Particle creation in expanding universes

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Abstract: In this work we explore a striking consequence of combining General Relativity and Quantum Field Theory, namely the creation of particles in expanding Universes. We first review a known simple 1+1 model and compute the amount of particle creation. Then we explore a generalization of this model regarding the kind of expansion and the diferences in the particle creation density.

I. INTRODUCTION

Two of the main cornerstones of the current description of the Universe are General Relativity and Quantum Field Theory. We don't have an experimentally tested theory where they are fully compatible, but we nevertheless have some results of combining them. In this work, we take advantatge of it and we study an important consequence of this pairing: particle creation due to an expansion of the Universe, a work first carried out by L. Parker ([1] and [2]).

We will first very briefly review formulation of Quantum Field Theory in curved spacetime, and will learn that notions of vacuum and particle number are no longer unique.

We will illustrate these findings with a simple known example found in Birrell and Davies [3], based on the work by Bernard and Duncan [4]. We will explicitly compute particle creation in a 1+1 model of expanding universe.

We will then consider a generalization of this model, studying what we will call an n-step expansion and present numerical evidence to support the conjecture that particle creation is maximal in the one-step case.

For simplicity, we consider a non self-interacting scalar field. Backreaction of the scalar field over the metric is not taken into account.

A. Different vacua, Bogolubov transformations and particle creation

As explained in [5], we start with a Lagrange density of a scalar field in curved spacetime. In order to quantize it, we start by considering the equation of motion. For a spacelike hypersurface Σ, we can define an inner product on solutions to this equation which is independent of the choice of Σ. In flat spacetime, the procedure now would be to define a complete orthonormal set of positive- and negative-frequency modes that form a basis for solutions in order to expand the field operator in terms of this solutions and creation and annihilation operators.

In a flat Minkowski spacetime we can find a timelike Killing vector, and we can define positive frequency modes with respect to it. From this, we can define a vacuum state and from it an entire Fock space. What is particular of this case, is that Poincaré's group of symmetries is privileged. Therefore, any inertial time coordinate is related by a Lorentz transformation, and all possible vacuum states are the same, and so is the number operator.

In a general spacetime, there will generically not be any timelike Killing vector. Hence, we will not in general be able to find mode solutions that separate into time-dependent and space-dependent factors, and thus we cannot classify modes as positive- or negative-frequency. We can find sets of basis modes, but there will be no way to prefer one over the rest, and the notion of vacuum and number operator will depend on the set chosen. Different observers will differ in their observations of the number of particles.

A general quantized scalar field $\phi(x)$ can be decomposed in a complete orthonormal set of mode solutions of the field equation like

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right], \tag{1}$$

with a vacuum state $|0\rangle$ defined by

$$a_i |0\rangle = 0, \quad \forall i, \tag{2}$$

and a Fock space. As carried out in Birrell and Davies [3], consider, then, a second complete orthonormal set of modes $\overline{u}_j(x)$. We can expand our field $\phi(x)$ in terms of our new choice of modes like

$$\phi(x) = \sum_j \left[\overline{a}_j \overline{u}_j(x) + \overline{a}_j^\dagger \overline{u}_j^*(x) \right]. \tag{3}$$

This decompositon gives a new vacuum state $|\overline{0}\rangle$ defined by

$$\overline{a}_j |\overline{0}\rangle = 0, \quad \forall j, \tag{4}$$

and consequently a new Fock space.

Given that both sets are complete, the new modes $\overline{u}_j$ can be expanded in terms of the old:

$$\overline{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right), \tag{5}$$

and conversely

$$u_i = \sum_j \left(\alpha_{ji}^* \overline{u}_j - \beta_{ji} \overline{u}_j^* \right). \tag{6}$$

These relations are known as Bogolubov transformations, and the matrices α_{ji}, β_{ji} are called Bogolubov coefficients. These coefficients also relate the creation and anihilation operators through

$$a_i = \sum_j \left(\alpha_{ji} \overline{a}_j + \beta_{ji}^* \overline{a}_j^\dagger \right), \quad \overline{a}_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \right). \tag{7}$$

Now, it can be seen that the two Fock spaces based on the two choices of modes u_i and $\overline{u}_j$ are different so long as $\beta_{ji} \neq 0$. An easy example of great value for our purposes is to see that the vacuum states are different. We can check this by observing that $|\overline{0}\rangle$ is not annihilated by a_i:

$$a_i|\overline{0}\rangle = \sum_j \left(\alpha_{ji} \overline{a}_j + \beta_{ji}^* \overline{a}_j^\dagger \right) |\overline{0}\rangle = \sum_j \beta_{ji}^* |\overline{1}_j\rangle \neq 0 \tag{8}$$

As a consequence, the expectation value of the operator for number of u_i modes, $N_i = a_i^\dagger a_i$, will give a non-zero value in $|\overline{0}\rangle$

$$\langle \overline{0} | N_i | \overline{0} \rangle = \sum_j |\beta_{ji}|^2, \tag{9}$$

which means that the vacuum of the $\overline{u}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode. That is what we will call particle creation.

II. A KNOWN SIMPLE MODEL

We consider the problem proposed in [3], based on [4]. We assume a 2-dimensional Robertson-Walker Universe, that is, with line element of the form

$$ds^2 = dt^2 - a^2(t) dx^2. \tag{10}$$

We can define a new time variable η such that $d\eta = \frac{dt}{a}$. Hence, we have

$$ds^2 = a^2(\eta) \left(d\eta^2 - dx^2 \right) := C(\eta) \left(d\eta^2 - dx^2 \right). \tag{11}$$

Such a metric is manifestly conformally flat, and we call $C(\eta)$ the conformal scale factor.

Our aim is to study a Universe in which the in and out regions are flat Minkowskian regions. A conformal scale factor is proposed of the form

$$C(\eta) = A + B \tanh(\rho\eta), \tag{12}$$

with A, B, ρ positive constants such that $A > B$ so the scale factor is positive. This conformal scale factor satisfies

$$C(\eta) = A \pm B, \quad \eta \to \pm\infty \tag{13}$$

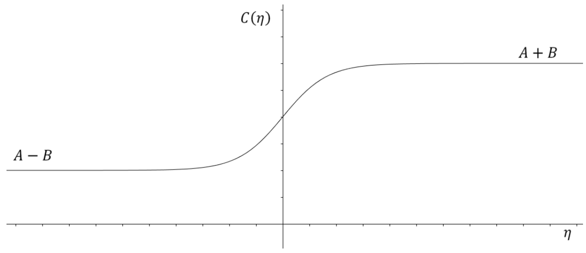

FIG. 1: $C(\eta) = A + B \tanh(\rho\eta)$. We can see that the in and out regions of the conformal scale factor give a Minkowskian flat space.

and so the *in* and *out* regions become flat Minkowskian regions (figure 1).

We will treat scalar fields without self-interactions. The fully covariant scalar field equation is then

$$\left[\Box + m^2 + \xi R \right] \phi(x) = 0 \tag{14}$$

where $\Box$ is the d'Alembertian operator ($\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$), m is the mass of the particle, ξ is the coupling to the metric and R is the Riemann curvature scalar. For the coupling we can distinguish two relevant cases: the minimal coupling $\xi = 0$ and the conformal coupling $\xi(n) = \frac{1}{4} \left(\frac{n-2}{n-1} \right)$, with n the dimension of the spacetime. In the particular case of the 2-dimensional problem we are solving, the minimal and the conformal couplings are equivalent. In this work, we set $\xi = 0$.

A general quantized scalar field $\phi(x)$ can be written like equation (1), where $u_i(x)$ is a complete orthonormal set of mode solutions of the field equation, the index i covers the set of quantities necessary to label the modes, and a_i, $a_i^\dagger$ are the annihilation and creation operators from which we can build our Fock space, with vacuum $|0\rangle$.

A. Simple model solution and evidence of particle creation

Following [3], we solve the equation in order to find the modes. Given that $C(\eta)$ is not a function of $\mathbf{x}$ (space), spatial translation is still a symmetry. From now on we will write $\mathbf{x} = x$. Therefore, we can separate variables

$$u_k(\eta, x) = S_k(x) \chi_k(\eta), \tag{15}$$

and we get to the differential equations

$$\frac{1}{\chi_k(\eta)} \frac{\partial^2 \chi_k(\eta)}{\partial \eta^2} + C(\eta) m^2 = \frac{1}{S_k(x)} \frac{\partial^2 S_k(x)}{\partial x^2} = -k^2 \tag{16}$$

with k a constant. This constant k is the reason why we labeled the modes previously. As we do not have any boundary condition for space, k can take any real value. The sign of it is chosen in order to have in the solutions an oscillator behavior rather than an exponential one. Fixing $C(\eta) = A + B \tanh(\rho\eta)$, we can find two linearly dependent solutions that have different behavior at the limits $\eta \to \pm\infty$. Both behaviors will be of plane waves, i.e., the one that a free particle would have in a flat Minkowskian spacetime such as our in and out regions. Therefore, we can write two normalized solutions for the modes $u_k^{in}(\eta, x)$ and $u_k^{out}(\eta, x)$, found in [3] with

$$\omega_{in} = \sqrt{k^2 + m^2(A - B)} \tag{17}$$
$$\omega_{out} = \sqrt{k^2 + m^2(A + B)} \tag{18}$$
$$\omega_{\pm} = \frac{1}{2}\left(\omega_{out} \pm \omega_{in}\right) \tag{19}$$

and labelled so as to match the respective limits

$$u_k^{in}(\eta, x) \to (4\pi\omega_{in})^{-1/2} e^{ikx - i\omega_{in}\eta}, \quad \eta \to -\infty \tag{20}$$

$$u_k^{out}(\eta, x) \to (4\pi\omega_{out})^{-1/2} e^{ikx - i\omega_{out}\eta}, \quad \eta \to +\infty \tag{21}$$

We clearly see that these mode solutions are different. They have been chosen for their asymptotic plane wave behavior, but they are solutions of the whole spacetime. Thus, as seen in (I.A), they define different vacuum states in the in and out regions and generally we will find a non-zero number of u_k^{out} modes in the in vacuum, i.e., particle creation. To see this, we need to find the Bogolubov coefficients. Using the linear properties of the hypergeometric functions described in [6], we can find

$$u_k^{in}(\eta, x) = \alpha_k u_k^{out}(\eta, x) + \beta_k u_{-k}^{out*}(\eta, x), \tag{22}$$

with

$$\alpha_k = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \frac{\Gamma(1 - (i\omega_{in}/\rho))\Gamma(-i\omega_{out}/\rho))}{\Gamma(-i\omega_{+}/\rho))\Gamma(1 - (i\omega_{+}/\rho))}, \tag{23}$$

$$\beta_k = \left(\frac{\omega_{out}}{\omega_{in}}\right)^{1/2} \frac{\Gamma(1 - (i\omega_{in}/\rho))\Gamma(i\omega_{out}/\rho))}{\Gamma(i\omega_{-}/\rho))\Gamma(1 + (i\omega_{-}/\rho))}, \tag{24}$$

with the relations $\alpha_{k,k'} = \alpha_k \delta_{k,k'}$, $\beta_{k,k'} = \beta_k \delta_{k,-k'}$.

B. Observations

As seen in (I.A), the value $|\beta_k|^2$ is the expected number of detected quanta in the mode k. If $\beta_k = 0$, then the in and out vacuums ($|0\rangle_{in}$ and $|0\rangle_{out}$ respectively) are the same state.

Provided that we stated the relations A and B are positive and $A - B > 0$ based on the assumption that the Universe has a positive scale factor, the only problem that we could face in expressions (23) and (24) is that we had m and k going to zero.

If we put $m = 0$ we find

$$\alpha_k = 1, \beta_k \to 0. \tag{25}$$

We see that there is no particle creation for $m = 0$. That is a manifestation of the conformal symmetry. When we have $m = 0$, we have extra symmetry in our problem, and it prevents particle creation.

On the other hand, if we fix m and take $k \to 0$, we find the same expected result.

Once we had studied the Bogolubov coefficients, we can compute $|\alpha_k|^2$ and $|\beta_k|^2$:

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_{+}/\rho)}{\sinh(\pi\omega_{out}/\rho)\sinh(\pi\omega_{in}/\rho)} \tag{26}$$

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_{-}/\rho)}{\sinh(\pi\omega_{out}/\rho)\sinh(\pi\omega_{in}/\rho)} \tag{27}$$

It can be checked that $|\alpha_k|^2 - |\beta_k|^2 = 1$, a general property of the Bogolubov coefficients.

III. STUDY OF THE NUMBER DENSITY OF CREATED QUANTA N

A. Preliminar considerations. Maximum N.

As seen in (I.A), and noticing that in our problem k is a real continuous variable, substituting the sums for integrals, N can be computed through the coefficient $|\beta_k|^2$ like

$$N = \int_{-\infty}^{+\infty} |\beta_k|^2 dk. \tag{28}$$

Then, we will first of all study the coefficient $|\beta_k|^2$. Performing a rather lengthy but straightforward computation, and given A, B positive, $A > B$, we get

$$\frac{d|\beta_k|^2}{dk} < 0. \tag{29}$$

Physically, it means that particles with a very high frequency do not feel the expansion. On the other hand, particles with a low frequency do feel it rather significantly.

Similarly to (29),

$$\frac{d|\beta_k|^2}{d\rho} > 0. \tag{30}$$

This tells us that in this Universe in expansion, the particle creation is larger as larger is the slope of the hyperbolic tangent of the conformal scale factor, i.e., as ρ increases. This last result leads to the idea of a possible maximum number density of quanta created in this expansion given when $\rho \to \infty$ (figure 2), which would be

$$N(\rho \to \infty) = \int_{-\infty}^{+\infty} |\beta_k(\rho \to \infty)|^2 dk, \tag{31}$$

provided that this integral converges.

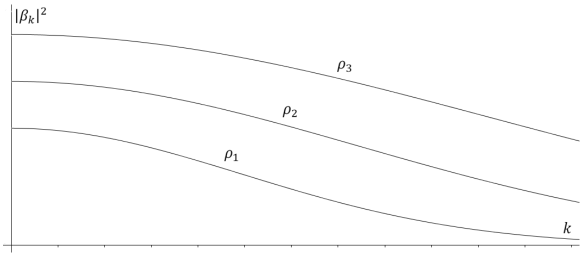

FIG. 2: $|\beta_k|^2(k)$ for three values of ρ such that $\rho_1 < \rho_2 < \rho_3$. We see that the sign of the derivative would lead to a maximum N, provided that the integral of $|\beta_k(\rho \to \infty)|^2$ converges.

B. Computation of N in the limit $\rho \to \infty$

The total number density of created quanta is computed as in equation (31). We will focus in the $\rho \to \infty$ limit. First of all, we see that

$$|\beta_k(\rho \to \infty)|^2 = \frac{\omega_-^2}{\omega_{out}\omega_{in}} = \frac{1}{4}\left(\frac{\omega_{out}}{\omega_{in}} + \frac{\omega_{in}}{\omega_{out}} - 2\right). \tag{32}$$

Hence

$$N(\rho \to \infty) = 2\int_0^{+\infty} |\beta_k(\rho \to \infty)|^2 dk$$

$$= \frac{m\sqrt{A}}{2}\int_0^{+\infty}\left(\sqrt{\frac{k_0^2+1+x}{k_0^2+1-x}} + \sqrt{\frac{k_0^2+1-x}{k_0^2+1+x}} - 2\right)dk_0, \tag{33}$$

where we defined $k_0 = \frac{k}{m\sqrt{A}}$ and $x = \frac{B}{A}$, $0 \le x < 1$. The resulting integral will be the cornerstone of our following discussion, and it can be solved in terms of complete elliptic integrals of the first and second kind. Explicitly,

$$N(\rho \to \infty) = m\sqrt{A}\frac{\mathbb{K}\left(\sqrt{\frac{2x}{1+x}}\right) - (1+x)\mathbb{E}\left(\sqrt{\frac{2x}{1+x}}\right)}{\sqrt{1+x}}. \tag{34}$$

This expression matches the facts that if $m = 0$ we did not have particle creation, and if $B = 0$, i.e., there is no real expansion, we are in the case $x = 0$ and again there is no particle creation.

An interesting feature easy to check of equation (33) is that it is symmetric with respect to x, and therefore so is (34). It means that whether we treat an expansion or a contraction, the number density of particles created would be the same.

C. Particle creation in multiple expansion periods of the $\rho \to \infty$ kind

We have computed $N(\rho \to \infty)$ in a Universe that has undergone a one-step expansion. We will denote it $N_1(\rho \to \infty)$. Now we are going to study this number in a Universe that has the same initial and final states but the expansion has n steps of the kind $\rho \to \infty$ (figure 3), stopping in the middle flat regions enough time to let the out modes to become plane waves. We will call it an n-step expansion. The question we want to address is whether the one-step expansion gives the largest number density of particle creation or it is accomplished with a larger number of expansions.

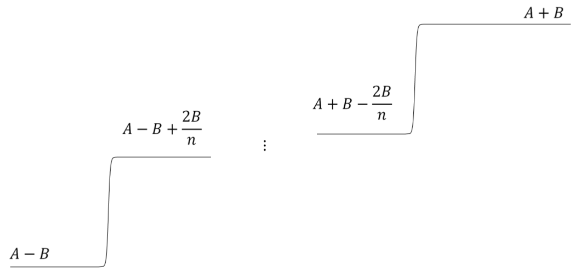

FIG. 3: Universe with the same initial and final states but with n equidistant additional expansions (n-step expansion). The expansions are separated enough in time to let the out modes to become plane waves.

For every j-step in the expansion, $1 \le j \le n$, we will have $u_k^{(j),in}$ and $u_k^{(j),out}$ modes, formally equal to those computed in section (I.C). The key argument in this procedure is to think that the expansions are separated enough in time for $u_k^{(j),out}$ modes to become plane waves. Then, in the second expansion, as they arrive from the "negative" side of time, these $u_k^{(j),out}$ modes become the corresponding $u_k^{(j+1),in}$ modes, as the frequency $\omega_{in}^{(j+1)}$ will be the same as $\omega_{out}^{(j)}$.

With all this stated, the number density of created particles in this n-step expansion, that we will note $N_n(\rho \to \infty)$, will be the sum of the $u_k^{(j),in}$ modes created in the middle flat region's vacuum of every j-step. Therefore, the total particle creation density in the n-step expansion is

$$N_n(\rho \to \infty) = \sum_{j=1}^{n}\int_{-\infty}^{+\infty} |\beta_k^{(j)}(\rho \to \infty)|^2 dk. \tag{35}$$

In analogy with the $N_1(\rho \to \infty)$ case, we can consider a particular scale factor for every j-th step in the expansion and perform the very same integral of equation (33) with a redefinition of the parameters. As a result, the number density of particle creation in an n-step expansion is given by the expression

$$N_n(\rho \to \infty) = m\sum_{j=1}^{n}\sqrt{A + B\left(\frac{2j-1-n}{n}\right)}$$
$$\times\left(\frac{\mathbb{K}\left(\sqrt{\frac{2x^{(j)}}{1+x^{(j)}}}\right) - (1+x^{(j)})\mathbb{E}\left(\sqrt{\frac{2x^{(j)}}{1+x^{(j)}}}\right)}{\sqrt{1+x^{(j)}}}\right) \tag{36}$$

with the condition $A > B$, and this determines the values of $x^{(j)} = \frac{B}{nA+B(2j-1-n)}$.

We can observe that, for every n-step expansion, either $m = 0$ or $B = 0$ give no particle creation.

D. Comparison of the particle creation in multiple expansion periods of the $\rho \to \infty$ kind

As we found a computable expression for $N_n(\rho \to \infty)$, we can compare its behavior for diferent expansions.

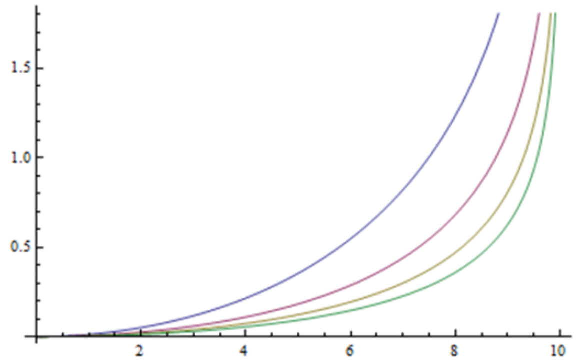

FIG. 4: $N_i(\rho \to \infty)$ for $i = 1, 2, 3, 4$ (from higher to lower) with $m = 1$, fixing $A = 10$ and varying $B \in [0, 10)$.

In view of the results obtained (figure 4), we can conjecture by numerical evidence

$$N_p(\rho \to \infty) > N_q(\rho \to \infty) \Leftrightarrow p < q. \quad (37)$$

A conclusion of this statement is that the maximum creation of particles occurs when the expansion takes place with only one step.

IV. CONCLUSIONS

- We mainly studied the 2-dimensional case in minimal (conformal) coupling with a very specific conformally flat metric. For massive scalar fields we have reviewed the argument for particle creation from a vacuum state. For massless scalar fields we found that no particle creation exists. Even more, the reason of this result resides in the fact that if we keep conformal coupling in higher dimensions this property is mantained. Conformal symmetry in such an expansion with Minkowskian in and out regions turns these regions indistinguishable and therefore they have the same vacuum state.
- In the situation where particle creation takes place, we have solved the problem for what we called an n-step expansion and computed the maximum density of particles created from the vacuum state. From this formula, numerical evidence suggests the conjecture

$$N_p(\rho \to \infty) > N_q(\rho \to \infty) \Leftrightarrow p < q,$$

which tells that the less steps the expansion takes place in, the more the particles created.

Acknowledgments

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