INSTANTANEOUS GRATIFICATION AND COMMON PROPERTY RESOURCE GAMES

André Reis Ferreira Pombinho

MSc in Economics, Universitat de Barcelona

Advisors: Jesús Marín Solano & Jorge Navas Ródenes

Universitat de Barcelona

ABSTRACT

We compute the cooperative and noncooperative solutions for sophisticated agents with Instantaneous-Gratification discounting in infinite horizon, as an extension of the work of Harris & Laibson (2013) and Zou, Chen & Wedge (2014). This research contributes to the existing literature to the extent that we compute multi-agent sophisticated solutions with Instantaneous-Gratification discounting in infinite time, and we place the results of the Instantaneous-Gratification model in context of the management of renewable natural resources. The conclusions withdrawn are applicable for resources of any kind and are suitable for settings where the temporal horizon is unlimited, but the duration of the short-run is large enough to dodge the future. The discussion of this work is useful for policy implementation towards exploitation of renewable natural resources under different forms of ownership.

KEYWORDS: Intertemporal choice, Instantaneous-Gratification discounting, dynamic games, cooperative solution, noncooperative solution, infinite horizon, sophisticated agent, renewable natural resources.

ACKNOWLEDGEMENTS: This accomplishment would have not been possible without the collaboration and support of many people. Foremost, I would like to express my profound gratitude to my thesis advisors, Jesús Marín Solano and Jorge Navas Ródenes, for their exceptional guidance. Despite their tight schedule, from beginning to end, they were always available to help me with any doubt about my research.

I would also like to thank my family. To my mother, for her encouragement and faith in all my projects, and to my grandparents and uncles, who inspire me to keep studying, especially, to my grandmother, a living example of lifelong learning. Thanks to Esther, the great woman I am fortunate to have next to me, for her support, her trust and for cheering me up in my cloudy moments.

Special thanks to Catalina Dunbar for her sponsorship.

Thanks to my colleagues, for their fellowship. Thanks to my coworkers from SAE LATAM, for their empathy and flexibility whenever I needed to change my working schedule to attend the studies. Thanks to Cristina López, for hiring me whilst I was studying, and for providing the structure required to combine this master with a side job.

Finally, I would like to acknowledge all the team of the School of Economics. It has been a wonderful experience to participate in this program. Thanks to Jordi Roca, for his willingness to help the students with all the issues they bring to his office.

1. INTRODUCTION

This research analysis integrates the thesis of the Master in Economics, imparted at the University of Barcelona, for the period 2015-2017. Within the field of Behavioral Economics, the scope of the study here presented, is to derive the equilibrium behavior of economic agents in regard of the intertemporal choices they are called to make.

Intertemporal Choice is the economic branch that appraises the impact that elections made by economic agents at different moments in time, have on their intertemporal welfare, contingent on the behavior they decided to follow. Overeating, smoking, working out, marriage or investing, are a few examples of a much broader spectrum of personal and business time-dependent resolutions with intertemporal effects. These decisions imply a present action, which has a flow of future outcomes and is associated with a current utility level. Eventually, any choice made today can trigger either positive (rewards) or negative (costs) future results.

Mathematically, future outcomes are discounted to be evaluated in the present. Discounting is the methodology followed to determine the present value of forthcoming consequences, which are downgraded at some discount rate. Discount rates are representative of agents' willingness to trade-off utility increments at different points in time (Thaler, 1981). Fisher (1930) said that individual discount rates are equal to interest rates because agents lend or borrow until their marginal rate of substitution, between current and future consumption, is equal to the interest rate.

Time discounting is a major issue in economics. It is not only a concern of Behavioral Economics but also of the Economic Growth Theory. For decades, Growth Theory has used time-consistent preferences, represented by exponential constant discount rates (Barro, 1999; Phelps & Pollak, 1968). Albeit, evidences are, that economic agents behave differently when they assess short- and long-run decisions, and that their preferences are not the same when they have to decide on rewards or costs (McClure, 2004). Therefore, we should debate whether exponential discounting is precise enough to epitomize human behavior and which alternatives could be approached.

When at a certain moment t, let us say March 1st of 2017, an agent declares her preference for one action, but when, at $t+1$, let us say Abril 1st of 2017, the same individual is asked again, in regard of the same action, and she clearly chooses a different option, we say that her intertemporal preferences are time inconsistent. Exponential discounting does not account for this inconsistency, as it assumes that all the consequences of present actions, are going to be discounted equally, no matter how far or close they are placed.

When we turn to non-constant discounting, the most developed breakthrough is the hyperbolic discounting. Hyperbolic discounting mirrors the intertemporal inconsistency of economic agents in intertemporal decision models by considering that agents are more impatient for immediate rewards than for later ones. This impatience is mathematically elucidated in the discount function, by associating a different weight to present and future decisions (the $\beta\delta$ preferences). Discrete-time hyperbolic discounting, illustrates the impatience individuals exhibit on their intertemporal preferences and has been used to study patterns of consumption, procrastination and addiction (Loewenstein & Thaler, 1989; O 'Donoghue & Rabin, 1999; O'Donoghue & Rabin, 2000).

The mainstream literature defines three profiles of behaviors regarding the time-inconsistency of agent's intertemporal preferences: *precommitment*, *sophisticated* or *naive*. The difference among these behaviors relies on the awareness of the individuals for the fact that their preferences may change over time, and their readiness to commit.

An individual who is aware of her time-inconsistent preferences, may decide to precommit or to be sophisticated, adopting her conduct in each moment. The first, leads to precommitment behaviors, such as saving plans, marriage and credits. The agent establishes an agreement where deviations are not contemplated and, thus, his behavior becomes invariable over time.

The sophisticated agent is aware of her weaknesses and decides in each moment what to do, taking into consideration that her preferences are going to change with time. At each moment, the agent reconsiders and redefines her strategy. The assumption of rationality underlying this behavior, together with its time consistency, makes of this solution the most studied in economics. The discussion below will hinge on the sophisticated agent.

Finally, the naive agent continuously procrastinates costs and hurry rewards, under the general belief that her lack of commitment is transitory and that she will behave well in the future.

In this paper, we are going to extend the analysis of a time-inconsistent discounting - the Instantaneous Gratification discounting (hereinafter IG), suggested by Harris & Laibson (2013) and further discussed by Zou, Chen, & Wedge (2014). The IG came to light in the field of nonexponential discounting, as an extension of the quasi-hyperbolic discounting, presented by Laibson (1997) as an extension of the hyperbolic discounting. The IG assumes that the transition from the present to the future occurs at a constant hazard rate λ that in the IG approaches the limiting case $\lambda \to \infty$.

Laibson (1997) analyzed the IG under a finite time-horizon framework for general assets and individual sophisticated agents, following Barro (1999) and Luttmer & Mariotti (2003). Here, we want to extend the analysis to an infinite horizon and for multi-agent cases. To do so, we are going to study the cooperative and noncooperative solutions. We believe the infinite time-horizon setting and the multiplayer approach, are consistent with environments such as the management of renewable natural resources where, for the characteristics of the resource, time-horizon restrictions are absent, and several owners can exploit the resource. This specification is relevant because we confine our research to cases in which the number of players is controlled, what is not possible to assure in common access resources. The analysis could easily be mimetic for any other resource of common property.

When we approach the exploitation of natural resources, often, the interests of the exploiter, under a profit maximization standpoint, and what is the best Maximum Sustainable Yield (MSY) for the resource, go in opposite directions. The aim is to withdraw conclusions on the equilibrium exploitation path for resources in settings where the temporal horizon is unlimited, but the duration of the short-run is large enough to dodge the future. The equilibrium exploitation path must be aligned with the MSY which is clearly defined in Clark (1990): "*based on a model of biological growth (…) assumes that at any given population below a certain level (…), a surplus production exists that can be harvested in perpetuity without altering the stock level.*" The economic agents here considered can be countries, companies, groups of companies or individuals.

The contribution of this research to the existing literature is twofold. From one side, the cooperative and noncooperative sophisticated solutions with Instantaneous-Gratification discounting in infinite time that we compute here, have not been analyzed yet, at least not that we are aware of. On the other side, we place the results of the Instantaneous-Gratification model in context of the management of renewable natural resources.

The next section provides an overview of the existing literature on intertemporal behaviors, specifically in what concerns time-inconsistent discounting. Section three introduces the methodology of this paper and section four presents some results for isoelastic utilities. Section five offers a discussion of the results obtained and section six concludes. The appendices compile all the calculations that for a matter of organization and ease of reading were left out of the main text.

2. LITERATURE REVIEW

Often, from the economic theory perspective, the complex and developed behavior of human beings is reduced to the maximization of a utility function subjected to a budget constraint. However, the rationing behind the economic assumptions of stability underlying the economic analysis, more specifically, in the aim of behavioral economics, the use of time-consistent discounting to study the present value of future utilities, has been contended since ever (Samuelson, 1937; Phelps & Pollak, 1968; Barro, 1999). The difficulty relies on finding those mathematic expressions that accurately replicate human behavior which is normally trimmed for the purpose of maximizing a certain utility. Samuelson (1937) on "*A note on Measurement of Utility*" correctly abridges the economic analysis framework stating: "*In order to arrive at the measurement of utility, essentially a subjective quantity, it is necessary to place the individual (homo economics) whose scale is south under certain ideal circumstances where his observable behavior will render open to unambiguous inference the form of the function which he is conceived of as maximizing*.", and he continues with the assumptions required for his research: "*During any specified period of time, the individual behaves so as to maximize the sum of all future utilities (…) This is in the nature of an axiom, or definition, not subject to proof in any empirical sense, since any and all types of observable behavior might conceivably result from such an assumption*.", alluding to the inconsistency of the assumption of constant discount rates on his third assumption: "*The individual discounts future utilities in some simple regular fashion which is known to us. (…) We assume in the first instance that the rate of discount of future utilities is a constant. This constant might of course be such that there is no time preference whatsoever, or even a premium on future utilities. This third assumption, unlike the previous two, is in the nature of a hypothesis, subject to refutation by the observable facts…*". Although, it is not until Strotz (1955) when the validity of constant discount rates is discussed and time-inconsistent preferences are analytically introduced: "*Special attention should be given, I feel, to a discount function (…) which differs from the logarithmically linear one in that it "over-values" the more approximate satisfaction relative to the more distant ones.*". Strotz (1955) identified three behaviors for agents with inconsistent time preferences: *spendthrift*, *precommit* and *consistent*. If the agent was aware of his problem of inconsistency, then she would be spendthrift; on the contrary, if the agent realized her problem, she would face two solutions – to precommit or to consistently plan her actions. For what Strotz declared: "*My own supposition is that most of us are "born" with discount functions of the sort considered here [non-constant], that precommitment is only occasionally a feasible strategy (…) and that we are taught to plan consistently…*".

Later on, Pollak (1968) revisited Strotz's analysis and labeled the *naive* individuals as the agents that do not recognize the problem of future misbehavior and the *sophisticated* individuals as those that do. His major criticism of Strotz's work was that for agents with inconsistent-intertemporal preferences, the conditions which determine the allocation of consumption and capital over time are different and while for the former, they coincide for the naive and the sophisticated behavior, the naive equilibrium path for the capital is different. Specifically, the difference between naive and sophisticated agents relies not on the way they allocate their consumption decisions, but on how they allocate their capital among their consumption decisions.

Empirical evidence of Strotz's research was reported on Thaler (1981) who dismissed Fisher (1930) and endorsed the idea that individuals' decisions tend to vary with the size of the reward (whether it is a profit or a cost) and the length of the delay.

O'Donoghue & Rabin (2000, 1999) pinpointed that time-consistent agents are self-controlled whereas time-inconsistent individuals face self-control problems. When an individual procrastinates every period without acknowledgment of her self-control problems and still believes that she will, eventually, manage to pick her most preferred choice, she is designated as naive. If the agent procrastinates but she perceives her inconsistency and acknowledge the risk of misbehaving due to her self-control problems, she is characterized as sophisticated. When costs are immediate, naive will execute the activity the latest as possible, completing the task too late whilst she will perform the assignment too soon if costs are delayed. On the contrary, when costs are instantaneous, aware of the chances of misbehaving, the sophisticated will act the soonest as possible, whereas she will do the task too late if costs are delayed. More generally, agents tend to under-indulge in activities with immediate costs and delayed benefits, and over-indulge in activities with immediate rewards and delayed costs (O'Donoghue and Rabin, 2000).

3. METHODOLOGY

In this section, we detail the notation that will be used throughout the paper and blueprint the problem we will solve in section 4.

In what follows, we start from a discrete-time analysis as in Phelps & Pollak (1968) and adopt the structure suggested in Harris & Laibson (2013) and further examined in Zou, Chen, & Wedge (2014).

In discrete time, the timeline is divided into several intervals, each of which are discounted at a specific discount rate. In continuous time, the intervals into which the timeline is divided are of such a tiny duration that the agent is continuously discounting at some discount rate. With quasi-hyperbolic time preferences, each interval of time is discounted at the rate $\beta\delta$, with $\beta, \delta \in (0,1]$. Present and future periods are discounted exponentially with the discount factor δ whilst future periods are further discounted with β, which is a "present-biased" indicator.

Following Harris & Laibson (2013), we denote τ as the length of the present, with $\tau_1, \tau_2, \tau_3, \dots, \tau_n$ representing the duration of each subsequent moment. τ is stochastic and is exponentially distributed with hazard rate $\lambda \in [0, \infty)$. If we let $\lambda \to 0$, the present is suppressed and when $\lambda \to \infty$, the future is neglected. The scope of this study is to outline the sophisticated solution, represented by the limiting case of $\lambda \to \infty$ of the IG discounting, in continuous time, for an infinite horizon.

We define t as the initial moment, T as the last period of the time horizon, here, $T \to \infty$, and s is any moment in between t and T, with $s_1, s_2, s_3, \dots, s_n$ describing successive moments in the interval $[t, \infty]$.

Figure 1: t and T are the initial and final moments of the timeline considered. τ represents the duration of the current moment which lasts from t to s, and is to be evaluated at t.

Let introduce $\rho(s)$, which represents the instantaneous discount rate at time s. The discount factor that evaluates a payoff at a certain moment, is exponentially discounted as

$$\theta(s-t)=\begin{cases} e^{-\int_t^s \rho(z-t)\,dz}, & (s-t)\le\tau \\ \beta e^{-\int_t^s \rho(z-t)\,dz}, & (s-t)>\tau \end{cases},$$

in particular, $\rho(s)=\rho$, if $s\le\tau$.

Harris & Laibson (2013) introduced a new model of time preferences: the Instantaneous-Gratification model, and developed their research for one consumer who had a stock of financial wealth and received a flow of labor income. Zou, Chen, & Wedge (2014) incorporated the IG preferences into the classical model of Merton (1971, 1969) with constant relative risk aversion (CRRA), in finite horizon. Here, we interpret the IG model under the framework of the management of renewable natural resources and compute the sophisticated solution in infinite horizon for one and several agents. These agents can be thought of as individuals, companies, institutions, governments or countries, who oversee a certain resource that replenishes over time by some natural process, such as forests, fisheries or agriculture.

Once the individual solutions are achieved, we will analyze the cooperative and noncooperative feedback solutions.

We designate J as the net payoff of the agent, so that, the functional to be maximized in the one-agent infinite-time problem, is

$$J=\int_t^{t+\tau} e^{-\rho(s-t)}u\big(c(s)\big)ds+\beta\int_{t+\tau}^{\infty} e^{-\rho(s-t)}u\big(c(s)\big)ds\,. \tag{1}$$

Let $c(s)$ be the equilibrium control rule and $w(s)$ the state trajectory of the problem. Then, $u(c,w)$ is the stream of payoffs obtained from applying the control c, given a state w at any moment $s, t\le s\le T$. Furthermore, $c(s)=\phi(w(s),s)$, where $c^*(s)=\phi(w(s),s)$ is the equilibrium strategy. The present-biased indicator, β, embodies a measure of how intense the resource is exploited in the short-run.

At any time $t\ge 0$ the resource owner extracts a certain amount of the resource stock, $c(s)>0$ and the resource refills at a rate $r>0$. Hence, the dynamics of the resource are

$$\dot{w}=rw-c\,. \tag{2}$$

Designate $V(w)$ as the value function:

$$V(w)=\int_t^{\infty}\theta(s-t)\,u(c^*,w)\;ds\,. \tag{3}$$

Finally, the Dynamic Programing Equation (henceforward DPE) for the single-agent problem is:

$$K(w)+\rho V(w)=\max_c\{u(c,w)+V'g(c,w)\}\,, \tag{4}$$

where

$$K(w) = \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\left[u\big(c^*(s;w)\big)\right]ds\right], \tag{5}$$

and $g(c,w)$ is the function that describes the equation of motion, $\frac{dw}{dt} = \dot{w} = g(c,w)$.

The problem above defined, first yields the result for the case of the stochastic hyperbolic discounting[1]. The solution for the IG discounting is retrieved by taking $\lambda \to \infty$ on the stochastic-hyperbolic discounting result.

In the multi-agent problem, agents can exploit the resource together or separately. For instance, think of a farm which is exploited by several farmers, for some, it might be more important to extract as much as possible in the current season while for others, it could be better to not surpass a certain level of extraction and assure that the field gets ready for the next harvest.

3.1. The Cooperative case

In the cooperative game, players joint strengths to exploit the resource. Therefore, they agree on personal parameters, such that, the duration of the present, τ, and the weight of future harvest, implied by β.

Let $J^C = \zeta J_1 + \gamma J_2$, with $\zeta, \gamma \geq 0$. If we assume $\zeta = \gamma = 1$, the net payoff of the resource, is expressed as:

$$J_1 + J_2 = J^C = \int_t^{t+\tau} e^{-\rho(s-t)}\left[u_1\big(c_1(s)\big) + u_2\big(c_2(s)\big)\right] ds$$
$$+\beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)}\left[u_1\big(c_1(s)\big) + u_2\big(c_2(s)\big)\right] ds\,, \tag{6}$$

contingent to the dynamics

$$\dot{w} = rw - c_1 - c_2\,. \tag{7}$$

The cooperative value function is $V_1 = V_2 = V;\ V^C = 2V$, and the DPE is defined as

$$\rho V^C(w) + K^C(w) = \max_{c_1,c_2}\left[[u_1(c_1) + u_2(c_2)] + V'^C(w)(rw - c_1 - c_2)\right], \tag{8}$$

where, $V^{c\prime}$, represents the value of the coalition between the agents, and $K(w)$ is expressed as

$$K^C(w) = \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}[u_1(c_1^*) + u_2(c_2^*)]ds\right]. \tag{9}$$

[1] See appendices for this result

In general, in the cooperative case with N agents, the problem to solve is:

$$\sum_{i=1}^{N} J_i = J^C = \int_t^{t+\tau} e^{-\rho(s-t)} \sum_{i=1}^{N} u_i\big(c_i(s)\big)\ ds + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)} \sum_{i=1}^{N} u_i\big(c_i(s)\big)\ ds\ , \tag{10}$$

subject to,

$$\dot{w} = rw - \sum_{i=1}^{N} c_i\ . \tag{11}$$

The value function is $V_i = V_j = \cdots = V_N = V; V^C = NV$, with the following DPE:

$$\rho V^C(w) + K^C(w) = \max_{c_1, c_2, \ldots, c_n} \left[\sum_{i=1}^{N} u_i(c_i) + V'^C(w) \left(rw - \sum_{i=1}^{N} c_i \right) \right], \tag{12}$$

where

$$K^C(w) = \lambda(1-\beta) \left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \sum_{i=1}^{N} u_i(c_i^*)\, ds \right]. \tag{13}$$

When the players are symmetric, meaning that $u_1 = u_2 = \cdots = u_N = u$, the problem is equivalent to:

i. $$NJ_i = J^C = \int_t^{t+\tau} e^{-\rho(s-t)}\, Nu\big(c(s)\big)\, ds + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)}\, Nu\big(c(s)\big)\, ds\ ; \tag{14}$$

ii. $$\dot{w} = rw - Nc\ ; \tag{15}$$

iii. $$\rho V^C(w) + K^C(w) = \max_c [Nu(c) + V'^C(w)(rw - Nc)]\ ; \tag{16}$$

iv. $$K^C(w) = \lambda(1-\beta) \left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} Nu(c^*) ds \right]. \tag{17}$$

3.2. The Noncooperative Solution

In the noncooperative game, the agents play independently and their strategies of extraction are unknown. Each player will maximize the following functional:

$$J_i = J^{NC} = \int_t^{t+\tau_i} e^{-\rho(s-t)}\, u_i\big(c_i(s)\big)\, ds + \beta_i \int_{t+\tau_i}^{\infty} e^{-\rho(s-t)}\, u_i\big(c_i(s)\big)\, ds, \tag{18}$$

with $i = 1,2,\ldots,N$. The dynamics of the model are

$$\dot{w} = rw - c_i - \sum_{j=1}^{N} \phi_j(w)\,, i, j = 1,2, \ldots, N;\ i \neq j\ . \tag{19}$$

The DPE is:

$$\rho V_i(w) + K_i(w) = \max_{c_i}\left[u_i(c_i) + V_i^{\prime NC}(w)\left(rw - c_i - \sum_{j=1}^{N}\phi_j(w)\right)\right], \tag{20}$$

with $i, j = 1,2, \dots, N;\ i \neq j$. Where,

$$K_i(w) = \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} u_i(c_i^*) ds\right], i = 1,2,..,N\ . \tag{21}$$

When the players are symmetric, $u_1 = u_2 = \cdots = u_N = u$, the problem is equivalent to:

i. $$J_i = J^{NC} = \int_t^{t+\tau_i} e^{-\rho(s-t)}\, u\big(c(s)\big)\, ds + \beta_i \int_{t+\tau_i}^\infty e^{-\rho(s-t)}\, u\big(c(s)\big)\, ds \tag{22}$$

ii. $$\dot{w} = rw - Nc \tag{23}$$

iii. $$\rho V_i^{NC}(w) + K_i^{NC}(w) = \max_c[u(c) + V_i^{\prime NC}(w)(rw - Nc)], \tag{24}$$

iv. $$K_i^{NC}(w) = \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} u(c^*) ds\right] \tag{25}$$

In the upcoming section, we will solve, analytically, the one-agent and the symmetric cooperative and noncooperative N-agent problems introduced up to this point.

4. RESULTS

To illustrate the results, we will use isoelastic utilities of the form:

$$u(c) = \begin{cases} \ln c(t)\,, & b = 1 \quad (26.1) \\ \dfrac{c^{1-b}}{1-b}, & b \neq 1 \quad (26.2) \end{cases},$$

as in Harris & Laibson (2013) and Zou, Chen, & Wedge (2014), where $\frac{1}{b}$ is the intertemporal elasticity of substitution. For a matter of simplicity, we narrow our analysis to the symmetric N-agent cases. To avoid burdening the reader with the calculations, we present them in the appendix. This section is subdivided into two major sections. In the first part, we present the results for the case of the logarithmic utility described by equation (26.1) and in section 4.2 we show the results for the power utility featured by (26.2).

4.1. Logarithmic Utility

Based on the form of the utility function, the general proposal for the value function is:

$$V_i(w) = A_i \ln(w) + B_i \,, i = 1, 2, \dots, N. \tag{27}$$

From where it follows that

$$V_i'(w) = \frac{A_i}{w} \,, i = 1, 2, \dots, N. \tag{28}$$

4.1.1. Single Agent

Equation (1) is to be maximized as

$$\max_c \int_t^{t+\tau} e^{-\rho(s-t)} \ln c + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)} \ln c \, ds \,, \tag{29}$$

subject to:

$$\dot{w} = rw - c \,. \tag{30}$$

The DPE equation is

$$\rho V(w) + K(w) = \max_c [\ln c + V'(w)(rw - c)] \,, \tag{31}$$

where,

$$K(w) = \lambda(1-\beta) \left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln c \, ds \right]. \tag{32}$$

After the calculations enclosed in appendix I, we arrive to proposition 1.

Proposition 1 The sophisticated agent with logarithmic utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^* = \frac{\rho}{\beta} w_0 \, e^{\left(r - \frac{\rho}{\beta}\right)t} \,; \tag{33}$$

ii. Consistent stock-level trajectory

$$w(t) = w_0 \, e^{\left(r - \frac{\rho}{\beta}\right)t} \,. \tag{34}$$

4.1.2. Several Agents

A. The cooperative case

We will now illustrate the N-agent cooperative case. We assume the agents are symmetric. Equation (14) is expressed as

$$NJ_i = J^C = \int_t^{t+\tau} e^{-\rho(s-t)} [N \ln c] \, ds + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)} [N \ln c] \, ds \,, \tag{35}$$

constrained to the resource dynamics, which are defined as

$$\dot{w} = rw - Nc \,. \tag{36}$$

The DPE is

$$\rho V^C(w) + K^C(w) = \max_c \left[[N \ln c] + V^{C\prime}(w)(rw - Nc) \right] , \tag{37}$$

where,

$$K^C(w) = \lambda(1-\beta) \left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \left[\ln c^{C^*} \right] ds \right] \tag{38}$$

With the inherent calculations that can be verified in appendix II, we come to the second proposition.

Proposition 2 The sophisticated cooperative solution for symmetric agents with logarithmic utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^{C^*} = \frac{\rho w_0 e^{\left(r - \frac{\rho N}{\beta - 1 + N}\right)t}}{\beta - 1 + N} \; ; \tag{39}$$

ii. Consistent stock-level trajectory

$$w(t) = w_0 e^{\left(r - \frac{\rho N}{\beta - 1 + N}\right)t} \,. \tag{40}$$

B. The Noncooperative case

Each agent will maximize his own objective, contingent to the extraction of the other players. Equation (22) is defined as

$$J_i = J^{NC} = \int_t^{t+\tau_i} e^{-\rho(s-t)} \ln c \; ds + \beta_i \int_{t+\tau_i}^{\infty} e^{-\rho(s-t)} \ln c \; ds \,, \tag{41}$$

subject to the dynamics of the resource defined by (23) as

$$\dot{w} = rw - Nc \,. \tag{42}$$

The DPE equation is

$$\rho V_i^{NC}(w) + K_i^{NC}(w) = \max_c \left[\ln c + V_i^{NC'}(w)(rw - Nc) \right] , \tag{43}$$

where,

$$K_i^{NC}(w) = \lambda(1-\beta_i) \left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln c^* \, ds \right] . \tag{44}$$

With the inherent calculations that can be verified in appendix III, we get to the following

Proposition 3 The sophisticated noncooperative solution for symmetric agents with logarithmic utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^{NC^*} = \frac{\rho w_0 e^{\left(r - \frac{\rho N}{\beta_i - 1 + N}\right) t^*}}{\beta_i} \quad ; \tag{45}$$

ii. Consistent stock-level trajectory

$$w^{NC}(t) = w_0 e^{\left(r - \frac{\rho N}{\beta_i}\right) t} \,. \tag{46}$$

4.2. Power Utility

Based on the form of the utility function, the general proposal for the value function is:

$$V_i(w) = h_i \frac{w(t)^{1-b_i}}{1-b_i} \,, i = 1, 2, \ldots, N. \tag{47}$$

From where it follows that

$$V_i'(w) = h_i w(t)^{-b_i} \,, i = 1, 2, \ldots, N. \tag{48}$$

4.2.1. Single Agent

Equation (1) is to be maximized as

$$\max_c \int_t^{t+\tau} e^{-\rho(s-t)} \frac{c^{1-b}}{1-b} + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)} \frac{c^{1-b}}{1-b}\, ds\,, \tag{49}$$

subject to:

$$\dot{w} = rw - c\,. \tag{50}$$

The DPE equation is

$$\rho V(w) + K(w) = \max_c [\frac{c^{1-b}}{1-b} + V'(w)(rw-c)]\,, \tag{51}$$

where,

$$K(w) = \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \frac{{c^*}^{1-b}}{1-b} ds\right]. \tag{52}$$

Through the calculations shown in appendix IV, we arrive to

Proposition 4 The sophisticated agent with power utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^* = \frac{(1-b)r-\rho}{1-\beta-b}\, w_0\, e^{\left(r-\frac{(1-b)r-\rho}{1-\beta-b}\right)t}\,; \tag{53}$$

ii. Consistent stock-level trajectory

$$w(t) = w_0\, e^{\left(r-\frac{(1-b)r-\rho}{1-\beta-b}\right)t}\,. \tag{54}$$

4.2.2. Several Agents

A. The Cooperative Solution

We will now illustrate the N-agent cooperative case. We assume symmetry among agents. Equation (14) is expressed as

$$NJ_i = J^C = \int_t^{t+\tau} e^{-\rho(s-t)} \left[N\frac{c^{1-b}}{1-b}\right] ds + \beta \int_{t+\tau}^{\infty} e^{-\rho(s-t)} \left[N\frac{c^{1-b}}{1-b}\right] ds\,, \tag{55}$$

constrained to the resource dynamics, which are defined as

$$\dot{w} = rw(t) - N\, c(t)\,. \tag{56}$$

The DPE is

$$\rho V^C(w) + K^C(w) = \max_c \left[\left[N \frac{c^{1-b}}{1-b}\right] + V'^C(w)(rw - N\,c)\right], \tag{57}$$

where,

$$K^C(w) \ = \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\left[\frac{{c^*}^{1-b}}{1-b}\right] ds\right] \tag{58}$$

With the inherent calculations that can be verified in appendix V, we come to the fifth proposition.

Proposition 5 The sophisticated cooperative solution for symmetric agents with power utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^{C^*} = \frac{-\rho + r(1-b)}{N(1-b-\beta)} w_0 \ e^{\left(r - \frac{-\rho + r(1-b)}{N(1-b-\beta)}\right)t}; \tag{59}$$

ii. Consistent stock-level trajectory

$$w^C(t) = w_0 \ e^{\left(r - \frac{-\rho + r(1-b)}{N(1-b-\beta)}\right)t}. \tag{60}$$

B. The Noncooperative Solution

Each agent will maximize his own objective, contingent to the extraction of the other players. Equation (22) is defined as

$$J_i = J^{NC} = \int_t^{t+\tau_i} e^{-\rho(s-t)} \frac{c^{1-b}}{1-b}\, ds + \beta_i \int_{t+\tau_i}^\infty e^{-\rho(s-t)} \frac{c^{1-b}}{1-b}\, ds\,, \tag{61}$$

subject to the dynamics of the resource defined by (23) as

$$\dot{w} = rw(t) - Nc(t) \tag{62}$$

The DPE is

$$\rho V_i^{NC}(w) + K_i^{NC}(w) = \max_c \left[\frac{c^{1-b}}{1-b} + V_i^{NC'}(w)(rw - Nc)\right], \tag{63}$$

where,

$$K_i^{NC}(w) = \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} \frac{c^{1-b}}{1-b} ds\right]. \tag{64}$$

With the inherent calculations that can be verified in appendix VI, we get to the following

Proposition 6 The sophisticated noncooperative solution for symmetric agents with power utility and instantaneous-gratification discounting is characterized by the following results:

i. Equilibrium extraction path

$$c^{NC^*} = N^{-\frac{1}{b}} \frac{-\rho + r(1-b)}{N\left(1 - b - N^{-\frac{1}{b}}\beta_i\right)} w_0 \, e^{\left[r - N^{\frac{b-1}{b}}\left(\frac{-\rho + r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}\right)\right]t}; \tag{65}$$

ii. Consistent stock-level trajectory

$$w^{NC}(t) = w_0 \, e^{\left[r - N^{\frac{b-1}{b}}\left(\frac{-\rho + r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}\right)\right]t}. \tag{66}$$

5. DISCUSSION

To illustrate the results obtained in the previous section, we are going to attribute values to the parameters β, r, ρ and b, to analyze the impact they have on the equilibrium extraction patch, c^*.

We are interested in comparing the log and the power utility for a single agent and then for the cooperative and the noncooperative solution. We follow Zou, Chen, & Wedge (2014) in the values chosen for the parameters, which are listed in table 1, and we set $w_0 = 1$.

Table I Values assigned to the parameters

Description	Symbol	Value assigned
Discount rate	ρ	0.046
Replenishing rate	r	0.5
Present-bias	β	1 0.8 0.5 0.25
Marginal elasticity	b	1.5 (single agent) 0.5 (several agents)

In figure 2, we can observe that stronger present - biased agents (lower β) extract at a greater rate in the beginning, but are overpassed by more patient agents in the medium - long run. In infinite horizon, in the long term, the extraction level of agents characterized by high values of β, thus more patient individuals, tend to coincide, whereas the extraction rate of impatient agents drops drastically with time. In the power utility case, patient agents are always better off, as impatient individuals are crossed by patient ones much earlier than in the logarithmic case.

In the cooperative case (figure 3), as expected, levels of individual extraction decrease with the number of players. In this scenario, the group is better off when all are impatient, regardless the form of the utility function. In the long term, impatient cooperative agents will still extract at a higher rate than patient ones.

In the noncooperative game (figure 4), since each player decides, unilaterally, which level of stock to extract at each moment, the number of players does not affect the rate of extraction. In the short-run, impatient resource owners represented by a logarithmic utility function are better off than patient agents with a power utility function. In the long term, all behaviors converge to the same extraction level.

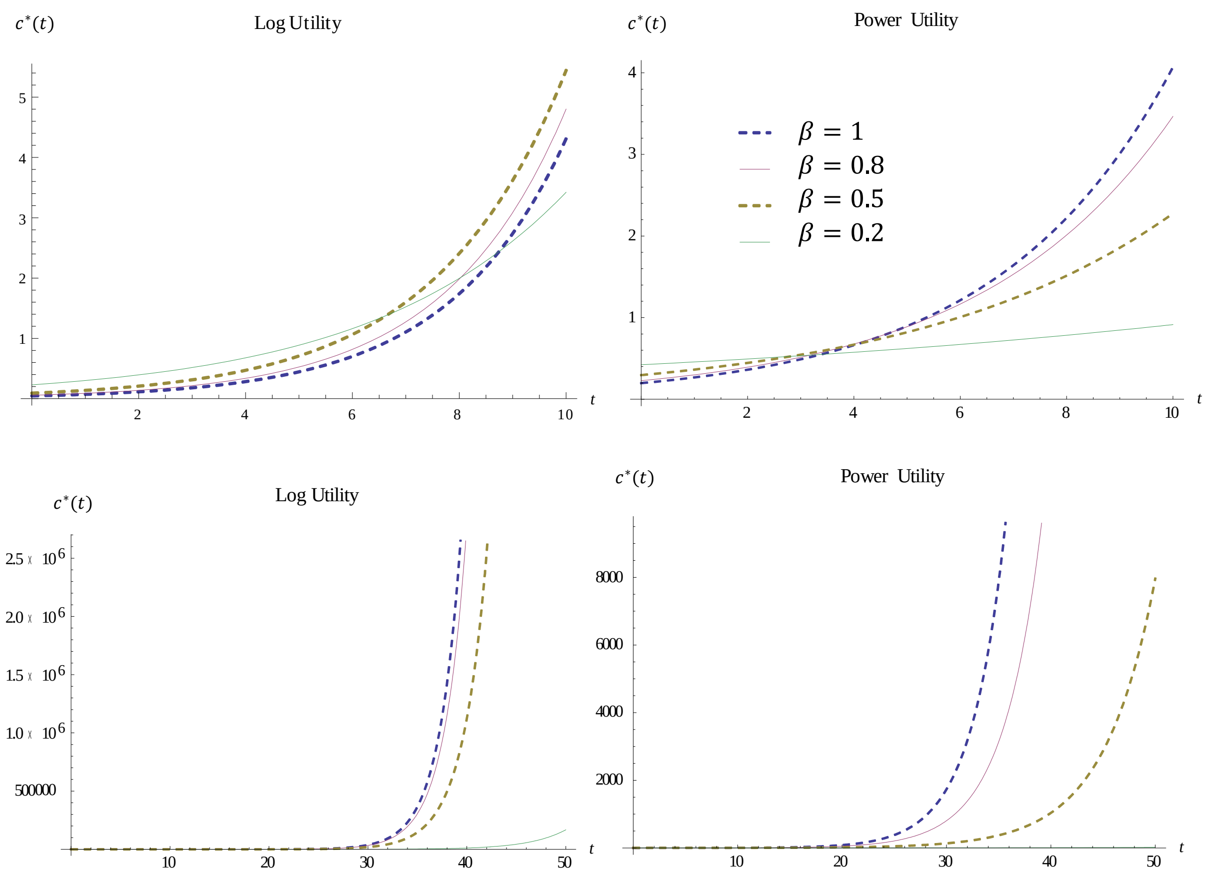

Figure 2 Impact of β and t on $c^*(t)$ of single sophisticated agents.

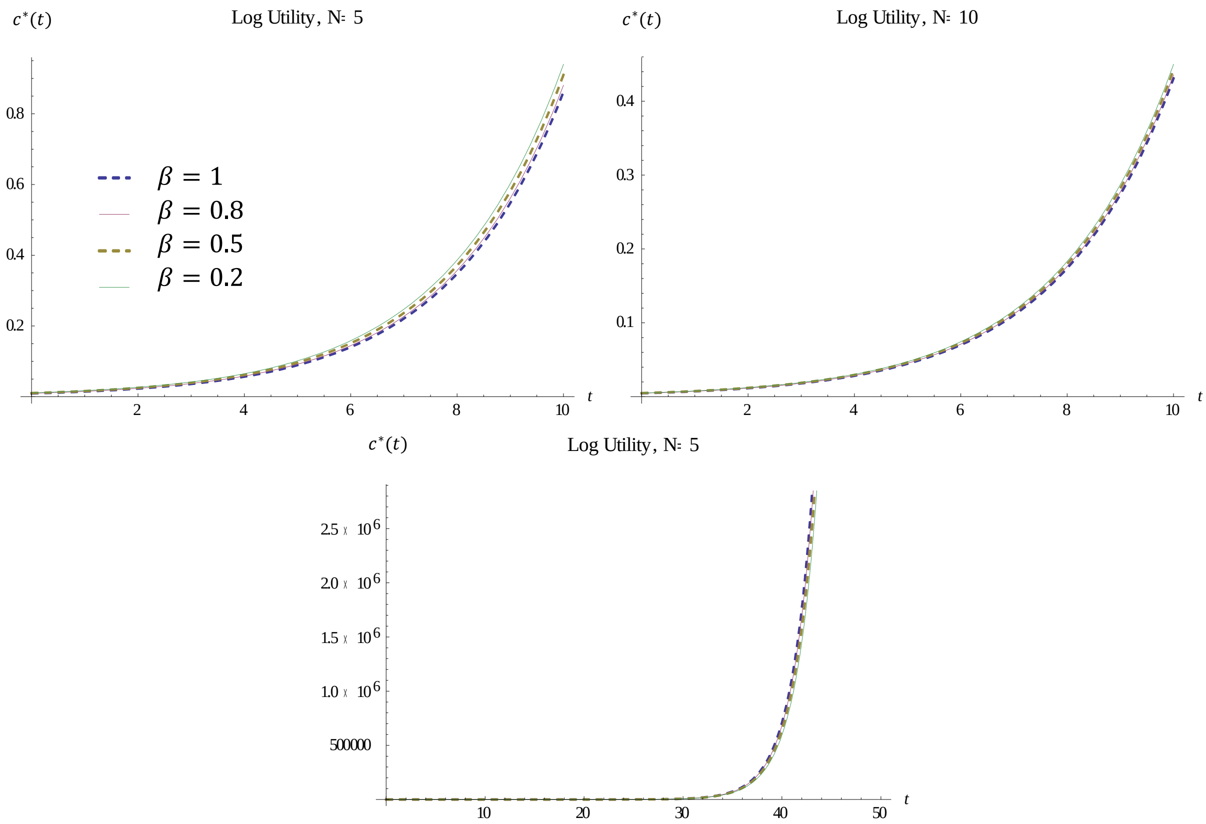

Figure 3: Impact of β, N and t on $c^*(t)$ of cooperative sophisticated agents.

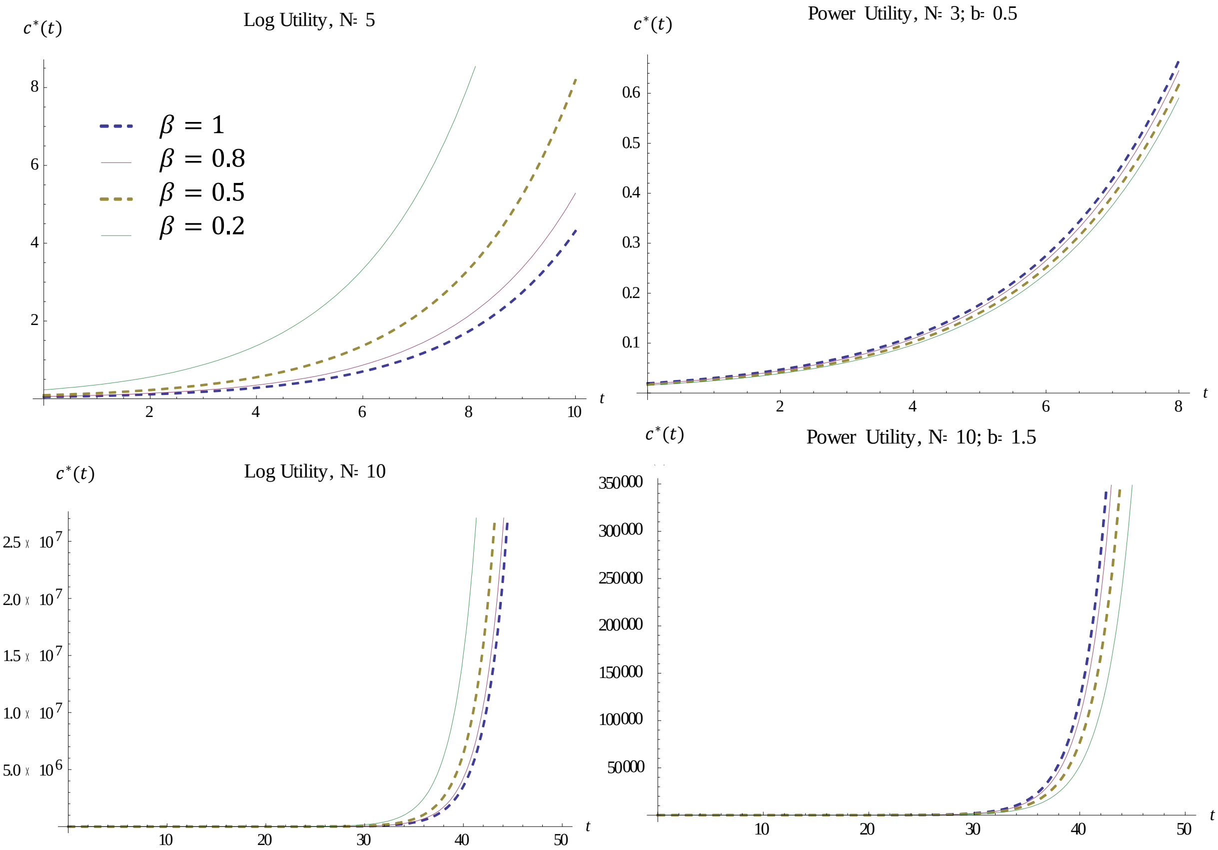

Figure 4*:* Impact of β, N and t on $c^*(t)$ of noncooperative sophisticated agents.

6. CONCLUSIONS

We have computed the sophisticated solution with Instantaneous-Gratification discounting in infinite horizon for a single agent, as an extension of Harris & Laibson (2013) and Zou, Chen, & Wedge (2014) and we have contributed to the existing literature by extending the model with more than one agent, studying both the cooperative and noncooperative solutions. In addition, the results were interpreted under the framework of natural resources management.

The discussion of this work may be useful for policy implementation towards exploitation of renewable natural resources under different forms of ownership.

In a resource to be exploited by a single agent, restrictions on impatient behavior are important to assure harvesting in the long run.

Contrariwise, in a resource that is to be exploited by several owners, who cooperate among them, the restrictions on impatient behavior could be relaxed, as our results suggest that the patient group will be worse off.

In the noncooperative game, the optimal policy will depend on the form of the utility function. For the logarithmic utility, agents extracting, virtually, all what the resource yields in each period, will be better off. Yet, for the power utility case, patient players will be more fortunate.

Unlike the timeline considered in this research, academic projects are handled in a finite time horizon. Thus, we leave room for improvements in future research. Despite the appealing feature the sophisticated behavior provides, the precommit and naive solutions embody other relevant behaviors too. As a matter of fact, the naive behavior is likely to be a finer portrayal of individuals' inborn conduct, whereas the precommit and sophisticated results could be the representative of learning or compulsory behaviors. Hence, it would be compelling, to integrate the analysis here performed, to compute the precommit and naive solutions in infinite time with Instantaneous-Gratification discounting.

Likewise, it would be noteworthy, to analyze the influence that the other parameters, such as the refilling rate and the coefficient b, have on the extraction capacity of the resources.

REFERENCES

Barro, R. J. 1999. "Ramsey Meets Laibson in The Neoclassical Growth Model." *The Quarterly Journal of Economics* 114 (4): 1125–52.

Clark, Colin Whitcomb. 1990. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*. *Pure and Applied Mathematics CN - QH705.C4 1990*. 2nd ed. New York: Wiley.

Fisher, I. 1930. "The Theory of Interest Rates." *New York*.

Harris, Christopher, and David Laibson. 2013. "Instantaneous Gratification." *Quarterly Journal of Economics*, no. 1999: 205–48. doi:10.1093/qje/qjs051.Advance.

Laibson, D. 1997. "Golden Eggs and Hyperbolic Discounting." *The Quarterly Journal of Economics* 112 (2): 443–78. doi:10.1162/003355397555253.

Loewenstein, George, and Richard H. Thaler. 1989. "Intertemporal Choice." *Journal of Economic Perspectives*. doi:10.1257/jep.3.4.181.

Luttmer, Erzo G J, and Thomas Mariotti. 2003. "Subjective Discounting in an Exchange Economy." *Journal of Political Economy* 111 (5): 959–89. doi:10.1086/376954.

McClure, S. M. 2004. "Separate Neural Systems Value Immediate and Delayed Monetary Rewards." *Science* 306 (5695): 503–7. doi:10.1126/science.1100907.

Merton, Robert C. 1971. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory* 3 (4): 373–413. doi:10.1016/0022-0531(71)90038-X.

Merton, Robert C. 1969. "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case." *The Review of Economics and Statistics* 51 (3): 247. doi:10.2307/1926560.

O'Donoghue, Ted, and Matthew Rabin. 2000. "The Economics of Immediate Gratification." *Journal of Behavioral Decision Making* 13 (2): 233–50. doi:10.1002/(SICI)1099-0771(200004/06)13:2<233::AID-BDM325>3.0.CO;2-U.

O 'Donoghue, Ted, and Matthew Rabin. 1999. "Doing It Now or Later." *Source: The American Economic Review* 89 (1): 103–24. http://www.jstor.org/stable/116981.

Phelps, E. S., and R. A. Pollak. 1968. "On Second-Best National Saving and Game-Equilibrium Growth." *The Review of Economic Studies* 35 (2): 185. doi:10.2307/2296547.

Pollak, R. A. 1968. "Consistent Planning." *The Review of Economic Studies* 35 (2): 201–8.

doi:10.2307/2296548.

Samuelson, P. a. 1937. “Note on Measurement of Utility.” *The Review of Economic Studies* 4 (2): 155–61. doi:10.1016/0001-6918(58)90027-1.

Strotz, R H. 1955. “Myopia and Inconsistency in Dynamic Utility Maximization.” *Source: The Review of Economic Studies* 23 (3): 165–80. http://www.jstor.org/stable/2295722.

Thaler, Richard H. 1981. “Some Empirical Evidence on Dynamic Inconsistency.” *Economic Letters*. doi:10.1016/0165-1765(81)90067-7.

Zou, Ziran, Shou Chen, and Lei Wedge. 2014. “Finite Horizon Consumption and Portfolio Decisions with Stochastic Hyperbolic Discounting.” *Journal of Mathematical Economics* 52 (May): 70–80. doi:10.1016/j.jmateco.2014.03.002.

APPENDIX: Calculations

In this section, we present in detail all the calculations behind each of the results proposed in the text.

Appendix I:

Calculations for proposition 1.

By maximizing the right side of the DPE respect to c, we obtain the optimal control of the resource, c^*:

$$\frac{1}{c} - V'(w) = 0$$

$$c = \frac{1}{V'(w)}$$

$$c^* = \frac{w}{A} \tag{1}$$

The consumption rate is given by the expression:

$$\frac{c^*(t)}{w(t)} = \frac{1}{A} \tag{2}$$

Now, we compute the time-consistent state of the stock of the resource.

$$\dot{w} = rw - c \overset{by\ (1)}{\Longleftrightarrow} \dot{w} = rw - \frac{w}{A} \Leftrightarrow \dot{w} - \left(r - \frac{1}{A}\right) w = 0$$

The above is a separable first-order linear differential equation that can easily be solved by any method that applies to this kind of equations. All these types of equations will be solved using the method of the integrating factor. This time, the integrating factor is $e^{-\left(r-\frac{1}{A}\right)t}$. Multiplying through, it yields:

$$e^{-\left(r-\frac{1}{A}\right)t}\left[\dot{w} - \left(r - \frac{1}{A}\right) w = 0\right] \Leftrightarrow e^{-\left(r-\frac{1}{A}\right)t}\dot{w} - e^{-\left(r-\frac{1}{A}\right)t}\left(r - \frac{1}{A}\right) w = 0$$

$$\Leftrightarrow \frac{d\left(e^{-\left(r-\frac{1}{A}\right)t} w\right)}{dt} = 0$$

$$\Leftrightarrow e^{-\left(r-\frac{1}{A}\right)t} w = D$$

Multiplying by $e^{\left(r-\frac{1}{A}\right)t}$, we obtain:

$$w(t) = e^{\left(r-\frac{1}{A}\right)t} D$$

Applying the boundary condition $w(0) = 0$:

$$w(0) = e^{\left(r-\frac{1}{A}\right)0} D = D$$

If we assume that $w(0) = w_0$, then $w_0 = D$, which yields:

$$w(t) = w_0\, e^{\left(r-\frac{1}{A}\right)t} \tag{3}$$

The above equation describes the optimal path of the stock of the resource over time. It can also be expressed for any moment s:

$$w(s) = w_t\, e^{\left(r-\frac{1}{A}\right)(s-t)}, s > t \tag{4}$$

Now, we replace the resulting expressions from above into $K(w)$:

$$\begin{aligned}
K(w) &= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} \ln c^*\, ds\right] \\
&= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} \ln\frac{w}{A} ds\right] \\
&= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}[\ln w - \ln A] ds\right] \\
&= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\left[\ln w_t + \left(r-\frac{1}{A}\right)(s-t)\right] ds - \right. \\
&\qquad \left. - \ln A \int_t^\infty e^{-(\lambda+\rho)(s-t)} ds\right] \\
&= \lambda(1-\beta)\left[(\ln w_t - \ln A)\int_t^\infty e^{-(\lambda+\rho)(s-t)} ds + \left(r-\frac{1}{A}\right)\right. \\
&\qquad \left.\times \int_t^\infty e^{-(\lambda+\rho)(s-t)}(s-t) ds\right]
\end{aligned} \tag{5}$$

Replacing $c^*, V(w), K(w)$ and V' into the DPE, we obtain:

$$\rho(A\ln w_t + B) + \lambda(1-\beta)\left[(\ln w_t - \ln A)\int_t^\infty e^{-(\lambda+\rho)(s-t)}\, ds + \left(r-\frac{1}{A}\right)\times \right.$$
$$\left. \times \int_t^\infty e^{-(\lambda+\rho)(s-t)}\,(s-t) ds\right] = \ln w(t) - \ln A + A\,r - 1$$

The next step is to impose that A is such, that it is the same for all values of $w(t)$. So that, we collect all the terms adjacent to w_t in the expression above, to obtain:

$$\rho A + \lambda(1-\beta)\int_t^\infty e^{-(\lambda+\rho)(s-t)}\, ds = 1\,,$$

from where we can extract the constant A:

$$A = \frac{\rho + \lambda\beta}{\rho(\lambda+\rho)} \tag{6}$$

The expression above provides a solution for the constant A for the hyperbolic discounting. Assuming the limiting case when λ approaches to infinite, we obtain the constant in the IG case:

$$A = \frac{\beta}{\rho} \tag{7}$$

By replacing the above expression into $c^*, w(t)$ and $\frac{c^*(t)}{w(t)}$, we will obtain the resulting equations for the IG case, which are:

$$c^* = \frac{\rho}{\beta} w_0 \, e^{\left(r - \frac{\rho}{\beta}\right)t} \tag{8}$$

$$w(t) = w_0 \, e^{\left(r - \frac{\rho}{\beta}\right)t} \tag{9}$$

$$\frac{c^*}{w(t)} = \frac{\rho}{\beta} \tag{10}$$

Appendix II

Calculations for proposition 2.

By maximizing the right side of the DPE, we obtain the optimal cooperative control, which is:

$$N \frac{1}{c} - V^{C'} N = 0$$

$$c^{C^*} = \frac{w^C}{A^C} \tag{11}$$

From where we extract the optimal cooperative extraction rate

$$\frac{c^{C^*}}{w^C(t)} = \frac{1}{A^C} \, . \tag{12}$$

Secondly, we compute the time-consistent state of the stock of the resource:

$$\dot{w} = rw - Nc^{C^*} \Leftrightarrow \dot{w} = rw - N\left(\frac{w^C}{A^C}\right) \Leftrightarrow \dot{w} - w^C\left(r - \frac{N}{A^C}\right) = 0 \tag{13}$$

Like what we did in last section, here the integrating factor is $e^{-\left(r-\frac{N}{A}\right)t}$. Following the same approach as for (3) and (4), we arrive to

$$w^C(t) = w_0 \, e^{\left(r - \frac{N}{A^C}\right)t} \, , \tag{14}$$

And to

$$w^C(s) = w_t \, e^{\left(r - \frac{N}{A^C}\right)(s-t)}, s > t \, . \tag{15}$$

Now, we replace the resulting expressions into $K^C(w)$:

$$
\begin{aligned}
K^C(w) &= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\ \ln c^*\, ds\right] \\
&= \ \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln\frac{w}{A}\, ds\right] \\
&= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}[\ln w(s) - \ln A] ds\right] \\
&= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln w(s)\, ds - \int_t^{\infty} e^{-(\lambda+\rho)(s-t)}(\ln A) ds\right] \\
&= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\left[\ln w_t + \left(r - \frac{N}{A}\right)(s-t)\right] ds - \ \ln A \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds\right] \\
&= \lambda(1-\beta)\left[(\ln w_t - \ln A)\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds + \left(r - \frac{N}{A}\right)\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\ (s-t) ds\right]
\end{aligned} \tag{16}
$$

Replacing $c^*, V^C(w), K^C(w)$ and $V^{C'}$ into the DPE, we obtain:

$$
\begin{aligned}
\rho(A\ln(w) + B) + \lambda(1-\beta)[(\ln w_t - \ln A)\times \\
\times \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds + \left(r - \frac{1}{AN}\right)\times \\
\times \int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\ (s-t) ds = \\
= \mathrm{N}[\ln \mathrm{w} - \ln \mathrm{A}] + Ar - N
\end{aligned}
$$

The next step is to impose that A^C is such, that it is the same for all values of w for all i. So that, we collect all the terms adjacent to w_t in the expression above:

$$
\rho A + \lambda(1-\beta)\int_t^{\infty} e^{-(\lambda+\rho)(s-t)}\ ds = N\ ,
$$

Solving the integral, we get

$$
\rho A + \lambda(1-\beta)\left[\frac{1}{\lambda + \ \rho}\right] = N\ .
$$

From where, the solution for the hyperbolic discounting is:

$$
A = \frac{N\rho + (\beta + N - 1)\lambda}{\rho(\rho+\lambda)} \tag{17}
$$

Like what was done before, taking the limiting case when λ approaches infinite, we obtain the constant for the IG model:

$$
A^C = \frac{\beta - 1 + N}{\rho} \tag{18}
$$

By replacing A^C into $c^{C*}, w(t)$ and $\frac{c^*(t)}{w(t)}$, we obtain the final expressions in the cooperative case when agents are symmetric and use a logarithmic utility with IG discounting:

$$c^{C^*} = \frac{\rho w_0 e^{\left(r - \frac{\rho N}{\beta - 1 + N}\right)t}}{\beta - 1 + N} \tag{19}$$

$$w^C(t) = w_0 e^{\left(r - \frac{\rho N}{\beta - 1 + N}\right)t} \tag{20}$$

$$\frac{c^{C^*}}{w^C(t)} = \frac{\rho}{\beta - 1 + N} \tag{21}$$

Appendix III

Calculations for proposition 3.

By maximizing the right side of the DPE respect to c, we obtain the optimal control:

$$\frac{1}{c} - NV_i^{NC'}(w) = 0$$

$$c^{NC^*} = \frac{w^{NC}(t)}{A_i N} \tag{22}$$

The time-consistent state of the stock of the resource is:

$$\dot{w} = rw - Nc^* \Leftrightarrow \dot{w} = rw - N\left(\frac{w}{A_i N}\right) \Leftrightarrow \dot{w} - w\left(r - \frac{1}{A_i}\right) = 0 \tag{23}$$

(25) is a separable first-order linear differential equation. Choosing $e^{-\left(r - \frac{1}{A}\right)t}$ as the integrating factor and multiplying through, as before, it yields:

$$w^{NC}(t) = w_0 \, e^{\left(r - \frac{1}{A_i}\right)t} \tag{24}$$

Note that now the constant A_i^{NC} is considered at the individual level. The above equation describes the optimal path of the stock of the resource over time. It can also be expressed for any moment s:

$$w^{NC}(s) = w_t \, e^{\left(r - \frac{1}{A_i}\right)(s-t)}, s > t \tag{25}$$

Now, we replace the resulting expressions from above into $K(w)$:

$$K_i^{NC}(w) = \lambda(1-\beta_i)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln c^* \, ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln \frac{w}{A_i N} ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} [\ln w - \ln A_i - \ln N] ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \ln w \, ds - \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} (\ln A_i - \ln N) ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \left[\ln w_t + \left(r - \frac{1}{A_i}\right)(s-t)\right] ds - \right.$$

$$\left. -(\ln A_i - \ln N) \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds\right]$$

$$= \lambda(1-\beta_i)[(\ln w_t - \ln A_i - \ln N) \times$$

$$\left. \times \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds + \left(r - \frac{1}{A_i}\right) \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} (s-t) ds\right] \quad (26)$$

Replacing $c^{*NC}, V^{NC}(w), K^{NC}(w)$ and V'^{NC} into the DPE, we obtain:

$$\rho(A_i \ln(w) + B_i) + \lambda(1-\beta_i) \times$$

$$\times \left[(\ln w_t - \ln A_i - \ln N) \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds + \left(r - \frac{1}{A_i}\right) \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} (s-t) ds\right] =$$

$$= \ln w - \ln A_i - \ln N + A_i r - 1 \quad (27)$$

The next step is to impose that A_i^{NC} is such, that it is the same for all values of $w$for all i. So that, we collect all the terms adjacent to w_t in the expression above:

$$\rho A_i^{NC} + \lambda(1-\beta_i) \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} ds = 1 \quad (28)$$

From where, the constant for the hyperbolic discount is:

$$A_i^{NC} = \frac{\rho + \lambda\beta_i}{\rho(\lambda+\rho)} \quad (29)$$

And as we did before, we take λ to infinite to get the IG constant:

$$A_i^{NC} = \frac{\beta_i}{\rho} \quad (30)$$

By replacing A_i into the original expressions we obtain the final expressions in the noncooperative case when agents are symmetric and use a logarithmic utility with instantaneous gratification discounting:

$$c^{NC^*} = \frac{\rho w_0 e^{\left(r - \frac{\rho N}{\beta_i - 1 + N}\right)t}}{\beta_i} \quad (31)$$

$$w^{NC}(t) = w_0 e^{\left(r-\frac{\rho N}{\beta_i}\right)t} \tag{32}$$

$$\frac{c^{NC^*}}{w^{NC}(t)} = \frac{\rho}{\beta_i} \tag{33}$$

Appendix IV

Calculations for proposition 4.

By maximizing the right side of the DPE, we obtain the optimal control, c^*:

$$c^{-b} - V'(w) = 0$$

$$c^* = h^{-\frac{1}{b}} w(t) \tag{34}$$

Secondly, we compute the time-consistent state of the stock of the resource. We have that:

$$\dot{w} = rw - c^* \Leftrightarrow \dot{w} = rw - h^{-\frac{1}{b}} w \Leftrightarrow \dot{w} - w\left(r - h^{-\frac{1}{b}}\right) = 0$$

In this case, we use $e^{-\left(r-h^{-\frac{1}{b}}\right)t}$ as the integrating factor. Multiplying through, it yields:

$$e^{-\left(r-h^{-\frac{1}{b}}\right)t}\left[\dot{w} - \left(r - h^{-\frac{1}{b}}\right) w(t) = 0\right]$$

$$\Leftrightarrow e^{-\left(r-h^{-\frac{1}{b}}\right)t}\dot{w} - e^{-\left(r-h^{-\frac{1}{b}}\right)t}\left(r - h^{-\frac{1}{b}}\right) w(t) = 0$$

$$\Leftrightarrow \frac{d\left(e^{-\left(r-h^{-\frac{1}{b}}\right)t} w(t)\right)}{dt} = 0$$

$$\Leftrightarrow e^{-\left(r-h^{-\frac{1}{b}}\right)t} w(t) = D \tag{35}$$

Multiplying (38) by $e^{\left(r-h^{-\frac{1}{b}}\right)t}$, we obtain:

$$w(t) = e^{\left(r-h^{-\frac{1}{b}}\right)t} D$$

Applying the boundary condition $w(0) = 0$:

$$w(0) = e^{\left(r-h^{-\frac{1}{b}}\right)0} D = D$$

If we assume that $w(0) = w_0$, then $w_0 = D$, which yields:

$$w(t) = w_0\, e^{\left(r - h^{-\frac{1}{b}}\right)t} \tag{36}$$

The above equation describes the optimal path of the stock of the resource over time. It can also be expressed for any moment s:

$$w(s) = w_t\, e^{\left(r - h^{-\frac{1}{b}}\right)(s-t)}, s > t \tag{37}$$

Now, we replace the resulting expressions from above into $K(w)$:

$$K(w) = \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \frac{c^{*1-b}}{1-b} ds\right]$$

$$= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \frac{\left(h^{-\frac{1}{b}} w\right)^{1-b}}{(1-b)} ds\right]$$

$$= \lambda(1-\beta)\left[\int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \frac{h^{\frac{b-1}{b}} w^{1-b}}{(1-b)} ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{\frac{b-1}{b}}}{1-b} \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} w^{1-b} ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{\frac{b-1}{b}}}{1-b} \int_t^{\infty} e^{-(\lambda+\rho)(s-t)} \left(w_t\, e^{\left(r - h^{-\frac{1}{b}}\right)(s-t)}\right)^{1-b} ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{\frac{b-1}{b}}}{1-b} w_t^{1-b} \int_t^{\infty} e^{(s-t)\left[-(\lambda+\rho)+\left(r-h^{-\frac{1}{b}}\right)(1-b)\right]} ds\right] \tag{38}$$

Replacing $c^*, V(w), K(w)$ and V' into the DPE, we obtain:

$$\rho\left(h \frac{w^{1-b}}{1-b}\right) + \left[\lambda(1-\beta)\left[\frac{h^{\frac{b-1}{b}}}{1-b} w_t^{1-b} \int_t^{\infty} e^{(s-t)\left[-(\lambda+\rho)+\left(r-h^{-\frac{1}{b}}\right)(1-b)\right]} ds\right]\right] =$$

$$= w^{1-b}\left[\frac{h^{\frac{b-1}{b}}}{1-b} + h\left(r - h^{-\frac{1}{b}}\right)\right]$$

The next step is to impose that h is such, that it is the same for all values of $w(t)$. So that, we collect all the terms adjacent to w^{1-b} from the expression above:

$$\rho\frac{h}{1-b}+\left[\lambda(1-\beta)\frac{h^{\frac{b-1}{b}}-1}{1-b}\int_t^{\infty}e^{(s-t)\left[-(\lambda+\rho)+\left(r-h^{-\frac{1}{b}}\right)(1-b)\right]}ds\right]$$

$$=\frac{h^{\frac{b-1}{b}}-1}{1-b}+h\left(r-h^{-\frac{1}{b}}\right)$$

$$\rho h+\left[\frac{\lambda(1-\beta)h^{\frac{b-1}{b}}}{(\lambda+\rho)-\left(r-h^{-\frac{1}{b}}\right)(1-b)}\right]=h\left[h^{-\frac{1}{b}}+(1-b)\left(r-h^{-\frac{1}{b}}\right)\right]$$

Since this is not a linear expression, the constant h cannot be computed straightforward as we did in the logarithmic case. Thus, for the power utility case, we are not able to offer the solution for the stochastic hyperbolic discounting. As we are interested in the solution when λ approaches ∞, we can analyze this limiting case in the expression above:

$$\lim_{\lambda\to\infty}\rho h+\left[\frac{\lambda(1-\beta)h^{\frac{b-1}{b}}}{(\lambda+\rho)-\left(r-h^{-\frac{1}{b}}\right)(1-b)}\right]-h\left[h^{-\frac{1}{b}}+(1-b)\left(r-h^{-\frac{1}{b}}\right)\right], \tag{39}$$

what brings us to the following expression:

$$h\left(-(\beta-1+b)h^{-1/b}+(b-1)r+\rho\right)=0. \tag{40}$$

Solving (42) for h, we obtain the constant in the IG case:

$$h=\left[\frac{1-\beta-b}{(1-b)r-\rho}\right]^b \tag{41}$$

And we replace h into them, we obtain the final expressions in the case of one agent with power utility and instantaneous gratification discounting:

$$c^*=\frac{(1-b)r-\rho}{1-\beta-b}\,w_0\,e^{\left(r-\frac{(1-b)r-\rho}{1-\beta-b}\right)t} \tag{42}$$

$$w(t)=w_0\,e^{\left(r-\frac{(1-b)r-\rho}{1-\beta-b}\right)t} \tag{43}$$

$$\frac{c^*(t)}{w(t)}=\frac{(1-b)r-\rho}{1-\beta-b} \tag{44}$$

Appendix V

Calculations for proposition 5.

By maximizing the right side of the DPE respect to c, we obtain the equilibrium control, c^{C^*}, for the cooperative case with power utility:

$$Nc^{-b} - NV^{c'} = 0$$

$$c^{C^*} = h^{-\frac{1}{b}}w \tag{45}$$

Computation of the time-consistent state of the stock of the resource:

$$\dot{w} = rw - Nc^* \Leftrightarrow \dot{w} = rw - N\left(h^{-\frac{1}{b}}w\right) \Leftrightarrow \dot{w} - w\left(r - Nh^{-\frac{1}{b}}\right) = 0 \tag{46}$$

$e^{-\left(r-Nh^{-\frac{1}{b}}\right)t}$ is the integrating factor. Multiplying through and following the same approach from previous sections, we obtain:

$$w^C(t) = w_0\, e^{\left(r-Nh^{-\frac{1}{b}}\right)t} \tag{47}$$

It can also be expressed for any moment s:

$$w^C(s) = w_t\, e^{\left(r-Nh^{C^{-\frac{1}{b}}}\right)(s-t)}, s > t \tag{48}$$

Now, we replace the resulting expressions from above into $K(w)$:

$$K^C(w) = \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{c(s)^{*1-b}}{1-b}ds\right]$$

$$= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{\left(h^{-\frac{1}{b}}w(s)\right)^{1-b}}{(1-b)}ds\right]$$

$$= \lambda(1-\beta)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{h^{-\frac{1-b}{b}}w(s)^{1-b}}{(1-b)}ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{-\frac{1-b}{b}}}{1-b}\int_t^\infty e^{-(\lambda+\rho)(s-t)}w(s)^{1-b}ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{-\frac{1-b}{b}}}{1-b}\int_t^\infty e^{-(\lambda+\rho)(s-t)}\left(w_t\, e^{\left(r-Nh^{-\frac{1}{b}}\right)(s-t)}\right)^{1-b}ds\right]$$

$$= \lambda(1-\beta)\left[\frac{h^{-\frac{1-b}{b}}}{1-b}w_t^{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-Nh^{-\frac{1}{b}}\right)(1-b)\right]}ds\right] \tag{49}$$

Replacing $c^{*C}, V^C(w), K^C(w)$ and V'^C into the DPE, we obtain:

$$\rho\left(h\frac{w(t)^{1-b}}{1-b}\right)+\left[\lambda(1-\beta)\left[\frac{h^{-\frac{1-b}{b}}}{1-b}w_t^{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-Nh^{-\frac{1}{b}}\right)(1-b)\right]}ds\right]\right]$$

$$=w(t)^{1-b}\left[N\frac{h^{\frac{b-1}{b}}}{1-b}+h\left(r-Nh^{-\frac{1}{b}}\right)\right] \tag{50}$$

We impose that h is such, that it is the same for all values of $w(t)$. So that, we collect all the terms adjacent to $w(t)$ in the expression above:

$$\rho\frac{h}{1-b}+\left[\lambda(1-\beta)\frac{h^{\frac{b-1}{b}}-1}{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-Nh^{-\frac{1}{b}}\right)(1-b)\right]}ds\right]$$

$$=N\frac{h^{\frac{b-1}{b}}-1}{1-b}+h\left(r-Nh^{-\frac{1}{b}}\right)$$

$$\rho h+\left[\frac{\lambda(1-\beta)h^{\frac{b-1}{b}}}{(\lambda+\rho)+\left(r-Nh^{-\frac{1}{b}}\right)(1-b)}\right]=h\left[Nh^{-\frac{1}{b}}+(1-b)\left(r-Nh^{-\frac{1}{b}}\right)\right]$$

Since this is not a linear expression, the constant h cannot be computed straightforward. As we are interested in the solution when λ approaches ∞, we can analyze this limiting case in the expression above:

$$\lim_{\lambda\to\infty}\rho h+\left[\frac{\lambda(1-\beta)h^{\frac{b-1}{b}}}{(\lambda+\rho)-\left(r-Nh^{-\frac{1}{b}}\right)(1-b)}\right]-h\left[Nh^{-\frac{1}{b}}+(1-b)\left(r-Nh^{-\frac{1}{b}}\right)\right] \tag{51}$$

What brings us to the following expression:

$$h\left(-N(\beta-1)h^{-\frac{1}{b}}+\rho-r+b\left(r-Nh^{-\frac{1}{b}}\right)\right)=0 \tag{52}$$

From where we can obtain the following expression for h:

$$h=\left[\frac{N(1-b-\beta)}{-\rho+r(1-b)}\right]^b \tag{53}$$

We can now replace h^C in the original expressions to obtain the final equations for the cooperative case with power utility.

$$c^{C^*}=\frac{-\rho+r(1-b)}{N(1-b-\beta)}w_0\,e^{\left(r-\frac{-\rho+r(1-b)}{N(1-b-\beta)}\right)t} \tag{54}$$

$$w^C(t)=w_0\,e^{\left(r-\frac{\rho+r(b-1)}{\beta-1-b}\right)t} \tag{55}$$

$$\frac{c^{C^*}}{w^C(t)} = \frac{-\rho + r(1-b)}{N(1-b-\beta)} \tag{56}$$

Appendix VI

Calculations for proposition 6.

By maximizing the right side of the DPE respect to c:

$$c^{-b} - NV_i^{NC\prime}0$$

$$c^{NC^*} = (Nh_i)^{-\frac{1}{b}} w(t) \tag{57}$$

From where, the consumption rate is given by the expression:

$$\frac{c^{NC^*}(t)}{w^{NC}(t)} = \left(Nh_i^{NC}\right)^{-\frac{1}{b}} \tag{58}$$

We compute the time-consistent state of the stock of the resource:

$$\dot{w} = rw(t) - N(Nh_i)^{-\frac{1}{b}} w(t) \Leftrightarrow \dot{w} - \left(r - N(Nh_i)^{-\frac{1}{b}}\right) w(t) = 0 \tag{59}$$

(64) is a separable first-order linear differential. Choosing $e^{-\left(r - N(Nh_i)^{-\frac{1}{b}}\right)t}$ as the integrating factor we obtain the following expressions:

$$w^{NC}(t) = w_0\, e^{\left(r - N(Nh_i)^{-\frac{1}{b}}\right)t} \tag{60}$$

The above equation describes the optimal path of the stock of the resource over time. It can also be expressed for any moment s:

$$w^{NC}(s) = w_t\, e^{\left(r - N(Nh_i)^{-\frac{1}{b}}\right)(s-t)}, s > t \tag{61}$$

Now, we replace the resulting expressions from above into $K(w)$:

$$K_i^{NC}(w) = \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{c(s)^{*1-b}}{1-b}ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{\left((Nh_i)^{-\frac{1}{b}}w(s)\right)^{1-b}}{(1-b)}ds\right]$$

$$= \lambda(1-\beta_i)\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\frac{(Nh_i)^{-\frac{1-b}{b}}w(s)^{1-b}}{(1-b)}ds\right]$$

$$= \lambda(1-\beta_i)\left[\frac{(Nh_i)^{-\frac{1-b}{b}}}{1-b}\int_t^\infty e^{-(\lambda+\rho)(s-t)}w(s)^{1-b}ds\right]$$

$$= \lambda(1-\beta_i)\left[\frac{(Nh_i)^{-\frac{1-b}{b}}}{1-b}\int_t^\infty e^{-(\lambda+\rho)(s-t)}\left(w_t\, e^{\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(s-t)}\right)^{1-b}ds\right]$$

$$= \lambda(1-\beta_i)\left[\frac{(Nh_i)^{-\frac{1-b}{b}}}{1-b}w_t^{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(1-b)\right]}ds\right] \quad (62)$$

Replacing $c^{*NC}, K^{NC}(w)$ and V'^{NC} into the DPE, we obtain:

$$\rho\left(h_i\frac{w(t)^{1-b}}{1-b}\right)+\left[\lambda(1-\beta_i)\left[\frac{(Nh_i)^{-\frac{1-b}{b}}}{1-b}w_t^{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(1-b)\right]}ds\right]\right] =$$

$$= w(t)^{1-b}\left[\frac{(Nh_i)^{\frac{b-1}{b}}}{1-b}+h\left(r-Nh_i^{-\frac{1}{b}}\right)\right]$$

The next step is to impose that h_i is such, that it is the same for all values of $w(t)$. So that, we collect all the terms adjacent to $w(t)$ in the expression above:

$$\rho\frac{h_i}{1-b}+\left[\lambda(1-\beta_i)\frac{(Nh_i)^{\frac{b-1}{b}}-1}{1-b}\int_t^\infty e^{(s-t)\left[-(\lambda+\rho)+\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(1-b)\right]}ds\right] =$$

$$= \frac{(Nh_i)^{\frac{b-1}{b}}-1}{1-b}+h_i\left(r-Nh_i^{-\frac{1}{b}}\right)$$

$$\rho h_i+\left[\frac{\lambda(1-\beta_i)(Nh_i)^{\frac{b-1}{b}}}{(\lambda+\rho)-\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(1-b)}\right] = h_i\left[N^{\frac{b-1}{b}}h_i^{-\frac{1}{b}}+(1-b)\left(r-Nh_i^{-\frac{1}{b}}\right)\right]$$

Since this is not a linear expression, the constant h cannot be computed straightforward.

As we are interested in the solution when λ approaches ∞, we can analyze this limiting case in the expression above:

$$\lim_{\lambda\to\infty} \rho h_i + \left[\frac{\lambda(1-\beta_i)(Nh_i)^{\frac{b-1}{b}}}{(\lambda+\rho)-\left(r-N(Nh_i)^{-\frac{1}{b}}\right)(1-b)}\right] - h_i\left[N^{\frac{b-1}{b}}h_i{}^{-\frac{1}{b}}+(1-b)\left(r-Nh_i^{-\frac{1}{b}}\right)\right] \tag{63}$$

What brings us to the following expression:

$$h_i(-(\beta_i-1)N(Nh_i)^{-\frac{1}{b}}+h_i^{-\frac{1}{b}}N\left(1-b-N^{-\frac{1}{b}}\right)+\rho+(b-1)r)=0 \tag{64}$$

From where we can obtain the following expression for h:

$$h_i=\left[\frac{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}{-\rho+r(1-b)}\right]^b \tag{65}$$

By replacing h_i into the original expressions we obtain the final expressions in the noncooperative case when agents are symmetric and use a logarithmic utility with instantaneous gratification discounting:

$$c^{NC^*}=N^{-\frac{1}{b}}\frac{-\rho+r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}w_0\, e^{\left[r-N^{\frac{b-1}{b}}\left(\frac{-\rho+r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}\right)\right]t} \tag{66}$$

$$w^{NC}(t)=w_0\, e^{\left[r-N^{\frac{b-1}{b}}\left(\frac{-\rho+r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)}\right)\right]t} \tag{67}$$

$$\frac{c^{NC^*}}{w^{NC}(t)}=N^{-\frac{1}{b}}\frac{-\rho+r(1-b)}{N\left(1-b-N^{-\frac{1}{b}}\beta_i\right)} \tag{68}$$