Using a threshold model to study latency states in complex networks

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Abstract: Many different systems present states of latency in their activity. In this work, we study how these latency states can appear in complex networks of interactions by running on them a dynamical threshold process to model activity. We find the thresholds values for what is most likely to find these states, the dependence with the network size and we make a brief description of how these systems develop. This work is an initial description of latency states in networks, which will be explored more in depth in future work.

I. INTRODUCTION

There are many situations in which we would like to predict the movement of the decisions made by individuals in a group. Responding to this desire in a general way is extremely difficult even if we restrict to binary decisions. However, and although simple, binary-choice models can illuminate the dynamics of a surprising range of decisions.

In particular, threshold models are based on the influence of others decisions in the choice of an individual to act. The key concept is that there is a threshold value which stays for the minimum number of neighbours who must make one decision before a given individual does so.

Threshold models can be applied in many different problems and fields. For instance, in toxicology, or the spread of diseases and the diffusion of innovations, [2][3].

Granovetter [1] illustrated the model with the example of a hypothetical crowd poised on the brink of a riot. Everyone involved is uncertain about the costs and benefits associated with rioting, but each member of the crowd is influenced by his peers, such that each of them can be characterized by some threshold rule: *I will join a riot only when sufficiently many others do; otherwise I will refrain.*

Interactions between individuals are then crucial and the concept of complex network of interactions [4][5] is useful to simulate the behavior of threshold models, or other dynamical processes, on structured populations. A complex network is a graph representation of a complex system with discrete interacting elements (nodes) which displays non-trivial topological features, like the small-world property or heterogeneity in the number of neighbours per node (degree). The links define the nodes that can directly influence each other, but collective behaviour emerges and the whole system is not equivalent to a mere superposition of individual behaviors or direct influences.

One feature of threshold model on networks which has not been explored so far is latency. This is interesting as there are many systems that present latency states. For example, serious diseases like HIV and cancers. For HIV, the term latency is used to describe the long asymptomatic period between initial infection and development of AIDS. After an individual contracts the disease, there is a normal immune response over a time scale of months, followed by a latency period that can last beyond 10 years, during which an individual's T-cell level slowly decreases with time. Finally, after the T-cell level falls below a threshold value, there is a final fatal phase that lasts 2-3 years [6].

The existence of latency periods is also well known for many forms of cancer. The formation of tumors is a complex process and cancers do not occur immediately after exposure to a causative agent and they usually take many years up to several decades to manifest clinically.

On the other hand, latency can refer to the period elapsed between the diagnosis of a primary tumor and the emergence of detectable metastatic lesions[8].

Another example is virus latency, the ability of a pathogenic virus to lie dormant (latent) within a cell. After initial infection, proliferation of virus particles ceases and a latent viral infection persists. However, the viral genome is not fully eradicated. The result of this is that the virus can reactivate and begin producing large amounts of viral progeny without the host being infected by new outside virus and stays within the host indefinitely, like Herpes virus [7].

Our goal in this work is to describe the mechanisms which control latency states and the length of latent periods in dynamical threshold processes on networks.

II. MODEL

In this attempt to understand latency states in complex systems, we will focus only on networks with a regular degree, FIG. 1a. The degree, k_i, of a node, i, is defined as its number of neighbors or, in other words, the number of links attached to it. This means that there is only one type of link to connect two nodes and every node has the same number of neighbors, $k_j = k$, $\forall j \in N$ (self-loops are not considered).

We want to work with networks as simple as possible to ensure maximum homogeneity, thus making sure that all the properties of the results are given by the characteristics of the dynamics, and not by inhomogeneities in the structure of the network. We focus only on the study

of regular degree networks with degrees of $k = 15$ and $k = 16$.

A. Dynamical Process

The threshold model that we use in this study is a variant of the threshold model introduced by González-Avella et al. [9].

The nodes have a binary state, and we make an analogy with spins. Every node can be in the up state ($s_i = +1$) or in the down state ($s_i = -1$) as we can see graphically in the illustration 1a.

We start setting up the initial condition by associating to each node a state up with probability p and state down with probability $1 - p$. In this particular case, the probability p is $\frac{1}{2}$, so we have the same probability to find a node with spin up or spin down in the initial state.

Now we can classify the nodes, until now indistinguishable, into two different types, active and inactive. This classification depends on the neighborhood of each node and the threshold value, T_h, that has to be in the range $0 \leq T_h \leq k$.

The activation state of the node i is given by:

$$n_i(t) = \theta \left(\sum_{j=1}^{N} \Gamma_{ij} \frac{1 - s_i(t)s_j(t)}{2} - T_h \right) \qquad (1)$$

Where $s_i(t)$ is the state of the node i at time t with two possible values ± 1, and Γ_{ij} is the adjacency matrix of connections in the network. $\Gamma_{ij} = 1$ if there is a link between nodes i and j, and $\Gamma_{ij} = 0$ if it is absent, $\forall i, j \in N$.

By equation 1, an active node at time t, $n_i(t) = 1$, is defined by having a number of neighbors in the opposite state (to its state) greater than or equal to the value of the threshold. If there are not enough neighbors in the opposite state, the node is marked as inactive, $n_i(t) = 0$.

As we can appreciate in the illustration 1b, if we focus on a node *(marked in green)* we see that its neighbors are those that can be reached through a link *(marked in blue)*. To determine the activity of a given node, we are interested in those neighboring nodes that have a contrary state to the one of the reference node *(marked in red)*. If the number of the red marked nodes is larger than the threshold value, then the reference node is fixed as an active node, $n_i(t) = 1$.

It is crucial to determine which are these active nodes in the network since only these will have the option to change their state in the development of the dynamical process. Once we have listed all the active nodes, then we can make a step forward in the simulation and flip the state of one of these active nodes.

The choice of the node that will change state is done randomly. Each active node has the same probability $(1/N_A)$ of being selected to flip its previous state, where this N_A is the number of active nodes in the network

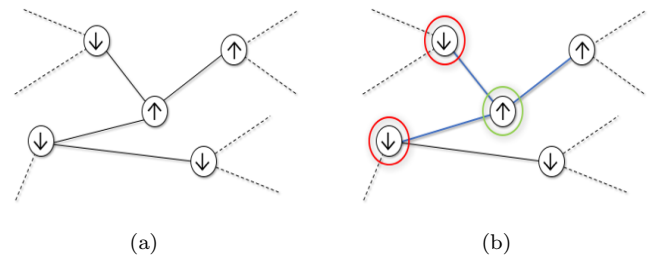(a)   (b)

Figure 1: In the illustration above we can see, on the left side, (a), how the spins distribution looks like on a piece of a degree regular network with $\langle k \rangle = 3$. On the right side, (b), how this distribution allows us to define the active and inactive nodes.

at that moment. Once a node is selected, its state is flipped and this forces us to check again all the network to determine which nodes now will be active and inactive. Whenever we do this calculation the time will be updated by $1/N_A$, where N_A is again the number of active nodes in the network at that moment. Obviously the realization will end when there are no more active nodes ($N_A = 0$), thus arriving at a steady state where there cannot be more changes.

III. RESULTS AND DISCUSSIONS

We study the development of active nodes over time to find latency states. At each time, the state of nodes is given by equation 1. We can see that if we are in cases where the threshold is very large or very small compared with the degree, the result is very predictable. Depending on whether $T_h \gg \frac{k}{2}$ or $T_h \ll \frac{k}{2}$ the results differ qualitatively.

We must remember that the model tells us that the initial probability of finding a state up or state down for a given node is the same and equal to one-half. When we are in the situation that the threshold value is $T_h \gg \frac{k}{2}$ say $T_h = 12$, the probability to have an active node is barely bigger than 0.2 percent, considering that the probability of having x neighbors with the opposite state goes like $(1/2)^x$. In this case, an active node will always become inactive after flipping. This flipped node will be too far from the threshold, so the possibility of having a sufficient number of active neighbors so that in the future this node will be active again is negligible.

In the opposite case, where the threshold value is $T_h \ll \frac{k}{2}$, some nodes will always remain active after changing sign. And also exists the possibility that this node, if now is inactive, that will be attached to sufficient active nodes whom can lead the reactivation of our previous node rewrite this sentence.

So, as shown in the Figure 2, these situations are those for thresholds between 16 or 15 to 12, for $T_h \gg \frac{k}{2}$, and thresholds between 5 to 1, for $T_h \ll \frac{k}{2}$. Obviously, in
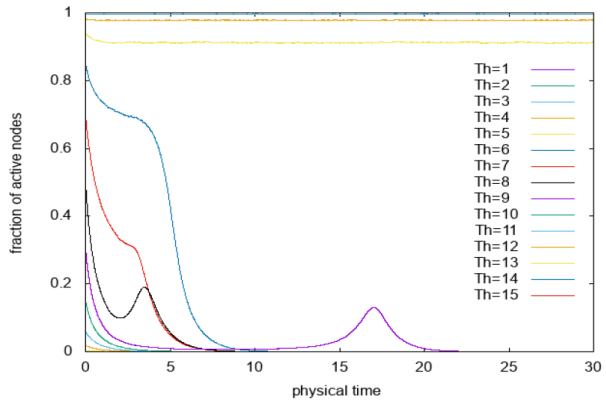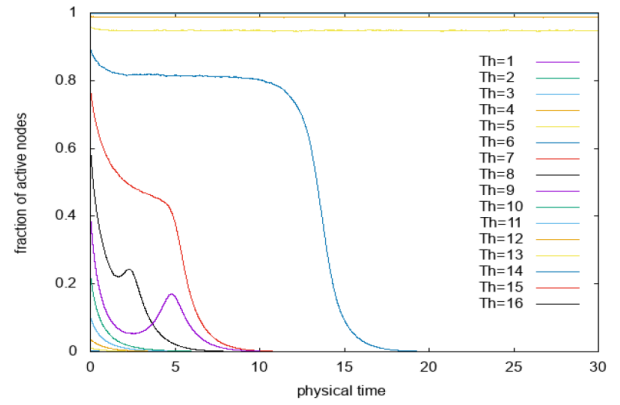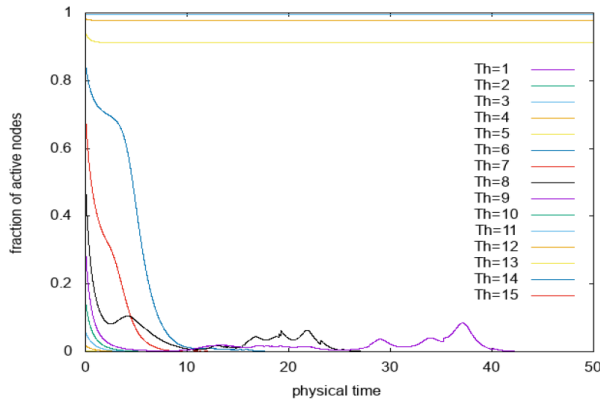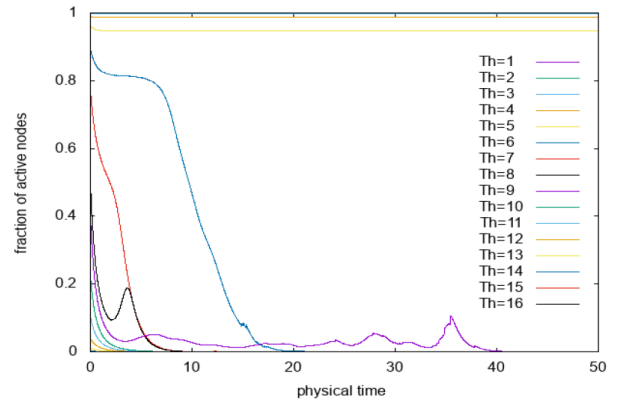
(a) Single realization for a regular degree network with $k = 15$

(b) Single realization for a regular degree network with $k = 16$

(c) Average of 100 realizations for a regular degree network with $k = 15$

(d) Average of 100 realizations for a regular degree network with $k = 16$

Figure 2: Representation of the fraction of active nodes, $\left(\frac{N_A}{N}\right)$, versus the physical time for every possible threshold, $(0 < T_h \leq k)$. For these graphs we have used a network of $N = 100000$.

these ranges of the threshold value latency periods are not observed.

Now we will focus on the study of values of the threshold close to $\frac{k}{2}$ where latency states appear.

We can see how for of $T_h \approx \frac{k}{2}$ in Figure 2, the number of active nodes has a very particular behavior. At the beginning, it decreases very steeply to a value much lower than the initial one. After this, only a few active nodes are maintained, with a fraction approximately constant, in a state of latency during a high period of time. And finally, the system has a peak before it dies.

These latency periods are not a timely fact due to a special distribution of the nodes. Because it is always repeated explain the preceding words better, we can observe it by looking at Figure 2c or 2d. In each of the realizations, the initial distribution has been changed. Neither it is a fact of the degree since for different degrees we obtain similar results, see again the Figures 2c and 2d.

The existence of latency periods can also be seen in Figure 3. There is a progressive increase in the probability of finding active simulations for ever greater times. A clear transition is shown between $T_h > 10$ where all the embodiments go to a frozen state and $T_h < 6$ where a steady state is reached, that is to say, their activity does never end. In the situation where $T_h > \frac{k}{2}$, whenever you flip an active node it becomes inactive, but it remains with a value close to the threshold.

In Figure 4, we introduce new variables that will be of interest. The vulnerable active nodes are those active nodes attached to exactly T_h nodes in the opposite state and there is at least one of these that is active. So if the node itself or one of its neighbors is selected to flip, the previously vulnerable active node will become inactive. Similarly, we introduce the vulnerable inactive nodes as those which are inactive with $T_h - 1$ neighbors with a contrary state and at least one neighbor active with the same state. So if the neighboring active node is selected it will become active.
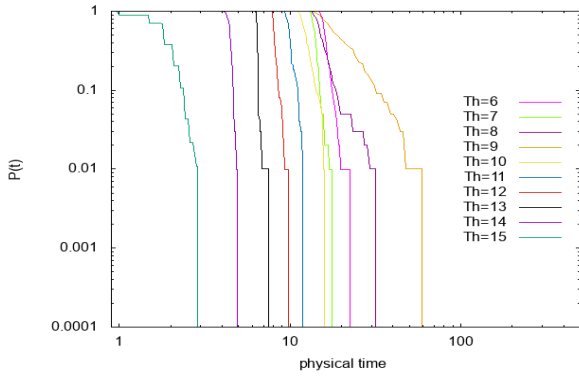
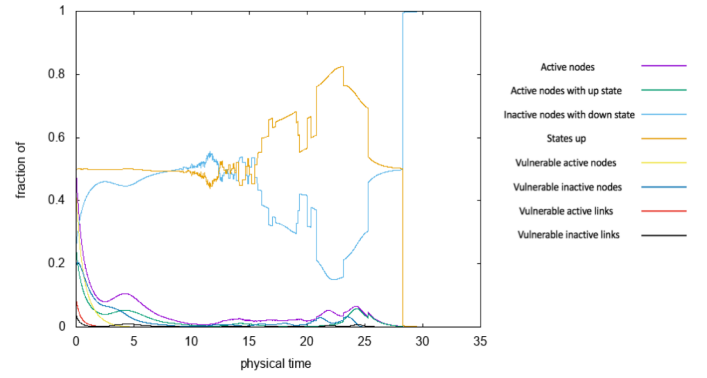(a) Cumulative distribution fiction of the live time for simulations with $k = 15$.
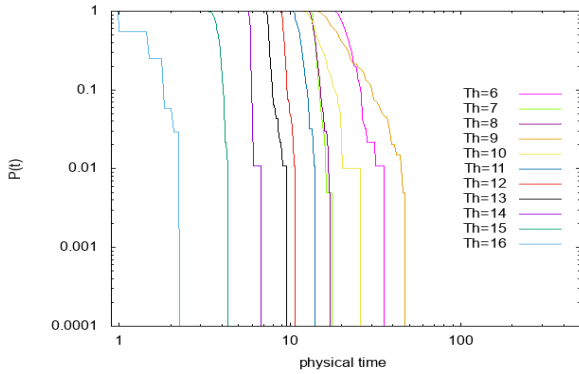
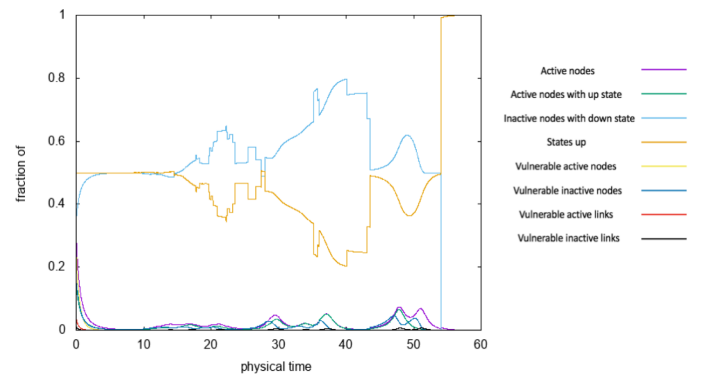(b) Cumulative distribution fiction of the live time for simulations with $k = 16$.

Figure 3: Representation of the cumulative distribution function, $P(t)$, of the life time of 100 realizations for every threshold value. This shows us the probability of finding a still active simulation after a given time. We only represent the results for ($T_h \geq 6$), because for the realizations with ($T_h < 6$) never end. For these graphs we also have use a regular degree network of $N = 100000$.

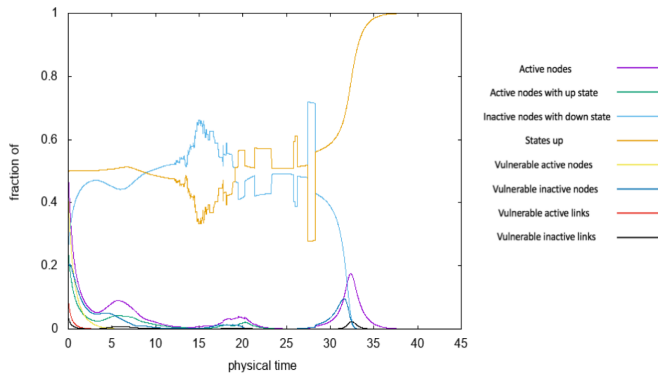(a) Average of 100 realizations with a fixed threshold value, $T_h = 8$.

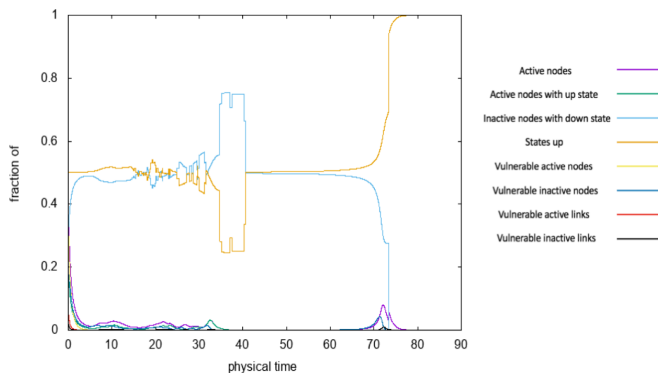(b) Average of 100 realizations with a fixed threshold value, $T_h = 9$.

Figure 4: Representation of the fraction of the active nodes, active nodes with up state, inactive nodes with down state, states up, vulnerable active nodes, vulnerable inactive nodes, vulnerable active links and the vulnerable inactive links. For these graphs we have use a regular degree network of $N = 100000$ and a fixed $k = 15$.

We will also define the vulnerable active links as those links between two active nodes with a contrary state. And the vulnerable inactive links as those links between two active nodes with the same state.

From Figure 4, we can see how there is a clear relationship between the peaks for the active nodes and the peaks related to the vulnerable inactive nodes. Since all the realizations have a peak in the vulnerable inactive nodes that precedes the peak of the active nodes and the subsequent end of this realization, we view this as somehow the number of vulnerable inactive nodes grow and surround the active nodes. Thus when an active node is selected to flip its state there will be a sudden rise of active nodes. The number of nodes that will become actives is at least equal to the number of inactive vulnerable neighbours that had the selected node.

This sudden rise of active nodes is shown in every final stage of the latency states, as we can see in the Figure 4. These new active nodes have just T_h neighbors against their state. This leads us to a state where the active nodes are linked to inactive nodes far from the threshold, so the change of an active node does not imply the creation of new activity. The active nodes will be depleted as they are being selected to flip the state.

Another feature we have seen is that these latency states do not seem to be affected by the finite size of the networks that we use. We have observed these non-monotonous states in networks between 1000 and 500000 nodes, see Figure 5.

(a) Average of 100 realizations with a fixed threshold and degree values, $k = 15$ and $T_h = 8$.

(b) Average of 100 realizations with a fixed threshold and degree values, $k = 16$ and $T_h = 9$.

Figure 5: Representation of the fraction of the active nodes, active nodes with up state, inactive nodes with down state, states up, vulnerable active nodes, vulnerable inactive nodes, vulnerable active links and the vulnerable inactive links. For these graphs we have use a regular degree network of $N = 500000$.

IV. CONCLUSIONS

The study of the dynamic threshold process in complex networks with a regular degree, in particular with a degrees $k = 15$ and $k = 16$, has allowed us to verify that for a certain range of threshold values ($T_h \approx \frac{k}{2}$) the activity of the system presents non-monotonous latency states. We have seen how these states are neither a particular fact of the degree of the network nor an effect due to the finite size of the network. Furthermore, we have noticed that in every realization the end of the latency state is preceded by a peak of vulnerable inactive nodes.

We could introduce a modification in the used model, in order to try to control the time span of the latency state to be able to extend it so that the system not always ends in frozen state but reached a constant dynamic state. Finally, we would also like to apply this modified model to scale-free complex network structures which are more descriptive of real networks.

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