On the Externality-free Shapley–Shubik Index

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Abstract

We address the problem of extending the Shapley–Shubik index to the class of simple games with externalities introduced in Alonso-Meijide et al. (2015). On the one hand, we provide bounds for any efficient, symmetric, and monotonic power index. On the other hand, we characterize the restriction of the externality-free value of de Clippel and Serrano (2008) to the class of games under study by adapting well-known properties.

Keywords: externality-free value; Shapley–Shubik index; partition function

1 Introduction

Since the seminal paper of Shapley and Shubik (1954) was published, the *a priori* assessment of the power possessed by each agent participating in a decision making body has been an important topic in game theory. Simple coalitional games can be used to describe these situations by attaching 1 to any coalition that is strong enough to pass a proposal and 0 to the rest. If power is understood as the ability of an agent to change the outcome of a ballot, then it is sensible to use the agent's contributions to coalitions to develop power indices. Thus, the value proposed by Shapley (1953) to distribute the surplus generated from the cooperation of agents in economic environments has been shown to also be valuable for evaluating the power in a legislature or committee.

In this paper, we aim to study the distribution of power in the presence of coalitional externalities. We will use a real example of the mechanism used by the parliament of the Basque autonomous region of Spain to elect the president of the regional government. Each party represented in the parliament can nominate a deputy for the presidency. At the voting stage, each member of the parliament can either abstain or vote for one of the proposed candidates. The candidate that gets more votes than any other (plurality rule) is the winner. That is, whether a coalition is winning or not depends on the behavior of the other parties. We argue that coalitional games with externalities (Thrall and Lucas, 1963) are an appropriate framework in which to study situations like these. Bolger (1986) has already employed games with externalities to study multi-candidate elections and has proposed several power indices. One of the main novelties of our approach is to consider a subclass of games with externalities that are monotonic. This class of games generalizes the simple games (without externalities) introduced by von Neumann and Morgenstern (1944). The aforementioned monotonicity property has recently been proposed in Alonso-Meijide et al. (2015) and it makes special sense in situations with negative externalities, such as the ones outlined previously.

The problem of extending the Shapley value to games with externalities was first tackled by Myerson (1977). More recently, the topic has attracted some attention and several alternative generalizations of the Shapley value have been proposed (Albizuri et al., 2005; Macho-Stadler et al., 2007; de Clippel and Serrano, 2008; McQuillin, 2009; Dutta et al., 2010, among others). The existence of so many different proposals can be explained by the difficult task of generalizing the concept of a contribution in the presence of externalities. Indeed, if we want to measure the change in the utility of a coalition when one of its members leaves it, then we should know which coalition the defecting agent will join, if any. Albizuri et al. (2005) assume that the agent can join any coalition and that all coalition configurations are equally likely. Macho-Stadler et al. (2007) generalize this approach by considering a probability distribution over the different events that could take place. However, de Clippel and Serrano (2008) argue that the intrinsic marginal contribution is originated by an agent who leaves a coalition to become a singleton. In a subsequent step, the agent could join any coalition, but the effect of this move should not be considered. The approach used by McQuillin (2009) differs from the rest because it is much broader and suggests that agents not participating in a coalition form a block. Finally, Dutta et al. (2010) follow the potential approach and study a family of values that contains the previous proposals.

In this paper, we aim to tackle the problem of extending the Shapley–Shubik index to simple games with externalities as devised by Alonso-Meijide et al. (2015). On the one hand, we adapt the properties used by Young (1985) in the popular characterization of the Shapley value[1] to our setting. These three properties—namely efficiency, symmetry, and monotonicity—allow for a variety of power indices when externalities are present, as is the case for general games in partition function (de Clippel and Serrano, 2008). By imposing these conditions, we derive simple bounds for the power index of an agent. On the other hand, we propose a natural way to generalize the properties used by Dubey (1975) to characterize the Shapley–Shubik index. Indeed, the monotonicity property of the games considered allows us to speak about minimal winning embedded coalitions. This kind of coalition enables us to define null and symmetric players while avoiding the use of contributions. We show that combining efficiency, symmetry, null player property, and the transfer property yields a unique power index. This index is the restriction of the value introduced by de Clippel and Serrano (2008) to simple games with externalities.

The rest of the paper is organized in four sections. The preliminaries describe some of the previous results and notations. In Section 3, we describe the bounds for an efficient, symmetric, and monotonic power index. Section 4 presents our characterization result and Section 5 concludes.

2 Preliminaries

Let N be a finite set ($|N| > 1$) of players, which we keep fixed henceforth. A *characteristic function* is a mapping $v\colon 2^N = \{S : S \subseteq N\} \to \mathbb{R}$, satisfying $v(\emptyset) = 0$. The set of characteristic functions is denoted by $\mathcal{CG}$. A *value* is a mapping f that assigns a unique vector $\mathsf{f}(v) \in \mathbb{R}^N$ to every $v \in \mathcal{CG}$. The *Shapley value* (Shapley, 1953), Sh, is defined for every $v \in \mathcal{CG}$ and $i \in N$ by[2]

$$\mathsf{Sh}_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \left[v\left(S \cup i\right) - v\left(S\right) \right].$$

The set of partitions of N is denoted by $\mathcal{P}(N)$.[3] An *embedded coalition* is a pair (S, P) where $P \in \mathcal{P}(N)$ and $S \in P$. We sometimes refer to S as the active coalition in P and we say

[1] Recall that Young's characterization is valid on the subclass of simple games and can, thus, also be considered as a characterization of the Shapley–Shubik index.

[2] We abuse notation slightly and write $T \cup i$ and $T \setminus i$ instead of $T \cup \{i\}$ and $T \setminus \{i\}$, respectively, for $T \subseteq N$ and $i \in N$. We use lowercase letters to denote the cardinality of a finite set.

[3] For convenience, we assume that the empty set is an element of every partition, even though we may omit writing it; that is, for every $P \in \mathcal{P}(N)$, $\emptyset \in P$.

that a player $i \in N$ belongs to an embedded coalition (S, P) when $i \in S$. The set of embedded coalitions is denoted by $\mathcal{EC}$; that is, $\mathcal{EC} = \{(S, P) : P \in \mathcal{P}(N) \text{ and } S \in P\}$. Given $P \in \mathcal{P}(N)$ and a nonempty coalition $S \subseteq N$, we let $P_{-S} \in \mathcal{P}(N \setminus S)$ denote the partition $P = \{T \setminus S : T \in P\}$.

A *partition function* is a mapping $v : \mathcal{EC} \to \mathbb{R}$ such that $v(\emptyset, P) = 0$ for every $P \in \mathcal{P}(N)$. The set of partition functions is denoted by $\mathcal{G}$. It is easy to see that $\mathcal{G}$ is a vector space over $\mathbb{R}$. Indeed, de Clippel and Serrano (2008) develop the following basis of this vector space $\left\{e_{(S,P)} : (S, P) \in \mathcal{EC} \text{ and } S \neq \emptyset\right\}$ where $e_{(S,P)} \in \mathcal{G}$ is defined for every $(T, Q) \in \mathcal{EC}$ by

$$e_{(S,P)}(T, Q) = \begin{cases} 1 & \text{if } S \subseteq T \text{ and } \forall T' \in Q \setminus T, \exists S' \in P \text{ such that } T' \subseteq S',^{4} \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we focus on the so-called simple games with externalities that were introduced in Alonso-Meijide et al. (2015). This subclass of games is a natural generalization of the class of simple games without externalities. To define monotonicity in the class $\mathcal{G}$, we consider the following notion of inclusion between embedded coalitions.

Definition 2.1. *Let $(S, P), (T, Q) \in \mathcal{EC}$. We say that (S, P) is contained in (T, Q) and write $(S, P) \sqsubseteq (T, Q)$ when $S \subseteq T$ and $\forall T' \in Q \setminus T$, $\exists S' \in P$ such that $T' \subseteq S'$.*

Note that whenever $S \neq \emptyset$, $(S, P) \sqsubseteq (T, Q)$ if and only if $e_{(S,P)}(T, Q) = 1$. According to this definition, an embedded coalition (S, P) is a subset of another embedded coalition (T, Q) if the active coalition in P is contained in the active coalitions in Q (i.e., $S \subseteq T$) and, moreover, the partition P_{-T} is coarser than $Q \setminus T$. Notice that both P_{-T} and $Q \setminus T$ are partitions of $N \setminus T$.

We are now in a position to introduce the class of games under study.

Definition 2.2. *A partition function $v \in \mathcal{G}$ is a* simple game (with externalities) *if it satisfies the following three conditions:*

i) For every $(S, P) \in \mathcal{EC}$, $v(S, P) \in \{0, 1\}$.

ii) $v(N, \{\emptyset, N\}) = 1$.

iii) If $(S, P), (T, Q) \in \mathcal{EC}$ is such that $(S, P) \sqsubseteq (T, Q)$, then $v(S, P) \leq v(T, Q)$.

An embedded coalition, $(S, P) \in \mathcal{EC}$, is winning *if $v(S, P) = 1$ and* losing *otherwise. The set of simple games is denoted by $\mathcal{SG}$.*

[4]As before, we may omit the braces and write $Q \setminus T$ instead of $Q \setminus \{T\}$ for every $T \in Q \in \mathcal{P}(N)$.

The monotonicity property defined in point *iii)* allows us to properly speak about minimal winning embedded coalitions. Let $v \in \mathcal{SG}$. A winning embedded coalition, $(S, P) \in \mathcal{EC}$, is *minimal* if every proper subset of it is a losing embedded coalition; that is, if $(T, Q) \sqsubset (S, P)$ implies that $v(T, Q) = 0$.[5] The set of all minimal winning embedded coalitions of the simple game v is denoted by $\mathcal{M}(v)$ and the subset of minimal winning embedded coalitions that contain a given player $i \in N$ is denoted by $\mathcal{M}_i(v)$; that is, $\mathcal{M}_i(v) = \{(S, P) \in \mathcal{M}(v) : i \in S\}$.

A player $i \in N$ is a *null player* in $v \in \mathcal{SG}$ if i does not belong to any minimal winning embedded coalition; that is, $\mathcal{M}_i(v) = \emptyset$. In contrast, a player $i \in N$ is a *veto player* in $v \in \mathcal{SG}$ if it belongs to every minimal winning embedded coalition; that is, $\mathcal{M}(v) = \mathcal{M}_i(v)$.

Two players i and j are *symmetric* in v if exchanging the two players in an embedded coalition in which either player participates does not change its worth. Formally, let $\pi : N \to N$ be defined by $\pi(i) = j$, $\pi(j) = i$, and for every $l \in N \setminus \{i, j\}$, $\pi(l) = l$. Then, i and j are symmetric in v if for every $(S, P) \in \mathcal{M}(v)$ such that $i \in S$ and $j \notin S$, $(\pi(S), \pi(P)) \in \mathcal{M}(v)$, where $\pi(S) = \{\pi(l) : l \in S\}$ and $\pi(P) = \{\pi(T) : T \in P\}$.

There are many ways in which a partition function $v \in \mathcal{G}$ can be converted into a characteristic function (see for instance Macho-Stadler et al., 2007). Any such conversion is based on each coalition's expectation of how the other agents are going to be organized. For instance, externalities in $v \in \mathcal{G}$ could be ruled out by using the associated *optimistic game*, which is the characteristic function defined by $v^*(S) = \max\limits_{(S,P)\in\mathcal{EC}} \{v(S, P)\}$ for every $S \subseteq N$. In contrast, the *pessimistic game* associated with v is defined by $v_*(S) = \min\limits_{(S,P)\in\mathcal{EC}} \{v(S, P)\}$ for every $S \subseteq N$. Note that whenever the partition function is monotonic, in particular when $v \in \mathcal{SG}$, the above maximum and minimum are attained at the finer and coarser partitions, respectively; that is, $v^*(S) = v(S, \{S, \{j\}_{j\in N\setminus S}\})$ and $v_*(S) = v(S, \{S, N \setminus S\})$. The two games that we borrow from de Clippel and Serrano (2008) and McQuillin (2009) play an important role henceforth.

Finally, it is convenient to consider the effect of a player's change of affiliation. Formally, given $v \in \mathcal{SG}$, $(S, P) \in \mathcal{EC}$, $i \in S$, and $T \in P \setminus S$, we define the *contribution of i to (S, P) with respect to T in v* by

$$c_i(v, S, P, T) = v(S, P) - v(S \setminus i, \{S \setminus i, T \cup i\} \cup P \setminus \{S, T\}).^6$$

Note that this contribution is always non-negative in a monotonic game. For games in $\mathcal{SG}$, the contribution is 1 if the affiliation change of i from T to S turns (S, P) into a winning embedded coalition, and 0 otherwise.

[5] A proper subset, $(T, Q) \sqsubset (S, P)$, is a subset $(T, Q) \sqsubseteq (S, P)$ satisfying $(T, Q) \neq (S, P)$.

[6] Note that T could be the empty set, i.e., we allow the possibility of player i becaming a singleton.

3 Monotonicity of solutions and bounds on the power

In this section we will focus on point-valued solution concepts for the class of games introduced in Definition 2.2. A *power index* is a mapping, f, that assigns a vector $\mathsf{f}(v) \in \mathbb{R}^N$ to every simple game $v \in \mathcal{SG}$, where each coordinate $\mathsf{f}_i(v)$ describes the power of player $i \in N$. Next, we present three interesting properties that we would like to impose on a power index, f.

Efficiency (EFF): $\sum_{i \in N} \mathsf{f}_i(v) = 1$ for every $v \in \mathcal{SG}$.

Symmetry (SYM): $\mathsf{f}_i(v) = \mathsf{f}_j(v)$ for every $(N, v) \in \mathcal{SG}$ and every pair $i, j \in N$ of symmetric players in v.

Monotonicity (MON): $\mathsf{f}_i(v) \leq \mathsf{f}_i(w)$ for every $v, w \in \mathcal{SG}$ and $i \in N$ such that for every $(S, P) \in \mathcal{EC}$ with $i \in S$ and every $T \in P \setminus S$, $c_i(v, S, P, T) \leq c_i(w, S, P, T)$.

The three of them are based on the well known properties that Young (1985) used to characterize the Shapley value. MON is the monotonicity property of de Clippel and Serrano (2008) restricted to simple games. It states that if a player's contribution vector weakly increases from one game to another, then so should her power. These three properties allow for a variety of solutions. In the next result, we derive simple bounds for any power index that satisfies these three conditions.

Proposition 3.1. *Let* f *be a power index satisfying* EFF*,* SYM*, and* MON*. Then, for every* $v \in \mathcal{SG}$ *and* $i \in N$*,*

$$\mathsf{f}_i(v) \in \left[\mathsf{Sh}_i(\underline{w}), \mathsf{Sh}_i(\overline{w})\right],$$

where $\underline{w}, \overline{w} \in \mathcal{CG}$ *are defined for every* $S \subseteq N \backslash i$ *by* $\underline{w}(S) = v^*(S)$*,* $\underline{w}(S \cup i) = \max\{v^*(S), v_*(S \cup i)\}$*,* $\overline{w}(S) = v_*(S)$*, and* $\overline{w}(S \cup i) = v^*(S \cup i)$*.*

Proof. Let $v \in \mathcal{SG}$ and $i \in N$.

First, observe that any characteristic function can be viewed as a partition function by just taking the worth of every $(S, P) \in \mathcal{EC}$ to be constant, for every $S \subseteq N$. Moreover, by the monotonicity of v and the definition of the two games above, $\underline{w}$ and $\overline{w}$ are monotonic (see item *iii)* in Definition 2.2). Then, since $v \in \mathcal{SG}$, $\underline{w}$ and $\overline{w}$ are simple games without externalities. Hence, $\mathsf{f}_i(\underline{w})$ and $\mathsf{f}_i(\overline{w})$ are well defined.

Next, let $S \subseteq N \setminus i$, $(S \cup i, P) \in \mathcal{EC}$, and $T \in P \setminus \{S \cup i\}$. If $\underline{w}(S \cup i) = v^*(S) = v\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right)$, then $c_i(\underline{w}, S, P, T) = 0$. Otherwise, if $\underline{w}(S \cup i) = v_*(S \cup i) = v\left(S \cup i, \{S \cup i, N \setminus (S \cup i)\}\right)$, then

$$\underline{w}(S \cup i) - \underline{w}(S) \leq v(S \cup i, P) - v(S, \{S, T \cup i\} \cup P \setminus \{S \cup i, T\}),$$

because $(S \cup i, \{S \cup i, N \setminus (S \cup i)\}) \sqsubseteq (S \cup i, P)$, $(S, \{S, T \cup i\} \cup P \setminus \{S \cup i, T\}) \sqsubseteq (S, \{S, \{j\}_{j \in N \setminus S}\})$, and v is monotonic. In any case, $c_i(\underline{w}, S, P, T) \leq c_i(v, S, P, T)$. Using the definition of $\overline{w}$ and the fact that v is monotonic, we also get $c_i(v, S, P, T) \leq c_i(\overline{w}, S, P, T)$. By MON, $\mathsf{f}_i(v) \in [\mathsf{f}_i(\underline{w}), \mathsf{f}_i(\overline{w})]$.

Finally, since both $\underline{w}$ and $\overline{w}$ are characteristic functions, using the characterization of Sh that was developed by Young (1985), we obtain $\mathsf{f}_i(\underline{w}) = \mathsf{Sh}_i(\underline{w})$ and $\mathsf{f}_i(\overline{w}) = \mathsf{Sh}_i(\overline{w})$ for every $i \in N$. □

For each player $i \in N$, the bounds are her Shapley value payoffs in two games. It should be emphasized that each of these games depends on the player under consideration. The game used to obtain the lower bound represents the worst case scenario for i; that is, the worth in the optimistic game for every coalition in which i does not participate and the worth in the pessimistic game for every coalition to which i belongs, as long as this value yields a monotonic game. Similarly, the game used to obtain the upper bound can be interpreted as the best case scenario from the perspective of player i. Indeed, the worth in the pessimistic game is used for every coalition not containing i and the worth in the optimistic game for coalitions that do contain i.

It is natural to wonder whether the bounds of Proposition 3.1 are tight. We are able to identify some cases in which they are. For each agent, there are games where the two bounds are attained by some power indices satisfying EFF, SYM, and MON. Since we have not yet introduced any particular power index, we postpone the precise statement of the claim to Section 5 and only present a remark that will facilitate this discussion.

Remark 3.1. *Let $i \in N$ be a veto player in $v \in \mathcal{SG}$. Then, $\underline{w} = v_*$ and $\overline{w} = v^*$. Indeed, since $\mathcal{M}(v) = \mathcal{M}_i(v)$ it follows that for every $S \subseteq N \setminus i$, $\underline{w}(S) = \overline{w}(S) = v^*(S) = v_*(S) = 0$, $\underline{w}(S \cup i) = \max\{0, v_*(S \cup i)\} = v_*(S \cup i)$, and $\overline{w}(S \cup i) = v^*(S \cup i)$.*

4 The externality-free Shapley–Shubik index

In this section, we present two additional properties that a power index, f, may be asked to satisfy. Again, they are adaptations of well known properties to the class of games under study.

Null player property (NPP): $\mathsf{f}_i(v) = 0$ for every $v \in \mathcal{SG}$ and every null player $i \in N$ in v.

Transfer property (TRA): $\mathsf{f}(v) + \mathsf{f}(w) = \mathsf{f}(v \vee w) + \mathsf{f}(v \wedge w)$ for every $v, w \in \mathcal{SG}$.[7]

As we will shortly see, imposing these properties singles out a unique power index. Next, as did Shapley and Shubik (1954) with the Shapley value, we consider the restriction of the externality-free value (de Clippel and Serrano, 2008) to our class of simple games.

Definition 4.1. *The* externality-free Shapley–Shubik index*,* $\mathsf{SS}^{\mathsf{EF}}$*, is the power index defined by* $\mathsf{SS}^{\mathsf{EF}}(v) = \mathsf{Sh}(v^*)$*, where* $v \in \mathcal{SG}$*.*

Finally, we present our main result.

Theorem 4.1. $\mathsf{SS}^{\mathsf{EF}}$ *is the only power index satisfying* EFF*,* NPP*,* SYM*, and* TRA*.*

Proof. *Existence:* We show that $\mathsf{SS}^{\mathsf{EF}}$ satisfies the four properties.

EFF. This follows from Definition 4.1 and the fact that Sh is efficient.

NPP. Let $i \in N$ be a null player in $v \in \mathcal{SG}$. We will see that $v^*(S \cup i) = v^*(S)$ for every $S \subseteq N \setminus i$. Suppose, in contrast, that there is a coalition $S \subseteq N \setminus i$ such that $v^*(S \cup i) \neq v^*(S)$. Then, by definition of v^*, $v\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right) \neq v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right)$. Taking into account the fact that $\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$ and the definition of v, we have that $v\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right) = 1$ and $v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) = 0$. Since i is a null player in v, $\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$ is not a minimal winning embedded coalition in (N, v). Let $(T, Q) \in \mathcal{M}(v)$ be such that $(T, Q) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$. Again, since i is a null player in v, $i \notin T$ or, equivalently, $T \subseteq S$. Then $(T, Q) \sqsubseteq \left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right)$, which contradicts the assumption that $v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) = 0$. That is, we have shown that i is a null player in the classical sense in the characteristic function v^*. Finally, since Sh satisfies the null player property (Shapley, 1953), $\mathsf{SS}_i^{\mathsf{EF}}(N, v) = 0$.

SYM. Let $i, j \in N$ be two symmetric players in $v \in \mathcal{SG}$ and let $S \subseteq N \setminus \{i, j\}$. Suppose that $\left(S \cup i, \left\{S \cup i, \{l\}_{l \in N \setminus (S \cup i)}\right\}\right)$ is a winning embedded coalition. We show that $\left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ is also a winning embedded coalition. Let $(T, Q) \in \mathcal{M}(v)$ be such that $(T, Q) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{l\}_{l \in N \setminus (S \cup i)}\right\}\right)$. If $i \notin T$, then $(T, Q) \sqsubseteq \left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ and we are done. Otherwise, suppose that $i \in T$. Let $\pi : N \to N$ be defined by $\pi(i) = j$, $\pi(j) = i$, and for every $l \in N \setminus \{i, j\}$, $\pi(l) = l$. Note that since i and j are symmetric players, $(\pi(T), \pi(Q)) \in \mathcal{M}(v)$. Moreover, $(\pi(T), \pi(Q)) \sqsubseteq \left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ because $\pi(T) = (T \setminus i) \cup j \subseteq S \cup j$. All in all, we have shown that i and j are symmetric players (in the classical sense) in the characteristic

[7] The games $v \vee w$ and $v \wedge w$ are defined for every $(S, P) \in \mathcal{EC}$ by, $(v \vee w)(S, P) = \min\{v(S, P), w(S, P)\}$ and $(v \wedge w)(S, P) = \max\{v(S, P), w(S, P)\}$.

function v^*. Finally, since Sh is symmetric (Shapley, 1953), the payoffs of i and j in v according to $\mathsf{SS}^{\mathsf{EF}}$ coincide.

TRA. Let $v, w \in \mathcal{SG}$. Then

$$\mathsf{SS}^{\mathsf{EF}}(v) + \mathsf{SS}^{\mathsf{EF}}(w) = \mathsf{Sh}(v^*) + \mathsf{Sh}(w^*) = \mathsf{Sh}(v^* \vee w^*) + \mathsf{Sh}(v^* \wedge w^*) = \mathsf{Sh}((v \vee w)^*) + \mathsf{Sh}((v \wedge w)^*)$$
$$= \mathsf{SS}^{\mathsf{EF}}(v \vee w) + \mathsf{SS}^{\mathsf{EF}}(v \wedge w),$$

where the first and last equalities hold by definition of $\mathsf{SS}^{\mathsf{EF}}$, the second is due to the fact that Sh satisfies the classic transfer property (Dubey, 1975), and the third follows from $v^* \vee w^* = (v \vee w)^*$ and $v^* \wedge w^* = (v \wedge w)^*$. Indeed, if $S \subseteq N$, then

$$(v^* \vee w^*)(S) = \max\{v^*(S), w^*(S)\} = \max\left\{v\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right), w\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right)\right\}$$
$$= (v \vee w)\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right) = (v \vee w)^*(S).$$

Exchanging the maximum with the minimum in this equation shows that $v^* \wedge w^* = (v \wedge w)^*$ and this concludes the existence part.

Uniqueness: Let f be a power index satisfying the four properties. We show that f is unique by induction on the number of minimal winning coalitions.

First, let $v \in \mathcal{SG}$ be such that $|\mathcal{M}(v)| = 1$. Then $\mathcal{M}(v) = \{(S, P)\}$ for some $(S, P) \in \mathcal{EC}$ and $v = e_{(S,P)}$. It is immediate to check that every $i \notin S$ is a null player in $e_{(S,P)}$. Then, by NPP, $\mathsf{f}_i\left(e_{(S,P)}\right) = 0$. Similarly, every two players in S are symmetric in $e_{(S,P)}$. Then, by SYM, they get the same payoff and by EFF we conclude that $\mathsf{f}_i\left(e_{(S,P)}\right) = \frac{1}{|S|}$ for every $i \in S$.

Second, suppose that f is uniquely determined for every $v \in \mathcal{SG}$ with $|\mathcal{M}(v)| < r$. Let $v \in \mathcal{SG}$ with $\mathcal{M}(v) = \left\{\left(S_1, P^1\right), \ldots, \left(S_r, P^r\right)\right\}$. Next, since v is monotonic, for every $(T, Q) \in \mathcal{EC}$,

$$v(T, Q) = \max_{(S,P) \in \mathcal{M}(v)} e_{(S,P)}(T, Q) = \max\left\{w(T, Q), e_{(S_r, P^r)}(T, Q)\right\},$$

where $w(T, Q) = \max_{k \in \{1, \ldots, r-1\}} e_{(S_k, P^k)}(T, Q)$. Since $v = w \vee e_{(S_r, P^r)}$, by TRA,

$$\mathsf{f}(w) + \mathsf{f}\left(e_{(S_r, P^r)}\right) = \mathsf{f}(v) + \mathsf{f}\left(w \wedge e_{(S_r, P^r)}\right).$$

Note that the two payoffs on the left-hand side of this equation are uniquely determined by the induction hypothesis. Then, it only remains to prove that the vector $\mathsf{f}\left(w \wedge e_{(S_r, P^r)}\right)$ is uniquely determined.

Third, for every $k \in \{1, \ldots, r-1\}$, we define the coalition $T_k = S_k \cap S_r$ and the partition $Q^k = \left\{U \cap V : U \in P^k \text{ and } V \in P^r\right\}$. Observe that $\left(T_k, Q^k\right) \in \mathcal{EC}$. We claim that

$\mathcal{M}\left(w \wedge e_{(S_r,P^r)}\right) = \left\{\left(T_k, Q^k\right) : k \in \{1,\dots,r-1\}\right\}$. Indeed, let $(T,Q) \in \mathcal{EC}$. Then

$$w \wedge e_{(S_r,P^r)}(T,Q) = 1 \Leftrightarrow \begin{cases} w(T,Q) = 1 \text{ and} \\ e_{(S_r,P^r)}(T,Q) = 1 \end{cases} \Leftrightarrow \begin{cases} \exists k \in \{1,\dots,r-1\} : (S_k, P^k) \sqsubseteq (T,Q) \text{ and} \\ (S_r, P^r) \sqsubseteq (T,Q) \end{cases}$$

$$\Leftrightarrow \exists k \in \{1,\dots,r-1\} : \begin{cases} S_k \cap S_r \subseteq T \text{ and} \\ Q \setminus T \text{ is finer than } P^k \setminus S_k \text{ and } P^r \setminus S_r \end{cases}$$

Notice that, by definition of Q^k, any partition which is finer than both $P^k \setminus S_k$ and $P^r \setminus S_r$ is necessarily finer than $Q^k \setminus T_k$. Thus, the previous statement is equivalent to

$$\exists k \in \{1,\dots,r-1\} : \begin{cases} T_k \subseteq T \text{ and} \\ Q \setminus T \text{ is finer than } Q^k \setminus T_k \end{cases} \Leftrightarrow \exists k \in \{1,\dots,r-1\} : (T_k, Q^k) \sqsubseteq (T,Q).$$

Since all of the previous statements are if and only if implications, we have shown the claim.

Fourth, and last, since $\left|\mathcal{M}\left(w \wedge e_{(S_r,P^r)}\right)\right| < r$, by the induction hypothesis $\mathsf{f}\left(w \wedge e_{(S_r,P^r)}\right)$ is unique and the proof is concluded. □

Remark 4.1. *The four properties of this characterization result are independent, as the following examples show.*

Let f^1 *be the power index defined by* $\mathsf{f}^1_i(v) = 0$ *for* $v \in \mathcal{SG}$ *and* $i \in N$*. Then* f^1 *satisfies* NPP*,* SYM*, and* TRA*, but not* EFF*.*

Let f^2 *be the power index introduced by McQuillin (2009) and defined as the Shapley value of the game in characteristic function* $v_*(S) = v\left(S; \{S, N \setminus S\}\right)$*. Then,* f^2 *satisfies* EFF*,* SYM*, and* TRA*, but not* NPP*.*

Let f^3 *be the power index that is defined as the weighted Shapley value (Kalai and Samet, 1987) of the characteristic function* v^**, where the weights are different for at least two players. Then,* f^3 *satisfies* EFF*,* NPP*, and* TRA*, but not* SYM*.*

The DP-externality power index defined in Alonso-Meijide et al. (2015) satisfies EFF*,* NPP*, and* SYM*, but not* TRA*.*

5 Concluding remarks

In this paper, we have specified the externality-free value of de Clippel and Serrano (2008) to a class of monotonic, $\{0,1\}$-valued games with externalities. We have shown that this is the only power index satisfying EFF, NPP, SYM, and TRA. The main appeal of our characterization is that the properties are stated without making explicit use of contributions. Instead, we use the participation of players in minimal winning embedded coalitions to determine null and symmetric players. In this way, we consider players who are never part of the active coalition in a minimal winning structure as null. Similarly, we consider players who can be replaced from the active coalition while keeping it minimal winning as symmetric.

To relate NPP with other properties in the literature, it can be shown that for every $v \in \mathcal{SG}$, $\mathcal{M}_i(v) = \emptyset$ if and only if for every $(S;P) \in \mathcal{EC}$ with $i \in S, v(S;P) = v\left(S;(P \setminus S) \cup \{S \setminus i, \{i\}\}\right)$. That is, a player is null when leaving the active coalition to remain singleton does not change the worth of an embedded coalition. This notion of a null player is equivalent to that of de Clippel and Serrano (2008) and to the type 2 dummy player of Dutta et al. (2010), for instance. It could be reasonably argued that a player should only be considered null when any coalitional move that she may make does not change the worth of an embedded coalition. This is the approach followed by McQuillin (2009),[8] which leads to a weaker version of the null player property. Identifying the family of indices that satisfy EFF, SYM, TRA, and the later version of the null player property is a topic for future research.

Finally, we would like to elaborate a little on the bounds identified in Proposition 3.1. Recall how Remark 3.1 shows that in games where there is a veto player (whose participation is necessary for any coalition to be winning), the two games that are used to describe the bounds are precisely the pessimistic and optimistic games. In other words, the lower bound corresponds to the value studied by McQuillin (2009) while the upper bound corresponds to the value introduced by de Clippel and Serrano (2008). Taking into account the fact that these two values satisfy EFF, SYM, and MON, we can conclude that under these circumstances the bounds are tight. However, this does not hold for every game. Indeed, the lower bound of Proposition 3.1 coincides with $\mathsf{SS}^{\mathsf{EF}}$ for null players but it is easy to find games for which $\mathsf{SS}^{\mathsf{EF}}$ is in the interior of the interval for some players.

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[8]This notion of a null player corresponds to the null player in the strong sense of de Clippel and Serrano (2008) and to the type 1 dummy player of Dutta et al. (2010), for instance.

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