

Identification of strongly correlated states of cold atoms

Author: Aleix López Pascual

Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Núria Barberán Falcón

Abstract: Using a small sample of cold bosonic atoms submitted to a strong artificial magnetic field, we obtain the ground state of the system in the lowest Landau level. Our goal is to identify the ground state obtained with the known Laughlin state. The calculations have been done by exact diagonalization of the corresponding Hamiltonian. The obtained ground state is a strongly correlated state with zero interaction. This state shows all of the characteristics familiar from the fractional quantum Hall effect.

I. INTRODUCTION

The quantum Hall effect is observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields. In that case, a stepwise dependence of the Hall resistance R_H on the magnetic field B is shown. Otherwise, the relation between R_H and B becomes strictly linear, as confirmed by Edwin Hall [1]. The proposed system is a many-body system with electrostatic interaction, so-called Coulomb interaction. This interaction plays an irrelevant role to the understanding of the integer quantum Hall effect (IQHE). Thus, the phenomenon can be interpreted as a single-particle effect. On the other hand, the essence of the fractional quantum Hall effect (FQHE) lies in the interaction. Therefore, it needs to be treated as a many-particle effect. This paper is focused on the FQHE.

Many-body systems with Coulomb interaction are quite complicated. Nevertheless, there is the possibility of removing most of the electron-electron interaction by replacing the system of interacting electrons with an equivalent system of quasiparticles, so-called composite fermions (CF). Therefore, the problem is transformed to a much simpler single particle problem of rather complex objects. Following the CF model, it is possible to define a wave function that describes the system. However, although these wave functions are very accurate, they are not an exact solution for the system of electrons under Coulomb interaction. The main problem is the Coulomb interaction [2].

In this paper, we propose a model that simulates the system of electrons with Coulomb interaction submitted to real magnetic fields. This model consists on the confinement of ultracold atoms in a rotating two dimensional trap, which generates an artificial magnetic (gauge) field. The purpose of the article is to identify the exact ground state of the proposed boson system in the lowest Landau level, which corresponds to the limit of very high magnetic fields, with the bosonic Laughlin state. The importance of this state is that it presents strong quantum correlations that cause the bosonic Laughlin state to show all of the characteristics familiar from the conventional FQHE.

This document is organized in the following way. In section II we present the QHE. In section III we introduce these new composite particles and the interpretation of the FQHE using them. In section IV we find the wave function of the system. In section V we describe the proposed boson system and the bosonic Laughlin state. In section VI we give our results and finally the conclusions are commented in section VII.

II. INTEGER QUANTUM HALL EFFECT

A. Two-dimensional electron systems

The QHE can only appear in 2D electron systems. Electrons in a high magnetic field are forced onto circular orbits, following the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Quantum mechanically, not all the orbits are allowed. There is a discrete set of orbits with their corresponding energy levels, the so-called Landau levels. Electrons can only reside at these energies, but not in the large energy gaps in between. The existence of these gaps is crucial for the occurrence of the quantum Hall effect. However, in three dimensions, the electrons in the direction of the magnetic field can have any amount of energy ($\vec{v} \times \vec{B} = 0$). Therefore, the energy gaps are filled up and hence eliminated, preventing the quantum Hall effect from occurring.

B. Quantization of the Hall resistance

The quantization of the Hall resistance is determined by the number of Landau levels that are completely filled. The filling factor is the number of filled Landau levels $\nu = \frac{N}{M}$ where N is the number of electrons and $M = \frac{eBA}{h}$ is the capacity (degeneracy) of each Landau level, where e is the elementary charge of an electron, h is Planck's constant and A is the area of the sample. The capacity indicates the number of single particle states with the same Landau energy. At low temperatures, all electrons try to fall into the energetically lowest available states. Consequently, given an electron density n , those magnetic fields

$B_\nu = \frac{n\hbar/e}{\nu}$, at which all electrons fill up an exact number of Landau levels, keeping all higher Landau levels exactly empty, will be the values at which the Hall resistance will show the quantized values $R_H = \frac{B_\nu}{ne} = \frac{\hbar}{\nu e^2}$.

C. Origin of plateaus

The origin for plateau formation arises from the energetic valleys and hillocks along the interface caused by impurities and the resulting localization of carriers. Another and no less important condition to show the IQHE is the existence of imperfections in the two-dimensional electron system. Instead, one would revert to Edwin Hall's straight line, since the plateaus would not be observed. These defects create localized states in the energy gap. Consequently, as long as the magnetic field is being reduced in order to fill more Landau levels, some of the electrons get trapped and isolated. The localized electrons no longer contribute to the density of carriers nor to the current, which means that n only takes into account the delocalized electrons. Therefore, a reduction of B implies a reduction of n . However, the most fascinating result is that they evolve at the same rate, providing the appearance of the plateau in the Hall resistance $R_H \propto \frac{B_\nu}{n}$.

III. COMPOSITE PARTICLES

Composite particles are quasiparticles composed by an electron and a number of magnetic flux quanta. These quanta $\phi_0 = \frac{\hbar}{e}$ are the elementary units in which a magnetic field interacts with a system of electrons. From the 2D point of view, the action of the magnetic field can be interpreted as the creation of vortices in our plane of electrons. Each vortex is associated with a magnetic flux quantum. Inside the vortices, electronic charge drops to zero. As a consequence, placing vortices onto electrons will cause the displacement of nearby electrons and the disappearance of the electronic charge. Thus, the electron system will reduce its electrostatic Coulomb energy.

The number of vortices forming the composite particle may vary. At least, each electron always needs to be surrounded by one vortex in order to satisfy the Pauli exclusion principle. This situation corresponds to the IQHE, where the Landau levels are completely filled by electrons and therefore electrons have no freedom to avoid one another.

In the FQHE, Landau levels are only partially occupied (fractional filling factors). This means that given a certain Landau level completely filled, the FQHE is achieved by applying stronger magnetic fields than the corresponding IQHE, since the capacity of the Landau level increases and therefore the level is partially occupied. Stronger magnetic fields provides more magnetic flux quanta and hence more vortices than electrons. Consequently, composite particles of more than one vortex

will be created.

Composite particles can show either fermion or boson behavior, depending on the number of attached flux quanta. An electron plus an even number of flux quanta becomes a composite fermion. An electron plus an odd number of flux quanta becomes a composite boson.

A. Composite bosons

In this case, all the external magnetic field is incorporated into the new particles via flux quantum attachment to the electrons. Thus, they reside in an apparently magnetic-field-free region. Since CBs behave like bosons, at very low temperatures they can be found in the same lowest energy state, generating the corresponding energy gap (Bose-Einstein condensate). As we have seen, this gap is essential for the QHE. Therefore, the quantization of the R_H is manifested.

B. Composite fermions

Similar to the previous case, the magnetic field is also incorporated into the new particles, so they reside in an apparently magnetic-field-free region. Nevertheless, in this case the particles cannot condense at the same energy level (Pauli exclusion principle). Instead, they fill up successively the sequence of lowest-lying energy states (Fermi Sea). Consequently, there is no energy gap and hence the quantized Hall resistance is not shown. This is precisely what happens with $\nu = 1/2$.

But what about filling factors near $1/2$? CFs still have two vortices, however, they no longer reside in a magnetic-field-free region, but in an effective magnetic field. Consequently, CFs experience the effect of the field and their motion becomes quantized into CF-Landau orbits as it did with electrons. They fill up the corresponding CF-Landau levels, generating CF-energy gaps. As a result, CFs exhibit an IQHE. However, this time Landau levels are filled by CFs instead of electrons. The quantization of the Hall resistance will arise exactly at the filling factors corresponding to the effective magnetic field at which the CF-Landau levels are completely filled.

This last CF model can be generalized for all FQHE states, including the CBs states. Instead of creating CBs, CFs in an effective magnetic field are always created. Thus, a direct analogy between the IQHE and the FQHE is established.

IV. FRACTIONAL QUANTUM HALL EFFECT

Composite fermions experience a reduced magnetic field $B^* = B - \frac{2pN\phi_0}{A}$, where $2p$ is the even number of magnetic flux quanta that composes the CF. As a consequence, the degeneracy of each CF-Landau level is also modified by $M^* = \frac{eA|B^*|}{h} = \frac{A|B^*|}{\phi_0} = |M - 2pN|$, which

implies that the filling factor of CFs is $\nu^* = \frac{N}{M^*}$. The FQHE is shown when some CF-Landau levels are completely filled, i.e. $\nu^* \in \mathbb{N}$. Therefore, a fractional filling factor of electrons can be understood as an integer filling factor of CFs. The relation between both is given by $\nu = \frac{\nu^*}{2p\nu^* \pm 1}$, $\nu^* \in \mathbb{N}$, where the $-$ sign corresponds to the situation when B^* points opposite to B . We conclude that the FQHE of electrons is equivalent to the IQHE of CFs.

Note, this last expression does not include filling factors with even denominators $\nu = 1/2, 3/2, 1/4\dots$. In these cases, all the magnetic field is exactly incorporated to the CFs. They reside in a magnetic-field-free region $B^* = 0$. Therefore, they do not present CF-Landau levels nor CF-energy gaps, so they cannot exhibit the quantization of the Hall resistance. However, it has been found that the $5/2$ and $7/2$ FQHE states show such quantization even though they should not. They should behave like the $1/2$ state, but they do not. The scenario, which currently remains unresolved, suggests that there could be higher-order electron-electron correlations than those of the CF model.

A. Microscopic theory

The analogy between the IQHE of electrons and the IQHE of CFs can be exploited in order to formulate a microscopic theory for the FQHE. The wave functions for the IQHE of electrons are known. Thus, as we know that there is an analogy, we can construct a wave function for the IQHE of CFs, which we have seen that can be identified with the FQHE of electrons at $\nu = \frac{\nu^*}{2p\nu^* \pm 1}$, $\nu^* \in \mathbb{N}$. This wave function is given by [2]

$$\Psi_{\nu^*/(2p\nu^* \pm 1)} = \prod_{j < k} (z_j - z_k)^{2p} \phi_{\pm\nu^*} \quad (1)$$

where $z_j = x_j - iy_j$ denotes the electron coordinates as a complex number, $\phi_{\pm\nu^*}$ is the Slater determinant wave function for ν^* filled Landau levels of electrons, and $\prod_{j < k} (z_j - z_k)^{2p}$ is the Jastrow factor. Multiplication by the Jastrow factor binds $2p$ vortices to each electron in $\phi_{\pm\nu^*}$ to convert it into CFs. Therefore, the whole wave function can be interpreted as ν^* filled Landau levels of composite fermions.

Eq. (1) is a generalization of the Laughlin's wave function, which was invented earlier for an explanation of the $\nu = 1/m$ FQHE states, m odd. Laughlin's wave function corresponds to $\nu^* = 1$ in Eq. (1), so acquires the physical meaning of one filled Landau level of composite fermions.

V. MODEL

For quantitative studies, the wave functions of Eq. (1) need to be projected into the lowest Landau level, which

corresponds to the limit of very high magnetic fields. Even though the resulting wave functions are not the exact solution for a system of electrons with Coulomb interaction [3]. The main problem of the study is the Coulomb interaction, so the purpose of our model is to simulate the system of electrons with Coulomb interaction submitted to real magnetic fields, from another system with the same physics but different interaction. One of the experimental possibilities to mimic these systems using dilute ultracold atoms, consists on the confinement of N neutral bosons with repulsive contact interaction in a rotating two dimensional trap, which rotation frequency plays the role of the magnetic field.

The total Hamiltonian of N bosonic atoms trapped in a rotating parabolic potential can be written as [3]

$$H = H_{sp} + V \quad (2)$$

where V is the two body interaction potential and H_{sp} is the sum of the single particle Hamiltonians given by

$$H_{sp} = \sum_{i=1}^N \left[\frac{p_i^2 + p_{zi}^2}{2M} + \frac{M}{2} (\omega_{\perp}^2 r_i^2 + \omega_z^2 z_i^2) - \Omega L_{zi} + W_i \right] \quad (3)$$

where $\vec{r} = (x, y)$, M is the mass of the atoms, ω_{\perp} and ω_z are the trap frequencies in the xy -plane and in the z -direction respectively, Ω is the rotation frequency of the system, L_z is the z -component of the angular momentum and W_i is the anisotropic potential due to the presence of impurities, which we are not going to consider. As we want to produce an effective 2D system, we impose $\omega_z \gg \omega_{\perp}$.

The artificial magnetic field along the z -direction generated from the rotation of the trap is $\vec{B} = \frac{2M\Omega c}{e} \hat{z}$. The corresponding vector potential satisfies $\vec{B} = \nabla \times \vec{A}$. This vector potential is not unique. Thus, there is some gauge freedom in the choice of the vector potential for a given magnetic field. The most convenient gauge for our purpose is the so-called symmetric gauge:

$$\vec{A} = (A_x, A_y) = \frac{B}{2}(-y, x) = \frac{M\Omega c}{e}(-y, x) \quad (4)$$

which breaks translational symmetry in both the x and the y directions, but it does preserve the rotational invariance around the z -direction. With all these considerations, Eq. (3) can be rewritten as

$$H_{sp} = \sum_{i=1}^N \left[\frac{(\vec{p} - \frac{e}{c}\vec{A})_i^2}{2M} + \frac{M}{2} (\omega_{\perp}^2 - \Omega^2) r_i^2 \right] \quad (5)$$

where the electronic charge $-e$ and the speed of light c are solely introduced for reasons of algebraic equivalence.

The two body interaction potential, which needs to simulate the Coulomb interaction of electrons, is expressed as a repulsive contact interaction characterized by

$$V = \frac{\hbar^2 g}{M} \sum_{i < j} \delta^{(2)}(\vec{r}_i - \vec{r}_j) \quad (6)$$

where $g = \sqrt{8\pi}a/\lambda_z$ is the dimensionless coupling parameter, a is the 3D scattering length and $\lambda_z = \sqrt{\hbar/M\omega_z}$.

The contact interaction is important when $\vec{r}_i = \vec{r}_j$. Thus, in order to reduce the interaction, atoms need to occupy different positions. This implies that the contact interaction is repulsive to reduce the energy of the system. As mentioned in section III, from the point of view of electrons, vortices were placed onto electrons causing the displacement of nearby electrons and the disappearance of the electronic charge to reduce its electrostatic Coulomb energy. Therefore, Eq. (6) is consistent with the Pauli exclusion principle and the Coulomb interaction.

We need to restrict the system to the lowest Landau level (LLL) regime in order to make quantitative studies. As we have seen in section III, this level is achieved in the strong magnetic-field limit, i.e. for large rotational frequencies Ω . The Hamiltonian projected onto the LLL can be written as

$$\hat{H} = \hbar(\omega_\perp - \Omega)\hat{L} + \hbar\omega_\perp\hat{N} + \hat{V} \quad (7)$$

where \hat{L} and \hat{N} are the total z-component angular momentum and particle number operators respectively. This Hamiltonian describes a many-particle system of N identical bosons. The multi-particle symmetric states of N identical bosons are given by

$$|\Phi\rangle \equiv |\alpha_1(z_1)\alpha_2(z_2)\dots\alpha_N(z_N)\rangle_S = \sqrt{\frac{\prod_\alpha n_\alpha!}{N!}} \cdot \sum_{P \in S_N} |\alpha_{P1}(z_1)\rangle|\alpha_{P2}(z_2)\rangle\dots|\alpha_{PN}(z_N)\rangle \quad (8)$$

where $|\alpha\rangle$ are the single-particle states and n_α is the number of times that each of the single-particle states appears in the multi-particle state ($\sum_\alpha n_\alpha = N$). The sum is taken over all different states under permutations P acting on N elements. The appropriate single-particle states that describe single particles in a 2D parabolic confinement potential are the so-called Fock-Darwin states, given by

$$|\psi\rangle \equiv |m\rangle = \frac{1}{\lambda\sqrt{\pi m!}} \left(\frac{z}{\lambda}\right)^m e^{-|z|^2/2\lambda^2} \quad (9)$$

where z are generalized complex coordinates $z = x + iy$, $\lambda = \sqrt{\hbar/M\omega_\perp}$ and m is the single-particle angular momentum. Note that the state is only characterized by the angular momentum.

A. Second quantization

The Hamiltonian (7) describes a many-particle system. Therefore, it is very useful to use the second quantized formalism and set the problem in the Fock space. From now on, we will consider λ , $\hbar\omega_\perp$ and ω_\perp as units of length, energy and frequency respectively. The kinetic contribution of the Hamiltonian is a one-body operator, so it can be written in second quantization as

$$\begin{aligned} \hat{H}_{kin} &= \sum_{ij} \langle \psi_i | \left(1 - \frac{\Omega}{\omega_\perp}\right) \hat{L} + \hat{N} | \psi_j \rangle a_i^\dagger a_j \\ &= \sum_{ij} \left[\int dz \psi_i^*(z) \left(\left(1 - \frac{\Omega}{\omega_\perp}\right) (-i\partial_\theta) + \mathbb{1} \right) \psi_j(z) \right] a_i^\dagger a_j \\ &= \sum_j \left[\left(1 - \frac{\Omega}{\omega_\perp}\right) m_j + 1 \right] a_j^\dagger a_j \end{aligned} \quad (10)$$

Instead, the interaction contribution of the Hamiltonian is a two-body operator, so

$$\begin{aligned} \hat{V} &= \frac{1}{2} \sum_{ijkl} \langle \psi_i \psi_j | \hat{V} | \psi_k \psi_l \rangle a_i^\dagger a_j^\dagger a_l a_k \\ &= \frac{g}{2} \sum_{ijkl} \left[\int dz_1 dz_2 \psi_i^*(z_1) \psi_j^*(z_2) \delta^{(2)}(z_1 - z_2) \psi_k(z_1) \psi_l(z_2) \right] \\ &\quad \cdot a_i^\dagger a_j^\dagger a_l a_k \\ &= \frac{g}{2} \sum_{ijkl} \frac{1}{\pi} \frac{\delta_{m_i+m_j, m_k+m_l}}{\sqrt{m_i! m_j! m_k! m_l!}} \frac{(m_i + m_j)!}{2^{m_i+m_j+1}} a_i^\dagger a_j^\dagger a_l a_k \end{aligned} \quad (11)$$

where we have used Eq. (6).

B. Bosonic Laughlin state

The Laughlin's wave function for electrons under Coulomb interaction is not exact. Instead, there is an exact wave function for the proposed boson system, the so-called bosonic Laughlin state [4]:

$$\Psi_{\mathcal{L}}(z_1, z_2, \dots, z_N) = \mathcal{N} \prod_{i < j}^N (z_i - z_j)^2 e^{-\sum |z_i|^2/2\lambda_\perp^2} \quad (12)$$

where \mathcal{N} is a normalization constant. It represents the exact ground state of a 2D boson system in the limit of very high magnetic fields (LLL) at total angular momentum $L = N(N-1)$. The Laughlin state is a strongly correlated state with zero probability to find two particles at the same place [4]. Therefore, the contact interaction (6) is zero. Consequently, the boson system presents a fermion-like behaviour in the Laughlin state.

From the point of view of the electron system, the bosonic Laughlin state corresponds to the $\nu = 1/3$ Laughlin state of electrons (1). In accordance with the CF model, this implies that all the electrons are attached to two vortices ($2p = 2$). Therefore, as we have seen in section III, all the electrons see zero electronic charge in the other electron positions. As a result, the Coulomb interaction is zero. Note that there is no unoccupied vortex in the Laughlin state. A vacant vortex would expand the system and would cause an increase in energy (excited state). Thus, the bosonic Laughlin state shows all of the characteristics familiar from the conventional FQHE [5, 6].

VI. RESULTS

In this section, we calculate the ground state of the boson system subjected to an artificial magnetic field in the LLL regime by performing exact diagonalization. Then we compare it with the bosonic Laughlin state (12). The result must be exact.

In order to do this, we have prepared an algorithm described as follows: we fix the number of particles N and the total angular momentum L . We construct a basis with the multi-particle Fock states. First, we identify all the possible configurations of the atoms that give us the angular momentum of interest, without repeating configurations. Then, we rewrite each configuration in a Fock state, which is characterized by the occupation number. The number of states in the basis determines the dimension of the Hilbert space in which we will diagonalize. We use the generated states to calculate the matrix elements $\langle \Phi_r | \hat{H} | \Phi_s \rangle$, where \hat{H} includes Eq. (10) and (11). We diagonalize it to obtain its eigenvalues and eigenstates. Finally, we identify the ground state. All the calculations are performed for $N = 3$ atoms, the corresponding Laughlin's angular momentum $L = N(N - 1) = 6$, coupling parameter $g = 1$ and rotational frequency $\Omega/\omega_{\perp} = 0.95$, which has to be large enough to ensure the LLL regime and so, the Laughlin state.

We compare the obtained ground state with Eq. (12). It can be shown that the overlap between both states is given by

$$\langle \Psi_{\mathcal{L}} | \Psi_{GS} \rangle = \mathcal{N} \left[\frac{\gamma_{006} \beta_{006}}{\nu_{006} \alpha_0 \alpha_6} + \frac{\gamma_{015} \beta_{015}}{\nu_{015} \alpha_0 \alpha_1 \alpha_5} + \dots \right] \quad (13)$$

where the sum is taken over all the Fock states of the basis, characterized by three subscripts that indicate the angular momentum of each atom. β are the coefficients of the GS obtained from the algorithm, $\nu = \sqrt{\frac{n_0! n_1! \dots n_6!}{N!}}$, $\alpha_m = \frac{1}{\sqrt{\pi m!}}$ and γ is the number of times that each of the Fock states appears in Eq. (12). The normalization constant has been calculated similarly. Performing detailed numerical calculations, we obtain $\mathcal{N} = 0.0615457$ and $\langle \Psi_{\mathcal{L}} | \Psi_{GS} \rangle = 1$.

VII. CONCLUSIONS

The FQHE is observed in 2D electron systems with Coulomb interaction subjected to low temperatures and strong magnetic fields. This many-body problem has not an exact analytical expression. However, replacing the system with CFs allows us to find a wave function that describes the system accurately. A particular case of this wave function, is the Laughlin's wave function, which describes the $\nu = 1/m$ FQHE states, m odd.

In order to simulate this system, we have proposed an ultracold boson system with repulsive contact interaction in a rotating two dimensional trap and we have found by performing exact diagonalization that its ground state in the limit of very high magnetic fields (LLL) at total angular momentum $L = N(N - 1)$ is exactly the bosonic Laughlin state. We have seen that, even in a bosonic system, the bosonic Laughlin state is a strongly correlated state with zero interaction that shows all of the characteristics familiar from the FQHE. As this state describes very accurately the system of electrons under Coulomb interaction, we conclude that the proposed boson system is a very good candidate to simulate the 2D electron system with Coulomb interaction submitted to real magnetic fields in the Laughlin state.

Acknowledgments

Special thanks to my advisor Núria Barberán, for her continued guidance and counsel. I would like to thank as well my family and friends for their encouragement.

-
- [1] Stormer, H. L., Rev. Mod. Phys. **71**, 875889 (1999).
 [2] Jain, J. K., Physica E **20**, 7988 (2003).
 [3] Barberán, N., Dagnino, D., García-March, M. A., Trombettoni, A., Taron, J. and Lewenstein, M., New J. Phys. **17**, 125009 (2015)
 [4] Juliá-Díaz, B., Dagnino, D., Günter, K. J., Graß, T., Bar-

- berán, N., Lewenstein, M. and Dalibard, J., Phys. Rev. A **84**, 053605 (2011)
 [5] Cooper, N. R., Advances in Physics **57**, 539 (2008)
 [6] Cooper, N. R. and Wilkin, N. K., Phys. Rev. B **60**, R16279 (1999)