A procedure to compute the nucleolus of the assignment game

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Abstract

The assignment game [6] is a model for a two-sided market where there is an exchange of indivisible goods for money and buyers or sellers demand or supply exactly one unit of the good. We give a procedure to compute the nucleolus of any assignment game, based on the distribution of equal amounts to the agents, until the game is reduced to less agents.

Keywords: Assignment game, core, nucleolus

1. Introduction

The main solution concept in cooperative game theory is the core. Its importance relies on the fact that any proposal of allocation within the core has certain stability since no subgroup of players can do better by splitting off. In case the core is large we can select an allocation in it. For assignment

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games, the core is always nonempty, and other well-known solutions, such as the Shapley value [5] are not usually in the core. The nucleolus [4] occupies a central position inside it, and has outstanding normative properties. In this paper we study how to compute the nucleolus of the assignment game, by giving the players in an egalitarian way certain amounts following a well-defined pace.

Assignment games were introduced by Shapley and Shubik [6] and describe bilateral markets, formed by a set of buyers, a set of sellers, and for every buyer and seller, a non-negative real number which is the potential profit obtained by them if they trade. The worth of any coalition is defined as the maximum profit obtained by matching buyers to sellers within the coalition, and an optimal matching of the market gives the maximum profit that the agents can obtain.

The core is defined as the set of allocations of the worth of the grand coalition such that no subcoalition can further improve upon. The nucleolus [4] is the unique core element that lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions. For assignment games Solymosi and Raghavan [7] provide an algorithm that computes the nucleolus of an arbitrary assignment game. Recently Llerena and Núñez [1] have characterized the nucleolus of a square assignment game from a geometric point of view, using the fact that the nucleolus is the midpoint for some specific segments inside the core. We use this geometric characterization to give a procedure to compute the nucleolus of an arbitrary square assignment game.

2. Preliminaries

An assignment market (M, M', A) is defined to be two disjoint finite sets: M, the set of buyers, and M', the set of sellers and a nonnegative matrix $A = (a_{ij})_{i\in M, j\in M'}$ which represents the profit obtained by each mixed-pair $(i,j) \in M \times M'$. To distinguish the j-th seller from the j-th buyer we will write the former as j' when needed. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers. The assignment market is called square whenever $|M| = |M'|$.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M_0' \subseteq M'$ such that $|M_0| = |M_0'| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. If for some buyer $i \in M$ there is no $j \in M'$ such that $(i,j) \in \mu$ we say that i is unmatched by μ and similarly for sellers. The set of all matchings from M to M' is represented by $\mathcal{M}(M, M')$. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for (M, M', A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for any $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}_A^*(M, M')$ the set of all optimal matchings. [6] associate any assignment market with a game in coalitional form $(M \cup M', w_A)$ called the *assignment game* where the worth of a coalition formed by $S \subseteq M$ and $T \subseteq M'$ is $w_A(S \cup T) = \max\limits_{\mu\in\mathcal{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$, and any coalition formed only by buyers or only by sellers gets zero.

The *core* of the assignment game, $C(w_A)$, is defined as those allocations $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ satisfying $u(M) + v(M') = w_A(M \cup M')$ and $u(S) + v(T) \geq w_A(S \cup T)$ for all $S \subseteq M$ and $T \subseteq M'$ where $u(S) = \sum_{i\in S} u_i$, $v(T) = \sum_{j\in T} v_j$, $u(\emptyset) = 0$ and $v(\emptyset) = 0$. It is always nonempty.

Given an optimal matching $\mu \in \mathcal{M}_A^*(M, M')$, the core of the assignment game can be easily described as the set of non-negative payoff vectors $(u, v) \in$

$\mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ such that

$$u_i + v_j \geq a_{ij} \text{ for all } i \in M,\ j \in M', \tag{1}$$

$$u_i + v_j = a_{ij} \text{ for all } (i,j) \in \mu, \tag{2}$$

and all agents unmatched by μ get a null payoff.

Now we define the nucleolus of an assignment game, taking into account that its core is always nonempty. Given an allocation in the core, $x \in C(w_A)$, define for each coalition $S \subseteq M \cup M'$ its excess as $e(S,x) := w_A(S) - \sum_{i \in S} x_i$. As it is known (see [7]) that the only coalitions that matter are the individual and mixed-pair ones, define the vector $\theta(x)$ of excesses of individual and mixed-pair coalitions arranged in a non-increasing order. Then the *nucleolus* of the game $(M \cup M', w_A)$ is the unique allocation $\nu(w_A) \in C(w_A)$ which minimizes $\theta(x)$ with respect to the lexicographic order over the set of core allocations. The lexicographic order $\geq_{lex}$ on $\mathbb{R}^d$, is defined in the following way: $x \geq_{lex} y$, where $x, y \in \mathbb{R}^d$, if $x = y$ or if there exists $1 \leq t \leq d$ such that $x_k = y_k$ for all $1 \leq k < t$ and $x_t > y_t$.

Llerena and Núñez [1] characterize the nucleolus of a square assignment game from a geometric point of view. The nucleolus is the unique core allocation that is the midpoint of some well-defined segments inside the core. To be precise they define the maximum transfer from a coalition to another coalition. Given any square assignment market (M, M', A), and two arbitrary coalitions of the same cardinality $\emptyset \neq S \subseteq M$, and $\emptyset \neq T \subseteq M'$, with $|S| = |T|$ they define:

$$\delta^A_{S,T}(u,v) := \min_{i \in S, j \in M' \setminus T} \{u_i, u_i + v_j - a_{ij}\}, \tag{3}$$

$$\delta^A_{T,S}(u,v) := \min_{j \in T, i \in M \setminus S} \{v_j, u_i + v_j - a_{ij}\}, \tag{4}$$

for any core allocation $(u,v) \in C(w_A)$.

It is easy to see that expression (3) represents the largest amount that can be transferred from players in S to players in T with respect to the core allocation (u,v) while remaining in the core, that is,

$$\delta^A_{S,T}(u,v) = \max\left\{\varepsilon \geq 0 \mid \left(u - \varepsilon 1^S, v + \varepsilon 1^T\right) \in C(w_A)\right\},$$

where 1^S and 1^T represent the characteristic vectors (for $S \subseteq \{1,\ldots,n\}, 1^S \in \mathbb{R}^n$ is such that $1^S_i = 1$, if $i \in S$, and zero otherwise) associated with coalition $S \subseteq M$ and $T \subseteq M'$, respectively.

Llerena and Núñez [1] prove that the nucleolus of a square assignment market is characterized as the unique core allocation $(u,v) \in C(w_A)$ such that

$$\delta^A_{S,T}(u,v) = \delta^A_{T,S}(u,v) \tag{5}$$

for any $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$. Notice that if $T \neq \mu(S)$ for some $\mu \in \mathcal{M}^*_A(M,M')$, then $\delta^A_{S,T}(u,v) = \delta^A_{T,S}(u,v) = 0$. Then, for this characterization we only check the case $T = \mu(S)$ for all optimal matchings. This is called the bisection property.

3. The procedure to compute the nucleolus of the assignment game

In this section we give a procedure to compute the nucleolus of an arbitrary assignment game. The main idea is to distribute some "dividends" to the players in such a way that we retain an assignment market, whose nucleolus gives the remaining worth to the agents. In it, we lower the entries in the matrix, until at least one optimal entry of some optimal matching is set to zero. Players involved in these entries will not receive any more dividends. In this way we associate a new game with, at least, one player less on each side.

If the assignment market is not square, we can add dummy players, i.e. null rows or columns, and compute its nucleolus. These players get zero at any core allocation, and in [3] is discussed that making the matrix square does not modify the nucleolus of the assignment game, if we drop at the end the null payoff to the added dummy agents. Therefore our procedure also applies to non-square assignment games.

The procedure we are going to present is based on two propositions. The first proposition is a direct consequence of the fact that the nucleolus of assignment games satisfies an adapted reduced game property [2]. It is stated without proof, because if one entry of some optimal matching is zero, agents optimally assigned get zero at all points of the core, and thus in the nucleolus.

Proposition 3.1. *Let* $A \in \mathrm{M}_m^+$ *be a square matrix and let* $\mu \in \mathcal{M}_A^*(M, M')$ *be an optimal matching such that* $a_{k\,\mu(k)} = 0$ *for some* $k \in M$*. Then, matrix* $A' \in \mathrm{M}_{m-1}^+$ *defined by:*

$$a'_{ij} = \max\{0, a_{ij} - a_{i\,\mu(k)} - a_{kj}\} \text{ for } i \in M \setminus \{k\} \text{ and } j \in M' \setminus \{\mu(k)\},$$

satisfies:

$$\begin{aligned} \nu_i(w_A) =& \nu_i(w_{A'}) + a_{i\,\mu(k)} \text{ for } i \in M \setminus \{k\}, \\ \nu_j(w_A) =& \nu_j(w_{A'}) + a_{kj} \text{ for } j \in M' \setminus \{\mu(k)\}, \text{ and} \\ \nu_k(w_A) =& \nu_{\mu(k)}(w_A) = 0. \end{aligned}$$

The second proposition states that decreasing all the entries in the assignment matrix following a determined pace, we could simplify the computation of the nucleolus. The decreasing rate of the matrix entries depends on whether they are part of an optimal matching or not. If they belong to optimal matching, the decreasing rate is twice the one corresponding to the entries that do not belong to an optimal matching. Once an entry has dropped to zero, it stays to zero.

To this end, we need some notation. Given a square assignment matrix $A \in \mathrm{M}_m^+$ we define the set of all entries that belong to some optimal matching,

$$H^A = \left\{(i,j) \in M \times M' \mid (i,j) \text{ belongs to some optimal matching in } A\right\}.$$

Consider now the minimum entry in matrix A that is in some optimal matching, and define

$$\alpha^A := \min\left\{\frac{a_{ij}}{2} \mid (i,j) \in H^A\right\}. \tag{6}$$

On the other hand, for $t \geq 0$, we introduce the following matrix A^t. Its entries are defined as:

$$a_{ij}^t = \begin{cases} \max\{0, a_{ij} - 2t\} & \text{for} \quad (i,j) \in H^A, \\ \max\{0, a_{ij} - t\} & \text{for} \quad (i,j) \notin H^A. \end{cases} \tag{7}$$

These entries a_{ij}^t are non-increasing functions of t, and for $t = 0, A^0 = A$. Now for each non-optimal matching, $\mu \in \mathcal{M}(M, M') \backslash \mathcal{M}_A^*(M, M')$, consider the following equation, in $t \geq 0$:

$$f_\mu^A(t) = w_A(M \cup M') - 2mt - \sum_{(i,j)\in\mu} a_{ij}^t = 0, \tag{8}$$

and denote $t_\mu^A \geq 0$ its unique solution. To see that this solution exists and is unique, note that since μ is not optimal, we have that $f_\mu^A(0) > 0$ because $w_A(M \cup M') > \sum_{(i,j)\in\mu} a_{ij}$; at $t = \infty$ the left-hand side is negative;

and function $f_\mu^A(t)$ is continuous, concave and piecewise linear with negative slope. Notice that if $t_\mu^A \leq \alpha^A$ we know that we will find at least two matchings in $A^{t_\mu^A}$ with the same worth. Then define

$$\beta^A := \min\left\{t_\mu^A \mid \mu \in \mathcal{M}\left(M, M'\right) \setminus \mathcal{M}_A^*\left(M, M'\right)\right\}. \tag{9}$$

Now we are in a position to state our second proposition.

Proposition 3.2. *Let $A \in \mathrm{M}_m^+$ be a square matrix and let α^A and β^A as in (6) and (9) Then, for each $\varepsilon \leq \min\{\alpha^A, \beta^A\}$, matrix $A^\varepsilon \in \mathrm{M}_m^+$ defined by (7) satisfies:*

$$\begin{aligned} \nu_i(w_A) =& \nu_i(w_{A^\varepsilon}) + \varepsilon \quad \textit{for } i \in M, \\ \nu_j(w_A) =& \nu_j(w_{A^\varepsilon}) + \varepsilon \quad \textit{for } j \in M'. \end{aligned}$$

Proof. Let $\nu(w_{A^\varepsilon}) = (u, v)$ be the nucleolus of the game w_{A^ε}. We must prove that $(u', v') = (u, v) + \varepsilon(1^M, 1^{M'})$ is the nucleolus of w_A.

Notice first that if $\mu^* \in \mathcal{M}_A^*(M, M')$ is an optimal matching for A then it is an optimal matching for A^ε. To see it, notice that since $\varepsilon \leq \alpha^A$, we have $\sum_{(i,j)\in\mu^*} a_{ij}^\varepsilon = w_A(M \cup M') - 2m\varepsilon$ and since $\varepsilon \leq \beta^A$, $w_A(M \cup M') - 2m\varepsilon \geq \sum_{(i,j)\in\mu} a_{ij}^\varepsilon$ for all $\mu \in \mathcal{M}(M, M') \setminus \mathcal{M}_A^*(M, M')$.

Fix one optimal matching $\mu \in \mathcal{M}_A^*(M, M')$, and notice that $(u', v') \in C(w_A)$ since for each $(i, j) \in M \times M'$ we have $u_i' + v_j' = u_i + v_j + 2\varepsilon \geq a_{ij}^\varepsilon + 2\varepsilon \geq a_{ij}$ and if $(i, j) \in \mu$ we have the equality. Now let $\emptyset \neq S \subseteq M$ be a coalition of buyers. We compute $\delta_{S,\mu(S)}^A(u', v') = \min\limits_{i\in S, j\in M'\setminus\mu(S)} \left\{u_i', u_i' + v_j' - a_{ij}\right\}$, and we distinguish two cases:

(a) if there exists $i \in S$ and $j \in M' \setminus \mu(S)$ such that $(i, j) \in H^A$ we have $\delta_{S,\mu(S)}^A(u', v') = 0$, and $\delta_{\mu(S),S}^A(u', v') = 0$,

(b) if for all $i \in S$ and $j \in M' \setminus \mu(S)$ we have $(i,j) \notin H^A$ we deduce

$$\begin{aligned}\delta^A_{S,\mu(S)}\left(u',v'\right) &= \min_{i \in S, j \in M' \setminus \mu(S)} \{u_i + \varepsilon, u_i + v_j + 2\varepsilon - a_{ij}\} \\ &= \varepsilon + \min_{i \in S, j \in M' \setminus \mu(S)} \{u_i, u_i + v_j - (a_{ij} - \varepsilon)\} \qquad (10)\end{aligned}$$

Notice now that, if $a_{ij} > \varepsilon$, we have $a_{ij} - \varepsilon = a^{\varepsilon}_{ij}$, and if $a_{ij} \leq \varepsilon$, we have $a^{\varepsilon}_{ij} = 0$ and $u_i + v_j - (a_{ij} - \varepsilon) \geq u_i + v_j \geq u_i$, and thus we can substitute in (10) expression $u_i + v_j - (a_{ij} - \varepsilon)$ by $u_i + v_j - a^{\varepsilon}_{ij}$, and we obtain:

$$\begin{aligned}\delta^A_{S,\mu(S)}\left(u',v'\right) &= \varepsilon + \min_{i \in S, j \in M' \setminus \mu(S)} \left\{u_i, u_i + v_j - a^{\varepsilon}_{ij}\right\} \\ &= \varepsilon + \delta^{A^{\varepsilon}}_{S,\mu(S)}\left(u,v\right).\end{aligned}$$

A similar argument proves that $\delta^A_{\mu(S),S}\left(u',v'\right) = \varepsilon + \delta^{A^{\varepsilon}}_{\mu(S),S}\left(u,v\right)$, and since (u,v) is the nucleolus of the game $w_{A^{\varepsilon}}$, we know that $\delta^{A^{\varepsilon}}_{S,\mu(S)}\left(u,v\right) = \delta^{A^{\varepsilon}}_{\mu(S),S}\left(u,v\right)$. Therefore it is clear that by (5) we have finished the proof. □

Remark 3.1. *Notice that, by applying Proposition 3.2, any optimal matching for A remains optimal for A^{ε} for $\varepsilon \leq \min\{\alpha^A, \beta^A\}$. Therefore in each step, for $\varepsilon = \min\{\alpha^A, \beta^A\}$, either we obtain at least one entry of an optimal matching equal to zero, and/or at least one more optimal matching.*

Remark 3.2. *If $\alpha^A < \beta^A$ we obtain, for $\varepsilon = \alpha^A$, that at least one entry in the optimal matching has been dropped to zero. If $\alpha^A > \beta^A$ we obtain, for $\varepsilon = \beta^A$, that at least we have another optimal matching.*

The iterated application of Proposition 3.1 and Proposition 3.2 increases the number of optimal matchings and/or reduces the number of players. In a finite number of steps we finish the procedure.

We will illustrate the application of the procedure to the non-square example in [7].

Example 3.1 (Section 7 in [7])**.** *Consider the following assignment market:* $M = \{1, 2, 3, 4\}$ *and* $M' = \{1', 2', 3', 4', 5'\}$*, and matrix*

$$A = \begin{pmatrix} 6 & 7 & 4 & 5 & 9 \\ 4 & 3 & 7 & 8 & 3 \\ 0 & 1 & 3 & 6 & 4 \\ 2 & 2 & 5 & 7 & 8 \end{pmatrix}.$$

In the first place, we add a dummy buyer, buyer 5, whose row is filled with zeroes. The optimal matching is denoted in boldface and by the boxes around the entries.

Therefore, the square matrix that we begin with is the following one:

$$A^{[0]} = \begin{pmatrix} 6 & \boxed{\mathbf{7}} & 4 & 5 & 9 \\ 4 & 3 & \boxed{\mathbf{7}} & 8 & 3 \\ 0 & 1 & 3 & \boxed{\mathbf{6}} & 4 \\ 2 & 2 & 5 & 7 & \boxed{\mathbf{8}} \\ \boxed{\mathbf{0}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 1: *Since there is one entry equal to zero in one optimal matching,* $a_{51} = 0$*, we apply Proposition 3.1, and players 5 and 1' leave the market.*

The new assignment market is $M = \{1, 2, 3, 4\}$ *and* $M' = \{2', 3', 4', 5'\}$ *and its matrix is:*

$$A^{[1]} = \begin{pmatrix} \boxed{\mathbf{1}} & 0 & 0 & 3 \\ 0 & \boxed{\mathbf{3}} & 4 & 0 \\ 1 & 3 & \boxed{\mathbf{6}} & 4 \\ 0 & 3 & 5 & \boxed{\mathbf{6}} \end{pmatrix}.$$

Recall that $a^{[1]}_{ij} = \max\{0, a_{ij} - a_{i1} - a_{5j}\}$*, for* $i = 1, 2, 3, 4$ *and* $j = 2, 3, 4, 5$.

Step 2: *Since there is no entry equal to zero in some optimal matching, we apply Proposition 3.2, and we distribute the players* $\varepsilon = \frac{1}{2}$ *which is exactly one half of the minimum entry in the unique optimal matching. In fact, in this case* $\beta^{A^{[1]}} = 1$ *and* $\min\{\alpha^{A^{[1]}}, \beta^{A^{[1]}}\} = \frac{1}{2}$. *The new assignment market is* $M = \{1, 2, 3, 4\}$ *and* $M' = \{2', 3', 4', 5'\}$ *and its matrix is:*

$$A^{[2]} = \begin{pmatrix} \boxed{\mathbf{0}} & 0 & 0 & 2\frac{1}{2} \\ 0 & \boxed{\mathbf{2}} & 3\frac{1}{2} & 0 \\ \frac{1}{2} & 2\frac{1}{2} & \boxed{\mathbf{5}} & 3\frac{1}{2} \\ 0 & 2\frac{1}{2} & 4\frac{1}{2} & \boxed{\mathbf{5}} \end{pmatrix}.$$

Notice that the optimal entries reduce their worth by $2\,\varepsilon = 2\,\frac{1}{2} = 1$, *whilst the non-optimal entries reduce their worth by* $\varepsilon = \frac{1}{2}$.

Step 3: *Since there is one entry equal to zero in one optimal matching, we apply Proposition 3.1, and players 1 and 2' leave the market.*

The new assignment market is $M = \{2, 3, 4\}$ *and* $M' = \{3', 4', 5'\}$ *and its matrix is:*

$$A^{[3]} = \begin{pmatrix} \boxed{\mathbf{2}} & 3\frac{1}{2} & 0 \\ 2 & \boxed{\mathbf{4\frac{1}{2}}} & \frac{1}{2} \\ 2\frac{1}{2} & 4\frac{1}{2} & \boxed{\mathbf{2\frac{1}{2}}} \end{pmatrix}.$$

Step 4: *Since there is no entry equal to zero in some optimal matching, we apply Proposition 3.2, and we must compute the limits in the statement of the proposition. In this case* $\alpha^{A^{[3]}} = 1$ *and* $\beta^{A^{[3]}} = \frac{1}{2}$. *Players receive* $\frac{1}{2}$ *and we obtain another optimal matching.*

The new assignment market is $M = \{2, 3, 4\}$ *and* $M' = \{3', 4', 5'\}$ *and its matrix is:*

$$A^{[4]} = \begin{pmatrix} \boxed{\mathbf{1}} & \boxed{\mathbf{3}} & 0 \\ \boxed{\mathbf{1\frac{1}{2}}} & \boxed{\mathbf{3\frac{1}{2}}} & 0 \\ 2 & 4 & \boxed{\mathbf{1\frac{1}{2}}} \end{pmatrix}.$$

Notice that this matrix has two optimal matchings.

Step 5: *Since there is no entry equal to zero in some optimal matching, we apply Proposition 3.2, obtaining* $\alpha^{A^{[4]}} = \frac{1}{2}$ *and* $\beta^{A^{[4]}} = \frac{1}{6}$*. Therefore we distribute* $\frac{1}{6}$ *to the players and we obtain several additional optimal matchings.*

The new assignment market is $M = \{2, 3, 4\}$ *and* $M' = \{3', 4', 5'\}$ *and its matrix is:*

$$A^{[5]} = \begin{pmatrix} \boxed{\mathbf{\frac{2}{3}}} & \boxed{\mathbf{2\frac{2}{3}}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{1\frac{1}{6}}} & \boxed{\mathbf{3\frac{1}{6}}} & 0 \\ \boxed{\mathbf{1\frac{5}{6}}} & \boxed{\mathbf{3\frac{5}{6}}} & \boxed{\mathbf{1\frac{1}{6}}} \end{pmatrix}.$$

Notice that we have obtained a new matrix with four optimal matchings.

Step 6: *Since there is one entry equal to zero in one optimal matching, we apply Proposition 3.1, and players 2 and 5' leave the market.*

The new assignment market is $M = \{3, 4\}$ *and* $M' = \{3', 4'\}$ *and its matrix is:*

$$A^{[6]} = \begin{pmatrix} \boxed{\mathbf{\frac{1}{2}}} & \boxed{\mathbf{\frac{1}{2}}} \\ \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \end{pmatrix}.$$

Step 7: *Since there are two entries equal to zero, one in each optimal matching, we apply Proposition 3.1, and all remaining players leave the market.*

In Table 3.1, we can see the payments to the agents in each step of the procedure. The sum of payments for each agent gives the nucleolus of game $w_{A^{[0]}}$*. The fact that a player is removed and leaves the market is denoted by a box.*

Since buyer 5 is a dummy player, which has been added at the beginning of the process to make square the original matrix, we can state the nucleolus

Table 1: The computation of the nucleolus of Example 3.1

Player	**1**	**2**	**3**	**4**	**5**	**1'**	**2'**	**3'**	**4'**	**5'**
Step 1	6	4	0	2	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	0	0	0	0
Step 2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Step 3	$\boxed{\mathbf{0}}$	0	$\frac{1}{2}$	0			$\boxed{\mathbf{0}}$	0	0	$2\frac{1}{2}$
Step 4		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$				$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Step 5		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$				$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Step 6		$\boxed{\mathbf{0}}$	0	$\frac{7}{6}$				$\frac{2}{3}$	$2\frac{2}{3}$	$\boxed{\mathbf{0}}$
Step 7			$\boxed{\frac{1}{2}}$	$\boxed{\mathbf{0}}$				$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	
TOTAL	$6\frac{1}{2}$	$5\frac{1}{6}$	$2\frac{1}{6}$	$4\frac{1}{3}$	0	0	$\frac{1}{2}$	$1\frac{5}{6}$	$3\frac{5}{6}$	$3\frac{2}{3}$

of the original example. The nucleolus is:

$$\nu(w_A) = \left(6\frac{1}{2},\ 5\frac{1}{6},\ 2\frac{1}{6},\ 4\frac{1}{3};\ 0,\ \frac{1}{2},\ 1\frac{5}{6},\ 3\frac{5}{6},\ 3\frac{2}{3}\right).$$

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