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Col·lecció d'Economia

Security Strategies and Equilibria in Multiobjective Matrix Games

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[1]Institutional support from research grants BEC 2002-00642 and SGR2001-0029 is gratefully acknowledged

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Abstract: Multiobjective matrix games have been traditionally analyzed from two different points of view: equilibrium concepts and security strategies. This paper is based upon the idea that both players try to reach equilibrium points playing pairs of security strategies, as it happens in scalar matrix games. We show conditions guaranteeing the existence of equilibria in security strategies, named security equilibria.
Key words: multiobjective matrix games, multicriteria games, security strategies, security equilibria, border strategies, Pareto equilibria.

Resum: Los juegos matriciales multiobjetivo han sido abordados en la literatura bajo dos enfoques diferenciados: busqueda de estrategias de seguridad y análisis de los pares de estrategias de equilibrio. En este trabajo se explora la idea de que, como ocurre en los juegos matriciales de un único objetivo, ambos jugadores intentan alcanzar pares de estrategias de equilibrio jugando sus estrategias de seguridad. Se exponen las condiciones que garantizan la existencia de estos equilibrios cuando ambos jugadores juegan estrategias de de seguridad. A estos equilibrios se les llamara equilibrios de seguridad.

1 Introduction

The well-known classical zero-sum games, or matrix games, have a natural extension if we consider vector payoffs, that is, multiple goals for both players. This kind of games has attracted limited attention in the game theory literature, but nevertheless it is possible to describe two lines of study. One of them is based upon the concept of equilibrium points; the other one uses the notion of Pareto optimality or efficiency.

In the line of identifying equilibrium points, Shapley [5] defines the concept of equilibrium in matrix games with vector payoffs and proves that the solution of a certain scalarized nonzero-sum game gives some of the equilibria of the original multiobjective game. Borm *et al.* [1] show that the Pareto equilibrium points of a nonzero-sum multiobjective game correspond to Nash equilibria of single-objective games derived from strictly positive weighted objective functions, and conversely. In addition to that, Borm *et al.* use a simple and graphic method to find the equilibrium points in nonzero-sum 2×2 multiobjective games. In general the set of equilibrium points contains more than one element.

On the other hand, Ghose and Prassad [4] define the notion of Pareto-optimal security strategies in multiobjective matrix games. The interpretation of that concept is that every player chooses his strategy considering the worst payoff he may incur in each objective separately. The Pareto-optimal security strategies of a player are characterized in [3], [2] and [6].

In the present paper we study multiobjective matrix games, based upon the idea that players can reach equilibrium points playing pairs of security strategies, in a similar way than in a scalar matrix game. We present security equilibrium concepts and we prove that in all cases there exists at least a pair of security strategies that simultaneously satisfy weak equilibrium properties. Finally we give sufficient conditions guaranteeing security equilibria existence.

The paper is organized as follows. Section 1 formulates the multiobjective matrix game and reviews equilibrium concepts. Section 2 presents the concept of security strategies and introduces a simple method to find them in 2×2 games. In Section 3 we combine both concepts in order to prove existence of weak security equilibria. We also analyze the existence of security equilibria in multiobjective 2×2 matrix games. Some proofs are in the appendix.

2 Some definitions and remarks

A two-person multiobjective $m \times n$ zero-sum game is given by $\Omega = (\Omega^1, \ldots, \Omega^r)$, a vector of r payoff matrices, each one with m rows and n columns. The first player chooses a row, and the second a column. The payoff for the first player is the vector of elements selected in each matrix, and the payoff for the second player is just the opposite. We denote by $MOG_0(m \times n, r)$ the set of these games.

The r components of Ω represent the matrix payoffs to player I in each of the objectives of the game. The elements in the payoff matrices will be denoted by $\Omega^k = (\omega_{ij}^k)$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where k denotes an arbitrary objective, $k = 1, 2, \ldots, r$.

Mixed strategy spaces for the players I and II are

$$X = \{\mathbf{x} \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \ldots, m\}$$

and

$$Y = \{\mathbf{y} \in \mathbb{R}^n \mid \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, 2, \ldots, n\}.$$

If there exists $i^* \in \{1, \ldots, m\}$ such that $x_{i^*} = 1$ and $x_i = 0$ for all $i \neq i^*$, the strategy $\mathbf{x} \in X$ is a pure strategy, and it will be represented by $s_{i^*}^I$. Otherwise, $\mathbf{x}$ is a proper mixed strategy. Similarly we will use $s_{j^*}^{II}$, $j^* \in \{1, \ldots, n\}$, for the pure strategies associated to the second player.

In the mixed extension, when I plays $\mathbf{x}$ and II plays $\mathbf{y}$, the payoff for I is $\mathbf{x}\Omega\mathbf{y}^t = (\mathbf{x}\Omega^1\mathbf{y}^t, \ldots, \mathbf{x}\Omega^r\mathbf{y}^t) \in \mathbb{R}^r$. For pure strategies, if I plays $\mathbf{x} = s_i^I$ and II plays $\mathbf{y} = s_j^{II}$, the k-objective payoff for I is ω_{ij}^k, and II obtains $-\omega_{ij}^k$.

We adopt the following notations. For any two vectors $a, b \in \mathbb{R}^r$, $a \geq b$ means $a_j \geq b_j$ for all $j = 1, \ldots, r$; $a > b$ means $a \geq b$ and $a \neq b$; and finally we write $a \gg b$ if $a_j > b_j$ for all $j = 1, \ldots, r$.

Let $U \subseteq \mathbb{R}^r$ and $u \in U$; then the vector u is undominated in U if the set $\{v \in U \mid v \geq u\}$ reduces to $\{u\}$. The vector u is weakly undominated in U if $\{v \in U \mid v \gg u\}$ is empty.

Given a strategy $\mathbf{y}^* \in Y$, the payoff set for player I is $P_I(\mathbf{y}^*) = \{\mathbf{x}\Omega\mathbf{y}^{*t} \mid \mathbf{x} \in X\} \subseteq \mathbb{R}^r$. Similarly, once fixed a strategy $\mathbf{x}^* \in X$, the payoff set for player II is $P_{II}(\mathbf{x}^*) = \{-\mathbf{x}^*\Omega\mathbf{y}^t \mid \mathbf{y} \in Y\} \subseteq \mathbb{R}^r$.

Strategies producing undominated payoff vectors for I in $P_I(\mathbf{y}^*)$ are called **best reply strategies** for player I:

$$BRS_I(\mathbf{y}^*) = \{\mathbf{x} \in X \mid \mathbf{x}\Omega\mathbf{y}^{*t} \textit{ is undominated in } P_I(\mathbf{y}^*)\}.$$

Similarly, the set of best reply strategies for player II is:

$$BRS_{II}(\mathbf{x}^*) = \{\mathbf{y} \in Y \mid -\mathbf{x}^*\Omega\mathbf{y}^t \textit{ is undominated in } P_{II}(\mathbf{x}^*)\}.$$

Then, a **Pareto equilibrium** is a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ satisfying $\mathbf{x}^* \in BRS_I(\mathbf{y}^*)$ and $\mathbf{y}^* \in BRS_{II}(\mathbf{x}^*)$. The set of Pareto equilibria is denoted by $PE(\Omega)$.

Notice that when we have got only one objective, that is to say $r = 1$, the Nash equilibrium is a particular case of Pareto equilibrium.

Similarly, a **weak Pareto equilibrium** is a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ such that $\mathbf{x}^* \in WBRS_I(\mathbf{y}^*)$ and $\mathbf{y}^* \in WBRS_{II}(\mathbf{x}^*)$, where

$$WBRS_I(\mathbf{y}^*) = \{\mathbf{x} \in X \mid \mathbf{x}\Omega\mathbf{y}^{*t} \textit{ is weakly undominated in } P_I(\mathbf{y}^*)\},$$

$$WBRS_{II}(\mathbf{x}^*) = \{\mathbf{y} \in Y \mid -\mathbf{x}^*\Omega\mathbf{y}^t \textit{is weakly undominated in } P_{II}(\mathbf{x}^*)\}.$$

The set of weak Pareto equilibria is denoted by WPE(Ω).

Pareto equilibria are studied in [1] in the context of multiobjective bimatrix games. In [5] the existence of Pareto equilibria and weak Pareto equilibria for multiobjective matrix games is analyzed. These results, applied to the framework of the present paper are summarized in Theorem 1 and their proof can be found in [1] and [5].

Let Δ_r be the unit simplex in $\mathbb{R}^r$, and $\overset{o}{\Delta}_r$ its relative interior,

$$\Delta_r = \{(\delta_1, \ldots, \delta_r) \in \mathbb{R}^r \mid \sum_{k=1}^{r} \delta_k = 1, \delta_k \geq 0, k = 1, 2, \ldots, r\},$$

$$\overset{o}{\Delta}_r = \{(\delta_1, \ldots, \delta_r) \in \mathbb{R}^r \mid \sum_{k=1}^{r} \delta_k = 1, \delta_k > 0, k = 1, 2, \ldots, r\}.$$

Then, any element of Δ_r, or $\overset{o}{\Delta}_r$, can be interpreted as a weighted system for the objectives or criteria of the original game. Therefore, given a weighted system α for player I and β for player II, we can associate a standard scalarization of the original multiobjective game as follows:

$$(\Omega(\alpha), \Omega(\beta)) = (\sum_{k=1}^{r} \alpha_k \Omega^k, -\sum_{k=1}^{r} \beta_k \Omega^k). \tag{1}$$

Notice that, in general, the bimatrix game given in (1) is not a zero-sum game.

Theorem 1 *Let it be $\Omega \in MOG_0(m \times n, r)$.*

1. *Let α and β be a pair of weighting vectors, and let $E((\Omega(\alpha), \Omega(\beta)))$ be the set of all (pure and mixed) Nash equilibria of the scalar bimatrix game $(\Omega(\alpha), \Omega(\beta))$ given by (1). Then:*

1.1 If $\alpha, \beta \in \overset{o}{\Delta}_r$*, then* $E(\Omega(\alpha), \Omega(\beta)) \subseteq PE(\Omega)$*,*

1.2 If $\alpha, \beta \in \Delta_r$*, then* $E(\Omega(\alpha), \Omega(\beta)) \subseteq WPE(\Omega)$*.*

2. Moreover:

2.1 If $(\mathbf{x}^*, \mathbf{y}^*) \in PE(\Omega)$*, then there exists a pair of vectors* $\alpha, \beta \in \overset{o}{\Delta}_r$*, such that* $(\mathbf{x}^*, \mathbf{y}^*) \in E(\Omega(\alpha), \Omega(\beta))$*.*

2.2 If $(\mathbf{x}^*, \mathbf{y}^*) \in WPE(\Omega)$*, then there exists a pair of vectors* $\alpha, \beta \in \Delta_r$*, such that* $(\mathbf{x}^*, \mathbf{y}^*) \in E(\Omega(\alpha), \Omega(\beta))$*.*

Existence of Pareto equilibria and weak Pareto equilibria are easily deduced from the existence of Nash equilibria in bimatrix games and the above theorem. Moreover, from Theorem 1

$$PE(\Omega) = \bigcup_{\alpha, \beta \in \overset{o}{\Delta}_r} E(\Omega(\alpha), \Omega(\beta)),$$

$$WPE(\Omega) = \bigcup_{\alpha, \beta \in \Delta_r} E(\Omega(\alpha), \Omega(\beta)).$$

As a consequence, in general, PE(Ω) is not a singleton.

3 Security strategies

The concept of Pareto-optimal security strategies is studied, among others, in [4], [3], [2] and [6]. In this section we work with security strategies, and we describe a new procedure to determine them in the 2×2 multiobjective matrix games.

The interpretation behind the concept of security strategies is that every player acts considering the worst payoff he or she may incur in each objective separately for a given strategy. The r-tuples we can define with these payoffs are named security level. Observe that this interpretation is inherent in the concept of maxmin and minmax payoffs for a scalar game.

We define the functions $f : X \to \mathbb{R}^r$ for player I and $g : Y \to \mathbb{R}^r$ for player II as

$$f(\mathbf{x}) = (f^1(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^r(\mathbf{x})) = (\min_{\mathbf{y} \in Y} \mathbf{x}\Omega^1\mathbf{y}^t, \ldots, \min_{\mathbf{y} \in Y} \mathbf{x}\Omega^r\mathbf{y}^t),$$

$$g(\mathbf{y}) = (g^1(\mathbf{y}), g^2(\mathbf{y}), \ldots, g^r(\mathbf{y})) = (\max_{\mathbf{x} \in X} \mathbf{x}\Omega^1\mathbf{y}^t, \ldots, \max_{\mathbf{x} \in X} \mathbf{x}\Omega^r\mathbf{y}^t).$$

Notice that $f(\mathbf{x})$ and $g(\mathbf{y})$ determine the guaranteed payoffs in each of the objectives to I and II respectively. By continuity and compacity of the domain, the previous functions are well defined.

Following the definition in [4], we now introduce the set of Pareto-optimal security strategies for player I and player II, which we name **security strategies**, as

$$X_S = \{\mathbf{x}^* \in X \mid \ \textit{if } \mathbf{x} \in X \textit{ is such that } f(\mathbf{x}^*) \leq f(\mathbf{x}), \textit{ then } f(\mathbf{x}^*) = f(\mathbf{x})\},$$

$$Y_S = \{\mathbf{y}^* \in Y \mid \ \textit{if } \mathbf{y} \in Y \textit{ is such that } g(\mathbf{y}^*) \geq g(\mathbf{y}), \textit{ then } g(\mathbf{y}^*) = g(\mathbf{y})\}.$$

Observe that X_S and Y_S are not empty since $f(\mathbf{x})$ and $g(\mathbf{y})$ are continuous vector functions defined in compact domains.

The strategies in X_S and Y_S guarantee undominated security levels to player I and II respectively and they are the maximal and the minimal elements of the ordered sets $(f(X), \leq)$ and $(g(Y), \geq)$.

A method to determine security strategies by means of scalarization of the original multiobjective game can be found in [3]. In [2] the sets of security strategies are obtained by means of solving multiobjective linear programs.

If we isolate the k objective from the original game $\Omega \in MOG_0(m \times n, r)$, the solutions to the optimization problem $Max_{\mathbf{x}\in X} f^k(\mathbf{x})$ are the equilibrium strategies for player I in the scalar zero-sum game Ω^k. We denote by $O(I, \Omega^k)$ the set of these solutions, and similarly we name $O(II, \Omega^k)$ the set of such solutions for player II, i.e. $Min_{\mathbf{y}\in Y} g^k(\mathbf{y})$.

Observe that, due to the continuity and the concavity of the functions $f^k(\mathbf{x})$, the sets $O(I, \Omega^k)$ are closed and convex.

In spite of the fact that the sets of security strategies are difficult to obtain in the general $m \times n$ case, and only some algorithms are known (see for instance [2] and [6]), we can show some general properties which will be useful later on.

If the intersection of the optimal sets of one player is non-empty it coincides with his set of security strategies. Moreover we can see, and this will be crucial in our analysis, that for the general case $m \times n$ at least one optimal strategy of every set $O(I, \Omega^k)$ is a security strategy as we prove in the next proposition.

Proposition 2 *For any game $\Omega \in MOG_0(m \times n, r)$ it holds:*

1. *If $\cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k) \neq \emptyset$, then $X_S = \cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k)$.*
2. *For any $k = 1, 2, \ldots, r$ we have $O(I, \Omega^k) \cap X_S \neq \emptyset$ and $O(II, \Omega^k) \cap Y_S \neq \emptyset$.*

Proof.

For the first statement, if $x \in \cap_{k\in\{1,\dots,r\}} O(I,\Omega^k)$ *then* $f^k(x) \geq f^k(x')$*, for any* $x' \in X$ *and* $k = 1, 2, \dots, r$*. Let us suppose that* $x^* \in X$ *exists such that* $f(x) \leq f(x^*)$*. Then, since* $x \in \cap_{k\in\{1,\dots,r\}} O(I,\Omega^k)$ *we have* $f(x) = f(x^*)$*, which implies* $x \in X_S$*.*

Assume now $x \in X_S$*. Since* $\cap_{k\in\{1,\dots,r\}} O(I,\Omega^k) \neq \emptyset$*, let* $x^* \in \cap_{k\in\{1,\dots,r\}} O(I,\Omega^k)$*. We have* $f^k(x) \leq f^k(x^*)$ *for* $k = 1, 2, \dots, r$*. Since* $x \in X_S$ *we obtain* $f^k(x) = f^k(x^*)$ *for* $k = 1, 2, \dots, r$*. So* $x \in \cap_{k\in\{1,\dots,r\}} O(I,\Omega^k)$*.*

For the second statement, let us suppose that, for some $k \in \{1, 2, \dots, r\}$*,* $O(I,\Omega^k) \cap X_S = \emptyset$*. Take* $\alpha = (\alpha_1, \dots, \alpha_r) \in \overset{o}{\Delta}_r$ *a vector of positive weights and let* $\mathbf{x}^* \in O(I,\Omega^k)$ *be a solution of* $Max_{\mathbf{x}\in O(I,\Omega^k)} h(\mathbf{x})$*, where* $h(\mathbf{x}) = \sum_{i=1}^r \alpha_i f^i(x)$*.*

Since $\mathbf{x}^* \in O(I,\Omega^k)$*, by assumption* $\mathbf{x}^* \notin X_S$*, which means that there exists* $\mathbf{x}' \in X$ *such that* $f(\mathbf{x}^*) \leq f(\mathbf{x}')$ *and* $f(\mathbf{x}^*) \neq f(\mathbf{x}')$*. There are two possible cases. If* $\mathbf{x}' \in O(I,\Omega^k)$*, then* $h(\mathbf{x}^*) < h(\mathbf{x}')$ *getting a contradiction. If not,* $\mathbf{x}' \notin O(I,\Omega^k)$*, but since* $\mathbf{x}^* \in O(I,\Omega^k)$ *this means* $f^k(\mathbf{x}^*) > f^k(\mathbf{x}')$ *which involves a contradiction with the fact that* $f(\mathbf{x}^*) \leq f(\mathbf{x}')$*.*

■

As a first consequence of the above proposition, we propose a direct method to determine the security strategies in the case of 2×2 games.

For 2×2 games it is possible to substitute the strategies $\mathbf{x}$ and $\mathbf{y}$ by their projections in [0,1]. That is to say, the strategies for I and II in the game $MOG_0(2 \times 2, r)$ are $\mathbf{x} = (x, 1-x)$ and $\mathbf{y} = (y, 1-y)$, where $x, y \in [0,1]$.

By means of the last notation it is possible to identify the sets $O(I,\Omega^k)$ as intervals $[a_I^k, b_I^k]$ in [0,1], where $a_I^k \leq b_I^k$.

Among the extremes of all the intervals $[a_I^k, b_I^k]$ we are going to select two specific strategies which will be necessary to determine the security strategies set.

Definition 3 *The* ***border strategies*** *of player I in the game* $\Omega \in MOG_0(2\times 2, r)$ *are the strategies* $a_I^\Omega = max_{k\in\{1,\dots,r\}} a_I^k$ *and* $b_I^\Omega = min_{k\in\{1,\dots,r\}} b_I^k$*.*

Similarly, $a_{II}^\Omega = max_{k\in\{1,\dots,r\}} a_{II}^k$ and $b_{II}^\Omega = min_{k\in\{1,\dots,r\}} b_{II}^k$ are the border strategies for player II, where $O(II,\Omega^k) = [a_{II}^k, b_{II}^k]$ for $k = 1, 2, \dots, r$.

Observe that we only need to calculate the r intervals $O(I,\Omega^k)$, $k = 1, 2, \dots, r$, to determine the border strategies. The border strategies will be precisely the extreme points of the set of security strategies, as we prove in the next theorem.

Proposition 4 *Let it be $\Omega \in MOG_0(2\times 2, r)$. The sets of security strategies for players in Ω are the intervals defined by their border strategies: $X_S = [min(a_I^\Omega, b_I^\Omega), max(a_I^\Omega, b_I^\Omega)]$ and $Y_S = [min(a_{II}^\Omega, b_{II}^\Omega), max(a_{II}^\Omega, b_{II}^\Omega)]$.*

Proof.

See Appendix.

■

Proposition 4 gives a direct and simple method to determine the security strategies in any 2×2 multiobjective game independently of the number of objectives. We review some examples to determine in an easy way the security strategies.

Example 5 *Consider the following game (see [4], example 4.1):*

$$\Omega = (\Omega^1, \Omega^2) = \left[\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right].$$

Solving the scalar games Ω^1 and Ω^2 we obtain the optimal sets: $O(I, \Omega^1) = [a_I^1, b_I^1] = \{1/3\}$ and $O(I, \Omega^2) = [a_I^2, b_I^2] = \{2/3\}$. Therefore, $X_S = [1/3, 2/3]$. Similarly $O(II, \Omega^1) = \{2/3\}$, $O(II, \Omega^2) = \{1/3\}$ and then $Y_S = [1/3, 2/3]$.

Example 6 *Consider the game*

$$\Omega = (\Omega^1, \Omega^2) = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} \right].$$

Solving the scalar games Ω^1 and Ω^2 we obtain the optimal sets: $O(I, \Omega^1) = [a_I^1, b_I^1] = \{1/2\}$ and $O(I, \Omega^2) = [a_I^2, b_I^2] = \{0\}$. So, $X_S = [a_I^\Omega, b_I^\Omega] = [0, 1/2]$. Similarly $O(II, \Omega^1) = \{3/4\}$, $O(II, \Omega^2) = \{1\}$ and then $Y_S = [3/4, 1]$.

Observe, from Proposition 4, that for the 2×2 case the security strategies form a convex set in the strategy space. Let us see that this is not the case in general.

Example 7 *Consider now the game:*

$$\Omega = (\Omega^1, \Omega^2) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \\ 2 & 3 & 0 \end{pmatrix} \right].$$

For this game $O(I, \Omega^1) = \{\mathbf{x}^1\} = \{(6/11, 3/11, 2/11)\}$ and $O(I, \Omega^2) = \{\mathbf{x}^2\} = \{(1/3, 1/3, 1/3)\}$. By Proposition 2, $\mathbf{x}^1, \mathbf{x}^2 \in X_S$. But for example $\mathbf{x}^c = 3\mathbf{x}^1/5 + 2\mathbf{x}^2/5 = (76/165, 49/165, 40/165)$ is not in X_S because it is dominated by $\mathbf{x}^* = (9/19, 7/19, 3/19)$ since $f(\mathbf{x}^c) = (0.46, 1.375) < f(\mathbf{x}^*) = (0.47, 1.42)$. Then X_S is not a convex set.

Finally, as a direct consequence, we characterize the situation when one player has a unique security strategy.

Corollary 8 *For any game* $\Omega \in MOG_0(m \times n, r)$ *the following statements are equivalent,*

1. $X_S = \{\mathbf{x}^*\}$,
2. $\cap_{k \in \{1,\ldots,r\}} O(I, \Omega^k) = \{\mathbf{x}^*\}$.

Proof.
Apply Proposition 2.
■

The condition $\cap_{k \in \{1,\ldots,r\}} O(I, \Omega^k) = \{\mathbf{x}^*\}$ tells us that the first player has a unique common equilibrium in all the scalar games Ω^k. The above corollary shows that this strategy $\mathbf{x}^*$ will also be a security strategy for this player. We could see these cases as pathological, since once player I is constrained to play strategy $\mathbf{x}^*$, the game reduces to a vector optimization problem for the second player. We name these games, i.e. games where $\cap_{k \in \{1,\ldots,r\}} O(J, \Omega^k)$ is a singleton for some player $J \in \{I, II\}$, **degenerated** and we exclude them from our analysis. From here onward $MOG_0^*(m \times n, r)$ denotes the class of **non-degenerated** multiobjective zero-sum games.

In this section we have analyzed security strategies, and if we argue over single-objective games we know that the pairs of security strategies are simultaneously equilibria of these games. For more than one objective, Pareto equilibria set and pairs of security strategies are not the same. It is possible to apply the method described in [1] to find the Pareto equilibria set for the game in Example 6: $PE(\Omega) = ([0, 1/2] \times \{1\}) \cup ((1/2, 1] \times [0, 1/2)) \cup ((1/2, 1] \times (3/4, 1])$ (see figure 1). So $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (1, 0)$ is an equilibrium of the game, but it is not formed by a pair of security strategies.

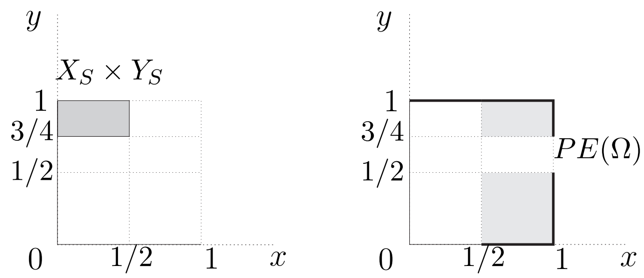

Figure 1: Efficient strategies and Pareto Equilibria for Example 6

4 Security equilibria

In a matrix game with only one objective, Nash equilibria are formed by pairs of simultaneous security strategies. This may not be the same in multiobjective matrix game as shown above.

However a similar behaviour of the players might lead to an equilibrium, that is to say, we look for equilibria formed by security strategies.

Definition 9 *Let it be $\Omega \in MOG_0^*(m\times n, r)$. A **security equilibrium** for Ω is a pair $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ satisfying $(\mathbf{x}^*, \mathbf{y}^*) \in PE(\Omega)$, $\mathbf{x}^* \in X_S$ and $\mathbf{y}^* \in Y_S$. The set of security equilibria for Ω is represented by SE(Ω).*

*Let it be $\Omega \in MOG_0^*(m\times n, r)$. A **weak security equilibrium** for Ω is a pair $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ satisfying $(\mathbf{x}^*, \mathbf{y}^*) \in WPE(\Omega)$, $\mathbf{x}^* \in X_S$ and $\mathbf{y}^* \in Y_S$. The set of weak security equilibria for Ω is represented by WSE(Ω).*

The following theorem shows that the set of weak security equilibria is always non-empty.

Theorem 10 *For any $\Omega \in MOG_0^*(m \times n, r)$, $WSE(\Omega) \neq \emptyset$.*

Proof.

Let Ω^k be one of the components of Ω and let $(\mathbf{x}, \mathbf{y})$ be a Nash equilibrium of Ω^k. Notice that any Nash equilibrium of Ω^k is a weak Pareto equilibrium of Ω. To see this take the weighting vectors $\alpha = \beta = (0, \ldots, ^{k)} 1, \ldots, 0)$ and apply Theorem 1.

Moreover, from Proposition 2 we know $O(I, \Omega^k) \cap X_S \neq \emptyset$ and $O(II, \Omega^k) \cap Y_S \neq \emptyset$. Combining both arguments we get $WSE(\Omega) \neq \emptyset$.

■

In general, the lack of convexity of the sets X_S and Y_S makes the analysis of existence of security equilibria difficult. And in spite of the existence of weak security equilibrium, it is not possible to guarantee the existence of security equilibrium for a given game $\Omega \in MOG_0^*(m \times n, r)$.

Nevertheless, in the 2×2 case, X_S and Y_S are convex sets. This allows to obtain some results for this kind of games. To this end, we classify the games $\Omega \in MOG_0^*(2 \times 2, r)$ into four categories. For three of them we show the existence of security equilibria. For the remaining one we show that security equilibrium might not exist.

Definition 11 *A game $\Omega \in MOG_0^*(2 \times 2, r)$ is **pure-solved** if, for some $k \in \{1, \ldots, r\}$, Ω^k has a unique Nash equilibrium and it is in pure strategies for each player.*

Definition 12 *A game $\Omega \in MOG_0^*(2\times 2, r)$ is* ***determinated*** *if it is not pure-solved and, for some player J, it holds that $\cap_{k\in\{1,\dots,r\}} O(J,\Omega^k) \neq \emptyset$.*

Any non-pure-solved and non-determinated game is named mixed-solved.

It is useful to introduce for any Ω^k, $k \in \{1,\dots,r\}$, the parameter $\delta^k := \omega_{11}^k + \omega_{22}^k - \omega_{12}^k - \omega_{21}^k$. In the special case of only two objectives, we can prove the following useful property.

Lemma 13 *For any mixed-solved biobjective game $\Omega \in MOG_0^*(2\times 2, 2)$ it holds:*

1. $\delta^1 \neq 0$ *and* $\delta^2 \neq 0$.
2. $X_S = [x^1, x^2] = [(\omega_{22}^1 - \omega_{21}^1)/\delta^1, (\omega_{22}^2 - \omega_{21}^2)/\delta^2]$,
   $Y_S = [y^1, y^2] = [(\omega_{22}^1 - \omega_{12}^1)/\delta^1, (\omega_{22}^2 - \omega_{12}^2)/\delta^2]$.

Proof.
See Appendix.
■

For more than two objectives we cannot guarantee that in a mixed-solved game all δ^k are not zero, as we can see in the next example.

Example 14 *Consider the game*

$$\Omega = \left[\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \right].$$

It is easy to see that $O(I,\Omega^1) = [0,1]$, $O(I,\Omega^2) = \{1/3\}$, $O(I,\Omega^3) = \{2/3\}$, $O(II,\Omega^1) = \{0\}$, $O(II,\Omega^2) = \{2/3\}$, $O(II,\Omega^3) = \{1/3\}$. Then the game is mixed-solved, but $\delta^1 = 0$.

Definition 15 *A mixed-solved game $\Omega \in MOG_0^*(2\times 2, r)$ is* ***positive*** *if there exists a mixed-solved subgame $\Omega^{kh} = (\Omega^k, \Omega^h) \in MOG_0^*(2\times 2, 2)$, for $h,k \in \{1,\dots,r\}$, $h \neq k$, such that $\delta^k \cdot \delta^h > 0$. Otherwise it is* ***negative****.*

The game in Example 14 is negative since the biobjective subgames formed by Ω^1 and Ω^2 or Ω^1 and Ω^3 are degenerated, and for the subgame formed by Ω^2 and Ω^3 it holds $\delta^2 \cdot \delta^3 < 0$.

In the case of biobjective mixed-solved games, by Lemma 13, $\delta^1 \neq 0$ and $\delta^2 \neq 0$. If moreover the game is negative, then $\delta^1 \cdot \delta^2 < 0$.

We have got a partition of $MOG_0^*(2\times 2, r)$ into four non empty categories: pure-solved, determinated, positive mixed-solved and negative mixed-solved. We are going to prove that for any negative mixed-solved game with only two objectives, $\Omega \in MOG_0^*(2\times 2, 2)$, the security equilibria set is empty.

Theorem 16 *If $\Omega \in MOG_0^*(2 \times 2, 2)$ is a negative mixed-solved game, then $SE(\Omega) = \emptyset$.*

Proof.

See Appendix.

■

Nevertheless, as we can check in Example 14, for more than two objectives the previous statement does not hold since the game is negative and $SE(\Omega) = [1/3, 2/3] \times [0, 1/3) \neq \emptyset$.

Now we show that for the remaining classes of multiobjective 2×2 matrix games the set of security equilibria is always non-empty regardless the number of objectives.

Theorem 17 *Let it be $\Omega \in MOG_0^*(2 \times 2, r)$.*

1. *If Ω is a pure-solved game, then $SE(\Omega) \neq \emptyset$.*
2. *If Ω is a determinated game, then $SE(\Omega) \neq \emptyset$.*
3. *If Ω is a positive mixed-solved game, then $SE(\Omega) \neq \emptyset$.*

Proof.

See Appendix.

■

Combining Theorems 16 and 17 we can characterize the existence of security equilibria in 2×2 matrix games with only two objectives.

Corollary 18 *Let it be $\Omega \in MOG_0^*(2 \times 2, 2)$. The set of security equilibria, $SE(\Omega)$, is non-empty if and only if Ω is not a negative mixed-solved game.*

Notice that we cannot completely characterize which multiobjective 2×2 matrix games have security equilibria. This is due to the fact that for negative games we can only guarantee the non-existence of security equilibrium in the biobjective case.

Finally, observe that the tools we have used are based on the convexity properties of the sets of security strategies in the 2×2 case. Since these properties may fail in the general case, we leave the study of the $m \times n$ case to subsequent works.

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A Proofs

Proof of Proposition 4: We analyze separately two cases:

1. Assume $\cap_{k\in\{1,\dots,r\}}O(I,\Omega^k)\neq\emptyset$. If $x\in\cap_{k\in\{1,\dots,r\}}O(I,\Omega^k)$, then $a_I^\Omega\leq x\leq b_I^\Omega$, for any $k=1,\dots,r$. Therefore, $a_I^\Omega=max_{k=1,\dots,r}a_I^k\leq b_I^\Omega=min_{k=1,\dots,r}b_I^k$ and $\cap_{k\in\{1,\dots,r\}}O(I,\Omega^k)=[a_I^\Omega,b_I^\Omega]$. By Proposition 2 we get the result.

2. Assume $\cap_{k\in\{1,\dots,r\}}O(I,\Omega^k)=\emptyset$. Notice that in this case $b_I^\Omega<a_I^\Omega$ and then we only have to prove $X_S=[b_I^\Omega,a_I^\Omega]$.

   Since $f^k(x)=min_{y\in[0,1]}\mathbf{x}\Omega\mathbf{y}^t=min\{\omega_{21}^k+x(\omega_{11}^k-\omega_{21}^k),\omega_{22}^k+x(\omega_{12}^k-\omega_{22}^k)\}$, then $f^k(x)$ is a polygonal concave function, defined as a minimum of two linear functions. Let us describe some properties that hold in this case where $\cap_{k\in\{1,\dots,r\}}O(I,\Omega^k)=\emptyset$:

   (a) If $b_I^k\neq 0$, then $f^k(x)$ is increasing in $[0,b_I^k]$, for all $k=1,\dots,r$.

(b) If $a_I^k \neq 1$, then $f^k(x)$ is decreasing in $[a_I^k, 1]$, for all $k = 1, \ldots, r$.

(c) If $b_I^\Omega = b_I^{k^*} \neq 1$, then $f^{k^*}(x)$ is strictly decreasing in $[b_I^\Omega, 1]$.

(d) If $a_I^\Omega = a_I^{k^*} \neq 0$, then $f^{k^*}(x)$ is strictly increasing in $[0, a_I^\Omega]$.

(e) If $b_I^\Omega \neq 0$, then the functions $f^k(x)$ are increasing in $[0, b_I^\Omega]$ for all $k = 1, \ldots, r$ and at least one of them is strictly increasing.

By (a) all functions $f^k(x)$ are increasing in $[0, b_I^\Omega]$. If none of them were strictly increasing then all of them would be constant in $[0, b_I^\Omega]$ and then $\cap_{k \in \{1,\ldots,r\}} O(I, \Omega^k) \neq \emptyset$.

(f) If $a_I^\Omega \neq 1$, then the functions $f^k(x)$ are decreasing in $[a_I^\Omega, 1]$ for all $k = 1, \ldots, r$ and at least one of them is strictly decreasing.

Let us prove that $X_S = [b_I^\Omega, a_I^\Omega]$. First notice that $b_I^\Omega \neq 1$ and $a_I^\Omega \neq 0$, otherwise we would be in case 1.

Let $x \in X_S$ be any arbitrary element. If $x \notin [b_I^\Omega, a_I^\Omega]$, then $x \in [0, b_I^\Omega)$ or $x \in (a_I^\Omega, 1]$.

In the first case, $x \in [0, b_I^\Omega)$, by (e) we obtain $f(x) \leq f(b_I^\Omega)$ and $f(x) \neq f(b_I^\Omega)$, which contradicts the fact $x \in X_S$. Similarly if $x \in (a_I^\Omega, 1]$ but now using (f). Therefore, $X_S \subseteq [b_I^\Omega, a_I^\Omega]$.

Let us prove the other inclusion. Let $x \in [b_I^\Omega, a_I^\Omega]$ be an arbitrary element. Suppose $x \notin X_S$, this implies the existence of $x^* \in [0,1], x \neq x^*$ such that $f(x) \leq f(x^*)$ and $f(x) \neq f(x^*)$. If $x \in (x^*, 1]$, by (c) we know the existence of $k^* \in \{1, \ldots, r\}$ such that $f^{k^*}(x)$ is strictly decreasing in $[b_I^\Omega, 1]$. Therefore $f^{k^*}(x^*) < f^{k^*}(x)$, which contradicts $f(x) \leq f(x^*)$.

If $x \in [0, x^*)$, by (d), there exists at least $k^* \in \{1, \ldots, r\}$ such that $f^{k^*}(x)$ is strictly increasing in $[0, a_I^\Omega]$, but then $f^{k^*}(x^*) < f^{k^*}(x)$, which contradicts $f(x) \leq f(x^*)$.

Proof of Lemma 13:

1. Assume that $\delta^1 = 0$ (similarly for $\delta^2 = 0$), and so $\omega_{11}^1 + \omega_{22}^1 = \omega_{12}^1 + \omega_{21}^1$. Without loss of generality suppose that $\omega_{11}^1 \geq \omega_{12}^1$, and then one of the next three possibilities holds:

   (a) If $\omega_{22}^1 < \omega_{12}^1$, then $\omega_{22}^1 < \omega_{12}^1 \leq \omega_{11}^1$, from where either $O(I, \Omega^1) = \{1\}$ and $O(II, \Omega^1) = \{0\}$, being Ω a pure-solved game, or $O(II, \Omega^1) = [0, 1]$, being Ω a determinated game, depending on whether $\omega_{11}^1 > \omega_{12}^1$ or $\omega_{11}^1 = \omega_{12}^1$.

(b) If $\omega^1_{22} > \omega^1_{12}$, since $\delta^1 = 0$, then $\omega^1_{12} < \omega^1_{22} \leq \omega^1_{21}$. Hence either $O(I, \Omega^1) = \{0\}$ and $O(II, \Omega^1) = \{0\}$, being Ω a pure-solved game, or $O(II, \Omega^1) = [0, 1]$, being Ω a determinated game, depending on whether $\omega^1_{11} > \omega^1_{12}$ or $\omega^1_{11} = \omega^1_{12}$.

(c) If $\omega^1_{22} = \omega^1_{12}$, since $\delta^1 = 0$, then $\omega^1_{11} = \omega^1_{21} \geq \omega^1_{22} = \omega^1_{12}$. Hence $O(I, \Omega^1) = [0, 1]$, and Ω is a determinated game.

2. By hypothesis neither Ω^1 nor Ω^2 have got a unique equlibrium in pure strategies for I and II simultaneously. If player I has got mixed strategies in $O(I, \Omega^1)$, then either $O(I, \Omega^1) = \{x^*\}$ or $O(I, \Omega^1) = [0, x^*]$ or $O(I, \Omega^1) = [x^*, 1]$ for some $x^* \in (0, 1)$. Since $f^1(x)$ is the minimum of two linear functions it holds that $\omega^1_{21}+(\omega^1_{11}-\omega^1_{21})x^* = \omega^1_{22}+(\omega^1_{12}-\omega^1_{22})x^*$, and then $x^* = (\omega^1_{22} - \omega^1_{21})/\delta^1$.

   If there is only one optimal pure strategy, $x^* = 0$ or $x^* = 1$, for player I in Ω^1, then by hypothesis on the game Ω, player II has got an interval of optimal strategies and this implies either $\omega^1_{21} = \omega^1_{22}$ for $x^* = 0$ or $\omega^1_{12} = \omega^1_{11}$ for $x^* = 1$. In both cases $x^* = (\omega^1_{22} - \omega^1_{21})/\delta^1$.

   In any case, the interval $O(I, \Omega^1)$ has got $x^* = (\omega^1_{22} - \omega^1_{21})/\delta^1$ as an extreme point. In a similar way, we can obtain $(\omega^2_{22}-\omega^2_{21})/\delta^2$ as extreme of $O(I, \Omega^2)$.

   Now, from Proposition 4 and taken into account that Ω is a mixed-solved game, we get $X_S = [x^1, x^2] = [(\omega^1_{22} - \omega^1_{21})/\delta^1, (\omega^2_{22} - \omega^2_{21})/\delta^2]$.

   The second part of the statement is reached in a similar way.

Proof of Theorem 16: From Lemma 13 we know that $Y_S = [y^1, y^2] = [(\omega^1_{22} - \omega^1_{12})/\delta^1, (\omega^2_{22} - \omega^2_{12})/\delta^2]$. Notice that $y^1 \neq y^2$, otherwise we are in the degenerated case (see Corollary 8). Then, without loss of generality, $y^1 < y^2$.

As the game is negative mixed-solved, it holds $\delta^1 \cdot \delta^2 < 0$. Let be $\delta^1 > 0$ and $\delta^2 < 0$ (the case $\delta^1 < 0$ and $\delta^2 > 0$ is similar).

Since $y^1 < y^2$, then $0 \leq y^1 < 1$ which means $0 \leq (\omega^1_{22} - \omega^1_{12})/\delta^1 < 1$. Therefore, since $\delta^1 > 0$, $\omega^1_{11} > \omega^1_{21}$ and $\omega^1_{22} \geq \omega^1_{12}$. And moreover we claim $\omega^1_{11} > \omega^1_{12}$. Suppose not (i.e. $\omega^1_{22} \geq \omega^1_{12} \geq \omega^1_{11} > \omega^1_{21}$), in this case $O(I, \Omega^1) = \{1\}$ and either $O(II, \Omega^1) = \{1\}$ if $\omega^1_{12} > \omega^1_{11}$ or $O(II, \Omega^1) = [0, 1]$ if $\omega^1_{22} = \omega^1_{12} = \omega^1_{11}$ or $O(II, \Omega^1) = [y^1, 1]$ if $\omega^1_{22} > \omega^1_{12} = \omega^1_{11}$. In the first case the game is pure-solved; in the other two cases, since $O(II, \Omega^1) = [0, 1]$, or $O(II, \Omega^1) = [y^1, 1]$ and $y^2 \in [y^1, 1]$ by hypothesis, the game is determinated, and in both cases we get a contradiction.

Summarizing, for $\delta^1 > 0$ we have got

$$\omega^1_{11} > \omega^1_{21},\ \omega^1_{22} \geq \omega^1_{12},\ \omega^1_{11} > \omega^1_{12}. \tag{2}$$

By a similar reasoning, as $\delta^2 < 0$,

$$\omega_{12}^2 > \omega_{22}^2,\ \omega_{21}^2 \geq \omega_{11}^2,\ \omega_{12}^2 > \omega_{11}^2. \tag{3}$$

Given $y^* \in Y$, we know that $P_I(y^*) = \{\mathbf{x}\Omega\mathbf{y}^{*t} \mid \mathbf{x} \in X\} = conv\{z_1, z_2\}$, being:

$$\begin{aligned} z_1 &= (1,0)\Omega\mathbf{y}^{*t} = (\omega_{12}^1 + y^*(\omega_{11}^1 - \omega_{12}^1), \omega_{12}^2 + y^*(\omega_{11}^2 - \omega_{12}^2)), \\ z_2 &= (0,1)\Omega\mathbf{y}^{*t} = (\omega_{22}^1 + y^*(\omega_{21}^1 - \omega_{22}^1), \omega_{22}^2 + y^*(\omega_{21}^2 - \omega_{22}^2)). \end{aligned} \tag{4}$$

For $y^* = y^1$ it holds $\omega_{12}^1 + y^1(\omega_{11}^1 - \omega_{12}^1) = \omega_{22}^1 + y^1(\omega_{21}^1 - \omega_{22}^1)$. Therefore player I can play any strategy $x \in [0,1]$ to obtain the highest possible payoff in the first objective.

If $y^* \in (y^1, 1]$ it holds $\omega_{12}^1 + y^*(\omega_{11}^1 - \omega_{12}^1) > \omega_{22}^1 + y^*(\omega_{21}^1 - \omega_{22}^1)$ since for $y^* = 1$ we have $\omega_{11}^1 > \omega_{21}^1$. Therefore player I obtains the highest payoff in the first objective by playing $x = 1$ if $y^* \in (y^1, 1]$.

Similarly, the highest payoff in the second objective for player I is $x = 1$ if $y^* \in [0, y^2)$ and $x \in [0,1]$ if $y^* = y^2$.

Combining the previous reasoning we have $BRS_I(y^*) = \{1\}$ for $y^* \in [y^1, y^2]$.

Therefore, if $SE(\Omega) \neq \emptyset$, then $x = 1 \in X_S = [x^1, x^2]$. If $x^1 = (\omega_{22}^1 - \omega_{21}^1)/\delta^1 = 1$ then $\omega_{11}^1 = \omega_{12}^1$ getting a contradiction with (2). If $x^2 = 1$ we have a contradiction with (3). As a consequence $SE(\Omega) = \emptyset$.

Proof of Theorem 17:

1. If $\Omega \in MOG_0^*(2 \times 2, r)$ is a pure-solved game, then $SE(\Omega) \neq \emptyset$. Without loss of generality let $x^1 = 1$ and $y^1 = 1$ be the unique equilibrium in the scalar game Ω^1. As a consequence we have: $\omega_{12}^1 \geq \omega_{11}^1 \geq \omega_{21}^1$.

We claim that the above inequalities are strict: $\omega_{12}^1 > \omega_{11}^1 > \omega_{21}^1$.

Notice that $\omega_{12}^1 = \omega_{11}^1 = \omega_{21}^1$ implies either $O(I, \Omega^1) = [0,1]$ or $O(II, \Omega^1) = [0,1]$ or both, which contradicts that Ω is a pure solved game.

Assume $\omega_{12}^1 = \omega_{11}^1 > \omega_{21}^1$. As by hypothesis $y^1 = 1$ is the unique solution for player II in Ω^1 we know that $y^1 = 1$ has to be the unique solution of the program $Min_{y\in[0,1]} max\{(0,1)\Omega^1(y, 1-y)^t, (1,0)\Omega^1(y, 1-y)^t\}$, that is to say $Min_{y\in[0,1]} max\{\omega_{22}^1 + y(\omega_{21}^1 - \omega_{22}^1), \omega_{12}^1\}$.

If $\omega_{22}^1 \leq \omega_{12}^1$, then $\omega_{22}^1 + y(\omega_{21}^1 - \omega_{22}^1) \leq \omega_{12}^1$ for all $y \in [0,1]$, since for $y = 0$ $\omega_{22}^1 \leq \omega_{12}^1$ and for $y = 1$ $\omega_{21}^1 < \omega_{12}^1$. Therefore the above program becomes $Min_{y\in[0,1]}\{\omega_{12}^1\}$, and [0,1] is the set of solutions. This implies a contradiction with the fact that $y^1 = 1$ is the unique solution.

If $\omega_{22}^1 > \omega_{12}^1$, the above two linear functions involved in the minimization program intersect, since for $y = 0$ $\omega_{22}^1 > \omega_{12}^1$ and for $y = 1$ $\omega_{21}^1 < \omega_{12}^1$. The intersection point $y^* = (\omega_{12}^1 - \omega_{22}^1)/(\omega_{21}^1 - \omega_{22}^1)$ is in (0,1). As a consequence, $O(II, \Omega^1) = [y^*, 1]$ giving a contradiction with the fact of y^* is the unique solution.

The remaining case can be analyze in the same way.

Once proved $\omega_{12}^1 > \omega_{11}^1 > \omega_{21}^1$, notice that $x^1 = 1$ is the best reply strategy to $y^* = 1$ for player I since $\omega_{11}^1 > \omega_{21}^1$. And since $\omega_{11}^1 < \omega_{12}^1$, $y^1 = 1$ is the best reply strategy to $x^* = 1$.

Therefore the unique pure equilibrium of Ω^1, $(x^1, y^1) = (1, 1)$, is a Pareto equilibrium of the game Ω.

Finally, by Proposition 2, $x^1 = 1 \in X_S$ and $y^1 = 1 \in Y_S$. As a consequence $(x^1, y^1) = (1, 1) \in SE(\Omega) \neq \emptyset$.

2. If $\Omega \in MOG_0^*(2 \times 2, r)$ is a determinated game, then $SE(\Omega) \neq \emptyset$. Assume first $\cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k) \neq \emptyset$ and $\cap_{k\in\{1,\ldots,r\}} O(II, \Omega^k) \neq \emptyset$. By Proposition 2 it holds that $X_S = \cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k)$ and $Y_S = \cap_{k\in\{1,\ldots,r\}} O(II, \Omega^k)$. Let it be $x^* \in X_S$ and $y^* \in Y_S$. Taken any $\alpha, \beta \in \overset{o}{\Delta}_r$, by Theorem 1, we obtain $(x^*, y^*) \in PE(\Omega)$. Therefore $(x^*, y^*) \in SE(\Omega)$.

Assume now $\cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k) \neq \emptyset$ and $\cap_{k\in\{1,\ldots,r\}} O(II, \Omega^k) = \emptyset$ (similarly for the other case).

By Corollary 8, the set $\cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k)$ is not a singleton. Then each $O(I, \Omega^k)$ has got at least two elements.

The sets $O(I, \Omega^k)$ are the solutions of the program

$$\max_{x\in[0,1]} \min\{\omega_{21}^k + x(\omega_{11}^k - \omega_{21}^k), \omega_{22}^k + x(\omega_{12}^k - \omega_{22}^k)\}. \tag{5}$$

If $O(I, \Omega^k)$ is not a singleton, then at least one of the two linear functions involves in (5), $l_1(x) = \omega_{21}^k + x(\omega_{11}^k - \omega_{21}^k)$ and $l_2(x) = \omega_{22}^k + x(\omega_{12}^k - \omega_{22}^k)$, has to be flat, i.e. $\omega_{11}^k = \omega_{21}^k$ or $\omega_{12}^k = \omega_{22}^k$, for any $k = 1, 2, \ldots, r$.

If $\omega_{11}^k = \omega_{21}^k$ (similarly for the other case), then one of the next three cases may accur:

$$\begin{array}{lc} \textbf{(A)} & \omega_{22}^k > \omega_{21}^k = \omega_{11}^k \\ \textbf{(B)} & \omega_{12}^k > \omega_{21}^k = \omega_{11}^k \\ \textbf{(C)} & \omega_{12}^k = \omega_{22}^k \leq \omega_{21}^k = \omega_{11}^k. \end{array}$$

To see it, notice that if $\omega_{11}^k = \omega_{21}^k$ and the other two entries of the matrix Ω^k are both above the value $\omega_{11}^k = \omega_{21}^k$, then both entries have to coincide. In other words, if $\omega_{11}^k = \omega_{21}^k \geq \omega_{12}^k, \omega_{22}^k$, then $l_1(x) \geq l_2(x)$ for any $x \in [0, 1]$, and the program (5) has a unique solution if $\omega_{12}^k \neq \omega_{22}^k$, which involves a contradiction with the fact that $O(I, \Omega^k)$ is not a singleton.

For the other case, $\omega_{12}^k = \omega_{22}^k$, by a a similar argument we obtain three possible cases:

$$\textbf{(D)} \qquad \omega_{21}^k > \omega_{22}^k = \omega_{12}^k$$

$$\textbf{(E)} \qquad \omega_{11}^k > \omega_{22}^k = \omega_{12}^k$$

$$\textbf{(F)} \quad \omega_{11}^k = \omega_{21}^k \leq \omega_{22}^k = \omega_{12}^k.$$

Since the set $O(II, \Omega^k)$ is the solution of the program

$$\min_{y\in[0,1]} \max\{\omega_{12}^k + x(\omega_{11}^k - \omega_{12}^k), \omega_{22}^k + x(\omega_{21}^k - \omega_{22}^k)\}.$$

It is easy to prove in cases (A) or (B), $O(II, \Omega^k) = \{1\}$, in cases (D) or (E), $O(II, \Omega^k) = \{0\}$ and in cases (C) or (F) $O(II, \Omega^k) = [0, 1]$.

Since $\cap_{k\in\{1,\ldots,r\}} O(II, \Omega^k) = \emptyset$ we deduce the existence of at least $k \neq h \in \{1, \ldots, r\}$ such that Ω^k satisfies (A) or (B) and Ω^h satisfies (D) or (E). So, by Proposition 2, we have $Y_S = [0, 1]$.

Now, we are going to study the following four cases:

1. If $X_S = [0, 1]$, since $Y_S = [0, 1]$, by Theorem 1, $SE(\Omega) \neq \emptyset$.

2. If $X_S = [0, b]$, being $0 < b < 1$.

   By Proposition 2 item 1 we have that for every scalar game Ω^k, $k \in \{1, \ldots, r\}$, $O(I, \Omega^k) = [0, b^k]$, $b \leq b^k \leq 1$. Moreover, as $X_S = \cap_{k\in\{1,\ldots,r\}} O(I, \Omega^k) = [0, b]$ there exists a scalar game Ω^k, $k \in \{1, \ldots, r\}$, satisfying $O(I, \Omega^k) = [0, b]$. As a consequence Ω^k satisfies conditions (A) or (D). Let us asume that Ω^k satisfies (A) (the case Ω^k satisfying (D) can be argued similarly).

   As $O(I, \Omega^k) = [0, b]$ being $0 < b < 1$ then condition (A) for Ω^k will be satisfied in the following strong form,

   $$\omega_{22}^k > \omega_{21}^k = \omega_{11}^k > \omega_{12}^k. \qquad (6)$$

   The value b can be achieved by solving (5). Since $\omega_{22}^k > \omega_{21}^k = \omega_{11}^k > \omega_{12}^k$ then the parameter b will be the strategy where $\omega_{21}^k + x(\omega_{11}^k - \omega_{21}^k) = \omega_{22}^k + x(\omega_{12}^k - \omega_{22}^k)$, or

   $$b = (\omega_{22}^k - \omega_{21}^k)/\delta^k. \qquad (7)$$

   The payoff for player I in the game Ω^k is:

   $$P_I^k(y*) = conv\{(1, 0)\Omega^k \begin{pmatrix} y^* \\ 1 - y^* \end{pmatrix}, (0, 1)\Omega^k \begin{pmatrix} y^* \\ 1 - y^* \end{pmatrix}\}$$

   $$= conv\{\omega_{12}^k + y^*(\omega_{11}^k - \omega_{12}^k), \omega_{22}^k + y^*(\omega_{21}^k - \omega_{22}^k)\}.$$

Player I obtains his highest payoff in objective k playing $x = 0$ if $y^* \in [0, 1)$ and any $x \in [0, 1]$ when $y^* = 1$. Therefore for any $y^* \in [0, 1)$

$$x^* = 0 \in BRS_I(y^*). \tag{8}$$

On the other hand, as $\cap_{k \in \{1,\ldots,r\}} O(II, \Omega^k) = \emptyset$ and Ω^k satisfies (6) which implies $\{1\} = O(II, \Omega^k)$, it must exist a second scalar game Ω^h, $h \neq k \in \{1, \ldots, r\}$, satisfying condition (D) or (E).

If Ω^h satisfies (D), $\omega_{21}^h > \omega_{22}^h = \omega_{12}^h$, then the payoff for player II in the objective h when player I plays $x^* = 0$ is

$$P_{II}^h(x^* = 0) = conv\{(0, 1)\Omega^h \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (0, 1)\Omega^h \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = conv\{\omega_{21}^h, \omega_{22}^h\}.$$

Since $\omega_{21}^h > \omega_{22}^h$ and player II is a minimizer we obtain

$$y^* = 0 \in BRS_{II}(x* = 0). \tag{9}$$

By (8) and (9) we obtain $(0, 0) \in PE(\Omega)$ and $(0, 0) \in X_S \times Y_S = [0, b] \times [0, 1]$, or equivalently $(0, 0) \in SE(\Omega)$.

If Ω^h satisfies (E), $\omega_{11}^h > \omega_{22}^h = \omega_{12}^h$, then $\omega_{21}^h \geq \omega_{22}^h = \omega_{12}^h$ since if $\omega_{22}^h = \omega_{12}^h > \omega_{21}^h$ then we should have $\omega_{11}^h > \omega_{22}^h = \omega_{12}^h > \omega_{21}^h$ which implies $O(I, \Omega^h) = [d, 1]$ with $0 < d < 1$ getting a contradiction with the fact that $O(I, \Omega^k) = [0, b^k]$, $0 < b \leq b^k \leq 1$, for any $k = 1, \ldots, r$. Therefore if $\omega_{21}^h > \omega_{22}^h = \omega_{12}^h$ we are in case (D) for the game Ω^h and the proof is finished. Only rest to analyze the case where Ω^h satisfies

$$\omega_{11}^h > \omega_{22}^h = \omega_{12}^h = \omega_{21}^h. \tag{10}$$

The payoff for player I in the objective Ω^h is given by:

$$\begin{aligned} P_I^h(y^*) &= conv\{(1, 0)\Omega^h \begin{pmatrix} y^* \\ 1 - y^* \end{pmatrix}, (0, 1)\Omega^h \begin{pmatrix} y^* \\ 1 - y^* \end{pmatrix}\} \\ &= conv\{\omega_{12}^h + y^*(\omega_{11}^h - \omega_{12}^h), \omega_{22}^h + y^*(\omega_{21}^h - \omega_{22}^h)\}. \end{aligned}$$

Player I obtains his highest payoff in objective h playing $x = 1$ when $y^* \in (0, 1]$ and any $x \in [0, 1]$ when $y^* = 0$.

By (8) and the above reasoning we obtain $[0, 1] = BRS_I(y^*)$ when $y^* \in (0, 1)$.

For player II we have:

$$P^k_{II}(x^*) = conv\{(x^*, 1-x^*)\Omega^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (x^*, 1-x^*)\Omega^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$$= conv\{\omega^k_{21} + x^*(\omega^k_{11} - \omega^k_{21}), \omega^k_{22} + x^*(\omega^k_{12} - \omega^k_{22})\}$$

and

$$P^h_{II}(x^*) = conv\{(x^*, 1-x^*)\Omega^h \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (x^*, 1-x^*)\Omega^h \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$$= conv\{\omega^h_{21} + x^*(\omega^h_{11} - \omega^h_{21}), \omega^h_{22} + x^*(\omega^h_{12} - \omega^h_{22})\}.$$

In the objective k, for $x^* \in (0, b)$ player II (the minimizer) obtains the lowest payoff playing $y = 1$ since for $x^* = 0$, $\omega^k_{22} > \omega^k_{21}$ and for $x^* = b$, it holds $\omega^k_{21} + x^*(\omega^k_{11} - \omega^k_{21}) = \omega^k_{22} + x^*(\omega^k_{12} - \omega^k_{22})$ (see (6) and (7)).

In the objective h, for $x^* \in (0, 1]$ player II obtains the lowest payoff playing $y = 0$ since for $x^* = 0$, $\omega^h_{22} = \omega^h_{21}$ and for $x^* = 1$ it holds $\omega^h_{11} > \omega^h_{12}$ (see (10)).

We have obtained $[0, 1] = BRS_{II}(x^*)$ for any $x^* \in (0, b)$.

Finally, notice that any $(x^*, y^*) \in (0, b) \times (0, 1) \subseteq X_S \times Y_S$ will satisfy $(x^*, y^*) \in SE(\Omega)$. As a consequence $SE(\Omega) \neq \emptyset$.

3. If $X_S = [a, 1]$, being $0 < a < 1$. $SE(\Omega) \neq \emptyset$ similarly to part 2.

4. If $X_S = [a, b]$, being $0 < a < b < 1$.

   By Proposition 4 and from $\cap_{k\in\{1,\dots,r\}} O(I, \Omega^k) \neq \emptyset$ there exist scalar games Ω^k and Ω^h, $k \neq h \in \{1, \dots, r\}$ satisfying $O(I, \Omega^k) = [0, b]$ and $O(I, \Omega^h) = [a, 1]$. As a consequence the game Ω^k has to satisfy one of the following conditions:

   (a) $\omega^k_{22} > \omega^k_{21} = \omega^k_{11} > \omega^k_{12}$ **(b)** $\omega^k_{21} > \omega^k_{22} = \omega^k_{12} > \omega^k_{11}$

   and the game Ω^h one of the following,

   (c) $\omega^h_{11} > \omega^h_{22} = \omega^h_{12} > \omega^h_{21}$ **(d)** $\omega^h_{12} > \omega^h_{21} = \omega^h_{11} > \omega^h_{22}$.

   We can obtain the value b from (5), being b the strategy x such that $\omega^k_{21} + x(\omega^k_{11} - \omega^k_{21}) = \omega^k_{22} + x(\omega^k_{12} - \omega^k_{22})$. It is to say $b = (\omega^k_{22} - \omega^k_{21})/\delta^k$. Similarly $a = (\omega^h_{22} - \omega^h_{21})/\delta^h$.

   Suposse that Ω^k satisfies (a)(the case Ω^k satisfying (b) can be argued similarly). We have to analyze two cases:

When Ω^k satisfies (a) and Ω^h satisfies (c).

In part 2 above we have seen that for the game Ω^k satisfying (a) player I obtains his highest payoff in the objective k playing $x = 0$ when $y^* \in [0, 1)$ and any $x \in [0, 1]$ when $y^* = 1$.

Similarly, when Ω^h satisfies (c) I obtains his highest payoff in the objective h playing any $x \in [0, 1]$ when $y^* = 0$ since

$$P_I^h(y^* = 0) = conv\{(1,0)\Omega^h \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (0,1)\Omega^h \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = conv(\omega_{12}^h, \omega_{22}^h)$$

and it holds $\omega_{12}^h = \omega_{22}^h$ and $x = 1$ when $y^* \in [0, 1)$ since

$$P_I^h(y^* = 1) = conv\{(1,0)\Omega^h \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (0,1)\Omega^h \begin{pmatrix} 1 \\ 0 \end{pmatrix}\} = conv(\omega_{11}^h, \omega_{21}^h)$$

and it holds $\omega_{11}^h > \omega_{21}^h$.

We have deduced $BRS_I^{kh}(y^*) = [0, 1]$ for any $y^* \in (0, 1)$ which implies for any $y^* \in (0, 1)$

$$BRS_I(y^*) = [0, 1]. \tag{11}$$

The payoff for player II in the scalar game Ω^k is

$$\begin{aligned} P_{II}^k(x^*) &= conv\{(x^*, 1-x^*)\Omega^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (x^*, 1-x^*)\Omega^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \\ &= conv\{\omega_{21}^k + y^*(\omega_{11}^k - \omega_{21}^k), \omega_{22}^k + y^*(\omega_{12}^k - \omega_{22}^k)\} \end{aligned}$$

being for $x^* = b$ the equality

$$\omega_{21}^k + x^*(\omega_{11}^k - \omega_{21}^k) = \omega_{22}^k + x^*(\omega_{12}^k - \omega_{22}^k).$$

Then, for $x^* = b$, player II obtains his lowest payoff in objective h playing any strategy $y \in [0, 1]$. The lowest payoff when player I plays $x^* \in [0, b)$ is obtained when player II plays $y = 1$ since $\omega_{21}^k < \omega_{22}^k$ and the lowest payoff when player I plays $x^* \in (b, 1]$ is obtaining when player II plays $y = 0$ since $\omega_{12}^k < \omega_{11}^k$.

In the objective h, for $x^* = a$ player II obtains his lowest payoff in objective k playing any strategy $y \in [0, 1]$. The lowest payoff when player I plays $x^* \in [0, a)$ is obtained when player II plays $y = 1$ since $\omega_{21}^h < \omega_{22}^h$ and the lowest payoff when player I plays $x^* \in (a, 1]$ is obtained when player II plays $y = 0$ since $\omega_{12}^h < \omega_{11}^h$.

We deduce $BRS_{II}^{kh}(x^*) = BRS_{II}(x^*) = [0,1]$ for $x^* \in (a,b)$. As a consequence (see also (11) $(a,b) \times (0,1) \subseteq SE(\Omega)$ and then $SE(\Omega) \neq \emptyset$.

When Ω^k satisfies (a) and Ω^h satisfies (d).

When Ω^h satisfies (d) player I (the maximizer) obtains his highest payoff in the objective h playing $x = 1$ when $y^* \in [0,1)$ and any $x \in [0,1]$ when $y^* = 1$, since

$$P_I^h(y^*) = conv\{(1,0)\Omega^h \begin{pmatrix} y^* \\ 1-y^* \end{pmatrix}, (0,1)\Omega^h \begin{pmatrix} y^* \\ 1-y^* \end{pmatrix}\}$$

$$= conv\{\omega_{12}^h + y^*(\omega_{11}^h - \omega_{12}^h), \omega_{22}^h + y^*(\omega_{21}^h - \omega_{22}^h)\}.$$

Therefore $BRS_I^{kh}(y^*) = BRS_I(y^*) = [0,1]$ for $y^* \in [0,1)$.

To calculate the best replay strategies set for player II we need to consider that if the scalar games Ω^k and Ω^h with $k \neq h \in \{1,\ldots,r\}$, satisfy conditions (a) and (d) then $O(II,\Omega^k) \cap O(II,\Omega^h) = \{1\}$. Since that, it must exist a third game Ω^j, $j \in \{1,\ldots,r\}$, $j \neq k and\, j \neq h$, such that Ω^j satisfies (D) or (E) and in this way we get $O(II,\Omega^j) = \{0\}$.

Let us assume first that Ω^j satisfies (D). Then $O(I,\Omega^j) = [0,b^k]$ with $b \leq b^k \leq 1$ as we have seen in part 2 above.

We are going to use the scalar games Ω^k and Ω^j to deduce $BRS_{II}(x^*)$.

The payoff for player II in the scalar game Ω^j is

$$P_{II}^j(x^*) = conv\{(x^*, 1-x^*)\Omega^j \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (x^*, 1-x^*)\Omega^j \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$$= conv\{\omega_{21}^j + x^*(\omega_{11}^j - \omega_{21}^j), \omega_{22}^j + x^*(\omega_{12}^j - \omega_{22}^j)\}.$$

Doing as before, we see that the lowest payoff in the objective j for player II when player I plays $x^* \in [0,b^k)$ it is obtained when player II plays $y = 0$, since $\omega_{21}^j > \omega_{22}^j$.

Let us assume now that Ω^j satisfies (E). Then $O(I,\Omega^j) = [a^k,1]$ with $0 \leq a^k \leq a$. And doing as before we reach that the lowest payoff in the objective j for player II when player I plays $x^* \in (a^k,1]$ it is obtained when player II plays $y = 0$, since $\omega_{11}^j > \omega_{12}^j$.

Considering the objectives k and j, we reach in both cases $BRS_{II}^{kj}(x^*) = BRS_{II}(x^*) = [0,1]$ for $x^* \in [a,b)$. As a consequence we get $(a,b) \times (0,1) \subseteq SE(\Omega)$ which implies $SE(\Omega) \neq \emptyset$.

3. If $\Omega \in MOG_0^*(2 \times 2, r)$ is a positive game, then $SE(\Omega) \neq \emptyset$. As $\Omega \in MOG_0^*(2 \times 2, r)$ is a positive mixed-solved game there exists a mixed-solved subgame $\Omega^{kh} = (\Omega^k, \Omega^h) \in MOG_0^*(2 \times 2, 2)$, with $k \neq h \in \{1, \ldots, r\}$, satisfying $\delta^k \cdot \delta^h > 0$. Let us assume without loss of generality k=1 and h=2 in order to use the same arguments as in Theorem 16.

From Lemma 13 we know that $Y_S = [y^1, y^2] = [(\omega_{22}^1 - \omega_{12}^1)/\delta^1, (\omega_{22}^2 - \omega_{12}^2)/\delta^2]$. Notice that $y^1 = y^2$ violates the non-degeneration hypothesis (see Corollary 8), and so we can assume $y^1 < y^2$.

As $\delta^1 \cdot \delta^2 > 0$ let us analyze the case $\delta^1 > 0$ and $\delta^2 > 0$ (the case $\delta^1 < 0$ and $\delta^2 < 0$ is similar).

Following the reasoning in the proof of Theorem 16, we know $\omega_{11}^1 > \omega_{21}^1$, $\omega_{22}^1 \geq \omega_{12}^1$ and $\omega_{11}^1 > \omega_{12}^1$.

Since $y^1 < y^2$ then $0 < y^2 \leq 1$ which means $0 < (\omega_{22}^2 - \omega_{12}^2)/\delta^2 \leq 1$. Therefore $\omega_{11}^2 \geq \omega_{21}^2$ and $\omega_{22}^2 > \omega_{12}^2$ since $\delta^2 > 0$. We claim that $\omega_{11}^2 > \omega_{12}^2$. If not (i.e. $\omega_{22}^2 > \omega_{12}^2 \geq \omega_{11}^2 \geq \omega_{21}^2$) $O(II, \Omega^2) = \{1\}$ and either $O(I, \Omega^2) = \{1\}$ if $\omega_{11}^2 > \omega_{21}^2$ or $O(I, \Omega^2) = [0, 1]$ if $\omega_{11}^2 = \omega_{21}^2$. In the first case Ω^{12} is a pure-solved game or determinated in the second, and both cases are out of our analisys. Summarizing: $\omega_{11}^2 \geq \omega_{21}^2$, $\omega_{22}^2 > \omega_{12}^2$ and $\omega_{11}^2 > \omega_{12}^2$.

From Theorem 16 we know that player I obtains the highest payoff in the first objective by playing $x = 1$ if $y^* \in (y^1, 1]$.

Reasoning as in the proof of Theorem 16, see (4), for $y^* = y^2$ it holds $\omega_{12}^2 + y^2(\omega_{11}^2 - \omega_{12}^2) = \omega_{22}^2 + y^2(\omega_{21}^2 - \omega_{22}^2)$. Therefore player I can play any strategy $x \in [0, 1]$ to obtain the highest payoff in the second objective.

If $y^* \in [0, y^2)$ it holds $\omega_{12}^2 + y^*(\omega_{11}^2 - \omega_{12}^2) < \omega_{22}^2 + y^*(\omega_{21}^2 - \omega_{22}^2)$ since for $y^* = 0$ we have $\omega_{12}^2 < \omega_{22}^2$. Therefore player I obtains the highest payoff in the second objective by playing $x = 0$.

From there we deduce that $BRS_I^{12}(y^*) = [0, 1]$ for $y^* \in (y^1, y^2)$. Similarly for player II we deduce that $BRS_{II}^{12}(x^*) = [0, 1]$ for $x^* \in (x^1, x^2)$, where $X_S = [x^1, x^2]$.

Assume now that $x^* \in (x^1, x^2)$, $y^* \in (y^1, y^2)$ and $x^* \notin BRS_I(y^*)$. Then there exists $x^+ \in X$ and $x^+ \neq x^*$ such that $x^+ \Omega y^{*t} > x^* \Omega y^{*t}$. As $x^* \in BRS_I^{12}(y^*)$ this implies that $x^+ \Omega^1 y^{*t} = x^* \Omega^1 y^{*t}$ and $x^+ \Omega^2 y^{*t} = x^* \Omega^2 y^{*t}$. Working with these last expressions, since $x^+ \neq x^*$ we reach $\delta^1 y^* + \omega_{12}^1 - \omega_{22}^1 = 0$ and $\delta^2 y^* + \omega_{12}^2 - \omega_{22}^2 = 0$ from where, by Lemma 13, $y^* = y^1 = y^2$ getting a contradiction with the fact that Ω^{12} is not a determinated game.

A similar argument produces $y^* \in BRS_{II}(x^*)$. From $x^* \in BRS_I(y^*)$ and $y^* \in BRS_{II}(x^*)$ we reach $(x^1, x^2) \times (y^1, y^2) \subseteq PE(\Omega)$ and then $SE(\Omega) \neq \emptyset$.