Heavy Hybrids Mesons in NRQCD: Fine and Hyperfine Structure

Author: Pere Solé Campreciós.
Advisor: Joan Soto
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Abstract: We study a Heavy Hybrid Meson system, which is composed of a heavy quark-antiquark pair and a non-trivial contribution coming from the gluon field. The framework of the study is the strong interaction, described by QCD, in the special case where the mass of the quarks is heavy, and much higher than the energies of the gluonic field, which will let us assume a non-Relativistic case and work with the Schrödinger Equation. At leading order, the potential does not depend on the mass and spin of the quarks, but some corrections related to the fine and hyperfine structure (the study will focus on the second one), will come at $1/m_Q$ order, and we will study them. Finally, we will use the results from the hyperfine structure to find relations between the different masses of states in the same spin multiplet.

I. Introduction

In Particle Physics, a Meson, is a bound system made of a quark-antiquark pair $(q-\bar{q})$, with a certain orbital angular momentum between them $L_{q\bar{q}}$ and a coupled spin $S_{q\bar{q}}$ (we will omit the subscripts from now on), moving under a static potential (no dependence on time) that matches with the symmetries and properties of the Strong Interaction, since we are working with quarks (P, C, T and Total Angular Momentum conservation). In this context, we define the states of particles, or an ensemble of them, with the notation J^{PC} , where J is the Total Angular Momentum of the system, P and C are the eigenvalues of the Parity and Charge Conjugation, respectively. This notation is convenient due to the conservation rules mentioned before. Applying it to a fermion-antifermion system, they pick the values:

$$P = (-1)^{L+1}$$
$$C = (-1)^{L+S} \qquad (1)$$

If we consider the pair with angular momentum L and total spin S we can create a lengthy list of states such as 0^{-+}, etc…. (corresponding to $S=0$ and L=0). Nevertheless, with the development of QCD, many theoretical physicists thought that the gluon field could perturbate the spin of the system in a non-trivial way, that, what we call Hybrids. In this point of view, a certain state $(J^{PC})_g$ is associated to the gluon field as well, considering it part of the system (in the usual Meson, this contribution is ignored). It is a result from QCD, that the intrinsic parity of the gluons is (-1). Now that we have the states of the Meson alone and the gluonic field, we can build Hybrid states combining both:

$(J^{PC})_g$	L	S	$(J^{PC})_{Total}$
1^{+-}	0	0	1^{--}
1^{+-}	1	1	$(0,1,2,3)^{+-}$

Table 1*: Possible hybrid states when the gluon is carrying one unit of angular momentum.*

Looking at Table 1, we can see that the state 0^{+-}could not be reached without the contribution of the gluonic field. Here, we see that this perturbation of the system brings new possible states with it. Since firstly we will work with L and L_g (the spin of the quarks will be coupled later on), the Angular Momentum is defined by $J = L + L_g$. The total one is $\mathcal{J} = J + S$ but, as we begin with a non-depending quark spin state potential, J will be conserved for the first part.

To describe this system, one must make use of QCD and Quantum Field Theory but, since the mass of the heavy quarks m_Q is much greater than the energy contribution of the gluons, we will be able to work on the Non-Relativistic frame because the gluons (the field), will respond immediately to the motion of $q-\bar{q}$. We can make an analogy between Electromagnetism and Electrodynamics in which, on the first one, as the energy of the particles is comparable to their rest mass (low speed particles) we consider the electromagnetic field created on a point to respond immediately to the motion of the source hence not existing a "reaction time", in opposition to Electrodynamics, where we consider a time for propagation of the electromagnetic wave (velocity of the light c). With this approximation, the gluonic field will remain in a stationary state given a position of the sources. Furthermore, we will use the Born-Oppernheimer approximation, which associates to each stationary state of the gluon field a stationary potential $V(r)$ that, added to the Non-Relativistic frame, the motion will be described by means of the Schrödinger Equation. Since the gluon can contribute with an arbitrary angular momentum, we will work on the lowest energy levels, given by the value $L_g = 1$, which is demonstrated in [3] and Figure 1. We can imagine this gluon field acting like a string between the components with a certain vibrational state N, then, as we increase this value, the energy contribution increases too. The stationary potentials with lowest energy on short distances, where we are focusing (the distance between $q-\bar{q}$), are $V_{\Sigma_u^-}$ and V_{Π_u} as we can see in Figure 1.

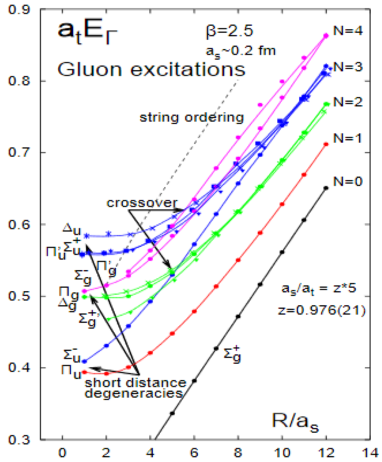

Figure 1: Low Lying Hybrids potentials in the static limit for heavy quarks. From [2].

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* Electronic address: pscamprecios1@gmail.com

Moreover, the fine correction is related with the interaction between the angular momentum of the quarks L, and the total angular momentum of the gluons. The hyperfine structure will consider the contribution of the coupling between the spin of the quarks and the spin of the gluon field. These elements will come at $1/m_Q$ correction order, and the hyperfine one will force the energy to split between the different quantum numbers of the system; in other words, it will break the degeneration that we have initially because, the term describing this part depends on the quarks spin and it will make a difference between these different states.

All these potentials, interaction and correction terms that will be added to the Lagrangian, as it is mentioned at the beginning, must fulfill the conservation of the discrete symmetries P, C, T (and conserve the Total Angular Momentum). Also, the potentials are describing the interactions between gluons and quarks, so they must agree with the postulates of QCD. The most important aspect in this context, is the confinement, stating that quarks cannot stand alone, therefore, they need to stay in groups. Because of this, when you have two quarks and you try to separate them, the energy stored in the time-space where the interaction takes place increases, until the energy of this gluon field is enough to create another pair of quarks. This statement forces the potentials to include lineal terms of r, because they need a term that increases energy with respect to r.

The first part of the work consists on verifying the Schrödinger equations (6), (7) and (8) of the Lower Lying Hybrid states to immediately, add the fine and hyperfine structure to the energy of the system. We will use the hyperfine ones, to find relations between the mass difference of states in the same spin multiplet.

II. Leading order: The Schrödinger Equation

In this section, we verify and review the results of [2] as a training for the Section III. This problem can be treated as a two-body system with two independent Hamiltonians, one for the centre of mass **R** and one for the relative position **r**. We will begin setting the position of the centre of mass at **R=0** and focusing on the relative position **r**. For the Low-Lying Hybrids, we are dealing with the angular momentum between the pair and the gluonic field contribution (we will take $L_g = 1$), it is natural then, to associate a vectorial wavefunction to this ensemble $\boldsymbol{H(R,r,t)}$**,** because of having the degree of freedom of the gluon (its spin 1). The Hamiltonian that describes this situation is:

$$h_{H_{ij}} = \left(-\frac{\nabla^2}{m_Q} + V_{\Sigma_u^-}\right)\delta_{ij} + \left(\delta_{ij} - \hat{r}^i\hat{r}^j\right)\left[V_{\Pi_u} - V_{\Sigma_u^-}\right] \quad (2)$$

Recall that the operator $\nabla^2 = \frac{1}{r}\partial_{rr}(r\,) - \frac{L^2}{r^2}$, the coordinate r is between the pair, the angular momentum term coincides with the L of the $q - \bar{q}$, not including the gluonic excitation state. The first part plays the same role as the central potential of the Hydrogen atom and the second one sets some off diagonal terms related to the gluonic field; also, this Hamiltonian does not depend on the spin of the quarks and will be invariant under spin transformations carrying, because of this, the degeneration mentioned in the Introduction, 4 for $\mathcal{J} \neq 0$ and 2 for $\mathcal{J} = 0$. If we express $\boldsymbol{H}$ using a basis of eigenfunctions of the Hamiltonian, following the notation [1], we have:

$$\boldsymbol{H} = \frac{1}{r}\left[P_0^+(r)\boldsymbol{\mathcal{Y}}_{00}^+ + \sum_{J=1}^{\infty}\sum_{M=-J}^{J}\sum_{L=J-1}^{J+1} P_J^L(r)\boldsymbol{\mathcal{Y}}_{JM}^L\right] \quad (3)$$

Where we are using the notation $L = 0, +, -$ in reference with the possible values of $L = J, J+1, J-1$; $\boldsymbol{\mathcal{Y}}_{JM}^L$, the vector spherical harmonics (see Appendix A)**,** fulfils:

$$\begin{array}{ll} J^2\boldsymbol{\mathcal{Y}}_{JM}^L = J(J+1)\boldsymbol{\mathcal{Y}}_{JM}^L & {L_g}^2\boldsymbol{\mathcal{Y}}_{JM}^L = 2\boldsymbol{\mathcal{Y}}_{JM}^L \\ L^2\boldsymbol{\mathcal{Y}}_{JM}^L = L(L+1)\boldsymbol{\mathcal{Y}}_{JM}^L & J_3\boldsymbol{\mathcal{Y}}_{JM}^L = M\boldsymbol{\mathcal{Y}}_{JM}^L \end{array} \quad (4)$$

And transforms under the discrete symmetries like:

$$\begin{array}{l} P: \boldsymbol{H}(\boldsymbol{R},\boldsymbol{r},t) \xrightarrow{yields} -\boldsymbol{H}(-\boldsymbol{R},-\boldsymbol{r},t) \\ C: \boldsymbol{H}(\boldsymbol{R},\boldsymbol{r},t) \xrightarrow{yields} -\sigma^2\boldsymbol{H}^T(\boldsymbol{R},-\boldsymbol{r},t)\sigma^2 \\ T: \boldsymbol{H}(\boldsymbol{R},\boldsymbol{r},t) \xrightarrow{yields} -\sigma^2\boldsymbol{H}(\boldsymbol{R},\boldsymbol{r},-t)\sigma^2 \end{array} \quad (5)$$

Now, the action of the Hamiltonian onto this wavefunction, using the relations (B1) and $\boldsymbol{\mathcal{Y}}_{JM}^L$ with $L = 0, \pm$ as the vector basis, the results are:

For J=0:

$$\left(-\frac{1}{m_Q}\partial_{rr} + \frac{2}{r^2 m_Q} + V_{\Sigma_u^-}\right)P_0^+ = EP_0^+ \quad (6)$$

Because of the coupling, the result $J = 0$ requires the value $L = 1 \xrightarrow{yields} L(L+1) = 2$.

For $J \neq 0$ we have two decoupled solutions:

$$\left(-\frac{1}{m_Q}\partial_{rr} + \begin{pmatrix} \frac{J(J-1)}{r^2 m_Q} & 0 \\ 0 & \frac{(J+1)(J+2)}{r^2 m_Q} \end{pmatrix} + V_{\Sigma_u^-} + V_q\begin{pmatrix} \frac{J+1}{2J+1} & \frac{\sqrt{J(J+1)}}{2J+1} \\ \frac{\sqrt{J(J+1)}}{2J+1} & \frac{J}{2J+1} \end{pmatrix}\right)\begin{pmatrix} P_J^- \\ P_J^+ \end{pmatrix} = E\begin{pmatrix} P_J^- \\ P_J^+ \end{pmatrix} \quad (7)$$

$$\left(-\frac{1}{m_Q}\partial_{rr} + \frac{J(J+1)}{r^2 m_Q} + V_{\Pi_u}\right)P_J^0 = EP_J^0 \quad (8)$$

The notation used for $L = 0, +, -$ is related to the possible values of the angular momentum $L = 0, L = J+1, L = J-1$ respectively, and V_q is the difference between the Hybrid potentials, the term $V_q = V_{\Pi_u} - V_{\Sigma_u^-}$.

III. $\frac{1}{m_Q}$ Corrections: Fine and Hyperfine Structure

The next step is the computation of the fine and hyperfine structures that appears at $1/m_Q$ order. To address this step, because we are adding another degree of freedom (the spin of the quarks), we will be needing a geometrical object able to define at the same time the spin of the gluon field, the spin of the quarks and the relative angular momentum. The only candidate in this case are the Tensorial Spherical Harmonics (see Appendix A). Now, the definition of the wave function will be as follows:

$$\begin{gathered} H^{j} = \frac{1}{\sqrt{2}}(H_0^j + \sigma^i H_1^{ji}) \\ H_1^{ij} = \sum_{M,L,J,\mathcal{J},\mathcal{M}} P_{1\ \mathcal{J}\mathcal{M}}^{LJ}(r) Y_{\mathcal{J}\mathcal{M}}^{ij\ LJ}(\hat{\boldsymbol{r}}) \end{gathered} \tag{9}$$

Coupling two spins ½ will result in $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ for the total spin S, which are the labels of the sub scripts, $\mathcal{J} = J + S$ is the Total Angular Momentum, including the spin of the quarks, and $\mathcal{M}$ its third component. Now that we have the objects to work with, we need to know if the candidates for the Spectrum splitting agree with the discrete symmetries (5). The only energy contributions to the Lagrangian density, which we verified that satisfy the discrete symmetries, from [2] are:

$$\begin{gathered} i\varepsilon^{ijk} V^S(r) tr({H^i}^\dagger [\sigma^k, H^j]) \\ i\varepsilon^{ijk} V^L(r) tr({H^i}^\dagger L^k H^j) \end{gathered} \tag{10}$$

We ought to remember that these are not the leading terms, they come with a $1/m_Q$ factor (the correction order) that attenuates them. We will begin with the hyperfine term, the one working with the interactions between the spins of the gluon and quark. If we develop the product using (9) we get:

$$\begin{aligned} &i\varepsilon^{ijk} V^S(r) tr({H^i}^\dagger [\sigma^k, H^j]) \\ &\qquad = -2\varepsilon^{ijk} V^S(r) \left[{H_1^{il}}^\dagger H_1^{jr} \varepsilon^{lkr} \right] \end{aligned} \tag{11}$$

We can contract the Levi-Civitta symbol to get:

$$\varepsilon^{ijk}\varepsilon^{lkr} = -\left(\delta_l^i \delta_r^j - \delta_l^j \delta_r^i\right) \tag{12}$$

And now we get two terms into the Lagrangian density to integrate:

$$\mathcal{L} = 2V^S(r) \left[{H_1^{ii}}^\dagger H_1^{jj} - {H_1^{ij}}^\dagger H_1^{ji} \right] \tag{13}$$

The development of this two terms is, respectively:

$$\begin{aligned} &\int {H_1^{ii}}^\dagger H_1^{jj} d\Omega = \\ &\int d\Omega \sum_{\substack{L,L',J,J',M,M',\mathcal{J},\mathcal{M},\mathcal{M}' \\ \mu,\mu',\nu,\nu'}} (P_{1\ \mathcal{J}\mathcal{M}'}^{L'J'} Y_{L'}^{M'-\mu'})^\dagger P_{1\ \mathcal{J}\mathcal{M}}^{LJ} Y_L^{M-\mu} \\ &C(L'1J'; M'-\mu',\mu') C(J'1\mathcal{J}; \mathcal{M}'-\nu',\nu') \\ &C(L1J; M-\mu,\mu) C(J1\mathcal{J}; \mathcal{M}-\nu,\nu) {\chi_{\mu'}}^{i^*} {\chi_{\nu'}}^{i^*} {\chi_\mu}^j {\chi_\nu}^j \end{aligned} \tag{14}$$

These $C(J_1, J_2, J_3; M_1 M_2)$ are the Clebsch-Gordan coefficient defined on Appendix A and following the notation of [1]. This one will be easier to calculate, because the labels of each wavefunction are contracted, meaning that it is a scalar and the relative angular momentum will fulfil $L = \mathcal{J} \xrightarrow{yields} L = + \Rightarrow J = -$ (the notation for J is the same we used for L). The other:

$$\begin{aligned} &\int {H_1^{ij}}^\dagger H_1^{ji} d\Omega = \\ &\int d\Omega \sum_{\substack{L,L',J,J',M,M',\mathcal{J},\mathcal{M},\mathcal{M}' \\ \mu,\mu',\nu,\nu'}} (P_{1\ \mathcal{J}\mathcal{M}'}^{L'J'} Y_{L'}^{M'-\mu'})^\dagger P_{1\ \mathcal{J}\mathcal{M}}^{LJ} Y_L^{M-\mu} \\ &C(L'1J'; M'-\mu',\mu') C(J'1\mathcal{J}; \mathcal{M}'-\nu',\nu') \\ &C(L1J; M-\mu,\mu) C(J1\mathcal{J}; \mathcal{M}-\nu,\nu) {\chi_{\mu'}}^{i^*} {\chi_{\nu'}}^{j^*} {\chi_\mu}^j {\chi_\nu}^i \end{aligned} \tag{15}$$

Both calculations were so complicated that we had to use the application Mathematica to compute them. If we express this results in matrix representation:

$$\begin{pmatrix} V^{+++} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V^{+00} & 0 & V^{0+0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V'^{+--} + V^{+--} & 0 & V'^{00-} + V^{00-} & 0 & V'^{-+-} + V^{-+-} & 0 & 0 \\ 0 & V^{+0+} & 0 & V^{0++} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V'^{+-0} + V^{+-0} & 0 & V^{000} + V'^{000} & 0 & V^{-+0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V^{0--} & 0 & V^{-0-} & 0 \\ 0 & 0 & V'^{+-+} + V^{+-+} & 0 & V^{00+} & 0 & V'^{-++} + V^{-++} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V^{0-0} & 0 & V^{-00} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V^{---} \end{pmatrix} \begin{pmatrix} P_{1\ \mathcal{J}\mathcal{M}}^{++} \\ P_{1\ \mathcal{J}\mathcal{M}}^{+0} \\ P_{1\ \mathcal{J}\mathcal{M}}^{+-} \\ P_{1\ \mathcal{J}\mathcal{M}}^{0+} \\ P_{1\ \mathcal{J}\mathcal{M}}^{00} \\ P_{1\ \mathcal{J}\mathcal{M}}^{0-} \\ P_{1\ \mathcal{J}\mathcal{M}}^{-+} \\ P_{1\ \mathcal{J}\mathcal{M}}^{-0} \\ P_{1\ \mathcal{J}\mathcal{M}}^{--} \end{pmatrix} = H_{HF} \begin{pmatrix} P_{1\ \mathcal{J}\mathcal{M}}^{++} \\ P_{1\ \mathcal{J}\mathcal{M}}^{+0} \\ P_{1\ \mathcal{J}\mathcal{M}}^{+-} \\ P_{1\ \mathcal{J}\mathcal{M}}^{0+} \\ P_{1\ \mathcal{J}\mathcal{M}}^{00} \\ P_{1\ \mathcal{J}\mathcal{M}}^{0-} \\ P_{1\ \mathcal{J}\mathcal{M}}^{-+} \\ P_{1\ \mathcal{J}\mathcal{M}}^{-0} \\ P_{1\ \mathcal{J}\mathcal{M}}^{--} \end{pmatrix} \tag{16}$$

Where,

$V^{-++} = -2V_S \frac{1}{(\mathcal{J}+1)(2\mathcal{J}+1)}$ $\quad V^{-00} = 2V_S \frac{1}{\mathcal{J}}$

$V^{+++} = -2V_S$ $\quad V^{---} = -2V_S$

$V^{+00} = -2V_S(\frac{1}{\mathcal{J}+1})$ $\quad V^{+0+} = V^{0+0} = -2V_S(\frac{\sqrt{\mathcal{J}(\mathcal{J}+2)}}{\mathcal{J}+1})$

$V^{+--} = -2V_S(\frac{1}{\mathcal{J}(2\mathcal{J}+1)})$ $\quad V^{00+} = V^{-+0} = 2V_S(\frac{(2\mathcal{J}+3)}{(\mathcal{J}+1)\sqrt{4\mathcal{J}(\mathcal{J}+2)+3}})$

$V^{000} = -2V_S(1-\frac{1}{\mathcal{J}}+\frac{1}{\mathcal{J}+1})$ $\quad V^{+-+} = V^{-+-} = -2V_S(\frac{\sqrt{4\mathcal{J}(\mathcal{J}+1)-3}}{2\mathcal{J}+1})$

$V^{0++} = 2V_S(\frac{1}{\mathcal{J}+1})$ $\quad V^{+-0} = V^{00-} = -2V_S(\frac{1}{\mathcal{J}}\sqrt{\frac{2\mathcal{J}-1}{2\mathcal{J}+1}})$

$V^{0--} = 2V_S \frac{1}{\mathcal{J}}$ $\quad V^{0-0} = V^{-0-} = -2V_S(\frac{\sqrt{\mathcal{J}^2-1}}{\mathcal{J}})$

In this Matrix of the Hyperfine Structure, we can observe the splitting of the spectrum because of the quark spin coupling.

For the Fine Structure, which is also a correction on (7) and (8), we can make some physical observations. To begin with, we can see that the operator does not act on the quark spin state, it means, that this term cannot connect states with different values of it and, as a consequence, the energy contribution of the integral will be independent from the coupling state. Using (9) and the Pauli's trace properties we get:

$$i\varepsilon^{ijk}V^L(r)tr\left(H^{i\dagger}L^kH^j\right) = 2i\varepsilon^{ijk}V^L(r)\left[H_0^{i\dagger}L^kH_0^j + H_1^{il\dagger}L^kH_1^{jl}\right] \quad (18)$$

Working with the notation (3) because of the independence on S, we have:

$$\varepsilon^{ijk}\int H_0^{i\dagger}L^kH_0^j d\Omega = \varepsilon^{ijk}\sum_{J,J',L,L',M,M'} P^L_{0\,JM}{}^{\dagger} P^L_{0\,JM}\int d\Omega\ \mathcal{Y}^{L'}_{J'M'}{}^{i\dagger} L^k \mathcal{Y}^L_{JM}{}^{j} \quad (19)$$

With the relations on Appendix A and B and computing it with Mathematica we get:

$$\begin{pmatrix} -2V^L(J+2) & 0 & 0 \\ 0 & -2V^L & 0 \\ 0 & 0 & -2V^L(J-1)\end{pmatrix}\begin{pmatrix} P^+_{0\,JM} \\ P^0_{0\,JM} \\ P^-_{0\,JM}\end{pmatrix} = H_F\begin{pmatrix} P^+_{0\,JM} \\ P^0_{0\,JM} \\ P^-_{0\,JM}\end{pmatrix} \quad (20)$$

This element is not as interesting as the hyperfine one because as we can see, it keeps the degeneration due to the quark spin state because it's Hamiltonian does not depend on them. It just shifts the states on the diagonal.

$$\begin{gathered} V'^{+--} = 2V_S\left(1-\frac{2}{2\mathcal{J}+1}\right) \\ V'^{000} = 2V_S \\ V'^{-++} = 2V_S\left(1+\frac{2}{2\mathcal{J}+1}\right) \\ V'^{00-} = V'^{+-0} = -2V_S\sqrt{\frac{2\mathcal{J}-1}{2\mathcal{J}+1}} \\ V'^{-+-} = V'^{+-+} = 2V_S\left(\frac{\sqrt{4\mathcal{J}(\mathcal{J}+1)-3}}{2\mathcal{J}+1}\right) \\ V'^{-+0} = V'^{00+} = 2V_S\left(\frac{-(2\mathcal{J}+3)}{\sqrt{4\mathcal{J}(\mathcal{J}+2)+3}}\right) \end{gathered} \quad (17)$$

IV. Mass predictions of the Hyperfine term:

Finally, we can use the hyperfine splitting for one more thing. In particle physics, there are Mesons for example, that even being constituted by the same particles, they differ on their mass. Physicists, being aware that the Strong Interaction has a dependence on the spin, associated this difference on the mass to the coupling state of the spin i.e. that states on the same multiplet (for $S=1 \Rightarrow M_S = -1,0,1$ the triplet) will have different masses. The way to construct this is, at leading order without corrections (7) and (8) tells us that the states with $L=J$ does not mix with other values of it, but $L=J\pm1$ do. We have:

$$\begin{gathered} M_{1\ J} - M_{0\ J} = (H_{HF}(\mathcal{J}))_{J\,J}A \\ M_{1\ J} - M_{0\ J} = \left(H_{HF}(\mathcal{J})\right)_{J+1\,J}B + \left(H_{HF}(\mathcal{J})\right)_{J-1\,J}C \end{gathered} \quad (21)$$

The first expression will be useful for $L=J$, since it has just 1 structure constant because of being decoupled as we mentioned before; the second one has two because of the mixing between the terms $L=J\pm1$ on the Schrödinger Equation (the structure constants holds for the same multiplet). The term $\left(H_{HF}(\mathcal{J})\right)_{LJ}$ are the elements of the hyperfine matrix and, $M^J_{1\ \mathcal{J}\mathcal{M}} - M_{0\ JM}$ is the difference between the masses mentioned above. Now, we are ready to solve the system (each matrix component is divided by V_S):

$$\begin{Bmatrix} M_{1\ J-1} - M_{0\ J} = V^{0++}A \\ M_{1\ J+1} - M_{0\ J} = V^{0--}A \\ M_{1\ J} - M_{0\ J} = (V^{000}+V'^{000})A \end{Bmatrix} \quad (22)$$

$$\begin{Bmatrix} M_{1\ J-1} - M_{0\ J} = V^{+++}B + (V'^{-++} + V^{-++})C \\ M_{1\ J+1} - M_{0\ J} = (V'^{+--} + V^{+--})B + V^{---}C \\ M_{1\ J} - M_{0\ J} = V^{+00}B + V^{-00}C \end{Bmatrix} \quad (23)$$

And the relations we get respectively are:

$$\begin{Bmatrix} \dfrac{M_{1\ J-1}-M_{0\ J}}{M_{1\ J+1}-M_{0\ J}}=\dfrac{(J+1)}{J} \\ \dfrac{M_{1\ J-1}-M_{0\ J}}{M_{1\ J}-M_{0\ J}}=\dfrac{J(J+1)}{(2J(J+1)-1)J} \end{Bmatrix} \quad (24)$$

$$\begin{Bmatrix} \dfrac{M_{1\ J-1}-M_{0\ J}}{M_{1\ J}-M_{0\ J}}=(J+1) \\ \dfrac{M_{1\ J+1}-M_{0\ J}}{M_{1\ J}-M_{0\ J}}=-J \end{Bmatrix} \quad (25)$$

Here we have the relations between the masses of the different states as a function of the Total Angular Momentum. We must mention that the parameter that marks the multiplet we are is J not $\mathcal{J}$, because the spin quark coupling is on J.

V. Conclusions

First, we have seen that if we want to describe Mesons in a more general context, it is necessary to include the degree of freedom of the gluonic field. At leading order, the values of the Hamiltonian are mixed between the states $L = J \pm 1$, but decouple for $L = J$. It carries a degeneration coming from both quark spin, S, and rotation symmetry, M_J. This degeneration, holds for the fine correction, which shifts the energy states of $L = J$ and $L = J \pm 1$ but, the hyperfine term breaks it (the quark spin part). The hyperfine matrix, tells us that the perturbation comes in groups of 1 (upper left and lower right boxes) and a group of 7 (the middle one). We conclude, that the shifted states are the ones on the corners, plus 7 combinations coming from the middle box . Since $\mathcal{J} \geq 0$, the sign of V_S will mark if these states are going to be shifted by a positive or negative amount of energy. If the states with higher $\mathcal{J}$ are shifted positively, it will mean that we have a normal coupling; if they are shifted negatively as $\mathcal{J}$ increases, we are in front a reverse one.

For the differences of the mass of states in the same spin multiplet, we see from (24) and (25) that they must follow a relation between them, like the Coupling $\vec{L}\vec{S}$ follows the Landé formula.

VI. Appendix

A. Tensor Spherical Harmonics

The notation used during all the development follows [1] and the definitions are, for the spin vector basis:

$$\boldsymbol{\chi}_{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} \quad \boldsymbol{\chi}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (A1)$$

$$C(J_1, J_2, J_3; M_1 M_2)$$

$$Y_{\mathcal{J}M}^{ij\,LJ} = \sum_{\nu=-1}^{1} C(J1\mathcal{J}; \mathcal{M}-\nu, \nu)\,\mathcal{Y}_{\mathcal{J}\,\mathcal{M}-\nu}^{L\ \ i}\,\chi_\nu{}^j$$
$$\mathcal{Y}_{J\,M}^{L\ \ i} = \sum_{\mu=-1}^{1} C(L1J; M-\mu, \mu)\,Y_L^{M-\mu}\,\chi_\mu{}^i \qquad (A2)$$

The order in the Clebsch-Gordan coefficients is as follows, the two first angular momentums coupled to give the third one.

B. Relations between elements and operators

Here we present a large list of relations that will be useful during the essay from [1]:

$$\begin{Bmatrix} \hat{\boldsymbol{r}}\boldsymbol{\mathcal{Y}}_{JM}^{L} = -C(J1L; 000)\,Y_L^{M-\mu} \\ \hat{\boldsymbol{r}}Y_L^M = -\sum_{L'} C(L1L'; 00)\,\boldsymbol{\mathcal{Y}}_{LM}^{L'} \end{Bmatrix} \qquad (B1)$$

$$\begin{Bmatrix} \boldsymbol{\chi_\mu}^\dagger \boldsymbol{\chi_\nu} = \delta_{\mu\nu} \\ \boldsymbol{\chi_\mu} \wedge \boldsymbol{\chi_\nu} = i\sqrt{2}C(111; \mu\nu)\,\boldsymbol{\chi_{\mu+\nu}} \\ \boldsymbol{\chi_\mu}\boldsymbol{\chi_\nu} = -\delta_\mu^{-\nu} \end{Bmatrix} \qquad (B2)$$

$$\varepsilon^{ijk}\left(\chi_\mu{}^{i\dagger}\chi_\nu{}^j\chi_\sigma{}^k\right) = i\sqrt{2}C(111; \nu\sigma)\,\delta_{\nu+\sigma}^{\mu} \qquad (B3)$$

Where the sub index is related to the 3 Pauli's matrixes, not the components. A relation between $L \leftrightarrow \sigma$:

$$L^k = \sum_{\rho=-1}^{1} (-1)^\rho\,\chi_{-\rho}{}^k L_\rho$$
$$L_\pm\,|\boldsymbol{LM}> = \mp\sqrt{\frac{1}{2}[L(L+1) - M(M\pm 1)]}\,|\boldsymbol{LM}\pm\boldsymbol{1}> \qquad (B4)$$

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