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A Note on Shapley's Convex Measure Games

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Abstract

L. S. Shapley, in his paper 'Cores of Convex Games', introduces Convex Measure Games, those that are induced by a convex function on $\mathbb{R}$, acting over a measure on the coalitions. But in a note he states that if this function is a function of several variables, then convexity for the function does not imply convexity of the game or even superadditivity. We prove that if the function is directionally convex, the game is convex, and conversely, any convex game can be induced by a directionally convex function acting over measures on the coalitions, with as many measures as players.

Keywords: Convex cooperative games, directional convexity, supermodularity, multilinear extension.
JEL Classification: C71

Resumen

L. S. Shapley introduce el concepto de Convex Measure Games en su artículo 'Cores of Convex Games'. Se trata de los juegos inducidos por una función convexa definida sobre $\mathbb{R}$, y que actúa sobre una medida en las coaliciones. Sin embargo, en una nota a pie de página señala que si esta función es una función de varias variables entonces la convexidad de la función no implica la convexidad del juego ni siquiera la superaditividad. Aquí se prueba que si la función es direccionalmente convexa, el juego es convexo, y que también cualquier juego convexo puede ser inducido por una función direccionalmente convexa que actúa sobre medidas sobre las coaliciones, con, como máximo, tantas medidas como jugadores.

1 Introduction

Shapley (1971) introduced the notion of convex game in his seminal paper. This kind of games present several appealing regularity properties. They are based in [what Shapley calls] convex set functions, or in other contexts, such as Combinatorial Optimization or Integer Programming, supermodular functions, what intuitively means that the incentives for joining a coalition increase as the coalition grows. That is, a game that presents increasing returns along with the coalition's size or increasing differences in the evaluation of the characteristic function. In his paper, Shapley introduces Convex Measure Games, as those that are induced by a convex function on $\mathbb{R}$, which acts over a measure on the coalitions. In an economic situation, this measure could be the initial distribution of a resource, for any player, and any coalition gets the sum of the resources of its members. This function could, then, be interpreted as a production function. He shows that not all convex games are convex measure games, nor are equivalent to such games. The question we want to solve is the note Shapley states in his paper, that if this function is a function of several variables, then convexity for the function does not imply convexity of the game or even superadditivity. The appopiate tool will be directionally convex functions, and the main result of this paper is to prove that any convex cooperative game can be represented as a measure game, by using as many measures (resources) as the number of players and combining them by a directionally convex production function.

2 Preliminaries and Notation

A T.U.-Cooperative Game in *coalitional form* is a pair (N, v), where $N = \{1, 2, ..., n\}$ is the set of *players* (we are assuming it is finite) and $v : 2^N \longrightarrow \mathbb{R}$ is a function defined over the subsets of N (*coalitions*), 2^N, such that $v(\emptyset) = 0$.

We use the notation $e^S \in \mathbb{R}^N$ to indicate the incidence vector ($01-$ vector) to the coalition $S \subseteq N$, i.e., $(e^S)_i = 1$ if $i \in S$, and 0 otherwise. If $S = \{i\}$, we will denote by e^i the corresponding $e^{\{i\}}$.

Convex games were introduced by L. S. Shapley (1971). This class of games is very interesting because of their properties with regard to stable sets, solution concepts, core properties and inheriting properties.

A game (N, v) is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, for all $S, T \in 2^N$. This is equivalent to the "snowballing" or "bandwagon" effect: $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$, for all $i \in N$ and all $S \subset T \subseteq N \backslash \{i\}$.

On the other side, a lattice X is a set with a partial order relation (a poset) such that every two elements, x and x' have a least upper bound (their join, $x \vee x'$) and a greatest lower bound (their meet, $x \wedge x'$).

The real function $f(x)$ defined on a lattice $(X, \vee, \wedge)$ is supermodular (see Topkis, 1998) if

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'') \quad \text{for all} \quad x', x'' \in X.$$

Notice that supermodularity is, for cooperative games, just the definition of convex game, assuming the lattice of coalitions $\left(2^N\right)$, where the maximum of two coalitions S and T is its union ($S \cup T$), and minimum is its intersection ($S \cap T$).

Convex measure games were introduced by Shapley (1971) in the following way:

Consider the game (N, v) defined by

$$v(S) = f(\mu(S)), \quad \text{for all } S \subseteq N,$$

where μ is a measure over the coalitions, and f a real function such that $f(0) = 0$. Recall that a measure μ is an additive (with respect to the union of disjoint sets) positive real-valued function, such that $\mu(\emptyset) = 0$. This measure μ may be interpreted as the initial distribution of one resource among the players, and f as a production function that gives the net worth that can be achieved with the resources.

For the case of one-variable production function, Shapley notes that convexity on f implies the convexity of the game v, but not all convex games can be described in this way with convenient function f and measure μ.

Moreover, Shapley points out in a note:

> "Curiously, if f is a function of several variables and μ is a vector of measures, then convexity of f does not imply convexity of v, or even superadditivity."

At this point, we can add that supermodularity of function f does not imply convexity of the game v. This is easy to see for a one-variable production function, since it is well known that any function of one variable is supermodular. For more than one variable we have the following example:

Example 1 *Consider the supermodular function* $f(x, y) = -x^2 - y^2$, *and consider two players:* 1 *with endowment* $(2, 1)$, *and* 2 *with endowment* $(1, 2)$. *Then the game associated to this function is:* $v(\emptyset) = 0$, $v(\{1\}) = -5$, $v(\{2\}) = -5$, $v(\{1, 2\}) = -18$, *and it is not a convex game.*

Therefore it remains open which should be the right concept on the production function that gives convexity on the class of cooperative measure games. By the previous comments, it has to be a generalization of convexity for one-variable functions, and we will introduce it in the next section.

3 Directional convexity

Let $\leq$ denote the usual order in $\mathbb{R}^m$, which makes componentwise comparisons: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $i = 1, 2, ..., m$. With this order $\mathbb{R}^m$ forms a lattice (which is the product of m chains), and in this lattice we have:

$$\mathbf{x} \wedge \mathbf{y} := (\min \{\mathbf{x}_1, \mathbf{y}_1\}, \min \{\mathbf{x}_2, \mathbf{y}_2\}, ..., \min \{\mathbf{x}_m, \mathbf{y}_m\}),$$

and

$$\mathbf{x} \vee \mathbf{y} := (\max \{\mathbf{x}_1, \mathbf{y}_1\}, \max \{\mathbf{x}_2, \mathbf{y}_2\}, ..., \max \{\mathbf{x}_m, \mathbf{y}_m\}).$$

Definition 2 *A function $f : S \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ is* directionally convex *if for any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in S$ such that $\mathbf{x} + \mathbf{y} = \mathbf{z} + \mathbf{t}$, with $\mathbf{z} \leq \mathbf{x} \leq \mathbf{t}$, and $\mathbf{z} \leq \mathbf{y} \leq \mathbf{t}$, it happens that*

$$f(\mathbf{x}) + f(\mathbf{y}) \leq f(\mathbf{z}) + f(\mathbf{t}).$$

Notice that in this definition $\mathbf{x}$ and $\mathbf{y}$ can be equal, and it is not necessary that the sum $\mathbf{x} + \mathbf{y}$ belongs to S.

The concept of directionally convex function is introduced and used in Shaked and Shanthikumar (1990), Meester and Shanthikumar (1993, 1999), and recently by Müller and Scarsini (2001) or Müller (2001). It has arised in the field of multivariate stochastic orders. Moreover, for twice differentiable functions, as Shaked and Shanthikumar (1990) prove, directional convexity is equivalent to the nonnegativity of all second partial derivatives:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) \geq 0, \quad \text{for all } i, j \in \{1, 2, \ldots, m\}.$$

Therefore, for $m > 1$, directional convexity over $\mathbb{R}^m$ neither implies, nor is implied by classical convexity.

For $m = 1$, it is obvious that directional convexity is equivalent to increasing first differences, what is called Wright-convexity (Roberts and Varberg, 1973), but this is not equivalent to convexity as is incorrectly assumed, for example, in Shaked and Shanthikumar (1990). The well-known construction of an additive function on the real line that is nowhere continuous (Hardy, Littlewood and Pólya, 1952) is an example, because convex functions are continuous in the relative interior of its domain. Then, it is easy to prove that convexity on f implies directional convexity of f. Adding continuity, the converse will also be true.

Now it is time to see that directional convexity is the concept we need to check convexity in the associated cooperative game:

Theorem 3 *Let $f : \mathbb{R}_+^m \longrightarrow \mathbb{R}$ be a directionally convex function, such that $f(0) = 0$, and $\mu_1, \mu_2, \ldots, \mu_m$ be measures over 2^N. Then, the game (N, v) defined as*

$$v(S) = f(\mu_1(S), \mu_2(S), \ldots, \mu_m(S)), \quad \textit{for all } S \subseteq N,$$

is a convex game.

Proof. For any measure μ, $\mu(\emptyset) = 0$, and $\mu(S) + \mu(T) = \mu(S \cap T) + \mu(S \cup T)$, for all $S, T \subseteq N$; moreover $\mu(S \cap T) \leq \mu(S), \mu(T) \leq \mu(S \cup T)$. Then convexity of v is obvious. ∎

4 Directionally convex measure games

In this section we will analyze the converse part of the above results. In fact we will show that any convex cooperative game can be expressed as a measure game for a suitable directionally convex function. Moreover, in our representation, we will need only as many measures as players. To this end, we consider any game (N, v) as a function defined on the unit cube: $v : \{0,1\}^N \longrightarrow \mathbb{R}$, with $v(0) = 0$, identifying each coalition $S \subseteq N$ with its incidence vector $\mathbf{e}^S \in \mathbb{R}^N$. We will need to extend this initial discrete function over the extreme points of the unit cube to the whole space $\mathbb{R}_+^n$ in such a way that if the original function is supermodular, then the extended function should be directionally convex.

Recall that for a given cooperative game $v : \{0,1\}^N \longrightarrow \mathbb{R}$, with $v(0) = 0$, it can be expressed in terms of the unanimity basis, where the coefficients are the unanimity cooordinates:

$$v = \sum_{S \in 2^N \setminus \emptyset} \lambda_S \, u_S.$$

The basis is formed by the unanimity games i.e. $u_T(S) = \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise} \end{cases}$, with $T \subseteq N, T \neq \emptyset$, and the coordinates or Harsanyi dividends are:

$$\lambda_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T).$$

Moreover, Owen's multilinear extension (MLE)(Owen, 1995), extended to the nonnegative orthant,

$$f_o : \mathbb{R}_+^N \longrightarrow \mathbb{R},$$

is defined, for all $\mathbf{x} \in \mathbb{R}_+^N$ by:

$$f_o^v(\mathbf{x}) = f_o^v(x_1, x_2, \dots, x_n) = \sum_{S \subseteq N} \left\{ \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right\} v(S) = \\ = \sum_{S \subseteq N} \left\{ \prod_{i \in S} x_i \right\} \lambda_S. \tag{1}$$

In Rafels and Ybern (1995) it is proved that f_o^v is supermodular on the unit cube, the lattice $\left([0,1]^N, \leq\right)$, if and only if the game v is convex (v supermodular). But Owen's Multilinear Extension does not preserve supermodularity, if we consider this extension in $\mathbb{R}_+^N$ instead of the unit cube. As an example, consider the following 3−player convex cooperative game defined by $v(S) = 0$ for $|S| = 1$, $v(S) = 2$ for $|S| = 2$, and $v(N) = 4$. Its MLE is $f_o(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3 - 2x_1x_2x_3$, and $\frac{\partial^2 f_o}{\partial x_1 \partial x_2}(x_1, x_2, x_3) = 2 - 2x_3$, which is not positive for $\mathbf{x} \in \mathbb{R}_+^3$, if $x_3 \geq 1$.

As a consequence, the above example shows that Owen's multilinear extension, considered in $\mathbb{R}_+^N$, it is not directionally convex on the class of convex cooperative games, and it cannot be used for our central result. As the reader may suspect, the problem in the above example arises from the negativity of a unanimity coordinate in the original convex game, $\lambda_N^v = -2$.

For a convex game where all of its unanimity coordinates are nonnegative, expression (1) will give us a positive answer. And in fact, Owen's MLE can be considered a "good" directionally convex extension of the original game only for this class of games, as the next theorem shows:

Theorem 4 *Let (N, v) be a cooperative game. Owen's multilinear extension (MLE),*

$$f_o^v : \mathbb{R}_+^N \longrightarrow \mathbb{R},$$

defined, for all $\mathbf{x} \in \mathbb{R}_+^N$ by:

$$f_o^v(\mathbf{x}) = f_o(x_1, x_2, \dots, x_n) = \sum_{S \subseteq N} \left\{ \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right\} v(S),$$

is a directionally convex extension of v, if and only if all unanimity coordinates of coalitions of size greater or equal than 2 are nonnegative, i.e. $\lambda_S \geq 0$ for all $|S| \geq 2$.

Proof. If all unanimity coordinates of coalitions of size greater or equal than 2 are nonnegative, i.e. $\lambda_S \geq 0$ for all $|S| \geq 2$ in the game (N, v), by using expression (1) we obtain:

$$\frac{\partial^2 f_o^v}{\partial x_i \partial x_j}(\mathbf{x}) = \begin{cases} 0 & \text{if } \quad i = j, \\ \sum\limits_{S \subseteq N \setminus \{i,j\}} \left\{ \prod\limits_{k \in S} x_k \right\} \lambda_{S \cup \{i,j\}} & \text{if } \quad i \neq j, \end{cases}$$

where a product over the empty set of indices is taken to be equal to 1.

Then we have obtained $\frac{\partial^2 f_o^v}{\partial x_i \partial x_j}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^N$ and all $i, j \in \{1, 2, \ldots, m\}$, what is equivalent to directional convexity for twice differentiable functions.

To see the 'only if' part, suppose that there is some coalition(s) of size greater or equal than 2 with negative coordinate, and pick a minimal size coalition $S' \subseteq N$ with $|S'| \geq 2$ and $\lambda_{S'} < 0$. Consider two different players in $S' : i, j \in S'$, $i \neq j$, and take a vector $\hat{\mathbf{x}} \in \mathbb{R}_+^N$ defined by $\hat{x}_k = t$ if $k \in S'$ and $\hat{x}_k = 0$ if $k \notin S'$. It is easy to check that

$$\frac{\partial^2 f_o^v}{\partial x_i \partial x_j}(\hat{\mathbf{x}}) = \sum_{S \subseteq S' \setminus \{i,j\}} t^{|S|} \; \lambda_{S \cup \{i,j\}},$$

which is a polynomial in t, with a negative coefficient of maximum degree (which is $\lambda_{S'} < 0$) and all other coefficients nonnegative, because if $S \subsetneq S' \setminus \{i, j\}$, then $\lambda_{S \cup \{i,j\}} \geq 0$. It is enough to take t as large as necessary to get a negative valuation for $\frac{\partial^2 f_o^v}{\partial x_i \partial x_j}(\hat{\mathbf{x}})$. ■

The above proof shows that supermodularity of Owen's multilinear extension in the whole space $\mathbb{R}_+^N$ is a characterization of nonnegativeness of all unanimity coordinates associated to coalitions of size greater or equal than 2 of the game.

Theorem 4 solves the converse problem that we want to analyze for this specific subclass of convex games.

Theorem 5 *Any convex cooperative game* (N, v) *with* $\lambda_S \geq 0$ *for all* $|S| \geq 2$ *is a directionally convex measure game.*

Proof. Consider $v : \{0,1\}^N \longrightarrow \mathbb{R}$, with $v(0) = 0$, and, for $i = 1, 2, \ldots, n$, define $\mu_i(S) = 1$ if $i \in S$, and 0 otherwise. Then for any $S \subseteq N$, $(\mu_1(S), \mu_2(S), \ldots, \mu_n(S)) = e^S \in \mathbb{R}_+^N$.

If we take Owen's MLE: $f_o^v : \mathbb{R}_+^N \longrightarrow \mathbb{R}$, we obtain that $v\left(e^S\right) = f_o^v(\mu_1(S), \mu_2(S), \ldots, \mu_n(S))$, and by theorem 4, f_o^v is a directionally convex function. Therefore, v is a directionally convex measure game. ■

As we know, a cooperative game can be convex without having all of its unanimity coordinates of size greater than two nonnegative, and therefore, a representation theorem remains open for the whole class of convex games. As the reader may suspect, we will need other kind of extension for cooperative games, and we will proceed in two steps. First we will analyze an extension procedure from $\{0,1\}^N$ to $\mathbb{N}^N$, and, second, from $\mathbb{N}^N$ to $\mathbb{R}_+^N$. The result of the combination of both will be a general extension from $\{0,1\}^N$ to $\mathbb{R}_+^N$ in such a way that, if the original cooperative game is convex (supermodular function), its extension to $\mathbb{R}_+^N$ will be a directionally convex function.

In order to reach these results, we will need to use some characterizations of nondifferentiable directionally convex functions defined in spaces $\mathbb{N}^N$ and $\mathbb{R}_+^N$, that are developed in the appendix of this paper.

Lemma 6 *Let $v : \{0,1\}^N \longrightarrow \mathbb{R}$ be a real-valued function defined on the vertices of the unit cube, with $v(0) = 0$, such that v is supermodular in the lattice $\left(\{0,1\}^N, \leq\right)$. Then, there is a directionally convex function*

$$f^v : \mathbb{N}^N \longrightarrow \mathbb{R},$$

with $f^v(0) = 0$, which is an extension of v.

Proof. For each $\mathbf{n} \in \mathbb{N}^N$, we will denote $S(\mathbf{n}) = \{i \in N \; ; \; n_i > 0\}$. Define, for any $\mathbf{n} \in \mathbb{N}^N$,

$$f^v(\mathbf{n}) = \left(\prod_{j \in S(\mathbf{n})} n_j\right) \left[v\left(\mathbf{e}^{S(\mathbf{n})}\right) - \sum_{j \in S(\mathbf{n})} v\left(\mathbf{e}^j\right)\right] + \sum_{j \in S(\mathbf{n})} n_j v\left(\mathbf{e}^j\right),$$

where the product over the empty set equals 1, and summation over the empty set equals 0.

This is an extension of the original function, because any vector $\mathbf{e}^S$ is a 01 vector, and $f^v\left(\mathbf{e}^S\right) = v\left(\mathbf{e}^S\right)$.

This function is directionally convex. To see it, we first calculate $\Delta_i f^v(\mathbf{n}) = f^v(\mathbf{n} + \mathbf{e}^i) - f^v(\mathbf{n})$, and two cases must be distinguished:

If $n_i = 0$, then $S(\mathbf{n} + \mathbf{e}^i) = S(\mathbf{n}) \cup \{i\}$, and

$$\Delta_i f^v(\mathbf{n}) = \left(\prod_{j \in S(\mathbf{n})} n_j\right) \left[v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right).$$

If $n_i > 0,$ then $S\left(\mathbf{n}+\mathbf{e}^i\right) = S\left(\mathbf{n}\right)$, and

$$\Delta_i f^v(\mathbf{n}) = \left(\prod_{\substack{j\in S(\mathbf{n})\\ j\neq i}} n_j\right)\left[v\left(\mathbf{e}^{S(\mathbf{n})}\right) - \sum_{j\in S(\mathbf{n})} v\left(\mathbf{e}^j\right)\right] + v\left(\mathbf{e}^i\right).$$

Now, to prove that f^v is directionally convex, for all $\mathbf{n}, \mathbf{n'} \in \mathbb{N}^N$ with $\mathbf{n} \leq \mathbf{n'}$, and all $i \in N$, we will check that $\Delta_i f^v(\mathbf{n}) \leq \Delta_i f^v(\mathbf{n'})$. For these vectors, $n_k \leq n'_k$, for all $k \in N$, and $S\left(\mathbf{n}\right) \subseteq S\left(\mathbf{n'}\right)$. We must distinguish also several cases:

If $n_i > 0,$ then $n'_i > 0$, and being v supermodular, $v\left(\mathbf{e}^{S(\mathbf{n})}\right) + \sum_{j\in S(\mathbf{n'})\setminus S(\mathbf{n})} v\left(\mathbf{e}^j\right) \leq v\left(\mathbf{e}^{S(\mathbf{n'})}\right)$. Then:

$$\begin{aligned}
\Delta_i f^v(\mathbf{n}) &= \left(\prod_{\substack{j\in S(\mathbf{n})\\ j\neq i}} n_j\right)\left[v\left(\mathbf{e}^{S(\mathbf{n})}\right) - \sum_{j\in S(\mathbf{n})} v\left(\mathbf{e}^j\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{\substack{j\in S(\mathbf{n'})\\ j\neq i}} n'_j\right)\left[v\left(\mathbf{e}^{S(\mathbf{n})}\right) - \sum_{j\in S(\mathbf{n})} v\left(\mathbf{e}^j\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{\substack{j\in S(\mathbf{n'})\\ j\neq i}} n'_j\right)\left[v\left(\mathbf{e}^{S(\mathbf{n'})}\right) - \sum_{j\in S(\mathbf{n'})} v\left(\mathbf{e}^j\right)\right] + v\left(\mathbf{e}^i\right) \\
&= \Delta_i f^v(\mathbf{n'}).
\end{aligned}$$

If $n_i = 0,$ **and** $n'_i = 0$ then $S\left(\mathbf{n}+\mathbf{e}^i\right) = S\left(\mathbf{n}\right)\cup\{i\}$, and $S\left(\mathbf{n'}+\mathbf{e}^i\right) = S\left(\mathbf{n'}\right)\cup\{i\}$. By the supermodularity of v,

$$v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) \leq v\left(\mathbf{e}^{S\left(\mathbf{n'}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n'})}\right),$$

and we get:

$$\begin{aligned}
\Delta_i f^v(\mathbf{n}) &= \left(\prod_{j\in S(\mathbf{n})} n_j\right)\left[v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{j\in S(\mathbf{n}')} n'_j\right)\left[v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{j\in S(\mathbf{n}')} n'_j\right)\left[v\left(\mathbf{e}^{S\left(\mathbf{n}'+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n}')}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right) \\
&= \Delta_i f^v(\mathbf{n}').
\end{aligned}$$

If $n_i = 0$, **and** $n'_i > 0$ then $S\left(\mathbf{n}+\mathbf{e}^i\right) = S(\mathbf{n}) \cup \{i\}$, and $i \in S\left(\mathbf{n}'+\mathbf{e}^i\right) = S(\mathbf{n}')$. The following inequalities hold:

$$v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) + \sum_{j\in S(\mathbf{n}')\setminus S(\mathbf{n}+\mathbf{e}^i)} v\left(\mathbf{e}^j\right) \leq v\left(\mathbf{e}^{S(\mathbf{n}')}\right)$$

and

$$\sum_{j\in S(\mathbf{n})} v\left(\mathbf{e}^j\right) \leq v\left(\mathbf{e}^{S(\mathbf{n})}\right),$$

which lead to:

$$\begin{aligned}
\Delta_i f^v(\mathbf{n}) &= \left(\prod_{j\in S(\mathbf{n})} n_j\right)\left[v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{\substack{j\in S(\mathbf{n}')\\ j\neq i}} n'_j\right)\left[v\left(\mathbf{e}^{S\left(\mathbf{n}+\mathbf{e}^i\right)}\right) - v\left(\mathbf{e}^{S(\mathbf{n})}\right) - v\left(\mathbf{e}^i\right)\right] + v\left(\mathbf{e}^i\right) \\
&\leq \left(\prod_{\substack{j\in S(\mathbf{n}')\\ j\neq i}} n'_j\right)\left[v\left(\mathbf{e}^{S(\mathbf{n}')}\right) - \sum_{j\in S(\mathbf{n}')} v\left(\mathbf{e}^j\right)\right] + v\left(\mathbf{e}^i\right) \\
&= \Delta_i f^v(\mathbf{n}').
\end{aligned}$$

And from the characterization of directionally convex functions (see appendix), the statement is proved.

■

Now we will proceed to extend a directionally convex function on $\mathbb{N}^N$ to $\mathbb{R}^N_+$. The procedure will be done by extending the discrete function by cells or "blocks", and for this reason several notions will be introduced.

Consider, for $\mathbf{z}^0 \leq \mathbf{z}^1$, $\mathbf{z}^0, \mathbf{z}^1 \in \mathbb{N}^N$ the following set, which is called a discrete rectangle:

$$R\left[\mathbf{z}^0, \mathbf{z}^1\right] = \left\{\mathbf{z} \in \mathbb{N}^N \ \mid \mathbf{z}^0 \leq \mathbf{z} \leq \mathbf{z}^1\right\}.$$

When $\|\mathbf{z}^1 - \mathbf{z}^0\|_\infty = \sup_{i \in N}\{|z_i^1 - z_i^0|\} \leq 1$, the convex hull of $R\left[\mathbf{z}^0, \mathbf{z}^1\right]$ is called a **cell** D in $\mathbb{R}^N_+$, and its infimum is $\mathbf{z}^0$. Notice that, for any cell D, all the points in $D \cap \mathbb{N}^N$ are the extreme points of D. They will be denoted by $extD$.

Any point $\mathbf{x}$ in $\mathbb{R}^N_+$ is contained in some cell, but it can be in the intersection of two or more cells. The intersection of two cells is either empty or another cell.

For each $\mathbf{x} \in \mathbb{R}^N_+$, we define the discrete neighbourhood (Miller, 1971) of $\mathbf{x}$:

$$N\left(\mathbf{x}\right) = \left\{\mathbf{z} \in \mathbb{N}^N \quad \mid \|\mathbf{z} - \mathbf{x}\|_\infty < 1\right\}.$$

Note that if $\mathbf{x}$ itself is in $\mathbb{N}^N$, then $N\left(\mathbf{x}\right) = \{\mathbf{x}\}$. The set $N\left(\mathbf{x}\right)$ has 2^m elements, where m is the number of coordinates of $\mathbf{x}$ that are not integers. The convex hull of $N\left(\mathbf{x}\right)$ is a cell, precisely, the intersection of all cells that contain $\mathbf{x}$. Moreover, its minimal element $\mathbf{x}^0 \in N\left(\mathbf{x}\right)$ is

$$\mathbf{x}^0 = \lfloor\mathbf{x}\rfloor = \left(\lfloor x_1\rfloor, \lfloor x_2\rfloor, \ldots, \lfloor x_n\rfloor\right),$$

where $\lfloor t\rfloor$ is the integer part of t (the largest integer smaller than or equal to t).

To extend a directionally convex function we will use the concept of weighted function, defined by Miller (1971) in the following way: a function $f : D \longrightarrow \mathbb{R}$ (where $D \subseteq \mathbb{R}^N$) is weighted if D is the convex hull of a discrete rectangle S, and for $\mathbf{x} \in D$, $f\left(\mathbf{x}\right) = \sum_{\mathbf{z} \in N(\mathbf{x})} w_z\left(\mathbf{x}\right) f\left(\mathbf{z}\right)$ where $w_z\left(\mathbf{x}\right)$ are called the weights and satisfy the conditions $\sum_{\mathbf{z} \in N(\mathbf{x})} w_z\left(\mathbf{x}\right) = 1$ and $w_z\left(\mathbf{x}\right) \geq 0$.

Now we can obtain the extension lemma:

Lemma 7 (Extension of a directionally convex function on $\mathbb{N}^N$ to $\mathbb{R}^N_+$) *Let $f : \mathbb{N}^N \longrightarrow \mathbb{R}$, with $f\left(0\right) = 0$, be a directionally convex function on $\mathbb{N}^N$. There is a continuous directionally convex function $\tilde{f} : \mathbb{R}^N_+ \longrightarrow \mathbb{R}$, which is an extension of f.*

Proof. Let $f : \mathbb{N}^N \longrightarrow \mathbb{R}$, with $f(0) = 0$, be a function on $\mathbb{N}^N$. We are going to define a function $\tilde{f} : \mathbb{R}_+^N \longrightarrow \mathbb{R}$, which is an extension of f.

Define the following weighted function, for any $\mathbf{x} \in \mathbb{R}_+^N$,

$$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{z} \in N(\mathbf{x})} w_z(\mathbf{x}) f(\mathbf{z}),$$

where $w_z(\mathbf{x}) = \prod_{i \in R(\mathbf{z})} \bar{x}_i \prod_{i \notin R(\mathbf{z})} (1 - \bar{x}_i)$ with $\bar{x}_i = x_i - x_i^0$ for all $i \in N$, and $R(\mathbf{z}) = \{i \in N \mid z_i - x_i^0 = 1\}$, and $\mathbf{x}^0$ is the minimal element of the cell $N(\mathbf{x})$.

The function $\tilde{f}$ is well defined, and these coefficients $w_z(\mathbf{x})$, $\mathbf{z} \in N(\mathbf{x})$ are nonnegative, and add up to 1. Moreover, function $\tilde{f}$ coincides with f on $\mathbb{N}^N$, since $N(\mathbf{x}) = \{\mathbf{x}\}$ if $\mathbf{x} \in \mathbb{N}^N$.

Notice that the value on an arbitrary point $\mathbf{x} \in \mathbb{R}_+^N$ is obtained as the corresponding multilinear (linear in each variable) interpolation on $N(\mathbf{x})$, in the spirit of Owen's multilinear extension. We claim that the function $\tilde{f}$ has the following properties:

1. $\tilde{f}$ should be defined independently of the cell we want to use for a given vector $\mathbf{x} \in \mathbb{R}_+^N$. Formally, if C is an arbitrary cell, and $\mathbf{x} \in C$, then

   $$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{z} \in extC} w_z(\mathbf{x}) f(\mathbf{z}),$$

   where $w_z(\mathbf{x}) = \prod_{i \in R(\mathbf{z})} \bar{x}_i \prod_{i \notin R(\mathbf{z})} (1 - \bar{x}_i)$ with $\bar{x}_i = x_i - z_i^0$ for all $i \in N$, and $R(\mathbf{z}) = \{i \in N \mid z_i - z_i^0 = 1\}$, and z^0 is the minimal element of the cell C.

   Observe that if $\mathbf{z} \in extC$ but $\mathbf{z} \notin N(\mathbf{x})$, the coefficient of $f(\mathbf{z})$ is zero, because in this case there is some index $k \in N$ such that $|z_k - x_k| = 1$. Then, $z_k^0 = x_k$ if $z_k = z_k^0 + 1$, or $z_k^0 = x_k - 1$ if $z_k = z_k^0$. In the first case $\bar{x}_k = x_k - z_k^0 = 0$, and $k \in R(\mathbf{z})$, and in the second case $\bar{x}_k = x_k - z_k^0 = 1$, and $k \notin R(\mathbf{z})$. In each case, the coefficient of $f(\mathbf{z})$ is zero.

2. $\tilde{f}$ is continuous in $\mathbb{R}_+^N$. Indeed, the function $\tilde{f}$ is continuous in each cell C of $\mathbb{R}_+^N$, and each neighborhood of a vector $\mathbf{x} \in \mathbb{R}_+^N$ can be partitioned in a finite number of neighborhoods, one in each cell $\mathbf{x}$ belongs to.

3. $\tilde{f}$ is linear in each coordinate inside a cell C. Formally: if C is a cell, and $\mathbf{x}^1, \mathbf{x}^2 \in C$ with $x_i^1 = x_i^2$ for all $i \in N,\ i \neq k$, then $\tilde{f}(\alpha \mathbf{x}^1 + (1-\alpha)\mathbf{x}^2) = \alpha \tilde{f}(\mathbf{x}^1) + (1-\alpha)\tilde{f}(\mathbf{x}^2)$ for any $\alpha \in [0,1]$.

4. If C is a cell, $i \in N$ and $\mathbf{x},\ \mathbf{x}+\varepsilon\mathbf{e}^i \in C$, then $\Delta_i^\varepsilon \tilde{f}(\mathbf{x}) = \tilde{f}(\mathbf{x}+\varepsilon\mathbf{e}^i) - \tilde{f}(\mathbf{x}) = \varepsilon\Delta_i^1 \tilde{f}(\mathbf{x}^{\{i\}})$, where $\mathbf{x}^S$ is defined as $x_j^S = x_j$ if $j \notin S$, and $x_j^S = \lfloor x_j \rfloor$ if $j \in S$. It is a direct consequence of 3.

5. If C is a cell, $i, j \in N$ and $\mathbf{x},\ \mathbf{x}+\varepsilon\mathbf{e}^i+\varepsilon'\mathbf{e}^j \in C$, then $\Delta_j^{\varepsilon'}\Delta_i^\varepsilon \tilde{f}(\mathbf{x}) = \varepsilon'\varepsilon\Delta_j^1\Delta_i^1 \tilde{f}(\mathbf{x}^{\{i,j\}})$. It is obtained by applying 4. twice.

6. For any $\mathbf{x} \in \mathbb{R}_+^N$, $\varepsilon, \delta \in \mathbb{R}$ with $0 \leq \delta \leq \varepsilon$ and $i \in N$, $\Delta_i^\varepsilon \tilde{f}(\mathbf{x}) = \Delta_i^{\varepsilon-\delta} \tilde{f}(\mathbf{x}+\delta\mathbf{e}^i)+\Delta_i^\delta \tilde{f}(\mathbf{x})$. It follows directly from the definition of $\Delta_i^\varepsilon \tilde{f}(\mathbf{x})$.

7. For any $\mathbf{x} \in \mathbb{R}_+^N$ and $i \in N$, $\Delta_i^1 \tilde{f}(\mathbf{x}) = \sum\limits_{\mathbf{z}\in N(\mathbf{x})} w_z(\mathbf{x})\, \Delta_i^1 f(\mathbf{z})$. It is a consequence of the following identities: $N(\mathbf{x}+\mathbf{e}^i) = N(\mathbf{x}) + \{\mathbf{e}^i\}$ and $w_{z+\mathbf{e}^i}(\mathbf{x}+\mathbf{e}^i) = w_z(\mathbf{x})$.

8. For any $\mathbf{x} \in \mathbb{R}_+^N$, $k \in \mathbb{N}$ and $i \in N$, $\Delta_i^k \tilde{f}(\mathbf{x}) = \sum\limits_{\mathbf{z}\in N(\mathbf{x})} w_z(\mathbf{x})\, \Delta_i^k f(\mathbf{z})$. It can be obtained by iterative application of 7.

9. For any $\mathbf{x} \in \mathbb{R}_+^N$ and $i, j \in N$, $\Delta_j^1\Delta_i^1 \tilde{f}(\mathbf{x}) = \sum\limits_{\mathbf{z}\in N(\mathbf{x})} w_z(\mathbf{x})\, \Delta_j^1\Delta_i^1 f(\mathbf{z})$. Similar to 7.

10. For any $\mathbf{x} \in \mathbb{R}_+^N$, $k, k' \in \mathbb{N}$ and $i, j \in N$, $\Delta_j^{k'}\Delta_i^k \tilde{f}(\mathbf{x}) = \sum\limits_{\mathbf{z}\in N(\mathbf{x})} w_z(\mathbf{x})\, \Delta_j^{k'}\Delta_i^k f(\mathbf{z})$. Similar to 8.

We must show now that $\tilde{f}$ is directionally convex.

For any $\mathbf{x} \in \mathbb{R}_+^N$, $i, j \in N$, and $k, k' \in \mathbb{N}$, by property 10, and directional convexity of f (see appendix), $\Delta_j^{k'}\Delta_i^k \tilde{f}(\mathbf{x}) \geq 0$.

For any $\mathbf{x} \in \mathbb{R}_+^N$, $i, j \in N$, and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, define the integer part of α and β : $a = \lfloor \alpha \rfloor$ and $b = \lfloor \beta \rfloor$, and the rest $\alpha' = \alpha - a$, $\beta' = \beta - b$, with $0 \leq \alpha', \beta' < 1$. Since $\alpha = a + \alpha'$ and $\beta = b + \beta'$, and applying 6 iteratively, we have:

$$\begin{aligned}
\Delta_j^\beta\Delta_i^\alpha \tilde{f}(\mathbf{x}) \quad = \quad & \Delta_j^b\Delta_i^a \tilde{f}(\mathbf{x}) + \\
& \Delta_j^b\Delta_i^{\alpha'} \tilde{f}(\mathbf{x}+a\mathbf{e}^i) + \\
& \Delta_j^{\beta'}\Delta_i^a \tilde{f}(\mathbf{x}+b\mathbf{e}^j) + \\
& \Delta_j^{\beta'}\Delta_i^{\alpha'} \tilde{f}(\mathbf{x}+a\mathbf{e}^i+b\mathbf{e}^j).
\end{aligned}$$

The first sumand is positive, as we have just seen. For the second, if $\mathbf{x}+a\mathbf{e}^i$ and $\mathbf{x}+a\mathbf{e}^i+\alpha'\mathbf{e}^i$ do not belong to the same cell, there is a number $\delta' \in \mathbb{R}$ with $0 \leq \delta' \leq \alpha'$ such that $\mathbf{x}+a\mathbf{e}^i$ and $\mathbf{x}+a\mathbf{e}^i+\delta'\mathbf{e}^i$ belong to the same cell, and so do $\mathbf{x}+a\mathbf{e}^i+\delta'\mathbf{e}^i$ and $\mathbf{x}+a\mathbf{e}^i+\alpha'\mathbf{e}^i$. Then if we apply properties 6, 4, 9 and 10, and directional convexity of f, this sumand is positive.

For the third and the latter just repeat the previous procedure.

We have proved that for any $\mathbf{x}\in\mathbb{R}_+^N$, $i,j\in N$, and $\alpha,\beta\in\mathbb{R}$, $\alpha,\beta>0$, $\Delta_j^\beta\Delta_i^\alpha\tilde{f}(\mathbf{x})\geq 0$, and by the characterization of directionally convex functions, given in the appendix, $\tilde{f}$ is directionally convex. ■

If we apply the preceding lemmas, we can state the main result of this paper:

Theorem 8 *Let (N,v) be a convex game. Then, there is a directionally convex function $f:\mathbb{R}_+^n\longrightarrow\mathbb{R}$, such that $f(0)=0$, and measures over $2^N:\mu_1,\mu_2,...,\mu_n$ such that:*

$$v(S)=f(\mu_1(S),\mu_2(S),\ldots,\mu_n(S)),\quad \textit{for all } S\subseteq N.$$

Proof. Define, for $i=1,2,\ldots,n$, $\mu_i(S)=1$ if $i\in S$, and 0 otherwise (each player has the control of one resource). Then $(\mu_1(S),\mu_2(S),\ldots,\mu_n(S))=\mathbf{e}^S\in\mathbb{R}_+^N$.

Define $f(\mathbf{e}^S)=v(S)$. This gives a function defined on the vertices of the unit cube. If v is a convex game, this is equivalent to function f being supermodular in the vertices of the unit cube. Apply now lemma 6 and lemma 7, and the result is proved. ■

As an illustration, look at an economic example quoted by Rosenmüller (1981), and assume there are two factors, labor and land. There are n players and each player $i\in N$ owns l_i units of labor force and c_i units of land. There is a function $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+, g(0)=0$ that gives the amount of crops per unit of land that can be harvested.

The production function is, then

$$v(S)=c(S)\cdot g(l(S))$$

assuming that $c(S)=\sum_{i\in S}c_i$, and $l(S)=\sum_{i\in S}l_i$. If we define the production function $f:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$ by $f(s,t)=sg(t)$, the game is convex if the function g is convex. Notice that in this case f is directionally convex.

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5 Appendix

For one-variable functions the usual definition of convex function defined over a convex subset (an extended interval) $S \subset \mathbb{R}$ is the following one: a function $f : S \subset \mathbb{R} \longrightarrow \mathbb{R}$ is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If S is not an extended interval, but another subset of $\mathbb{R}$, one should require this condition for $\lambda \in [0, 1]$, and such that $\lambda x + (1 - \lambda)y \in S$. For a function defined on an extended interval of the integer set, that is, $S = (a, b) \cap \mathbb{Z}$, f is convex if and only if f has nonnegative second differences (Marshall and Olkin, 1979), i.e.,

$$f(x + 2) - 2f(x + 1) + f(x) \geq 0,$$

for all $x \in \mathbb{Z}$ such that $a < x, x + 2 < b$.

To give a characterization of directionally convex functions, let us define the difference operator:

Definition 9 *For a function $f : S \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ define the difference operator $\Delta_i^\varepsilon f(\mathbf{x}) := f(\mathbf{x}+\varepsilon\mathbf{e}^i) - f(\mathbf{x})$, for all $i \in \{1, 2, \ldots, m\}$ and all $\mathbf{x} \in S$ such that $\mathbf{x}+\varepsilon\mathbf{e}^i \in S$, where $\mathbf{e}^i$ is the $i-$th unit vector and $\varepsilon \in \mathbb{R}_+$.*

We will use $\Delta_i f(\mathbf{x})$ instead of $\Delta_i^1 f(\mathbf{x})$, if no confusion arises.

Theorem 10 *Let $f : \mathbb{N}^m \longrightarrow \mathbb{R}$ be a real-valued function. Then the following statements are equivalent:*

(i) *f is directionally convex.*

(ii) *For all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{N}^m$ with $\mathbf{x}_1 \leq \mathbf{x}_2$ and all $\mathbf{y} \in \mathbb{N}^m$,*

$$f(\mathbf{x}_1+\mathbf{y}) - f(\mathbf{x}_1) \leq f(\mathbf{x}_2+\mathbf{y}) - f(\mathbf{x}_2).$$

(iii) *For all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{N}^m$ with $\mathbf{x}_1 \leq \mathbf{x}_2$, and for all $i \in \{1, 2, \ldots, m\}$,*

$$\Delta_i f(\mathbf{x}_1) \leq \Delta_i f(\mathbf{x}_2).$$

(iv) *For all* $\mathbf{x} \in \mathbb{N}^m$, *and for all* $i, j \in \{1, 2, \ldots m\}$,

$$\Delta_j \Delta_i f(\mathbf{x}) \geq 0.$$

(v) *For all* $\mathbf{x} \in \mathbb{N}^m$, *for all* $k, k' \in \mathbb{N}$, *and for all* $i, j \in \{1, 2, \ldots m\}$,

$$\Delta_j^{k'} \Delta_i^k f(\mathbf{x}) \geq 0.$$

(vi) f *is supermodular and convex in each coordinate over* $\mathbb{N}$, *all other coordinates held fixed.*

Proof. See Shaked and Shanthikumar (1990), Proposition 2.1. Equivalence of (i) and (ii) is immediate, and also equivalence of (ii), (iii), (iv) and (v) by repeated iteration. But (ii) implies supermodularity, because for any $\mathbf{z}$ and $\mathbf{t}$, consider $\mathbf{z} \vee \mathbf{t}$ and $\mathbf{z} \wedge \mathbf{t}$, and then take $\mathbf{x}_1 = \mathbf{z} \wedge \mathbf{t}$, $\mathbf{x}_2 = \mathbf{t}$ and $\mathbf{y} = \mathbf{z} - (\mathbf{z} \wedge \mathbf{t}) \geq 0$. And the convexity in each coordinate results immediately if $\mathbf{x}_1$and $\mathbf{x}_2$ differ only in the $i-$th coordinate. To see that (vi) implies (iii), we can consider two cases: if $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same $i-$ th coordinate, it is obvious applying supermodularity to $\mathbf{x}_1 + \mathbf{e}^i$ and $\mathbf{x}_2$. If they do not have the same $i-$ th coordinate, consider the auxiliary vector $\hat{\mathbf{x}}_1$, defined by $(\hat{\mathbf{x}}_1)_k = (\mathbf{x}_1)_k$ if $k \neq i$ and $(\hat{\mathbf{x}}_1)_i = (\mathbf{x}_2)_i$, and first apply convexity in coordinate i for $\mathbf{x}_1$ and $\hat{\mathbf{x}}_1$, and then the previous case with $\hat{\mathbf{x}}_1$ and $\mathbf{x}_2$. ■

We can state also equivalent characterizations for functions defined on another subset of $\mathbb{R}^m$ that we use: $\mathbb{R}_+^m$.

Theorem 11 *Let* $f : \mathbb{R}_+^m \longrightarrow \mathbb{R}$ *be a real-valued function. Then the following statements are equivalent:*

(i) f *is directionally convex.*

(ii) *For all* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}_+^m$ *with* $\mathbf{x}_1 \leq \mathbf{x}_2$ *and all* $\mathbf{y} \in \mathbb{R}_+^m$,

$$f(\mathbf{x}_1 + \mathbf{y}) - f(\mathbf{x}_1) \leq f(\mathbf{x}_2 + \mathbf{y}) - f(\mathbf{x}_2).$$

(iii) *For all* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}_+^m$ *with* $\mathbf{x}_1 \leq \mathbf{x}_2$, *for all* $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ *and for all* $i \in \{1, 2, \ldots, m\}$,

$$\Delta_i^\varepsilon f(\mathbf{x}_1) \leq \Delta_i^\varepsilon f(\mathbf{x}_2).$$

(iv) *For all* $\mathbf{x} \in \mathbb{R}_+^m$, *for all* $\varepsilon, \varepsilon' \in \mathbb{R}$, $\varepsilon, \varepsilon' > 0$ *and for all* $i, j \in \{1, 2, \ldots m\}$,

$$\Delta_j^{\varepsilon'} \Delta_i^\varepsilon f(\mathbf{x}) \geq 0.$$

Proof. Immediate following the lines of the previous proof. ■