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Acyclicity in Algebraic K-theory

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ABSTRACT

The central topic of this work is the concept of acyclic spaces in topological K-theory and their analogues in algebraic K-theory. We start by describing topological K-theory and some basic results, such as representability by a spectrum. Next we discuss algebraic K-theory and some of its properties, including Swan's theorem, followed by the topological tools required to construct higher algebraic K-theory by means of Quillen's plus-construction. Finally, we describe a class of rings whose algebraic K-theory groups vanish in all dimensions. In fact each ring R admits a cone CR with $K_i(CR) = 0$ for all i and a suspension SR that is used to define negative K-theory groups of R in analogy with the topological case.

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Contents

Introduction

The core of what is now known as K-theory originated with the works of Whitehead and Grothendieck around 1950. The common denominator of both works is the study of algebraic invariants in their respective areas. Whitehead was interested in creating algebraic invariants that would allow to classify homotopy equivalences in topology, while Grothendieck was interested in generalizing the Riemann-Roch theorem in algebraic geometry. The distillation of ideas of Whitehead and Grothendieck and their application in the case of topological vector bundles resulted in topological K-theory, through the works of Atiyah, Bott and Hirzebruch. In algebra, the application of these ideas to projective modules and general linear groups led to the origins of algebraic K-theory around 1960.

Topological K-theory and algebraic K-theory are closely related, as we shall see. Our initial goal was to search for examples of acyclic rings, that is, rings R whose algebraic K-theory groups $K_i(R)$ vanish for all i. A subsequent research objective would be to compare the lattices of K^*-acyclics and K_*-acyclics both in topological K-theory and in algebraic K-theory, starting from results proved by Hovey [10] in the topological setting. This study probably requires background on motivic homotopy theory and goes beyond the scope of the present essay.

This work is divided in two parts. In Part I we discuss topological K-theory. The first chapter begins with basic definitions about vector bundles and we explain some of their properties, the basic operations between them and ways to construct them. We define the first topological K-theory group and show some of its properties. In the second chapter we define the negative K-theory groups both for reduced and unreduced K-theory. We also put into play the basic computational tools by means of long exact sequences of groups. Then we show that K-theory is a generalized cohomology theory according to the Eilenberg-Steenrod axioms. We also define the spectrum of complex K-theory and state Bott's periodicity theorem. Finally we compute the complex K-theory groups of some topological spaces like the real projective plane or the complex projective plane.

Part II consists of two chapters. In the first one we give the definition and basic properties of the Grothendieck and Bass-Whitehead groups for rings with unit; we provide a characterization of Grothendieck groups by means of idempotents, and compute K_0 for commutative and local rings. We also give a proof of the Serre-Swan theorem, that relates $K^0(X)$ and $K_0(C(X))$, where X is a compact Hausdorff space and $C(X)$ is the ring of continuous complex-valued functions on X. We finish this chapter by computing K_1 of fields. In the last chapter we give the construction of higher algebraic K-theory groups by means of Quillen's plus-construction, and compute the algebraic K-theory groups for finite fields. Next we define infinite sum rings and show that they are K-acyclic. In fact every ring R can be embedded into a K-acyclic ring CR, called its cone. In analogy with topological K-theory one can then define a ring SR, called suspension of R, which can be used to define negative algebraic K-theory groups.

Part I

Topological K-theory

Chapter 1

Vector Bundles and First Notions of K-theory

Vector bundles are the starting point for topological K-theory and they are generalizations of vector spaces. We begin describing the notion of a family of vector spaces. In this chapter we follow the expositions given in [3] and [7], but see also [9].

1.1 Definition and properties

Definition 1.1.1. Let X be a topological space. A *family of (complex) vector spaces* over X is a topological space E, together with

(i) a continuous map $p : E \longrightarrow X$;

(ii) a (complex) vector space structure on each $E_x := p^{-1}(x)$, compatible with the topology on E.

Such a family is denoted by $\xi = (E, p, X)$. The map p is called the *projection map*, E is called the *total space*, and E_x is called the *fiber* of ξ at the point x. Most of the time we shall simply refer to the family E, letting the rest of the data be implicit.

Compatibility just means that multiplication by scalars $\mathbb{C} \times E \longrightarrow E$ and addition $E \times_X E \longrightarrow E$ are continuous, where $E \times_X E := \{(e_1, e_2) \in E \times E : p(e_1) = p(e_2)\}$. Notice also that we can add elements only if they lie on the same fiber. The dimension of E_x is called the *rank* of the family at x, and will be denoted by $\operatorname{rank}_x(E)$. The rank of E is defined as

$$\operatorname{rank}(E) := \sup\{\operatorname{rank}_x(E) : x \in X\}.$$

Example 1.1.2.

- Consider $E = X \times \mathbb{C}^n$ together with the projection map $X \times \mathbb{C}^n \longrightarrow X$. This is the *trivial family of rank* n, and when the space X is understood, we will denote it simply by $\underline{n}$

- Consider $X = \mathbb{C}$ and let e_1, e_2 be the standard basis for $\mathbb{C}^2$. Let

$$E = \{(x, ze_1) \mid x \in \mathbb{Q}, t \in \mathbb{C}\} \cup \{(x, ze_2) \mid x \in X \backslash \mathbb{Q}, z \in \mathbb{C}\} \subseteq X \times \mathbb{C}^2.$$

  By the preceding example, $X \times \mathbb{C}^2 \longrightarrow X$ is a family of vector spaces, and E becomes a sub-family of vector spaces under the same operations.

A morphism of families is, roughly speaking, a map that preserves fibers. The next definition states it precisely.

Definition 1.1.3. A *morphism* from one family $p\colon E \longrightarrow X$ to another family $p'\colon E' \longrightarrow X$ is a continuous map $\varphi\colon E \longrightarrow E'$ that satisfies

(i) $p' \circ \varphi = p$;

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

(ii) for each $x \in X$ the induced maps $\varphi_x : E_x \longrightarrow E'_x$ are linear.

We say that φ is an *isomorphism* if it is bijective and φ^{-1} is continuous, and that E and E' are *isomorphic* if there exists an isomorphism φ between them.

If Y is a subspace of X, we write $E|_Y$ for the restriction $p^{-1}(Y)$. Clearly the restriction $p|_Y\colon E|_Y \longrightarrow Y$ is a family over Y. We shall denote it by $E|_Y$ and call it the restriction of E to Y. This can be seen as a family induced by the inclusion $i : Y \longrightarrow X$ and the next definition generalizes this for any continuous map from Y to X.

Definition 1.1.4. Given a family (E, p, X) and a continuous map $f : Y \longrightarrow X$, the *induced family* $(f^*E, f^*(p), Y)$ is given by f^*E as the subspace of $Y \times E$ consisting of all points (y, e) such that $f(y) = p(e)$, with addition and multiplication given by $(y, e) + (y, e') = (y, e + e')$ and $r \cdot (y, e) = (y, re)$.

If the map f in the above definition is an inclusion map, then sending each $e \in E$ into the corresponding $(p(e), e)$ we have a map that is clearly an isomorphism $E|_Y \cong f^*(E)$. If f is not an inclusion map, given $y \in Y$, there is a natural map of vector spaces $(f^*E)_y \longrightarrow E_{f(y)}$ which is an isomorphism.

Definition 1.1.5. A *(complex) vector bundle* over X is a family of vector spaces $p\colon E \longrightarrow X$ such that every point $x \in X$ has a neighborhood $U \subseteq X$, an $n \in \mathbb{Z}_{\geqslant 0}$, and an isomorphism of families of vector spaces

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "p"'] & U \times \mathbb{C}^n \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$

The isomorphism in the above diagram is called a *local trivialization.* Usually one simply says that a vector bundle is a family of vector spaces that is *locally trivial.* A vector bundle of constant rank n will be denoted by $\xi : \mathbb{C}^n \longrightarrow E \xrightarrow{p} X$.

The n appearing in the previous definition depends on the point x. It is called the *rank* of the vector bundle at x. The rank is constant on the connected components of X. Vector bundles of rank 1 are often called *line bundles.*

We will assume throughout that our base spaces are connected. If $X = \bigsqcup_\alpha X_\alpha$ is disjoint union of path components, then a vector bundle E over X is by definition a collection of vector bundles E_α over each X_α, and the rank of each E_α may be different. Assuming that our base spaces are connected simplifies the discussions, and all the arguments can be extended to the non-connected case in a straightforward way.

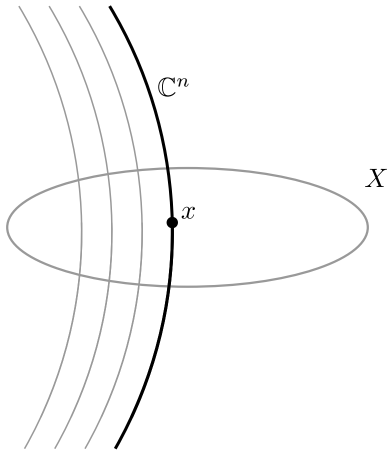

Figure 1.1: Vector Bundle

Example 1.1.6. The most important example of a vector bundle is the tangent bundle $\pi : TM \longrightarrow M$ of a smooth manifold.

Definition 1.1.7. A *morphism of vector bundles* is just a morphism of the underlying families of vector spaces. The induced family of a vector bundle is called the *pullback bundle*. The category of vector bundles is denoted **Vect**, and the category of vector bundles over a fixed base space X is denoted by $\mathbf{Vect}_X$.

Proposition 1.1.8. *Given a morphism φ of vector bundles*

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

Then φ is an isomorphism if and only if it is a linear isomorphism on each fiber, i.e.

$$\varphi|_{E_x} : E_x \xrightarrow{\cong} E'_x \qquad \textit{for all } x \in X$$

Proof. If φ has an inverse φ^{-1}, it restricts to an isomorphism on each fiber. Conversely, suppose that $E = X \times \mathbb{C}^n$ and $E' = X \times \mathbb{C}^m$ are trivial vector bundles and that $\varphi : E \to E'$ restricts to an isomorphism on each fiber. By the exponential law for spaces, we have homemorphisms of spaces (with respect to the compact-open topology)

$$\mathrm{map}(X \times \mathbb{C}^n, X \times \mathbb{C}^m) \cong \mathrm{map}(X \times \mathbb{C}^n, \mathbb{C}^m) \cong \mathrm{map}(X, \mathrm{map}(\mathbb{C}^n, \mathbb{C}^m))$$

where the left-hand side denotes maps over X. When we restrict attention to morphisms of vector bundles on the left-hand side, we get a homeomorphism

$\square$

1.2 Sections

Definition 1.2.1. Given a family of vector spaces $p : E \longrightarrow X$, a *section* of p is a map $s : X \longrightarrow E$ such that $p \circ s = id_X$, the identity map of X. Hence, a section assigns to each $x \in X$ a vector in the fiber $p^{-1}(x)$. The set of sections is denoted $\Gamma(E)$, an it becomes a vector space using pointwise addition and multiplication in the fibers of E. A collection of sections $s_1, \ldots, s_k$ is *linearly independent* if the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent in E_x for every $x \in X$.

Since vector bundles are locally trivial families of vector spaces, a section of a vector bundle can be described locally by a vector valued function on the base space. Given a family of vector spaces, it will be useful to have some criterion to decide when it is trivial

Proposition 1.2.2. *Let $p : E \longrightarrow X$ be a family of vector spaces of constant rank n. Then the family is trivial if and only if there is a linearly independent collection of sections $s_1, \ldots, s_n$.*

Proof. It is clear that $X \times \mathbb{C}^n$ has such sections, and any vector bundle isomorphism takes linearly independent sections to linearly independent sections. Conversely, if $s_1, \ldots, s_k$ are linearly independent sections of $p : E \longrightarrow X$, then the map

$$\varphi : X \times \mathbb{C}^n \longrightarrow E \tag{1.1}$$

$$(x, \lambda_1, \ldots, \lambda_n) \longmapsto \sum \lambda_i s_i(x) \tag{1.2}$$

is an isomorphism on each fiber and hence an isomorphism of vector bundles. □

Given a map $\varphi : E \longrightarrow E'$ of vector bundles over X, then neither Ker φ nor Coker φ are necessarily vector bundles. For example, let $X = [-1, 1]$ and let $E = \underline{1}$. Define $\varphi : E \longrightarrow E$ as multiplication by t on the fibers E_t.

1.3 Operations on vector bundles

There are several canonical constructions that can be applied to vector spaces, and we expect to extend almost all of them to vector bundles. As we shall see it is not hard to do this, the only subtle part is how to define the topology that the resulting vector bundles should carry. We will define these topologies locally, and check continuity also locally.

Direct Sum

Given two vector bundles $\xi : \mathbb{C}^n \longrightarrow E_1 \xrightarrow{p_1} X$ and $\xi' : \mathbb{C}^m \longrightarrow E_2 \xrightarrow{p_2} X$, we consider the subspace

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 : p_1(e_1) = p_2(e_2)\}$$

of $E_1 \times E_2$, together with the map $p_\oplus : E_1 \oplus E_2 \longrightarrow X$ defined as $p_\oplus(e_1, e_2) = p_1(e_1)$. It is easy to check that we have a vector bundle $\xi \oplus \xi' : \mathbb{C}^n \oplus \mathbb{C}^m \longrightarrow E_1 \oplus E_2 \xrightarrow{p_\oplus} X$. Given local trivializations $\phi_1 : p_1^{-1}(U) \longrightarrow U \times \mathbb{C}^n$ and $\phi_2 : p_2^{-1}(V) \longrightarrow V \times \mathbb{C}^m$, the map $p_1 \times p_2 : E_1 \times E_2 \longrightarrow X \times X$ is a vector bundle because $\phi_1 \times \phi_2 : p_1^{-1}(U) \times p_2^{-1}(V) \longrightarrow U \times V \times X \times X$ is a local trivialization for $E_1 \times E_2$, and hence $p_1 \times p_2$ is the pullback along the diagonal map $\delta : X \longrightarrow X \times X$. We will call this operation the direct sum or internal Whitney sum of E_1 and E_2. The rules for vector addition and multiplication are the evident ones. Notice also that the fiber of $E_1 \oplus E_2$ over a point x is simply $(E_1)_x \oplus (E_2)_x$.

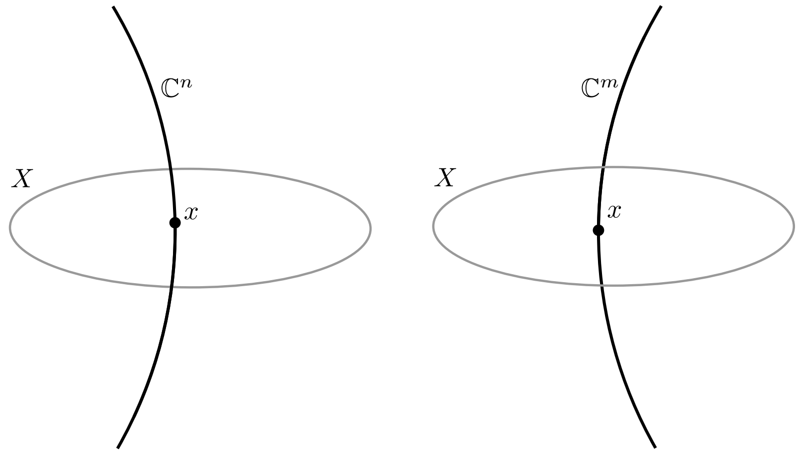

Figure 1.2: Direct sum

External Whitney sum

Given vector bundles $\xi : \mathbb{C}^n \longrightarrow E \xrightarrow{p} X$ and $\eta : \mathbb{C}^m \longrightarrow F \xrightarrow{q} Y$, we consider $E \boxplus F := E \times F$ as a topological space with the product topology, and the map $p \boxplus q : E \boxplus F \longrightarrow X \times Y$ given by $(p \boxplus q)(e, f) = (p(e), q(f))$. In this way we have a vector bundle $\xi \boxplus \eta : \mathbb{C}^n \oplus \mathbb{C}^m \longrightarrow E \boxplus F \xrightarrow{p} X \times Y$ whose fiber over an element (x, y) is $E_x \oplus F_y$.

Tensor product

Given two vector bundles $\xi : \mathbb{C}^n \longrightarrow E \xrightarrow{p} X$ and $\xi' : \mathbb{C}^m \longrightarrow E' \xrightarrow{p'} X$, consider the set

$$E \otimes E' = \{(x, v) \mid x \in X, v \in E_x \otimes E'_x\}$$

Is clear how to the define addition and scalar multiplication in the fibers, but it is not so clear how to topologize $E \otimes E'$. Given $x \in X$ and U a neighborhood of x over which both are trivializable. We choose local trivializations $\phi : U \times \mathbb{C}^n \longrightarrow E|_U$ and $\phi' : U \times \mathbb{C}^m \longrightarrow E'|_U$. We extend by linearity the map $\Phi : U \times (\mathbb{C}^n \times \mathbb{C}^m) \longrightarrow (E \times E')|_U$, given by sending $(u, v \otimes w) \mapsto (u, \phi(u, v) \otimes \phi'(u, w))$. This map is is bijective, since local trivializations are isomorphisms, and gives linear isomorphisms when restricted to each fiber. Finally we give $(E \times E')|_U$ the topology induced by this map, with this topology the vector space operations are continuous.

1.4 Transition functions

Definition 1.4.1. Given a vector bundle $\xi : \mathbb{C}^n \longrightarrow E \xrightarrow{p} X$ with local trivializations

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[dr, "p"'] & U_\alpha \times \mathbb{C}^n \arrow[d, "pr_1"] \\ & U_\alpha \end{tikzcd}$$

that restrict to vector space isomorphisms $h_\alpha|_{E_x} : E_x \xrightarrow{\cong} \{x\} \times \mathbb{C}^n$. A *transition function* is a map

$$f_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(\mathbb{C}^n)$$

given by $f_{\beta\alpha}(x) := h_\beta|_{E_x}(h_\alpha|_{E_x})^{-1}$.

Remark 1.4.2. $\mathrm{GL}(\mathbb{C}^n)$ is a topological space with the topology inherited as a subspace of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.

Given trivializing open sets U_α, U_β and U_γ and the associated transition functions, on the triple intersection they satisfy $f_{\gamma\beta} \circ f_{\beta\alpha} = f_{\gamma\beta}$, this is known as the *cocycle condition*, to see this one just has to consider the following diagram

$$\mathbb{C}^n \xrightarrow{h_\alpha^{-1}} E_x \xrightarrow{h_\beta} \mathbb{C}^n \xrightarrow{h_\beta^{-1}} E_x \xrightarrow{h_\gamma} \mathbb{C}^n$$

Proposition 1.4.3. *Given an open cover $\{U_\alpha\}$ of a connected topological space X, assume we are given maps*

$$f_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(\mathbb{C}^n)$$

that satisfy the cocycle condition. Then there is a complex vector bundle $\xi : \mathbb{C}^n \longrightarrow E \xrightarrow{p} X$ with transition functions $f_{\beta\alpha}$.

Proof. Define $E = (\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n)/\sim$, where for every $x \in U_\alpha \cap U_\beta$ we have $(x, v) \sim (x, w)$ if and only if $w = f_{\beta\alpha}(x)(v)$. The cocycle condition implies that $f_{\beta\alpha} = f_{\beta\alpha}^{-1}$. Thus, if $(x, v) \sim (x, w)$, $v = f_{\alpha\beta}(x)(w)$, so $\sim$ is symmetric. Transitivity follows in a similar way and thus $\sim$ is an equivalence relation. Define $p : E \longrightarrow X$ by $p([x, v]) = x$. Then the map $U_\alpha \times \mathbb{C}^n \longrightarrow \bigsqcup_\alpha U_\alpha \times \mathbb{C}^n \longrightarrow E$ can be factored as

$$\begin{tikzcd} U_\alpha \times \mathbb{C}^n \arrow[r] \arrow[dr] & E \arrow[d] \\ & E|_{U_\alpha} \end{tikzcd}$$

And $p : E \longrightarrow X$ is vector bundle with transition functions $f_{\alpha\beta}$. ☐

1.5 Paracompact spaces

We will summarize some results about paracompact spaces that we will need in the next section to classify vector bundles.

Definition 1.5.1. A Hausdorff space X is *paracompact* if for each open cover $\{U_\alpha\}$ of X there is a partition of unity $\{\varphi_\beta\}$ subordinated to it, i.e., there are maps $\varphi_\alpha : X \longrightarrow I$ that satisfy the following:

(i) Each φ_α has its support contained in some U_β

(ii) Each $x \in X$ has a neighborhood in which only finitely many φ_β are nonzero

(iii) $\sum_\beta \varphi_\beta = 1$

Compact Hausdorff spaces, CW-complexes and metric spaces are all examples of paracompact spaces.

Definition 1.5.2. An open cover $\{U_\alpha\}$ is *locally finite* if for any $x \in X$ there is an open neighbourhood V_x such that each V_β is a disjoint union of open sets, each contained in some U_α

Theorem 1.5.3. *A space X is paracompact if and only if it is a Hausdorff space and every open cover has a locally finite open refinement.*

Proof. See [9, p. 35]. □

Lemma 1.5.4. *Let X be a paracompact space. If $\{U_\alpha\}$ is an open cover, there is a countable open cover $\{V_\beta\}$ such that each V_β is a disjoint union of open sets and it is contained in some $\{U_\alpha\}$.*

Proof. See [9, p. 37]. □

1.6 Classification of vector bundles

Definition 1.6.1. The set of n-dimensional vector subspaces of $\mathbb{C}^k$ is called the Grasmmanian, and we denote it by $\mathrm{Gr}_n(\mathbb{C}^k)$. It is a topological space with the quotient topology given by

$$\begin{aligned} \mathbb{C}^k \times \cdots \times \mathbb{C}^k &\longrightarrow \coprod_{0 \leqslant n \leqslant k} \mathrm{Gr}_n(\mathbb{C}^k) \\ (v_1, \ldots, v_n) &\longmapsto \langle v_1, \ldots, v_n \rangle. \end{aligned}$$

We also denote by $BU(n) := \mathrm{Gr}_n(\mathbb{C}^\infty) := \bigcup_{k \geqslant n} \mathrm{Gr}_n(\mathbb{C}^k)$, and define

$$EU(n) := \{(V, v) \mid V \subset \mathbb{C}^k \text{ for some } k, \dim_{\mathbb{C}} V = n, v \in V\}.$$

The notation $EU(n)$ comes from the unitary group

$$U(n) = \{\text{matrices } B \in M(n, \mathbb{C}) \mid B\bar{B}^t = Id\}.$$

$U(n)$ is a compact Lie group and the colimit along the inclusion maps $A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, denoted by $U = \mathrm{colim}_n\, U(n) = \bigcup_{n \geqslant 1} U(n)$, is an infinite-dimensional topological group.

We have a vector bundle $\gamma_n^{\mathbb{C}}$:

$$\begin{aligned} \gamma_n^{\mathbb{C}} : \mathbb{C}^n \longrightarrow EU(n) &\xrightarrow{p} BU(n) =: \mathrm{Gr}_n(\mathbb{C}^\infty) \\ (V, v) &\longmapsto V \end{aligned}$$

The vector bundle $\gamma_n^{\mathbb{C}}$ is called the *tautological vector bundle* or also *universal bundle*. Every complex vector bundle of rank n is a pullback of $\gamma_n^{\mathbb{C}}$

$BU(n)$ is called the *classifying space* of complex vector bundles of rank n.

Proposition 1.6.2. *Let $p\colon E \to X$ be a (complex) vector bundle of rank n, where X is paracompact. There exists a map $f\colon X \to BU(n)$ and an isomorphism of vector bundles over X so that $E \cong f^*EU(n)$*

Proof. We can assume that $p : E \longrightarrow X$ has trivializations $\varphi_\alpha\colon p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times V$ with $\{U_\alpha\}$ locally finite and countable. Let $h_\alpha\colon X \longrightarrow [0,1]$ be a partition of unity with respect to $\{U_\alpha\}$ and define $g_\alpha\colon E \longrightarrow V$ by $g_\alpha|_{p^{-1}(U_\alpha)} = (h_\alpha \circ p)\cdot(\pi_2 \circ \varphi_\alpha)$. , where $\pi_2\colon U_\alpha \times V \longrightarrow V$ is the projection map, and $g_\alpha = 0$ else. The map g_α is continuous since $\overline{h_\alpha^{-1}(0,1]} \subseteq U_\alpha$. Choose an isomorphism $\Sigma_\alpha V \cong \mathbb{C}^\infty$ (I is countable) and define $g = \sum_\alpha g_\alpha\colon E \longrightarrow \Sigma_\alpha V \cong \mathbb{C}^\infty$. Then g is well defined since $\{U_\alpha\}$ is locally finite. We now claim that g maps each $EU(x)$ isomorphically onto V. This is so since if $h_\alpha(x) \neq 0$ then for any $e \in EU(x)$, $g(e) = \Sigma_\alpha g_\alpha(e) = (\Sigma_\alpha h_\alpha(x))\cdot \pi_2\varphi_\alpha(e) = \pi_2(\varphi_\alpha(e)) \in V$. Define $f\colon B \longrightarrow BU(n)$ via $f(b) = g(EU(x))$. We consider the pullback

$$\begin{tikzcd}
f^*(EU(n)) \arrow[r] \arrow[d] & EU(n) \arrow[d] \\
X \arrow[r] & BU(n)
\end{tikzcd}$$

□

Then $f^*(EU(n))$ consists of triples (x, V, v) such that g maps $EU(x)$ isomorphically onto $V \subseteq \mathbb{C}^\infty$. Thus, the map $E \longrightarrow f^*(EU(n))$ given by the isomorphism $g\colon EU(x) \xrightarrow{\cong} V$ on every fiber E_x is an isomorphism of vector bundles.

We need the following lemmas

Lemma 1.6.3. *Let X be paracompact. A vector bundle $p\colon E \longrightarrow X \times I$ whose restrictions over $X \times [0,t]$ and over $X \times [t,1]$ are trivial is trivial as well.*

Proof. Let $h_0\colon E_0 := E|_{(X\times[0,t])} \xrightarrow{\cong} X\times[0,t]\times V$ and $h_1\colon E_1 := E|_{(X\times[t,1])} \xrightarrow{\cong} X\times[t,1]\times V$ be isomorphism to trivial bundles. The maps h_0 and h_1 may not agree on $E|_{(X\times\{t\})}$ so we cannot glue yet them. Define an isomorphism $h_{01}\colon X\times[t,1]\times V \longrightarrow X\times[t,1]\times V$ by duplicating the map $h_0h_1^{-1}\colon X\times\{t\}\times V$ on each slice $X\times\{s\}\times V$ for $t \leqslant s \leqslant 1$, and set $\overline{h}_1 := h_{01}h_1$. Then $\overline{h}_1$ is an isomorphism of bundles and agrees with h_0 on $E|_{(X\times\{t\})}$. We can now glue together h_0 and h_1 to get the desired. □

Lemma 1.6.4. *For every vector bundle $p\colon E \longrightarrow X \times I$ there is an open cover $\{U_\alpha\}$ such that each restriction $E|_{(U_\alpha\times I)} \longrightarrow U_\alpha \times I$ is trivial.*

Proof. For each $x \in X$, take open neighbourhoods U_x with $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $E|_{U_x\times[t_{i-1},t_i]} \longrightarrow U_x \times [t_{i-1}, t_i]$ is trivial. This can be done since for each (x,t) we can find an open neighbourhood of the form $U_x \times J_t$, where J_t is an open interval, over which E is trivial; if we then fix x then the collection $\{J_t\}$ covers I and we can take a finite subcover $J_1, \ldots, J_{k+1}$ and choose $t_i \in J_i \cap J_{i+1}$; this way E remains trivial over $U_x \times [t_{i-1}, t_i]$. Now, by Lemma 1.6.3, E is trivial over $U_x \times I$. □

Theorem 1.6.5. *Let X be paracompact and let $p\colon E \longrightarrow X \times I$ be a vector bundle. Then $E|_{X\times\{0\}} \cong E|_{X\times\{1\}}$.*

Proof. By Lemma 1.6.4, take an open cover $\{U_\alpha\}$ of X such that $E|_{(U_\alpha \times I)}$ is trivial. Assume first that X is compact. Then we can take a cover of the form $\{U_i\}_{i=1}^n$. Take a partition of unity $\{h_i \colon X \longrightarrow I\}_{i=1}^n$ subordinated to $\{U_i\}$. For $i \geqslant 0$, set $g_i = h_1 + \cdots + h_i$ where $g_0 = 0$ and $g_n = 1$, let $X_i = \text{Graph}(g_i) \subseteq X \times I$ be the graph of g_i and let $p_i \colon E_i \longrightarrow X_i$ be the restriction of E to X. The map $X_i \longrightarrow X_{i-1}$ given by $(x, g_i(x)) \longmapsto (x, g_{i-1}(x))$ is a homeomorphism, and since $E|_{(U_i \times I)}$ is trivial,

$$\begin{tikzcd}
E|_{X_i \cap (U_i \times I)} \arrow[r, dashed, "\cong"] \arrow[d] & E|_{X_{i-1} \cap (U_i \times I)} \arrow[d] \\
X_i \cap (U_i \cap I) \arrow[r, "\cong"] & X_{i-1} \cap (U_i \times I)
\end{tikzcd}$$

the dashed arrow in the above diagram exists. Since outside U_i , h_i equals zero and $E|_{X_i \cap U_i^c} = E|_{X_{i-1} \cap U_i^c}$, we obtain an isomorphism of vector bundles over different base spaces

$$f_i \colon E|_{X_i} \xrightarrow{\cong} E|_{X_i}.$$

□

The composition $f = f_1 \circ \cdots \circ f_n$ is then an isomorphism from $E|_{X_n} = E|_{X \times \{1\}}$ to $E|_{X_0} = E|_{X \times \{0\}}$. Assume now X is paracompact. Take a countable cover $\{V_i\}_i$ such that each V_i is a disjoint union of open sets, each of them contained in some U_α. This means that E is trivial over each $V_i \times I$. Let $\{h_i \colon X \longrightarrow I\}$ be a partition of unity subordinated to $\{V_i\}$, and set as before $g_i = h_1 + \cdots + h_i$ and $p_i \colon E_i \longrightarrow X_i := \text{Graph}(g_i)$ the restriction. As before we obtain isomorphisms $f_i \colon E_i \xrightarrow{\cong} E_{i+1}$. The infinite composition $f = f_1 \circ f_2 \circ \cdots$ is well defined since for every point, almost all f_i's are the identity. As before f is an isomorphism from $E|_{X \times \{1\}}$ to $E|_{X \times \{0\}}$.

Corollary 1.6.6. *A homotopy equivalence of paracompact spaces $f \colon X \longrightarrow Y$ induces a bijection $f^* \colon \mathrm{Vect}^n_{\mathbb{C}} Y \xrightarrow{\cong} \mathrm{Vect}^n_{\mathbb{C}} X$*

Proof. If g is a homotopy inverse of f, then $f^* \circ g^* = id^* = id$ and $g^* \circ f^* = id^* = id$. □

Theorem 1.6.7. *Let X be paracompact. Then the pullback along $\gamma_n^{\mathbb{C}} \colon EU(n) \longrightarrow BU(n)$ induces a bijection*

$$\begin{aligned}
[X, BU(n)] &\xrightarrow{\cong} \mathrm{Vect}_n^{\mathbb{C}} X \\
[f] &\longmapsto f^* EU(n)
\end{aligned}$$

Definition 1.6.8. The vector bundle $\gamma_n^{\mathbb{C}} \colon \mathbb{C}^n \longrightarrow EU(n) \longrightarrow BU(n)$ is called the *universal vector bundle of rank n*

The universal vector bundle admits an inner product, induced from an inner product on $\mathbb{C}^\infty$. Since every vector bundle of rank n is obtained as a pullback along $\gamma_n^{\mathbb{C}}$, we deduce that any vector bundle admits an inner product which is obtained by pulling back the one on $\gamma_n^{\mathbb{C}}$.

Proposition 1.6.9. *Let X be a paracompact space. Then any n-dimensional bundle can be embedded in a trivial infinite dimensional bundle.*

Proof. See [9, p. 29] □

Corollary 1.6.10. *If X is compact Hausdorff, any n-dimensional vector bundle can be embedded in a trivial (finite dimensional) vector bundle.*

Proof. For $k > n$,

$$\mathrm{Gr}_n(\mathbb{C}^k) \subseteq \mathrm{Gr}_n(\mathbb{C}^{k+1}) \subseteq \cdots \subseteq \bigcup_{k>n} \mathrm{Gr}_n(\mathbb{C}^k) = BU(n).$$

Since X is compact, the classifying map $X \longrightarrow BU(n)$ factors as

$$X \xrightarrow{\overline{f}} \mathrm{Gr}_n(\mathbb{C}^d) \xrightarrow{i} \mathrm{Gr}_n(\mathbb{C}^d) = BU(n)$$

and by Lemma 1.6.3 for pullbacks we get that $f^*EU(n) \cong f'^*EU(n,d)$. We thus get

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr] & f'^*EU(n) \arrow[r, "\overline{f'}"] \arrow[d] & EU(n,d) \arrow[r] \arrow[d] & \mathrm{Gr}_n(\mathbb{C}^d) \times \mathbb{C}^d \\ & X \arrow[r, "f'"] & \mathrm{Gr}_n(\mathbb{C}^d) & \end{tikzcd}$$

i.e, an embedding of E in a trivial bundle. □

1.7 Group completion

From now on all our spaces are compact Hausdorff, this includes for example all finite CW-complexes. Let X be a connected space. We denote by $\mathrm{Vect}^n_{\mathbb{C}} X$ the set of isomorphism classes of n-dimensional vector bundles over X. We write $\mathrm{Vect}^*_{\mathbb{C}} = \bigoplus_{n\geqslant 0} \mathrm{Vect}^n_{\mathbb{C}} X$, where by convention $\mathrm{Vect}^0_{\mathbb{C}} X = *$. The direct sum of vector bundles induces an abelian monoid structure on $\mathrm{Vect}^*_{\mathbb{C}} X$. We can further extend this by setting, for a non-connected space $X = \bigsqcup_\alpha X_\alpha$ (a disjoint union of path components), $\mathrm{Vect}\, X = \prod_\alpha \mathrm{Vect}^*_{\mathbb{C}} X_\alpha$ with the ordinary abelian monoid structure.

Let A be an abelian monoid. A *group completion* of A is an abelian group $K(A)$ together with a map of abelian monoids $\alpha = \alpha_A \colon A \to K(A)$ such that for any abelian group A' and any map of abelian monoids $\rho\colon A \to A'$, there exists a unique map of abelian groups $\overline{\rho}\colon K(A) \to A'$ that makes commutative the following diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\rho"'] & K(A) \arrow[d, dashed, "\exists!\overline{\rho}"] \\ & A' \end{tikzcd} \tag{1.3}$$

If $K(A)$ exists is unique up to isomorphism. We construct $K(A)$ for an arbitrary abelian monoid $(A, \oplus)$. Let $F(A)$ be the free abelian group generated by the elements of A and let $E(A) \subseteq F(A)$ be the subgroup generated by elements of the form $a + a' - a \oplus a'$, where $+ = +_{F(A)}$. The quotient $K(A) := F(A)/E(A)$ is an abelian group which together with the obvious map $\alpha\colon A \longrightarrow K(A)$ and satisfies the universal property of (1.3). Alternatively we can define $K(A)$ as follows:

Let $\triangle\colon A \longrightarrow A \times A$ be the diagonal map. The quotient $K(A) = (A \times A)/\triangle(A)$ inherits an abelian monoid structure which has inverses since $[a, a] = 0$. We think of an element $[a, b]$ of $K(A)$ as a formal difference $a - b$ where $[a, b] = [a', b']$ if and only if $a \oplus b' = a' \oplus b$. We set $\alpha_A\colon A \longrightarrow K(A)$ by $a \longmapsto [a, 0]$. Since $K(A)$ is functorial in A, we get for any map of abelian monoids $\rho : A \longrightarrow B$ a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "\rho"'] & K(A) \arrow[d, "K(\rho)"] \arrow[dl, dashed, "\overline{\rho}"'] \\ B \arrow[r, "\alpha_B"] & K(B) \end{tikzcd}$$

If B is an abelian group, α_B is an isomorphism so that $\overline{\rho} := \alpha_B^{-1} \circ K(\rho)$ satisfy the universal property.

Definition 1.7.1. The K-group of a connected space X is defined as

$$K(X) = KU(X) = K(\mathrm{Vect}^*_{\mathbb{C}}(X), \oplus)$$

If $X = \coprod_\alpha X_\alpha$ is a disjoint union of path components, we set

$$K(X) = K(\mathrm{Vect}(X)),$$

By the construction of the group completion, the elements of $K(B)$ can be described as formal differences $[E] - [F]$ of isomorphism classes of vector bundles. The elements of $K(B)$ are called *virtual vector bundles.*

If E is a vector bundle over X, there is $n \in \mathbb{N}$ and an embedding $E \to \underline{n}$. We can take the orthogonal complement $E^\perp$ of E with respect to $\underline{n}$. This is done like the other operations on vector bundles, Strictly, $(-)^\perp$ is not a functor on finite dimensional vector spaces but rather a topological functor on finite dimensional vector spaces, embedded in some ambient vector space. The induced functor on suitable vector bundles is constructed in the same way as before. It follows that for any E there is an $n \in \mathbb{N}$ such that $E \oplus E^\perp \cong \underline{n}$.

Suppose that $[E] - [F] \in K(X)$, and let G be a vector bundle such that $F \oplus G$ is trivial. Then

$$[E] - [F] = [E] + [G] - ([G] - [F]) = [E \oplus G] - [\underline{n}]$$

Thus every element in $K(X)$ is of the form $[H] - [\underline{n}]$. Suppose $[E] = [F]$ in $K(X)$. Then $([E], [F]) = ([G], [G])$ for some G so that $E \oplus G \cong F \oplus G$. Let G' be such that $G \oplus G' \cong \underline{n}$. Then $E \oplus \underline{n} \cong F \oplus \underline{n}$. We would like to view all trivial as one (trivial) element. We thus make the following definition.

Definition 1.7.2. Two vector bundles E and F over X are said to be *stably equivalent* if there are $n, m \in \mathbb{N}$ such that $E \oplus \underline{n} \cong F \oplus \underline{m}$. We denote by $\cong_S$ the equivalence relation of stably equivalent vector bundles, and let $S\,\mathrm{Vect}^*_{\mathbb{C}}\, X = \mathrm{Vect}^*_{\mathbb{C}} / \cong_S$.

Suppose now X is pointed, i.e, equipped with a map $* \longrightarrow X$. We obtain an *augmentation map* $\epsilon : K(X) \longrightarrow K(*) \cong Z$.

Definition 1.7.3. The *reduced K-theory* of a pointed space $(X, *)$ is defined as:

$$\widetilde{K}(X) = \ker(\epsilon : K(X) \longrightarrow K(*)).$$

The map $\epsilon : K(X) \longrightarrow K(*)$ is given by $[E] \longmapsto \dim E$. It follows that $\widetilde{K}(X)$ consists of elements of the form $[E] - [F]$, where $\dim E = \dim F$.

Remark 1.7.4. The map $X \longrightarrow *$ gives a natural splitting $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.

Proposition 1.7.5. *Let $(X, *)$ be a pointed compact space. Then $S\operatorname{Vect}^* X$ is an abelian group, and there is an isomorphism*

$$S\operatorname{Vect}^* X \cong \widetilde{K}(X).$$

Proof. $S\operatorname{Vect}^* X \cong \widetilde{K}(X)$ is an abelian monoid under direct sum and has inverses since the isomorphism $E \oplus E^{\perp} \cong \underline{n}$ implies $[E]^{-1} \cong [E^{\perp}]$. The natural surjection

$$\operatorname{Vect}^*_{\mathbb{C}} X \longrightarrow S\operatorname{Vect}^* X$$

is a map into an abelian group. By the universal property of the group completion implies there exists $\overline{\rho}$ in the following diagram which must be also surjective:

$$\begin{tikzcd}
\operatorname{Vect}^*_{\mathbb{C}} X \arrow[r, "\alpha"] \arrow[dr] & K(X) \arrow[r] \arrow[d, dashed, "\overline{\rho}"] & \widetilde{K}(X) \arrow[dl, dashed, "f"] \\
& S\operatorname{Vect}^* X/\cong_S &
\end{tikzcd}$$

In the above diagram, the map $K(X) \longrightarrow \widetilde{K}(X)$ is given by $[E] \longmapsto [E] - [\underline{\dim E}]$. Recall that the elements in $\widetilde{K}(X)$ are of the form $[E] - [F]$ with $\dim E = \dim F$. Since $\overline{\rho}(\underline{n}) = 0$, we get a factorization of ρ through the map $f\colon \widetilde{K}(X) \longrightarrow S\operatorname{Vect}^* X$ given by $[E] - [F] \longmapsto [E]_S - [F]_S$. The map f is surjective since ρ is. To prove injectivity of f, we construct a left inverse. The map $\operatorname{Vect}^* X \longrightarrow K(X) \longrightarrow \widetilde{K}(X)$ given by $[E] \longmapsto [E] - [\underline{n}]$ respects $\cong_S$ and hence induces a map $j\colon S\operatorname{Vect}^* X \longrightarrow \widetilde{K}(X)$. If $[E] - [F] \in \widetilde{K}(X)$ then $j(f([E] - [F])) = [E] - [\underline{n}] - ([F] - [\underline{n}])$ since $\dim E = \dim F$. We see that $j \circ f = id$, so f is an isomorphism. □

1.8 Relative K-groups

If $X = X' \bigsqcup X'' \in$ **CHaus**, we have $\operatorname{Vect}_* X = \operatorname{Vect}_* X' \oplus \operatorname{Vect}^* X''$. Since $\oplus$ is the coproduct in **AbMon** and **AbGrp**, and K is a left adjoint, $K(X) = K(X') \oplus K(X'')$. Let $(-)^*$ be the left adjoint to the forgetful functor $\mathbf{CHaus}_* \to \mathbf{CHaus}$ from pointed compact Hausdorff spaces (and pointed maps) to compact Hausdorff spaces. It is given by $X_+ = X \sqcup \{*\}$. We then have

$$\widetilde{K}(X^+) = \ker(\epsilon : K(X) \oplus K(*) \to K(*)) = K(X)$$

For an inclusion $i : X' \to X$ in **CHaus** we make the following definition

Definition 1.8.1. The *relative K-groups* of a pair $Y \subseteq X \in$ **CHaus** are

$$K(X, Y) = \widetilde{K}(X/Y)$$

where the base point is taken to be Y/Y

We have $K(X, \varnothing) = \widetilde{K}(X_+) = K(X)$, so our definition specializes to the old one in the degenerate case. Our aim now is to establish an exact sequence of the form

$$K(X, Y) \longrightarrow K(X) \longrightarrow K(Y)$$

1.9 Construction of vector bundles over quotients

We assume $Y \subseteq X \in \mathbf{CHaus}$ is a pair and denote by $q\colon X \to X/Y$ the quotient map. Suppose that $p\colon E \to X$ is a vector bundle which is trivial over Y. Let $\alpha : E|_Y \xrightarrow{\cong} Y \times V$ be a trivialization and let $\pi\colon Y \times V \to V$ be the projection. Define an equivalence relation on $E|_Y$ by setting $e \sim e'$ if and only if $\pi(\alpha(e)) = \pi(\alpha(e'))$, and extend this relation by the identity to E. Let $E/\alpha = E/\sim$ be the quotient space and set

$$\overline{p} : E/\alpha \longrightarrow X/Y$$

by $\overline{p}([e]) = q(p(e))$. Note that $\overline{p}$ is well-defined since if $e \neq e'$, $e \sim e'$ only if $p(e), p(e') \in Y$. In fact $e \sim e'$ only if they are in a different fiber, which means that we collapsed all the fibers parametrized by Y into a single fiber. Thus $\overline{p} : E/\alpha \to X/Y$ has a fiber isomorphic to V over every point. We would like to show that $\overline{p} : E/\alpha \to X/Y$ is in fact a vector bundle.

Lemma 1.9.1. *If $E \longrightarrow X$ is trivial over a closed subspace $Y \subseteq X$, then there exists an open neighbourhood $Y \subseteq U \subseteq X$ over which E is still trivial.*

Take such an open $Y \subseteq U$ and a trivialization $(\varphi_1, \varphi_2) : E|_U \xrightarrow{\cong} U \times V$. Then this induces a trivialization $\varphi : (E/\alpha)|_U = (E|_U)/\alpha \to (U/Y) \times V$, given by $\varphi([e]) = (q\varphi_1(e), \varphi_2(e))$. This is a local trivialization of E/α around $Y/Y \in X/Y$. Around $x \in X\backslash Y$ we have an open neighbourhood $U \subseteq X\backslash Y$ so that we can use the same local trivializations of $E \to X$ (restricted to U) to get a trivialization of $E/\alpha \to X/Y$. We deduce that $E/\alpha \to X/Y$ is a vector bundle.

Lemma 1.9.2. *If $Y \subseteq X$ is a closed subspace, then any trivialization $\alpha\colon E|_Y \cong \underline{n}$ on Y of a vector bundle $p\colon E \longrightarrow X$ defines a vector bundle $E/\alpha \longrightarrow X/Y$ on the quotient X/Y.*

Chapter 2

Higher K-theory Groups

We start by introducing some notation and topological constructions in order to define the higher K-groups of a space and see some properties of them. We have defined $K(X)$ as the group completion of the abelian monoid of isomorphism classes of vector bundles over X. $K(X)$ is $K^0(X)$ in an infinite sequence of abelian groups $K^n(X)$ for $n \in \mathbb{Z}$. Our aim is to see that this sequence defines a cohomology theory in the sense of Eilenberg-Steenrod.

2.1 Notation and basic constructions

Let **Top** denote the category of compact Hausdorff spaces and $\mathbf{Top}_*$ the category of *pointed* compact Hausdorff spaces. By $\mathbf{Top}^2$ we denote the category of *compact pairs*, that is, the objects are pairs of spaces (X, Y), where X is compact Hausdorff and $Y \subseteq X$. There are functors

$$\mathbf{Top} \longrightarrow \mathbf{Top}^2 \qquad\qquad \mathbf{Top}^2 \longrightarrow \mathbf{Top} \tag{2.1}$$

$$X \longmapsto (X, \varnothing) \qquad\qquad (X, Y) \longrightarrow X/Y \tag{2.2}$$

where the basepoint in the quotient X/Y is Y/Y. If $Y = \varnothing$, then $X/\varnothing = X_+$ is the space X with a disjoint basepoint.

For a space X in **Top** we denote by $K(X)$ the group completion of $\mathrm{Vect}_{\mathbb{C}} X$ and for a pointed space X in **Top**, the *reduced* K-theory group $\widetilde{K}(X)$ is

$$\ker(i\colon K(X) \to K(x_0)) = \mathbb{Z},$$

where i^* is the map induced by the inclusion of the basepoint $i\colon x_0 \to X$. There is a short exact sequence

$$0 \longrightarrow \ker i^* = \widetilde{K}(X) \longrightarrow K(X) \xrightarrow{i^*} K(x_0) \longrightarrow 0$$

which has a section c^* induced by the unique map $c : X \longrightarrow x_0$. It gives a natural splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$. We also have that $K(X) = \widetilde{K}(X_+)$ for every X in **Top**. Thus $\widetilde{K}(X)$ defines a contravariant functor from $\mathbf{Top}_*$ to abelian groups. For a compact pair (X, Y), we define $K(X, Y) = \widetilde{K}(X/Y)$. So $K(-,-)$ is a contravariant functor from $\mathbf{Top}^2$ to abelian groups.

Recall that the *smash product* of two pointed spaces is defined as the quotient $X \wedge Y = X \times Y/X \vee Y$, where $X \vee Y = X \times \{x_0\} \cup \{x_0\} \times Y$ is the wedge of X and Y, that is, the disjoint union glued by the base points.

For a pointed space X in $\mathbf{Top}_*$ the *reduced suspension* ΣX is $S^1 \wedge X$. The n-th reduced suspension of X is $\Sigma^n X = S^n \wedge X$.

2.2 Negative K-groups

We use the reduced suspension to define negative K-groups for spaces, pointed spaces and pairs of spaces.

Definition 2.2.1. For $n \geqslant 0$

$$\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X) \qquad \text{for } X \in \mathbf{Top}_* \tag{2.3}$$

$$\widetilde{K}^{-n}(X,Y) = \widetilde{K}(X/Y) \cong \widetilde{K}(\Sigma^n(X/Y)) \qquad \text{for } (X,Y) \in \mathbf{Top}^2 \tag{2.4}$$

$$\widetilde{K}^{-n}(X) = \widetilde{K}^{-n}(X,\varnothing) = \widetilde{K}(\Sigma^n(X_+)) \qquad \text{for } X \in \mathbf{Top}_* \tag{2.5}$$

$\widetilde{K}^{-n}(-)$, $K^{-n}(-,-)$ and $K^{-n}(-)$ are contravariant functors for every $n \geqslant 0$ from $\mathbf{Top}_*, \mathbf{Top}^2$ and $\mathbf{Top}$ respectively, to abelian groups.

Given X in $\mathbf{Top}$. The *cone on* X is the quotient $CX = X \times I/X \times \{0\}$. The cone CX has a natural basepoint given by $X \times \{0\}$, and that defines a functor $C : \mathbf{Top} \longrightarrow \mathbf{Top}_*$. The space CX/X is called the *unreduced suspension* of X.

If X is a pointed space, we have an inclusion $Cx_0/x_0 \cong I \longrightarrow CX/X$ and the quotient space is obtained by collapsing I in CX/X is the reduced suspension ΣX. Since I is a closed contractible subspace of CX/X, we have that $\mathrm{Vect}_{\mathbb{C}}(CX/X) \cong \mathrm{Vect}_{\mathbb{C}}((CX/X)/I)$. Hence $K(CX/X) \cong K(\Sigma X)$ and $K(CX,X) = \widetilde{K}(CX/X) \cong \widetilde{K}(\Sigma X)$.

For a compact pair (X,Y) we define $X \cup CY$ to be the space obtained by identifying $Y \subseteq X$ with $Y \times \{1\}$ in CY. There is a natural homeomorphism $X \cup CY/X \cong CY/Y$. Thus, if Y is a pointed space we have that

$$K(X \cup CY, X) = \widetilde{K}(CY,Y) \cong \widetilde{K}(\Sigma Y) = \widetilde{K}^{-1}(Y)$$

2.3 Exact sequences of K-groups

Now we relate the K-groups of a pair (X,Y) with the K-groups of X and Y.

Lemma 2.3.1. *Let (X,Y) be a compact pair in $\mathbf{Top}^2$ and let $i\colon Y \longrightarrow X$ and $j\colon (X,\varnothing) \longrightarrow (X,Y)$ be the canonical inclusions. There exists an exact sequence*

$$K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y)$$

Proof. The composition $(Y,\varnothing) \xrightarrow{i} (X,\varnothing) \xrightarrow{j} (X,Y)$ factors through the zero group (Y,Y). Applying K^0 yields a commutative diagram

$$\begin{tikzcd} K^0(X,Y) \arrow[rr, "i^*\circ j^*"] \arrow[dr] & & K^0(Y) \\ & K^0(Y,Y) = \widetilde{K}^0(Y/Y) = 0 \arrow[ur] & \end{tikzcd}$$

So, $i^* \circ j^* = 0$ and hence $\operatorname{im} j^* \subseteq \ker i^*$. Suppose now that $\xi \in \ker i^*$. We can represent ξ as a difference $[E] - [\underline{n}]$, where E is a vector bundle over X. Since $i^*(\xi) = 0$, $i^*(\xi) = [E|_Y] - [\underline{n}] = 0$. So, $[E|_Y] = [\underline{n}]$ in $K^0(Y)$. There is an $m \geqslant 0$ such that then

$$\alpha\colon (E \oplus \underline{m})|_A \cong \underline{n} \oplus \underline{m}.$$

So, we have a vector bundle that is trivial in Y. By Lemma 1.9.2 we have a vector bundle $(E \otimes \underline{m}/\alpha$ over X/Y. Now $\eta = [(E \oplus \underline{m})/\alpha] - [\underline{n} - \underline{m}]$. Observe that η lies in $\widetilde{K}^0(X/Y)$, since the rank of $(E \oplus \underline{m})/\alpha$ in the component of the basepoint is $n + m$. So

$$j^*(\eta) = [E \oplus \underline{m}] - [\underline{n} - \underline{m}] = [E] - [\underline{n}] = \xi,$$

Thus $\ker i^* \subseteq \operatorname{im} j^*$. ☐

Corollary 2.3.2. *Let (X, Y) be a compact pair in $\mathbf{Top}^2$ and Y in $\mathbf{Top}_*$. There is an exact sequence*

$$K^0(X, Y) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(Y)$$

Proof. We have natural isomorphisms $K^0(X) \cong \widetilde{K}^0(X) \oplus K^0(*)$ and $K^0(Y) \cong \widetilde{K}^0(Y) \oplus K^0(*)$, thus the following diagram commutes.

$$\begin{tikzcd}
 & \widetilde{K}^0(X) \arrow[r] \arrow[d] & \widetilde{K}^0(Y) \arrow[d] \\
K^0(X,Y) \arrow[r] \arrow[ur, dashed] & \widetilde{K}^0(X) \oplus K^0(*) \arrow[r] \arrow[d] & \widetilde{K}^*(Y) \oplus K^0(*) \arrow[dl] \\
 & K^0(*) &
\end{tikzcd}$$

The central row and the columns are exact. Now, any element in $K^0(X, Y)$ goes to zero in $K^0(*)$ so there is a map $K^0(X, Y) \longrightarrow \widetilde{K}^0(X)$ that makes the diagram commutative. It is immediate from Lemma 2.3.1 that the required sequence is exact. ☐

Proposition 2.3.3. *Let (X, Y) be a compact pair of spaces and Y in $\mathbf{Top}_*$. Then there is a natural exact sequence:*

$$\widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} K^0(X, Y) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(Y)$$

Proof. We need to check exactness of the three subsequences of three terms. Exactness of $K^0(X, Y) \longrightarrow \widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(Y)$ is given by Corollary 2.3.2. To prove exactness at $\widetilde{K}^{-1}(Y) \longrightarrow K^0(X, Y) \longrightarrow \widetilde{K}^0(X)$ we consider the pair of spaces $(X \cup CY, X)$. Applying Corollary 2.3.2 we get an exact sequence

$$\begin{tikzcd}
K^0(X \cup CY, X) \arrow[r, "m^*"] \arrow[d, "\theta"', "\cong"] & \widetilde{K}^0(X \cup CY) \arrow[r, "k^*"] & \widetilde{K}^0(X) \\
\widetilde{K}^{-1}(Y) \arrow[r, dashed, "\delta"] & \widetilde{K}^0(X/Y) \arrow[u, "p^*"', "\cong"] \arrow[ur, "j^*"'] &
\end{tikzcd} \tag{2.6}$$

Since CY is contractible, the quotient map $p \colon X \cup CY \longrightarrow X/Y$ induces an isomorphism on $\widetilde{K}^0$ and moreover $k^* \circ p^* = j^*$, which follows directly from the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[dr, "j"'] & X \cup CY \arrow[d, "p"] \\
 & X \cup CY/CY \cong X/Y
\end{tikzcd}$$

We define the connecting homomorphisms $\delta = (p^*)^{-1} \circ m^* \circ \theta^{-1}$. We denote the respective cones C_1Y and C_2X to distinguish between them. Now we apply Corollary 2.3.2 to the pair $(X, C_1Y \cup C_2X, X \cup C_1Y)$. We get the exact sequence:

$$\begin{tikzcd}
K^0(X \cup C_1Y \cup C_2X, X \cup C_1Y) \arrow[r] & \widetilde{K}^0(X \cup C_1Y \cup C_2X) \arrow[r] \arrow[d, dash, "\cong"] & \widetilde{K}^0(X \cup C_1Y) \\
& \widetilde{K}^0((X \cup C_1Y \cup C_2X)/C_2X) \arrow[d, equal] & \\
& K^0((X \cup C_1Y)/X) \arrow[d, equal] & \widetilde{K}^0(X/Y) \arrow[uu, "p^*"', "\cong"] \\
& K^0(X \cup C_1Y, X) \arrow[uur, bend right, "m^*"] \arrow[d, dash, "\cong"', "\theta"] & \\
& \widetilde{K}^{-1}(Y) \arrow[uur, dashed, bend right, "\delta"'] &
\end{tikzcd}$$

Using the definition of δ given in the previous step, we can check that the composition in the square on the right is indeed δ. For the left part of the diagram we have a square as follows:

$$\begin{tikzcd}
K^0(X \cup C_1Y \cup C_2X, X \cup C_1Y) \arrow[r] \arrow[d, equal] & \widetilde{K}^0(X \cup C_1Y \cup C_2X) \arrow[d, dash, "\cong"] \\
\widetilde{K}^0((X \cup C_1Y \cup C_2X)/C_2X) \arrow[d, equal] & \widetilde{K}^0(C_1Y/Y) \arrow[d, dash, "\cong"] \\
\widetilde{K}^0(C_2X/X) \arrow[d, dash, "\cong"] & \widetilde{K}^{-1}(Y) \\
\widetilde{K}^{-1}(X) \arrow[ur, dashed, bend right] &
\end{tikzcd}$$

Now, we would like the dashed arrow that makes the diagram commutative to be i^* to conclude the proof. Consider the following diagram

$$\begin{tikzcd}
& X \cup C_1Y \cup C_2X \arrow[r] \arrow[dl] & C_1Y/Y \arrow[r] & \Sigma Y \arrow[d, "T"] \\
C_2X/X \arrow[d] & C_1Y \cup C_2Y \arrow[l] \arrow[r] \arrow[u] \arrow[ur] & C_2Y/Y \arrow[r] & \Sigma Y \arrow[dlll, bend left=10, "1 \wedge i"] \\
\Sigma X & & &
\end{tikzcd}$$

which induces the following commutative diagram,

$$\begin{tikzcd}
& \widetilde{K}^0(X \cup C_1Y \cup C_2X) \arrow[d, "\cong"'] & \widetilde{K}^0(C_1Y/Y) \arrow[l, "\cong"'] \arrow[dl, "\cong"'] & \widetilde{K}^{-1}(Y) \arrow[l, "\cong"'] \\
\widetilde{K}^0(C_2X/X) \arrow[r] \arrow[ur] & \widetilde{K}^0(C_1Y \cup C_2Y) & \widetilde{K}^0(C_2Y/Y) \arrow[l, "\cong"'] & \widetilde{K}^{-1}(Y) \arrow[l, "\cong"'] \arrow[u, "T^*"'] \\
\widetilde{K}^{-1}(X) \arrow[u, "\cong"'] \arrow[urrr, bend right=10, "i^*"'] & & &
\end{tikzcd}$$

Inserting this diagram into diagram (2.6), we can check that the latter commutes if the dashed arrow is $T^* \circ i^*$. So in the end, we get an exact sequence

$$\widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} \widetilde{K}^0(X,Y).$$

Since $-i^*$ and i^* have both the same kernel and image, we can replace $-i^*$ by i^* and we still have an exact sequence. This completes the proof. □

Corollary 2.3.4. *If (X,Y) is a compact pair and $Y \in \mathbf{Top}_*$, then there is a long exact sequence*

$$\cdots \to \widetilde{K}^{-2}(X) \xrightarrow{i^*} \widetilde{K}^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} \widetilde{K}^{-1}(X) \xrightarrow{i^*}$$
$$\xrightarrow{i^*} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(Y)$$

Proof. Replace in the exact sequence of Proposition 2.3.3 the compact pair (X,Y) by $(\Sigma^n X, \Sigma^n Y)$ for $n = 1, 2, \ldots$ □

Corollary 2.3.5. *If (X,Y) is a compact pair, then there is a long exact sequence*

$$\cdots \longrightarrow K^{-2}(X) \xrightarrow{i^*} K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*}$$
$$\xrightarrow{i^*} K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y)$$

Proof. Apply Corollary 2.3.4 to the pair (X_+, Y_+). Recall that $\widetilde{K}^{-n}(X_+) = K^{-n}(X)$. □

2.4 K-theory as a cohomology theory

In 1945 Eilenberg and Steenrod introduced an axiomatic approach to cohomology theory by abstracting the fundamental properties that any cohomology theory should satisfy.

Definition 2.4.1. A *cohomology theory* h^* on $\mathbf{Top}_*$ (or any nice subcategory like compact pairs, pairs of CW- complexes, etc.) is a collection of *contravariant* functors

$$h^n : \mathbf{Top}^2 \to \mathbf{Ab}, \quad n \in \mathbb{Z}$$

where $\mathbf{Ab}$ denotes the category of abelian groups, and natural transformations

$$\delta^n : h^n \circ R \to h^{n+1}$$

where $R : \mathbf{Top}^2 \to \mathbf{Top}^2$ is the functor that sends (X,Y) to $(Y, \varnothing)$ and f to $f|_Y$, satisfying the following axioms

(i) *Homotopy invariance.* If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in \mathbb{Z}$

(ii) *Excision.* For every pair (X,Y) and $U \subseteq Y$ such that the closure $\overline{U}$ is contained in the interior Y^o, the inclusions $(X \backslash U, Y \backslash U) \to (X,Y)$ induces an isomorphism

$$h^n(X \backslash U, Y \backslash U) \cong h^n(X,Y), \quad \text{for every } n \in \mathbb{Z}$$

(iii) *Exactness.* For every pair (X, Y) in $\mathbf{Top}^2$ and $\mathbf{Top}_*$ there is an exact sequence

$$\widetilde{h}^n(X \cup CY) \xrightarrow{j^*} \widetilde{h}^n(X) \xrightarrow{i^*} \widetilde{h}^n(Y) \text{ for every } n \in \mathbb{Z}$$

where $i : Y \longrightarrow X$ and $j : X \longrightarrow X \cup CY$ denote the canonical inclusions.

We need the following theorem to show that K-theory is a generalized cohomology theory.

Theorem 2.4.2 (Bott periodicity I)**.** *If X is a compact Hausdorff space, then*

$$\widetilde{K}(X) \cong \widetilde{K}(\Sigma^2 X).$$

Proof. See [4] or [9, p. 51]. □

Recall that we have defined the negative K-groups in 2.2.1 by

$$\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X).$$

Thus, we have an isomorphism $\beta\colon \widetilde{K}^{-n}(X) \xrightarrow{\cong} \widetilde{K}^{-n-2}(X)$ for all $n \geqslant 0$. Since for any space $X \in \mathbf{Top}$ we have that $K^{-n}(X) = \widetilde{K}^{-n}(X_+)$, there is also an isomorphism $K^{-n}(X) \cong K^{-n-2}(X)$ in the unreduced case. Thus, for a space X in $\mathbf{Top}_*$ we can define:

$$\widetilde{K}^{2n}(X) := \widetilde{K}^0(X), \qquad \widetilde{K}^{2n+1}(X) := \widetilde{K}^{-1}(X), \qquad \text{for every } n \in \mathbb{Z}.$$

And similarly, for any pointed space X in $\mathbf{Top}_*$, we define

$$K^{2n}(X) = K^0(X), \qquad K^{2n+1}(X) = K^{-1}(X), \qquad \text{for every } n \in \mathbb{Z}.$$

The results about exact sequences that we have seen can be extended to all the integers. In particular, we can extend the long exact sequence of Corollary 2.3.5 to an infinite long exact sequence on the right. Exactness for K^* is Corollary 2.3.5 and for $\widetilde{K}^*$ it follows from Corollary 2.3.2. The excision axiom follows from $K^n(X, Y) = \widetilde{K}(X/Y)$.

Theorem 2.4.3. *K-theory and reduced K-theory are a generalized cohomology theory and a reduced cohomology theory respectively.*

Corollary 2.4.4. *Let X and Y in $\mathbf{Top}_*$. Then $\widetilde{K}^{-n}(X \vee Y) \cong \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$ for every $n \in \mathbb{Z}$.*

Proof. We have pointed inclusions $i_1 : X \longrightarrow X \vee Y$ and $i_2 : Y \longrightarrow X \vee Y$, and surjections $r_1 : X \vee Y \longrightarrow X$ and $r_2 : Y \longrightarrow X \vee Y$. They satisfy that $r_1 \circ i_1 = id_X$ and $r_2 \circ i_2 = id_Y$. So taking $\widetilde{K}^{-n}$ we have maps

$$\widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y) \xrightarrow{r_1^* + r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{(i_1^*, i_2^*)} \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$$

such that $(i_1^*, i_2^*) \circ (r_1^* + r_2^*) = id$, so (i_1^*, i_2^*) is surjective. To see that it is also injective, let $\xi \in \ker(i_1^*, i_2^*)$. Then $i_1^*(\xi) = 0$ and $i_2^*(\xi) = 0$. Now, consider the pair $(X \vee Y, X)$ and apply Corollary 2.3.4. We get an exact sequence

$$K^{-n}(X \vee Y, X) = \widetilde{K}^{-n}((X \vee Y)/X) \cong \widetilde{K}^{-n}(Y) \xrightarrow{r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{i_1^*} \widetilde{K}^{-n}(X).$$

Since $\xi \in \ker i_1^*$, there exists an element $\eta \in \widetilde{K}^{-n}(Y)$ such that $r_2^*(\eta) = \xi$. However, $\eta = i_2^* \circ r_2^*(\eta) = i_2^*(\xi) = 0$, and therefore $\eta = 0$ and $\xi = 0$ too. □

Corollary 2.4.5. *Let (X, Y) in $\mathbf{Top}^2$ and Y in $\mathbf{Top}_*$. If Y is contractible, then*

$$\widetilde{K}^{-n}(X/Y) \cong \widetilde{K}^{-n}(X) \quad \text{for every } n \geqslant 0.$$

Proof. By Corollary 2.3.4, since Y is contractible, then $\Sigma^n Y$ is also contractible, thus $\widetilde{K}^{-n}(Y) = 0$ for each $n \geqslant 0$. □

Corollary 2.4.6. *Let X and Y in $\mathbf{Top}_*$ and Y be a retract of X. Then*

$$\widetilde{K}^{-n}(X) \cong K^{-n}(X, Y) \oplus \widetilde{K}^{-n}(Y)$$

for every $n \geqslant 0$.

Proof. Since Y is a retract of X, there exists a map $r\colon X \longrightarrow Y$ such that $r \circ i = id_Y$, where i denotes the inclusion. Then, $i^* \circ r^* = id$ and therefore i^* is injective. The map δ that appears in the long exact sequence in Corollary 2.3.2 factors through the zero map. Since r^* is a section, we have split short exact sequences.

$$0 \longrightarrow K^{-n}(X, Y) \xrightarrow{j^*} \widetilde{K}^{-n}(X) \xrightarrow{i^*} (Y) \longrightarrow 0,$$

So $\widetilde{K}^{-n}(X) \cong K^{-n}(X, Y) \oplus K^{-n}(Y)$. □

Corollary 2.4.7. *Let X and Y in $\mathbf{Top}_*$. Then the projection maps $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ and the quotient map $q : X \times Y \to X \times Y/X \vee Y = X \wedge Y$ induce an isomorphism*

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y).$$

for every $n \geqslant 0$.

Proof. The map $X \longrightarrow X \times Y$ that sends x to (x, y_0) and the projection pr_1 we can see that X is a retract of $X \times Y$. By Corollary 2.4.6 we have that

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \times Y, X) \oplus \widetilde{K}^{-n}(X).$$

Now, $K^{-n}(X \times Y, X) = \widetilde{K}(X \times Y/X)$. Since Y is a retract of $X \times Y/X$, applying Corollary 2.4.6 again, we obtain

$$\widetilde{K}^{-n}(X \times Y/X) \cong K^{-n}(X \times Y/X, Y) \oplus \widetilde{K}^{-n}(Y).$$

Since $K^{-n}(X \times Y/X, Y) = \widetilde{K}^{-n}(X \vee Y)$, we are done. □

By Bott's periodicity, all of the previous corollaries hold for K^n and $\widetilde{K}^n$ for $n \in \mathbb{Z}$.

2.5 The external product for reduced K-theory

Given two vector bundles $\xi : \mathbb{C}^n \to E \to X$ and $\xi' : \mathbb{C}^m \to E' \to X$, their tensor product $\xi \otimes \xi' : \mathbb{C}^{n \cdot m} \to E'' \to X$ is well defined and satisfies $\operatorname{rank}(\xi \otimes \xi') = \operatorname{rank}(\xi) \cdot \operatorname{rank}(\xi')$, since the product $\otimes$ is distributive over the sum $\oplus$. With this operation, $K^0(X)$ becomes a ring. We can define an external product

$$\begin{aligned} K^0(X) \otimes K^0(Y) &\xrightarrow{\mu} K^0(X \times Y) \\ [\xi] \otimes [\eta] &\longmapsto [(p_1)^* \xi \otimes (p_2)^* \eta] \end{aligned}$$

Let $[\xi] \in \widetilde{K}^0(X)$ and $[\eta] \in \widetilde{K}^0(Y)$. Then from the following commutative diagram

$$\begin{tikzcd}
K^0(\{x_0\} \times Y) & K^0(X \times Y) \arrow[l] & K^0(X) \arrow[l, "p_1^*"'] \arrow[dll] \\
K^0(x_0) \arrow[u] & &
\end{tikzcd}$$

it follows that $(p_1)^*(\xi)$, that lies in $K^0(X \times Y)$, restricts to zero in $K^0(\{x_0\} \times Y)$. Similarly $(p_2)^*(\eta)$ restricts to zero in $K^0(X \times \{y_0\})$. So $p_1^*(\xi) \cdot p_2^*(\eta)$ restricts to zero in $K^0(X \vee Y)$ and hence, it lies in the kernel of $K^0(X \times Y) \to K^0(*)$, which is $\widetilde{K}^0(X \vee Y)$. By Corollary 2.4.7 there is a split short exact sequence

$$0 \longrightarrow \widetilde{K}^0(X \wedge Y) \longrightarrow \widetilde{K}^0(X \times Y) \longrightarrow \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \cong \widetilde{K}^0(X \vee Y) \longrightarrow 0$$

Since $p_1^*(\xi) \cdot p_2^*(\eta)$ lies in $\widetilde{K}^0(X \times Y)$ and it is zero in $\widetilde{K}^0(X \times Y)$, it lies in the kernel of the third map in the above sequence, which is $\widetilde{K}^0(X \wedge Y)$. So we have defined a map

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y).$$

This map is in fact the restriction of the exterior product on K^0 as we can see in the following diagram;

$$\begin{array}{ccccccccc}
K^0(X) \otimes K^0(Y) & \cong & (\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y)) & \oplus & \widetilde{K}^0(X) & \oplus & \widetilde{K}^0(Y) & \oplus & \mathbb{Z} \\
\downarrow & & \downarrow & & \| & & \| & & \| \\
K^0(X \times Y) & \cong & \widetilde{K}^0(X \wedge Y) & \oplus & \widetilde{K}^0(X) & \oplus & \widetilde{K}^0(Y) & \oplus & \mathbb{Z}
\end{array} \tag{2.7}$$

The first isomorphism is obtained by using $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$, and similarly for Y, and the isomorphism on the second row is obtained by using Corollary 2.4.7. We can replace X by $\Sigma^n X$ and Y by $\Sigma^m Y$ in (2.7) to obtain a pairing

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}^{-m}(Y) \to \widetilde{K}^{-n-m}(X \wedge Y)$$

If X and Y are in **Top**, we can replace X by X_+ and Y by Y_+ in the previous pairing to obtain a pairing

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{n-m}(X \wedge Y)$$

in the unreduced case.

2.6 *K*-theory groups of the spheres

The sphere S^k can be decomposed as the union of the upper and lower hemisphere. Since each hemisphere is contractible, every vector bundle on S^k restricts to a trivial bundle on each of the hemispheres. A vector bundle on S^k can be determined by a map from the intersection of the two hemispheres to $\mathrm{GL}(n, \mathbb{C})$.

Definition 2.6.1. A *clutching function* for S^k is a map $f \colon S^{k-1} \longrightarrow \mathrm{GL}(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible matrices with complex coefficients.

Every clutching function $f\colon S^{k-1} \longrightarrow \mathrm{GL}(n,\mathbb{C})$ gives rise to a vector bundle E_f over S^k of rank n. We define

$$E_f = (D^- \times \mathbb{C}^n) \cup_{S^{k-1}\times\mathbb{C}^n} (D^+ \times \mathbb{C}^n),$$

where $D^- = \{(x_1,\dots,x_{k+1}) \in S^k \mid x_{k+1} \leqslant 0\}$ is the lower hemisphere, and $D^+ = \{(x_1,\dots,x_{k+1}) \in S^k \mid x_{k+1} \geqslant 0\}$ is the upper hemisphere. If $x \in S^{k-1}$, then we identify (x,v) in $D^- \times \mathbb{C}^n$ with $(x, f(x)v)$ in $D^+ \times \mathbb{C}^n$. If f is homotopic to g, then $E_f \cong E_g$.

Proposition 2.6.2. *There is an isomorphism* $\mathrm{Vect}^n_{\mathbb{C}}(S^k) \cong [S^{k-1}, \mathrm{GL}(n,\mathbb{C})]$ *for every* $n,k \geqslant 1$.

Proof. See [3, p. 24]. □

Lemma 2.6.3. *The group* $\mathrm{GL}(n,\mathbb{C})$ *is path-connected for every* $n \geqslant 1$.

Proof. For $n = 1$ is trivial since $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C}\backslash\{0\} \cong \mathbb{R}^2\backslash\{0\}$. Let $n \geqslant 2$ and let $M \in \mathrm{GL}(n,\mathbb{C})$. Let J be the Jordan canonical form of M

$$J = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_k \end{pmatrix}, \quad \text{where } J_i = \begin{pmatrix} \lambda_i & 1 & 1 \\ \vdots & \ddots & 1 \\ 0 & \cdots & \lambda_i \end{pmatrix}.$$

There exists an invertible matrix Q such that $M = QJQ^{-1}$. For each $\lambda_i \in \mathbb{C}$, let $\gamma_i\colon I \longrightarrow \mathbb{C}$ be a path from λ_i to 1 that does not pass through the origin. Let $J(t)$ be a matrix obtained from J by replacing λ_i by γ_i and multiplying by $(1-t)$ all elements above the diagonal. Now define the path $\gamma\colon I \longrightarrow \mathrm{GL}(n,\mathbb{C})$ by $\gamma(t) = QJ(t)Q^{-1}$. This path satisfies that $\gamma(0) = QJ^{-1}Q = M$ and $\gamma(1) = QQ^{-1} = 1 \in \mathrm{GL}(n,\mathbb{C})$. □

Corollary 2.6.4. *Every complex vector bundle over* S^1 *is trivial. In particular,* $K^0(S^1) \cong \mathbb{Z}$.

Proof. By Proposition 2.6.2 , $\mathrm{Vect}^1_{\mathbb{C}}(S^k) \cong [S^{k-1}, \mathrm{GL}(1,\mathbb{C})] \cong [S^{k-1}, S^1]$, since $\mathrm{GL}(1,\mathbb{C}) \cong U(1) = S^1$. The sphere S^{k-1} is simply connected for $k > 2$, hence any map $S^{k-1} \longrightarrow S^1$ factors through the universal cover $\mathbb{R} \longrightarrow S^1$. Since $\mathbb{R}$ is contractible, any map is homotopic to a constant map, and any two constant maps on S^1 are homotopic because S^1 is path-connected. Thus $[S^{k-1}, S^1]$ has only one element. □

2.7 The complex K-theory spectrum

Let $\widetilde{h}^*$ be a reduced cohomology theory. We restrict ourselves to cohomology theories defined on pointed CW-complexes and we assume that they are additive,i.e, they satisfy the *wedge axiom*:

$$\widetilde{h}^n\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} \widetilde{h}^n\,(X_i)$$

We will make use of the following theorem:

Theorem 2.7.1 (Brown representability)**.** *Every reduced cohomology theory on the category of basepointed* CW*-complexes and base-point preserving maps has the form* $\widetilde{h}^n(K) = [X, E_n]$ *for some* Ω*-spectrum* $\{E_n\}$.

Proof. See [1, p. 406] □

For every $n \in \mathbb{Z}$ the functor $\widetilde{h}^n$ satisfies the conditions of the Brown representability theorem. There is a unique (up to homotopy) pointed connected CW-complex L_n and a natural equivalence

$$\widetilde{h}^n \xrightarrow{\cong} [X, L_n]_*$$

for every pointed connected CW-complex X. Let $E_n = \Omega L_{n+1}$, recall that Ω is right adjoint to the suspension functor Σ. For any X the suspension is connected, so

$$\widetilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}]_*.$$

Since $\widetilde{h}^n(X)$ is a reduced cohomology theory, $\widetilde{h}^{n+1}(\Sigma X) \cong \widetilde{h}^n(X)$, so

$$\widetilde{h}^*(X) \cong \widetilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}]_* \cong [X, \Omega L_{n+1}] = [X, E_n]_*$$

where the third isomorphism is given by the adjunction between Σ and Ω. Thus, we can associate to $\widetilde{h}^*$ the family of pointed CW-complexes $\{E_n\}_{n\in\mathbb{Z}}$ which for any pointed CW-complex X satisfies:

$$[X, E_n]_* \cong \widetilde{h}^n(X)\widetilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}]_* \cong [X, E_{n+1}]_*.$$

This implies that there is a homotopy equivalence

$$E_n \xrightarrow{\simeq} \Omega E_{n+1}.$$

Definition 2.7.2. A *spectrum*, or an Ω-*spectrum* is a sequence of pointed CW-complexes $\{E_n\}_{n\in\mathbb{Z}}$ together with homotopy equivalences

$$\epsilon : E_n \longrightarrow \Omega E_{n+1}$$

for every $n \in \mathbb{Z}$.

So we have proved the following:

Theorem 2.7.3. *Every additive reduced cohomology theory $\widetilde{h}^*$ on pointed CW-complexes determines an Ω-spectrum $\{E_n\}_{n\in\mathbb{Z}}$ such that $\widetilde{h}^n(X) = [X, E_n]_*$ for every $n \in \mathbb{Z}$.*

The converse is also true. Let $\{E_n\}_{n\in\mathbb{Z}}$ be an Ω-spectrum and define

$$\widetilde{E}^n(X) = [X, E_n].$$

Then $\widetilde{E}^*$ is a reduced cohomology theory. It is homotopy invariant and the suspension isomorphism is given by

$$\widetilde{E}^{n+1}(\Sigma X) = [\Sigma X, E_{n+1}]_* \cong [X, \Omega E_{n+1}]_* \xrightarrow{(\epsilon_n)_*^{-1}} .[X, E_n]_* = \widetilde{E}^n(X)$$

This also implies that $\widetilde{E}^n(-)$ takes values in abelian groups since

$$\widetilde{E}^n(X) \cong \widetilde{E}^{n+2}(\Sigma^2 X) = [\Sigma^2 X, E_{n+2}]_*$$

and $[\Sigma^2(-), -]$ is always an abelian group . To prove exactness, consider a pair (X, Y) and the sequence

$$Y \xrightarrow{i} X \xrightarrow{j} X \cup CY.$$

This gives an exact sequence

$$[X \cup CY, Z]_* \xrightarrow{j^*} [X, Z]_* \xrightarrow{i^*} [Y, Z]_*$$

for every Z. Taking $Z = E_n$ gives the required exact sequence

$$\widetilde{E}^n(X \cup CY) \xrightarrow{j^*} \widetilde{E}^n(X) \xrightarrow{i^*} \widetilde{E}^n(Y).$$

$\widetilde{E}^*$ is also additive since

$$\widetilde{E}^n\left(\bigvee_{i\in I} X_i\right) = [\bigvee_{i\in I} X_i, E_n] \cong \prod_{i\in I}[X_i, E_n]_* = \prod_{i\in I} \widetilde{E}^n(X).$$

So we have proved the following:

Theorem 2.7.4. *If $\{E_n\}_{n\in\mathbb{Z}}$ is an Ω-spectrum, then the functors $\widetilde{E}^n$ defined as $\widetilde{E}^n = \widetilde{E}^n(X) = [X, E_n]_*$ for every $n \in \mathbb{Z}$ form an additive reduced cohomology theory on pointed CW-complexes.*

Example 2.7.5. Let G be an abelian group and let $K(G, n)$ be the associated Eilenberg-MacLane space. This space is characterized (up to homotopy) by the property that $\pi_n(K(G, n)) \cong G$ if $k = n$ and zero if $k \neq n$. There is a homotopy equivalence

$$K(G, n) \xrightarrow{\cong} \Omega K(G, n + 1).$$

The spaces $K(G, n)$ define an Ω-spectrum HG called the *Eilenberg-MacLane spectrum* associated to G. It is defined as $(HG)_n = K(G, n)$ for $n \geqslant 0$ and zero for $n < 0$. The cohomology theory that it describes

$$\widetilde{HG}^n(X) = [X, K(G, n)]_* \cong \widetilde{H}^n(X; G)$$

for $n \geqslant 0$ corresponds to singular cohomology with coefficients in G, see [8, p. 453].

2.8 The spectrum KU

Recall from 1.6.1 that the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^k)$ consists of all n-dimensional vector subspaces of $\mathbb{C}^k$. The canonical inclusion $\mathbb{C}^k \to \mathbb{C}^{k+1}$ that sends $(v_1, \ldots, v_n)$ to $(v_1, \ldots, v_n, 0)$ induces maps

$$i_k : \mathrm{Gr}_n(\mathbb{C}^k) \longrightarrow \mathrm{Gr}_n(\mathbb{C}^{k+1}).$$

Theorem 2.8.1. *Let $X \in$ **Top**. There is a natural bijection $[X, BU(k)] \cong \mathrm{Vect}^k_{\mathbb{C}}(X)$ that sends f to the pullback $f^*(E_k(\mathbb{C}^n))$.*

If we apply Theorem 2.8.1 with $k + 1$ and $X = BU(k)$, we obtain a bijection

$$[BU(k), BU(k + 1)] \cong \mathrm{Vect}^{k+1}_{\mathbb{C}}(BU(k)).$$

So, taking on the right-hand side the vector bundle $E_k(\mathbb{C}^\infty) \oplus \underline{1}$ over $BU(k)$ gives a map

$$i_k : BU(k) \longrightarrow BU(k + 1)$$

such that $i_k^*(E_{k+1}(\mathbb{C}^\infty) \cong E_k(\mathbb{C}^\infty) \oplus \underline{1}$. We define $BU = \operatorname{colim}_k\{BU(k), i_k\}$ as the colimit of the sequence given by the inclusion maps i_k.

Let $d : \mathrm{Vect}_{\mathbb{C}}(X) \to [X, \mathbb{N}]$ be the function that assigns to a vector bundle $p : E \to X$ the function $d_E : X \to \mathbb{N}$ defined as $d_E(X) = \dim p^{-1}(x)$. The set $[X, \mathbb{N}]$ has an abelian monoid structure defined using the one on $\mathbb{N}$ such that d is a map of abelian monoids. Consider the natural inclusion $[X, \mathbb{N}] \to [X, \mathbb{Z}]$, which is in fact the group completion of $[X, \mathbb{N}]$. By the universal property of the group completion there is a map $\overline{d} : K^0(X) \to [X, \mathbb{Z}]$ and a commutative square

$$\begin{tikzcd}
\mathrm{Vect}_{\mathbb{C}}(X) \arrow[r, "d"] \arrow[d] & {[X, \mathbb{N}]} \arrow[d] \\
K^0(X) \arrow[r, dashed, "\overline{d}"] & {[X, \mathbb{Z}]}
\end{tikzcd}$$

We will denote $\widehat{K}(X) = \ker \overline{d}$.

Proposition 2.8.2. *There is a split short exact sequence*

$$0 \to \widehat{K}(X) \to K^0(X) \to [X, \mathbb{Z}] \to 0.$$

In particular, $K^0(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}]$.

Proof. Let $f \colon X \longrightarrow \mathbb{N}$. Since X is compact, $f(X)$ is compact in $\mathbb{N}$ and hence finite. So suppose that $f(X) = \{n_1, \ldots, n_r\}$. Then $X = X_1 \sqcup \cdots \sqcup X_r$, where each $X_i = f^{-1}(n_i)$. We define a bundle over X by taking trivial bundles $\underline{n_i}$, at each X_i. This defines a map $\varphi \colon [X, \mathbb{N}] \longrightarrow \mathrm{Vect}_{\mathbb{C}} X$ that satisfies $d \circ \phi = id$. Now, using the universal property of the group completion, there exists a map $\overline{\varphi} \colon [X, \mathbb{Z}] \longrightarrow K^0(X)$ that satisfies $\overline{d} \circ \overline{\varphi} = id$. The map $\overline{\varphi}$ a section of the map $[X, \mathbb{Z}] \longrightarrow 0$, thus the sequence splits. □

Corollary 2.8.3. *If $X \in \mathbf{Top}_*$ is connected, then $\widehat{K}(X) \cong \widetilde{K}^0(X)$.*

Proof. Consider the following commutative diagram of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \widehat{K}(X) \arrow[r] \arrow[d] & K^0(X) \arrow[r, "\overline{d}"] \arrow[d, equal] & {[X, \mathbb{Z}]} \arrow[r] \arrow[d, "i^*"'] & 0 \\
0 \arrow[r] & \widetilde{K}^0(X) \arrow[r] & K^0(X) \arrow[r] & {[*, \mathbb{Z}]} \arrow[r] & 0
\end{tikzcd}$$

where $i \colon * \longrightarrow X$ is the inclusion of the basepoint. If X is connected, then i^* is an isomorphism and hence $\widehat{K}(X) \cong \widetilde{K}^0(X)$. □

Consider the sets $\mathrm{Vect}^k_{\mathbb{C}}(X)$ and define for every $k \geqslant 0$

$$t_k \colon \mathrm{Vect}^k_{\mathbb{C}} X \longrightarrow \mathrm{Vect}^{k+1}_{\mathbb{C}} X \tag{2.8}$$
$$[E] \longmapsto [E \oplus \underline{1}] \tag{2.9}$$

and denote by $\mathrm{Vect}_s X = \operatorname{colim}\{\mathrm{Vect}^k_{\mathbb{C}} X, t_k\}$ the colimit of the sequence given by the maps t_k.

Proposition 2.8.4. *For every $X \in \mathbf{Top}$ we have that* $\mathrm{Vect}_s(X) \cong \widehat{K}(X)$

Proof. For each $k \geqslant 0$ consider the maps $\varphi_k\colon \operatorname{Vect}^k_{\mathbb{C}} X \longrightarrow \widehat{K}(X)$ given by $\varphi_k([E]) = [E] - [\underline{k}] \in \widehat{K}(X)$. Then $\varphi_{k+1} \circ t_k([E]) = \varphi_k([E])$ for every k, so by the universal property of the colimit, there is a map $\varphi\colon \operatorname{Vect}_s X \longrightarrow \widehat{K}(X)$ and a commutative triangle

$$\begin{tikzcd} \operatorname{Vect}^k_{\mathbb{C}} X \arrow[r] \arrow[d, "\varphi_k"'] & \operatorname{Vect}_s X \arrow[dl, "\varphi"] \\ \widehat{K}(X) & \end{tikzcd}$$

Since for each vector bundle E there is a vector bundle E' such that $E \oplus E' \cong \underline{n}$ for some n, it follows that φ is bijective. □

Proposition 2.8.5. *For every $X \in$ **Top** there is an isomorphism $\widehat{K}(X) \cong [X, BU]$.*

Proof. By Theorem 2.8.1, $\operatorname{Vect}^k_{\mathbb{C}} X \cong [X, BU(k)]$. The maps $t_k\colon \operatorname{Vect}^k_{\mathbb{C}} X \longrightarrow \operatorname{Vect}^{k+1}_{\mathbb{C}} X$ and $i_k\colon BU(k) \longrightarrow BU(k+1)$ are compatible with this isomorphism. We get an isomorphism after taking colimits

$$\operatorname{colim}_k \operatorname{Vect}^k_{\mathbb{C}} X \cong \operatorname{colim}_k [X, BU(k)].$$

The left hand side is $\operatorname{Vect}_s X$, which by Proposition 2.8.4 is isomorphic to $\widehat{K}(X)$. Since X is compact and the maps i_k are embeddings, the right-hand side is isomorphic to $[X, \operatorname{colim}_k BU(k)] = [X, BU]$. □

Corollary 2.8.6. *If $X \in$ **Top**, then $K^0(X) \cong [X, BU \times \mathbb{Z}]$. If $X \in$ **Top**$_*$ and X is connected, then $\widetilde{K}^0(X) \cong [X, BU]$.*

Proof. By Propositions 2.8.2 and 2.8.5 we have

$$K^0(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}] \cong [X, BU] \oplus [X, \mathbb{Z}] \cong [X, BU \times \mathbb{Z}].$$

The second part follows from Corollary 2.8.3 and Proposition 2.8.5 □

Corollary 2.8.7. *Let $X \in$ **Top**$_*$ such that the inclusion $i : * \to X$ is a cofibration (e.g , if X is a CW-complex). Then $\widetilde{K}^0(X) \cong [X, BU \times Z]_*$.*

Proof. We need to show that $[X, BU \times \mathbb{Z}]_*$ is the kernel of the map

$$K^0(X) \cong [X, BU \times \mathbb{Z}]_* \xrightarrow{i^*} [*, BU \times \mathbb{Z}] \cong K^0(X).$$

Let $j\colon [X, BU \times \mathbb{Z}]_* \longrightarrow BU \times \mathbb{Z}]$ be the natural inclusion. If $f \in [X, BU \times \mathbb{Z}]_*$, then $i^*(j(f))$ is zero in $[*, BU \times \mathbb{Z}]$. So $[X, BU \times \mathbb{Z}]_* \subseteq \ker i^*$. To prove the converse, let $g \in [X, BU \times \mathbb{Z}]$ and suppose that $i^*(j(g))$ is zero. Since BU is connected, there is a homotopy between the basepoint of BU and $g_1(x_0)$, where x_0 denotes the basepoint of X. So we can build a homotopy

$$\alpha\colon \{x_0\} \times I \longrightarrow BU \times \mathbb{Z}$$

between $(g_1(x_0), 0)$ and $(*, 0)$, where $*$ is the basepoint of BU. Now consider the following diagram

$$\begin{tikzcd} X \times \{0\} \cup \{x_0\} \times I \arrow[r, "{(g,\alpha)}"] \arrow[d] & BU \times \mathbb{Z} \\ X \times I \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

Since $* \longrightarrow X$ is a cofibration, there is a lifting H, giving a homotopy between $g(x) = H(x,0)$ and $H(x,1)$ which is a pointed map, since $H(x_0,1) = \alpha(x_0,1) = (*,0)$. $\square$

The family of spaces $E_{2n} = BU \times \mathbb{Z}$ and $KU_{2n+1} = \Omega BU$ for $n \in \mathbb{Z}$ have the property that

$$KU_{2n-1} = \Omega BU = \Omega(BU \times \mathbb{Z}) = \Omega KU_{2n}.$$

By Bott's periodicity (Theorem 2.4.2), we know that $\widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2 X)$, hence Corollary 2.8.7 shows that

$$[X, BU \times \mathbb{Z}]_* \cong \widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2 X) \cong [X, \Omega^2(BU \times \mathbb{Z})]_* = [X, \Omega^2 BU]_*$$

for every pointed (CW-complex) X. So $KU_{2n} = BU \times \mathbb{Z} \simeq \Omega^2 BU = \Omega KU_{2n+1}$.

The sequence $\{KU_n\}_{n \in \mathbb{Z}}$ defines an Ω-spectrum called the *complex K-theory spectrum*, and hence a reduced cohomology theory by Theorem 2.7.4. If X is a pointed finite CW-complex, then $\widetilde{KU}^0(X) \cong \widetilde{K}^0(X)$.

The existence of a homotopy equivalence $BU \times \mathbb{Z} \cong \Omega^2 BU$ is equivalent to Bott periodicity.

Theorem 2.8.8 (Bott periodicity II)**.** *There is a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega^2 BU$*

Proof. See [5]. $\square$

2.9 Acyclic spaces in topological K-theory

An important consequence of the fact that $\widetilde{K}^*$ is a generalized cohomology theory is the following:

Proposition 2.9.1. *If $j \colon X \longrightarrow Y$ is an inclusion of cell complexes such that the induced morphism*

$$j^* \colon \widetilde{K}^n(Y) \longrightarrow \widetilde{K}^n(X)$$

is an isomorphism for all n, then $\widetilde{K}^n(Y/X) = 0$ for all n, i.e, Y/X is K^*-acyclic.

Every continuous map between cell complexes is equivalent to an inclusion in the following sense:

Given any inclusion $X \xrightarrow{f} Y$, there exists a cell complex which we denote by $Zf \simeq Y$ and an inclusion $X \xrightarrow{j} Zf$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Zf \arrow[ur, "h"', "\simeq"] & \end{tikzcd}$$

where h is a homotopy equivalence. The quotient $Zf/X =: Cf$ is called the *cone of f* or the *cofiber of f*. More generally, the following proposition holds.

Proposition 2.9.2. *If $f \colon X \longrightarrow Y$ is a continuous map between cell complexes such that*

$$f^* : \widetilde{K}^n(Y) \longrightarrow \widetilde{K}^n(X)$$

is an isomorphism for all n, then $\widetilde{K}(Cf) = 0$ for all n.

There are examples of maps that induce isomorphisms in $\widetilde{K}^*$; see [15, p. 414] and [17, p. 70].

2.10 Computation of some K-groups

We compute the complex K-theory groups of $\mathbb{R}P^2$ and $\mathbb{C}P^n$

Proposition 2.10.1. *The complex K-theory groups of the real projective plane are;*

$$K^0(\mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2, \qquad \widetilde{K}^0(\mathbb{R}P^2) \cong \mathbb{Z}/2,$$
$$K^1(\mathbb{R}P^2) = 0, \qquad \widetilde{K}^1(\mathbb{R}P^2) = 0.$$

Proof. To compute the complex K-theory groups of $\mathbb{R}P^2$ we start by computing the groups $K^*(S^2)$. Recall the formulas:

$$\widetilde{K}^n(X) \cong \widetilde{K}^n(X) \oplus K^n(*) \tag{2.10}$$

$$\begin{cases} \widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X), \\ K^{-n}(X) = \widetilde{K}^{-n}(X_+). \end{cases} \tag{2.11}$$

We obtain

$$\widetilde{K}^{-1}(S^1) := \widetilde{K}^0(S^2) \cong \mathbb{Z},$$
$$\widetilde{K}^0(S^1) \cong [S^1, BU] = \pi_1(BU) \cong \pi_0(U) = 0,$$
$$\widetilde{K}^0(S^1) \cong [S^1, BU] \cong \pi_1(BU) \cong \pi_0(U) = 0.$$
$$\widetilde{K}^0 S^2) \cong [S^2, BU] = \pi_2(BU) \cong \pi_1(BU) \cong \pi_1(U) \cong \pi_1(U(1)) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Where we recall that $[S^2, BU] = [\Sigma S^1, BU] \cong [S^1, \Omega BU]$,

$$\begin{cases} \widetilde{K}^0(*) = 0, \\ \widetilde{K}^1(*) = 0, \end{cases} \qquad \begin{cases} K^0(*) = 0, \\ K^1(*) = 0. \end{cases} \tag{2.12}$$

$$\begin{cases} \widetilde{K}^0(S^1) = 0, \\ \widetilde{K}^1(S^1) = \mathbb{Z}, \end{cases} \qquad \begin{cases} K^0(S^1) = \mathbb{Z} = K^0(*), \\ K^1(S^1) = \mathbb{Z}. \end{cases} \tag{2.13}$$

$$\begin{cases} \widetilde{K}^0(S^2) = \mathbb{Z}, \\ \widetilde{K}^1(S^2) = 0, \end{cases} \qquad \begin{cases} K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z} = \widetilde{K}^0(S^2) \oplus K^0(*), \\ K^1(S^2) = 0. \end{cases} \tag{2.14}$$

We apply the Mayer-Vietoris sequence to compute $K^*(\mathbb{R}P^2)$. See Figure 2.1; denote by $U_1 = D \simeq *$ and by $U_2 = \mathbb{R}P^2 \backslash \{p\} \simeq S^1$. Then $U_1 \cap U_2 \cong S^1$ and $U_1 \cup U_2 = \mathbb{R}P^2$. We thus have inclusion maps $i_1 \colon U_1 \cap U_2 \longrightarrow U_1 \simeq *$ and $i_2 \colon U_1 \cap U_2 \longrightarrow U_2 \simeq S^1$. The inclusion $S^1 \xrightarrow{i_2} S^1$ has degree 2, since we have an isomorphism $K^0(S^1) \xrightarrow{\cong} K^0(S^1)$ and the map $\widetilde{K}^0(S^2) \longrightarrow \widetilde{K}^0(S^2)$ induced by the suspension $S^2 \xrightarrow{\Sigma i_2} S^2$ has degree 2. The other inclusion $S^1 \xrightarrow{i_1} *$ induces $K^0(U_1) \xrightarrow{i_1^*} K^0(U_1 \cap U_2)$, that is, $K^0(*) \xrightarrow{\cong} K^0(S^1)$, so it must correspond to multiplication by 0. We can write down the corresponding Mayer-Vietoris sequence;

$$\begin{tikzcd} K^0(\mathbb{R}P^2) \arrow[r] & K^0(*) \oplus K^0(S^1) \arrow[r] & K^0(S^1) \arrow[d, "\triangle"] \\ K^1(S^1) \arrow[u, "\triangle"] & K^1(*) \oplus K^1(S^1) \arrow[l] & K^1(\mathbb{R}P^2) \cong K^{-1}(\mathbb{R}P^2) \arrow[l] \end{tikzcd}$$

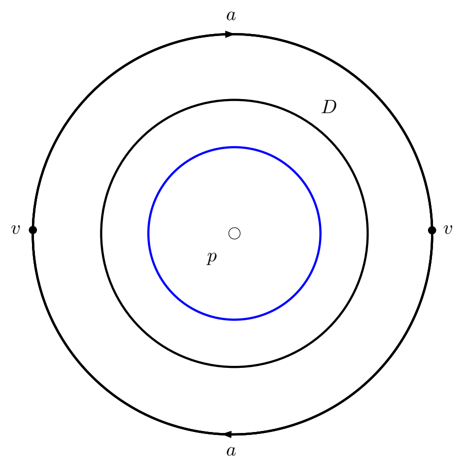

Figure 2.1: Real projective plane $\mathbb{R}P^2$

and by the previous computations we obtain;

$$\begin{tikzcd}
& & \mathbb{Z}=\langle e_1-e_2\rangle \arrow[dr, bend left] & & & \\
& \mathbb{Z}\oplus\mathbb{Z}/2 = K^0(\mathbb{R}P^2) \arrow[ur, bend left] \arrow[rr] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] \arrow[dd] & 0 \arrow[ddl] \\
\mathbb{Z}/2 \arrow[ur, bend left] & & & & & \\
& \mathbb{Z} \arrow[uu] \arrow[ul, bend left] & & \mathbb{Z} \arrow[ll, "\cdot 2"'] & K^1(\mathbb{R}P^2)=0 \arrow[l] \arrow[dl, bend left] & \\
& & & 0 \arrow[u, bend left] & &
\end{tikzcd}$$

Thus,

$$K^0(\mathbb{R}P^2) \cong \mathbb{Z}\oplus\mathbb{Z}/2, \qquad \widetilde{K}^0(\mathbb{R}P^2) \cong \mathbb{Z}/2,$$
$$K^1(\mathbb{R}P^2) = 0, \qquad \widetilde{K}^1(\mathbb{R}P^2) = 0.$$

$\square$

Proposition 2.10.2. *If X is a finite cell complex with n cells, then $K^*(X)$ is a finitely generated group with at most n generators. If all the cells of X have even dimension then $K^1(X) = 0$ and $K^0(X)$ is free abelian with one basis element for each cell.*

Proof. We show this by induction on the number of cells. The complex X is obtained from a subcomplex by attaching a k-cell for some k. For the pair (X,Y) we have an exact sequence $\widetilde{K}(X/Y) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}^*(Y)$. Since $X/Y = S^k$, we have $\widetilde{K}(X/Y) \cong \mathbb{Z}$, and exactness implies that $\widetilde{K}^*(X)$ requires at most one more generator than $\widetilde{K}^*(Y)$. The first term of the exact sequence $K^1(X/Y) \longrightarrow K^1(X) \longrightarrow K^1(Y)$ is zero if all cells of X are of even dimension, so induction on the number of cells implies that $K^1(X) = 0$. Then there is a short exact sequence

$$0 \longrightarrow \widetilde{K}^0(X/Y) \longrightarrow \widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(Y) \longrightarrow 0$$

with $\widetilde{K}^0(X/Y) \cong \mathbb{Z}$. By induction $\widetilde{K}(Y)$ is free, so this sequence splits, hence

$$K^0(X) \cong \mathbb{Z} \oplus K^0(Y).$$

$\square$

Corollary 2.10.3. *The complex K-theory groups of $\mathbb{C}P^n$ are;*

$$K^0(\mathbb{C}P^2) \cong \mathbb{Z}^{n+1}, \tag{2.15}$$
$$K^1(\mathbb{C}P^n) = 0. \tag{2.16}$$

Proof. Since the complex projective plane $\mathbb{C}P^n$ has a cell structure with one cell in each dimension $0, 2, 4, \cdots, 2n$, by Proposition 2.10.2 we are done. $\square$

Part II

Algebraic K-theory

Chapter 3

K_0 and K_1 of a Ring

Finitely generated R-modules are vector spaces when R is a field, and these have well defined notions of basis and dimension. In them, dimension is the only isomorphism invariant, and therefore the monoid of isomorphism classes of finitely generated R-modules is isomorphic to the additive monoid $\mathbb{N}$. Taking the group completion of this monoid we obtain the additive group $\mathbb{Z}$ as an algebraic invariant of such modules. For a ring R, not necessarily a field, this procedure can be done similarly taking the group completion of isomorphism classes of finitely generated projective R-modules. Projective modules are a natural generalization of free modules and the starting point of algebraic K-theory. We shall recall briefly some definitions and properties about them, following [14], but the reader can also find complete and self contained treatments in the books [2] and [19].

3.1 Projective modules

In this section we assume R is a ring and all our modules are, unless we state the contrary, left R-modules.

An R-module is called *projective* if it is a direct summand of a free R-module. If M is projective, then any R-module isomorphic to M is also projective, and an R-module P is projective if and only if there exists a free R-module F and R-linear maps

$$\begin{tikzcd} F \arrow[r, shift left, "g"] & P \arrow[l, shift left, "h"] \end{tikzcd}$$

such that $g \circ h = id_P$. Also, an R-module P is projective if and only if there exists an R-module Q for which $P \oplus Q$ is free. We have the following characterization of being projective by means of maps between modules.

Proposition 3.1.1. *The following conditions on a R-module P are equivalent:*

(i) P is projective.

(ii) For each diagram of R-linear maps

$$\begin{tikzcd} & P \arrow[d, "j"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

with exact row, there is an R-linear map $h : P \longrightarrow M$ with $g \circ h = j$.

(iii) Every surjective R-linear map $g : M \longrightarrow P$ has an R-linear right inverse.

(iv) Every short exact sequence or R-linear maps

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

splits.

Proof. This can be found in [14, p. 53] □

If F is a free R-module with basis B, each $m \in F$ has a unique expression

$$m = \sum_{b \in B} c(b, m)b,$$

where $c(b, m) \in R$ and $c(b, m) = 0$ for all but finitely many $b \in B$. For each $b \in B$, the map

$$\begin{aligned} b^* : F &\longrightarrow R \\ m &\longmapsto c(b, m) \end{aligned}$$

is called *projection to the b-coordinate* and is R-linear. This provides a function

$$\begin{aligned} (\ )^* : B &\longrightarrow \mathrm{Hom}_R(F, R) \\ b &\longmapsto b^* \end{aligned}$$

A *projective basis* of an R-module M is any function

$$\begin{aligned} (\ )^* : S &\longrightarrow \mathrm{Hom}_R(M, R) \\ s &\longmapsto s^*, \end{aligned}$$

where $S \subseteq M$, and where, for each $m \in M$

(i) $s^*(m) = 0$ for all but finitely many $s \in S$, and

(ii) $m = \sum_{s \in S} s^*(m)s$.

We say that such a set S is a *generating set* of M, and there is also the following characterization of projectivity

Proposition 3.1.2. *An R-module P is projective if and only if P has a projective basis. If P is projective, then every generating set S of P is the domain of a projective basis.*

And finally, for finitely generated projective R-modules we have this particularly simple description:

Proposition 3.1.3. *Let P be an R-module and n be a positive integer. The following conditions are equivalent:*

(i) P is projective and generated by n elements.

(ii) P is isomorphic to a direct summand of R^n.

(iii) P is isomorphic to the R-module generated by the rows of an idempotent matrix in $M(n, R)$.

(iv) There exists an R-module Q with $P \oplus Q \cong R^n$.

For a proof of Proposition 3.1.3 see [14, p. 55]. It will be used thoroughly.

3.2 The Grothendieck group K_0

We denote by $\operatorname{Proj} R$ the set of isomorphism classes of finitely generated projective R-modules. $\operatorname{Proj} R$ is an abelian monoid with the direct sum $\oplus$ operation and the 0-module as the identity. Like we did in the topological setting with $K^0(X)$, we take the group completion of this monoid.

Definition 3.2.1. Let R be a ring with unit. The *Grothendieck group of* R is the group completion of $\operatorname{Proj} R$, and we denote it by $K_0(R)$.

Recall that this group completion can be seen has the quotient $\mathcal{F}/\mathcal{R}$ where $\mathcal{F}$ is the free abelian group on $\operatorname{Proj} R$ and $\mathcal{R}$ is the subgroup generated by elements of the form

$$[P \oplus Q] - [P] - [Q], \quad P, Q \in \operatorname{Proj} R$$

If P and Q are finitely generated projective R-modules, we say that P is *stably isomorphic* to Q if for some $n \in \mathbb{N}$ we have $R^n \oplus P \cong R^n \oplus Q$. The following assertion is an immediate consequence of the abstract properties of the group completion

Lemma 3.2.2. *Let P and Q be finitely generated projective R-modules. Then $[P] = [Q]$ if and only if P is stably isomorphic to Q.*

Example 3.2.3 (Division rings)**.** Let R be a division ring. Every finitely generated R-module is a finite-dimensional vector space, thus, for each generator $[P] \in K_0(R)$ we have that $[P] = [R^n] = n[R]$, where n is the dimension of P. From this we see that $K_0(R)$ is generated by $[R]$. If there exists $m \in \mathbb{N}$ that satisfies $[0] = m[R] = [R^m]$, by Lemma 3.2.2, there exists $n \in \mathbb{N}$ for which

$$R^n \oplus 0 \cong R^n \oplus R^m$$

and thus

$$n = \dim_R(R^n \oplus 0) = \dim_R(R^n \oplus R^m) = n + m$$

Therefore, $m = 0$ and $K_0(R) \cong \mathbb{Z}$, since $[R]$ does not have finite order in $K_0(R) = \langle [R] \rangle$.

Given a ring R, the existence of a well defined rank function is crucial in order to have non-trivial K_0 groups, as we shall see in Proposition 3.4.2. By well defined rank function we mean that $R^m \cong R^n$ implies $m = n$.

Example 3.2.4 (Grothendieck group of the integers)**.** Any finitely generated $\mathbb{Z}$-module can be decomposed in a direct sum of its torsion and torsion-free part. The torsion-free part consists of finitely many copies of $\mathbb{Z}$, and its torsion part consists of finitely many primary cyclic groups. Thus, every finitely generated projective $\mathbb{Z}$-module is torsion-free, since being projective implies being torsion-free, so $K_0(\mathbb{Z})$ is cyclic and generated by $[\mathbb{Z}]$. Since the rank of a $\mathbb{Z}$-module is well defined, in fact we have that $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Let$\varphi : R \to S$ be a ring homomorphism, we can view S as an (R, S)-bimodule by defining $sr = s\varphi(r)$ for all $r \in R$ and $s \in S$. The additive functor $S \otimes_R \square : {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$, when restricted to finitely generated projective R-modules, maps them to finitely generated projective S-modules, and then we can define $K_0(\varphi) : K_0(R) \to K_0(S)$ by $[P] \mapsto [S \otimes_R P]$. In particular if $P = R^n$ is a finitely generated free R-module, we have $K_0(\varphi)([R^n]) = [S \otimes_R R^n] = [S^n]$, and thus, K_0 is a functor from rings to abelian groups.

One might wonder why we restrict ourselves to finitely generated projective modules in the construction of the Grothendieck group. The reason is that if we allow countable generated modules, the Grothendieck group would always be trivial.

Remark 3.2.5 (Eilenberg swindle)**.** If P is a countably generated projective module over a non-trivial ring R, and $P \oplus Q$ is free, then

$$F := \bigoplus_{i \geqslant 1} (P \oplus Q)$$

is also free, and $P \oplus F \cong P \oplus ((P \oplus Q) \oplus (P \oplus Q) \oplus \dots) \cong P \oplus ((Q \oplus P) \oplus (Q \oplus P) \oplus \dots) \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong F$ so we would thus have that $P \oplus F \cong F$ and then $[P] + [F] = [F]$, leading to $[P] = [0]$ for any P.

3.3 K_0 from idempotents

If R is a ring with unit, we denote by $M(n, R)$ the $n \times n$ matrix ring over R, and by $\mathrm{GL}(n, R)$ the group of $n \times n$ invertible matrices over R. For any $n \in \mathbb{N}$, we can consider the inclusions $M(n, R) \subset M(n+1, R)$ and $\mathrm{GL}(n, R) \subset \mathrm{GL}(n+1, R)$ given respectively by

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

Notice that the first inclusion is a non-unital ring homomorphism. Taking the colimit along these inclusions, we define $M(R) = \bigcup_{n=1}^{\infty} M(n, R)$ and $\mathrm{GL}(R) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, R)$.

An *idempotent* is an element $e \in R$ that satisfies $e^2 = e$. If $e \in R$ is an idempotent, we can consider the R-module $P = Re$. This module can be decomposed as $R = Re \oplus R(1-e)$, thus, it is projective. Conversely, given a decomposition $R = P \oplus Q$, there are unique elements $e \in P$ and $f \in Q$, such that $e + f = 1 \in R$. The elements e and $f = 1 - e$ are idempotent, and $ef = fe = 0$. Thus, we have a bijective correspondence between idempotent elements of R and decompositions $R \cong P \oplus Q$.

If P is a finitely generated projective R-module, we have $P \oplus Q = R^n$ for some $n \in \mathbb{N}$. The composition $e : R^n \xrightarrow{\alpha} P \xrightarrow{\beta} R^n$ is an idempotent endomorphism of the ring R^n, where α is the projection onto P and β is the inclusion $p \mapsto (p, 0)$. This idempotent can be identified with a matrix in $M(n, R)$. The image $e(R^n)$ of e is $P \oplus \{0\} \cong P$.

We denote the set of idempotent elements of $M(n, R)$ by $\mathrm{Idem}(n, R)$, and we have

$$\mathrm{Idem}(R) = \bigcup_{n=1}^{\infty} \mathrm{Idem}(n, R) \subset M(R)$$

The *block sum* of matrices p and q is defined as the matrix

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

set $\mathrm{Idem}(R)$ is closed under the block sum operation, and if e is conjugate to p by u, an f is conjugate to q by v, we have

$$e \oplus f = (u \oplus v)(p \oplus q)(u \oplus v)^{-1}$$

For each class $[P] \in \operatorname{Proj} R$, we can associate an idempotent $e \in \operatorname{Idem}(n, R)$ for some n. Different idempotent matrices can give rise to to the same isomorphism classes of projective modules, and to compute $K_0(R)$ from idempotent matrices we need to describe the equivalence relation between them that corresponds to isomorphism of the corresponding modules.

Lemma 3.3.1. *Let $p \in \operatorname{Idem}(n, R)$ and $q \in \operatorname{Idem}(m, R)$. The corresponding finitely generated projective R-modules, $R^n p$ and $R^m q$ are isomorphic if and only if we can enlarge the sizes of p and q (padding with zeroes in the right lower right-hand corner) so that they have the same size $N \times N$ and are conjugate under the group* $\operatorname{GL}(N, R)$

Proof. $\boxed{\Leftarrow}$ Adding zeroes if necessary, we can assume that p and q are of the same size, so there exists a matrix $u \in \operatorname{GL}(N, R)$ such that $upu^{-1} = q$. Now, right multiplication by u induces an isomorphism from $R^N q$ to $R^N p$, $\boxed{\Rightarrow}$ Let $p \in \operatorname{Idem}(n, R)$ and $q \in \operatorname{Idem}(m, R)$, assume that $\alpha : R^n p \to R^m q$ is an isomorphism. We can extend α to an R-module homomorphism $R^n \to R^m$ taking $\alpha = 0$ on the complementary module $R^n(1-p)$ and viewing the image $R^m q$ as embedded in R^m. Similarly, α^{-1} extends to an R-module homomorphism $\beta : R^m \to R^n$ that maps to 0 the elements in $R^m(1-q)$. Under this extensions, α is given by multiplication on the right by an $n \times m$ matrix a, and β is given by multiplication on the right by an $m \times n$ matrix b. We have the relations $ab = p$, $ba = q$, $a = pa = aq$ and $b = qb = bp$. The matrix $\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$ is invertible, as its square is $1_n \oplus 1_m$, and it conjugates $p \oplus 0$ to $0 \oplus q$, since

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} = \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

and the matrix $0 \oplus q$ is conjugate to $q \oplus 0$ by a permutation matrix. □

Lemma 3.3.1 allows us to give another description of the Grothendieck group.

Theorem 3.3.2. *For any ring R, the monoid* $\operatorname{Proj} R$ *can be identified with the set of conjugation orbits of* $\operatorname{GL}(R)$ *on* $\operatorname{Idem}(R)$. *The semigroup operation is induced by block sum.*

$$(p, q) \longmapsto \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

The Grothendieck group $K_0(R)$ is the group completion of this semigroup.

Proof. By Lemma 3.3.1, $\operatorname{Proj} R$ is isomorphic to the orbit space $\operatorname{Idem}(R)/\operatorname{GL}(R)$, now recall that the group completion is a functorial construction. □

Theorem 3.3.3 (Morita invariance)**.** *Let R be a ring. For any positive integer $n \in \mathbb{N}$, there is a natural isomorphism*

$$K_0(R) \xrightarrow{\cong} K_0(M(n, R))$$

Proof. It follows from Theorem 3.3.2. By the usual identification of $M(m, M(n, R))$ with $M(m \cdot n, R)$, is clear that $\operatorname{Idem}(M(n, R)) = \operatorname{Idem}(R)$ and $\operatorname{GL}(M(n, R)) = \operatorname{GL}(R)$. □

The Grothendieck functor is continuous in the following sense

Theorem 3.3.4. *Let $(R_\alpha)_{\alpha\in I}, (\theta_{\alpha\beta}; R_\alpha \longrightarrow R_\beta)_{\alpha<\beta}$ be a directed system of rings and let $R = \varinjlim R_\alpha$ be the direct limit of the system. Then*

$$K_0(R) \cong \varinjlim K_0(R_\alpha).$$

Proof. See [18, p. 9] □

We are ready to give our first example of a ring whose Grothendieck group vanishes.

Example 3.3.5 (A ring with vanishing K_0)**.** We make use of Theorem 3.3.2 to construct a ring R for which all projective modules are stably isomorphic to one another, and hence, for which $K_0(R) = 0$. Let $\mathbb{F}$ be a field and let V be an infinite-dimensional vector space over $\mathbb{F}$. Consider the ring $R = \mathrm{End}_{\mathbb{F}}(V)$. Let $p, q \in \mathrm{Idem}(R)$, they are idempotents in $M(n, R)$ for some $n \in \mathbb{N}$. Consider $p \oplus 1 \oplus 0$ and $q \oplus 1 \oplus 0$ in

$$M(n+2, R) \cong \mathrm{End}_{\mathbb{F}}(\mathbb{F}^{n+2}) \otimes_{\mathbb{F}} R \cong \mathrm{End}_{\mathbb{F}}(\mathbb{F}^{n+2}) \otimes_{\mathbb{F}} \mathrm{End}_{\mathbb{F}}(V) \cong \mathrm{End}_{\mathbb{F}}(V^{n+2}) \cong \mathrm{End}_{\mathbb{F}}(V) \cong R,$$

Since V is infinite-dimensional, V^{n+2} and V have the same dimension over $\mathbb{F}$. Now

$$0 \leqslant \mathrm{rank}\, p \leqslant \dim_{\mathbb{F}}(V^n) = n \dim(V) = \dim(V)$$

Thus $\dim V \leqslant \mathrm{rank}(p \oplus 1 \oplus 0) \leqslant \dim V + \dim V$ and $\mathrm{rank}((p \oplus 1 \oplus 0) = \dim V$. Similarly, $\mathrm{rank}(q \oplus 1 \oplus 0)) = \dim V$, and

$$\begin{aligned} \mathrm{rank}((1 \oplus 1 \oplus 1)(p \oplus 1 \oplus 0)) &= \mathrm{rank}((1-p) \oplus 0 \oplus 1) = \dim V, \\ \mathrm{rank}((1 \oplus 1 \oplus 1)(q \oplus 1 \oplus 0)) &= \mathrm{rank}((1-q) \oplus 0 \oplus 1) = \dim V. \end{aligned}$$

Since $p \oplus 1 \oplus 0$ and $q \oplus 1 \oplus 0$ are idempotent endomorphisms of a vector space with same rank and corank, they are conjugate. Hence, $p \oplus 1 \oplus 0 \simeq q \oplus 1 \oplus 0$ and hence $[p] = [q]$ in $K_0(R)$.

As a result of the characterization by idempotents of K_0 we can prove easily the following assertion.

Proposition 3.3.6. *Let $R = R_1 \times R_2$ be a cartesian product of rings. There is a natural isomorphism $K_0(R_1 \times R_2) \cong K_0(R_1) \oplus K_0(R_2)$*

Proof. There is an isomorphism $M(R) \cong M(R_1) \times M(R_2)$ given by $(r_{ij}) \longmapsto (p_1(r_{ij}), p_2(r_{ij}))$. It sends $\mathrm{GL}(R_1 \times R_2)$ to $\mathrm{GL}(R_1) \times \mathrm{GL}(R_2)$ and $\mathrm{Idem}(R_1 \times R_2)$ to $\mathrm{Idem}(R_1) \times \mathrm{Idem}(R_2)$, so it induces an isomorphism $K_0(R_1 \times R_2) \cong K_0(R_1) \times K_0(R_2)$.

□

3.4 K_0 of commutative and local rings

The ring of integers $\mathbb{Z}$ is an initial object in the category of rings with unit. For any ring with unit R , we denote by $i : \mathbb{Z} \longrightarrow R$ the unique ring homomorphism which sends $1 \in \mathbb{Z}$ to the unit element of R. It induces a ring homomorphism $i_* : K_0(\mathbb{Z}) \longrightarrow K_0(R)$. By the properties of the group completion, $i_*(K_0(\mathbb{Z}))$ is the subgroup of $K_0(R)$ generated by all finitely free R-modules. If V is an infinite-dimensional vector space over a field $\mathbb{F}$, the ring $R = \mathrm{End}_{\mathbb{F}}(V)$ satisfies $R \cong R \oplus R$ as R-modules, thus, in this case, $i_*(K_0(\mathbb{Z})) = \{0\}$. In fact we have seen in Example 3.3.5) that $K_0(R) = \{0\}$. The group $i_*(K_0(\mathbb{Z}))$ measures in some sense the non-trivial part of $K_0(R)$.

Definition 3.4.1. Let R be a ring with unit. The group

$$\widetilde{K}_0(R) = \operatorname{coker} i_*(K_0(\mathbb{Z})) = K_0(R)/i_*(K_0(\mathbb{Z}))$$

is called the *reduced Grothendieck group* of R

Proposition 3.4.2. *Let R be a ring with unit, all of whose finitely generated projective R-modules are free and which has a rank function; that is $R^m \cong R^n$ implies $n = m$. Then $K_0(R) \cong \mathbb{Z}$. In particular, $K_0(R) \cong \mathbb{Z}$ if R is a division ring, a principal ideal domain, a local ring, or $\mathbb{F}[x_1, \ldots, x_n]$ for any field $\mathbb{F}$.*

Proof. Consider the function $r : \operatorname{Proj} R \to \mathbb{N}$ induced by the rank, that is, $r(R^n) = n$. This map is well-defined, and it is a semigroup map, since

$$r(V \oplus W) = r(V) + r(W)$$

It is an insomorphism because we are assuming that two free modules P and Q are isomorphic if and only if $r(P) = r(Q)$. Since group completion is a functor, r induces an isomorphism $r_* : K_0(R) \to \mathbb{Z}$. Each of the rings mentioned satisfy the hypothesis. For division rings we have seen it previously in Example 3.2.3, for principal ideal domains see [20, p. 638], for local rings see [20, p. 887] and for polynomial rings in several variables over a field, it follows from Quillen-Suslin Theorem, see [19, p. 209]. □

3.5 The Serre-Swan Theorem

Given a compact Hausdorff space X and a vector bundle over X, we prove that the set of sections $\Gamma(X, E)$ is a finitely generated projective module over the ring of continuous complex-valuated functions $C(X)$, and every finitely generated projective $C(X)$-module arises, up to isomorphism, from this construction. This is known as the Serre-Swan theorem. In fact, there is an equivalence between the category of finitely generated projective modules over $C(X)$ and the category of complex vector bundles on X (see [23, p. 267]). The group K^0 of a compact Hausdorff space, and the Grothendieck group of $C(X)$ are thus isomorphic. This is a nice result that relates topological and algebraic K-theory.

Proposition 3.5.1. *The set of sections $\Gamma(X, E)$ of a (complex) vector bundle $E \xrightarrow{p} X$ is a finitely generated projective module over the ring of continuous complex-valued functions when X is compact Hausdorff.*

Proof. For every point $x \in X$ we have a neighborhood $U \subseteq X$ and a local trivialization $p^{-1}(U) \cong U \times \mathbb{C}^n$ for some $n \in \mathbb{Z}_{\geqslant 0}$. We have functions constant functions $e_j : U \to \mathbb{C}^n$ for $1 \leqslant j \leqslant n$, determined by the standard basis of $\mathbb{C}^n$. These functions generate the sections of this vector bundle over the ring of complex-valued continuous functions. By compactness, we have a finite open covering $\{U_i\}$ of X. If we choose a partition of unit $\{f_i\}$ subordinated to the finite covering $\{U_i\}$. Given a section e_j corresponding to the open U_i, we consider $e_{ij}(x) = e_j(x)f_i(x)$, this is a section supported in U_i, and can be extended to all X just by defining it as identically zero on $X \backslash U_i$. By construction, these e_{ij} generate $\Gamma(X, E)$ as a $C(X)$-module. Thus, $\Gamma(X, E)$ is finitely generated. Now, to see that $\Gamma(X, E)$ is projective, we choose a set of generators of $\Gamma(X, E)$ as a $C(X)$-module. Let $\{s_j\}_{j=1}^k$ be this set of generators, we can construct a vector bundle morphism φ between the trivial bundle $X \times \mathbb{C}^k \xrightarrow{\pi_1} X$ and E, by sending

$$(x, v_1, \ldots, v_k) \longmapsto \sum_{j=1}^{k} v_j s_j(x).$$

Since for each $x \in X$, the $s_j(x)$ span $p^{-1}(x)$, the morphism φ is surjective on each fiber. The subbundle defined fiberwise by $E'_x = \ker \varphi_x$ will be denoted E'. It is locally trivial, since it is trivial over any open subset where E is trivial. Now, we must check that

$$\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, X \times \mathbb{C}^k) \cong \mathbb{C}^k$$

To see this, it suffices to show that $E \oplus E' \cong X \times \mathbb{C}^k$. Choose an hermitian metric on E and consider also the induced hermitian metric on $X \times \mathbb{C}^k$ that comes from the usual inner product of $\mathbb{C}^k$. With respect to this metrics, the morphism φ has and adjoint φ^* which satisfies the relation

$$\langle \varphi v, w \rangle = \langle v, \varphi^* w \rangle$$

Since φ is surjective on each fiber, φ^* will be injective on each fiber, and its image will be the orthogonal complement of $E = \ker \varphi$. Thus, φ^* gives an isomorphism of vector bundles from E to $E'^{\perp}$, showing that $E \oplus E' \cong X \times \mathbb{C}^k$ □

Proposition 3.5.2. *If X is a compact Hausdorff space and P is a finitely generated projective module over $R = C(X)$, then $P \cong \Gamma(X, E)$ for some vector bundle $E \xrightarrow{p} X$.*

Proof. As usual, $P \oplus Q \cong R^n$ for some n. Notice that $C(X)^n \cong C(X, \mathbb{C}^n)$, thus, the elements of P are continuous functions from X to $\mathbb{C}^n$. We consider the set

$$E = \{(x, v_1, \ldots, v_n) \in X \times \mathbb{C}^n : \exists s \in P \text{ such that } s(x) = (v_1, \ldots, v_n)\}$$

and the map $E \to X$ given by $p(x, v_1, \ldots, v_n) = x$. Is clear that $\Gamma(X, E) = P$, so we only have to check that $E \xrightarrow{p} X$ is a vector bundle. First we check that it is locally trivial. Let $e^1, \ldots, e^r \in P$ such that $e^1(x), \ldots, e^r(x)$ are a basis for each fiber $E_x = p^{-1}(x)$ of $\mathbb{C}^n$. This elements are vector-valued functions, so we write $e^i = (e^i_1, \ldots, e^i_n)$. Since they are linearly independent, we can choose indices $1 \leqslant j_1 < \cdots < j_r \leqslant n$, so that

$$e = \det \begin{pmatrix} e^1_{j_1} & e^1_{j_2} & \cdots & e^1_{j_r} \\ \vdots & & & \vdots \\ e^r_{j_1} & e^r_{j_2} & \cdots & e^r_{j_r} \end{pmatrix}$$

is distinct from 0 at zx. Similarly, we may choose elements $f^1, \ldots, f^{n-r} \in Q$ such that $f^1(x), \ldots, f^{n-r}(x)$ form a basis for the image of Q in $\mathbb{C}^n$ at x. Notice that the dimensions must be complementary, since $P \oplus Q = R^n \cong C(X, \mathbb{C}^n)$. Now we can consider a determinant f that we build in a similar way as we did for e. Since the determinants are continuous, in some neighborhood U of x, both are distinct from zero. At any point y of this neighborhood $e^1(y), \ldots, e^r(y)$ are linearly independent, and generate a free submodule of P rank r. Similarly, $f^1(y), \ldots, f^{n-r}(y)$ are linearly independent and generate a free submodule of Q rank $n - r$. Now the dimensions imply that these must exhaust P and Q, so both P and Q are trivial over U. □

3.6 The Bass-Whitehead group K_1

Given a ring with unit R, we want to construct another algebraic invariant associated to this ring. Our invariant should be an abelian group, as is $K_0(R)$. There is a group that arises in a natural way from the ring R, which is the infinite general linear group $\mathrm{GL}(R)$, and is also strongly related with K_0, as we have seen. This is highly non-abelian, but we obtain an abelian group $K_1(R)$, just by considering the quotient with respect to the commutator subgroup. It turns out that this commutator subgroup has a very nice property, it is generated by the infinite elementary matrices. Recall from 3.3 the definitions of $\mathrm{GL}(R)$ and $M(R)$.

Definition 3.6.1. An invertible matrix $M \in \mathrm{GL}(n, R)$ is called an *elementary matrix* if M has the form $I + ae_{ij}$, where I is the identity matrix, $a \in R$, $i \neq j$, and e_{ij} is the matrix with 1 in the (i, j)-th position and 0 elsewhere. We denote by $E(n, R)$ the subgroup of $\mathrm{GL}(n, R)$ generated by all $n \times n$ elementary matrices, and by $E(R)$ the colimit of $E(n, R)$ with respect to the inclusion $\mathrm{GL}(n, R) \hookrightarrow \mathrm{GL}(n + 1, R)$.

Lemma 3.6.2. *The elementary matrices over a ring R satisfy the relations*

$$\begin{aligned}
&(i) && e_{ij}(a) \cdot e_{ij}(b) = e_{ij}(a + b); \\
&(ii) && e_{ij}(a) \cdot e_{kl}(b) = e_{kl}(b) \cdot e_{ij}(a), && j \neq k \text{ and } i \neq l; \\
&(ii) && e_{ij}(a) \cdot e_{jk}(b) \cdot e_{ij}(a)^{-1} \cdot e_{jk}(b)^{-1} = e_{ik}(ab), && i, j, k \text{ distinct}; \\
&(iii) && e_{ij}(a) \cdot e_{ki}(b) \cdot e_{ij}(a)^{-1} \cdot e_{ki}(b)^{-1} = e_{kj}(-ba), && i, j, k \text{ distinct}.
\end{aligned}$$

Furthermore, any upper-triangular or lower triangular matrix with 1's on the diagonal belongs to $E(R)$.

Proof. It is a matter of multiply and check. For the last assertion, it is known from linear algebra that any upper or lower triangular matrix can be reduced by elementary operations to the identity, and these operations correspond to multiplication by elementary matrices. □

Lemma 3.6.3. *For any $A \in \mathrm{GL}(n, R)$,*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(2n, R)$$

Proof. It follows from Lemma 3.6.2 and the identity

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The first three factors of the right hand side lie in $E(2n, R)$, and the last factor on the right lies also in $E(2n, R)$ since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

□

Proposition 3.6.4 (Whitehead's Lemma)**.** *If R is a ring with unit, then*

$$E(R) = [E(R), E(R)] = [\mathrm{GL}(R), \mathrm{GL}(R)]$$

Proof. The first equality follows from (*iii*) of Lemma 3.6.2. To check the second one, we compute

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

By Lemma 3.6.3 and the first equality, we have

$$[\mathrm{GL}(R), \mathrm{GL}(R)] \subseteq E(R) = [E(R), E(R)] \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)]$$

□

Definition 3.6.5. Let R be a ring with unit. The *Bass-Whitehead group of* R, denoted by $K_1(R)$, is the group

$$\mathrm{GL}(R)_{\mathrm{ab}} = \mathrm{GL}(R)/E(R)$$

Any ring homomorphism $\varphi : R \to S$ induces, by entrywise application, a ring homomorphism $M(R) \to M(S)$, which restricts to a group homomorphism from $\mathrm{GL}(R)$ to $\mathrm{GL}(S)$, and hence from $\mathrm{GL}(R)_{\mathrm{ab}}$ to $\mathrm{GL}(S)_{\mathrm{ab}}$. In this way, we have a functor from rings to abelian groups, which is called the *Bass-Whitehead functor*.

The group operation in $K_1(R)$ can be described in two different ways. As a quotient group, given two invertible matrices $A, B \in \mathrm{GL}(n, R)$, the product of the corresponding classes $[A]$ and $[B]$ in $K_1(R)$ is just $[A] \cdot [B] = [AB]$. On the other hand, we can consider the block sum of A and B,

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Since $B \oplus B^{-1} \in E(R)$ by Lemma 3.6.3, and the matrices AB and $AB \oplus 1$ are identified in $\mathrm{GL}(R)$, we have

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix},$$

That is,

$$[A \oplus B] = [(AB \oplus 1)(B^{-1} \oplus B)] = [AB \oplus 1] = [AB].$$

One may also intepret $K_1(R)$ as the group of canonical forms for invertible matrices over R under elementary row or column operations (in the usual sense of linear algebra). For if $A \in M(n, R)$, $e_{ij}(a)A$ is the matrix obtained from A by adding a times the j-th row to the i-th row (an elementary row operation), and $Ae_{ij}(a)$ is the matrix obtained from A by adding a times the i-th column to the j-th column (an elementary column operation). Vanishing of $K_1(R)$ for instance, would mean that every matrix in $\mathrm{GL}(R)$ can be row-reduced or column-reduced to the identity matrix

Proposition 3.6.6 (Morita invariance)**.** *Let R be a ring with unit. For any positive integer $n \in \mathbb{N}$, there is a natural isomorphism*

$$K_1(R) \xrightarrow{\cong} K_1(M(n, R))$$

Proof. Recall that in Theorem 3.3.3 we have seen that $\mathrm{GL}(M(n,R) \cong \mathrm{GL}(R)$. We must check that this isomorphism also identifies $E(M(n,R))$ with $E(R)$. Note that elementary matrices in $M(n,R)$, when viewed as a matrices in R, are upper triangular, thus, by Lemma 3.6.2, we have that $E(M(n,R)) \subseteq E(R)$. Conversely, the image of the generators of $E(M(n,R))$ generate $E(R)$, since it contains all elementary matrices, except the ones with an entry in some slot of an $n \times n$ identity matrix on the diagonal. But if $e_{ij}(a)$ is such a matrix, then $e_{(i+n)j}(1)$ and $e_{i(i+n)}(a)$ are not, and we have the relation

$$e_{ij}(ab) = e_{ik}(a)e_{kj}(b)e_{ik}(a)^{-1}e_{kj}(b)^{-1}.$$

Taking $k = i+n$ and $b = 1$, we see that $E(R) \subseteq E(M(n,R))$. So we obtain an isomorphism

$$\mathrm{GL}(M(n,R))/E(M(n,R)) \longrightarrow \mathrm{GL}(R)/E(R)$$

□

There are analogues of Theorems 3.3.4 and 3.3.6 for the Bass-Whitehead functor. We prove below the analogue for the latter.

Proposition 3.6.7. *Let $R = R_1 \times R_2$ be a cartesian product of rings. There is a natural isomorphism $K_1(R_1 \times R_2) \cong K_1(R_1) \times K_1(R_2)$*

Proof. Since $\mathrm{GL}(R_1 \times R_2) \cong \mathrm{GL}(R_1) \times \mathrm{GL}(R_2)$, this isomorphism maps $E(R_1 \times R_2)$ to $E(R_1) \times E(R_2)$. □

Now we show an example of a ring whose Bass-Whitehead group is trivial.

Example 3.6.8 (A ring with vanishing K_1)**.** Let V be an infinite-dimensional vector space over a field $\mathbb{F}$, and let $R = \mathrm{End}_{\mathbb{F}}(V)$. V is isomorphic to an infinite direct sum of copies of itself. For any $A \in \mathrm{GL}(R)$, we can form the infinite block sum of A with itself, which we denote by $\infty \cdot A$.

$$\infty \cdot A = \begin{pmatrix} A & 0 & 0 & \\ 0 & A & 0 & \\ 0 & 0 & A & \\ & & & \ddots \end{pmatrix}$$

It is an element of $\mathrm{GL}(R)$. Now, by an Eilenberg Swindle argument, similar to the one we have seen in Example 3.3.5, the matrices $A \oplus (\infty \cdot A)$ and $(\infty \cdot A)$ are conjugate. Hence, the matrix A represents the identity in $K_1(R)$.

3.7 K_1 of commutative and local rings

Proposition 3.7.1. *If R is a commutative ring and $R^\times = \mathrm{GL}(1,R)$ is its group of units, then the determinant $\det : \mathrm{GL}(n,R) \longrightarrow R^\times$ extends to a split surjection $\mathrm{GL}(R) \longrightarrow R^\times$, and thus it gives a surjection $K_1(R) \longrightarrow R^\times$.*

Proof. Since $\det(A \oplus 1) = \det(A)$, the determinants on $\mathrm{GL}(n,R)$ are compatible with the usual embeddings of $\mathrm{GL}(n,R)$ in $\mathrm{GL}(m,R)$ for $n < m$. We obtain an homomorphism $\mathrm{GL}(R) \longrightarrow R^\times$, because $\det(AB) = \det(A)\det(B)$. This homomorphism must factor throught a map $\mathrm{GL}(R)_{\mathrm{ab}} \longrightarrow R^\times$ since the group of units is commutative. The splitting is defined by the inclusion $R^\times = \mathrm{GL}(1,R) \hookrightarrow \mathrm{GL}(R)$. □

Given a commutative ring R, we denote by $SL(n,R) \subseteq \mathrm{GL}(n,R)$ the matrices of determinant 1 in $\mathrm{GL}(n,R)$. As in the definitions of $\mathrm{GL}(R)$ and $E(R)$, we denote by $SL(R)$ the colimit of $SL(n,R)$ with respect to the usual inclusion maps. It is called the *special linear* group of R. Since elementary matrices have determinant 1, we have $E(R) \subseteq SL(R)$. We denote by $SK_1(R)$ the quotient $SL(R)/E(R)$. For fields, we have the following proposition.

Proposition 3.7.2. *If $\mathbb{F}$ is a field, then $SK_1(\mathbb{F})$ is trivial, i.e, the determinant induces an isomorphism* $\det : K_1(\mathbb{F}) \longrightarrow \mathbb{F}^\times$.

Proof. Given a matrix $A \in \mathrm{GL}(n,\mathbb{F})$ for some $n \in \mathbb{N}$. By performing elementary row operations that only add a multiple of a row to another, we can reduce the matrix A to a diagonal matrix $D = \mathrm{diag}(d_1,\ldots,d_n) = E_1A$, where the matrix $E_1 \in \mathrm{GL}(n,\mathbb{F})$ is a product of elementary matrices of type $e_{ij}(a)$. Since $\det(D) = \det(E_1A) \neq 0$, all the diagonal entries of D are non-zero. Now, consider the matrix $\begin{pmatrix} d_n & 0 \\ 0 & d_n^{-1} \end{pmatrix}$. By Lemma 3.6.3, it is in $E(2,\mathbb{F})$, and thus we see that the matrix $D_n = \mathrm{diag}(1,\ldots,1,d_n,d_n^{-1})$ is in $E(n,\mathbb{F})$. In the same way

$$D_{n-1} = \mathrm{diag}(1,\ldots,1,d_{n-1}d_n,d_{n-1}^{-1}d_n^{-1},1) \in E(n,\mathbb{F})$$

Iterating this process we obtain

$$E_2 = D_1D_2\ldots D_n \in E(n,\mathbb{F})$$

with $E_2E_1A = E_2D = \mathrm{diag}(d,1,\ldots,1)$, where $d = d_1d_2\ldots d_n = \det(D) = \det(A)$. Thus, if $A \in SL(n,\mathbb{F})$ we have that $A = E_1^{-1}E_2^{-1} \in E(n,\mathbb{F})$, so $SL(n,\mathbb{F}) \subseteq E(n,\mathbb{F})$, giving $E(n,\mathbb{F})) = SL(n,\mathbb{F})$ □

Proposition 3.7.3. *Let R be a commutative ring with unit. Then*

$$K_1(R) \cong R^\times \oplus SK(1,R).$$

Proof. See [18, p. 64] □

Corollary 3.7.4. *If $\mathbb{F}$ is a field, then*

$$K_1(\mathbb{F}) \cong \mathbb{F}^\times.$$

More generally, the following theorem holds.

Theorem 3.7.5. *Let R be a local ring or an euclidean domain. Then*

$$K_1(R) \cong R^\times.$$

Proof. See [18, p. 69-74] □

Example 3.7.6 (K_1 of the ring $\mathbb{Z}/n\mathbb{Z}$)**.** Let $n \in \mathbb{N}, n \neq 1$, and let $p_1^{e_1}\cdots p_k^{e_k}$ be the prime factorization of n. By the Chinese remainder theorem, we have the following isomorphism of rings.

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{e_k}\mathbb{Z}.$$

Now, by Theorem 3.6.7,

$$K_1(\mathbb{Z}/n\mathbb{Z}) \cong K_1(\mathbb{Z}/p_1^{e_1}\mathbb{Z}) \oplus \cdots \oplus K_1(\mathbb{Z}/p_k^{e_k}\mathbb{Z}).$$

The rings $\mathbb{Z}/p_i^{e_i}\mathbb{Z}$ are local with maximal ideal $p_i\mathbb{Z}/p_i^{e_i}\mathbb{Z}$ for each i, and Theorem 3.7.5 gives

$$K_1(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/p_1^{e_1}\mathbb{Z})^\times \oplus \cdots \oplus (\mathbb{Z}/p_k^{e_k}\mathbb{Z})^\times \tag{3.1}$$

See [12, p. 44] for a concrete description of the groups of units that appear in (3.1) above.

Chapter 4

Higher K-theory and K-acyclic Rings

In Part I, we have seen that (complex) topological K-theory is a generalized cohomology theory associated to the spectrum $KU = \{BU, \Omega BU, BU, \Omega BU, \dots\}$. As topological groups, we know that $U \cong \mathrm{GL}(\mathbb{C})$. One might try to define higher algebraic K-groups for a ring R using the classifying space of the group $\mathrm{GL}(R)$. The classifying space of a topological group G is the quotient of a weakly contractible space by a free action of G (see [21, p. 107] for more details). For a discrete group G, its classifying space is a path-connected topological space whose fundamental groups are trivial, except the first one, $\pi_1(BG) \cong G$. By results from homology theory of groups (see [6, p. 36]), this implies that the homology of the group G and the singular homology groups of the space BG are isomorphic:

$$H_*(G; A) \cong H_*(BG; A)$$

for all coefficients A. Hence, the correspondence $G \longrightarrow BG$ and $X \longrightarrow \pi_1(X)$ satisfies $\pi_1(BG) \cong G$, but in general, it is not true that the classifying space of the fundamental group of a space X is homotopically equivalent to X. Since, in principle, $\mathrm{GL}(R)$ is a discrete group, all the homotopy groups of $B\,\mathrm{GL}(R)$ are 1, except $\pi_1(BG) \cong G$.

Quillen proposed a construction that associates to a topological space X another space X^+ whose fundamental group is that of X, but in which a certain subgroup of $\pi_1(X)$ has been "killed". This construction is known as the *plus-construction*. It will allow us to define algebraic K-functors for any $i \geqslant 1$, and we will see that for $i = 1$ coincides with the Bass-Whitehead group.

4.1 Quillen's plus-construction

Recall a group G is called *perfect* if $G_{\mathrm{ab}} = 0$ or equivalently, if $G = [G, G]$. Any group has a maximal perfect subgroup which is called the *perfect radical* and we denote it by $\mathcal{P}G$. Since commutators are sent to commutators by group homomorphisms, the perfect radical is preserved by automorphisms, in particular by inner automorphisms, thus it is a normal subgroup. Note that if G is an abelian group, then $[G, G] = 1$ and $\mathcal{P}G = \{1\}$, since for any perfect subgroup $S \subset G$ we have $S = [S, S] = \{1\}$.

Theorem 4.1.1. *Let X be a connected CW-complex with basepoint x_0 chosen from the 0-skeleton. Suppose that $\mathcal{P}\pi_1(X) = \pi_1(X)$, that is, $\pi_1(X)$ is perfect. Then, there exists a connected CW-complex X^+ and a pointed continuous map $q : X \longrightarrow X^+$ such that* *(i)* $\pi_1(X^+) = \{1\}$;

(ii) $q_* : H_i(X;\mathbb{Z}) \xrightarrow{\cong} H_i(X^+;\mathbb{Z})$ *for all* $i \in \mathbb{N}$.

Proof. We can assume that $\mathcal{P}\pi_1(X) \neq \{1\}$. Let $\{\alpha_i\}_{i\in I}$ be a set of base-point preserving maps that generate $\pi_1(X)$

$$\begin{aligned} \alpha_i : S^1 &\longrightarrow X \\ (1,0) &\longmapsto x_0 \end{aligned}$$

By means of this maps, we adjoin 2-cells to X for each i to obtain

$$X' = X \cup_{\{\alpha_i\}} \{e_i^2\}$$

This space is simply connected since, by the Seifert-van Kampen theorem, we have

$$\pi_1(X') = \pi_1(X) \Big/ \lhd\, \alpha_i \rhd_{i\in I} = \{1\}$$

Therefore, the Hurewicz map $h : \pi_2(X') \to H_2(X')$ is an isomorphism. Consider the following diagram associated to the inclusion $q' : X \hookrightarrow X'$

$$\begin{tikzcd} \pi_2(X') \arrow[r] \arrow[d, "h"', "\simeq"] & \pi_2(X',X) \arrow[r] \arrow[d] & \pi_1(X) \arrow[d] \\ H_2(X') \arrow[r, "j"] & H_2(X',X) \arrow[r] & H_1(X) \end{tikzcd}$$

The first homology group is $H_1(X) = 0$, since we are assuming that $\pi_1(X)$ is perfect and $\pi_1(X)_{\mathrm{ab}} = H_1(X)$. By construction, $H_2(X',X)$ is a free abelian group, generated by the cells (e_i^2). Since j is surjective and the Hurewicz map is an isomorphism, we can choose maps $\beta_i : S^2 \to X'$ such that $j \circ h[\beta_i] = (e_i^2)$. As before, we attach 3-cells e_i^3 to X' to obtain

$$X^+ = X' \cup_{\{\alpha_i\}} \{e_i^3\}$$

By the Seifert-van Kampen theorem, the fundamental group $\pi_1(X^+)$ is trivial. Now we must see that the inclusion $q : X \hookrightarrow X^+$ is an isomorphism in homology. For this, it suffices to prove that the relative homology $H_*(X^+,X;\mathbb{Z})$ is trivial. We have a chain complex of the form

$$\cdots \to 0 \to C_3(X^+,X) \xrightarrow{d} C_2(X^+,X) \to 0 \to \cdots$$

where the groups $C_3(X^+,X)$ and $C_2(X^+,X)$ are free abelian, generated by (e_i^3) and (e_i^2) respectively. Thus, it suffices to show that $d(e_i^3) = (e_i^2)$. By construction, the boundary of (e_i^3) is the image of the map β_i, so

$$j \circ h[\beta_i] = (e_i^2) \in H_2(X',X) = C_2(X',X) = C_2(X^+,X).$$

Thus, the complex $C_*(X^+,X)$ is acyclic. ☐

Recall that the mapping cylinder M_f of a map $f : X \longrightarrow Y$ is the quotient space of the disjoint union $(X \times I) \sqcup Y$, obtained by identifying each $(x,1) \in X \times I$ with $f(x) \in Y$ (see [8, p. 2])

Theorem 4.1.2 (Quillen)**.** *Let X be a connected CW-complex, with basepoint x_0, say, chosen from the 0-skeleton. Let N be a perfect normal subgroup of $\pi_1(X)$. Then, there exists a connected CW-complex X^+, depending on N, and a pointed continuous map $q : X \to X^+$ such that*

(i) $q_* : \pi_1(X) \longrightarrow \pi_1(X^+)$ *is the quotient map* $\pi_1(X) \longrightarrow \pi_1(X)/N$

(ii) $q_* : H_i(X;A) \xrightarrow{\cong} H_i(X^+;A)$ *for any i and for any A, where A is a $\pi_1(X^+)$-module*

(iii) *q is universal up to homotopy, i.e., for any $f : X \longrightarrow Y$ that satisfies (i) and (ii), there is a unique map g such that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X^+ \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

with $g \circ q \simeq f$.

Proof. We shall only prove part (*i*). For part (*ii*) see [22, p. 18], and for part (*iii*) see [13, p. 31].

Let $p : \widetilde{X} \to X$ be the covering space that corresponds to N. We can apply Theorem 4.1.1 with respect to $\widetilde{X}$ to form $\widetilde{X}^+$, since $\pi_1(\widetilde{X}) = N$ is perfect.

The space X^+ is defined as the disjoint union of $\widetilde{X}^+$ and the mapping cylinder M_p, by identifying the copies of $\widetilde{X}$ in these two spaces. We have a commutative diagram given by the inclusion maps

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{X}^+ \arrow[d] \\ M_p = X \arrow[r] & X^+ \end{tikzcd}$$

By the Seifert-van Kampen theorem, the induced map $\pi_1(X) \to \pi_1(X^+)$ is surjective and its kernel is the normal subgroup generated by N. Notice that, since X^+/M_p is homeomorphic to $\widetilde{X}^+/\widetilde{X}$, we have $H_*(X^+, M_p) = H_*(\widetilde{X}^+, \widetilde{X}) = 0$, so the map $X \to X^+$ induces an isomorphism on homology with coefficients in $\mathbb{Z}$. □

Example 4.1.3. Let X be a homology n-sphere, that is, a path-connected space whose homology groups are

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0, n \qquad \text{for } n > 1 \\ \mathbb{Z} & \text{for } i = n \end{cases}$$

Since $H_1(X;\mathbb{Z}) = \pi_1(X)_{\mathrm{ab}}$ is trivial, the fundamental group $\pi_1(X)$ is perfect. The space X^+ is simply connected, and $H_i(X^+;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \cong H_i(S^n;\mathbb{Z})$ for all i. Since X^+ is simply connected, Hurewicz's theorem (see [8, p. 366]) tells us that X^+ is in fact $(n-1)$-connected and $\pi_n(X^+) \cong H_n(X^+;\mathbb{Z}) \cong \mathbb{Z}$. If $f\colon S^n \longrightarrow X^+$ represents a generator of $\pi_n(X^+)$, then f induces isomorphisms in all homology groups since the homomorphism $f_*\colon H_n(S^n) \longrightarrow H_n(X^+)$ sends the canonical class to the image of $[f]$ under the Hurewicz map, and therefore Whitehead's theorem (see [8, p. 346]) implies that f is a homotopy equivalence. Hence $X^+ \simeq S^n$.

4.2 Higher algebraic K-theory

Let R be a ring with unit and let $B\,\mathrm{GL}(R)$ be its classifying space. We know that $E(R)$ is the perfect radical of $\mathrm{GL}(R)$, so we can take the plus-construction of $B\,\mathrm{GL}(R)$ and define

Definition 4.2.1. The algebraic K-groups of R are

$$K_n(R) = \pi_n(B\,\mathrm{GL}(R)^+), \quad n \geqslant 1$$

Any homomorphism of rings with unit $f : R \longrightarrow S$ induces a homomorphism $\mathrm{GL}(f) : \mathrm{GL}(R) \longrightarrow \mathrm{GL}(S)$, and therefore, a continuous map $B\,\mathrm{GL}(R) \longrightarrow B\,\mathrm{GL}(S)$. We define $K_n(f) = \pi_i(B\,\mathrm{GL}(f)^+)$. They are covariant functors from the category of rings with unit to the category of abelian groups. By Theorem 4.1.2, the space $B\,\mathrm{GL}(R)^+$ is connected and

$$\pi_1(B\,\mathrm{GL}(R)^+) \cong \pi_1(B\,\mathrm{GL}(R))/E(R) \cong \mathrm{GL}(R)/E(R)$$

Thus, $K_1(R) = \pi_1(B\,\mathrm{GL}(R)^+)$ and the Bass-Whitehead group $K_1(R)$ given in 3.6.5 coincide. So far we have only defined the groups K_0 and K_1, but there are also groups K_2 due to Milnor, which coincide with the corresponding group as defined above, and therefore suggest that Quillen's construction is correct. However, for K_0 the construction "fails", since $B\,\mathrm{GL}(R)^+$ is path connected (for $B\,\mathrm{GL}(R)$ is), and the group $\pi_0(B\,\mathrm{GL}(R)^+)$ has just one element. However, this can be corrected replacing the space $B\,\mathrm{GL}(R)^+$ by $K_0(R) \times B\,\mathrm{GL}(R)^+$, where we view $K_0(R)$ as a topological group with the discrete topology. Regardless, we will just consider K_i for $i \geqslant 1$ and K_0 independently. There is another construction of higher algebraic K-functors for exact categories, known as the Q-construction, which is also due to Quillen and does not have this drawback, see for example [22, p. 38] or [25, p. 347].

4.3 K-theory of finite fields

The higher K-theory groups are hard to compute for certain rings, and one of the major achievements of Quillen, which was also part of his motivation to define the higher K-groups as we have seen, was his computation of all the K-groups for finite fields. We give an outline of this computation, assuming the following theorem

Theorem 4.3.1 (Quillen). *Let $\mathbb{F}_q$ be a finite field, $q \geqslant 2$. There are maps*

$$B\,\mathrm{GL}(\mathbb{F}_q)^+ \to BU \xrightarrow{\psi^q - id} BU$$

that induce the following long exact sequence on homotopy groups

$$\begin{array}{l}\cdots \longrightarrow \pi_n(B\,\mathrm{GL}(\mathbb{F}_q)^+) \longrightarrow \pi_n(BU) \xrightarrow{(\psi^q - id)_*} \pi_n(BU) \longrightarrow \\ \longrightarrow \pi_{n-1}(B\,\mathrm{GL}(\mathbb{F}_q)^+) \longrightarrow \pi_{n-1}(BU) \xrightarrow{(\psi^q - id)_*} \pi_{n-1}(BU) \cdots \longrightarrow \\ \longrightarrow \pi_1(B\,\mathrm{GL}(\mathbb{F}_q)^+) \longrightarrow \pi_1(BU) \xrightarrow{(\psi^q - id)_*} \pi_1(BU)\end{array} \tag{4.1}$$

Proof. See [16, p. 582]. □

The maps $\psi^k : BU \to BU$, $k \geqslant 0$, are called *Adams operations*, they induce

$$\begin{aligned} \varphi^k_* : \pi_{2i}(BU) &\longrightarrow \pi_{2i}(BU) \\ a &\longmapsto k^i \cdot a \end{aligned}$$

Proposition 4.3.2. *$BU \xrightarrow{\psi^q - id} BU$ induces $\pi_{2i}(BU) \xrightarrow{q^i - 1} \pi_{2i}(BU)$.*

Proof. Let f and g be two maps from BU to BU:

$$\begin{tikzcd} BU \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & BU \end{tikzcd}$$

Consider their homotopy classes at $[BU, BU]$. By Bott's periodicity, $[BU, BU] \cong [BU, \Omega U]$, so there is an homotopy equivalence between BU and ΩU, which we denote by h. The loopspace ΩU is a group with the operation given by concatenation of loops. Now, for any $x \in BU$, we define $(h \circ f + h \circ g)(x) = (h \circ f)(x) * (h \circ f)(x)$.

$$\begin{tikzcd} BU \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[rr, bend right=50, "h\circ f + h\circ g"'] & BU \arrow[r, "\simeq", "h"'] & \Omega U \end{tikzcd}$$

This operation is commutative, since in fact $[BU, BU] \cong [BU, \Omega^2 BU]$ is an abelian group.

Recall that for any spaces A, X, we have $[\Sigma A, X] \cong [A, \Omega X]$, and also for any space X, $\pi_n(\Omega X) = [S^n, \Omega X]$. This allows us to define the sum in $\pi_*(X)$ for any space X, since

$$\pi_m(X) = [S^m, X] \cong [S^m, \Omega X] \cong [S^{m-1}, \Omega X] \cong [S^{m-2}, \Omega^2 X], \qquad \text{for } n \geqslant 2$$

So, given $\alpha, \beta \in \pi_n(\Omega X)$, the sum $\alpha + \beta$ is defined also as concatenation of paths

$$(\alpha + \beta)(x) = \alpha(x) * \beta(x) \qquad \forall x \in S^n$$

Now we want to check that $(f + g)_* = f_* + g_*$. Given $\alpha \in \pi_n(BU)$, then $f_*(\alpha) \in \pi_n(BU)$ is represented by $f \circ \alpha$

$$\begin{tikzcd} BU \arrow[r, "f"] & BU \\ S^n \arrow[u, "\alpha"] \arrow[ur, "f\circ\alpha"'] & \end{tikzcd}$$

We must check that for every $\alpha \in \pi_n(BU)$, the induced maps $(f + g)_*(\alpha) = f_*(\alpha) + g_*(\alpha)$, and for this, it suffices to check that

$$h \circ (f + g) \circ \alpha = (h \circ f \circ \alpha) + (h \circ g \circ \alpha)$$

which is immediate since the sum on the right hand side corresponds to concatenation of loops in ΩU. Now ,

$$\begin{aligned} (\psi^q)_*(a) &= q^i \cdot a \qquad && \forall a \in \pi_{2i}(BU) \\ id_*(a) &= a && \forall a \in \pi_{2i}(BU) \end{aligned}$$

So finally, $(\psi^q - id)_*(a) = q^i a - a = (q^i - 1)a$ □

Theorem 4.3.3 (Quillen)**.** *For every finite field $\mathbb{F}_q$ and $n \geqslant 1$, we have*

$$K_n(\mathbb{F}_q) = \pi_n(BGL(\mathbb{F}_q)^+) \cong \begin{cases} \mathbb{Z}/(q^i - 1) & \text{for } n = 2i - 1, \\ 0 & \text{for } n \text{ even }. \end{cases}$$

Proof. By the long exact sequence (4.1) from Theorem 4.3.1, to compute the first K-groups of $\mathbb{F}_q$, we just replace the maps that appear in the exact sequence (4.2). These are given by multiplication by $(q^i - 1)$, by Proposition 4.3.2. Recall that

$$\pi_i(BU) = \begin{cases} \mathbb{Z} & \text{for } i \text{ even}, \\ 0 & \text{for } i \text{ odd}. \end{cases}$$

We obtain the exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & K_4(\mathbb{F}_q) \arrow[r] & \pi_4(BU) = \mathbb{Z} \arrow[r] & \pi_4(BU) = \mathbb{Z} \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-6ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& K_3(\mathbb{F}_q) \arrow[r] & \pi_3(BU) = 0 \arrow[r] & \pi_3(BU) = 0 \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-6ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& K_2(\mathbb{F}_q) \arrow[r] & \pi_2(BU) = \mathbb{Z} \arrow[r] & \pi_2(BU) = \mathbb{Z} \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-6ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& K_1(\mathbb{F}_q) \arrow[r] & \pi_1(BU) = 0 \arrow[r] & \pi_1(BU) = 0
\end{tikzcd} \tag{4.2}$$

Then,

$$K_4(\mathbb{F}_q) = \ker(\cdot(q^2 - 1) = 0, \quad K_3(\mathbb{F}_q) = \operatorname{coker}(\cdot(q^2 - 1)) = \mathbb{Z}/(q^2 - 1),$$
$$K_2(\mathbb{F}_q) = \ker(\cdot(q - 1)) = 0, \quad K_1(\mathbb{F}_q) = \operatorname{coker}(\cdot(q - 1)) = \mathbb{Z}/(q - 1).$$

Now apply induction to finish the proof. ∎

4.4 *K*-acyclic rings

In this section we provide examples of rings for which all K-groups vanish. They are called *infinite sum rings*. Their definition is due to Wagoner; see [24]. These rings have several properties in analogy with Examples 3.3.5 and 3.6.8, where conjugation played an important role. Recall that a group G is called *quasi-perfect* if $[G, G]$ is perfect.

Definition 4.4.1. Let G be a quasi-perfect group, and let

$$\oplus : G \times G \to G$$

be a group homomorphism, which we denote by $(g, g') \mapsto g \oplus g'$. We say that $(G, \oplus)$ is a *direct sum group* if it satisfies

(i) For any finite set $g_1, \ldots, g_n$ of elements of $[G, G]$ and any $g \in G$ there exist $h \in [G, G]$ such that $gg_ig^{-1} = hg_ih^{-1}$ for $1 \leqslant i \leqslant n$

(ii) For any finite set $g_1, \ldots, g_n$ of elements of G there exist elements c and d of G such that

$$c(g_i \oplus 1)c^{-1} = d(1 \oplus g_i)d^{-1}, \quad 1 \leqslant i \leqslant n$$

A map $f : G \to G'$ is a morphism of direct sum groups if, in addition to being a homomorphism, satisfies

$$f(g \oplus g') = f(g) \oplus f(g'), \quad \forall g, g' \in G$$

Lemma 4.4.2. *Let $f : G \longrightarrow G$ be an automorphism of a discrete group G, such that for any finite set $g_1, \ldots, g_k \in G$ there is an element $h \in G$ such that $f(g_i) = hg_ih^{-1}$ for $1 \leqslant i \leqslant k$. Then the induced map*

$$f_* : H_*(BG) \longrightarrow H_*(BG)$$

is the identity.

Proof. See [24, p. 352]. □

Definition 4.4.3. A direct sum group $(G, \oplus)$ is called *flabby* if there exists a homomorphism $\tau : G \to G$, such that for any finite set $g_1, \ldots, g_n \in G$, there is an element $c \in G$ such that

$$c(g_i \oplus \tau(g_i))c^{-1} = \tau(g_i), \quad 1 \leqslant i \leqslant n. \tag{4.3}$$

Proposition 4.4.4. *If G be a flabby group, then $H_n(BG; \mathbb{Z}) = 0$ for all $n > 0$*

Proof. The map $\oplus : G \times G \to G$ induces a ring structure on $H_*(BG; \mathbb{Z})$, where the multiplication, denoted by $\oplus$, is given by

$$H_*(BG; \mathbb{Z}) \otimes H_*(BG; \mathbb{Z}) \xrightarrow{\approx} H_*(B(G \times G); \mathbb{Z}) \xrightarrow{\oplus} H_*(BG; \mathbb{Z}).$$

We denote by $1 \in H_0(BG; Z)$ the generator determined by the map of the standard 0-simplex to the basepoint. This generator is a unit for the multiplication, since for any $z \in H_*(BG; \mathbb{Z})$ the correspondences

$$z \mapsto z \oplus 1 \tag{4.4}$$
$$z \mapsto 1 \oplus z \tag{4.5}$$

are induced respectively by the group homomorphisms

$$g \mapsto g \oplus e \tag{4.6}$$
$$g \mapsto e \oplus g \tag{4.7}$$

The group homomorphisms (4.6) and (4.7) induce the identity on homology, since conjugation induces the identity on $H_*(BG; \mathbb{Z})$ by [6, Proposition 6.2, p. 48]. Now consider the maps induced by the algebraic diagonal map $\triangle$ and by τ given below.

$$\triangle : G \to G \times G \tag{4.8}$$
$$\triangle_* : H_*(BG; \mathbb{Z}) \to H_*(BG; \mathbb{Z}) \otimes H_*(BG; \mathbb{Z}) \tag{4.9}$$

$$\tau : G \to G \tag{4.10}$$
$$\tau_* : H_*(BG, \mathbb{Z}) \to H_*(BG; \mathbb{Z}) \tag{4.11}$$

Fix $z \in H_n(BG;\mathbb{Z})$, which we can assume is represented by a chain $\sum_i k_i(g_1^i, \ldots, g_n^i)$. We claim that

$$\tau_*(z) = \oplus \circ (id \times \tau_*) \circ \triangle_*(z)$$

The left hand side of this equation is represented by $\sum_i k_i(\tau(g_1^i), \ldots, \tau(g_n^i))$ and the right hand side of this equation is represented by $\sum_i k_i(g_1^i \oplus \tau(g_1^i), \ldots, g_n^i \oplus \tau(g_n^i))$. Since G is flabby, these chains are conjugate and therefore, homologous. Now we prove by induction that $H_*(BG;\mathbb{Z}) = 0$ for $n > 0$.

Assume that $H_i(BG;\mathbb{Z}) = 0$ for $0 < i < n$, and let $z \in H_n(BG;\mathbb{Z})$. Then

$$\triangle_*(z) = z \otimes 1 + 1 \otimes z + \sum_i u_i \otimes v_i$$

where $0 < \deg u_i < n$ and $0 < \deg v_i < n$. By the inductive hypothesis, we know that $u_i = v_i = 0$. Hence

$$\tau_*(z) = \oplus(id \times \tau_*)\triangle_*(z) = z \oplus 1 + 1 \oplus \tau_*(z) = z + \tau_*(z)$$

and therefore $z = 0$. □

Proposition 4.4.5. *For any ring with unit, the group* $\mathrm{GL}(R)$ *is a direct sum group.*

Proof. The group $\mathrm{GL}(R)$ is quasi-perfect, since the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is the group $E(R)$ of elementary matrices, which is perfect by Proposition 3.6.4. The group $\mathrm{GL}(n, R)$ is also quasi-perfect for $n \geqslant 3$

Let $\mathbb{P}$ denote the positive integers, we partition $\mathbb{P}$ into two disjoint subsets $\mathbb{P} = P_1 \cup P_2$ and choose bijections $\alpha : \mathbb{P} \to P_1$ and $\beta : \mathbb{P} \to P_2$. For any $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathrm{GL}(R)$, we define

$$A \oplus B = \left(a_{\alpha(i),\alpha(j)}\right) \cdot \left(b_{\beta(i),\beta(j)}\right)$$

To check condition (i) in Definition 4.4.1, take $g_1, \ldots, g_n \in [\mathrm{GL}(R), \mathrm{GL}(R)]$ and let $g \in \mathrm{GL}(R)$. Choose an integer k such that $g, g_1, \ldots, g_n \in \mathrm{GL}(k, R)$, now we take

$$h = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in E(2k, R)$$

□

Definition 4.4.6. A *sum ring* is a ring with unit R together elements $\alpha_0, \alpha_1, \beta_0, \beta_1 \in R$ that satisfy the following identities:

$$\alpha_0\beta_0 = \alpha_1\beta_1 = 1$$
$$\beta_0\alpha_0 + \beta_1\alpha_1 = 1$$

Given a sum ring R, we can define a ring homomorphism $\boxplus : R \times R \to R$ given by $a \boxplus b = \beta_0 a \alpha_0 + \beta_1 b \alpha_1$, which is a unit-preserving homomorphism.

Proposition 4.4.7. *If* $(R, \boxplus)$ *is a sum ring, then* $\mathrm{GL}(R)$ *is a direct sum group with operation given by* $A \oplus B = (a_{ij}) \boxplus (b_{ij})$.

Proof. We must check only condition (ii) in definition 4.4.1. To see this it suffices to show that for $A, B \in \mathrm{GL}(n, R)$, there is an invertible matrix $Q \in \mathrm{GL}(3n, R)$ such that

$$Q^{-1}\begin{pmatrix} A \oplus B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3n, R).$$

We denote by $D_n(x)$ the $n \times n$ diagonal matrix with x along the diagonal. Just take

$$Q = \begin{pmatrix} D_n(\beta_0) & D_n(\beta_1) & 0 \\ 0 & 0 & D_n(\alpha_0) \\ 0 & 0 & D_n(\alpha_1) \end{pmatrix},$$

where

$$Q^{-1} = \begin{pmatrix} D_n(\alpha_0) & 0 & 0 \\ D_n(\alpha_1) & 0 & 0 \\ 0 & D_n(\beta_0) & D_n(\beta_1) \end{pmatrix}.$$

□

Definition 4.4.8. Let $(R, \boxplus)$ be a sum ring, we say that it is an *infinite sum ring* if there exists an identity preserving ring homomorphism $\sigma : R \to R$ such that $a \boxplus \sigma(a) = \sigma(a)$ for any $a \in R$.

Corollary 4.4.9. *If $(R, \boxplus, \sigma)$ is an infinite sum ring, then $B\,\mathrm{GL}(R)^+$ is contractible*

Proof. $\mathrm{GL}(R)$ is flabby, with

$$\tau : \mathrm{GL}(R) \to \mathrm{GL}(R) \tag{4.12}$$
$$A = (a_{ij}) \mapsto \tau(A) = (\sigma(a_{ij})) \tag{4.13}$$

By Proposition 4.4.4, the space $B\,\mathrm{GL}(R)$ is acyclic. Thus, by Theorem 4.1.2 the space $B\,\mathrm{GL}(R)^+$ is also acyclic and therefore, contractible. □

4.5 Cone and suspension of a ring

The notion of cone ring will allow us to provide examples of infinite sum rings.

Let R be a ring with unit. Any function $f : \mathbb{N} \times \mathbb{N} \to R$ can be considered as a "$\mathbb{N} \times \mathbb{N}$-matrix" A_f with infinitely many rows and columns:

$$A_f = \begin{pmatrix} a_{11} = f(1,1) & a_{12} = f(1,2) & \cdots \\ a_{21} = f(2,1) & a_{22} = f(2,2) & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

Definition 4.5.1. The ring of infinite matrices with entries in R such that each row and each column has at most finitely many non-zero entries is called the *cone ring* of R, and we denote it by CR. The ideal of matrices with at most finitely many non-zero entries is denoted by $mR \subset CR$. The ring $SR = CR/mR$ is called the *suspension* of the ring R.

We have the identities $CR = C\mathbb{Z} \otimes_{\mathbb{Z}} R$, $mR = m\mathbb{Z} \otimes_{\mathbb{Z}} R$ and $SR = S\mathbb{Z} \otimes_{\mathbb{Z}} R$ for any ring R with unit.

Let R be a ring (not necessarily with unit) that satisfies the following condition:

For any finite set $a_1, \ldots, a_n \in R$, there exists an idempotent $r \in R$ so that

$$ra_i = a_i r = a_i, \quad 1 \leqslant i \leqslant n \tag{4.14}$$

Any ring with unit satisfies the condition (4.14). If R satisfies (4.14), then mR, CR and SR also satisfy (4.14).

Let E and F be free R-modules based on countable sets $\{e_\alpha\}$ and $\{f_\beta\}$ respectively. An R-linear transformation $h : E \to F$ is called *locally finite* provided that for each f_β there are at most finitely many e_α such that f_β appears in $h(e_\alpha)$ with a non-zero coefficient. If $h(e_\alpha) = \sum f_\beta \cdot r_{\beta\alpha}$, then h is locally finite if and only if the matrix $(r_{\beta\alpha})$ is locally finite, in the sense that each row and each column or $(r_{\beta\alpha})$ has at most finitely many non-zero terms. The ring of all locally finite transformations of E to itself will be denoted by $\mathrm{End}_{loc}(E, R)$. Note that $\mathrm{End}_{loc}(E, R)$ and $\mathrm{End}_{loc}(F, R)$ are isomorphic if there is a bijection between the bases $\{e_\alpha\}$ and $\{f_\beta\}$.

Proposition 4.5.2. *For any ring with unit R, the cone ring CR is an infinite sum ring*

Proof. We must construct an identity preserving ring homomorphism $\sigma : CR \to CR$ such that for any $r \in CR$ it satisfies $r \oplus \sigma(r) = \sigma(r)$. Instead of working with CR, we identify the cone with the ring $\mathrm{End}_{loc}(E, R)$ of locally finite R-linear maps, where E is a free right R-module with countable basis $\{e_j^k\}$ and $1 \leqslant i, j < \infty$. We partition the basis $\{e_j^k\}$ into two disjoint infinite subsets, $\{e_j^k\} = A_0 \cup A_1$. Let $\beta_i : \{e_j^k\} \to B_i$, $i = 0, 1$, be any two bijections and let $\beta_i \in \mathrm{End}_{loc}(E, R)$ be the corresponding locally finite matrix, i.e, having in each row and in each column at most finitely many non-zero entries. Define $\alpha_i \in \mathrm{End}_{loc}(E, R)$ for $i = 0, 1$ by

$$\alpha_i(e_j^k) = \begin{cases} \beta_i^{-1}(e_j^k) & \text{if } e_j^k \in A_i, \\ 0 & \text{otherwise.} \end{cases}$$

This gives us a a sum structure on $\mathrm{End}_{loc}(E, R)$, and hence on the cone CR. The are many sum structures, but we fix the following one: choose β_0 to be a bijection of $\{e_j^k\}$, $1 \leqslant j < \infty$, onto $\{e_1^k\}$, and define β_1 by $\beta_1(e_j^k) = e_{j+1}^k$. Let α_0 and α_1 be as above. To make $\mathrm{End}_{loc}(E, R)$ into an infinite sum ring, recall that $E = \bigoplus E_j$, where E_j is the free submodule of E spanned by $\{e_j^k\}$ for $1 \leqslant k < \infty$. Define $\sigma : \mathrm{End}_{loc}(E, R) \to \mathrm{End}_{loc}(E, R)$ as follows. Let $h \in \mathrm{End}_{loc}(E, R)$ and $e_j^k \in E$. Then

$$\sigma(h)(e_j^k) := \beta_1^{j-1}\beta_0 h \alpha_0 \alpha_1^{j-1}(e_j^k).$$

For $j = 1$ we have

$$\beta_0 h \alpha_0(e_1^k) = \sigma(h)(e_1^k), \tag{4.15}$$

and for $j > 1$

$$\begin{aligned} \beta_1\sigma(h)\alpha_1(e_j^k) &= \beta_1\sigma(h)(e_{j-1}^k = \\ &= \beta_1(\beta_1^{j-1}\beta_0 h \alpha_0 \alpha_1^{j-1}(e_{j-1}^k)) = \\ &= \beta_1^j \beta_0 h \alpha_0 \alpha_1^j(e_j^k) = \sigma(h)(e_j^k). \end{aligned} \tag{4.16}$$

Therefore, $h \oplus \sigma(h) = \sigma(h)$ for any $h \in \mathrm{End}_{loc}(E, R)$. □

The suspension SR of a ring R is a sum ring because it is the homomorphic image of CR, but in general it is not an infinite sum ring. The two categories of sum rings and infinite sum rings are closed under the following operations:

(i) pullbacks of two morphisms;

(ii) forming the monoid ring $R[M]$ where R is a ring and M is a monoid;

(iii) taking the cone and suspension of a ring.

To show that the algebraic K-groups of a cone ring vanish for all $i \geqslant 0$ we need the following theorem.

Theorem 4.5.3 (Fundamental theorem of algebraic K-theory)**.** *For every ring R there is a natural sum decomposition:*

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus NK_1(R) \oplus NK_1(R),$$

where $NK_1(R)$ is the cokernel of the natural map $K_1(R) \longrightarrow K_1(R[t])$.

Proof. See [25, p. 225]. □

Theorem 4.5.4. *Let R be a ring with unit. The cone ring CR is K-acyclic, i.e.,*

$$K_i(CR) = 0 \quad \forall i \geqslant 0$$

Proof. The cone ring CR is an infinite sum ring by Proposition 4.5.2, and Corollary 4.4.9 implies that $K_i(CR) = 0$ for all $i > 0$. In particular, $K_i(CR[t,t^{-1}]) = 0$ for all $i > 0$, since forming the monoid ring of an infinite sum ring is again an infinite sum ring. Now Theorem 4.5.3 implies that

$$0 = K_1(R[t,t^{-1}]) \supset K_1(CR) \oplus K_0(CR) \oplus NK_1(CR) \oplus NK_1(CR)$$

Thus $K_0(CR) = 0$ and CR is K-acyclic. □

The suspension satisfies the following nice property.

Proposition 4.5.5. *If R is a ring with unit, then*

$$K_i(SR) = K_{i-1}(R) \quad \text{for } i \geqslant 1$$

We will need the following theorem to prove Proposition 4.5.5

Theorem 4.5.6. *Let*

$$1 \longrightarrow G_1 \xrightarrow{i} G_2 \xrightarrow{r} G_3 \longrightarrow 1$$

be an exact sequence of direct sum groups such that

(i) *G_3 is perfect*

(ii) *G_3 acts trivially on the homology of G_1*

Then the sequence

$$BG_1^+ \xrightarrow{i^+} BG_2^+ \xrightarrow{r^+} BG_3^+$$

is a homotopy fibration, where the plus-construction is taken with respect to the commutator subgroup.

Proof. See [11, p. 83] □

Proof of Proposition 4.5.5. Recall that for every ring R we have defined $SR = CR/mR$ where R is the ideal of CR that consists of the matrices that have at most a finite number of coefficients. There is an exact sequence of rings

$$0 \longrightarrow mR \longrightarrow CR \longrightarrow SR \longrightarrow 0$$

Note that mR is an ideal and CR and SR are rings. The sequence induces a long exact sequence of groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{GL}(mR) \arrow[r] & \mathrm{GL}(CR) \arrow[r, "\varphi"] \arrow[d, equal] & \mathrm{GL}(SR) & \\ 0 \arrow[r] & \mathrm{GL}(mR) \arrow[r] & \mathrm{E}(CR) \arrow[r, "\phi"] & \mathrm{E}(SR) \arrow[r] & 0 \end{tikzcd} \tag{4.17}$$

The map φ is not exhaustive but ϕ is, since $K_1(CR) = 0$ implies that $\mathrm{GL}(CR) = \mathrm{E}(CR)$. This long exact sequence of groups induces a fibration

$$B\,\mathrm{GL}(mR) \longrightarrow B\,\mathrm{GL}(CR) \longrightarrow B\,\mathrm{E}(SR)$$

and the sequence

$$B\,\mathrm{GL}(mR)^+ \longrightarrow B\,\mathrm{GL}(CR)^+ \longrightarrow B\,\mathrm{E}(SR)^+$$

is again a fibration by Proposition 4.5.6. Now $B\,\mathrm{GL}(CR)^+ \simeq *$ since CR is K-acyclic. Thus

$$\Omega_0 B\,\mathrm{E}(SR)^+ \simeq B\,\mathrm{GL}(mR)^+ \xrightarrow{\simeq} B\,\mathrm{GL}\,R^+.$$

where Ω_0 denotes the connected component of the basepoint and the homotopy equivalence $B\,\mathrm{GL}(mR)^+ \simeq B\,\mathrm{GL}\,R^+$ follows from the fact $mR \cong m(mR)$.

Then for $i \geqslant 2$:

$$K_{i-1}(R) = \pi_{i-1} B\,\mathrm{GL}\,R^+ \cong \pi_{i-1}\Omega B\,\mathrm{E}(SR)^+ \cong \pi_i B\,\mathrm{E}(SR)^+ = K_i(SR).$$

where $\pi_i B\,\mathrm{E}(SR)^+ = K_i(SR)$ follows from the fact that $B\,\mathrm{E}(SR)^+$ is the universal cover of $B\,\mathrm{GL}(SR)^+$, i.e, they have the same homotopy groups except π_1 ; see [24, p. 357]. Hence $\pi_i B\,\mathrm{E}(SR)^+ \cong \pi_i B\,\mathrm{GL}(SR)^+$ for all $\geqslant 2$.

For $i = 1$,

$$K_1(SR) = \pi_1(B\,\mathrm{GL}(SR)^+ = \mathrm{GL}(SR)/\,\mathrm{E}(SR) = K_0(R)$$

because $K_0(R) \times B\,\mathrm{GL}\,R^+ \simeq \Omega B\,\mathrm{GL}(SR)^+$. See [24, p. 357, Proposition 3.2]. □

This fact provides a definition of

Definition 4.5.7. Let R be a ring with unit. The *negative K-groups* are

$$K_{-i}(R) = K_1(S^{i+1}R) \text{ for } i \geqslant 0,$$

where $S^{i+1}(R) = S(S^i R)$.

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