Analogue Gravity in Hyperbolic Metamaterials

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(Dated: June 12, 2017)

Abstract: Metamaterials are artificial materials whose electromagnetic parameters can be properly designed to achieve many interesting properties not found in natural materials. With a convenient design they demonstrate an analogy between their electrodynamics and general relativity phenomena. In this work we discuss the unusual optical properties of *hyperbolic metamaterials* and show that the wave equation for light propagation in these media is formally equivalent to the Klein-Gordon equation for a scalar field. Based on these results, the analogue gravity in hyperbolic metamaterials is considered to model an expanding Universe in a (2+1)-dimensional de Sitter spacetime.

I. INTRODUCTION

Einstein's General Relativity has become a great success in explaining gravity at a macroscopic scale. It did not only adjust to observations with an astounding precision but it also predicted effects that could not be observed until many years later. One of the latest of these observed effects are the gravitational waves, which remained undetected until September 2015 [1]. Some other effects predicted by General Relativity have not been detected yet or are extremely hard to study due to our strong limitation in low energy experimentation and astronomical observations. However, the emergent *Analogue Gravity* studies offer a promising way around this difficulty. One of the systems that mimic General Relativity phenomena are the metamaterials. These materials have electric and magnetic properties that can be engineered by subwavelength structure design, making them exhibit unusual optical properties such as negative refraction, subwavelength resolution lenses, cloacking, etc. [2, 3]. These properties take place due to metamaterials can have one or both electric permitivity (ε) and magnetic permeability (μ) with negative sign. If both ε and μ are simultaneously negative then the group and phase velocities ($\vec{v_g}, \vec{v_p}$) are antiparallel [4] and, therefore, the refraction index n is negative while it is positive for common materials ($\varepsilon, \mu > 0$).

$$n = \pm\sqrt{\varepsilon\mu} \tag{1}$$

On the other hand, materials with ε sign opposite to μ have imaginary refraction index. All of these possibilities open up a new branch of optics that is very useful not only for modern technology development but also to mimic other physical systems. Actually, it has been recently proven that propagation of electromagnetic fields in curved spacetime is equivalent to propagation in flat spacetime in a certain anisotropic media [5, 6]. Based on this, researchers have recently studied some particular general relativity phenomena like black holes[6–8], dark energy effects [9] and universe inflation[10, 11] by engineering metamaterials that mimic them.

In this work, we want to model an analogue inflationary universe in metamaterials. To do that, we are going to work with *hyperbolic metamaterials*, which are metamaterials with hyperbolic dispersion determined by their $\tilde{\varepsilon}$ or $\tilde{\mu}$ tensors. One of the diagonal components of these tensors is opposite sign to the two others, turning them to be the "ultra-anisotropic limit of traditional uniaxial crystals" [3]. This allows us to engineer hyperbolic metamaterials that, when illuminated with monochromatic light, the stationary pattern of light behaves like it was propagating through a (2+1)-dimensional spacetime [11]. The light traveling through this kind of metamaterial "sees" two of the material's axes like space-like variables while the other can be interpreted as a time-like variable. All of these concepts are addressed in section II.

II. MAXWELL EQUATIONS IN HYPERBOLIC METAMATERIALS

A. Wave equation and unusual properties

To study how electromagnetic waves propagate through a hyperbolic metamaterial we begin by writing Maxwell equations for a nonmagnetic media. We also take into account that this material is electrically neutral everywhere ($\sigma = \rho = 0$).

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \vec{B} = 0 \tag{2a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2b}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{2c}$$

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Where $\vec{D} = \tilde{\varepsilon}\vec{E}$. If we compute the curl of equation (2b) and assume that we illuminate the material with a monochromatic light source (single-frequency ω) we obtain the wave equation

$$\nabla^2\vec{E} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) + \frac{\omega^2}{c^2}\vec{D} = 0 \tag{3}$$

It is easy to see that for a plane wave and an anisotropic media ($\vec{E} = \vec{E}^0 e^{-i(\vec{k}\vec{r}-\omega t+\varphi_0)}$, $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$) we obtain the eigenvalue equation [12]:

$$\begin{pmatrix} \frac{\omega^2}{c^2}\varepsilon_x - k_y^2 - k_z^2 & k_xk_y & k_xk_z \\ k_xk_y & \frac{\omega^2}{c^2}\varepsilon_y - k_x^2 - k_z^2 & k_yk_z \\ k_xk_z & k_yk_z & \frac{\omega^2}{c^2}\varepsilon_z - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_y \end{pmatrix} = 0 \tag{4}$$

This equation determines how the light disperse through the media. Notice that this equation corresponds to the well known Fresnel equation [13] and, for a given direction of propagation, this last one has two different solutions for the phase velocity v_p and, therefore, two different solutions for the refraction index, n_e and n_o (for the extraordinary and ordinary waves respectively).

As we noticed in the introduction, hyperbolic metamaterials are basically uniaxial materials with the product $\varepsilon_\parallel\varepsilon_\perp < 0$ (where $\varepsilon_\parallel$ is the electric permitivity in the direction of the axis of symmetry and $\varepsilon_\perp$ is the permitivity in the directions perpendicular to that axis). This makes their dispersion relation to be different from the one for conventional uniaxial materials ($\varepsilon_\parallel\varepsilon_\perp > 0$). If we define $\gamma^2 \equiv \varepsilon_\parallel/\varepsilon_\perp$ and solve now the Maxwell equations for an uniaxial medium (with $\varepsilon_z \equiv \varepsilon_\parallel$) we get the equation

$$\begin{aligned}&\left[-\left(k_x^2+k_y^2+k_z^2\right)+\frac{\omega^2}{c^2}\varepsilon_\perp\right]^2\\ &\times\left[-\left(k_x^2+k_y^2+\gamma^2k_z^2\right)+\frac{\omega^2}{c^2}\varepsilon_\parallel\right]=0\end{aligned} \tag{5}$$

Notice that we obtain two possible wave solutions, for ordinary and extraordinary waves respectively[14].

The ordinary wave solution yields

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\varepsilon_\perp \tag{6a}$$

$$E_z^0 = 0 \tag{6b}$$

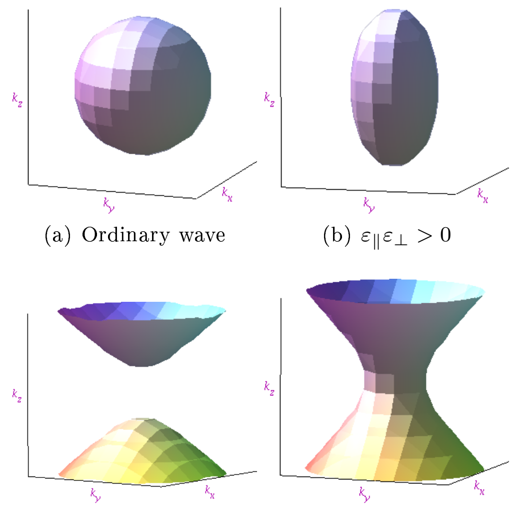
This relation determines that ordinary waves' isofrequencies surfaces are spheres both for conventional and hyperbolic materials. It is in the study of extraordinary waves' behavior that the differences appear.

For the extraordinary wave solution we get the following relations.

$$\frac{k_x^2+k_y^2}{\varepsilon_\parallel} + \frac{k_x^2}{\varepsilon_\perp} = \frac{\omega^2}{c^2} \tag{7a}$$

$$E_z^0 \neq 0 \tag{7b}$$

Comparing with equation (6a), this new relation shows that hyperbolic metamaterials and conventional materials behave differently. While for $\varepsilon_\parallel\varepsilon_\perp > 0$ media the isofrequency surface turns out to be an ellipsoid, for $\varepsilon_\parallel\varepsilon_\perp < 0$ the isofrequency surfaces are hyperboloids, hence the name *hyperbolic*. As it is shown in figure 1, if we choose $\varepsilon_\parallel < 0$ then the surfaces are two-fold hyperboloid (*type I* hyperbolic metamaterial) while for $\varepsilon_\perp < 0$ the surface is a one-fold hyperboloid (*type II*).



(a) Ordinary wave (b) $\varepsilon_\parallel\varepsilon_\perp > 0$ (c) $\varepsilon_\parallel < 0$ (type I) (d) $\varepsilon_\parallel > 0$ (type II)

FIG. 1: Isofrequency surfaces given by $\omega(\vec{k}) = constant$.

As a consequence of these hyperboloidal isofrequency surfaces, waves with arbitrarily large wave-vectors keep propagating in time while for conventional materials the limited isofrequency surface makes them becoming evanescent. This is why hyperbolic metamaterials have no diffraction limit and, therefore, they can be used as perfect lenses for subwavelength imaging [3].

Another property of this medium compared to the conventional materials is the highly directional light emission. The directions of energy flow and propagation of electromagnetic waves, determined by the group velocity $\vec{v_g}(\omega)$ and the Poynting vector $\vec{S}$ respectively, are perpendicular to the isofrequency surfaces. Because of this, light travels through the metamaterial along a cone whose axis matches with the axis of symmetry [12]. This allows us to approximate with a good accuracy the light propagating through the metamaterial as a ray and to make the analogy between this rays and geodesics in general relativity.

B. Electromagnetic Klein-Gordon equation

From now on we are going to study only the propagation of the extraordinary wave ($E_z^0 \neq 0, \varepsilon_z \equiv \varepsilon_\parallel$). Let us rewrite equation (3), for the z component of the electric field, assuming that we illuminate the metamaterial by monochromatic laser light with frequency ω_0 and naming $\varphi \equiv E_z$.

$$\nabla^2 \varphi - \frac{\partial}{\partial z}\left(\vec{\nabla}\cdot\vec{E}\right) + \frac{\omega_0^2}{c^2} D_z = 0 \tag{8}$$

Assuming $\varepsilon_\parallel < 0$ and $\varepsilon_\perp > 0$, equation (8) reads

$$-\frac{1}{\varepsilon_\perp}\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{|\varepsilon_\parallel|}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = \frac{\omega_0^2}{c^2}\varphi \tag{9}$$

This equation has the same functional form as the Klein-Gordon equation for a massive scalar field in the Minkowsky spacetime, where the z variable behaves as a *time-like variable* ($z \equiv \tau$). Despite of this, we are not interested in studying a flat spacetime but a curved one. The Klein-Gordon equation for scalar field in a spacetime with a generic metric defined by $g_{\mu\nu}$ is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) = \frac{m^2c^2}{\hbar^2}\phi \tag{10}$$

To find the analogy between this equation and Maxwell equations we must not force the electric permitivity to be constant in all the medium. Actually, it is the easiness to control the ε variation of hyperbolic metamaterials what allows us to model general relativity phenomena. Thus, if we allow the electric permitivity components to vary in the direction of light propagation, we get the new equation for a material with a non-constant $\tilde{\varepsilon}$ tensor.

$$-\frac{1}{\varepsilon_\perp}\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{|\varepsilon_\parallel|}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + \left(\frac{1}{\varepsilon_\perp^2}\frac{\partial\varepsilon_\perp}{\partial z} - \frac{2}{\varepsilon_\perp\varepsilon_\parallel}\frac{\partial\varepsilon_\parallel}{\partial z}\right)\frac{\partial\varphi}{\partial z} + \left(\frac{1}{\varepsilon_\perp^2\varepsilon_\parallel}\frac{\partial\varepsilon_\perp}{\partial z}\frac{\partial\varepsilon_\parallel}{\partial z} - \frac{1}{\varepsilon_\perp\varepsilon_\parallel}\frac{\partial^2\varepsilon_\parallel}{\partial z^2}\right)\varphi = \frac{\omega_0^2}{c^2}\varphi \tag{11}$$

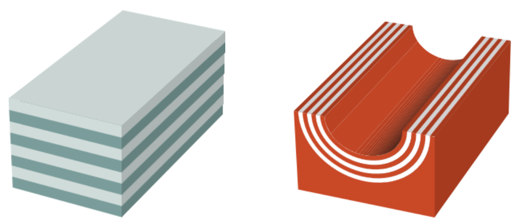
In section III we discuss how $\varepsilon_\parallel$ and $\varepsilon_\perp$ must behave in order to model an expanding universe with our metamaterial. Notice that, up to now, we have been working all the time in Cartesian coordinates but we can get new versions of this equation by changing the coordinates system. If we solve Maxwell equations in cylindrical coordinates making z to be the time-like variable ($\varepsilon_z \equiv \varepsilon_\parallel < 0$) the new equation will be

$$-\frac{1}{\varepsilon_\perp}\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{|\varepsilon_\parallel|}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right) + \left(\frac{1}{\varepsilon_\perp^2}\frac{\partial\varepsilon_\perp}{\partial z} - \frac{2}{\varepsilon_\perp\varepsilon_\parallel}\frac{\partial\varepsilon_\parallel}{\partial z}\right)\frac{\partial\varphi}{\partial z} + \left(\frac{1}{\varepsilon_\parallel\varepsilon_\perp^2}\frac{\partial\varepsilon_\perp}{\partial z}\frac{\partial\varepsilon_\parallel}{\partial z} - \frac{1}{\varepsilon_\parallel\varepsilon_\perp}\frac{\partial^2\varepsilon_\parallel}{\partial z^2}\right)\varphi - \frac{1}{\varepsilon_\parallel r}\frac{\partial\varphi}{\partial r} = \frac{\omega_0^2}{c^2}\varphi \tag{12}$$

It is worth notice that both equation (11) and (12) determine the stationary pattern of light propagating through the same kind of hyperbolic metamaterial. This metamaterials are called "layered" hyperbolic metamaterials and their structure is made by intercalating metallic and dielectric layers as it is shown in figure 2(a). Changing the width of this layers, we can control the electric permitivity in the direction perpendicular to the layers (in our equations, $\varepsilon_\parallel$). The other two components of the electric permitivity, the ones parallel to the layers, are equal ($\varepsilon_\perp$ in our notation).

Although we can perfectly model the inflationary universe using a metamaterial with layered structure, it is worth to tell that cylindrical structured metamaterial (see figure 2(b)) can also be used to make some studies in analogue gravity if we make the radial coordinate to be the time-like variable. In this case, we should neglect the derivatives of ε_i along r in order to obtain something that can be compared to the Klein-Gordon equation. Therefore, the electric permitivity must be nearly constant. Another consideration we can make to simplify our final expression is, that at large enough r we can neglect the terms $\frac{\partial E_r}{\partial r}$, $\frac{\partial E_\theta}{\partial \theta}$ and $\frac{E_r}{r^2}$ [11]. Under these simplifications our expression results in

$$-\frac{1}{\varepsilon'_\perp}\frac{\partial^2\varphi_\omega}{\partial r^2} + \frac{1}{\left|\varepsilon'_\parallel\right|}\left(\frac{\partial^2\varphi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\varphi_\omega}{\partial\theta^2}\right) = \frac{\omega_0^2}{c^2}\varphi_\omega \tag{13}$$



(a) Layered (b) Cylindrical

FIG. 2: Some hyperbolic metamaterials structures

III. KLEIN-GORDON EQUATION FOR AN EXPANDING UNIVERSE

As we have seen in the previous chapter, the solution of Maxwell equations for a hyperbolic metamaterial with non-constant dielectric permitivity lead to equations that are similar to the Klein-Gordon equation. We just need to know how the $\varepsilon_\perp$ and $\varepsilon_\parallel$ should behave to model an expanding universe. To do so, we must first write down explicitly the Klein-Gordon equation for the (2+1) metric that we are going to studying.

Consider an expanding universe with the FLRW (Friedman-Lemaître-Robertson-Walker) metric

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right] \tag{14}$$

Where $a(t)$ is the *scale factor* and χ is the radial coordinate. The Klein-Gordon equation for this metric is given by

$$\frac{m^2c^2}{\hbar^2}\phi = -\frac{\partial^2\phi}{\partial t^2} + \frac{1}{a^2(t)}\left[\frac{\partial^2\phi}{\partial\chi^2} + \frac{1}{S_k^2(\chi)}\frac{\partial^2\phi}{\partial\Omega^2}\right] - \frac{2}{a(t)}\frac{da(t)}{dt}\frac{\partial\phi}{\partial t} + \frac{1}{a^2(t)S_k(\chi)}\frac{dS_k(\chi)}{d\chi}\frac{\partial\phi}{\partial\chi} \tag{15}$$

Comparing it with equation (12) we conclude that any metric can be modeled with the metamaterial as far as this metric has a flat geometry. That is $k = 0$, and therefore $S_k(\chi) = \chi$. We also see that the medium properties and the scale factor are related by $\frac{\varepsilon_\parallel}{\varepsilon_\perp} = -a^2(t) \equiv -a^2(z)$. Therefore, the problem is reduced to study only the relation between the material and the scale factor. Of all possible FLRW metrics we propose the study of the *de Sitter* one. This metric is of a great interest since it is relevant for the inflation stage and it is characterized by being spatially flat and having a scale factor $a(t) = \exp(\sqrt{\Lambda/3}t)$. Finally, the Klein-Gordon equation for this metric is

$$-\frac{\partial^2\phi}{\partial t^2} + \frac{1}{e^{2\sqrt{\Lambda/3}t}}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - 2\sqrt{\frac{\Lambda}{3}}\left(\frac{\partial\phi}{\partial t}\right) = \frac{m^2c^2}{\hbar^2}\phi \tag{16}$$

We have expressed it in Cartesian coordinates because it will make easier the further computation.

IV. METAMATERIAL IMPLEMENTATION

In the last section we have seen that the electric permitivity components must obey the relation $\frac{\varepsilon_\parallel}{\varepsilon_\perp} = -a^2(z)$, but this relation does not establish exactly how each of this components must behave. Actually, depending in our choice we can have two different cases. On the one hand we can get the equivalent to the Klein-Gordon equation for a massive particle while on the other hand, making a different choice for the ε_i functional form, we obtain the equivalent equation for massless particles.

With the choice $\varepsilon_\parallel = -\varepsilon' \exp(Az)$ and $\varepsilon_\perp = \varepsilon$ (where ε and ε' are positive constants and $A = 2\sqrt{\Lambda/3}$) we get the version of the Klein-Gordon equation for massive particles if we introduce the new wave function $\psi \equiv (-\varepsilon_\parallel)^{1/2}\varphi$ [10].

$$-\frac{\partial^2\psi}{\partial z^2} + \frac{\varepsilon}{\varepsilon' e^{Az}}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - A\frac{\partial\psi}{\partial z} = \left(\frac{\varepsilon\omega^2}{c^2} + \frac{A^2}{4}\right)\psi \tag{17}$$

As we can see, the right side of the equation is a constant multiplied by the field and, therefore, can be directly related to the right side of equation (16). The only difference between both equations is the scaling factor ε/ε' accompanying the x and y derivatives.

On the other hand, if we take $\varepsilon_\perp = \varepsilon \exp(-Az)$ and $\varepsilon_\parallel = -\varepsilon'$, we obtain, in the limit of large Az, the Klein-Gordon version for massless particles (photons)

$$-\frac{\partial^2\varphi}{\partial z^2} + \frac{\varepsilon}{\varepsilon' e^{Az}}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) - A\frac{\partial\varphi}{\partial z} = \frac{\omega^2}{c^2 e^{Az}}\varphi \approx 0 \tag{18}$$

This last behavior of the materials properties can be engineered using layered hyperbolic metamaterials, with the layers oriented perpendicular to the z axis. By using the Maxwell-Garnett approximation, the effective electric components for such materials are expressed as [12]

$$\varepsilon_\perp = p\varepsilon_m + (1-p)\,\varepsilon_d \tag{19}$$

$$\varepsilon_\parallel = \left(\frac{p}{\varepsilon_m} + \frac{1-p}{\varepsilon_d}\right)^{-1} \tag{20}$$

Where p is the fraction of metal phase in a period, $\varepsilon_d > 0$ is the electric permitivity of the dielectric layer and $\varepsilon_m < 0$ is the one of metallic layer. By changing p with z we can get the desired electric. This can be done by gradually increasing the metallic layer thickness while keeping the dielectric one constant. The expression we are looking for is then [10]

$$p = \frac{d_m}{d_d + d_m} = \frac{\varepsilon_d - e^{-Az}}{\varepsilon_d + \varepsilon_m} \tag{21}$$

This relation will be true for large Az and $\varepsilon_d > -\varepsilon_m$ since it produces the behavior of $\varepsilon_\perp \sim e^{-Az}$ while $\varepsilon_\parallel = \varepsilon_m\varepsilon_d/(\varepsilon_m + \varepsilon_d - e^{-Az}) \approx constant < 0$.

V. EIKONAL EQUATION AND GEODESICS

In the geometrical optics limit, $\lambda \to 0$, light propagation is easier to study since it follows ray trajectories. A ray is nothing more than the normal to the wave surface of constant phase and its equation of motion is called the *eikonal equation* [15]. We assume that equation (18) has a solution of the form

$$\varphi = \varphi_0 e^{iS(x,y,z)} \tag{22}$$

Where φ_0 is a slow-varying function while S, known as eikonal, is a fast-varying function. Then, by solving equation (18) taking into account that $\frac{\partial S}{\partial x^i} \gg 0$ and making $\varepsilon/\varepsilon' = 1$ we obtain finally the eikonal equation for our metamaterial:

$$-\left(\frac{\partial S}{\partial z}\right)^2 + \frac{1}{e^{Az}}\left\{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right\} = 0 \tag{23}$$

Notice that it has no difference with the eikonal equation in a gravitational field for the de Sitter metric [16]

$$\partial^\alpha S \partial_\alpha S = 0 \tag{24}$$

We can clearly see that since both, the metamaterial and the de Sitter universe, have the same eikonal equation, light will follow the same trajectories in both media, as far as the geometrical optics approximation holds in the metamaterial. We can obtain this trajectories by solving the corresponding Hamiltonian equation.

$$\mathcal{H} = -P_z^2 + e^{-Az} P_r^2 \tag{25}$$

Where $p_\alpha = \partial_\alpha S$ and $P_r^2 = P_x^2 + P_y^2$. Therefore, the trajectories are given by

$$\begin{aligned} z(\lambda) &= 2A^{-1} \ln\left(A\lambda + C\right) + B_z \\ r(\lambda) &= 2P_r e^{-B_z} A^{-1} \left(A\lambda + C\right)^{-1} + B_r \end{aligned} \tag{26}$$

Where λ is the affine parameter and P_r, C, B_z and B_r are constants. This solution, apart from being the solution for ray trajectories in the metamaterial, is also the geodesic of a photon in a (2+1)-dimensional spacetime if we do the analogy $z \to t$.

VI. CONCLUSIONS

In this work I have exposed why metamaterials are good to mimic general relativity phenomena. Specifically, in my case, I have modeled a hyperbolic metamaterial that is the analogue version of the de Sitter spacetime. This has been done because Maxwell equations in the hyperbolic metamaterial present the same functional form as Klein-Gordon equation for a (2+1) metric. This unusual optical behavior of the medium is possible due to the isofrequency surfaces are hyperboloids and they allow us to interpret one of the directions as a time-like variable. This makes the propagation though the material extremely directional and with an extraordinary wave component that does not become evanescent.

To conclude this work, I have computed the trajectories of light rays using the eikonal approximation in the metamaterial media. Since in both, metamaterial and de Sitter spacetime, the eikonal equation has exactly the same form, we can ensure that the trajectories computed are equivalent to the geodesics of a massless particle in a de Sitter universe.

All of this allows us to study the inflationary universe by just experimenting with a material in an optical laboratory. This is very helpful since we can model some of the general relativity phenomena that are too hard to detect and, therefore, to study.

It is also worth notice that hyperbolic metamaterials are no longer only artificial man-made media, since they have recently been found in nature [17].

Acknowledgments

I want to thank my advisor, Dr. Oleg Bulashenko, for his help and advice, as well as the freedom he has given to me and the confidence he has shown in my work.

I also thank my parents, my sister Anna, my 'brother' Marc and of course Édel, for their incessant support during this 4 years and for trusting me when I did not.

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