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Sequential decisions in allocation problems

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Abstract

In the context of cooperative TU-games, and given an order of players, we consider the problem of distributing the worth of the grand coalition as a sequential decision problem. In each step of the process, upper and lower bounds for the payoff of the players are required related to successive reduced games. Sequentially compatible payoffs are defined as those allocation vectors that meet these recursive bounds. The core of the game is reinterpreted as a set of sequentially compatible payoffs when the Davis-Maschler reduced game is considered (Th.1). Independently of the reduction, the core turns out to be the intersection of the family of the sets of sequentially compatible payoffs corresponding to the different possible orderings (Th.2), so it is in some sense order-independent. Finally, we analyze advantageous properties for the first player.

Resum

Dins el context de jocs cooperatius i problemes de repartiment de guanys, l'article proposa realitzar aquest repartiment de forma seqüencial, on el pagament a cada jugador s'assigna un darrera l'altre i seguint un ordre. El procés consisteix a fixar successivament, a cada pas del procés, una fita inferior i superior que delimiten el possible pagament al jugador analitzat, i a "reduir" el joc un cop assignat el pagament al jugador. Els pagaments seqüencialment compatibles són aquells que compleixen aquestes fites definides de forma recurrent. El nucli del joc es reinterpreta aleshores com el conjunt de pagament seqüencials compatibles amb la reducció del joc à la Davis-Maschler. Independentment del tipus de reducció utilitzat, el nucli és exactament la intersecció de tota la família de conjunts de pagaments seqüencialment compatibles relatius als diferents ordres possibles (Th.2). D'aquesta manera diem que el nucli és independent de l'ordre. Finalment, a l'article s'analitza les avantatges del jugador que figura en primer lloc en l'ordre fixat.

Key words: TU-game, Sequential allocation, reduced game, core.

JEL Classification: C71

1 Introduction

One of the main goals of cooperative game theory is to describe fair methods for allocating the joint profit arising from cooperation between agents. A preliminary normative task of the theory is to describe the possible allocation vectors according to criteria related to equity or justice: that is the core, the bargaining sets or the stable sets. All these solutions propose distributions that may be accepted simultaneously by all players. However, sometimes decisions regarding payoff allocations are not taken as a one-shot decision but as a sequence of decisions. In this study we introduce the idea of *sequential payoffs* in set-solutions defined for transferable utility cooperative games (TU-games).

The concept of a sequential payoff scheme has already been used in the analysis of point-solutions for TU-games. A first analysis was given implicitly by Shapley (1972), who introduced the value for cooperative games, which is the average of all the marginal worth vectors. These vectors are usually interpreted as sequential payoffs in which each player receives his marginal contribution to the set of predecessors with respect to a fixed order given on the player set. Therefore, as in the Shapley value, the sequential analysis can help to propose and analyze solutions for the cases where there is no apparent reason to discriminate players. At this point it is interesting to mention the reduced marginal worth vectors introduced by Núñez and Rafels (1998), where the marginal contributions of the players are evaluated with respect to successive reduced games.

In some models, cooperation must take into account a certain order of players. One direction of the analysis comes when the cooperative phenomenon is performed sequentially in time. This approach can be viewed as a source of sequentiality and has been considered as an argument in the discussion of the concepts of recursive core (Becker and Chakrabarti, 1995), sequential core (Gale, 1978, 1982) and strong sequential core (Predtetchinski et al., 2002).

In our approach the allocation decision problem will be performed step by step, with players taking part in a natural sequential way. This point of view is closely related to the recent works of Moulin (2000) and Potters and Sudhölter (1999). Moulin (2000) studies in depth priority rules and other asymmetric methods for rationing problems. His work is mainly devoted to the study and axiomatization of point-solutions for special problems which have their translation into the class of cooperative TU-games. From his work it is clear that the term sequentiality gives far from a unique outcome, so sequential set-solutions should then be introduced. Potters and Sudhölter focus on the airport cost games and the axiomatization of point-solutions. In their analysis, sequentiality appears implicitly as the criterion for determining the payoff to the first player and the iterative application of consistency determines the whole payoff vector.

The central idea of sequentiality we present in this paper relies on three main aspects. The first one is that the assignment process is made following an order and so, whenever we analyze the payoff of some player, we know the payoff of his predecessors and, more importantly, we do not need to know at that moment the payoff of the players who follow him.

Second, at each step of the process the payoff to the current player is chosen between some upper and lower bounds. The upper bound will be a marginal contribution of the player and the lower bound will be his individual worth of a suitable reduced game.

Third, each time a payoff to a player is accepted (i.e. if it passes the reasonability test), and before the next player in the list is analyzed, the worth of coalitions "still in the game" are reevaluated according to the already fixed payoffs; technically we say that we reduce the game. At this point we note that for the reduction operator we will adopt the α-max generic reduction inspired in the works of Thomson (1990, 1996) who analyzes (weak) consistency properties of solutions. These three aspects will comprise the concept of the *set of sequentially compatible payoffs* with respect to an order.

This paper is organized as follows. In Section 2, we review the core from a sequential point of view and reinterpret it as a set of sequentially compatible payoffs when the Davis and Maschler reduction method is adopted. In Section 3, we define the main concept of the paper, the *set of sequentially compatible payoffs* with respect to an order on the set of players, which turns out to be a compact polyhedron in between the core and the imputation set. Hence, for some selection of the reduced game, the *set of sequentially compatible payoffs* may be an alternative whenever the core of the game is empty. In section 4 we present our main results. The first result (Theorem 1) states that under the Davis and Maschler reduction, all the sets of sequentially compatible payoffs coincide with the core regardless of the prescribed order on the player set we fix. This result by itself can be viewed as a new description for the classical core concept. The second result (Theorem 2) states that for any arbitrary α-max reduction the intersection of all the sequentially compatible payoff sets depending on the orders on the players is always the core of the original game. This result has an interesting consequence since it states that only the core is order-independent (Corollary 2). The third result (Theorem 3) aims to solve a natural but dual question: which allocations can be supported by a sequential argument? Curiously, we will see that any imputation can be supported in this way if the game is totally balanced. In Section 5 we will analyze advantage properties for players depending on their positions in the order, and finally, in Section 6 we present some concluding remarks.

Before starting the analysis, let us establish our notation. By the set of natural numbers $\mathbb{N}$ we will denote the universe of potential players. By $N \subseteq \mathbb{N}$ we will denote a finite set of players, in general $N = \{1, 2, \ldots, n\}$. For any coalition $S \subseteq N$, $|S|$ represents its cardinality and 2^N the power set of N. The symbol $S \subset T$ is used for the strict inclusion, i.e. $S \subseteq T$ and $S \neq T$.

A cooperative game with transferable utility is a pair (N, v) where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\varnothing) = 0$, is the characteristic function of the game. If no confusion arises we will denote a cooperative TU-game by its characteristic function v and G^N will be the set of all cooperative TU-games on N. Given $\varnothing \neq S \subseteq N$ and $v \in G^N$, v_S will represent the subgame which results of the characteristic function to the subsets of S.

Let $\mathbb{R}^S$, $\varnothing \neq S \subseteq N$ stand for the real-valued linear space of vectors, $x = (x_i)_{i \in S}$. Given $x \in \mathbb{R}^N$ and $\varnothing \neq S \subseteq N$, $x(S) := \sum_{i \in S} x_i$ and $x_{|S} := (x_i)_{i \in S}$. We assume $x(\varnothing) = 0$.

Let $I^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ be the set of efficient vectors, also called preimputations, and $I(N, v) := \{x \in I^*(N, v) \mid x_i \geq v(\{i\}) \text{ for } i = 1, \ldots, n\}$ be the set of imputations of the game. We will denote by $C(N, v)$ the core of the game $v \in G^N$ defined by $C(N, v) := \{x \in I(N, v) \mid x(S) \geq v(S), \text{ for all } S \subseteq N\}$.

A game with a non-empty core is called balanced and, if the game and all its subgames have non-empty core, the game is called totally balanced.

2 Reviewing the core

The idea behind the core of a game is to distribute the total profit by trying at least to satisfy the justified demands of any potential subcoalitions of players. Another point of view - in fact historically the first one - was given by Gillies (1959) who defined the core as the set of undominated preimputations (for more details see also Rafels and Tijs, 1997). In this section we will give another interpretation based on sequential payments.

To fix ideas, let us suppose that $v \in G^N$ and we fix an ordering of players in N, denoted by $\sigma = (i_1, i_2, \ldots, i_n)$, to implement the sequentiality process. Hence i_1 is the first player, i_2 is the second player and so on. Let us now take a distribution in the core of the game, say $x \in C(N, v)$. By definition of the core, it holds that the payoff of any player, and in particular the payoff of player i_1, is between his individual worth and his marginal contribution to the grand coalition. Formally,

$$v(\{i_1\}) \leq x_{i_1} \leq v(N) - v(N \setminus \{i_1\}). \tag{1}$$

Notice that these bounds also hold for the rest of players but could be sharpened if the payoff for player i_1 had been announced in advance. If this happens, the reference game to establish these bounds is not the original game but what is known as the *reduced game*.

Since the payoff for some players is fixed (we say that these players are in fact out of the game), reducing the game means that players who are still in the game reevaluate their worth by taking into account not only potential subcoalitions with each other, but also coalitions that might include players out of the game, and bearing in mind that these players will claim what they have been promised.

The first system of reduction was introduced by Davis and Maschler (1965). Suppose players in $T \subset N$ are still in the game and players in $N \setminus T$ are out of the game, as they have been assigned the payoff given by the components of the vector $x \in \mathbb{R}^N$. Then, the *reduced game* on T at x is defined as

$$r_x^T(v)(S) := \max_{Q \in 2^{N \setminus T}} \{v(S \cup Q) - x(Q)\} \tag{2}$$

for all $\varnothing \neq S \subset T$, with $r_x^T(v)(\varnothing) := 0$ and $r_x^T(v)(T) := v(N) - x(N \setminus T)$. This last equality means that the amount to be distributed is exactly what is left by the players who are out of the game.

Peleg (1986) characterized the core axiomatically using, among other properties, a consistency axiom. The standard consistency property says that for any $\varnothing \neq T \subseteq N$, if $x \in C(N, v)$, it should hold $x_{|T} \in C(T, r_x^T(v))$. If $T = N \setminus \{i_1\}$ we have

$$x_{|N\setminus\{i_1\}} \in C(N \setminus \{i_1\}, r_x^{N\setminus\{i_1\}}(v)). \tag{3}$$

Once again, by the definition of the core, it holds that

$$r_x^{N\setminus\{i_1\}}(v)(\{i_2\}) \leq x_{i_2} \leq r_x^{N\setminus\{i_1\}}(v)(N \setminus \{i_1\}) - r_x^{N\setminus\{i_1\}}(v)(N \setminus \{i_1, i_2\}). \tag{4}$$

Repeating the reduction process, it is easy to check that, for any fixed order $\sigma = (i_1, i_2, \ldots, i_n)$, any core element $x \in C(N, v)$ meets the following inequalities:

$$\begin{aligned}
&v(\{i_1\}) \leq x_{i_1} \leq v(N) - v(N \setminus \{i_1\}),\\
&r_x^{N\setminus\{i_1\}}(v)(\{i_2\}) \leq x_{i_2} \leq r_x^{N\setminus\{i_1\}}(v)(N \setminus \{i_1\}) - r_x^{N\setminus\{i_1\}}(v)(N \setminus \{i_1, i_2\}),\\
&r_x^{N\setminus\{i_1, i_2\}}(v)(\{i_3\}) \leq x_{i_3} \leq r_x^{N\setminus\{i_1, i_2\}}(v)(N \setminus \{i_1, i_2\})\\
&\qquad\qquad\qquad\qquad\qquad\qquad - r_x^{N\setminus\{i_1, i_2\}}(v)(N \setminus \{i_1, i_2, i_3\}),\\
&\vdots\\
&r_x^{\{i_n\}}(v)(\{i_n\}) \leq x_{i_n} \leq r_x^{\{i_n\}}(v)(\{i_n\}) - r_x^{\{i_n\}}(v)(\varnothing).
\end{aligned} \tag{5}$$

Notice that the payoff to the players and their bounds are obtained sequentially starting from the original game and following the reduction process as soon as players are given their payoff. As we will see in Theorem 1, all inequalities in (5) are necessary and sufficient to recover the core.

3 The set of sequentially compatible payoffs

From the above interpretation of the core, a sequential payoff scheme will consist of an iterative process where the cooperative game is reduced each time the payoff to a player is assigned. The Davis and Maschler reduced game makes sense when it is plausible to take into account all coalitions of players. However, in some situations it would be interesting to incorporate other possibilities or restrictions in the model. For example, imagine that the set of predecessors to a given player acts as a block. In this case, when we reduce the game a dichotomous situation will appear: a coalition can go alone or join with all the predecessors, but not with a subgroup, paying them their corresponding payoffs. Another possibility is to think on an unfavoured set of agents $N^* \subseteq N$ such that we only want to cooperate with those predecessors belonging to this group. This and other interesting possibilities can be found in Elster (1992).

Our model will incorporate this kind of exogenous information by just allowing for a more general family of reduced games than the original one given by Davis and Maschler. This will be done by using the concept of *admissible subgroup correspondence* inspired in the works of Thomson (1990, 1996).

Definition 1 *Let $\alpha : 2^N \longrightarrow 2^N$ be a correspondence associating to every subset $Q \subseteq N$ a list of subgroups of Q. Then, we say that α is an* **admissible subgroup correspondence** *on N if and only if, for any $Q \subseteq N$, we have $\alpha(Q) \subseteq 2^Q$ where $\varnothing \in \alpha(Q)$.*

The interpretation of $\alpha(Q) \subseteq 2^Q$, for $Q \subseteq N$, is that it lists the admissible coalitions of Q. This is the main reason for justifying that the empty set is always an admissible coalition. We shall denote by $\mathcal{A}$ the set of all admissible subgroup correspondences. Notice that we can define a partial order in $\mathcal{A}$ by means of the natural order inclusion. Formally, given α and α' in $\mathcal{A}$, we say that $\alpha \leq \alpha'$ if and only if, for all $Q \subseteq N$, $\alpha(Q) \subseteq \alpha'(Q)$.

The admissible subgroup correspondence will be used (see Definition 2) when a subset $N \setminus T$ of players, $T \subset N$, has been paid, and then $\alpha(N \setminus T)$ will describe which coalitions of $N \setminus T$ are admissible to join players of T.

Associated to any admissible subgroup correspondence we can introduce the corresponding α-max reduction, which is no more than the reduced game *à la* Davis and Maschler but taking into account the information given by the correspondence α.

Definition 2 *Let $v \in G^N$, $\alpha \in \mathcal{A}$, $x \in \mathbb{R}^N$ and $\varnothing \neq T \subset N$. The α-***max reduced game** *relative to T at x is defined as the cooperative game $(T, r_x^{T,\alpha}(v))$ where*

$$r_x^{T,\alpha}(v)(S) := \begin{cases} 0 & \text{if } S = \varnothing, \\ \max\limits_{Q \in \alpha(N \setminus T)} \{v(S \cup Q) - x(Q)\} & \text{if } \varnothing \neq S \subset T, \\ v(N) - x(N \setminus T) & \text{if } S = T. \end{cases}$$

The interpretation is that, given a payoff vector $x \in \mathbb{R}^N$, the worth of a coalition S in the α-max reduced game relative to T at x, $\varnothing \neq T \subset N$, is evaluated under the assumption that S can ensure the cooperation of any admissible subgroup Q not overlapping with T, $Q \in \alpha(N \setminus T)$, provided that each member of Q receives his original payoff given by the vector x. The worth of a proper coalition S, $S \neq \varnothing$, will be the largest net worth $v(S \cup Q) - x(Q)$ for some admissible coalition Q.

Notice that the classical Davis and Maschler reduced game is a particular case when we take $\alpha(Q) = 2^Q$, for all $Q \subseteq N$, as an admissible subgroup correspondence. We denote this admissible subgroup correspondence by α_{DM} and, if no confusion arises, we will maintain the standard notation $r_x^T(v)$ instead of $r_x^{T,\alpha_{DM}}(v)$.

The minimal admissible subgroup correspondence is $\alpha(Q) = \varnothing$, for all $Q \subseteq N$. Hence, the associated α-max reduced game is the subgame except, eventually, for the efficiency;

following Thomson (1996) we will name it *projected reduced game*. We denote this admissible subgroup correspondence by α_P. Notice that, $\alpha_P \leq \alpha \leq \alpha_{DM}$, for all $\alpha \in \mathcal{A}$.

Example 1 *Dichotomous reduction.* This reduction is defined by

$$\alpha_d(Q) = \{\varnothing, Q\}, \text{ for all } Q \subseteq N.$$

It explains the idea that any coalition may stand alone or join with the whole group of players Q. The corresponding α_d-max reduced game was already used in Núñez and Rafels (1998) to analyze consistency properties for the extreme core points.

Example 2 N^**-reduction.* The argument of this reduction relies on the possibility that, prior to the game, it should be plausible to select some of the agents as a fixed reference admissible group. Formally, let $N^* \subseteq N$ be an arbitrary subset of $N = \{1, \ldots, n\}$. The admissible subgroup correspondence associated to N^* is defined by

$$\alpha_{N^*}(Q) := 2^{Q \cap N^*}, \text{ for all } Q \subseteq N.$$

Notice that when $N^* = N$ we obtain $\alpha_N = \alpha_{DM}$ and for $N^* = \varnothing$ we have $\alpha_\varnothing = \alpha_P$.

Other examples of admissible subgroup correspondences can be given by taking into account several important aspects of the coordination of players: communication, hierarchies, geographical areas, or the size of the subgroups.

Now we can define formally what we understand by a sequential cooperative problem.

Definition 3 *A sequential cooperative problem is a four-tuple (N, v, α, σ), where (N, v) is a cooperative game, α is an admissible subgroup correspondence on N and σ is an arbitrary order on the player set N.*

An order $\sigma = (i_1, \ldots, i_n)$ on the player set N where, $|N| = n$, is a bijection from $\{1, \ldots, n\}$ to N. From now on, we will interpret σ as follows: $\sigma(1) = i_1$ means that player i_1 is the first player, $\sigma(2) = i_2$ means that player i_2 is the second player, and so on. We will denote by $\mathcal{S}_N$ the set of all orderings on N. Given $\sigma = (i_1, \ldots, i_n)$ we define the set of predecessors of player $i_k \in N$ with respect to σ by $P_k^\sigma := \{i_1, \ldots, i_{k-1}\}$ where $k = 2, \ldots, n$, and $P_1^\sigma := \varnothing$. By $F_k^\sigma := \{i_k, i_{k+1}, \ldots, i_n\}$, for $k \in \{1, \ldots, n\}$, we will denote the set of followers of player i_k in N, including i_k, with respect to σ. Notice that $F_k^\sigma = N \backslash P_k^\sigma$ for any $k = 1, \ldots, n$.

Finally, we can define the set of sequentially compatible payoffs. Notice that the definition is no more than a recursive method that imposes at each step the *marginal* bounds to the payoffs of players.

Definition 4 *Let (N, v, α, σ) be an arbitrary sequential cooperative problem. The* **set of sequentially compatible payoffs** *with respect to σ, denoted by $SC_\alpha^\sigma(N, v)$, is the set of vectors $x \in \mathbb{R}^N$ such that*

$$r_x^{F_k^\sigma,\alpha}(v)(\{i_k\}) \leq x_{i_k} \leq r_x^{F_k^\sigma,\alpha}(v)(F_k^\sigma) - r_x^{F_k^\sigma,\alpha}(v)(F_k^\sigma \backslash \{i_k\}),$$

for all $k \in \{1, \ldots, n\}$.

Notice that for the first player i_1 we just require that

$$v(\{i_1\}) \leq x_{i_1} \leq v(N) - v(N \setminus \{i_1\}),$$

which can be justified by a *stand-alone* principle and a *non-subsidy* principle (Moulin, 1988) since the final allocation will be efficient (see the proof of Proposition 1). After this, the game is reduced and the same two criteria are applied for the rest of the players.

We will now look for a new expression of the set of sequentially compatible payoffs in terms of linear inequalities, similar to the classical expression of the core. To do this we will need to associate a hypergraph (Berge, 1973) to any sequential cooperative problem.

Definition 5 *Let α be an admissible subgroup correspondence on N and $\sigma \in \mathcal{S}_N$. We define the* **sequential hypergraph** $\mathcal{H}_\alpha^\sigma \subseteq 2^N$ *as*

$$\mathcal{H}_\alpha^\sigma := \{\{i_k\} \cup Q,\ F_{k+1}^\sigma \cup Q \text{ for all } Q \in \alpha(P_k^\sigma),\ k = 1, \ldots, n-1\}.$$

The sequential hypergraph is formed by the union of any admissible coalition Q of the set of predecessors of an arbitrary player i_k, $Q \in \alpha(P_k^\sigma)$, and the corresponding player $\{i_k\}$ or all his strict followers, F_{k+1}^σ.

As an illustration, the reader may check that if we take the projected reduction $\alpha = \alpha_P$, then for any $\sigma = (i_1, \ldots, i_n)$ its sequential hypergraph is formed by all the individual coalitions and the chains formed by deleting players following the order given by σ. Formally,

$$\mathcal{H}_{\alpha_P}^\sigma = \{\{i_k\}, \{i_{k+1}, \ldots, i_n\} \mid k = 1, \ldots, n-1\}. \tag{6}$$

In general $\mathcal{H}_\alpha^\sigma$ will select some special coalitions of N. Let us point out that all the individual coalitions will belong to $\mathcal{H}_\alpha^\sigma$ for any $\alpha \in \mathcal{A}$ and any $\sigma \in \mathcal{S}_N$, since we have imposed $\varnothing \in \alpha(Q)$ for any $Q \subseteq N$ and $\alpha \in \mathcal{A}$. The relevance of the above hypergraph is given in the next proposition.

Proposition 1 *For any sequential cooperative problem (N, v, α, σ), the set of sequentially compatible payoffs is*

$$SC_\alpha^\sigma(N, v) = \{x \in I^*(N, v) \mid x(S) \geq v(S) \text{ for all } S \in \mathcal{H}_\alpha^\sigma\}.$$

Moreover, the set of sequentially compatible payoffs is a compact and convex polyhedral set satisfying

$$C(N,v) \subseteq SC_\alpha^\sigma(N,v) \subseteq I(N,v).$$

Proof: An allocation x belongs to $SC_\alpha^\sigma(N,v)$ if and only if

$$r_x^{F_k^\sigma,\alpha}(v)(\{i_k\}) \leq x_{i_k} \leq r_x^{F_k^\sigma,\alpha}(v)(F_k^\sigma) - r_x^{F_k^\sigma,\alpha}(v)(F_k^\sigma \backslash \{i_k\}) \text{ for all } k = 1, \ldots, n.$$

By using the expression of the α-max reduction, the above inequalities can be split into

$$\begin{aligned} \max_{Q \in \alpha(P_k^\sigma)} \{v(\{i_k\} \cup Q) - x(Q)\} \leq x_{i_k} \text{ and} & \\ x_{i_k} \leq v(N) - x(P_k^\sigma) - \max_{Q \in \alpha(P_k^\sigma)} \{v(F_{k+1}^\sigma \cup Q) - x(Q)\}, & \end{aligned} \tag{7}$$

for $k = 1, \ldots, n-1$, and

$$r_x^{F_n^\sigma,\alpha}(v)(\{i_n\}) \leq x_{i_n} \leq r_x^{F_n^\sigma,\alpha}(v)(F_n^\sigma) - r_x^{F_n^\sigma,\alpha}(v)(F_n^\sigma \backslash \{i_n\}). \tag{8}$$

Since $F_n^\sigma = \{i_n\}$, we obtain $F_n^\sigma \backslash \{i_n\} = \varnothing$ and then $x_{i_n} = r_x^{F_n^\sigma,\alpha}(v)(\{i_n\})$. Moreover, by definition of the reduced game, $r_x^{F_n^\sigma,\alpha}(v)(\{i_n\}) = v(N) - x(N \backslash \{i_n\})$, and so inequalities in (8) reduce to

$$x_{i_n} = v(N) - x(N \backslash \{i_n\}).$$

Therefore, any $x \in SC_\alpha^\sigma(N,v)$ is efficient and then $v(N) - x(P_k^\sigma) = x(F_k^\sigma)$ for $1 \leq k < n$. Using this fact, (7) is equivalent to

$$\max_{Q \in \alpha(P_k^\sigma)} \{v(\{i_k\} \cup Q) - x(Q)\} \leq x_{i_k} \leq x(F_k^\sigma) - \max_{Q \in \alpha(P_k^\sigma)} \{v(F_{k+1}^\sigma \cup Q) - x(Q)\},$$

for $1 \leq k < n$. The left hand inequalities for x_{i_k} are equivalent to

$$x(\{i_k\} \cup Q) \geq v(\{i_k\} \cup Q), \text{ for all } Q \in \alpha(P_k^\sigma)$$

and the right hand ones to

$$x(F_{k+1}^\sigma \cup Q) \geq v(F_{k+1}^\sigma \cup Q), \text{ for all } Q \in \alpha(P_k^\sigma).$$

Therefore, $x \in SC_\alpha^\sigma(N, v)$ is equivalent to

$$SC_\alpha^\sigma(N, v) = \{x \in I^*(N, v) \mid x(S) \geq v(S) \text{ for all } S \in \mathcal{H}_\alpha^\sigma\}.$$

A direct consequence of the above equality is that the sequential core is a convex polyhedral set which includes the classical core, $C(N, v) \subseteq SC_\alpha^\sigma(N, v)$. To prove compactness and that the imputation set includes the set of sequentially compatible payoffs, we only have to take into account that the individual coalitions belong to the sequential hypergraph, i.e. $\{i_k\} \in \mathcal{H}_\alpha^\sigma$ for any $k = 1, \ldots, n$, $\alpha \in \mathcal{A}$ and $\sigma \in \mathcal{S}_N$. □

By this characterization, it is easy to find examples where the set of sequentially compatible payoffs is empty. Nevertheless, notice that if the original game is balanced, i.e. $C(N, v) \neq \varnothing$, then all sequentially compatible payoff sets are non-empty, whatever $\alpha \in \mathcal{A}$ and $\sigma \in \mathcal{S}_N$ we fix.

Moreover, given two admissible subgroup correspondences $\alpha, \alpha' \in \mathcal{A}$, if they are comparable, i.e. $\alpha \leq \alpha'$, then for any order $\sigma \in \mathcal{S}_N$ we have $\mathcal{H}_\alpha^\sigma \subseteq \mathcal{H}_{\alpha'}^\sigma$, which implies the corresponding reverse inclusion between the set of sequentially compatible payoffs, $SC_{\alpha'}^\sigma(N, v) \subseteq SC_\alpha^\sigma(N, v)$.

The above proposition also connects the set of sequentially compatible payoffs with the work of Faigle (1989), which analyzed the case of games with restricted cooperation.

From this connection it is easy to develop the Shapley-Bondareva algebraic conditions that characterize the non-emptiness of a specific set of sequentially compatible payoffs. In fact, we only need to work with balanced collections formed by coalitions on the sequential hypergraph associated.

Remark 1 Notice that the definition of a sequentially compatible payoff set implies that, at each step of the sequential analysis, we reduce the same n-player game v and the payoff vector we start from is always x. Another intuitive approach to the sequential analysis could be, at each step of the process, to reduce the reduced game obtained in the previous step. This process involves a smaller and smaller set of players, and with respect to a payoff vector with fewer and fewer coordinates. However, this approach turns out to be just a particular case of the one we have adopted, which is the main reason for following this approach.

4 The main results

Once we have introduced the concept of a set of sequentially compatible payoffs we will look for its properties. First we will show that under the Davis and Maschler reduction the set of sequentially compatible payoffs coincide with the core, regardless of the order we fix on the player set.

Theorem 1 *For any game $v \in G^N$ and any order σ on N, we have*

$$C(N, v) = SC^{\sigma}_{\alpha_{DM}}(N, v).$$

The proof is straightforward taking into account proposition 1 and the fact that, for any $\sigma = (i_1, \ldots, i_n)$, $\mathcal{H}^{\sigma}_{\alpha_{DM}} = 2^N \backslash \{\varnothing, N\}$.

The above result states an interesting *order-independence* of the set of sequentially compatible payoffs if we use the Davis and Maschler reduced game. Nevertheless, this property is lost when other reduced games are used (see corollary 2).

A consequence of the above theorem is a sort of recursive characterization of the core that emphasizes an interesting feature of it, already stated by Driessen (1985).

Corollary 1 *Let $v \in G^N$, $x \in \mathbb{R}^N$, $i \in N$ and $T := N \setminus \{i\}$. Then,*

$$x \in C(N,v) \Leftrightarrow \begin{cases} v(\{i\}) \leq x_i \leq v(N) - v(N \setminus \{i\}) \text{ and} \\ x_{|T} \in C(T, r_x^T(v)), \end{cases}$$

where $r_x^T(v)$ is the DM-reduced game relative to T at x.

Notice that from the above recursive result we obtain a way to analyze core-selection solutions. Roughly speaking, first-player marginality plus first-player consistency implies core selection. This fact was used by Potters and Sudhölter (1999) to analyze point-solutions for the class of airport cost games.

Another consequence of Theorem 1 is that, in general, the bounds imposed in the definition of the set $SC^\sigma_\alpha(N,v)$, for any $\alpha \in \mathcal{A}$, are not always attainable. By Theorem 1, $SC^\sigma_{\alpha_{DM}}(N,v) = C(N,v)$ and it is well known that a balanced game may not attain the individual worths of some players or/and their marginal contributions. Therefore, bounds imposed in the definition of the sequentially compatible payoff set may not be attainable. In particular, the initial bounds for the first player, $v(\{i_1\}) \leq x_{i_1} \leq v(N) - v(N \setminus \{i_1\})$ can be modified during the complete sequential analysis.

In addition, if we describe the inequalities of the set of sequentially compatible payoffs, $SC^\sigma_{\alpha_{DM}}(N,v)$, we will obtain a complete description of the core by giving explicit inequalities for the individual payoffs, x_i, $i \in N$, only depending on the payoffs to the predecessors.

First notice that, by Theorem 1, the core can be described, for any $\sigma = (i_1, \ldots, i_n)$, as those vectors $x \in \mathbb{R}^N$ such that

$$r_x^{F_k^\sigma}(v)(\{i_k\}) \leq x_{i_k} \leq r_x^{F_k^\sigma}(v)(F_k^\sigma) - r_x^{F_k^\sigma}(v)(F_k^\sigma \setminus \{i_k\}),$$

for $k = 1, \ldots, n$, where $r_x^T(v)$ is the DM-reduced game on T at x. Hence, for any $1 \leq k < n$,

$$\begin{aligned} r_x^{F_k^\sigma}(v)(\{i_k\}) &= \max_{Q \in 2^{P_k^\sigma}} \{v(\{i_k\} \cup Q) - x(Q)\} \text{ and} \\ r_x^{F_k^\sigma}(v)(F_k^\sigma) - r_x^{F_k^\sigma}(v)(F_k^\sigma \setminus \{i_k\}) &= v(N) - x(P_k^\sigma) - \max_{Q \in 2^{P_k^\sigma}} \{v(F_{k+1}^\sigma \cup Q) - x(Q)\} \\ &= \min_{Q \in 2^{P_k^\sigma}} \{v(N) - v(F_{k+1}^\sigma \cup Q) - x(P_k^\sigma \setminus Q)\}, \end{aligned}$$

and so we obtain a core description in terms of the efficient allocations that satisfy

$$\max_{Q \in 2^{P_k^\sigma}} \{v(\{i_k\} \cup Q) - x(Q)\} \leq x_{i_k} \leq \min_{Q \in 2^{P_k^\sigma}} \{v(N) - v(F_{k+1}^\sigma \cup Q) - x(P_k^\sigma \setminus Q)\},$$

for $k = 1, \ldots, n-1$.

Let us point out that to limit the payoff to player i_k, only the payoffs to his predecessors are taken into account. As an illustrative example notice that for $N = \{1, 2, 3\}$ and for any $\sigma = (i_1, i_2, i_3)$ we are describing the core as those $x \in \mathbb{R}^N$ such that

$$\begin{aligned} v(\{i_1\}) \leq\ & x_{i_1} \leq v(N) - v(N \setminus \{i_1\}), \\ \max \left\{ \begin{array}{l} v(\{i_2\}), \\ v(\{i_1, i_2\}) - x_{i_1} \end{array} \right\} \leq\ & x_{i_2} \leq \min \left\{ \begin{array}{l} v(N) - v(N \setminus \{i_2\}), \\ v(N) - v(N \setminus \{i_1, i_2\}) - x_{i_1} \end{array} \right\} \\ \text{and}\ \ & x_{i_3} = v(N) - x_{i_1} - x_{i_2}. \end{aligned}$$

From this recursive description of the core it should be possible to introduce new point-solution concepts: for example, the sequential point-solution assigning to the players a half of their range in the core, or a fixed proportion of these ranges. We leave these matters for future studies.

As a second main result we will analyze the intersection of the different sequentially compatible payoff sets corresponding to all possible orders on N. Therefore, we look for those imputations which satisfy all possible sequentiality criteria, with the reduction $\alpha \in \mathcal{A}$ fixed. Obviously, by Theorem 1 the intersection mentioned above will coincide with the core for the Davis and Maschler reduction since each of the sequentially compatible payoff sets coincides with the core itself. The next theorem will show that this will always be the case, whatever reduction $\alpha \in \mathcal{A}$ we fix.

Theorem 2 *For any cooperative game* (N, v) *and any* $\alpha \in \mathcal{A}$*, we have*

$$\bigcap_{\sigma \in \mathcal{S}_N} SC_\alpha^\sigma(N, v) = C(N, v).$$

Proof: By Proposition 1 we know that $C(N, v) \subseteq \bigcap_{\sigma \in \mathcal{S}_N} SC_\alpha^\sigma(N, v)$. Let $x \in \bigcap\limits_{\sigma \in \mathcal{S}_N} SC_\alpha^\sigma(N, v)$ be an arbitrary element of the intersection. Once again by Proposition 1, we know that $x \in I(N, v)$. In order to prove that $x \in C(N, v)$, let $S \subseteq N$ be an arbitrary sub-coalition, $|S| \geq 2$, $S \neq \varnothing, N$. Now take $\sigma^* \in \mathcal{S}_N$ where players in S enter the last positions, i.e. $\sigma^* = (i_1, \ldots, i_{n-s}, i_{n-s+1}, \ldots, i_n)$ and $S = \{i_{n-s+1}, \ldots, i_n\}$. By hypothesis, $x \in SC_\alpha^{\sigma^*}(N, v)$, which implies $x_{i_{n-s}} \leq x(F_{n-s}^{\sigma^*}) - \max_{Q \in \alpha(P_{n-s}^{\sigma^*})}\{v(F_{n-s+1}^{\sigma^*} \cup Q) - x(Q)\}$. Since $\alpha \in \mathcal{A}$, we know that $\varnothing \in \alpha(P_{n-s}^{\sigma^*})$ and then $x_{i_{n-s}} \leq x(F_{n-s}^{\sigma^*}) - v(F_{n-s+1}^{\sigma^*})$, or, equivalently, $v(F_{n-s+1}^{\sigma^*}) \leq x(F_{n-s+1}^{\sigma^*})$, which implies that $x \in C(N, v)$. $\square$

The above theorem also states that if we replace the Davis and Maschler reduction by an arbitrary one, $\alpha \in \mathcal{A}$, the core $C(N, v)$ splits into the family $\{SC_\alpha^\sigma(N, v)\}_{\sigma \in \mathcal{S}_N}$. Moreover, if we combine Theorem 1 and 2 we can obtain an interesting new feature of the core: the core could be viewed as an order-independent sequentially compatible solution.

Definition 6 *Let α be an admissible subgroup correspondence on N. The set of sequentially compatible payoffs is order-independent if*

$$SC^{\sigma}_{\alpha}(N,v) = SC^{\sigma'}_{\alpha}(N,v), \text{ for all } v \in G^N, \text{ and all } \sigma, \sigma' \in \mathcal{S}_N.$$

Corollary 2 *The core is the only order-independent set of sequentially compatible payoffs.*

Proof: By Theorem 1 we know that $SC^{\sigma}_{\alpha_{DM}}(N,v) = C(N,v)$ for any $\sigma \in \mathcal{S}_N$. Then let us suppose that there exists $\alpha \neq \alpha_{DM}$ such that for any $\sigma, \sigma' \in \mathcal{S}_N, \sigma \neq \sigma'$ it holds that $SC^{\sigma}_{\alpha}(N,v) = SC^{\sigma'}_{\alpha}(N,v)$, and so $\bigcap_{\sigma' \in \mathcal{S}_N} SC^{\sigma'}_{\alpha}(N,v) = SC^{\sigma}_{\alpha}(N,v)$, for any $\sigma \in \mathcal{S}_N$. But by Theorem 2, this intersection is the core of the game. □

For the last result in this section we will study the behavior of the union of the above family of sets. By Proposition 1, any set of sequentially compatible payoffs is a subset of the imputation set of the original game. Therefore, fixing $\alpha \in \mathcal{A}$, we have $\bigcup_{\sigma \in \mathcal{S}_N} SC^{\sigma}_{\alpha}(N,v) \subseteq I(N,v)$. Moreover, since the projected reduction process satisfies $\alpha_P \leq \alpha$, for any $\alpha \in \mathcal{A}$, we also know that $\bigcup_{\sigma \in \mathcal{S}_N} SC^{\sigma}_{\alpha}(N,v) \subseteq \bigcup_{\sigma \in \mathcal{S}_N} SC^{\sigma}_{\alpha_P}(N,v) \subseteq I(N,v)$, for any $\alpha \in \mathcal{A}$.

The last inclusion could be strict, as we will show later in an example, but in some cases we will have an equality. The next theorem states that, for a relatively large class of cooperative games, any imputation can be supported by a sequential approach by using the projected reduction.

Theorem 3 *Let (N,v) be a totally balanced game. Then, we have*

$$\bigcup_{\sigma \in \mathcal{S}_N} SC^{\sigma}_{\alpha_P}(N,v) = I(N,v).$$

Proof: By Proposition 1, $\cup_{\sigma\in\mathcal{S}_N} SC^{\sigma}_{\alpha_P}(N,v) \subseteq I(N,v)$. For the reverse inclusion, first notice that the totally-balancedness hypothesis implies

$$v(S) \geq \frac{1}{|S|-1}\sum_{i\in S} v(S\setminus\{i\}),\ \text{ for all } S\subseteq N \text{ with } |S|\geq 2. \tag{9}$$

Then, let $x\in I(N,v)$ and suppose that, for all $i\in N$, $x_i > v(N)-v(N\backslash\{i\})$. By efficiency, $v(N) > n\,v(N) - \sum_{i\in N} v(N\backslash\{i\})$ or, equivalently, $\sum_{i\in N} v(N\backslash\{i\}) > (n-1)v(N)$, which contradicts (9). Therefore, there is a player $i_1\in N$ such that $v(\{i_1\}) \leq x_{i_1} \leq v(N) - v(N\backslash\{i_1\})$. Now consider the α_P-max reduced game relative to $N\backslash\{i_1\}$ at x,

$$r_x^{N\backslash\{i_1\},\alpha_P}(v)(S) = \begin{cases} 0 & \text{if } S=\varnothing,\\ v(S) & \text{if } \varnothing\neq S\subset N\backslash\{i_1\},\\ v(N)-x_{i_1} & \text{if } S=N\backslash\{i_1\}. \end{cases}$$

From the expression of this reduced game and (9), it is straightforward to check that

$$r_x^{N\backslash\{i_1\},\alpha_P}(v)(S) \geq \frac{1}{|S|-1}\sum_{i\in S} r_x^{N\backslash\{i_1\},\alpha_P}(v)(S\setminus\{i\}), \tag{10}$$

for all $S\subseteq N\backslash\{i_1\}$ with $|S|\geq 2$, as for $S\subset N\backslash\{i_1\}$ the reduced game is just the subgame and for $S = N\setminus\{i_1\}$, since $x_{i_1}\leq v(N)-v(N\backslash\{i_1\})$, we get

$$v(N)-x_{i_1} \geq v(N\backslash\{i_1\}) \geq \frac{1}{n-2}\sum_{i\in N\backslash\{i_1\}} v(N\setminus\{i_1,i\}),$$

where the last inequality follows from (9). On the other hand, if $x\in I(N,v)$, then $x_{|N\backslash\{i\}} \in I(N\backslash\{i\}, r_x^{N\backslash\{i\},\alpha_P}(v))$. Hence, by repeating the above reasoning we know that there is a player, say $i_2\in N\backslash\{i_1\}$, such that

$$r_x^{N\backslash\{i_1\},\alpha_P}(v)(\{i_2\}) \leq x_{i_2} \leq r_x^{N\backslash\{i_1\},\alpha_P}(v)(N\backslash\{i_1\}) - r_x^{N\backslash\{i_1\},\alpha_P}(v)(N\backslash\{i_1,i_2\}).$$

Finally, following the same argument, and taking into account that the projected reduction has the transitive property, i.e. for any game (N,v), all $x\in\mathbb{R}^N$ and all $\varnothing\neq S\subset T\subseteq N$,

$| T | \geq 2$,

$$r_{x_{|T}}^{S,\alpha_P}(r_x^{T,\alpha_P}(v)) = r_x^{S,\alpha_P}(v),$$

we can find an order $\sigma \in \mathcal{S}_N$, with $\sigma(1) = i_1$, $\sigma(2) = i_2, \ldots$, such that $x \in SC_{\alpha_P}^{\sigma}(N, v)$, and the desired result is obtained. □

The next example shows that the totally-balancedness condition of the game is not necessary for obtaining the same result.

Example 3 Let (N, v) be the five-person balanced game defined by:

$$v(\{1,2\}) = v(\{3,4\}) = 5, \;\; v(\{1,2,5\}) = v(\{3,4,5\}) = 2.5$$
$$v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 6.5$$
$$v(\{1,2,3,5\}) = v(\{1,2,4,5\}) = v(\{1,3,4,5\}) = v(\{2,3,4,5\}) = 3,$$
$$v(\{1,2,3,4\}) = 9, \;\; v(N) = 15 \text{ and } v(S) = 0 \text{ for the other coalitions.}$$

The game is not totally balanced because the subgame associated to $S = \{1,2,3,4\}$ is not balanced ($v(\{1,2\}) + v(\{3,4\}) = 10 > v(\{1,2,3,4\}) = 9$). However, it satisfies condition (9) in the proof of the Theorem, which is sufficient to obtain equality between the imputation set and the union of the sequentially compatible payoff sets.

To end this section, let us see an example where the union of the sequentially compatible payoff sets corresponding to the projected reduction does not coincide with the imputation set.

Example 4 Let (N, v) be the symmetric four-person game where $v(\{i\}) = 0$ for all $i \in \{1,2,3,4\}$, $v(\{i,j\}) = 100$, $v(\{i,j,k\}) = 125$ and $v(N) = 180$, for all $i, j, k \in N = \{1,2,3,4\}$ such that $i < j < k$. The imputation $(45, 45, 45, 45) \notin SC_{\alpha_P}^{\sigma}(N, v)$ for any

$\sigma \in \mathcal{S}_N$ since for any pair of players we have $x_i + x_j = 90 < v(\{i,j\}) = 100$ (see Proposition 1).

5 First-player advantage property

From the above results we can see that if we use a notion of reduced game other than the Davis and Maschler notion, i.e. $\alpha \neq \alpha_{DM}$, $\alpha \in \mathcal{A}$, then the set $SC^{\sigma}_{\alpha}(N,v)$ is order-dependent. This leads us to look for advantage properties for players depending on their positions in the order. To do this, let us first introduce some notation.

For a given order $\sigma = (i_1, \ldots, i_n)$ and $k = 2, \ldots, n-1$ we will denote $\sigma^k = (i_2, \ldots, i_k,\, i_1,\, i_{k+1},\, \ldots,\, i_n)$ and $\sigma^n = (i_2,\, i_3,\, \ldots,\, i_n,\, i_1)$. The interpretation is clear: σ^k, $k = 2, \ldots, n$, represents switches in the position of the original first player in σ without changing the sequence of the remaining players. We will identify σ with σ^1 whenever it is needed.

To analyze advantage properties for players depending on their positions we introduce, as a criterion for comparing potential payoffs, the idea that players prefer more rather than less. With this assumption in mind, let us associate to any sequential cooperative problem (N, v, α, σ), with a non-empty sequential compatible payoff set what we call the **maximal sequential rule** denoted by $\bar{x}^{\sigma,\alpha}(v) \in \mathbb{R}^N$.

The **maximal sequential rule** $\bar{x}^{\sigma,\alpha}(v) \in \mathbb{R}^N$ can be described as follows: for all $k = 1, \ldots, n$,

$$\bar{x}^{\sigma,\alpha}_{i_k}(v) = \max_{\substack{x \in SC^{\sigma}_{\alpha}(N,v) \\ x_{i_1} = \bar{x}^{\sigma,\alpha}_{i_1}(v) \\ \vdots \\ x_{i_{k-1}} = \bar{x}^{\sigma,\alpha}_{i_{k-1}}(v)}} \{x_{i_k}\}. \tag{11}$$

Notice that the first player in the given order $\sigma = (i_1, i_2, \ldots i_n)$ maximizes his potential gains over the set of sequentially compatible payoffs. Then,

$$\bar{x}^{\sigma,\alpha}_{i_1}(v) = \max_{x \in SC^{\sigma}_{\alpha}(N,v)} \{x_{i_1}\}.$$

In his turn, the second player will maximize his payoff by taking into account that $x_{i_1} = \bar{x}^{\sigma,\alpha}_{i_1}(v)$ for the first player. Then,

$$\bar{x}^{\sigma,\alpha}_{i_2}(v) = \max_{\substack{x \in SC^{\sigma}_{\alpha}(N,v) \\ x_{i_1} = \bar{x}^{\sigma,\alpha}_{i_1}(v)}} \{x_{i_2}\}.$$

We then repeat the process until we reach the last player. Notice that this last player is in fact a payoff-taker agent: his payoff is just what is left by the rest of the players, $\bar{x}^{\sigma,\alpha}_{i_n}(v) = v(N) - (\bar{x}^{\sigma,\alpha}_{i_1}(v) + \ldots + \bar{x}^{\sigma,\alpha}_{i_{n-1}}(v))$.

The maximal allocation rule is well-defined and it is easy to see that it is always an extreme point of the compact polyhedron $SC^{\sigma}_{\alpha}(N,v)$.

Moreover, the maximal sequential rule can be interpreted as a kind of priority rule for the initial sequential cooperative problem (in some cases it will coincide with a marginal worth vector, as we will see in the proof of Proposition 2).

Definition 7 *The α-max reduction, $\alpha \in \mathcal{A}$, has the* **first-player advantage property** *for a given game $v \in G^N$ if for all $\sigma = (i_1, i_2 \ldots, i_n)$ and all $k = 1, \ldots, n$ we have :*

1) $SC_{\alpha}^{\sigma^k}(N, v) \neq \varnothing$ and

2) $\bar{x}_{i_1}^{\sigma,\alpha}(v) \geq \bar{x}_{i_1}^{\sigma^k,\alpha}(v)$.

If the reduction has this property, the first player does not have an incentive to move to another position. Notice that $\bar{x}_{i_1}^{\sigma^k,\alpha}(v)$ is the maximum that player i_1 could obtain by going in position $k \in \{1, \ldots, n\}$ (after paying their corresponding maxima to his predecessors).

Remember at this point that if a game is balanced then all its sequentially compatible payoff sets are non-empty, which implies that the above property can be checked, at least, in a general class of games. Moreover, conditions weaker than balancedness can also guarantee the non-emptiness of the sequentially compatible payoff set. The reader may check that the condition $v(S) + \sum_{i \in N \setminus S} v(\{i\}) \leq v(N)$, for all $S \subseteq N$, is a necessary and sufficient condition to guarantee the non-emptiness of all projected sequentially compatible payoff sets.

As an example of reduction which has the first-player advantage property we have the case of the Davis and Maschler reduction on the class of balanced games. This is a direct consequence of theorem 1 and the justification is left to the reader.

On the other hand, the projected reduction does not in general have this advantage property. To check this, let us take the following balanced and superadditive 3-player game: $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = 3$, $v(\{2,3\}) = 5$ and $v(\{1,2,3\}) = 7$. By Proposition 1 it is easy to check that, if $\sigma = (1,2,3)$ then $\sigma^2 = (2,1,3)$ and

$$SC_{\alpha_P}^{\sigma}(N, v) = \{x \in I(N, v) \mid x_2 + x_3 \geq v(\{2,3\}) = 5\}$$

and

$$SC^{\sigma^2}_{\alpha_P}(N,v) = \{x \in I(N,v) \mid x_1 + x_3 \geq v(\{1,3\}) = 3\}.$$

Therefore, $\bar{x}^{\sigma}_{\alpha_P}(v) = (2,\, 5,\, 0)$ and $\bar{x}^{\sigma^2}_{\alpha_P}(v) = (3,\, 4,\, 0)$, which shows that the first player $i_1 = 1$ obtains an extra unit if he is *so kind* as to allow player 2 to be the first. Moreover, this extra unit is taken from the payoff of the second player.

The negative result shown by this example could also be explained by the next proposition. Recall that a game (N,v) is 0-monotonic if $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$ for all $S \subseteq T \subseteq N$.

Proposition 2 *On the class of 0-monotonic cooperative games the following statements are equivalent:*

1) The projected reduction α_P has the first-player advantage property.

2) For any player $i \in N$, $v(N) - v(N \setminus \{i\}) \geq \max_{S \subseteq N,\, i \in S} \{v(S) - v(S \setminus \{i\})\}$.

Proof: By Proposition 1, it is easy to see that for $\alpha = \alpha_P$ and $\sigma = (i_1,\, i_2,\, \ldots,\, i_n)$,

$$SC^{\sigma}_{\alpha_P}(N,v) = \left\{ \begin{array}{l} x \in I(N,v) \text{ such that} \\ \begin{array}{rrrrl} x_{i_2}+ & x_{i_3}+ & \ldots + & x_{i_{n-1}}+ & x_{i_n} \geq v(\{i_2,\, \ldots,\, i_n\}) \\ & x_{i_3}+ & \ldots + & x_{i_{n-1}}+ & x_{i_n} \geq v(\{i_3,\, \ldots,\, i_n\}) \\ & & \ddots & & \\ & & & x_{i_{n-1}}+ & x_{i_n} \geq v(\{i_{n-1}, i_n\}) \end{array} \end{array} \right\}. \tag{12}$$

Moreover, if $v \in G^N$ is 0-monotonic, then $SC^{\sigma}_{\alpha_P}(N,v) \neq \varnothing$ for any $\sigma \in \mathcal{S}_N$. This can be explained as follows: for 0-monotonic games and for any order $\sigma = (i_1,\, i_2,\, \ldots,\, i_n)$, the vector $z^{\sigma}(v) \in \mathbb{R}^N$ defined as $z^{\sigma}_{i_1}(v) = v(N) - v(N \setminus \{i_1\})$, $z^{\sigma}_{i_2}(v) = v(N \setminus \{i_1\}) - v(N \setminus \{i_1, i_2\})$, $\ldots$, $z^{\sigma}_{i_n}(v) = v(\{i_n\})$ is an imputation of the game. Furthermore, looking at the description of the set of sequentially compatible payoffs (see (12)), it follows straightforwardly that $z^{\sigma}(v) \in SC^{\sigma}_{\alpha_P}(N,v)$ for any 0-monotonic game.

We now that for a given $\sigma \in \mathcal{S}_N$, the vector $z^\sigma(v)$ is the maximal sequential rule, i.e. $\bar{x}^{\sigma,\alpha_P}(v) = z^\sigma(v)$. Let us prove this. Since $z^\sigma(v) \in SC^\sigma_{\alpha_P}(N,v)$, the maximum that the first player can obtain in $SC^\sigma_{\alpha_P}(N,v)$ is $\bar{x}^{\sigma,\alpha_P}_{i_1}(v) = v(N) - v(N \setminus \{i_1\})$. Now, for any $x \in SC^\sigma_{\alpha_P}(N,v)$ with $x_{i_1} = v(N) - v(N \setminus \{i_1\})$, we have $x_{i_2} + \ldots + x_{i_n} = v(N) - (v(N) - v(N \setminus \{i_1\})) = v(\{i_2,\ldots,i_n\})$. Since $x_{i_3} + \ldots + x_{i_n} \geq v(\{i_3,\ldots,i_n\})$, we obtain $x_{i_2} \leq v(\{i_2,\ldots,i_n\}) - v(\{i_3,\ldots,i_n\})$. Finally, from the fact that $z^\sigma(v) \in SC^\sigma_{\alpha_P}(N,v)$, we have $\bar{x}^{\sigma,\alpha_P}_{i_2}(v) = v(\{i_2,\ldots,i_n\}) - v(\{i_3,\ldots,i_n\}) = z^\sigma_{i_2}(v)$. With a similar argument for the rest of players (we omit details) we finally see that $\bar{x}^{\sigma,\alpha_P}(v) = z^\sigma(v)$.

From the above facts, the equivalence stated in the proposition can be straightforwardly deduced and it is left to the reader. □

As a first consequence of this result we can state an interesting property for convex games (Shapley, 1972).

Corollary 3 *On the domain of convex games, any α-max reduction, $\alpha \in \mathcal{A}$, has the first-player advantage property.*

Proof: By Proposition 2, as convex games are 0-monotonic and satisfy condition 2) of that proposition, the projected reduction α_P has the first player advantage property for this class of games. Moreover, as we have seen in the proof of Proposition 2 and given an ordering $\sigma = (i_1, i_2, \ldots, i_n)$, the maximal sequential rule coincides with the vector $z^\sigma(v) \in \mathbb{R}^N$ defined as $z^\sigma_{i_1}(v) = v(N) - v(N \setminus \{i_1\})$, $z^\sigma_{i_2}(v) = v(N \setminus \{i_1\}) - v(N \setminus \{i_1, i_2\})$, $\ldots$, $z^\sigma_{i_n}(v) = v(\{i_n\})$. Moreover, this vector is just a vector of marginal contributions of the game v so, as the game is convex, it belongs to its core (see Shapley, 1972), i.e. $z^\sigma(v) \in C(N,v)$. Therefore, $\bar{x}^{\sigma,\alpha_P}(v) = \bar{x}^{\sigma,\alpha_{DM}}(v) = z^\sigma(v)$.

Now, from the inclusion relation $C(N,v) \subseteq SC^{\sigma}_{\alpha}(N,v) \subseteq SC^{\sigma}_{\alpha_P}(N,v)$, we have $\bar{x}^{\sigma,\alpha}(v) = z^{\sigma}(v)$, for all $\alpha \in \mathcal{A}$ and all $\sigma \in \mathcal{S}_N$. Finally, from the convexity of the game, it follows that $\bar{x}^{\sigma,\alpha}_{i_1}(v) \geq \bar{x}^{\sigma^k,\alpha}_{i_1}(v)$, and the proof is done. □

For other classes of games, such as assignment games (Shapley and Shubik, 1972), the first-player advantage property still holds if we impose an additional requirement to the α-max reduction. In this sense, we say that $\alpha \in \mathcal{A}$ is *upper dichotomous* if, for any $Q \subseteq N$, $Q \in \alpha(Q)$, where the name comes from the fact that $\alpha_d \leq \alpha$. For instance, the Davis-Maschler reduction is upper-dichotomous, but this is not the case of the projected reduction.

Theorem 4 *On the class of assignment games any upper-dichotomous reduction has the first-player advantage property.*

Proof: It is well known that, given an assignment game (N,v), for any player $i \in N$ there exists a payoff vector x in the core of the game such that $x_i = v(N) - v(N \setminus \{i\})$ (the marginal contribution of player i is attained in the core, (see, for instance, Roth and Sotomayor, 1990). As the core is in any set of sequentially compatible payoffs, i.e. $C(N,v) \subseteq SC^{\sigma}_{\alpha}(N,v)$ for any $\alpha \in \mathcal{A}$ and $\sigma \in \mathcal{S}_N$, the marginal contribution of any player i will also be attainable in any set of sequentially compatible payoffs corresponding to an assignment game. This implies that, given an assignment game (N,v) and a fixed ordering of players $\sigma = (i_1, i_2, \ldots, i_n)$, the maximal sequential rule will assign to player i_1 at least his marginal contribution. In fact, it will assign exactly the marginal contribution as it is an upper bound in the definition of the set $SC^{\sigma}_{\alpha}(N,v)$. Hence, $\bar{x}^{\sigma,\alpha}_{i_1}(v) = v(N) - v(N \setminus \{i_1\})$.

At this point, since α is upper-dichotomous and by the description of the set of sequentially compatible payoffs given in Proposition 1, notice we have $x(N \setminus \{i\}) \geq v(N \setminus \{i\})$

for all $x \in SC^{\sigma}_{\alpha}(N, v)$, $\sigma \in \mathcal{S}_N$ and $i \in N$. Therefore, by efficiency, $x_i \leq v(N) - v(N \setminus \{i\})$ for all $i \in N$. Hence, any movement of player i_1 to other positions will not benefit him, as $\bar{x}^{\sigma^k,\alpha}_{i_1}(v) \leq v(N) - v(N \setminus \{i_1\})$ for all $k = 2, \ldots, n$. $\square$

Notice that assignment games do not meet condition 2) of Proposition 2, so the projected reduction will not preserve the first-player advantage property in this class of games. The next example shows this point.

Example 5 Let (N, v) be the assignment game associated to the assignment matrix

$$\begin{array}{cc} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left(\begin{array}{cc} 4 & 5 \\ 1 & 3 \end{array} \right) \end{array}$$

where $\{1, 2\}$ is the set of buyers and $\{3, 4\}$ is the set of sellers. In this case $v(N) - v(N \setminus \{1\}) = 7 - 3 = 4$ and $v(\{1, 4\}) - v(\{4\}) = 5$. This implies that if we take the orderings $\sigma = (1,\ 2,\ 3,\ 4)$ and $\sigma^3 = (2,\ 3,\ 1,\ 4)$, we have $\bar{x}^{\sigma,\alpha_p}_1(v) = 4 < \bar{x}^{\sigma^3,\alpha_p}_1(v) = 5$, where player 1 will take advantage to move to the third position.

6 Concluding remarks

This paper has studied the problem of sequential allocation decisions. The set of sequentially compatible payoffs describes which allocation vectors are accepted according to an interative application, at each step of the process, of the stand-alone principle and the non-subsidy principle.

This perspective opens up several lines of research. First of all, point-solution concepts could be analyzed within this sequential analysis. In this sense, a sequential solution would be defined by a rule or a criterion that assigns payoffs to players following a fixed order. In this paper, the sequential maximal rule is just one assymetric example of this kind of solutions. Furthermore, from any assymetric rule, an associated rule can be derived by taking the average of the assymetric solutions corresponding to the different orders. From this perspective, not only could new solutions be defined but old well-known solutions could be reviewed.

Secondly, the iterative process performed suggests a strategic analysis of a sequential non-cooperative game in which players take decisions in the given order (to leave or not to leave the game, to accept or not to accept a payoff). Are the equilibria of such a game consistent with the set of sequentially compatible payoffs? Regarding this question it is interesting to read the paper by Moldovanu and Winter (1995), which analyzes core allocations by a dynamic process of payoff vector proposals.

Finally, a natural extension is to apply sequential analysis to the case of non-transferable utility games. Several interesting questions then arise. For example, how does one define the iterative process and would the same general results still hold (in particular the order-independence of the core)?

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