower than if both compete in equal footing.

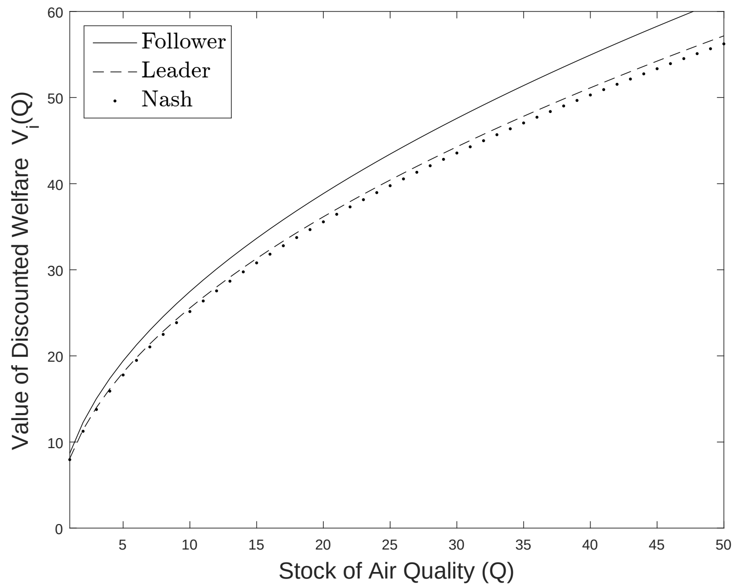

Figure 2: Model 1: Individual Value

In addition, Figure 2 shows that individually both are also better off in the MSE than in the MPNE regardless of their roles. Although the welfare they obtain for a given level of air quality is different depending on the roles they assume, being the leader who obtains a lower welfare value. Hence both prefer to not compete in equal footing but none of them wants to be the leader. This result is in accordance with other dynamic models such as Pohjola (1983).

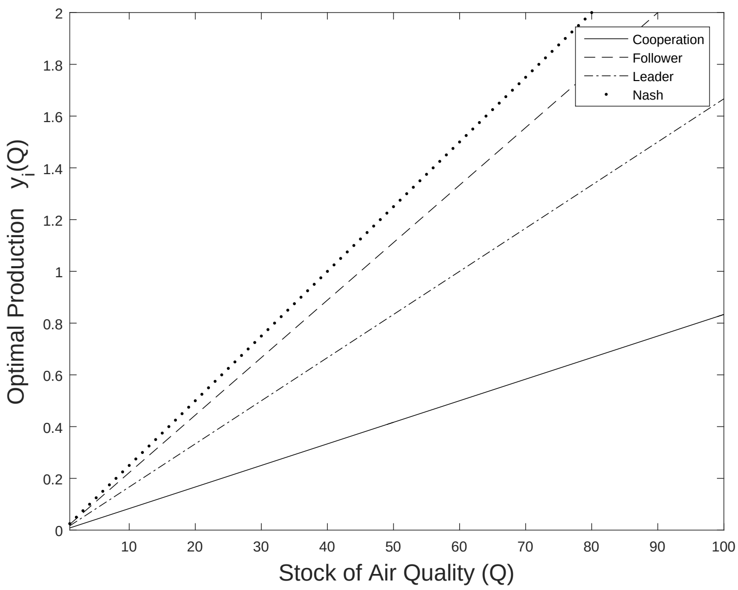

Figure 3: Model 1: Individual Production

Once we have compared the welfare outcomes, we analyse the individual strategies that produce these results, that is, the level of production each country chooses depending on the stock of air quality. Recall that in this graphical analysis we take the level of air quality as given, then the results obtained here do not take into account the different evolution of the state variable along time. In Figure 3 we see that the relation of the production choice among the three equilibria has changed totally in comparison to the relation in the welfare value analysis. The production is higher in the MPNE than in MSE, and in the cooperative equilibrium is the lowest, but in the case of the MSE the follower is who produces more, and that is the reason why his welfare is higher than the leader's.

3.2 Model 2: When a Country Neglects The Air Quality Problem

In this second model, one of the countries does not value the air quality in her welfare function. From the air quality perspective, we saw that all the equilibria in model 1 Pareto dominate all the second model equilibria, in which the two MSE obtain better outcomes than in the MPNE. Furthermore, we found that when the country that does not value the level of air quality (player 2) is the leader, the evolution of the air quality stock is better than if she is the follower.

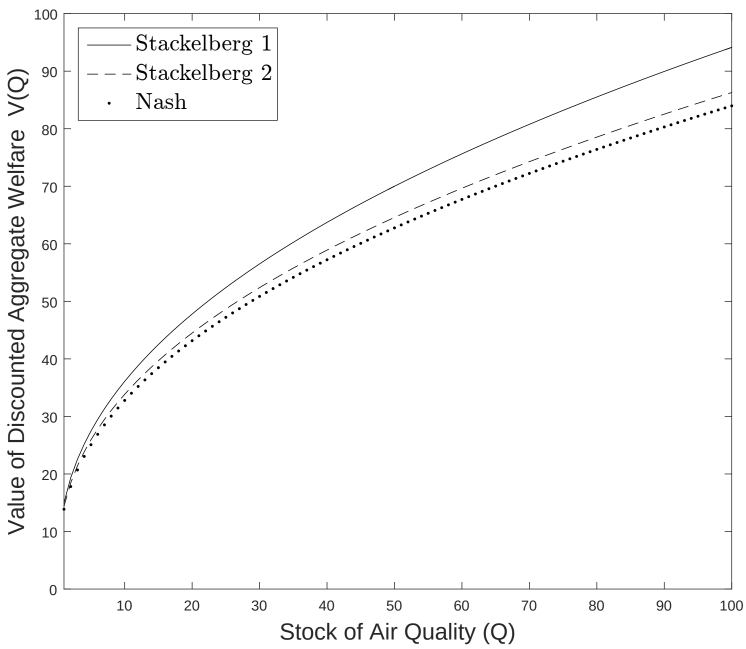

Figure 4: Model 2: Aggregate Value

From the countries' aggregate welfare perspective, Figure 4 shows the same relation we found for the evolution of the air quality stock. The MSE where the leader does not value the air quality (Stackelberg 1) yields higher aggregate welfare values than when the roles change, and both of them Pareto dominate the MPNE.

Figure 5 shows that the welfare of both players is higher when they are the follower than when they are the leader. Nonetheless, the welfare of player 1 is always higher than for the player 2 and the higher the level of air quality, the higher the difference between the welfare of the players.

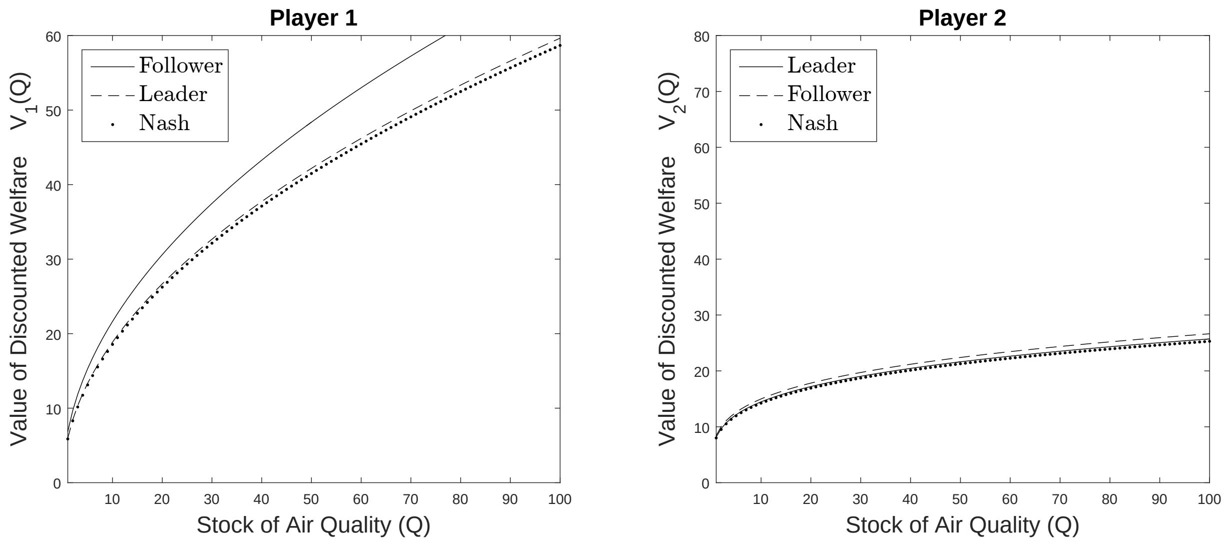

Figure 5: Model 2: Individual Value

In the MSE, through the reaction function, she (the leader) realizes that her production has a negative indirect effect on her welfare. Then, her optimal strategy is to produce less than in the MPNE, motivating the follower to reduce his production as well.

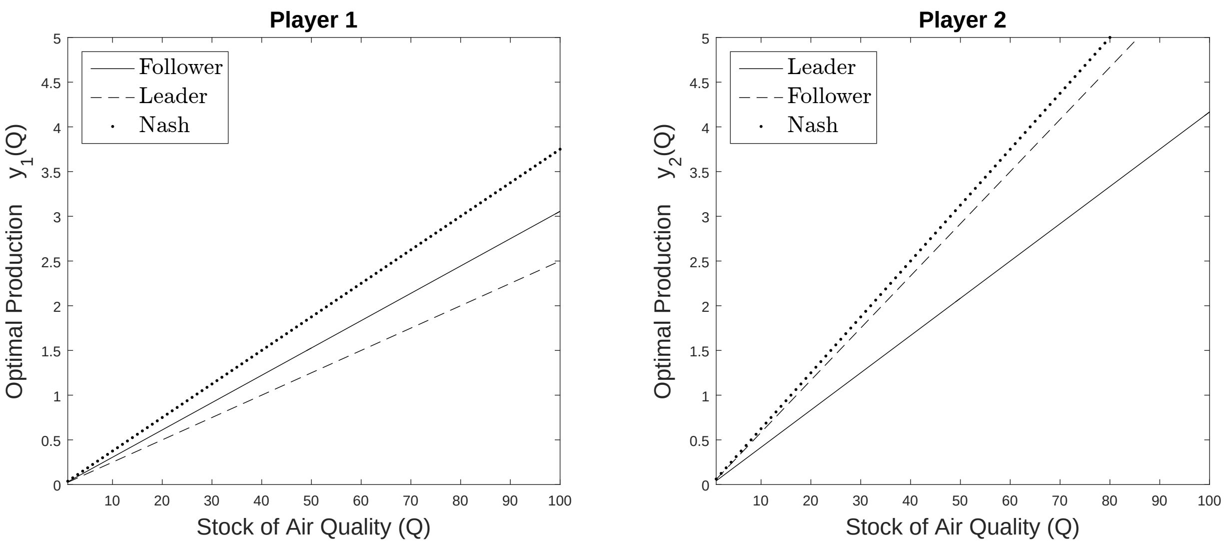

Figure 6: Model 2: Individual Production

As we commented in the introduction, the players' available information about the state variable has an important role in the properties of the equilibria, but the difference between Stackelberg and Nash competition is also the information that each player has about the other player's behaviour. In Nash competition none of the players knows how the other will behave and both adapt their strategies observing how the state variable evolves. In contrast, in Stackelberg competition the leader has information about the follower and therefore she can predict and manipulate his behaviour by announcing her strategy in advance. In our model, we have seen that both (symmetric) countries are better off in the MSE than in the MPNE, although both agents prefer to be the follower.

4 Conclusions

In this paper we have analysed the different strategic behaviour of the countries according to different scenarios of competition or cooperation. We found that the Cooperative Solution Pareto dominates all other equilibria from the perspective of the air quality evolution and from the aggregate welfare valuation. If there is no social planner or countries cannot agree to cooperate, a non-cooperative game is played, in which case, the MSE produces better outcomes for both the air quality evolution, the aggregate and the individual welfare values than the MPNE. We found that in the MSE both agents prefer to be the follower, it could imply that the game is in a stalemate, i.e., in a position where neither agent wants to take the initiative and announce his strategy first. However, in different economic context, either market power or political factors determine natural leaders and followers. We also show that when one of the countries does not value the stock of air quality, regardless of the type of competition the evolution of air quality gets worse, and therefore the welfare of the country that still value the level of air quality decreases drastically. In our analysis we have assumed total symmetry between the countries' welfare functions, but it would be interesting to analyse the countries strategies for different degrees of asymmetry. For simplicity in the derivation of the equilibriums, we confined our analysis to linear strategies, however by extending to non-linear strategies better outcomes could be achieved as showed for the Stackelberg equilibrium by Shimomura and Xie (2008).

We have not addressed the possibility of regulation as an instrument to achieve the Social optimum, or the possibility of transferable welfare which will make easier an agreement to achieve a Pareto efficient equilibrium.

Appendix

A Model 1

Nondegenerate Markovian Stackelberg Equilibrium

To solve (4) first we solve the right hand side

$$\frac{\partial\{\cdots\}}{\partial y_1} = 0 \qquad \Rightarrow \qquad \alpha n_1 y_1^{\alpha}\beta_1^{\alpha}Q^{\alpha} - V_1'(Q)n_1 = 0,$$

$$y_1 = \left(\frac{V_1'(Q)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}}, \tag{A.1}$$

then the HJB becomes

$$r_1V_1(Q) = n_1\left[\left(\frac{V_1'(Q)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}}\beta_1 Q\right]^{\alpha} +$$

$$+ V_1'(Q)\left[-n_1\left(\frac{V_1'(Q)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}} - n_2 bQ + kQ\right],$$

$$r_1 V_1(Q) = n_1 \left(\frac{V_1'(Q)}{\alpha \beta_1^\alpha Q^\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left[\beta_1^\alpha Q^\alpha (1-\alpha) \right] - V_1'(Q) \left[n_2 b Q - kQ \right].$$

Now we make the conjecture that the value function for player 1 is of the form

$$V_1(Q) = AQ^{2\alpha} \qquad \text{and therefore} \qquad V_1'(Q) = 2\alpha A Q^{2\alpha-1},$$

where A is a constant.

$$r_1 A Q^{2\alpha} = n_1 \left(\frac{2\alpha A Q^{2\alpha-1}}{\alpha \beta_1^\alpha Q^\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left[\beta_1^\alpha Q^\alpha (1-\alpha) \right] -$$

$$- 2\alpha A Q^{2\alpha-1} \left[n_2 b Q - kQ \right],$$

$$r_1 A Q^{2\alpha} = n_1 \left(\frac{2A}{\beta_1^\alpha} \right)^{\frac{\alpha}{\alpha-1}} \beta_1^\alpha Q^{2\alpha} (1-\alpha) - 2\alpha A Q^{2\alpha} \left[n_2 b - k \right],$$

$$n_1 \left(\frac{2}{\beta_1} \right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) A^{\frac{1}{\alpha-1}} = r_1 + 2\alpha (n_2 b - k),$$

$$A = \left(\frac{\beta_1}{2} \right)^\alpha \left(\frac{r1 + 2\alpha (n_2 b - k)}{n_1 (1-\alpha)} \right)^{\alpha-1}, \tag{A.2}$$

introducing the value of A in (A.1)

$$y_1(t,b) = \left(\frac{V_1'(Q)}{\alpha \beta_1^\alpha Q^\alpha} \right)^{\frac{1}{\alpha-1}} = \left(\frac{2\alpha \left(\frac{\beta_1}{2}\right)^\alpha \left(\frac{r1+2\alpha(n_2 b-k)}{n_1(1-\alpha)} \right)^{\alpha-1} Q^{2\alpha-1}}{\alpha \beta_1^\alpha Q^\alpha} \right)^{\frac{1}{\alpha-1}},$$

and it yields the reaction function (5). In order to solve the HJB equation for player 2 we follow the same procedure, first we solve the right hand side of the equation and we obtain the first order condition

$$y_2 = \left(\frac{V_2'(Q)}{\gamma \beta_2^\gamma Q^\gamma (1-\alpha)} \right)^{\frac{1}{\gamma-1}}, \tag{A.3}$$

substituting it in (6) we get

$$r_2 V_2(Q) = n_2 \left(\frac{V_2'(Q)}{\gamma \beta_2 Q (1-\alpha)} \right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - V_2'(Q) \cdot \frac{r_1 - 2k}{2(1-\alpha)} \cdot Q.$$

Now we make a conjecture of the functional form of the player 2 value function

$$V_2(Q) = BQ^{2\gamma} \qquad \Rightarrow \qquad V_2'(Q) = 2\gamma B Q^{2\gamma-1},$$

then, the previous equation becomes

$$r_2 B Q^{2\gamma} = n_2 \left(\frac{2\gamma B Q^{2\gamma-1}}{\gamma \beta_2 Q (1-\alpha)} \right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - 2\gamma B Q^{2\gamma-1} \cdot \frac{r_1 - 2k}{2(1-\alpha)} \cdot Q,$$

$$r_2 B = n_2 \left(\frac{2B}{\beta_2 (1-\alpha)} \right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - 2\gamma B \cdot \frac{r_1 - 2k}{2(1-\alpha)},$$

$$r_2 = n_2 \left(\frac{2}{\beta_2 (1-\alpha)} \right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) B^{\frac{1}{\gamma-1}} - 2\gamma \cdot \frac{r_1 - 2k}{2(1-\alpha)},$$

and finally we obtain

$$B = \left(\frac{\beta_2 (1-\alpha)}{2} \right)^{\gamma} \left(\frac{r_2(1-\alpha) + \gamma r_1 - 2\gamma k}{(1-\alpha)(1-\gamma) n_2} \right)^{\gamma - 1}.$$

Substituting this value in the conjecture done for player 2 we obtain (9), which we use in the first order condition of player 2 HJB equation and obtain (8), then we also have the value for b and we can obtain the production for player 1, and his value function.

Markov Perfect Nash Equilibrium

Solving the right hand side of (12)

$$\frac{\partial \{\cdots\}}{\partial y_1} = 0 \qquad \Rightarrow \qquad \alpha n_1 y_1^{\alpha} \beta_1^{\alpha} Q^{\alpha} - V_1'(Q) = 0,$$

$$y_1 = \left(\frac{V_1'(Q)}{\alpha \beta_1^{\alpha} Q^{\alpha}} \right)^{\frac{1}{\alpha-1}} \equiv \phi_1(Q). \tag{A.4}$$

Similarly for (13) we get:

$$y_2 = \left(\frac{V_2'(Q)}{\gamma \beta_2^{\gamma} Q^{\gamma}} \right)^{\frac{1}{\gamma-1}} \equiv \phi_2(Q), \tag{A.5}$$

then, back in the HJB

$$r_1 V_1(Q) = n_1 \left[\left(\frac{V_1'(Q)}{\alpha \beta_1^{\alpha} Q^{\alpha}} \right)^{\frac{1}{\alpha-1}} \beta_1 Q \right]^{\alpha} +$$

$$+ V_1'(Q) \left[-n_1 \left(\frac{V_1'(Q)}{\alpha \beta_1^{\alpha} Q^{\alpha}} \right)^{\frac{1}{\alpha-1}} - n_2 \left(\frac{V_2'(Q)}{\gamma \beta_2^{\gamma} Q^{\gamma}} \right)^{\frac{1}{\gamma-1}} + kQ \right],$$

$$r_1 V_1(Q) = n_1 \left(\frac{V_1'(Q)}{\alpha \beta_1^{\alpha} Q^{\alpha}} \right)^{\frac{\alpha}{\alpha-1}} \left[\beta_1^{\alpha} Q^{\alpha} (1-\alpha) \right] -$$

$$- V_1'(Q) \left[n_2 \left(\frac{V_2'(Q)}{\gamma \beta_2^{\gamma} Q^{\gamma}} \right)^{\frac{1}{\gamma-1}} - kQ \right].$$

We make a guessing about the functional form of the value function for player 1 and player 2 respectively

$$V_1(Q) = A Q^{2\alpha} \qquad \text{and therefore} \qquad V_1'(Q) = 2\alpha A Q^{2\alpha - 1},$$

$$V_2(Q) = B Q^{2\gamma} \qquad \text{and therefore} \qquad V_2'(Q) = 2\gamma B Q^{2\gamma - 1}.$$

Inserting them into the previous equation we get:

$$r_1AQ^{2\alpha} = n_1\left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_1^\alpha Q^\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left[\beta_1^\alpha Q^\alpha\left(1-\alpha\right)\right] -$$

$$-\alpha AQ^{2\alpha-1}\left[n_2\left(\frac{2\gamma BQ^{2\gamma-1}}{\gamma\beta_2^\gamma Q^\gamma}\right)^{\frac{1}{\gamma-1}} - kQ\right],$$

$$r_1AQ^{2\alpha} = n_1\left(\frac{2A}{\beta_1^\alpha}\right)^{\frac{\alpha}{\alpha-1}}\beta_1^\alpha Q^{2\alpha}\left(1-\alpha\right) - 2\alpha AQ^{2\alpha}\left[n_2\left(\frac{2B}{\beta_2^\gamma}\right)^{\frac{1}{\gamma-1}} - k\right],$$

$$r_1A = n_1\left(\frac{2A}{\beta_1^\alpha}\right)^{\frac{\alpha}{\alpha-1}}\beta_1^\alpha\left(1-\alpha\right) - 2\alpha A\left[n_2\left(\frac{2B}{\beta_2^\gamma}\right)^{\frac{1}{\gamma-1}} - k\right],$$

$$n_1\left(\frac{2}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}}\left(1-\alpha\right)A^{\frac{1}{\alpha-1}} = r_1 + 2\alpha\left[n_2\left(\frac{2B}{\beta_2^\gamma}\right)^{\frac{1}{\gamma-1}} - k\right],$$

$$A = \left(\frac{\beta_1}{2}\right)^\alpha\left(\frac{r1 + 2\alpha\left[n_2\left(\frac{2B}{\beta_2^\gamma}\right)^{\frac{1}{\gamma-1}} - k\right]}{n_1\left(1-\alpha\right)}\right)^{\alpha-1}.$$

Now, we go back to the HJB of player 2

$$r_2V_2(Q) = n_2\left[\left(\frac{V_2'(Q)}{\gamma\beta_2^\gamma Q^\gamma}\right)^{\frac{1}{\gamma-1}}\beta_2 Q\right]^\gamma +$$

$$+ V_2'(Q)\left[-n_1\left(\frac{V_1'(Q)}{\alpha\beta_1^\alpha Q^\alpha}\right)^{\frac{1}{\alpha-1}} - n_2\left(\frac{V_2'(Q)}{\gamma\beta_2^\gamma Q^\gamma}\right)^{\frac{1}{\gamma-1}} + kQ\right],$$

and following the same logic, with the guessing we have made previously we get:

$$r_2B = n_2\left(\frac{2B}{\beta_2}\right)^{\frac{\gamma}{\gamma-1}}\left(1-\gamma\right) - 2\gamma B\left[n_1\left(\frac{2A}{\beta_1^\alpha}\right)^{\frac{1}{\alpha-1}} - k\right].$$

With the result of A:

$$r_2B = n_2\left(\frac{2B}{\beta_2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma-\alpha}{(1-\alpha)}\right) - \gamma B\left(\frac{r1-2k}{(1-\alpha)}\right),$$

$$n_2\left(\frac{2}{\beta_2}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma-\alpha}{(1-\alpha)}\right)B^{\frac{\gamma}{\gamma-1}} = r_2 + \gamma\left(\frac{r1-2k}{(1-\alpha)}\right),$$

$$B = \left(\frac{\beta_2}{2}\right)^\gamma\left[\frac{r_2\left(1-\alpha\right)+\gamma r_1 - 2\gamma k}{n_2\left(1-\gamma-\alpha\right)}\right]^{\gamma-1}.$$

Then A becomes

$$A = \left(\frac{\beta_1}{2}\right)^\alpha\left(\frac{r_1\left(1-\gamma\right)+r_2\alpha - k2\alpha}{\left(1-\gamma-\alpha\right)n_1}\right)^{\alpha-1}.$$

And finally, substituting it in the value functions we obtain (16) and (17), and in (A.4) and (A.5) we have that the optimal strategies for player 1 and for player 2 are respectively (14) and (15).

Social Optimum

In (19) we obtain the first order conditions

$$\frac{\partial\{\cdots\}}{\partial y_1} = 0 \qquad \Rightarrow \qquad \alpha n_1 y_1^{\alpha-1}\beta_1^{\alpha}Q^{\alpha} - n_1 V'(E) = 0,$$

$$y_1 = \left(\frac{V'(E)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}}, \tag{A.6}$$

$$\frac{\partial\{\cdots\}}{\partial y_2} = 0 \qquad \Rightarrow \qquad \gamma n_2 y_2^{\gamma-1}\beta_2^{\gamma}Q^{\gamma} - n_2 V'(E) = 0,$$

$$y_2 = \left(\frac{V'(E)}{\gamma\beta_2^{\gamma}Q^{\gamma}}\right)^{\frac{1}{\gamma-1}}, \tag{A.7}$$

$$rV(E) = n_1\left[\left(\frac{V'(E)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}}\beta_1 Q\right]^{\alpha} + n_2\left[\left(\frac{V'(E)}{\gamma\beta_2^{\gamma}Q^{\gamma}}\right)^{\frac{1}{\gamma-1}}\beta_2 Q\right]^{\gamma} -$$

$$- V'(E)\left[n_1\left(\frac{V'(E)}{\alpha\beta_1^{\alpha}Q^{\alpha}}\right)^{\frac{1}{\alpha-1}} + n_2\left(\frac{V'(E)}{\gamma\beta_2^{\gamma}Q^{\gamma}}\right)^{\frac{1}{\gamma-1}} - kQ\right],$$

$$rV(E) = n_1\left(\frac{V'(E)}{\alpha\beta_1 Q}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) + n_2\left(\frac{V'(E)}{\gamma\beta_2 Q}\right)^{\frac{\gamma}{\gamma-1}}(1-\gamma) + V'(E)kQ.$$

In order to solve analytically the cooperative problem we assume that $\gamma = \alpha$, hence as in the other problems we can make a guessing about the functional form of the value function

$$V(E) = AQ^{2\alpha} \qquad \Rightarrow \qquad V'(E) = 2\alpha AQ^{2\alpha-1}.$$

Then, from the previous equation we get

$$rAQ^{2\alpha} = n_1\left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_1 Q}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) +$$

$$+ n_2\left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_2 Q}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) + 2\alpha AQ^{2\alpha-1}kQ,$$

$$rAQ^{2\alpha} = Q^{2\alpha}n_1\left(\frac{2A}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) +$$

$$+ n_2 Q^{2\alpha}\left(\frac{2A}{\beta_2}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) + 2\alpha AQ^{2\alpha}k,$$

$$r = A^{\frac{1}{\alpha-1}} \left(\frac{2}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}} n_1(1-\alpha) + A^{\frac{1}{\alpha-1}} \left(\frac{2}{\beta_2}\right)^{\frac{\alpha}{\alpha-1}} n_2(1-\alpha) + 2\alpha k,$$

$$r = A^{\frac{1}{\alpha-1}}(1-\alpha)2^{\frac{\alpha}{\alpha-1}} \left[\left(\frac{1}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}} n_1 + \left(\frac{1}{\beta_2}\right)^{\frac{\alpha}{\alpha-1}} n_2\right] + 2\alpha k,$$

$$A = \left(\frac{\beta_1\beta_2}{2}\right)^{\alpha} \left[\frac{r-2\alpha k}{(1-\alpha)\left(\beta_1^{\frac{\alpha}{\alpha-1}} n_1 + \beta_2^{\frac{\alpha}{\alpha-1}} n_2\right)}\right]^{\alpha-1}.$$

With the value of A we obtain the value function of the coalition which substituting in (A.6) and (A.7), we obtain (20) and (21), respectively.

B Model 2

Nondegenerate Markovian Stackelberg Equilibrium: The Leader Neglects

To solve (26) we obtain the first-order condition from the left hand side

$$\frac{\partial\{...\}}{\partial y_2} = 0 \qquad \Rightarrow \qquad y_2 = \left(\frac{V_2'(Q)}{\gamma(1-\alpha)}\right)^{\frac{1}{\gamma-1}}, \tag{B.1}$$

that substituting in the HJB equation yields

$$r_2 V_2(Q) = n_2 \left(\frac{V_2'(Q)}{\gamma(1-\alpha)}\right)^{\frac{\gamma}{\gamma-1}} - (1-\gamma) - V_2'(Q) \cdot \frac{r_1 2k}{2(1-\alpha)} \cdot Q.$$

Now we make the guessing about the functional form of $V_2(Q)$

$$V_2(Q) = BQ^{\gamma} \qquad \Rightarrow \qquad V_2'(Q) = \gamma B Q^{\gamma-1},$$

then, the previous equation becomes

$$r_2 B Q^{\gamma} = n_2 \left(\frac{\gamma B Q^{\gamma-1}}{\gamma(1-\alpha)}\right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - \gamma B Q^{\gamma-1} \cdot \frac{r_2 - 2k}{2(1-\alpha)} \cdot Q,$$

$$r_2 B = n_2 \left(\frac{B}{(1-\alpha)}\right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - \gamma B \cdot \frac{r_2 - 2k}{2(1-\alpha)},$$

$$r_2 = n_2 \left(\frac{1}{(1-\alpha)}\right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) B^{\frac{1}{\gamma-1}} - \gamma \cdot \frac{r_2 - 2k}{2(1-\alpha)},$$

and it yields

$$B = (1-\alpha)^{\gamma} \left(\frac{r_2 2(1-\alpha+\gamma r_1 - 2\gamma k}{2(1-\alpha)(1-\gamma)n_2}\right)^{\gamma-1},$$

that introduced in the conjecture for $V_2(Q)$ give us the value function for player 2, and then we can obtain from (B.1) the production for the leader (27), which give us the value for b and therefore substituting in the reaction function of the follower and on the value of A obtained in (A.2) we get (28) and (29) respectively.

Nondegenerate Markovian Stackelberg Equilibrium: The Leader Worries

Since the objective functions are asymmetric not only in the parameters but in the functional form we cannot obtain the Stackelberg equilibrium when the roles change from the previous analysis, then we have to repeat the process to reach the equilibrium when player 1 is the leader and player 2 is the follower. From the right hand side of the follower HJB equation (32)

$$\frac{\partial \{...\}}{\partial y_2} = 0 \qquad \rightarrow \qquad n_2 \gamma y_2^{\gamma-1} - n_2 V_2'(Q) = 0,$$

$$y_2 = \left(\frac{V_2'(Q)}{\gamma}\right)^{\dfrac{1}{\gamma - 1}}, \tag{B.2}$$

the HJB equation becomes

$$r_2 V(Q) = n_2 \left(\frac{V_2'(Q)}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - V_2'(Q) Q (n_1 b - k).$$

As in the other MSE, we make a guessing about the functional form of the follower's value function

$$V_2(Q) = BQ^\gamma \qquad \Rightarrow \qquad V_2'(Q = \gamma B Q^{\gamma-1},$$

then, the previous equation is

$$r_2 B Q^\gamma = n_2 \left(\frac{\gamma B Q^{\gamma-1}}{\gamma}\right)^{\frac{\gamma}{\gamma-1}} (1-\gamma) - \gamma B Q^{\gamma-1} Q (n_1 b - k),$$

$$r_2 B = n_2 B^{\frac{\gamma}{\gamma-1}} (1-\gamma) - \gamma B (n_1 b - k),$$

$$r_2 = n_2 B^{\frac{1}{\gamma-1}} (1-\gamma) - \gamma (n_1 b - k),$$

and finally it yields

$$B = \left(\frac{r_2 + \gamma(n_1 b - k)}{(1-\gamma) n_2}\right)^{\gamma-1},$$

which depends not only on the parameter but also on the strategy of the leader through b. With this value in (B.2) we obtain the reaction function of the follower (33). Now we solve the leader's problem, which in this case is the player 1. As we have done in the other analysis we begin with the HJB equation of the leader (34) and we obtain the first-order condition

$$\frac{\partial \{...\}}{\partial y_1} = 0 \qquad \Rightarrow \qquad \alpha n_1 y_1^{\alpha-1} \beta_1^\alpha Q^\alpha - \frac{n_1}{1-\gamma} V_1'(Q) = 0,$$

$$y_1 = \left(\frac{V_1'(Q)}{\alpha \beta_1^\alpha Q^\alpha (1-\gamma)}\right)^{\frac{1}{\alpha-1}}, \tag{B.3}$$

the HJB equation becomes

$$r_1 V_1(Q) = n_1 \left(\frac{V_1'(Q)}{\alpha \beta_1^\alpha Q^\alpha (1-\gamma)}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) - V_1'(Q) \cdot \frac{r_2 - k}{(1-\gamma)} \cdot Q,$$

following the same logic we used in the other analysis we make a conjecture about the functional form of the leader's value function

$$V_1(Q) = AQ^{2\alpha} \qquad \Rightarrow \qquad V_1'(Q) = 2\alpha AQ^{2\alpha-1},$$

then, the previous equation is

$$r_1 AQ^{2\alpha} = n_1 \left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_1 Q(1-\gamma)}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) - 2\alpha AQ^{2\alpha-1} \cdot \frac{r_2-k}{1-\gamma} \cdot Q,$$

$$r_1 A = n_1 \left(\frac{2A}{\beta_1(1-\gamma)}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) - 2\alpha A \cdot \frac{r_2-k}{1-\gamma},$$

$$r_1 = n_1 \left(\frac{2}{\beta_1(1-\gamma)}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) A^{\frac{1}{\alpha-1}} - 2\alpha \cdot \frac{r_2-k}{1-\gamma},$$

$$A = \left(\frac{\beta_1(1-\alpha)}{2}\right)^{\alpha} \left(\frac{r_1(1-\gamma)+2\alpha r_2 - 2\alpha k}{(1-\alpha)(1-\gamma)n_1}\right)^{\alpha-1}.$$

Once we have obtained the value of A we can obtain $V_1(Q)$, which introduced in (B.3) give us (35) the optimal strategy for the leader, and then with the value of b we can also solve the problem for the follower, obtaining (36) and (38).

Markov Perfect Nash Equilibrium

We begin solving the right hand side of (40)

$$\frac{\partial \{...\}}{\partial y_1} = 0 \qquad \Rightarrow \qquad \alpha n_1 \beta_1^{\alpha} y_1^{\alpha-1} Q^{\alpha} - n_1 V_1'(Q) = 0,$$

$$y_1 = \left(\frac{V_1'(Q)}{\alpha\beta_1^{\alpha} Q^{\alpha}}\right)^{\frac{1}{\alpha-1}} \equiv \phi_1(Q), \tag{B.4}$$

and of (41)

$$\frac{\partial \{...\}}{\partial y_2} = 0 \qquad \Rightarrow \qquad \gamma n_2 y_2^{\gamma-1} - n_2 V_2'(Q) = 0,$$

$$y_2 = \left(\frac{V_2'(Q)}{\gamma}\right)^{\frac{1}{\gamma-1}} \equiv \phi_2(Q). \tag{B.5}$$

Back into the HJB equation for player 1 with (B.4) and (B.5) we have

$$r_1 V_1(Q) = n_1 \left(\frac{V_1'(Q)}{\alpha\beta_1 Q}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha) -$$

$$- V_1'(Q) n_2 \left(\frac{V_2'(Q)}{\gamma}\right)^{\frac{1}{\gamma-1}} + V_1'(Q) kQ.$$

As usual we make a conjecture about the functional form of the players' value functions

$$V_1(Q) = AQ^{2\alpha} \qquad \Rightarrow \qquad V_1'(Q) = 2\alpha AQ^{2\alpha-1},$$

$$V_2(Q) = BQ^\gamma \qquad \Rightarrow \qquad V_2'(Q) = \gamma BQ^{\gamma-1},$$

then the HJB equation for player 1 becomes

$$r_1AQ^{2\alpha} = n_1\left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_1 Q}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) -$$

$$- 2\alpha AQ^{2\alpha-1}n_2\left(\frac{\gamma BQ^{\gamma-1}}{\gamma}\right)^{\frac{1}{\gamma-1}} + 2\alpha AQ^{2\alpha-1}kQ,$$

$$r_1A = n_1\left(\frac{2A}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}}(1-\alpha) - 2\alpha An_2B^{\frac{1}{\gamma-1}} + 2\alpha Ak,$$

$$r_1 = n_1\left(\frac{2}{\beta_1}\right)^{\frac{\alpha}{\alpha-1}}A^{\frac{1}{\alpha-1}}(1-\alpha) - 2\alpha n_2B^{\frac{1}{\gamma-1}} + 2\alpha k,$$

and finally it yields

$$A = \left(\frac{\beta_1}{2}\right)^\alpha\left(\frac{r_1 - 2\alpha k + 2\alpha n_2B^{\frac{1}{\gamma-1}}}{n_1(1-\alpha)}\right)^{\alpha-1}.$$

Now we have the value of A but it depends on B, to obtain the equilibrium we solve the HJB equation for player 2 with (B.4) and (B.5)

$$r_2V_2(Q) = n_2\left[\left(\frac{V_2'(Q)}{\gamma}\right)^{\frac{1}{\gamma-1}}\right]^\gamma +$$

$$+ V_2'(Q)\left[-n_1\left(\frac{V_1'(Q)}{\alpha\beta_1^\alpha Q^\alpha}\right)^{\frac{1}{\alpha-1}} - n_2\left(\frac{V_2'(Q)}{\gamma}\right)^{\frac{1}{\gamma-1}} + kQ\right],$$

and with the guessing we made previously

$$r_2BQ^\gamma = n_2\left[\left(\frac{\gamma BQ^{\gamma-1}}{\gamma}\right)^{\frac{1}{\gamma-1}}\right]^\gamma +$$

$$+\gamma BQ^{\gamma-1}\left[-n_1\left(\frac{2\alpha AQ^{2\alpha-1}}{\alpha\beta_1^\alpha Q^\alpha}\right)^{\frac{1}{\alpha-1}} - n_2\left(\frac{\gamma BQ^{\gamma-1}}{\gamma}\right)^{\frac{1}{\gamma-1}} + kQ\right],$$

$$r_2BQ^\gamma = n_2B^{\frac{\gamma}{\gamma-1}}Q^\gamma(1-\gamma) - \gamma BQ^\gamma n_1\left(\frac{2A}{\beta_1^\alpha}\right)^{\frac{1}{\alpha-1}} + \gamma BQ^\gamma k,$$

$$r_2B = n_2B^{\frac{\gamma}{\gamma-1}}(1-\gamma) - \gamma Bn_1\left(\frac{2A}{\beta_1^\alpha}\right)^{\frac{1}{\alpha-1}} + \gamma Bk,$$

introducing the value of A obtained before we have

$$r_2B = n_2B^{\frac{\gamma}{\gamma-1}}\left(1-\gamma-\frac{\gamma\alpha}{1-\alpha}\right) - \gamma B\frac{r_1-2\alpha k}{2(1-\alpha)} + \gamma Bk,$$

$$r_2 = n_2 \left(\frac{1-\alpha-\gamma}{1-\alpha} \right) B^{\frac{1}{\gamma-1}} - \gamma \frac{r_1 - 2\alpha k}{2(1-\alpha)} + \gamma k,$$

and finally it yields

$$B = \left(\frac{r_2(1-\alpha) + \frac{\gamma}{2} r_1 - \gamma k}{(1-\alpha-\gamma) n_2} \right)^{\gamma - 1}.$$

With this result we can obtain

$$A = \left(\frac{\beta_1}{2} \right)^{\alpha} \left[\frac{r_1(1-\gamma) + 2r_2\alpha - 2k\alpha}{(1-\alpha-\gamma) n_1} \right]^{\alpha-1}, \tag{B.6}$$

and now, substituting in the guessing of the value functions we get (44) and (45), which with the first-order conditions (B.4) and (B.5) we have (42) and (43) respectively.

References

Cornes, R., Van Long, N., and Shimomura, K. (2001). Drugs and pests: Intertemporal production externalities. *Japan and the World Economy*, 13(3), 255-278.

De-Paz, A., Marín-Solano, J., and Navas, J. (2013). Time-consistent equilibria in common access resource games with asymmetric players under partial cooperation. *Environmental Modeling and Assessment*, 18(2), 171-184.

Dockner, E., and Van Long, N. (1993). International pollution control: cooperative versus non-cooperative strategies. *Journal of Environmental Economics and Management*, 25(1), 13-29.

Dockner, E., Jørgensen, S., Van Long, N., and Sorger, G., 2000. *Differential Games in Economics and Management Science*. Cambridge University Press, Cambridge, UK.

Jørgensen, S., Martín-Herrán, G., and Zaccour, G. (2010). Dynamic games in the economics and management of pollution. *Environmental Modeling and Assessment*, 15(6), 433-467.

Pohjola, M. (1983). Nash and Stackelberg solutions in a differential game model of capitalism. *Journal of Economic Dynamics and Control*, 6, 173-186.

Van Long, N. (1992). Pollution control: A differential game approach. *Annals of Operational Research*, 37, 283-296.

Van Long, N. (2011). Dynamic games in the economics of natural resources: a survey. *Dynamic Games and Applications*, 1(1), 115-148.

Van Long, N., and Shimomura, K. (1998). Some results on the Markov equilibria of a class of homogeneous differential games. *Journal of economic behavior and organization*, 33(3), 557-566.

Van der Ploeg, F., and de Zeeuw, A. (1992). International aspects of pollution control. *Environmental and Resource Economics*, 2(2), 117-139.

Shimomura, K., and Xie, D. (2008). Advances on Stackelberg open-loop and feedback strategies. *International Journal of Economic Theory*, 4(1), 115-133.