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The Kepler conjecture

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Abstract

Kepler's conjecture asserts that the highest possible density an arrangement of congruent balls can have is the one of the *face-centered cubic packing*. That is, the pyramid arrangement of balls on a square base, or on a triangular base, like oranges are usually arranged at fruit stands. In this project, we study the proof of this problem presented by Thomas Hales in 1998.

It will be obvious that in some parts (specially in the end) we do not go into detail when we study the properties of the elements that take place in the proof. The reason is that the notation gets very cumbersome as we go along and the study of these details will not give us a better understanding of the proof. They are necessary steps to prove the conjecture, but our aim is to understand the proof as a whole and to see what strategy Thomas Hales followed.

It is also important to note that a big part of the proof relies in computer calculations. All the programs and algorithms can be found online on the documentation of the *Flyspeck project*. It took years to finish and verify this part of the proof (the project was finally completed on August 2014) and we will not study this part of the proof.

Chapter 1

Introduction

1.1 History: why, when and how

Thomas Harriot (1560-1621) was the mathematical assistant of the english nobleman Sir Walter Raleigh (1552-1618). During the end of the 1590's he was preparing his ships for an expedition, and he asked Harriot to find a formula that could tell him how many cannonballs where stacked in a pile just by looking at it. He easily found the solution to this problem but he could not leave it there: it made him wonder what was the most efficient way of stacking cannonballs. And this was not an easy problem. He decided to ask for help to one of the most important mathematicians of the time: his colleague in Prague, Johannes Kepler.

It did not take him a lot of time to realise that the best way to pack three-dimensional spheres was the face-centered cubic packing (FCC), a pyramid-like arrangement used to stack oranges at fruit stands. He published his work in 1611 in a little booklet named *The Six-Cornered Snowflake*, where he asks himself why single snowflakes, before they become entangled with other snowflakes, always fall with six corners. This has to do with the hexagonal-closed packing (HCP), which has the same density as the FCC, and he ends up explaining the method of packing balls as tightly as possible. The thing is that he never proved this statement and that is how the Kepler conjecture was born.

It was not until 1831 when Gauss managed to prove that the FCC is the densest *lattice* packing (that is a packing that comes from a lattice construction) in three dimensions, but the general conjecture remained open for many decades.

After a few decades, in 1907, William Jackson Pope and William Barlow published a paper in the *Journal of the Chemical Society* where they showed that there are infinite different arrangements to pack spheres in the most efficient manner, that is with the FCC's density, but they still could not prove the conjecture.

The first promising approached arrived in 1953, after van der Waerden and Schütte solved the Newton-Gregory problem from 1694, which consists in how many congruent balls can be arranged to touch a given ball. Newton, who was right, said that the maximum was twelve balls, but Gregory claimed that the number was thirteen. After reading their work, Fejes Tóth, a hungarian mathematician, managed to linked this problem with the Kepler Conjecture and suggested that a similar proof could be achieved.

Finally, in 1998, the american mathematician Thomas Hales gave a full proof of the conjecture making extensive use of computer calculations. It was very hard to verify the validity of his proof. It was finally published in 2003 in *The Annals of Mathematics* with a note stating that after a team of twelve reviewers had spend four years trying to verify it, there were still some parts of the paper that had not been checked and they where 99% certain that it was correct. In January of 2003, Hales launched the "Flyspeck project" ("Formal Proof of Kepler") in an attempt to use computers to automatically verify every step of the proof. In wasn not until 2014 that the project was finally closed and the conjecture was officially 100% proven. The documentation on the project can be accessed online in [6]

1.2 Brief description of the proof

In order to be able to prove the Kepler Conjecture we have to reduce the problem to one involving only a finite number of spheres. We will start by defining what a packing is and then we will study the conditions that we need so we can reduce the number of spheres.

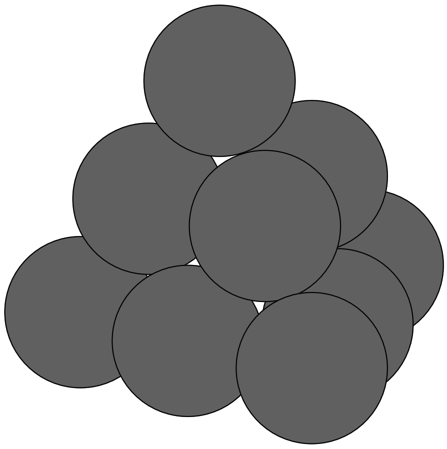
A **packing** is an arrangement of congruent balls (of radius 1 by convention) in the Euclidean three dimensional space that do not overlap. Usually it is represented as the set of centers of the balls in it. The distance between two of its points is at least 2. When no further balls can be added in a packing, it is said to be a **saturated packing**.

Definition 1.1 *The set $V \subset \mathbb{R}^3$ is a* ***packing*** *if*

$$\forall u, v \in V; \|u - v\| < 2 \Rightarrow (u = v)$$

V is ***saturated*** *if $\forall p \in \mathbb{R}^3$, there exists some $u \in V$ such that $\|u - p\| < 2$.*

The ***FCC packing*** *is the pyramid arrangement of balls on a square base or on a triangular base.*



The density of a packing in a certain region of space is: the volume occupied by the balls in the region, divided by the volume of the region. The purpose of a finite region is preventing the volumes from becoming infinite.

Definition 1.2 *The* ***density of a packing*** V ***within the bounded region of space*** *centered in p, with radius r, is*

$$\delta(V,p,r) = \frac{vol\Big(B(p,r) \cap (\bigcup_{v \in V} B(v,1))\Big)}{vol\big(B(p,r)\big)}$$

Which is the volume occupied by the balls (all of unit radius) of the packing that are in this region divided by the volume of the region.

And the ***density*** *of the packing V in all* $\mathbb{R}^3$ *is the density in the region as its radius tends to infinity:*

$$\delta(V) = \lim_{r \to \infty} \delta(V,p,r)$$

Conjecture 1.3 The **Kepler conjecture** asserts that the density of the FCC packing is the greatest possible density any packing can have in the three-dimensional Euclidean space.

As we will see later, the FCC-packing has density $\pi/18$. However, computing the density of a packing is not always an easy task. In definition 1.2, the region that we used to compute it was a ball, but we can also consider other types of regions, like the Voronoi cells of the spheres of the packing. The same can be done with different partitions of space, as we will see in the last chapter. We are also going to use the Voronoi cells to reduce the number of spheres.

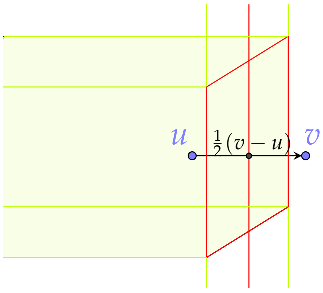
In order to understand what a Voronoi cell is, we need to know what the half-space of two points is:

Definition 1.4 *For $u, v \in \mathbb{R}^3$, the **half-space** $A_+(u,v)$ is the region of space separated by the orthogonal plane to $\overrightarrow{uv}$ passing through $\frac{1}{2}(v-u)$ that contains u:*

$$A_+(u,v) = \{p \in \mathbb{R}^3 : 2(v-u)\cdot p \quad \leq \quad \|v\|^2 - \quad \|u\|^2\}$$

Example 1.5 *If u=(0,0,0) and v=(1,0,0), the red region marks the orthogonal plane and the green region marks the half-plane*

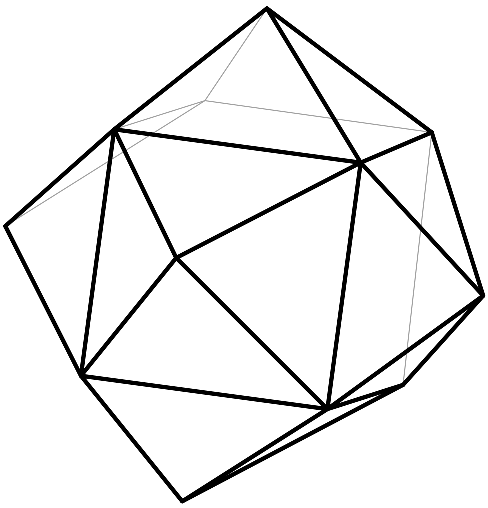
$$A_+(u,v) = \{(x,y,z) \ : \ 2(1,0,0)\cdot(x,y,z) \leq 1-0\} = \{(x,y,z) \ : \ 2x \leq 1\}$$



Definition 1.6 *The **Voronoi cell** $\Omega(V,u)$ is the intersection of the half-spaces $A_+(u,v)$ as v runs over $V \setminus \{u\}$.*

In $\mathbb{R}^3$ the Voronoi cells of the elements of a packing are polyhedrons. Because of that, when V is a saturated packing, the Voronoi cell $\Omega(V,v)$ of every $v \in V$ is compact, convex and mesurable.

It can be seen that the Voronoi cell of the FCC packing is a rhombic dodecahedron. It can be constructed by taking a cube and placing a pyramid in all its facets, as shown in the illustration below. It's volume is $4\sqrt{2}$.



We will go into further detail in following chapters. We will also study other geometric decompositions of space, but right now this is all we need to know about it so we can reduce the number of spheres.

We will define now a type of functions that are very important in the proof of the conjecture. They are defined for a packing. Given below, there is a lemma that explains that the existence of such functions implies that the particular packing for which they are defined holds the Kepler conjecture.

Definition 1.7 *Let $V \subset \mathbb{R}^3$ be a packing and $V(0,r) := V \cap B(0,r)$.*
We say that $G : V \to \mathbb{R}$ is ***negligible*** *if there exists a constant c such that $\forall r \geq 1$*

$$\sum_{v \in V(0,r)} G(v) \leq cr^2$$

and that G is ***FCC-compatible*** *if $\forall v \in V$*

$$4\sqrt{2} \leq vol(\Omega(V,v)) + G(v)$$

Where
$4\sqrt{2}$ is the volume of the Voronoi cell in the FCC-packing,
$vol(\Omega(V,v))$ is the volume of the Voronoi cell in the packing V,
$G(v)$ is used as the adjustment to correct the error.

Lemma 1.8 *If there exists a* negligible, FCC-compatible function $G : V \to \mathbb{R}$ *for a* satured packing V, *then there exists a constant $c = c(V)$ such that $\forall r \geq 1$,*

$$\delta(V,0,r) \leq \frac{\pi}{18} + \frac{c}{r}$$

The term $\frac{c}{r}$ is the error that comes from the boundary effect of the container of radius r holding the balls. This is because $\frac{\pi}{18} + \frac{c}{r}$ is computed using the volume of all the balls that have the center inside the container, but some of this balls will not be completely contained in it. As a result, the parts of the balls that are left outside give $\frac{c}{r}$, where c is an appropiately choosen constant that depends on the packing.

When the radius tends to infinity, the error tends to 0. Then, the inequality implies the Kepler conjecture, because if there exists a negligible, FCC-compatible function $G : V \to \mathbb{R}$,

$$\delta(V) \leq \frac{\pi}{18}$$

This means that it is enough to find a negligible FCC-compatible function G to prove the conjecture.

We will also study hypermaps and the relationship between them and packings. Hypermaps are abstractions of plane graphs that were used in the proof of the four-color theorem in order to avoid the use of the Jordan curve theorem. There is a chapter dedicated to them.

The next stage will be explaining other decomposition of space: Roger simplices and Marchal cells. Thanks to these decompositions, and to the Marchal conjecture, we are able to find an inequality that, if true for a packing, implies the existence of the negligible FCC-compatible function. The inequality has the form $\mathcal{L}(W) \leq 12$ where W is a saturated packing.

We want to prove that this inequality holds for every saturated packing, and we will do that by contradiction. We will assume that there is a packing that doesn't hold the inequality (then it holds $\mathcal{L}(W) > 12$) and we will study its properties. We will assing a hypermap to this packing and, because of its properties, call it a tame hypermap.

At this point, the proof relies heavily in computer computation. Using a computer program, Hales generated an explicit list, enumerating tame hypermaps up to isomorphism (that is all the hypermaps whose packing does not hold $\mathcal{L}(W) \leq 12$). This was a long list of approximately 25.000 hypermaps, and each hypermap had a nonlinear optimization problem associated to maximaze $\mathcal{L}(W)$. All these problems can be solved in the form of several linear optimization problems. For all the hypermaps in the list, it is checked that $\mathcal{L}(W) < 12$ and hence the contradiction is found.

This implies that the inequality holds for every saturated packing, hence that there exists a negligible FCC-compatible function for each on of them. This implies that the Kepler conjecture is true for any possible packing in $\mathbb{R}^3$.

Chapter 2

General background

As we said in the abstract, the notation tends to get cumbersome at some points of the proof. This is why we include this chapter of general background. Most of the concepts explained in this section are constructions build from basic concepts. They might seem a little stilted, but they are part of the foundation of the proof and they are basic for the understanding of it. We have tried to give as many illustrations as possible so it is easier and more intuitive to understand these concepts.

Definition 2.1 *The **affine hull** of a set $S \subset \mathbb{R}^N$, aff(S), is the smallest linear variety containing S.*

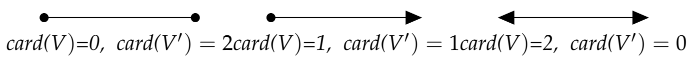
*The **affine** of two finit subsets of R^N, V={$v_1, ..., v_k$} and $V' = \{v_{k+1}, ..., v_n\}$ is*

$$aff_{\pm}(V, V') = \{t_1 \mathrm{v}_1 + \cdots + t_n \mathrm{v}_n \ : \ t_1 + \cdots + t_n = 1, \ \pm t_j \geq 0, \ for\ j > k\}$$

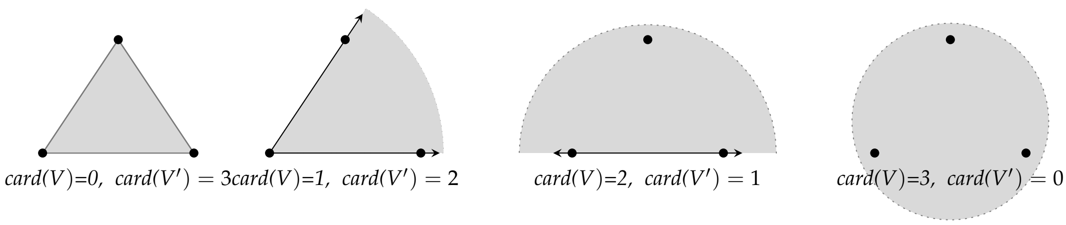
$$aff^0_{\pm}(V, V') = \{t_1 \mathrm{v}_1 + \cdots + t_n \mathrm{v}_n \ : \ t_1 + \cdots + t_n = 1, \ \pm t_j > 0, \ for\ j > k\}$$

Example 2.2 *The affine hull of two sets allows us to define, in a single concept, several geometric elements.*

For example, when card(V)+card(V') $= 2$, the set $aff_+(V, V')$ is a segment, a ray or a line:



When it is 3, it is a simplex, a blade, a half-plane or a plane:



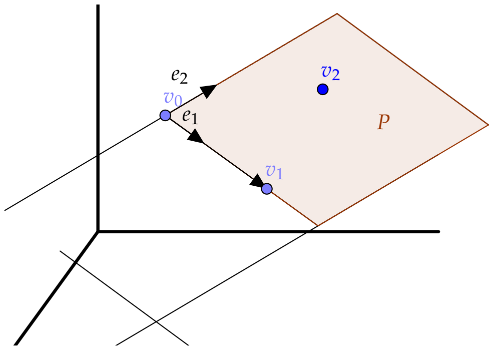
Definition 2.3 *A tuple (e_1, e_2, e_3) of vectors in $\mathbb{R}^3$ is a*

frame *if it is an orthonormal base,*

positive *if $e_1 \wedge e_2 = e_3$,*

adapted *to (v_0, v_1, v_2) if $e_1 = \frac{v_1 - v_0}{\|v_0 - v_1\|}$ and $e_2 \in aff_+^0(\{v_0, v_1\}, v_2)$, which is the half-plane P of $\langle v_0, v_1, v_2 \rangle$ separated by $\langle v_0, v_1 \rangle$ such that $v_2 \in P$.*

P is shown in the next illustration, as well as the choice of e_2.

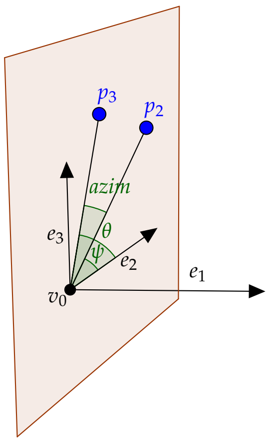


The next definition is the azimuth angle of four points. This might seem an unnatural angle to define, but we will come across it several times in next chapters.

Definition 2.4 *Let $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$. Let (e_1, e_2, e_3) be an positive frame adapted to (v_0, v_1, v_2).*

Let p_2 and p_3 be the projections of v_2 and v_3 respectively into the plane $\langle e_2, e_3 \rangle$ that traverses v_0.

*Considering the base $\langle e_2, e_3 \rangle$, we take polar coordinates in that plane and call ψ and θ the polar angles of p_2 and p_3. Then, we define the **azimuth angle** $azim(v_0, v_1, v_2, v_3) = \theta - \psi$.*



Remark 2.5 *Note that $azim(v_0, v_1, v_2, v_3)$ is the angle between the planes $\langle v_0, v_1, v_2 \rangle$ and $\langle v_0, v_1, v_3 \rangle$. Because of this, it does not depend on the frame $\langle e_1, e_2, e_3 \rangle$, but we define it to fix an orientation so we know which angle to choose between the two determined by the planes.*

The next definitions will help us get to definition 2.8: the azimuth cycle. It is an important concept to understand because it will help us understand important parts of the proof.

Definition 2.6 *Let V be a finite set of points in $\mathbb{R}^3$ and (v_0, v_1) an ordered pair of distinct points in $\mathbb{R}^3$. We say that V is **cyclic** with respect to (v_0, v_1) if:*

1. All parallel lines to $\overrightarrow{v_0v_1}$ through a point $w \in V$ do not contain other points of V:

$$\forall u, w \in V, \mathrm{h} \in \mathbb{R},\ u = w + \mathrm{h}(v_1 - v_0) \Rightarrow u = w.$$

2. The line through u and v does not meet any points of V:

$$\langle v_0, v_1 \rangle \cap V = \{\emptyset\}.$$

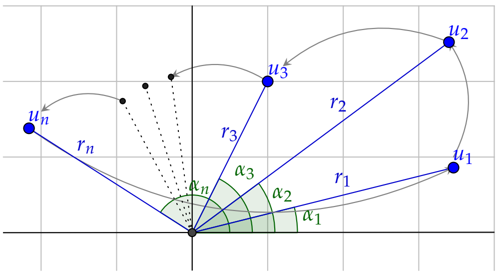
Definition 2.7 *The **polar cycle** is a cyclic permutation σ' on a set of vectors U in the plane that traverses them in order of increasing angle.*

In other words, let $U = \{u_1, ..., u_n\}$ with $u_i = (r_i)_{\alpha_i}$ $\forall i$ in polar coordinates. We reorder the elements of U so that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n$$

As a result, the polar cycle permutes the elements of U as follows:

$$u_1 \to u_2 \to \cdots \to u_n \to u_1$$



The azimuth cycle with respect to (v_0, v_1) is the polar cycle adapted to $\mathbb{R}^3$: the elements are permuted in space according to how they are permuted by the polar cycle when they are projected into an specific plane (the orthogonal plane to (v_0, v_1). The formal definition is as follows:

Definition 2.8 *Let $v_0 \neq v_1 \in \mathbb{R}^3$. Let $V \subset \mathbb{R}^3$ be a finite set that is cyclic with respect to (v_0, v_1).*

Select a $p \in \mathbb{R}^3$ such that $\{v_0, v_1, p\}$ is not collinear and let $\{e_1, e_2, e_3\}$ be the corresponding positive, adapted, frame.

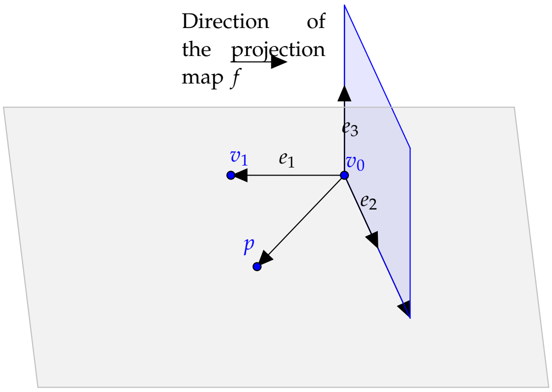
*We define the **azimuth cycle with respect to** (v_0, v_1) as the function $\sigma : V \to V$ that holds $f\sigma(u) = \sigma' f(u)$ where*

- *σ' is the polar cycle and*

- *f is the projection map* $f : v_0 + xe_2 + ye_3 + ze_1 \mapsto (x, y)$ *that projects all the elements of V into the orthogonal plane to $\overrightarrow{v_0 v_1}$ through v_0.*

Explained in other words, let $W = \{w_1, ..., w_n\} \subset \mathbb{R}^3$. To know how the azimuth cycle with respect to (v_0, v_1) will permute the elements of W, we neet to project them into the orthogonal plane to $\overrightarrow{v_0 v_1}$ through v_0.

We need W to be cyclic with respect to (v_0, v_1) because if it were not, the projection map f would not be injective



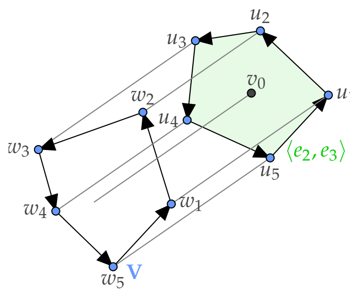
Once all the elements have been projected, we permute the projections with the polar cycle σ'.

The permutations of the elements of W by the azimuth cycle follow the permutations of its projections by the polar cycle. This means that if u_i is the projection of w_i and the polar cycle permutes as:

$$u_1 \to u_2 \to \cdots \to u_n \to u_1$$

then the azimuth cycle permutes as:

$$w_1 \to w_2 \to \cdots \to w_n \to w_1$$



And finally, the next definition is going to give us a general background for when we study polyhedrons in chapter 4.

Definition 2.9 *Let P be a convex set.*

A ***face of P*** *is a convex set $F \subset P$ such that any segment of P that intersects F in the interior, is contained in F.*

A face F is ***proper*** *if $F \neq \varnothing, P$.*

An ***extreme point*** *is an element $v \in P$ such that $\{v\}$ is a face of affine dimension zero.*

An ***edge*** *is a face of P of affine dimension one.*

A ***facet*** *of P is a proper face of affine dimension dim aff(P) - 1.*

Chapter 3

Hypermaps

In this chapter we will introduce hypermaps, an abstraction of plane graphs. Gonthier used them as a basic combinatorial structure in his proof of the four-color theorem. They helped him change many topological arguments into combinatorial arguments. When Hales found out about this, he changed many parts of the proof, which helped to formalize the argument.

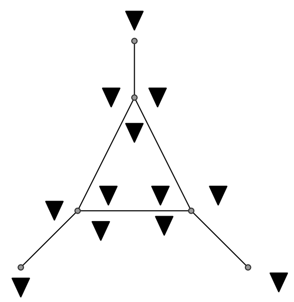
We start this chapter with a few definitions and then we will study various tranformations that can be done to a hypermap called walkups.

3.1 Definitions

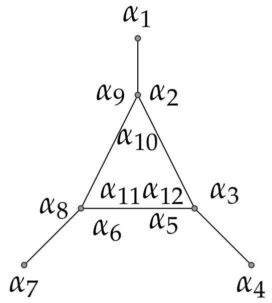
Definition 3.1 *A* ***hypermap*** (D, e, n, f) *is a finite set* D *of elements called* darts *together with three permutations* $e, f, n : D \to D$ *called* edge map*,* *face map and* node map *that compose to the identity in the following order:*

$$enf = I_D$$

Suppose we have the plane graph where ▼ are the angles between the edges. Then D is the set of all ▼.

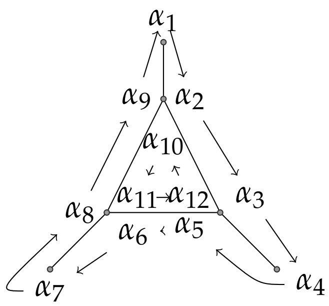


We can also identify the ▼ with α_i to make easier the understanding of the map functions associated with the hypermap.



Definition 3.2 *The **face map** function **f** cycles counterclockwise around the angles of each face:*

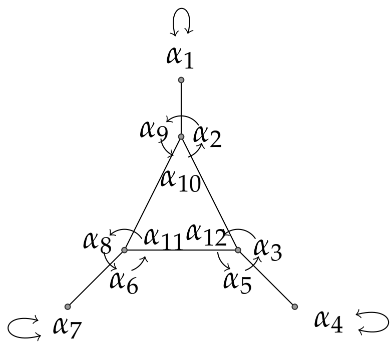
$$\left(\begin{array}{ccccccccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 11 & 12 & 10 \end{array}\right)$$



With orientation $\circlearrowright$ in the exterior face and $\circlearrowleft$ in the interior face.

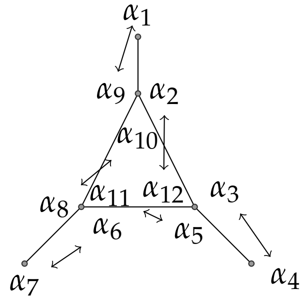
Definition 3.3 *The **node map** function **n** rotates counterclockwise around the angles of each edge:*

$$\left(\begin{array}{c|c|c|ccc|ccc|ccc} 1 & 7 & 4 & 2 & 9 & 10 & 6 & 11 & 8 & 3 & 12 & 5 \\ 1 & 7 & 4 & 9 & 10 & 2 & 11 & 8 & 6 & 12 & 5 & 3 \end{array}\right)$$



Definition 3.4 *The **edge map** function **e** pairs angles at opposite ends of each node. It might be helpful to think of it as $e = f^{-1}n^{-1}$:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 9 & 12 & 4 & 3 & 11 & 7 & 6 & 10 & 1 & 8 & 5 & 2 \end{pmatrix}$$



To simplify the notation, D will denote the hypermap (D, e, n, f) from now on.

We will now give a list of definitions that define different properties of hypermaps.

Definition 3.5 *Let D be a hypermap, and let* $S = \{e, n, f\}$.

*A **path*** P *from* x_0 *to* x_{k-1} *with steps in S is a list of darts* $[x_0, ..., x_{k-1}]$ *such that* $x_{i+1} = h_i x_i$ *for some* $h_i \in S$. *For example* $1 \xrightarrow{f} 2 \xrightarrow{n} 9 \xrightarrow{n} 10 \xrightarrow{e} 8$ *is a path in the hypermap above.*

We define a relation $x \sim_S y \Leftrightarrow$ *a path runs from x to y with steps in S. A **combinatorial component** of D is an equivalence class of the relation* $\sim_S$. *Let #c denote the number of combinatorial components of D.*

*A **node** of D is the orbit of a dart* $x \in D$ *under n. The number of orbits of the permutation n (#n) is the number of nodes of the hypermap. The nodes of the hypermap above are {1}, {2,9,10}, {6,8,11}, {3,5,12}, {4}, {7}.*

*A **face** is a orbit under* f. *The number of faces is the number of orbits* $\#f$ *of* f. *The faces of the hypermap above are {1,2,3,4,5,6,7,8,9} and {10,11,12}.*

*An **edge** is a orbit under e. The number of edges is the number of orbits #e of e. The edges of the hypermap above are {1,9}, {2,12}, {5,11}, {6,7}, {8,10}, {3,4}.*

*D is **simple** if the intersection of each face with each node contains at most one dart. It is easy to see that this is not the case of the hypermap above because* $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \cap \{2, 9, 10\} = \{2, 9\}$ *contains two darts.*

*D is **connected** if* $\#c = 1$.

*D is **plain** when e is an involution on D:* $e^2 = I_D$ *(which is the case, by definition, of the planar graph).*

*D is **degenerate** if it is a fixed point of one of the maps* e, n, f*; otherwise it is **non-degenerate**. The hypermap above is degenerate because* $n(1) = 1$.

With all of this it is easier to understand why we said that a hypermap is an abstraction of a graph since some of this properties are somehow similar to graphs properties.

3.2 Walkup

There are various operations to transform one hypermap into another. The simpliest is called a *single walkup*: it deletes one dart and it constructs permutations that skip the deleted dart. It reforms the edge, face and node maps to produce a hypermap on the reduced set of darts.

One of the three map functions will be determined by the other two in order to keep the relation $enf = I_D$. This function will determine which kind of walkup we have: an edge walkup, a face walkup or a node walkup. The formal definition is given below.

Definition 3.6 *The* ***edge walkup*** *at a dart $x \in D$ of a hypermap D is the hypermap $W_e = (D', e', n', f')$ where:*

$$D' = D \setminus x$$

$$f'(y) = \begin{cases} f(x) & \text{if } f(y) = x \\ f(y) & \text{otherwise} \end{cases}$$

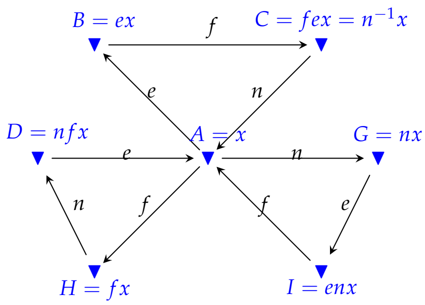
$$n'(y) = \begin{cases} n(x) & \text{if } n(y) = x \\ n(y) & \text{otherwise} \end{cases}$$

$$e'(y) = (n'f')^{-1}(y)$$

The definitions of ***face walkup*** *and* ***node walkup*** *are analogous.*

It can be seen that at a degenerate dart, all the walkups are the same $W = W_e = W_n = W_f$.

Example 3.7 *Let D be a hypermap. The next drawing illustrates a fragment of it:*



If we do a face walkup at x, we have to eliminate $A = x$ and redefine the three map functions:

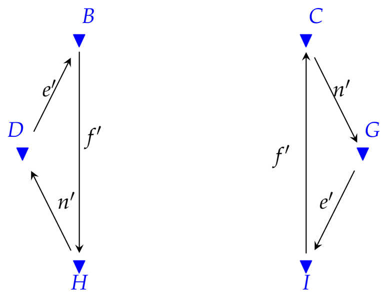
$$n'(y) = \begin{cases} n(x) & \text{if } n(y) = x \\ n(y) & \text{otherwise} \end{cases} \qquad e'(y) = \begin{cases} e(x) & \text{if } e(y) = x \\ e(y) & \text{otherwise} \end{cases} \qquad f'(y) = (e'n')^{-1}(y)$$

Which in this case means that

$$n'(C) = G \qquad n'(H) = D \qquad e'(G) = I \qquad e'(D) = B$$

$$f'(B) = (e'n')^{-1}(B) = H \qquad f'(I) = (e'n')^{-1}(I) = C$$

The fragment of the hypermap after the walkup:



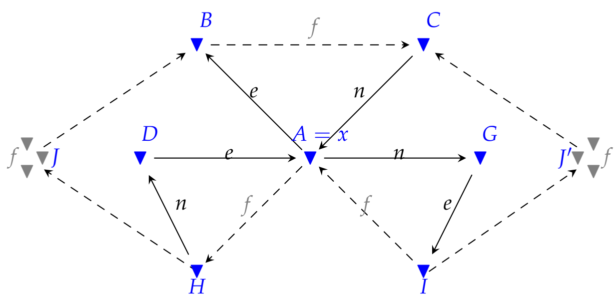
Definition 3.8 *Let D be a hypermap, $h = e, n$, or f and W_h the hypermap obtained after the walkup at $x \in D$. We say that this walkup* ***merges*** *when it joins two orbits of h. That is, the orbit $O(h', y)$ of some $y \in D'$ under h' (according to the choice of h) has the form:*

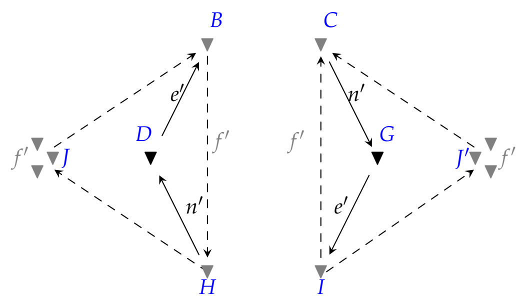
$$O(h', y) \cup x = O(h, x) \cup O(h, y)$$

where $y \notin O(h, x)$.
It ***splits*** *when there are distinct orbits O_1, O_2 under h' in the hypermap G' such that*

$$O_1 \cup O_2 \cup x = O(h, x)$$

Example 3.9 *Following the previous example, if we look at a bigger fragment of the hypermap, so we can see the orbit of f (the dashed arrows). It is clear that in this case the orbit splits:*





Following the notation of the previous definition,
$O_1 = \{B, H, J\} \quad O_2 = \{C, J', I\} \quad O(h, x) = O(h, A) = \{B, H, J, C, J', I, A\}$

We will not proof the following lemma, as it is not important in the proof of the conjecture, but it will give us an idea on when a walkup merges or splits orbits.

Lemma 3.10 *Let D be a hypermap and $h \in \{e, n, f\}$. Let W_h be a non-degenerate walkup at a dart $x \in D$. For some $y \in D$, $(h, y) \in \{(f, e(x)), (e, n(x)), (n, f(x))\}$.*

$$W_n \text{ merges} \Leftrightarrow x \text{ and } y \text{ lie in distinct orbits}$$

It is also possible to do a double walkup:

Definition 3.11 *A **double walkup** is the composite of two walkups under the same function. The two darts of the two walkups have to be the members of the same orbit (it has to have at least cardinality two). If they weren't, the second walkup would be forced to be degenerate.*

3.3 Planarity

Planarity will be an important concept because the hypermaps that are involved in the proof of the conjecture are all planar. That is why we are going to study some of the properties of planar hypermaps.

Definition 3.12 *A hypermap is **planar** when the Euler relation holds:*

$$\#n + \#e + \#f = \#D + 2\#c$$

where $\#n, \#e, \#f, \#D$ and $\#c$ are the number of nodes, edges, faces, darts and combinatorial components of the hypermap.

Lemma 3.13 *Let H be a connected plain planar hypermap such that: every edge has cardinality two and that every node has cardinality at least 3. Then*

$$\#D \leq (6\#f - 12).$$

Proof:

In a connected plain planar hypermap, $\#D = 2\#e$ and $\#c = 1$. Because of this the Euler relation becomes

$$\#f + \#n + \#e - \#D = 2 \Rightarrow \#f + \#n = \frac{1}{2}\#D + 2 \Rightarrow 6\#f - 12 = \#D + 2(\#D - 3\#n),$$

so it is enough to show that

$$\#D \geq 3\#n$$

and this inequality follows directly by assuming that every node has cardinality at least 3.

□

Definition 3.14 *The planar index of a hypermap D measures the distance of a hypermap from planarity. It is defined by the formula:*

$$\iota = \#f + \#e + \#n - \#D - 2\#c$$

Definition 3.15 *A hypermap is called **planar** when it has index zero.*

We are not going to proof the following lemma because walkups are not involved in the proof of the conjecture, but it will give us a better understanding on walkups and hypermaps.

Lemma 3.16 *Let D' be the result of the face walkup W at $x \in D$, a nondegenerate dart, of the hypermap D. W changes the cardinality of some orbits as follows:*

$$\#f' = \#f + split_f = \begin{cases} \#f + 1 & \text{if } W \text{ splits} \\ \#f - 1 & \text{if } W \text{ merges} \end{cases}$$

$$\#e' = \#e \qquad \#n' = \#n \qquad \#D' = \#D - 1$$

$$\#c' = \#c + split_c = \begin{cases} \#c + 1 & \text{if } e(x) \text{ and } f^{-1}(x) \text{ belong to different } c \text{ after } W \\ \#c & \text{otherwise} \end{cases}$$

And the planar index of D' is then:

$$\iota' = \iota + 1 + split_f - 2split_c$$

As we said before, the index measures the departure of the hypermap from planarity. Since with every walkup we are deleting a dart, the new hypermap is closer to being planar that the original. This is why:

Lemma 3.17 *Let D be a hypermap with planar index ι . If ι' is the planar index of D', then $\iota \leq \iota'$.*

Lemma 3.18 *The planar index of a hypermap is never positive.*

Proof: A walkup never decreases the index and a sequence of walkups will lead to an empty hypermap, which has index zero.

□

Planar hypermaps have the maximum index, thus the following lemma:

Lemma 3.19 *Walkups take planar hypermaps to planar hypermaps.*

Chapter 4

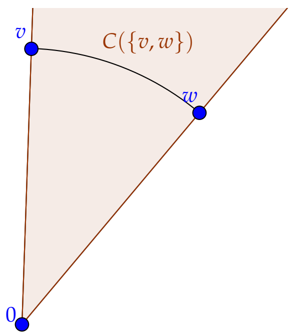
Fans

In this chapter we introduce fans, a geometric object to which we can associate a hypermaps and which is also related to sphere packings: it determines a set of points in $\mathbb{R}^3$ which can be interpreted as the set of centers of a packing.

4.1 Definitions

Notation: When $\epsilon \in \mathbb{R}^3$, we will write $C(\epsilon)$ for the affine hull $\text{aff}_+(\{0\}, \{\epsilon\})$, which is the half-line $0 + t \cdot \overrightarrow{0v}$, t$\in \mathbb{R}^+$.

When $\epsilon = \{v, w\}$, $C(\{v, w\}) = \text{aff}_+(\{0\}, \{v, w\})$ is the section of the plane $\langle 0, v, w \rangle$ enclosed by the lines $\overrightarrow{0v}$ and $\overrightarrow{0w}$.



Definition 4.1 *Let $V \subset \mathbb{R}^3$ be a finite nonempty set and let E be a set of unordered pairs of elements of V.*

*(V,E) is said to be a **fan** if the following properties hold:*

1. $0 \notin V$.

2. If $\{v, w\} \in E$, then $0 \notin \langle v, w \rangle$.

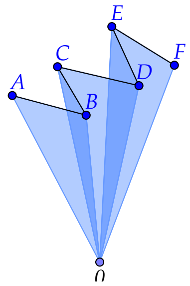
4. For all $\epsilon, \epsilon' \in E \cup V$,

$$C(\epsilon) \cap C(\epsilon') = C(\epsilon \cap \epsilon').$$

*When $\epsilon \in E$, $C(\epsilon)$ is called a **blade** of the fan.*

Example 4.2 *A fan (V,E) with six nodes, and five blades:*

$V = \{A, B, C, D, E, F\}$
$E = \{\{A, B\}, \{B, C\}, \{C, D\}, \{D, E\}, \{E, F\}\}$



It is easy to see that

Lemma 4.3 *If (V, E) is a fan, then for every $E' \subset E$, (V, E') is also a fan.*

Lemma 4.4 *Let (V, E) be a fan. For each $v \in V$, the set $E(v) = \{w \in V; \{v, w\} \in E\}$ is cyclic with respect to $(0, v)$.*

Proof
1. If $w \in E(v)$, then $0 \notin \langle v, w \rangle$.
2. If $w \neq w' \in E(v)$, then $C\{v, w\} \cap C\{v, w'\} = C\{v\}$.
This implies that $E(v)$ is cyclic.

□

Remark 4.5 *Let (V,E) be a fan.*
1. The pair (V, E) is a graph with nodes V and edges E.
2. Since $E(v)$ is cyclic with respect to (0,v), for each $v \in V$ we have an azimuth cycle $\sigma(v) = \sigma_v : E(v) \to E(v)$ of E(v) with respecto to (0,v).
If $E(v)$ is a singleton, $\sigma(v) = \sigma_v$ is the identity.

Example 4.6 *If we take the fan from example 4.2 and set v=B, E(v)={A,C}.*
Therefore, $\sigma_B : A \mapsto C$ and $C \mapsto A$.
If E(v) had a greatest cardinality, we would have to compute the polar angle of the elements of E(v) in the othogonal plane of $\overrightarrow{0B}$ through 0 to know how to permute the elements.

4.2 Hypermaps

In this section we show how to associate a hypermap with a fan. With this, we will be able to associate a sphere packing with a hypermap.

Let (V,E) be a fan. The set of darts D of the associated hypermap is the union of two sets:

$$D_1 = \{(v,w)\ :\ \{v,w\} \in E\}, \text{ formed by nonisolated darts,}$$
$$D_2 = \{(v,v)\ :\ v \in V, E(v) = \varnothing\}, \text{ formed by isolated darts, and}$$
$$D = D_1 \cup D_2$$

We need to define the permutations e, n and f.

On D_1:

$$\begin{aligned} n(v,w) &= (v,\ \sigma_v(w)) \\ f(v,w) &= (w,\ \sigma_w^{-1}(v)) \\ e(v,w) &= (w,v) \end{aligned}$$

On D_2 we define them as degenerate:

$$n(v) = f(v) = e(v) = v$$

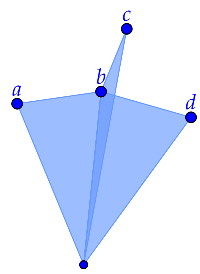
Where σ_v is the azimuth cycle of E(v) with respecto to (0,v) (as explained in Remark 4.5).

Then $hyp(V,E) = (D,e,n,f)$.

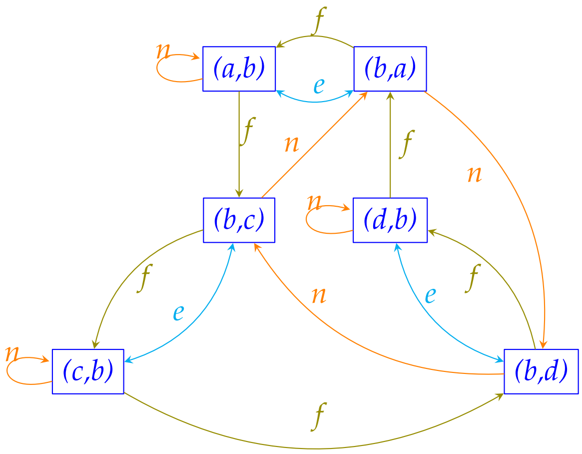
Example 4.7 *It's easier to understand with an example. Let (V,E) be a fan:*

$$V = \{a,b,c,d\}$$
$$E = \{(a,b),(b,d),(b,c)\}$$

Then the hypermap associated to (V,E) is:



hyp(V,E) = (D, e, n, f), where
$D = D_1 = \{(a,b),(b,a),(b,d),(d,b),(b,c),(c,b)\}$
$E(a) = \{b\}\ ,\ E(b) = \{a,d,c\}\ ,\ E(c) = \{b\}\ ,\ E(d) = \{b\}$
$\sigma_a, \sigma_c, \sigma_d :\ b \mapsto b\ ,\ \sigma_b :\ \begin{array}{ccc} a & \mapsto & d \\ d & \mapsto & c \\ c & \mapsto & a \end{array}$



Lemma 4.8 *Let (V,E) be a fan. Let $D = D_1 \cup D_2$ and $hyp(V,E) = (D,e,n,f)$. Then*

1. *hyp(V,E) is a plain hypermap.*
2. *e and f have no fixed points in D_1.*
3. *For every pair of distinct nodes, at most one edge meets both.*
4. *The two darts of an edge of D_1 lie at different nodes.*

Proof

$$e(n(f(v,w))) = e(n(w,\sigma_w^{-1}(v))) = e(w,v) = (v,w)$$

So $hyp(V,E)$ is a hypermap.

1. Seeing that $hyp(V,E)$ is plain is an elementary calculation: we only need to check if e is an involution on D.

$$e^2 = e(e(v,w)) = e(w,v) = (v,w) = Id_D$$

2. Because $v \notin E(v)$, there are no fixed points in D_1 under e. The argument that f has no fixed points is similar.

3. The next step is to show that for every two distinct nodes, there is at most one edge meeting both. This means that: for $i \in \mathbb{N}$, if there exists a $j \in \mathbb{N}$ such that

$$n^i(e(x)) = e(n^j(x)) \Rightarrow n^j(x) = x$$

Let $x = (v,w) \in D_1$. Then, by definition, $n^j(x) = (v, \sigma_v^j(w))$, and then $e(n^j(x)) = (\sigma_v^j(w), v)$.

On the other hand, $e(x) = (w,v)$ and $n^i(e(x)) = (w, \sigma_w^i(v))$.

If we impose that $n^i(e(x)) = e(n^j(x)) \Rightarrow w = \sigma_v^j(w) \Rightarrow n^j(x) = (v, \sigma_v^j(w)) = (v,w) = x$.

4. Finally, each dart of an edge lies on a different node. That is, $e(x) \neq n^i(x)$ for $x \in D_1$. As we can see:

$e(v,w) = (w,v), w \in E(v)$ and $n^i(v,w) = (v, \sigma_v^i(w)), v \notin E(v)$, then $\forall i$ they are different.

□

4.3 Planarity

In this section we are going to give a characteristation of how are the fans that have a planar hypermap associated. They are called conforming hypermaps and are very important in the proof of the conjecture.

Definition 4.9 *Let (V,E) be a fan. Let* $\mathbf{X}$ *= X(V,E) be the union of the blades* $X = \bigcup_{\epsilon \in E} C(\epsilon)$.

Let $\mathbf{Y}$ *= Y(V,E) be the complement* $Y = \mathbb{R}^3 \setminus X$.

Lemma 4.10 *Let (V,E) be a fan with hypermap H = (D,e,n,f). There is a natural bijection between the nodes of H and V that sends the node (orbit of n) containing the dart* $(v,*)$ *to* $v \in V$.

Proof

The function n is defined as $n(v,w) = (v,\ \sigma_v(w))$, therefore the orbits of n are defined by the chosen v. This means that there is only one orbit of n containing the dart $(v,*)$ and hence the bijection $(v,*) \leftrightarrow v$.

□

Example 4.11 *Take the fan (V,E) of the example 4.6, where* $V = \{a,b,c,d\}$ *and the set of darts of the associated fan is* $D = \{(a,b),(b,a),(b,d),(d,b),(b,c),(c,b)\}$.

The orbits of n (nodes of the hypermap) are $\{(a,b), \Big((b,a),(b,d),(b,c)\Big), (c,b), (d,b)\}$.

The natural bijection will then be

$$\begin{array}{ccc} a & \leftrightarrow & (a,b) \\ b & \leftrightarrow & ((b,a),(b,d),(b,c)) \\ c & \leftrightarrow & (c,b) \\ d & \leftrightarrow & (d,b) \end{array}$$

Definition 4.12 *We write* $\textbf{node}(x) \in V$ *for the node corresponding to a dart* $x \in D$ *under the bijection of the previous lemma.*

Definition 4.13 *Let (V,E) be a fan and (D,e,n,f)=hyp(V,E). Let* $x = (v,w) \in D$ *be a dart. Define the azimuth angle of x as:*

$$azim(x) = \begin{cases} azim(0,v,w,\sigma(v,w)) & \text{if } card(E(v)) > 1 \\ 2\pi & \text{otherwise.} \end{cases}$$

We can find the definition of $azim(v_1,v_2,v_3,v_4)$ *in definition 2.4.*

Definition 4.14 *A fan (V,E) is* ***fully surrounded*** *if* $azim(x) < \pi$ *for all darts in the hypermap of (V,E).*

We are going now to define a class of fans that we will need in the last chapter. The faces of the hypermaps associated to this fans have a geometrical description given by open half-spaces. The definition of this kind of fans is very thecnical, and we will not go into detail explaining its properties.

After that, we will announce two lemmas that we will admit without showing, because the proof is very cumbersome. They will help us reduce the number of properties that need to be check to know if a fan is in this class or not.

Definition 4.15 *Let (V,E) be a fan with hypermap (D,e,n,f). The fan is* ***conforming*** *if the following properties hold:*

1. (V,E) is fully surrounded.

2. There is a bijection between the faces of the hypermap and the topoogical components of Y.

3. For every face F, the topological component U_F *is the intersection of the open half-spaces*

$$aff_+^0(\{0, node(x), node(fx)\}, node(f^{-1}x)) \text{ as } x \text{ runs over } F.$$

4. For every F, the intersection $B(0,r) \cap U_F$ *is measurable.*

5. For every face F, if $x, y \in F$, $x \neq y$ *with corresponging nodes* $node(x), node(y) \in V$*, then node(x) and node(y) are not parallel. This means that* $\langle node(x) \rangle \cap \langle node(x) \rangle \neq \emptyset$.

Lemma 4.16 *Let (V,E) be a conforming fan. Then hyp(V,E) is simple and planar.*

Lemma 4.17 *Every fully surrounded fan is conforming.*

4.4 Polyhedrons

This section shows that a bounded polyhedron in $\mathbb{R}^3$ with nonempty interior determines a fan (and consequently a hypermap).

Definition 4.18 *A* ***polyhedron*** *is the intersection of a finite number of closed half-spaces in* $\mathbb{R}^n$.

Lemma 4.19 *(Krein-Milman) Every compact convex set* $P \subset \mathbb{R}^n$ *is the convex hull of its set of extreme points.*

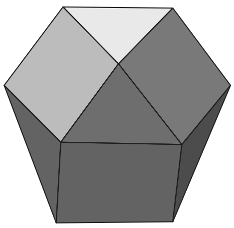
Remark 4.20 *A polyhedron is then closed and convex. If we considere a polyhedron that is bounded, then it will be a compact and convex set and according to Krein-Milman theorem, it will be equal to the convex hull of its set of extreme points.*

Definition 4.21 *Let P be a bounded polyhedron.* $\boldsymbol{V}_p$ *is the set of extreme points of P and* $\boldsymbol{E}_p$ *is the set of pairs {v,w} of extreme points such that conv{v,w} is an edge of P.*

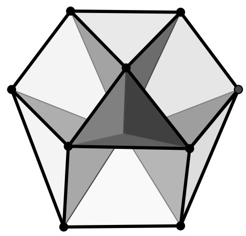
It is not hard to see that

Lemma 4.22 *Let P be a bounded polyhedron in* $\mathbb{R}^3$ *with the interior point 0. Then* (V_p, E_p) *is a fan.*

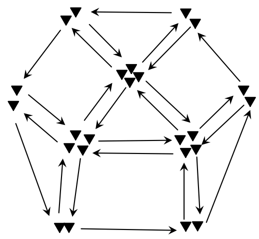
As the following illustration shows:



The polyhedron



The associated fan



The face permutation of the front half of the associated hypermap

Definition 4.23 *Let P be a bounded polyhedron and let* (V_p, E_p) *be the associated fan. For every facet F of P, we define* $\mathbf{W}_F = \{\boldsymbol{t} \cdot p \ : \ p \in ri(F),\ t \neq 0,\ t \in \mathbb{R}^+\}$ *where ri(F) is the relative interior of F.*

Lemma 4.24 *Let P be a bounded polyhedron and let* (V_p, E_p) *be the associated fan. The union of the sets* W_F *is the topological component of* (V_p, E_p)*:*

$$\bigcup W_F = Y(V_p, E_p).$$

Proof:

Select any $p \in Y(V_P, E_P)$. As 0 lies in the interior of the bounded polyhedron, we can multiply p by a positive scalar **t** so that $\mathbf{t}{\cdot}p$ is in the boundary of P, and hence in a facet F. Once in the facet, it can be in its relative interior or in its relative boundary.

If it is the first case, $\mathbf{t}{\cdot}p \in ri(F)$, then $p \in W_F$, as desired.

Otherwise, $\mathbf{t}{\cdot}p$ lies in the relative boundary of F. The facets of a three dimensional polyhedron have dimension two, and the facets forming the relative boundary of F have dimension one. These faces are edges of P. Thus, $\mathbf{t}{\cdot}p$ lies in an edge of P, and that implies $p \in X(V_P, E_P)$, which can't be true because we supposed that $p \in Y(V_P, E_P)$. It follows that the sets W_F are the topological components of $Y(V_P, E_P)$.

□

Lemma 4.25 *Let P be a bounded polyhedron with 0 as an interior point. Then $azim(x) < \pi$ for every dart x in the associated hypermap $hyp(V_P, E_P)$.*

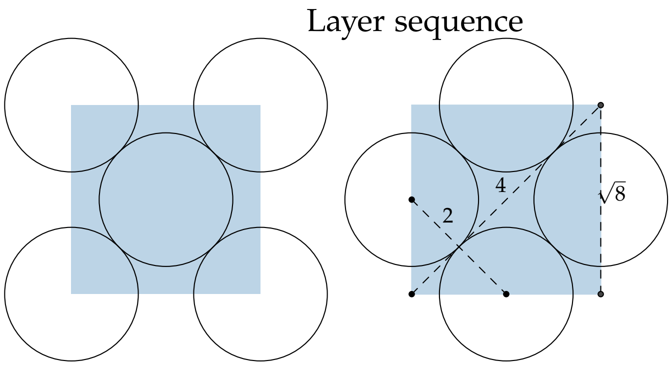
This lemma shows that the fan (V_P, E_P) is fully surrounded. Hence, by lemma 4.17, all of the properties of conforming fans apply to this fan.

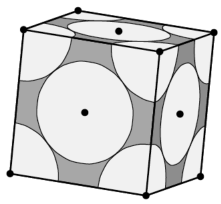
Chapter 5

The proof

The Kepler conjecture asserts that no packing of congruent balls in $\mathbb{R}^3$ has density greater than the density of the FCC packing. We begin this last chapter by computing the value of this density.

The FCC packing is obtained from a cubic lattice. We divide the space into cubes, then insert a ball at each of the eight vertices of each cube and then insert another ball at the center of the facets of each cube. The name face centered cubic comes from this construction.





This is what one of the cubes looks like.

For convenience, to compute the density of the FCC packing we will consider balls of unit radius. This means that the distance between the centers of two balls is at least 2. It follows that the diagonal of each facet is then 4, therefore the edge of each cube is $\sqrt{8}$.

Because all the cubes are the same, the density of the packing as a whole is equal to the density within a single cube. As explained in the introduction, this density is the volume occupied by the balls that are in the cube divided by the volume of the cube.

The cube has volume $\sqrt{8}^3$ and contains a total of four balls: half of a ball along each of the six facets and one eighth of a ball at each of the eight corners. Therefore, the density within one cube is four times the volume of a ball divided by the volume of the cube:

$$\frac{4\frac{4\pi}{3}}{\sqrt{8}^3} = \frac{\pi}{\sqrt{18}}$$

We aim to prove the conjecture by contradiction.

However, it is not always an easy task to compute the density of a packing. In the case of the FCC packing it was simple, because it is obtained from a cubic lattice, but in any other case we would have to compute the limit in definition 1.2, $\delta(V) = \lim_{r\to\infty} \delta(V, p, r)$.

This is why we present the proof in this four steps:

1. We will reduce the problem to one involving only a finite number of balls. To do so, we will define the decomposition of space into Voronoi cells and use the strategy that we explained in the introduction. With this, the problem will be reduced to prove the existence of a negligible FCC-funtion for a finite packing (that we are going to call W).

The next step will be then introducing Roger simplices and Marchal cells. Thanks to this decompositions, and to the Marchal conjecture, we are able to find an inequality that, if true for a finite packing W, implies the existence of such a function.

As we will explain below, the concrete inequality is $\mathcal{L}(W) \leq 12$ for every finite packing with centers in $B(0, 2.52)$.

We will then assume that W is a counterexample to this inequality and therefore satisfies $\mathcal{L}(W) > 12$. We want to see that it is impossible that such a W exists.

2. We will study some properties of W and we will say that a packing that possesses all these attributes is a *contravening* packing. This particular packings will be essential in steps 3 and 4.

3. We will study the hypermap H associated to W. A detailed study of the properties of H leads to a long list of properties that all hypermaps associated to a packing like W must possess. These properties define what a *tame hypermap* is.

4. A computer generates an explicit list, enumerating tame hypermaps up to isomorphism. Each tame hypermap H gives rise to a nonlinear optimization problem to maximize $\mathcal{L}(W)$ that can be reduced to several linear optimization problems. All of them have been solved by computer and in every case, after branching into subcases, the maximum of $\mathcal{L}(W, 0)$ is stricly less than 12. This is the contradiction that proves the Kepler conjecture, because we had supposed that W verified $\mathcal{L}(W) > 12$.

5.1 Decompositions of space

We start with the first step. In this section we will explain Voronoi cells and Roger simplices and give the necessary background to get to the inequality $\mathcal{L}(W \leq 12$.

First of all, we define again what a Voronoi cell is:

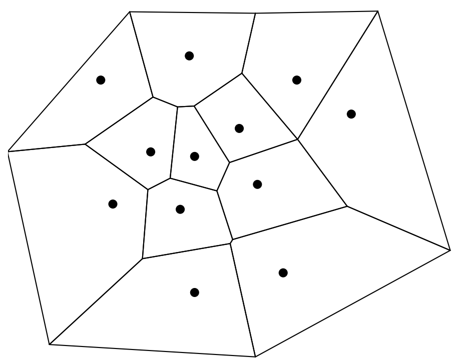
Definition 5.1 *The* ***Voronoi cell*** $\Omega(V, u)$ *is the intersection of the half-spaces* $A_+(u, v)$ *as* v *runs over* $V \setminus \{u\}$.

Recall that we defined half-space in definition 1.4.

The next property is a standard fact on Voronoi cells which we are not going to prove here.

Lemma 5.2 *If* V *is a saturated packing, then*

$$\mathbb{R}^3 = \bigcup_{v \in V} \Omega(V, v)$$



Example of Voronoi cells of a packing in $\mathbb{R}^2$

Remark 5.3 *It can be seen that each Voronoi cell is a bounded polyhedron and hence it is compact, convex and mesurable.*

We recall that we are able to define what a negligible, FCC-compatible function is thanks to the Voronoi cells. We recall now definition 1.7:

Let $V \subset \mathbb{R}^3$ *be a packing and* $V(0, r) := V \cap B(0, r)$.
We say that $G : V \to \mathbb{R}$ *is* ***negligible*** *if there exists a constant* c *such that* $\forall r \geq 1$

$$\sum_{v \in V(0,r)} G(v) \leq cr^2$$

and that G is ***FCC-compatible*** *if* $\forall v \in V$

$$4\sqrt{2} \leq \mathrm{vol}(\Omega(V,v)) + G(v)$$

Where
$4\sqrt{2}$ *is the volume of the Voronoi cell in the FCC-packing,*
$\mathrm{vol}(\Omega(V,v))$ *is the volume of the Voronoi cell in the packing* V*,*
$G(v)$ *is used as the adjustment to correct the error.*

Recall lemma 1.8 as well:

If there exists a negligible, FCC-compatible function $G : V \to \mathbb{R}$ *for a* satured packing V*, then there exists a constant* $c = c(V)$ *such that* $\forall r \geq 1$*,*

$$\delta(V,0,r) \leq \frac{\pi}{18} + \frac{c}{r}$$

We are then able to consider a packing with a finite number of balls: from a packing V with an infinite number of balls, we considere only the ones that are held in a finite container and call it the packing W. Thanks to lemma 1.9 we have a bound on the density of W and, if we expand the radius of the container to infinity, we have also one for the density of V. This can only be done if a negligible FCC-compatible function exists, and that is why our strategy will be to prove its existence.

As we explained in the introduction of the chapter, the next step is introducing Roger simplices and Marchal cells. We need some definitions first.

Notation: We write

$$\underline{u} = [u_0; ...; u_k]$$

for the ordered list of elements u_i with $i = 0, ..., k$ and

$$d_j\underline{u} = [u_0; ...; u_j]$$

for a sublist of $\underline{u}$ of lenght $j + 1$.

Definition 5.4 *Let V be a saturated packing. We use the notation* $\Omega(V,*)$*, where * is a set or list of points, to denote the intersection of Voronoi cells. If* $* = W \subset V$ *is a set of points*

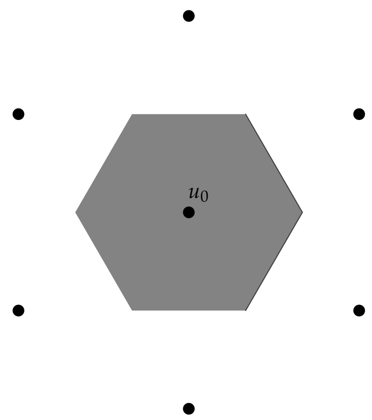
$$\Omega(V,W) = \bigcap\{\Omega(V,u) : u \in W\}$$

And if $* = [u_0; \ldots; u_k]$ *is an ordered list*

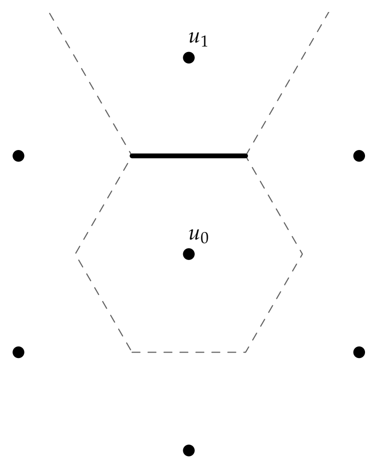
$$\Omega(V,[u_0;\ldots;u_k]) = \Omega(V,\{u_0;\ldots;u_k\})$$

Example 5.5 *Let* $V = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ *be a packing. Set* $\underline{u} = [u_0; u_1; u_2]$.

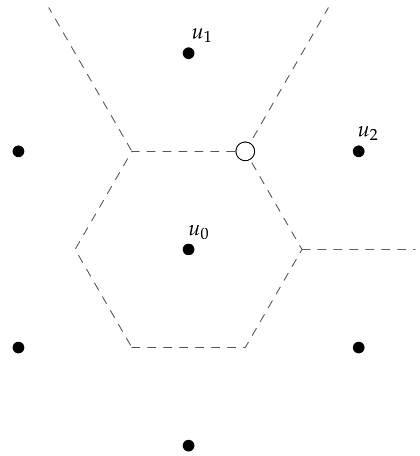
$\Omega(V, d_0\underline{u}) = \Omega(V, [u_0])$ *is the Voronoi cell of* u_0*, as shown in the picture below:*



$\Omega(V, d_1\underline{u}) = \Omega(V, [u_0; u_1])$ *is the black segment. It is the intersection of the Voronoi cells of* u_0 *and* u_1 *(drawn in dashed lines).*



$\Omega(V, d_2\underline{u}) = \Omega(V, [u_0; u_1; u_2])$ *is the white point. It is the intersection of the Voronoi cells of* u_0*, of* u_1 *and of* u_2 *(drawn in dashed lines).*



Definition 5.6 *Let $k = 0, 1, 2, 3$. We define $\underline{V}(k)$ as the set of lists*

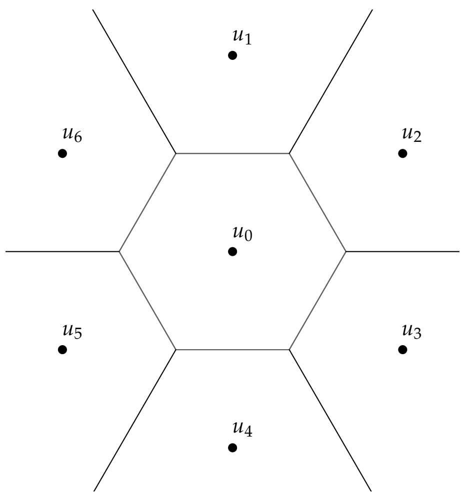
$$\underline{V}(k) = \{\underline{u} \;:\; dim\Big(aff(\Omega(V, d_j\underline{u}))\Big) = 3 - j;\;\; \forall j,\;\; 0 < j \leq k\}.$$

For $k > 3$, we set $\underline{V}(k) = \emptyset$.

Example 5.7 *If we consider the packing $V = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ from the previous example,*

$$\underline{V}(0) = \{[u_0], [u_1], [u_2], [u_3], [u_4], [u_5], [u_6]\}$$

because all of them have Voronoi cells.



On the other hand,

$$\underline{V}(1) = \{[u_0; u_1], [u_0; u_2], [u_0; u_3], [u_0; u_4], [u_0; u_5], [u_0; u_6], [u_1; u_2],$$

$$[u_2; u_3], [u_3; u_4], [u_4; u_5], [u_5; u_6], [u_1; u_6]\}$$

where all the lists are constructed with two elements u_i, u_j such that the Voronoi cells at this two points share a facet.

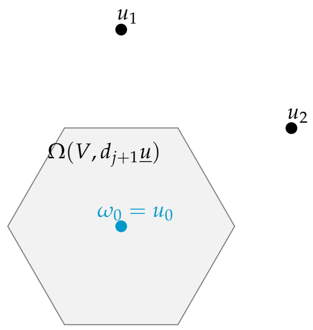
Each Roger simplex is given by the convex hull of its set of extreme points. We note these points as $\omega_j(V, \underline{u})$ and define them as follows:

Definition 5.8 *Let V be a satured packing and let $\underline{u} \in \underline{V}(k)$ for some k. We define the* ***extreme points of*** $\underline{\mathbf{u}}$*, $\omega_j = \omega_j(V, \underline{u}) \in \mathbb{R}^3$, by recursion over $j \leq k$:*

$$\omega_0 = u_0$$

$$\omega_{j+1} = \text{ the closest point to } \omega_j \text{ on } \Omega(V, d_{j+1}\underline{u}) \quad \text{for } j = 0, \ldots$$

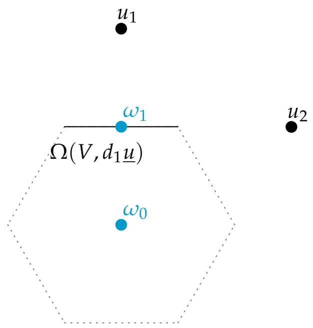
Example 5.9 *:*
This is an example in $\mathbb{R}^2$ for $u = [u_0; u_1; u_2]$.
The first step is $\omega_0 = u_0$.



And for the points ω_1 and ω_2 we use the recursive formula given in the definition:

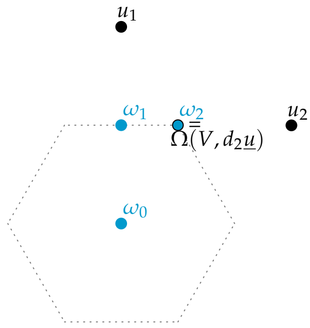
$$\omega_1 = \text{ the closest point to } \omega_0 \text{ on } \Omega(V, d_1\underline{u})$$

Where $\Omega(V, d_1\underline{u})$ is just a line:



$$\omega_2 = \text{ the closest point to } \omega_2 \text{ on } \Omega(V, d_2\underline{u})$$

$\Omega(V, d_2\underline{u})$ is just a point, thus $\omega_2 = \Omega(V, d_2\underline{u})$.

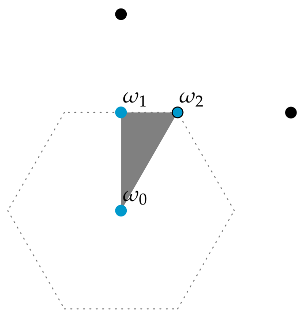


Definition 5.10 *Let $V \subset \mathbb{R}^3$ be a saturated packing. The set $R(\underline{u})$ for $\underline{u} \in \underline{V}(k)$*

$$R(\underline{u}) = conv\{\omega(d_0\underline{u}), \omega(d_1\underline{u}), ..., \omega(d_k\underline{u})\}$$

*is called the **Rogers simplex of $\underline{u}$**.*

Example 5.11 *In the previous example, $R(\underline{u}) = conv\{\omega_0, \omega_1, \omega_2\}$:*



We give now a definition that we are going to need as background:

Definition 5.12 *Let $S \subset \mathbb{R}^3$. A point p is the **circumcenter** of S if it is an element of the affine hull of S that is equidistant from every $v \in S$.*

*If S has a circumcenter p, then the distance $\|p - v\|$ for all $v \in S$ is the **circumradius** of S.*

If $\underline{u} := [u_0; ...; u_k]$ is a list of points in $\mathbb{R}^3$, then $h(\underline{u})$ is the circumradius of its point set.

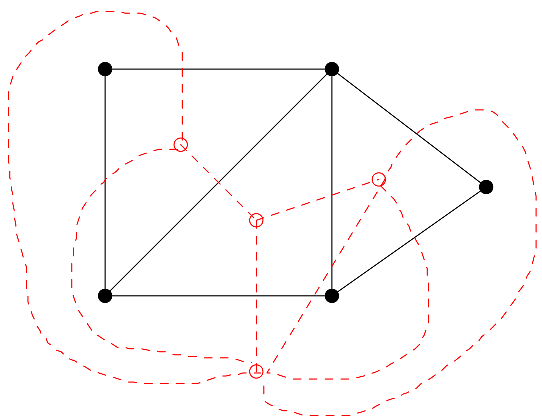
The most important is that we note that when there are only two elements, $h(u_0, u_1) = \frac{dist(u_0,u_1)}{2}$.

We are going to explain now what Delaunay simplices are because we are going to need them to explain what Marchal cells are.

There is a very well known variant of Roger simplices called **Delaunay simplices**. We will not give a formal definition because, in order to do so, we would need a background much bigger than the one we are giving, and it would not be relevant for the proof of the conjecture. Nevertheless, we are going to give an idea of what they are in $\mathbb{R}^2$:

Let V be a packing in $\mathbb{R}^2$. We can interpret the Voronoi cells of V as a graph G. The Delaunay decomposition of V would be the dual graph of G.

The dual graph of a graph G has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. We can see in it in the illustration below, where there is a graph painted with straight lines and its dual graph in dashed lines.



We move up now to Marchal cells. They are also a variant of Roger simplices, but much more sofisticated and thus we are not going to give a formal definition either but we are going to discuss them informally.

There are five different kinds of Marchall cells:
0- and 1-cells, which are parts of a Voronoi cell,
2- and 3-cells, which are gradations between Voronoi cells and Delaunay simplices,
4-cell, which are Delaunay simplices.

The advantage that Marchal cells have over the other decompositions of space is that they adapt more to each packing. Depending on the arrangement of the balls, different types of cells are going to be defined between two balls.

For example, the 0- and 1-cell of $\underline{u}$ are only defined when $\sqrt{2} \leq h(\underline{u})$. In vague terms, when the balls are closer to each other, more detailed is needed in the partition of space, thus Delaunay simplices are defined because they will adapt with more precision than Voronoi cells. If that is not the case, then Voronoi cells are defined.

Because of its definition, Marchal cells lead to the best boundary for the density of a packing.

All in all, we are now ready to find the discused inequality $\mathcal{L}(V) \leq 12$.

5.1.1 The inequality

In the proof of the conjecture, as a first step to get to the inequality explained in the introduction, Hales defines a polynomial function M and a function γ (in terms of the volumes, solid angles and dihedral angles of Marchal cells). He then announces Marchal's conjecture:

Conjecture 5.13 *(Marchal) For any packing V and any $u \in V$*

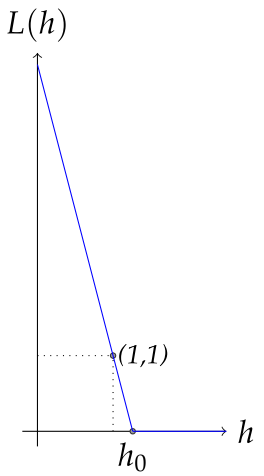
$$\sum_{v \in V \setminus \{u\}} M(h(u,v)) \leq 12$$

We do not give these definitions here because they are rather complex and they are not used in the proof of the conjecture. Instead, we will focus our attention in a variant of Marchal's conjecture that uses a linear function L instead of the polynomial function M. This variant will be enought to help us prove the existence of the negligent FCC-compatible function.

Notation: The constant $1,26$ will appear several times in the rest of the chapter and thus we give it a name $\mathbf{h}_0 = 1,26$.

Definition 5.14 *We define L as*

$$L(h) = \begin{cases} \frac{h_0 - h}{h_0 - 1} & h \leq h_0 \\ 0 & h \geq h_0 \end{cases}$$



With that we can now announce the variant of the Marchal conjecture that is the inequality explained in the introduction. Recall that $h(u_0, u_1) = \frac{dist(u_0,u_1)}{2}$.

Lemma 5.15 *Consider the function*

$$\mathcal{L}(V, u_0) = \sum_{u_1 \in V \,:\, h(u_0,u_1) \leq h_0} L(h(u_0, u_1))$$

Then, for any saturated packing V and any $u_0 \in V$,

$$\mathcal{L}(V, u_0) \leq 12$$

The proof of this lemma relies on many computer calculations and, as a conseguence, we are not going to give in the details. However, there are some non-computer parts. That is where we are going to focus our attention. As we explained in the introduction, the rest of the chapter is a study of the inequality and hence of its proof.

Lemma 5.16 *The inequality of the previous lemma implies that for every saturated packing V, there exists a negligible FCC-compatible function $G \to \mathbb{R}$.*

It can be seen that the negligible FCC-compatible function that this lemma talks about is the function γ that we mentioned in the beginning of this section. The proof of this lemma is shown in [1] and it is not hard to follow, but since we have not given the definition of γ and some Marchal cells properties are also involved, we are going to accept the lemma without proving it.

As we have seen before, the existence of such function implies the Kepler conjecture. What we want to do now is prove that the inequality $\mathcal{L}(V, u_0) \leq 12$ is always true.

Definition 5.17 *Let $\mathcal{B}$ be the annulus $\overline{B}(0, 2h_0) \setminus B(0, 2)$, where $\overline{B}(0, 2h_0)$ is the closed ball of center 0 and radius $2h_0 = 2,52$.*

The following lemma is one of the keys to proving the conjecture. Until now, we have been building a background and with this we start the real proof. As we said before, we our strategy is to prove the conjecture by contradiction, and this is the first step in that direction:

Lemma 5.18 *If the Kepler conjecture is false, then there exists a finite packing $W \subset \mathcal{B}$ such that*

$$\mathcal{L}(W, 0) = \sum_{w \in W} L(h(0, w)) > 12$$

Proof:

If the Kepler conjecture is false, inequality $\mathcal{L}(W, u_0) \leq 12$ does not hold for some packing W and some $u_0 \in W$. We only need to change the subindices of the inequality. To do so, we translate W to $W - u_0$ and u_0 to 0 (therefore $u_0 = 0 \in W$) and replace W with the finite subset $W \cap \mathcal{B}$ (which will impose the condition in the subindex of the sum). It follows that the packing is a finite subset of $\mathcal{B}$.

□

Recall that $h(u_0, u_1) = \frac{dist(u_0,u_1)}{2}$, and it is clear that then $h(0, u) = \frac{\|u\|}{2}$. We can rewrite the previous inequation as

$$\sum_{w \in W} L(\frac{\|w\|}{2})) > 12$$

5.2 Contravening packing

The following step to proving the conjecture will be studing some of the properties of the packing W mentioned on the last lemma of the previous section. We want to know how this packing is and see if it really exists. We will start by computing its cardinality in lemma 5.20, but in order to understand its proof we need this two definitions first:

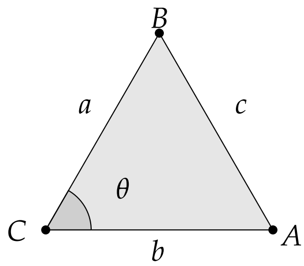
Definition 5.19 *Let $r, r' \in \mathbb{R}$ such that $2 \leq r \leq r'$. We say that a packing $V \in \mathbb{R}^3 \setminus B(0,2)$ is **weakly saturated** with parameters (r,r') if for every $p \in \mathbb{R}^3$ such that*

$$2 \leq \|p\| \leq r'$$

there exists an element $u \in V$ such that

$$\|u - p\| < r$$

Definition 5.20 *Let ABC be a triangle like in the figure*



We define

$$arc(a, b, c) = \theta$$

which, by the law of cosines is the same thing as

$$\theta = arccos(\frac{a^2 + b^2 - c^2}{2ab})$$

Now we announce the lemma that is going to give us the cardinality of W. This is one of the most important lemmas in the proof of the conjecture. With it, we will know exactly how many balls are in W and it will give us a best understanding of the packing. The proof has many steps and almost of the background that we have given up until this point is involved. We give the highlights of the proof so the notation does not get very cumbersome and we have tried to separate the steps clearly so it is easier to understand.

Lemma 5.21 *If $W \subset \mathcal{B} = \overline{B}(0, 2h_0) \setminus B(0,2)$ is a finite packing that satisfies*

$$\mathcal{L}(W) = \sum_{w \in W} L\Big(\frac{\|w\|}{2}\Big) > 12$$

then the cardinality of W is 13, 14 or 15.

Sketch of the proof:

The proof consists on finding a top boundary and an inferior boundary on the cardinality of W.

Finding the inferior boundary is easy, and it is the first step:

We want to see that the packing W contains more than twelve points. This is easy, because by definition

$$L(h) \leq 1$$

and we are supposing that W holds the inequality

$$\mathcal{L}(W) > 12.$$

Hence, W contains more than 12 points.

And now the second step: finding a top boundary on the cardinality of W. This is not an easy task and it takes the rest of the proof to see that the top boundary is 16 (hence the packing has cardinality 13, 14 or 15).

We will start by adding as many points as necessary so the packing becomes weakly satured with parameters $(r, r') = (2, 2h_0)$.

This means that for every $p \in \mathbb{R}^3$ with

$$2 \leq \|p\| \leq r'$$

then there exists a $u \in V$ such that

$$\|u - p\| < r$$

all in all, it is enought to show that this enlarged set has cardinality strictly less than 16.

For the following step we set the function

$$\mathbf{g(h)} = \arccos(\frac{h}{2}) - \frac{\pi}{6}$$

and the value

$$\mathbf{h}_i = \frac{\|u_i\|}{2}$$

whom is $\leq h_0 = 1.26$.

We then consider the spherical discs D_i of radii $g(h_i)$, centered at $\frac{u_i}{\|u_i\|}$ on the unit sphere.

We know that these discs do not overlap because, as we show in the next illustration,

$$\text{arc}(h_1, h_2, 1) = \alpha \geq g(h_1) + g(h_2)$$

for all $h_1, h_2 \in [1, h_0]$.

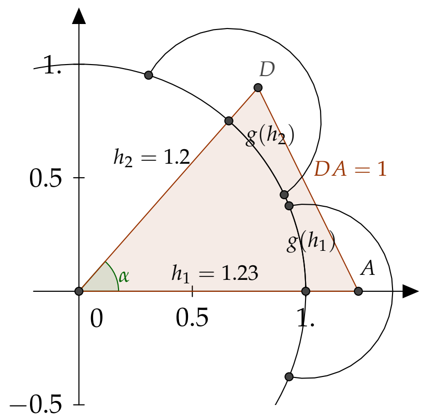
Note that we have the following equalities:

$$\text{dist}(0, A) = h_1,$$

$$\text{dist}(0, D) = h_2,$$

$$\text{dist}(DA) = 1.$$

Remark that we chose DA to be 1 because this would be the extreme case where the balls in the packing are tangent (the distance between the centers would be 2).



As we can see, the spherical discs do not overlap.

We extend a plane through the circular boundary of each D_i. This planes are the boundaries of half-spaces containing the origin. If we intersect them all, we obtain a polyhedron P.

Because the packing is weakly satured and the half-spaces contain the origin, this polyhedron is bounded.

For the next step, we define the sets:

V_p = {extreme points of P}
E_p ={{v, w}; $v, w \in V_p$ such that conv{v, w} is an edge of P}

As lemma 4.22 announces, (V_p, E_p) is the fan associated to P.

It is important to note that there are natural bijections between the following sets:

1. The elements of the packing $V = \{u_1, ..., u_N\}$.
2. The facets of P.
3. The set of topological components of $Y(V_p, E_p)$. Recall that that is the set of W_F, as lemma 4.24 shows.
4. The set of faces in the hypermap hyp(V_p, E_p). Recall that we defines the hypermap associated to a fan in section 4.2.

We proceed to prove some of them:
- The bijection between sets 2 and 3 is given by lemma 4.24.
- Thanks to lemma 4.25 we know that the fan (V_p, E_p) is fully surrounded. Hence, by lemma 4.17 it is also conforming. The definition of conforming gives the bijection between the last two sets.

Remark that because of the bijection between 1 and 4, the hypermap hyp(V_p, E_p) has as many faces as elements in the packing V. Hence, the hypermap has N faces.

Because of the bijection between 2 and 4, the number of edges of the facet i of P and the cardinality of the corresponding face in hyp(V_p, E_p) are the same. We call it k_i.

The next step is defining the function

$$\text{reg}(g(h_i), k_i) = 2\pi - 2k_i(\arcsin(\cos(g(h_i)) \cdot \sin(\tfrac{\pi}{k_i})))$$

which is an inferior boundary of the solid angle of the topological component W_i of $Y(V_p, E_p)$. We will not go into detail about the properties of the solid angle of W_i, but as it is shown in [1], the equality

$$4\pi = \sum_i sol(W_i)$$

holds.

By a computer calculation:

$$\text{reg}(g(h), k) \geq c_0 + c_1 k + c_2 L(h), \quad \text{for all } k = 3, 4, ...; \quad 1 \leq h \leq h_0 = 1.26$$

where

$$c_0 = 0.591, \; c_1 = -0.0331, \; c_2 = 0.506.$$

By lemma 4.22, the sum $\sum_i k_i$ is the number of darts in $\text{hyp}(V_p, E_p)$, and by lemma 3.13

$$\#\text{darts} = \sum_i k_i \leq (6N - 12) = 6 \; \#\{\text{faces}\} \; - 12.$$

Summing over i we obtain an estimate on N:

$$\begin{aligned} 4\pi &= \sum_{i=0}^{N} sol(W_i) \\ &\geq \sum_{i=0}^{N} reg(g(h_i), k_i) \\ &\geq c_0 N + c_1 \sum_{i=0}^{N} k_i + c_2 \sum_{i=0}^{N} L(h_i) \\ &\geq c_0 N + c_1(6N - 12) + c_2 12 \end{aligned}$$

It follows easily that the following inequality is equivalent:

$$\frac{4\pi + 12c_1 - 12c_2}{c_0 + 6c_1} \geq N$$

$$15,5382 \geq N$$

$$16 > N$$

To sum up: we saw that it was enough to verify that the

□

The value of $\mathcal{L}$ depends only on the norms $\|w\|$, and L is a decreasing function. Because of this, any rearrangement of the points of W that does not increase the norms, strengthens the inequality. In sum, we know now that W has cardinality 13, 14 or 15.

Now we are going to show two more properties of this packing that will lead us to the definition of contravening packing. As we said in the introduction of this chapter, this is going to be an important definition for the following steps.

Definition 5.22 *Let* $V \subset \mathcal{B} = \overline{B}(0, 2h_0) \setminus B(0, 2)$ *be a packing. We define* $E_{std} = E_{std}(V)$ *and* $E_{ctc} = E_{ctc}(V)$ *as*

$$E_{std} = \{\{v, w\} \subset V \ : \ 0 < \|v - w\| \leq 2h_0 = 2.52\}$$

$$E_{ctc} = \{\{v, w\} \subset V \ : \ \|v - w\| = 2\} \subset E_{std}$$

It is not hard to see that:

Lemma 5.23 *Let* $V \subset \mathcal{B}$ *be a packing. If* $E = E_{std}$ *or* $E = E_{ctc}$*, then* (V, E) *is a fan.*

Definition 5.24 *The fans* (V, E_{std}) *and* (V, E_{ctc}) *are called* ***the standard fan*** *and* ***the contact fan****, respectively.*

Definition 5.25 *Let (V, E) be a fan. As we defined in chapter 4,* $v \in V$ *is said to be* ***isolated*** *in the fan if E(v) is empty.*

We say that $v \in V$ *is* ***surrounded*** *in the fan if the cardinality of E(v) is three or more.*

And finally, gathering all the properties that we have studied, we define contravening packing:

Definition 5.26 *A finite packing V is a* ***contravening packing*** *if it satisfies the following properties:*

1. $V \subset \mathcal{B} = \overline{B}(0, 2h_0) \setminus B(0, 2)$
2. $\mathcal{L}(V) > 12$
3. *The cardinality of V is 13, 14 or 15.*
4. *Every node of the standard fan* (V, E_{std}) *is surrounded.*
5. *Every node v in the contact fan* (V, E_{ctc}) *that is not surrounded, satisfies* $\|v\| = 2$*.*

This is a summary of all the properties of a packing that does not satisfy the Kepler conjecture. In the next section we are going to see the properties of the hypermap associated to this packing.

5.3 Tame hypermap

In this section we are going to give the formal definition of tame hypermap. It consists in a list of properties that a hypermap has to follow in order to be tame. This properties have been obtained by studing the hypermap associated to the packing W that does not satisfy the Kepler conjecture. It is not necessary to go into the details of the definition. However we are going to note property number 8, that asserts that a tame hypermap has 13, 14 or 15 nodes. This is a consequence of lemma 5.20.

Definition 5.27 *A hypermap is **tame** if it satisfies the following conditions:*

1. (planar) The hypermap is plain and planar.

2. (simple) The hypermap is connected and simple. In particular, each intersection of a face with a node contains at most one dart.

3. (nondegenerate) The edge map e has no fixed points.

4. (no loops) The two darts of each edge lie in different nodes.

5. (no double joins) At most one edge meets any two (not necessarily distinct) nodes.

6. (face count) The hypermap has at least three faces.

7. (face size) The cardinality of each face is at least three and at most six.

8. (node count) There are thirteen, fourteen, or fifteen nodes.

9. (node size) The cardinality of every node is at least three and at most seven.

10. (node types) If a node has type (p, q, r) with $p + q + r \geq 6$ and $r \geq 1$, then (p, q, r) = (5, 0, 1).

11. (weights) There exists an admissible weight assignment of total weight less than tgt = 1.541.

The reason why we are not going to study this definition is that Hales developed an algorithm that generates all restricted hypermaps with a given bound on the number of nodes. With it he could generate all the different hypermaps with associated packing W (that is, all tame hypermaps).

He implemented and executed the algorithm and the result was an explicit list of about 25.000 tame hypermps.

5.4 Linear programming

This is the last part of the proof where we are going to find a contradictino that will lead to the conclusion that the Kepler conjecture is true. All of the showings in this section are done by computer and as a result it is a short section.

First of all, we associate a to each tame hypermap H on the list a configuration space $\mathcal{W}_H$ of all contravening packings $W \in \mathcal{B}$, whose standard fan (recall definition 5.23) is isomorphic to H.

The problem we face is a nonlinear optimization problem that asks for the maximum of

$$\mathcal{L}(W) = \sum_{w \in W} L\Big(\frac{\|w\|}{2}\Big)$$

over all $W \in \mathcal{W}_H$.

We found the solution to this problem in the following theorem:

Theorem 5.28 *Let H be a tame hypermap that appears in the explicit list.*
Let $W \in \mathcal{W}_H$. Then

$$\mathcal{L}(W) < 12.$$

Proof:
Each nonlinear optimization problem can be partitioned into several linear optimization problems that can be solved by computer. About 50, 000 linear programs arise and after solving all of them, it is seen that the maximum of $\mathcal{L}(W)$ is strictly less than 12.

$\square$

We have then found a contradicction, because when we started studing this packings W we supposed that $\mathcal{L}(W) > 12$.

Ware then able to affirm that the Kepler conjecture holds: no packing of congruent balls in the three-dimensional Euclidean space has density greater than the face-centered cubic packing (of density $\frac{\pi}{\sqrt{18}}$).

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