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Interiority of the Optimal Population Growth Rate with Endogenous Fertility

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Abstract

This paper analyzes the issue of the interiority of the optimal population growth rate in a two-period overlapping generations model with endogenous fertility. Using Cobb-Douglas utility and production functions, we show that the introduction of a cost of raising children allows for the possibility of the existence of an interior global maximum in the planner's problem, contrary to the exogenous fertility case.

Key words: Overlaping generations models, endogenous fertility, optimal population growth
JEL classification: J13, J18

Resumen

Este artículo analiza el problema de la interioridad de la tasa de crecimiento óptima de la población en un modelo de generaciones solapadas de dos períodos con fecundidad endógena. Usando funciones de utilidad y de producción de tipo Cobb-Douglas, se muestra que la introducción de un coste de tener hijos permite la posibilidad de que exista un máximo global interior en el problema planificador, contrariamente a lo que sucede en el caso con fecundidad exógena.

1 Introduction

Samuelson (1975) analyzed the question of the optimal growth rate for population in the classical model of two overlapping generations (OLG) à la Diamond. This analysis resulted in the so-called 'Serendipity Theorem', according to which only if population happened to grow at its optimal rate in the decentralized economy, its steady state equilibrium would reach the golden rule. Samuelson's analysis was followed by a criticism –Deardorff (1976)– concerning the interiority of the optimal solution in the planner's problem. In particular, Deardorff showed that if both the utility and the production function were of the typical Cobb-Douglas type, there would be no interior rate of population growth that maximized the utility of the representative agent at the steady state; and the same would happen for several other specifications of preferences and technology. A few years later, Michel and Pestieau (1993) analyzed this issue considering more general utility and production functions, of the CES type. They concluded that the planner's choice of the population growth rate would be interior only if either the two types of consumption or the two production factors were complements. All these papers consider the agents' fertility decision as being exogenous.

While some papers have developed models with endogenous fertility in which the optimal rate of population growth was analyzed[1], none of these has focused its attention on the issue of the interiority of the first best solution. As an exception, Schweizer (1996) deals with this problem. This author compares the optimal allocation of OLG models that consider optimal population growth with the optimal allocation of local public goods models. The main point of the paper is to insist on the fact that the planner's first order conditions are not sufficient to guarantee the existence of an interior global

[1]See, for example, Eckstein and Wolpin (1985) and Bental (1989).

maximum.

This paper investigates in what cases the optimal population growth rate is interior when fertility is endogenous in Samuelson's model. For simplicity, we restrict our analysis to the case of a Cobb-Douglas specification of utility and technology. We find that the possibility of an interior optimal solution exists and try to understand the driving forces behind this result.

In Section 2 we revise the issue of the optimum growth rate for population in an exogenous fertility framework, as analyzed by Samuelson in 1975. Using specific utility and production functions, we analyze whether the planner's choice of population growth is interior. Section 3 applies the same analysis in a framework with endogenous fertility –by introducing both a cost and a taste for children– and compares the results with those previously obtained. Our results show that, while the taste for children does not play any role in avoiding a corner solution, the introduction of a cost of children makes it possible to have an interior global maximum of utility. We conclude with some final remarks in Section 4.

2 Samuelson's Model and the Problem of the Interiority of the Optimal Solution

Samuelson (1975) derives the optimal population growth rate in the basic two-period overlapping generations model à la Diamond (1965). Individuals in this model live for two periods: in the first, they work; in the second, they are retired. The size of the generation born in period t is given by N_t. An individual born in t consumes an amount c_t when young and an amount d_{t+1} when old. Preferences are given by the utility function $U(c_t, d_{t+1})$, with derivatives $U_c(c_t, d_{t+1}) \equiv \frac{\partial U(c_t,d_{t+1})}{\partial c_t} > 0$ and $U_d(c_t, d_{t+1}) \equiv \frac{\partial U(c_t,d_{t+1})}{\partial d_{t+1}} > 0$. Samuelson assumes $U(\cdot)$ to be quasi-concave.

The economy produces a single good using two factors of production: capital and labor. At time t, there are N_t workers supplying N_t units of labor. Population grows at an exogenous constant rate $n-1$, so that[2]:

$$\frac{N_{t+1}}{N_t} = n \qquad \forall t$$

The production function, given by:

$$Y_t = F(K_t, N_t)$$

exhibits constant returns to scale. Thus it can be expressed as $y_t = f(k_t)$, with k_t the capital-labor ratio. Moreover, it is increasing and concave, so that $f'(k_t) > 0$ and $f''(k_t) < 0$. Capital depreciates at a rate $\delta \in [0,1]$ each period. At the end of the period, non-depreciated capital can be either consumed or invested.

The resource constraint of the economy can be expressed as:

$$F(K_t, N_t) + (1-\delta)K_t = c_t N_t + d_t N_{t-1} + K_{t+1}$$

Resources available in period t, given by the left-hand side of the expression, are devoted to consumption of the two generations alive in t and to investment in future capital. Dividing everything by N_t, we obtain the resource constraint in per capita terms of the present working generation:

$$f(k_t) + (1-\delta)k_t = c_t + \frac{d_t}{n} + nk_{t+1}$$

To derive the social optimum, Samuelson assumes that the planner maximizes the utility function of the representative individual subject to the resource constraint at the steady state, with respect to c, d, k and n:

$$\max_{c,d,n,k} U(c,d)$$

[2]Some authors, including Samuelson, consider $\frac{N_{t+1}}{N_t} = 1+n$, so that n is the rate of population growth. Using our notation, n is the number of children of the representative individual. This latter approach is used in most papers with endogenous fertility.

subject to:

$$f(k) + (1-\delta)k = c + \frac{d}{n} + nk \tag{1}$$

An interior solution is characterized by the following first order conditions:

$$\frac{U_c(c^*, d^*)}{U_d(c^*, d^*)} = n^* \tag{2}$$

$$\frac{d^*}{(n^*)^2} = k^* \tag{3}$$

$$f'(k^*) + 1 - \delta = n^* \tag{4}$$

The first condition gives the optimal allocation between consumption of the young and consumption of the old. Equation (3) determines the optimal population growth rate by equalizing the marginal benefit of additional population growth to its marginal cost. The first one is given by the left-hand side of the expression, and has been called the *intergenerational transfer effect*. It captures the fact that, when the population grows, there are more working individuals to support each retired person. On the other hand, the marginal cost of higher population growth, given by the right-hand side of the equation, has been called the *capital dilution effect*. According to this effect, when the population grows, the stock of capital must be expanded in order for the same capital-labor ratio to be maintained. Finally, equation (4) determines the optimal level of capital per capita, which is defined by the golden rule.

If the optimal solution is interior, these three equations together with the resource constraint of the economy (1) determine the optimal values of consumption, the capital-labor ratio and the population growth rate, c^*, d^*, k^* and $n^* - 1$.

A year after Samuelson published his article on the optimal growth rate for population, Deardorff (1976) published a comment on the former in the same review. Deardorff's article argued that, for a wide range of utility and production functions, Samuelson's problem did not have an interior global maximum. For instance, for the typical double Cobb-Douglas case, the optimal population growth rate turned out to be either zero or infinite. In the following, we analyze this specific case and try to clarify what underlies the results.

Consider the following log-linear utility function and Cobb-Douglas production function:

$$U(c_t, d_{t+1}) = \log c_t + \beta \log d_{t+1}$$

$$f(k_t) = Ak_t^{\alpha}$$

where $\beta \in [0,1]$ is the subjective discount factor, A is a technological parameter and $\alpha \in [0,1]$ is a parameter that represents the share of income that goes to capital earnings. Although the utility function is concave in each of its arguments, the planner's objective is not necessarily globally concave. Using these functions, equations (1)-(4) can be rewritten as:

$$\frac{1/c}{\beta/d} = n \tag{5}$$

$$\frac{d}{n^2} = k \tag{6}$$

$$A\alpha k^{\alpha-1} + 1 - \delta = n \tag{7}$$

$$Ak^{\alpha} + (1-\delta)k = c + \frac{d}{n} + kn \tag{8}$$

The solution is slightly different if one assumes $\delta = 1$ or $\delta < 1$.

2.1 Total Depreciation of Capital

Let's start by assuming that capital totally depreciates in the production process. In such a case, we can isolate k in (7) as:

$$k = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

and d in (5) as:

$$d = \beta c n \tag{10}$$

Substituting these two expressions in (8) and isolating c we obtain:

$$c = \frac{1}{1+\beta}\frac{1-\alpha}{\alpha}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} n \tag{11}$$

Equation (6) can then be expressed only as a function of n:

$$\frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \tag{12}$$

where the term $\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$ cancels out.

Hence the previous set of equations will only be satisfied simultaneously if the following condition on parameters holds by chance:

$$\alpha = \frac{\beta}{1+2\beta} \tag{13}$$

Otherwise, the first order conditions of the planner's problem have no solution. Note also that, if (13) holds, the set of first order conditions is satisfied for any value of c, d, k and n.

To understand the intuition behind this result, we will analyze how the marginal benefit and the marginal cost of higher population growth behave. As explained in the previous section, the marginal benefit of n is given by the intergenerational transfer effect (d/n^2), while the marginal cost is given by the capital dilution effect (k). Using the specific functions defined above, we can express these two effects as a function of n as:

$$IT = \frac{d}{n^2} = \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$$

$$KD = k = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$$

where IT refers to the intergenerational transfer effect and KD to the capital dilution effect. By taking the first and second derivatives, we conclude that both effects are always decreasing and convex as a function of n:

$$\begin{aligned}
\frac{\partial IT}{\partial n} &= -\frac{\beta}{\alpha(1+\beta)}(A\alpha)^{\frac{1}{1-\alpha}} n^{\frac{-2+\alpha}{1-\alpha}} < 0 \quad \forall n > 0 \\
\frac{\partial^2 IT}{\partial n^2} &= \frac{\beta(2-\alpha)}{\alpha(1+\beta)(1-\alpha)}(A\alpha)^{\frac{1}{1-\alpha}} n^{\frac{-3+2\alpha}{1-\alpha}} > 0 \quad \forall n > 0 \\
\frac{\partial KD}{\partial n} &= -\frac{1}{1-\alpha}(A\alpha)^{\frac{1}{1-\alpha}} n^{\frac{-2+\alpha}{1-\alpha}} < 0 \quad \forall n > 0 \\
\frac{\partial^2 KD}{\partial n^2} &= \frac{(2-\alpha)}{(1-\alpha)^2}(A\alpha)^{\frac{1}{1-\alpha}} n^{\frac{-3+2\alpha}{1-\alpha}} > 0 \quad \forall n > 0
\end{aligned}$$

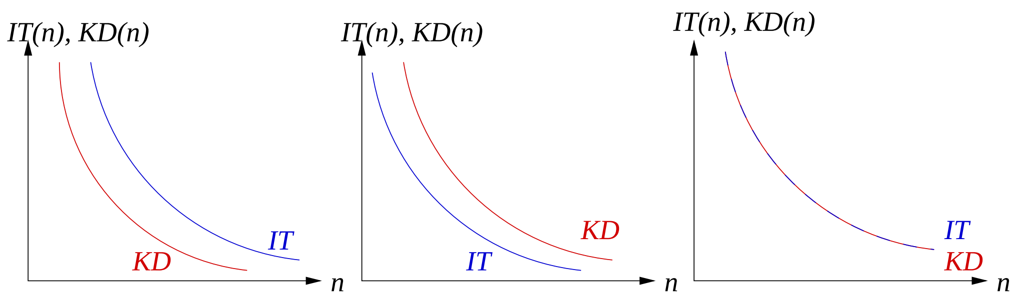
Moreover, by subtracting one effect from the other, we obtain:

$$IT - KD = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \frac{\beta - \alpha(1+2\beta)}{\alpha(1+\beta)}$$

the sign of which does not depend on the size of n. Depending on the sign of the expression $\beta - \alpha(1+2\beta)$, one of the two effects will always dominate the other, $\forall n > 0$. In particular, if this expression is positive, the intergenerational transfer effect dominates, implying that the optimal n is infinite; if it is negative, the capital dilution effect dominates, implying that the optimal n is zero. In the case $\alpha = \frac{\beta}{1+2\beta}$, the expression is zero and the two effects cancel out.

These results are illustrated graphically in Figure 1. For the intergenerational transfer effect to dominate the capital dilution, we need α to be low enough and β to be high enough; in other words, we need labor to be important enough in the production process, and we need individuals not to discount the value of future consumption too much. Then the marginal benefit of higher population growth will always be higher than its marginal cost, as shown in panel (a). On the other hand, if labor is not very important in production and future consumption is discounted considerably, population growth will be less valued and the marginal cost will dominate the marginal benefit, as depicted in panel (b). Panel (c) shows the case in which the marginal cost exactly compensates the marginal benefit of higher population growth.

Figure 1: The capital dilution and intergenerational transfer effects for $\delta = 1$



(a) $\alpha < \frac{\beta}{1+2\beta}$ (b) $\alpha > \frac{\beta}{1+2\beta}$ (c) $\alpha = \frac{\beta}{1+2\beta}$

In terms of utility, when the intergenerational transfer effect dominates, the marginal benefit of n is greater than its marginal cost, so utility increases in n. The opposite is true when the capital dilution effect dominates, so utility decreases. Finally, when the two effects cancel each other out, utility does not change as we vary n. To complement the analysis on the shape of the utility function, we shall study the limit cases. Substituting (10) and (11) in the utility function, we obtain the indirect utility function as a function of n:

$$\begin{aligned} U(c(n), d(n)) \;\; &\equiv \;\; V(n) = \log\left(\frac{1-\alpha}{\alpha(1+\beta)}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} n\right) + \\ &+ \beta \log\left(\frac{\beta(1-\alpha)}{\alpha(1+\beta)}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} n^2\right) \end{aligned}$$

The limits of this function can be easily obtained by grouping terms together:

$$V(n) = \log\left(\frac{1-\alpha}{\alpha(1+\beta)}\left[\frac{\beta(1-\alpha)}{\alpha(1+\beta)}\right]^{\beta}(A\alpha)^{\frac{1+\beta}{1-\alpha}}\,n^{1+2\beta-\frac{1+\beta}{1-\alpha}}\right) \tag{14}$$

then

$$\begin{aligned}
\lim_{n\to 0} V(n) &= \begin{cases} -\infty & \text{if } \beta-\alpha(1+2\beta)>0 \\ +\infty & \text{if } \beta-\alpha(1+2\beta)<0 \end{cases} \\
\lim_{n\to \infty} V(n) &= \begin{cases} +\infty & \text{if } \beta-\alpha(1+2\beta)>0 \\ -\infty & \text{if } \beta-\alpha(1+2\beta)<0 \end{cases}
\end{aligned}$$

In the case $\beta-\alpha(1+2\beta)=0$, $V(n)$ is constant at the following value:

$$V(n) = \log\left(\beta^{\beta}\left[\frac{1-\alpha}{\alpha(1+\beta)}\right]^{1+\beta}(A\alpha)^{\frac{1+\beta}{1-\alpha}}\right)$$

Taking the derivative of (14) and rearranging:

$$V'(n) = \frac{1}{n}\left[\frac{\beta-\alpha(1+2\beta)}{1-\alpha}\right]$$

which has the following limits:

$$\begin{aligned}
\lim_{n\to 0} V'(n) &= \begin{cases} +\infty & \text{if } \beta-\alpha(1+2\beta)>0 \\ -\infty & \text{if } \beta-\alpha(1+2\beta)<0 \end{cases} \\
\lim_{n\to \infty} V'(n) &= 0
\end{aligned}$$

The shape of the indirect utility function is depicted in Figure 2 for the three cases differentiated above.

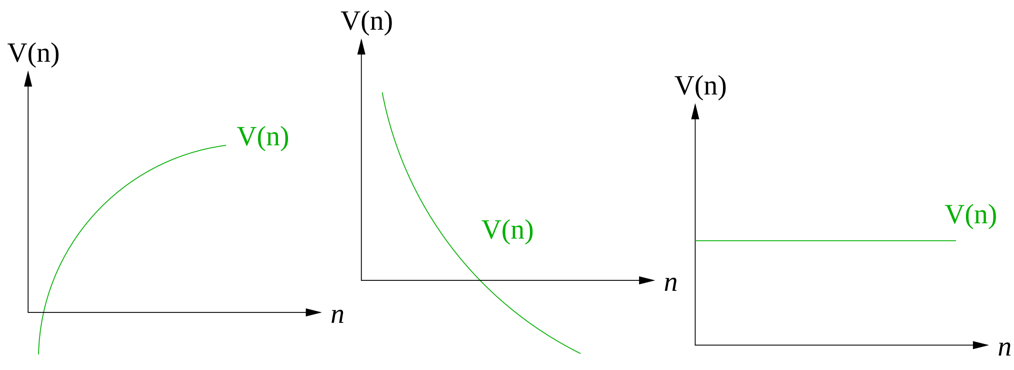
Thus, in this case, with $\delta = 1$, there is no interior solution to the planner's problem. Depending on the values of α and β, the optimal rate of population growth that maximizes utility can be ∞, 0 or any $n > 0$.

2.2 Partial Depreciation of Capital

Let's now turn to the case in which the depreciation rate is strictly smaller than 1[3]. In such a case, there is a solution to the planner's set of first order

[3]This is the case considered by Deardorff (1976).

Figure 2: Shape of the indirect utility function $V(n)$ for $\delta = 1$



(a) $\alpha < \frac{\beta}{1+2\beta}$ (b) $\alpha > \frac{\beta}{1+2\beta}$ (c) $\alpha = \frac{\beta}{1+2\beta}$

conditions:

$$\overline{n} = \frac{\beta(1-\alpha)(1-\delta)}{\beta - \alpha(1+2\beta)} \tag{15}$$

Note that this solution is valid as long as $\alpha < \frac{\beta}{1+2\beta}$, which guarantees that $\overline{n}$ is positive and finite.

The intergenerational transfer and the capital dilution effects are given by:

$$IT = \frac{d}{n^2} = \frac{\beta}{1+\beta}\frac{1-\alpha}{\alpha}\left(\frac{A\alpha}{n+\delta-1}\right)^{\frac{1}{1-\alpha}}\left(\frac{n+\delta-1}{n}\right)$$

$$KD = k = \left(\frac{A\alpha}{n+\delta-1}\right)^{\frac{1}{1-\alpha}}$$

which, as before, are decreasing and convex in n. Subtracting the two effects, we obtain:

$$IT - KD = \left(\frac{A\alpha}{n+\delta-1}\right)^{\frac{1}{1-\alpha}} \frac{[\beta - \alpha(1+2\beta)]\, n - \beta(1-\alpha)(1-\delta)}{n\alpha(1+\beta)} \tag{16}$$

The indirect utility function is in this case:

$$\begin{aligned} V(n) &= \log\left(\frac{1-\alpha}{\alpha(1+\beta)}\left(\frac{A\alpha}{n+\delta-1}\right)^{\frac{1}{1-\alpha}}(n+\delta-1)\right)+ \\ &\quad +\beta\log\left(\frac{\beta(1-\alpha)}{\alpha(1+\beta)}\left(\frac{A\alpha}{n+\delta-1}\right)^{\frac{1}{1-\alpha}}(n+\delta-1)\,n\right) \end{aligned}$$

and its derivative:

$$V'(n)=\frac{[\beta-\alpha(1+2\beta)]\,n-\beta(1-\alpha)(1-\delta)}{n(1-\alpha)(n+\delta-1)}$$

with the following limit values[4]:

$$\begin{aligned} \lim_{n\to 1-\delta} V(n) &= +\infty \\ \lim_{n\to\infty} V(n) &= \begin{cases} -\infty & \text{if } \beta-\alpha(1+2\beta)<0 \\ +\infty & \text{if } \beta-\alpha(1+2\beta)>0 \\ I & \text{if } \beta-\alpha(1+2\beta)=0 \end{cases} \end{aligned}$$

$$\begin{aligned} \lim_{n\to 1-\delta} V'(n) &= -\infty \\ \lim_{n\to\infty} V'(n) &= 0 \end{aligned}$$

where I is the following constant:

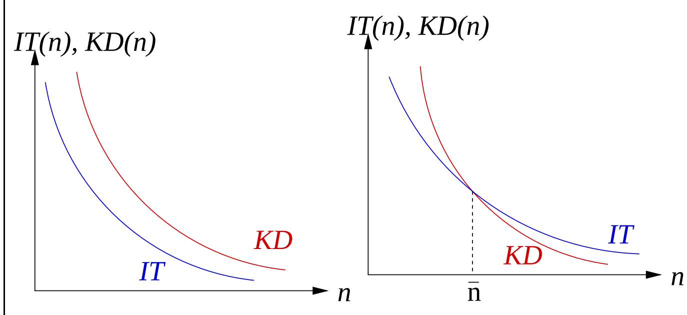
$$I\equiv\log\left[\left(\frac{1}{1+\beta}\frac{1-\alpha}{\alpha}(A\alpha)^{\frac{1}{1-\alpha}}\right)^{1+\beta}\beta^{\beta}\right]$$

Hence there are two possibilities:

- If $\alpha \geq \frac{\beta}{1+2\beta}$, (16) is always negative; thus the capital dilution effect always dominates the intergenerational transfer. Moreover, (15) is not a finite non-negative number, so there is no critical point that satisfies the set of first order conditions. This case is depicted in panel (a) of Figures 3 and 4.

[4]Note that in this case the lower bound for n is $1-\delta$, since $n>1-\delta$ is required to have a positive capital-labor ratio.

Figure 3: The capital dilution and intergenerational transfer effects for $\delta < 1$

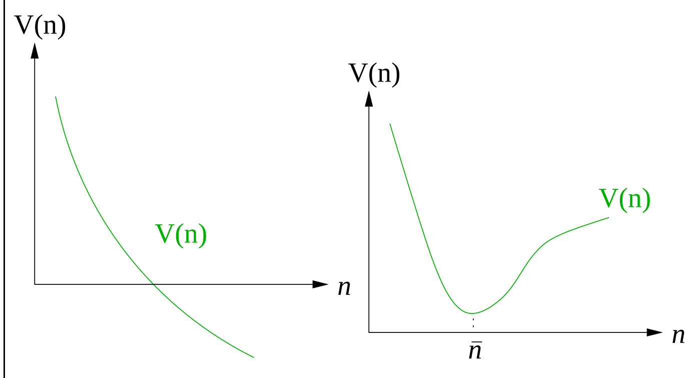


(a) $\alpha \geq \frac{\beta}{1+2\beta}$ (b) $\alpha < \frac{\beta}{1+2\beta}$

- If $\alpha < \frac{\beta}{1+2\beta}$, the effect that dominates depends on the size of n: for $n < \overline{n}$, the capital dilution effect dominates; for $n > \overline{n}$, the intergenerational transfer effect dominates; finally, if $n = \overline{n}$, both effects cancel each other out. In the first case, utility is decreasing in n since the marginal cost is greater than the marginal benefit, while the opposite is true in the second case. Thus the indirect utility function is U-shaped, as can be seen in panel (b) of Figure 4, and the solution to the planner's first order conditions is in fact a minimum.

Therefore, with $\delta < 1$, as in the case of total depreciation of capital, there is never an interior maximum for n. However, there exists the possibility of having an interior minimum, which reminds us of the importance of checking the second order conditions of the planner's problem. Compared with the previous case of total depreciation, when capital is not fully depreciated in the production process the capital dilution effect is relatively stronger than before, so that, even if labor is very important in production and future consumption is not discounted much, for very low n the marginal cost will dominate and utility will decrease in n.

Figure 4: Shape of the indirect utility function $V(n)$ for $\delta < 1$



(a) $\alpha \geq \frac{\beta}{1+2\beta}$ (b) $\alpha < \frac{\beta}{1+2\beta}$

Deardorff generalized his results and showed that, with standard preferences[5], there would be no interior solution to the planner's problem whenever the production function was unbounded, in the sense that output could be made infinitely large by greatly increasing the capital-labor ratio[6]. This happens in the Cobb-Douglas case as well as in the constant elasticity of substitution (CES) production function with substitutability between factors of production. Michel and Pestieau (1993) analyze, in the case $\delta = 1$, the shape of the indirect utility function when the utility and production functions are of the CES type. They conclude that, in order to have an interior global maximum, there must be complementarity between labor and capital in production, as in Deardorff's analysis. Alternatively, if the production function is Cobb-Douglas, complementarity between first and second period consumption in preferences is required. In all other cases, the optimum population growth rate is a corner solution.

[5] In the sense that utility increases monotonically in both of its arguments.

[6] The reason is that utility can always increase by reducing n, since this raises k and thus consumption.

3 Optimal Population Growth with Endogenous Fertility

We will now extend the previous model to allow for the number of children to be chosen by individuals –endogenizing population growth– and see how the results change. We will focus on the double Cobb-Douglas case and will consider only the case of total depreciation of capital.

To endogenize fertility in Diamond's model of overlapping generations, two new features will be introduced in the basic model. First, we will suppose individuals derive utility from having descendants –that will be the benefit of having children; and second, we will assume children are costly. Consequently, the number of children per individual, $n_t \equiv \frac{N_{t+1}}{N_t}$, will now be a decision variable and hence will no longer be constant over time.

Concerning the first new aspect of the model, we will assume, from now on, that individual preferences can be represented by the following utility function:

$$U_t(n_t, c_t, d_{t+1}) = \gamma \log(n_t) + (1-\gamma)\left[\log(c_t) + \beta \log(d_{t+1})\right] \tag{17}$$

where $\gamma \in [0, 1]$ is a parameter reflecting the taste for children. The first derivatives of the utility function with respect to each argument are thus positive. We assume absence of altruism, in the sense that agents do not value the utility of their children. Observe that, if $\gamma = 0$, individuals do not like children. In that case, they would only have descendants if there was an investment motive for having them, for instance if children supported their parents when old[7], or if pension benefits were linked somehow to the number

[7] See for example Wigger (1999), who derives a model in which gifts are given by children to their parents due to the existence of backward altruism; similarly, Cigno (1993) and Rosati (1996) assume children support their parents because this is a family rule that everyone obeys.

of children[8].

The cost of children can be introduced in several ways. We will include both a fixed monetary cost e per child and a time cost z per child; by making one of these two parameters equal to zero, the model is left with only one type of cost of children. The inclusion of a time cost of children implies the endogeneity of the labor supply, because it introduces a trade-off between working in the labor market and raising children. Let's call L_t the amount of labor supplied to the firms by all individuals in period t. Each individual devotes a share zn_t of their time to raising children and a share $(1 - zn_t)$ to working in the market. Assuming each agent has an endowment of 1 unit of time, the production function can now be written as[9]:

$$F(K_t, L_t) = F(K_t, (1 - zn_t)N_t)$$

With the Cobb-Douglas specification and in per-capita terms:

$$f(k_t, l_t) = Ak_t^{\alpha} l_t^{1-\alpha} \tag{18}$$

where $l_t = 1 - zn_t$ is the labor supply per individual, and k_t is no longer the capital-labor ratio but the stock of capital per capita. Note that the production function depends now on two arguments: capital and labor; the latter depends negatively on the number of children per individual.

3.1 The Planner

As in Samuelson (1975), optimality is defined here as the allocation that maximizes the steady state utility of the representative individual subject to the resource constraint:

$$\max_{c,d,n,k} U(n, c, d) = \gamma log(n) + (1 - \gamma) [log(c) + \beta log(d)] \tag{19}$$

[8]As proposed in Bental (1989), Sinn (1997) and Abío and Patxot (2001).
[9]With exogenous labor supply, $L_t = N_t$.

subject to

$$Ak^{\alpha}l^{1-\alpha} = c + \frac{d}{n} + en + nk \tag{20}$$

with $l = 1 - zn$.

The first order conditions for an interior solution are:

$$\frac{d^*}{\beta c^*} = n^* \tag{21}$$

$$\frac{d^*}{(n^*)^2} + \frac{\gamma}{1-\gamma}\frac{c^*}{n^*} = k^* + e + zA(1-\alpha)\left(\frac{k^*}{l^*}\right)^{\alpha} \tag{22}$$

$$A\alpha\left(\frac{l^*}{k^*}\right)^{1-\alpha} = n^* \tag{23}$$

As could be expected, the equation that determines the optimal number of children, (22), is different than before. The marginal benefit of children now includes two terms: the intergenerational transfer effect and the marginal utility of children, expressed in terms of first-period consumption. The marginal cost, on the other hand, is the sum of three terms: the capital dilution effect, the monetary cost per child and the loss in production due to the time cost of children.

An interior optimal solution at the steady state is a vector of variables $(c^*, d^*, n^*, k^*, l^*)$ satisfying equations (20)-(23) that maximizes the utility function in (19) subject to the resource constraint (20).

3.2 Is There an Interior Optimal Solution?

To obtain clear results, the analysis is undertaken assuming the existence of only one type of cost of children.

3.2.1 A Monetary Cost of Children

Let's first assume $e > 0$ and $z = 0$, so there is only a fixed monetary cost per child. As in the previous section, we can obtain the marginal benefit and the marginal cost of children in terms of n:

$$MB = \frac{\gamma+\beta(1-\gamma)}{(1-\gamma)(1+\beta)}\left[\frac{1-\alpha}{\alpha}\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} - e\right]$$

$$MC = e + \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$$

where MB refers to the marginal benefit and MC to the marginal cost; and verify that both are decreasing and convex in n. To know which of them dominates, we subtract them and obtain:

$$MB - MC = \frac{\left[\gamma+\beta(1-\gamma)-\alpha\left[1+2\beta(1-\gamma)\right]\right]\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} - e\alpha\left[1+2\beta(1-\gamma)\right]}{\alpha(1-\gamma)(1+\beta)} \tag{24}$$

the sign of which depends on n, α, β and γ. Define $\widetilde{\alpha}$ as:

$$\widetilde{\alpha} \equiv \frac{\gamma+\beta(1-\gamma)}{1+2\beta(1-\gamma)}$$

- If $\alpha \geq \widetilde{\alpha}$, (24) is always negative no matter the value of n, thus the marginal cost of children is always greater than their marginal benefit and utility is always decreasing in n. This case is depicted in panel (b) of Figures 5 and 6.

- If $\alpha < \widetilde{\alpha}$, the expression in (24) can be positive, negative, or zero. Let $\overline{n}$ be the value of n that makes it equal to zero:

$$\overline{n} \equiv A\alpha\left[\frac{\gamma+\beta(1-\gamma)-\alpha\left[1+2\beta(1-\gamma)\right]}{e\alpha\left[1+2\beta(1-\gamma)\right]}\right]^{1-\alpha}$$

which is also the value that solves the set of first order conditions. Then, when $n < \overline{n}$, (24) will be positive. When $n > \overline{n}$, (24) will become negative. In the first case, the marginal benefit will dominate the marginal cost of children and utility will be increasing in n, while the opposite happens in the second case. When $n = \overline{n}$, the marginal benefit of children equals their marginal cost and utility achieves a maximum. This case is represented graphically in panel (a) of the same figures.

Figure 5: Marginal benefit and marginal cost of children for $z = 0$ or $e = 0$

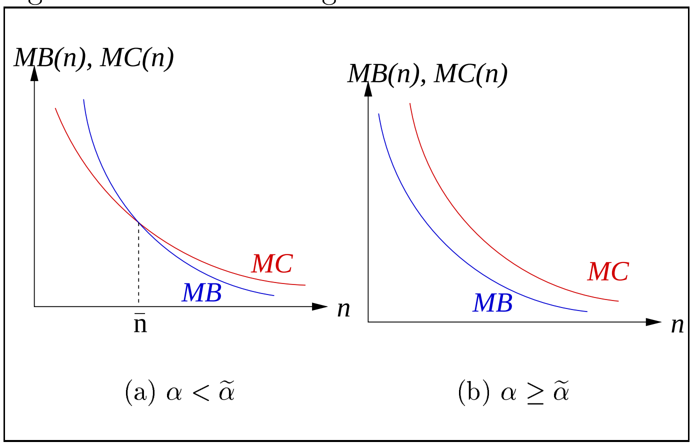
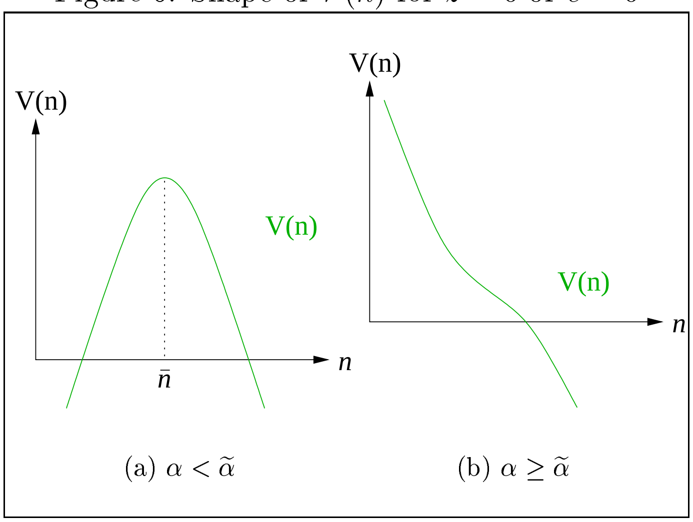


Figure 6: Shape of $V(n)$ for $z = 0$ or $e = 0$



The limit values of $V(n)$ and $V'(n)$ are[10]:

$$\begin{aligned}
\lim_{n\to 0} V(n) &= \begin{cases} -\infty & \text{if } \alpha < \widetilde{\alpha} \\ +\infty & \text{if } \alpha > \widetilde{\alpha} \\ I & \text{if } \alpha = \widetilde{\alpha} \end{cases} \\
\lim_{n\to n^{\max}} V(n) &= -\infty \\
\lim_{n\to 0} V'(n) &= \begin{cases} +\infty & \text{if } \alpha < \widetilde{\alpha} \\ -\infty & \text{if } \alpha > \widetilde{\alpha} \\ 0 & \text{if } \alpha = \widetilde{\alpha} \end{cases} \\
\lim_{n\to n^{\max}} V'(n) &= -\infty
\end{aligned}$$

where I is the following constant:

$$I \equiv \log\left(\beta^{\beta(1-\gamma)}A^{1+2\beta(1-\gamma)}\frac{(1-\gamma)^{(1+\beta)(1-\gamma)}\left[\gamma+\beta(1-\gamma)\right]^{\gamma+\beta(1-\gamma)}}{\left[1+2\beta(1-\gamma)\right]^{1+2\beta(1-\gamma)}}\right)$$

and

$$n^{\max} = A\left(\frac{1-\alpha}{e}\right)^{1-\alpha}\alpha^{\alpha}$$

So, in this case, there is the possibility of having an interior global maximum, under the conditions that the share of capital in income is low enough, the subjective discount factor is high enough, and the taste for children is high enough. Note that, if there were no taste for children ($\gamma = 0$), these results would not be altered; there would still be an interior global maximum of utility for low values of α and high values of β.

3.2.2 A Time Cost of Children

Let's now assume children are costly only in terms of time, so $e = 0$ and $z > 0$. After verifying that both the marginal cost and marginal benefit are

[10]In this case, the existence of a cost of children imposes an upper limit on the maximum feasible amount of descendants of an individual. Since more children imply less resources available for present consumption, this limit is determined by making consumption in the first period –and hence in the second– equal to zero. We denote this maximum number of descendants by $n^{\max}$.

decreasing and convex as a function of n, we obtain the difference between the two as:

$$MB - MC = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \frac{\Gamma - \Theta z n}{\alpha(1-\gamma)(1+\beta)} \tag{25}$$

where

$$\Gamma \equiv \gamma + \beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma)\right]$$

and

$$\Theta \equiv 1 + 2\beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)\right]$$

It is easy to prove that $\Gamma < \Theta$. The sign of (25) crucially depends, as before, on the size of α relative to those of β and γ, as well as on the value of n. We can define the following critical values of α:

$$\widetilde{\alpha}_1 \equiv \frac{\gamma + \beta(1-\gamma)}{1 + 2\beta(1-\gamma)} = \widetilde{\alpha}$$

$$\widetilde{\alpha}_2 \equiv \frac{1 + 2\beta(1-\gamma)}{1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)}$$

with $\widetilde{\alpha}_1 < \widetilde{\alpha}_2$, and the value of n that solves the first order conditions:

$$\overline{n} \equiv \frac{\gamma + \beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma)\right]}{z\left[1 + 2\beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma) + (1-\gamma)(1+\beta)\right]\right]}$$

This time there are three possible cases:

- For a sufficiently small share of capital in income ($\alpha < \widetilde{\alpha}_1$), both Γ and Θ are positive; hence the marginal benefit of children exceeds their marginal cost for $n < \overline{n}$, while the opposite is the case for $n > \overline{n}$. This case corresponds to panel (a) of the previous Figures 5 and 6.

- For moderate values of α ($\widetilde{\alpha}_1 \leq \alpha \leq \widetilde{\alpha}_2$), Θ is positive but Γ is negative; thus the marginal cost always dominates the marginal benefit, as can be seen in panel (b) of the same figures.

- If α is high enough ($\alpha > \widetilde{\alpha}_2$), both Γ and Θ are negative; so again the sign of the expression in (25) seems to depend on n; however, since $\Gamma < \Theta$ and $zn \leq 1$, the expression will always be negative, as in the previous case. Note that $\widetilde{\alpha}_2 > 1/2$, so under the typical assumption that the share of capital in income is lower than one half this case is automatically ruled out.

The limit values of the indirect utility function and its derivative are:

$$\begin{aligned}
\lim_{n\to 0} V(n) &= \begin{cases} -\infty & \text{if } \alpha < \widetilde{\alpha}_1 \\ +\infty & \text{if } \alpha > \widetilde{\alpha}_1 \\ I & \text{if } \alpha = \widetilde{\alpha}_1 \end{cases} \\
\lim_{n\to n^{\max}} V(n) &= -\infty \\
\lim_{n\to 0} V'(n) &= \begin{cases} +\infty & \text{if } \alpha < \widetilde{\alpha}_1 \\ -\infty & \text{if } \alpha > \widetilde{\alpha}_1 \\ -z(1+\beta)(1-\gamma) & \text{if } \alpha = \widetilde{\alpha}_1 \end{cases} \\
\lim_{n\to n^{\max}} V'(n) &= -\infty
\end{aligned}$$

where I is the following constant:

$$I \equiv \log\left(\beta^{\beta(1-\gamma)} A^{1+2\beta(1-\gamma)} \frac{(1-\gamma)^{(1+\beta)(1-\gamma)} \left[\gamma+\beta(1-\gamma)\right]^{\gamma+\beta(1-\gamma)}}{\left[1+2\beta(1-\gamma)\right]^{1+2\beta(1-\gamma)}}\right)$$

This time $n^{\max}$ is given by $1/z$, which is the value that makes present consumption equal to zero.

Observe that, again, the interior global maximum exists if α is low enough, β and γ high enough. If $\gamma = 0$, it still exists as long as the conditions on α and β are satisfied.

3.2.3 The Role of the Cost and the Taste for Children

In the two cases above, we have seen that the possibility of an interior global maximum would still exist if individuals did not like children. So it seems that it is the cost of children, be it monetary or in terms of time, that is crucial in determining whether there is an interior solution in the planner's problem. If there were no cost of children ($e = z = 0$), the only difference with Samuelson's model would be the taste for children, $\gamma > 0$. In such a case, it is easy to demonstrate that there will never be an interior solution to the planner's problem, since:

$$MB - MC = \left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \frac{\gamma + \beta(1-\gamma) - \alpha\left[1 + 2\beta(1-\gamma)\right]}{\alpha(1-\gamma)(1+\beta)}$$

the sign of which does not depend on n. The reason is the following: the marginal utility of children, with log-linear preferences, depends on n in the same proportion as the intergenerational transfer and the capital dilution effects do. So, depending on the parameter values, either the marginal benefit or the marginal cost will dominate independently of the value of n, as was the case when there was no taste for children.

With a cost of children, the condition for having an interior solution is the same as the one for having the intergenerational transfer effect dominate the capital dilution in Samuelson's model: labor must be sufficiently important in the production process (α sufficiently low), and future consumption must not be discounted too much by individuals (β sufficiently high). Yet, the question remains as to why the utility function eventually decreases under the presence of a cost of children. Recall that both the intergenerational transfer and the capital dilution effects are decreasing in n. When a cost per child is introduced, this cost does not decrease in n as much as the other effects, because we assume that the first child costs the same as the second, and so on[11]. In this way the marginal cost of children eventually dominates for a sufficiently high fertility rate. Thus in the case where the

[11] In the case of a time cost of children, the marginal cost of a child in terms of loss in production is decreasing in n due to general equilibrium effects, since this cost is proportional to the per-capita capital stock. However, it is not so decreasing in n as the capital dilution effect.

intergenerational transfer effect dominates the capital dilution and hence utility is increasing in n –at least for low values of n– at some point the cost of children will dominate and utility will start decreasing. Introducing a constant cost per child is therefore a way of introducing an upper bound for the choice of n, avoiding the potential *repugnant* solution of having an infinite rate of population growth as the optimal solution for the economy.

4 Final Remarks

We have seen that the endogenization of fertility, and in particular the introduction of a cost of children, can eliminate the problem of the non-existence of an interior global maximum in Samuelson's problem of finding the optimal population growth rate for an overlapping generations economy. If the cost of children was endogenous –i.e. chosen by the parents– the optimum would again be a corner solution, as parents would choose to have an infinite amount of descendants and to invest a minimum amount of resources on each of them. However, this problem disappears once we consider the human capital that is accumulated when investing in a child's education, with its positive effects on productivity growth[12].

It should not be concluded, from this theoretical analysis, that there exists an optimal population growth rate that each society must try to attain. As Samuelson (1976) claims, "An important purpose of the original analysis was not so much to enable society to identify n^* and normatively to move to it, as to learn what is implied for society's net welfare potentialities by the post-1957 drop in birth rates". After identifying the problem of the non-interiority of the optimal solution, one of his conclusions was that society should "reduce the fears that declining population growth makes old-age

[12] See, for example, Peters (1995).

security more difficult". Indeed, if n^* were equal to 0, utility would increase as fertility falls, until an infinite level of utility was attained when population tends to disappear. Our analysis with endogenous fertility shows that this result might no longer be true and confirms the conclusion of many other studies developed in the last decades: the demographic transition challenges the future finances of the social security system, and by making the provision of old age consumption more difficult, it may well have negative welfare effects on society.

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