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Single-valued solutions for the Böhm-Bawerk horse market game

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Abstract: Single–valued solutions for the case of two–sided market games without product differentiation, also known as Böhm–Bawerk horse market games, are analyzed. The nucleolus is proved to coincide with the τ–value, and is thus the midpoint of the core. Moreover a characterization of this set of games in terms of the assignment matrix is provided.
Key words: Assignment game, horse market game, core, nucleolus, τ–value, Shapley value.
JEL: C71, C78

Resum: En aquest treball s'analitzen solucions puntuals per a mercats a dues bandes amb productes homogenis, també coneguts com mercats de cavalls de Böhm–Bawerk. Es demostra que el nucleolus coincideix amb el valor de tau i és el punt mig del core. A més, es dona una caracterització d'aquests jocs en termes de la matriu d'assignació.

1 Introduction

The Böhm–Bawerk horse market game (Böhm–Bawerk, 1923) is a model for a two–sided market with homogeneous goods, and is thus a particular case of an assignment game where there is no product differentiation.

The assignment game was introduced by Shapley and Shubik (1972) as a model for a two–sided market with transferable utility. Let M be a finite set of buyers and M' a finite set of sellers, and let us denote by m and m' their cardinalities. We may think of the formal model of assignment games as arising from a situation where each seller $j \in M'$ has an object for sale which he valuates at $c_j \in \mathbf{R}_+$ (reservation price of seller j), $\mathbf{R}_+$ being the set of non negative real numbers, while each buyer $i \in M$ wants exactly one indivisible object and places a value of $h_{ij} \in \mathbf{R}_+$ on the object offered by seller j, $h_i = (h_{ij})_{j \in M'}$. Then, if $h = (h_i)_{i \in M}$ and $c = (c_j)_{j \in M'}$, a matrix $A = A(h,c) = (a_{ij})_{(i,j) \in M \times M'}$ is defined, where $a_{ij} = \max\{h_{ij} - c_j, 0\}$ are the potential gains from the trade between i and j. We will denote by $\mathcal{M}_{m \times m'}(\mathbf{R}_+)$ the set of non negative matrices with m rows and m' columns.

A matching (or assignment) between M and M' (or a matching for A) is a subset μ of $M \times M'$ such that each $k \in M \cup M'$ belongs to at most one pair in μ. We will denote by $\mathcal{M}(A)$ the set of matchings of A. We say a matching μ is optimal if for all $\mu' \in \mathcal{M}(A)$, $\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij}$ and will denote by $\mathcal{M}^*(A)$ the set of optimal matchings. When trying to allocate the profit obtained by an optimal matching among the agents, cooperative game theory plays an important role.

A transferable utility cooperative game is a pair (N, v), where the set $N = \{1, 2, \ldots, n\}$ is its finite player set and $v : 2^N \longrightarrow \mathbf{R}$ its characteristic function satisfying $v(\emptyset) = 0$. A payoff vector will be $x \in \mathbf{R}^n$ and, for every coalition $S \subseteq N$, we shall write $x(S) := \sum_{i \in S} x_i$ the payoff to coalition S (where $x(\emptyset) = 0$). An imputation is a payoff vector x that is efficient, $x(N) = v(N)$ and individually rational, which means each player $i \in N$ receives at least the individual worth $v(i)$. The set of all imputations of a game (N, v) is denoted by $I(v)$. The core of the game (N, v) is a set–solution concept which consists of those payoff vectors which allocate the worth of the grand coalition in such a way that every other coalition receives at least its worth by the characteristic function: $C(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$. The core is a bounded convex polyhedron and thus the set of extreme points, $Ext(C(v))$, is finite. A single–valued solution concept for TU games selects for any game (N, v) an efficient payoff $\alpha(v) \in \mathbf{R}^n$. Examples of single–valued solutions are the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969) and the τ–value (Tijs, 1981).

The marginal contribution of player $i \in N$ in the game v, $b_i^v = v(N) -$

$v(N \setminus \{i\})$ is an upper bound for player i's payoff in the core of the game. In general this upper bound may not be attained. However, there are balanced games with the property that all players can attain their marginal contribution in the core. This is the case of assignment games.

The above two–sided market can be described by means of a cooperative game where the player set consists of the union $M \cup M'$ of the sets of buyers and sellers, $n = m + m'$ being the cardinality of the player set. The profits of mixed–pair coalitions, $\{i,j\}$ where $i \in M$ and $j \in M'$, are $w_A(i,j) = a_{ij} \geq 0$ and the matrix A also determines the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, $\mathcal{M}(S,T)$ being the set of matchings between S and T. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Shapley and Shubik proved that the core of the assignment game $(M \cup M', w_A)$ is nonempty and coincides with the set of stable outcomes. This means that the core can be represented in terms of any optimal matching μ of $M \cup M'$ by

$$C(w_A) = \left\{ (u,v) \in \mathbf{R}^{M\times M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M\,;\, v_j \geq 0, \text{ for all } j \in M' \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu\,. \end{array} \right\} \tag{1}$$

Moreover, the core has a lattice structure with two special extreme core allocations: the buyers–optimal core allocation, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the sellers–optimal core allocation, $(\underline{u}, \overline{v})$, where each seller does. Notice that when agents on one side of the market obtain their maximum core payoff, the agents on the opposite side obtain their minimum core payoff, as the joint payoff of an optimally matched pair is fixed: $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$ if $(i,j) \in \mu$.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with her marginal contribution:

$$\overline{u}_i = w_A(N) - w_A(N \setminus \{i\}) \text{ and } \overline{v}_j = w_A(N) - w_A(N \setminus \{j\})\,. \tag{2}$$

The two foregoing extreme core allocations of the assignment game are not, in general, the only ones. In Núñez and Rafels (2003a) the extreme core allocations of the assignment game are proved to coincide with the set of reduced marginal worth vectors. These vectors are inspired by the marginal worth vectors. For each ordering $\theta = (i_1, i_2, \ldots, i_{n-1}, i_n)$, the reduced marginal worth vector $rm_\theta^{w_A}$ is a vector in $\mathbf{R}^n$ where each player receives her marginal contribution to her set of predecessors, and a reduction of the game is performed in each step (Núñez and Rafels, 1998): $(rm_\theta^{w_A})_{i_n} = b_{i_n}^{w_A}$ and, for all $1 \leq k < n$, $(rm_\theta^{w_A})_{i_k} = b_{i_k}^{w_A^{i_n i_{n-1} \cdots i_{k+1}}}$. To complete the definition of these vectors, as in each step only one player leaves the game, it only remains to say that the game $w_A^{i_n}$ is no more than the reduced game *à la* Davis and Maschler on coalition $N \setminus \{i_n\}$ and at the payoff $b_{i_n}^{w_A}$.

In the present paper we will focus on a two–sided market without product differentiation. This particular case is known as the *Horse Market of Böhm–Bawerk* (1891) and is also studied from the viewpoint of game theory in Shapley and Shubik (1972). In this market, each seller has one horse for sale and each buyer wishes to buy one horse and places the same valuation on all the horses available, as they are all alike. The data of the market are thus given: let $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m'}$ be the reservation prices of the sellers and $h_1 \geq h_2 \geq \cdots \geq h_m \geq 0$ the valuations of the buyers. If $h_i < c_j$, no transaction is possible between these two agents, but whenever $h_i \geq c_j$, agents i and j can trade and obtain a joint profit of $h_i - c_j$. Thus, the assignment matrix describing this market is $a_{ij} = \max\{h_i - c_j, 0\}$.

In section 2, given an arbitrary assignment matrix we would like to determine whether it represents a Böhm–Bawerk horse market. Take matrices $A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ to illustrate the problem. The assignment games related to both matrices have the same segment as a core, $C(w_{A_1}) = C(w_{A_2}) = [(1,1,1;0,0,0),(0,0,0;1,1,1)]$. The first matrix represents a well known horse market (in fact a symmetric glove market), taking for instance valuations $h_1 = h_2 = h_3 = 1$ and $c_1 = c_2 = c_3 = 0$. The second matrix does not represent a Böhm–Bawerk horse market. If A_2 were the matrix of such a market, $h_1 - c_1 = 1$, $h_2 - c_1 = 1$ and $h_3 - c_1 = 1$, which implies $h_1 = h_2 = h_3$. But on the other hand, $h_1 - c_2 \leq 0$ and $h_2 - c_2 = 1$ which contradicts $h_1 = h_2$.

The following question arises. Given an arbitrary assignment matrix, how can we recognize, merely by inspecting the matrix entries, if it represents a very particular market like the Böhm–Bawerk horse market? To answer this question we will develop an idea already present in the work of Shapley and Shubik (1972), who point out that a property of the assignment matrix of these particular markets is that *in each* 2×2 *submatrix with nonzero entries, the sums of the diagonals are equal.* This property is not enough to characterize the matrices defining a Böhm–Bawerk horse market, as it is easy to check that matrix A_2 satisfies the above property.

Following this analysis, those Böhm–Bawerk horse market games with the core reduced to only one point are characterized in section 3.

The second objective of the paper, which is addressed in section 4, is to make a cooperative analysis of the Böhm–Bawerk horse market. If we look at this market as a cooperative TU game, what do the classical solutions in this framework recommend for these special market games? We analyze the three main single valued solutions (the Shapley value, the nucleolus and the τ–value) and conclude that they have a strong tendency to recommend the midpoint of the core. In fact, we prove that the nucleolus, which is always a core allocation, and the τ–value, which also belongs to the core of the assignment game (Núñez and Rafels, 2003b), do always coincide with the midpoint of the core segment. This result is not surprising, as there does not seem to be any reason to discriminate one side of the market from the other.

The case of the Shapley value is different, as it generally lies outside the core of the assignment game. Nevertheless, we prove that whenever the Shapley value of a Böhm–Bawerk horse market is a core allocation, it is the midpoint of the core and thus coincides with the two previous solution concepts. All these results capture the idea that without any external information about the bargaining capabilities of the players, the theory of cooperative games predicts mean competitive price equilibrium. As Böhm–Bawerk says, if we only have one buyer and one seller, and the transaction of the good is possible between them, the price of the object will move in a segment. Depending on their bargaining capabilities, the seller may force a price near the highest price or the buyer will force a price approaching the lowest price, but with similar bargaining capabilities the price will be fixed somewhere near the middle price. Therefore, our aim in this paper is to show that, in this model, the middle competitive price can be viewed as a focal point, supported by all the classical solutions in the cooperative game theory.

2 The matrix of a Böhm–Bawerk horse market

In this section we characterize those non negative matrices defining a Böhm–Bawerk horse market. The characterization will be given in terms of all 2×2 submatrices and so first we need to characterize when such a matrix defines a Böhm–Bawerk horse market. Let us first define what we mean by this.

Definition 1 *A matrix A, with set of rows M and set of columns M', defines a Böhm–Bawerk horse market if and only if there exist* $h_1, \ldots, h_m \in$

$\mathbf{R}_+$ *and* $c_1, \ldots, c_{m'} \in \mathbf{R}_+$ *such that* $a_{ij} = \max\{h_i - c_j, 0\}$ *, for all* $i \in M$ *and* $j \in M'$ *.*

Notice that the property of defining such a market is invariant under permutation of rows or columns. Secondly, given an arbitrary assignment matrix, if one side of the market reduces to only one agent then that matrix always represents a Böhm–Bawerk horse market. Thus the simplest case we need to study is that of 2×2 matrices.

When analyzing Böhm–Bawerk horse markets in the case 2×2, a special type of optimal matching will be introduced, which in fact can be defined in the general framework of assignment games, regardless of the cardinality of each side of the market. Let $(M \cup M', w_A)$ be an assignment game; an optimal matching $\mu \in \mathcal{M}^*(A)$ is said to be *singular* when there exists a mixed pair coalition with worth zero and its agents are optimally paired by μ, that is to say, there exists $(i,j) \in \mu$ such that $a_{ij} = 0$.

The following lemma characterizes those 2×2 non negative matrices that correspond to a Böhm–Bawerk horse market. Notice that a non negative 2×2 matrix has either two optimal matchings or only one.

Lemma 2 *A matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbf{R}_+)$ *defines a Böhm–Bawerk horse market if and only if whenever* A *has only one optimal matching, this is singular.*

Proof: If A is a Böhm–Bawerk horse market, either A is positive, and in that case there exist $h \in \mathbf{R}^2_+$ and $c \in \mathbf{R}^2_+$ such that $a_{ij} = h_i - c_j$ for all $i, j \in \{1,2\}$ which implies $a_{11} + a_{22} = a_{12} + a_{21}$ and A has two optimal matchings, or it has some null elements. If A has a null row or column, then trivially if A has only one optimal matching, this one is singular. Assume now that A has only one null element which is a_{i_1, j_1} for some $(i_1, j_1) \in \{1,2\} \times \{1,2\}$. Let us denote by i_2 and j_2 the remaining buyer and seller. Then $h_{i_1} - c_{j_1} \leq 0$ and $a_{i_1 j_1} + a_{i_2 j_2} = h_{i_2} - c_{j_2} \geq h_{i_1} - c_{j_1} + h_{i_2} - c_{j_2} = a_{i_1 j_2} + a_{i_2 j_1}$ and $\mu = \{(i_1, j_1), (i_2, j_2)\}$ is an optimal matching which is singular.

To prove the converse statement assume without loss of generality, that $a_{11} \geq \max\{a_{12}, a_{21}\}$ and define $h_1 = a_{11}$, $h_2 = a_{21}$, $c_1 = 0$ and $c_2 = a_{11} - a_{12}$. If A has two optimal matchings, we have $a_{11} + a_{22} = a_{12} + a_{21}$ and then $h_2 - c_2 = a_{21} - (a_{11} - a_{12}) = a_{22}$. Consequently, $a_{ij} = \max\{h_i - c_j, 0\}$ for all $i, j \in \{1,2\}$.

If A has only one optimal matching and it is singular, this must be $\{(1,1),(2,2)\}$ and $a_{22} = 0$. Then $a_{11} > a_{12} + a_{21}$; taking the same valuations as above $a_{22} = \max\{h_2 - c_2, 0\}$ and A defines a Böhm–Bawerk horse market. $\square$

As a consequence, matrices defining a Böhm–Bawerk horse market are, up to possible permutations of buyers or sellers, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{11} + a_{22} = a_{12} + a_{21}$, but also $\begin{pmatrix} a_{11} & 0 \\ a_{22} & 0 \end{pmatrix}$ or $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}$ with $a_{11} \geq a_{12} + a_{21}$.

The following theorem shows that to check if a given non negative matrix defines a Böhm–Bawerk horse market, we only have to analyze all its 2×2 submatrices.

Theorem 3 *Let $A \in \mathcal{M}_{m \times m'}(\mathbf{R}_+)$. The matrix A defines a Böhm–Bawerk horse market if and only if every 2×2 submatrix defines a Böhm–Bawerk horse market.*

Proof: The "only if" part is straightforward as any submatrix of a Böhm–Bawerk horse market is also a Böhm–Bawerk horse market. To prove the "if" part, let us assume, without loss of generality, that rows and columns have been ordered in such a way that $a_{1j} \geq a_{1j+1}$ for all $j \in \{1, \ldots, m' - 1\}$, $a_{i1} \geq a_{i+11}$ for all $i \in \{1, \ldots, m-1\}$ and, moreover, $a_{11} \geq a_{ij}$ for all $i \in M$ and $j \in M'$. Notice that this can always be achieved.

We first claim that, under the assumption that all 2×2 submatrices define Böhm–Bawerk markets, the above ordering implies that, for all $i \in M$ and $j \in M'$, $a_{ij} \geq a_{ij'}$ for all $j' \geq j$ and $a_{ij} \geq a_{i'j}$ for all $i' \geq i$.

We prove the first inequality of the claim (the second one is proved analogously). Take $j' > j$ and consider the matrix $A' = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}$. As this matrix defines a Böhm–Bawerk market, if $a_{1j} = 0$, then $a_{ij} = 0$ follows from lemma 2. But on the other side, as $a_{1j} \geq a_{1j'}$, we obtain $a_{1j'} = 0$ and since matrix $\begin{pmatrix} a_{11} & a_{1j'} \\ a_{i1} & a_{ij'} \end{pmatrix}$ is a Böhm–Bawerk market, we obtain from lemma 2 that $a_{ij'} = 0$ and thus $a_{ij} \geq a_{ij'}$.

If $a_{1j} > 0$ we will first see that $a_{1j} \geq a_{ij}$. As this is obvious when $a_{ij} = 0$, let us assume $a_{ij} > 0$. Then, by lemma 2, we obtain $a_{11} + a_{ij} = a_{1j} + a_{i1}$, which from $a_{11} \geq a_{i1}$ implies $a_{1j} \geq a_{ij}$.

Now take matrix $A'' = \begin{pmatrix} a_{1j} & a_{1j'} \\ a_{ij} & a_{ij'} \end{pmatrix}$. If $a_{ij'} = 0$, then trivially $a_{ij} \geq a_{ij'}$. If $a_{ij'} > 0$, from lemma 2, $a_{1j} + a_{ij'} = a_{ij} + a_{1j'}$ which, as $a_{1j} \geq a_{1j'}$, implies $a_{ij} \geq a_{ij'}$.

After proving the claim, which implies that whenever $a_{ij} = 0$ then $a_{kl} = 0$ for all $k \geq i$ and $l \geq j$, we define valuations for buyers and sellers which show that A is a Böhm–Bawerk horse market.

Define $h_i = a_{i1}$ for all $i \in M$ and $c_j = a_{11} - a_{1j}$ for all $j \in M'$. If $a_{ij} > 0$, then $A' > 0$ and from lemma 2

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = \max\{a_{ij}, 0\} = a_{ij} .$$

If $a_{ij} = 0$ then, by lemma 2, $a_{11} \geq a_{1j} + a_{i1}$, which means

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = 0 = a_{ij} .$$

□

Our first remark is that, by using the above characterization, it is easy to recognize when a matrix defines a Böhm–Bawerk horse market. For instance, by inspection of all its 2×2 submatrices, we conclude that matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 8 & 10 & 15 & 13 \\ 0 & 0 & 5 & 3 \\ 3 & 5 & 10 & 8 \end{pmatrix}$ defines such a market, while the matrix $A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ analyzed in the introduction is not a Böhm–Bawerk horse market, as the submatrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has only one optimal matching, and it is not a singular one.

More consequences follow from theorem 3. The addition of two matrices defining Böhm–Bawerk horse markets might not be a Böhm–Bawerk horse market, as the following example shows:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

Nevertheless, if we restrict to positive matrices, the property of being a Böhm–Bawerk horse market is preserved by the addition.

Moreover, the class $\mathcal{BB}_{M \cup M'}$ of matrices defining Böhm–Bawerk horse markets is closed by the usual topology in $\mathbf{R}^{m \times m'}$. From theorem 3, it is enough to check that the class of 2×2–Böhm–Bawerk horse markets is closed in $\mathbf{R}^4$. To see this, we only have to decompose the above class as a finite union of subclasses: those matrices with two optimal matchings, those with a null first row, those with a null second column, those with only one null entry which is a_{ij},... etc. It is easy to prove that each one of these subsets is closed.

3 Some remarks about the core

It is already known from Shapley and Shubik (1972) that the core of the Böhm–Bawerk horse market game consists of a segment, in which the buyer–optimal and the seller–optimal core allocations are the extreme points. Moreover, in absence of product differentiation, all transactions take place at the same price. This means that there exists an interval of prices $[\underline{p}, \overline{p}]$ and $(u, v) \in C(w)$ if and only if there exists $p \in [\underline{p}, \overline{p}]$ such that

$$u_i = h_i - p \text{ and } v_j = p - c_j \tag{3}$$

if buyer i and seller j are involved in some transaction, while the remaining agents receive a zero payoff. As happens in an arbitrary assignment game, the core coincides with the set of competitive equilibria. Then, $[\underline{p}, \overline{p}]$ is the set of competitive prices. In order to give an expression of these extreme prices, some notations are fixed.

In the sequel, and until the end of the paper, given a Böhm-Bawerk horse market, we assume without loss of generality that $h_1 \geq h_2 \geq \cdots \geq h_m \geq 0$ and $0, \leq c_1 \leq c_2 \leq \cdots \leq c_{m'}$, and then $a_{ij} \geq a_{i'j'}$ for all $i' \geq i$ and $j' \geq j$. From this it follows that, whenever $a_{ij} = 0$ then $a_{kl} = 0$ for all $k \geq i$ and $l \geq j$.

Then, an optimal matching is $\mu = \{(i,i) \mid i \in \{1, \ldots, r\}\}$ where $r = \min\{m, m'\}$. If $s = \max\{i \in \{1, 2, \ldots, r\} \mid h_i - c_i \geq 0\}$, then agent k, with $k \leq s$ will be said to be an **active player**. In fact $\mu = \{(i,i) \mid i \in \{1, \ldots, s\}\}$ is also an optimal matching as $a_{ii} = 0$ for all $s < i \leq r$. Notice that active buyers or sellers can interchange their partners by μ and we still obtain an optimal matching.

From Moulin (1995), the maximum and minimum competitive prices are

$$\underline{p} = \max\{h_{s+1}, c_s\} \text{ and } \overline{p} = \min\{h_s, c_{s+1}\}, \tag{4}$$

where s denotes the last active agent on each side of the market and we define $h_{m+1} = -\infty$ and $c_{m'+1} = \infty$.

In some cases, these two extreme prices will coincide and then the core reduces to one unique point. This means not only that all transactions take place at the same price, but also that this price is fixed. This happens when $\underline{p} = \overline{p}$, but we would like to recognize this situation just by looking at the corresponding matrix. Notice that, in the more general framework of assignment games, no characterization of those games with core reduced to only one point is known, although some necessary condition can be given.

Given a Böhm–Bawerk horse market game $(M \cup M', w_A)$, with s the last active player on each side of the market, let us consider the matrix

$$A_s = \begin{pmatrix} a_{s\,s} & a_{s\,s+1} \\ a_{s+1\,s} & a_{s+1\,s+1} \end{pmatrix},$$

where $a_{s\,s+1} = a_{s+1\,s+1} = 0$ if $s = m'$ and $a_{s+1\,s} = a_{s+1\,s+1} = 0$ if $s = m$. That means that if $s = m'$ we complete the market with seller $s+1$ with valuation c_{s+1} large enough such that $a_{is+1} = 0$ for all $i \in M$. And if $s = m$ we complete the market with buyer $s+1$ with valuation h_{s+1} small enough such that $a_{s+1j} = 0$ for all $j \in M'$. Just by inspection of this matrix it is possible to determine whether the Böhm–Bawerk horse market game has only one core allocation.

Proposition 4 *Let $(M \cup M', w_A)$ be a Böhm–Bawerk horse market game. Then, $C(w_A)$ has only one point if and only if A_s has two optimal matchings.*

PROOF: Assume $C(w_A)$ has only one point, then $\underline{p} = \overline{p}$. If $\underline{p} = \max\{h_{s+1}, c_s\} = h_{s+1}$, then $\overline{p} = \min\{h_s, c_{s+1}\}$ must be attained at h_s, otherwise we would obtain $h_{s+1} = c_{s+1}$, which contradicts the definition of s. Then $\underline{p} = h_{s+1}$ and $\overline{p} = h_s$ and by assumption $h_s = h_{s+1}$ which implies $a_{s\,s} = a_{s+1\,s}$ and $a_{s\,s+1} = a_{s+1\,s+1}$, and A_s has two optimal matchings.

If $\underline{p} = c_s$ then either $\overline{p} = h_s$ which by the assumption means $a_{ss} = 0$ and A_s is the null matrix, or $\overline{p} = c_{s+1}$. In this second case, as also by the above assumption $c_s = c_{s+1}$, we obtain $a_{s\,s} = a_{s\,s+1}$ and $a_{s+1\,s} = a_{s+1\,s+1}$ and again A_s has two optimal matchings.

Conversely, if A_s has two optimal matchings, several cases will be considered. If $a_{s\,s} = 0$ then $h_s = c_s$ and, from $c_s \leq c_{s+1}$ and $h_s \geq h_{s+1}$ follows $\underline{p} = c_s = h_s = \overline{p}$. If $a_{s\,s} > 0$ and $a_{s+1\,s} = 0$, as $a_{s+1\,s+1} = 0$ by definition of A_s, the existence of two optimal matchings implies $a_{s\,s} = a_{s\,s+1} > 0$ and then $c_s = c_{s+1}$ which, from $h_{s+1} < c_{s+1} = c_s \leq h_s$, leads to $\underline{p} = c_s$ and $\overline{p} = c_{s+1}$. A similar argument is used in the case $a_{s\,s+1} = 0$.

It only remains to be seen what would happen if all entries in A_s except for $a_{s+1\,s+1}$, were positive. But in this case, from the existence of two optimal matchings in A_s follows $h_s - c_s = (h_s - c_{s+1}) + (h_{s+1} - c_s)$ and $h_{s+1} = c_{s+1}$ which contradicts the definition of s. □

It is now easy to see that the Böhm–Bawerk horse market, defined by matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 8 & 10 & 15 & 13 \\ 0 & 0 & 5 & 3 \\ 3 & 5 & 10 & 8 \end{pmatrix}$ in the introduction of the paper, has only one core point, as after reordering the player set we get $\begin{pmatrix} 15 & 13 & 10 & 8 \\ 10 & 8 & 5 & 3 \\ 5 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.
Following the proof of theorem 3, we may define the valuations of this horse market as $h_i = a_{i1}$ for all $i \in M$ and $c_j = a_{11} - a_{1j}$ for all $j \in M'$. Then $s = 3$ and matrix $A_s = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has two optimal matchings.

It is well known that a glove market (i.e. $a_{ij} = c \geq 0$ for all $i \in M$ and all $j \in M'$) with a different number of agents on each side of the market has a core which reduces to only one point. This also follows easily when applying the above proposition to these games.

4 Single–valued cooperative solutions

For games with a non empty core, the nucleolus is always a core allocation, while the Shapley value, which is the average of the marginal worth vectors, may lie outside the core. In fact, in the case of the assignment game, and also in Böhm–Bawerk horse market games, the Shapley value often produces an allocation that is not in the core.

The τ–value of an assignment game $(M \cup M', w_A)$ is the midpoint between the buyers–optimal and the sellers–optimal core allocations, $\tau(w_A) = \frac{1}{2}(\overline{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \overline{v})$ (Núñez and Rafels, 2003b) and therefore is always a core allocation.

From equation (3), and the worth of $\underline{p}$ and $\overline{p}$ in (4), it follows easily that the buyers–optimal core allocation is related to the minimum competitive price:

$$\begin{array}{ll} \overline{u}_i = h_i - \underline{p} & \text{for } i \text{ an active buyer} \\ \underline{v}_j = \underline{p} - c_j & \text{for } j \text{ an active seller}\,, \end{array} \tag{5}$$

while the sellers–optimal core allocation is related to the maximum competitive price:

$$\begin{array}{ll} \underline{u}_i = h_i - \overline{p} & \text{for } i \text{ an active buyer} \\ \overline{v}_j = \overline{p} - c_j & \text{for } j \text{ an active seller}\,. \end{array} \tag{6}$$

Now it is straightforward to obtain an expression for the τ–value of the Böhm–Bawerk horse market game. Being a core allocation, $\tau_i(w_A) = 0$ for all $i > s$. For all $i \in M$, $i \leq s$,

$$\tau_i(w_A) \;\; = \;\; \frac{1}{2}\overline{u}_i + \frac{1}{2}\underline{u}_i = h_i - \frac{1}{2}(\max\{h_{s+1}, c_s\} + \min\{h_s, c_{s+1}\})$$

and for all $i \in M'$, $i \leq s$,

$$\tau_i(w_A) \;\; = \;\; \frac{1}{2}\underline{v}_i + \frac{1}{2}\overline{v}_i = \frac{1}{2}(\max\{h_{s+1}, c_s\} + \min\{h_s, c_{s+1}\}) - c_i\,,$$

taking into account the above convention $h_{m+1} = -\infty$ and $c_{m'+1} = \infty$.

The price associated to the τ–value is thus the middle competitive price.

We will now compute the nucleolus of a Böhm–Bawerk horse market game. To this end, let us first recall the definition of the nucleolus $\eta(w_A)$ of a cooperative game (N, w_A) due to Schmeidler (1969). For all imputation x of (N, w_A), and given any coalition $S \subseteq N$, the excess of coalition S with respect to x is $e(S, x) = w_A(S) - x(S)$. Now, for each imputation x, let us define the vector $\theta(x) \in \mathbf{R}^{2^n-2}$ of excesses of all non trivial coalitions at x, in decreasing order. That is to say, for all $k \in \{1, \ldots, 2^n - 2\}$, $\theta(x)_k = e(S_k, x)$, where $\{S_1, \ldots, S_k, \ldots, S_{2^n-2}\}$ is the set of all non empty coalitions in N different from N, and $e(S_k, x) \geq e(S_{k+1}, x)$.

Then the nucleolus of the game (N, w_A) is the imputation $\eta(w_A)$ which minimizes $\theta(x)$ with the lexicographic order, over the set of imputations: $\theta(\eta(w_A)) \leq_{Lex} \theta(x)$ for all $x \in I(w_A)$. It is easy to see that whenever the game has a non empty core, the nucleolus belongs to the core.

In the assignment game, only individual player coalitions and all mixed–pair coalitions play a role in the computation of the nucleolus, and Solymosi and Raghavan (1994) give an algorithm to locate this solution.

Granot and Granot (1992) characterize the nucleolus of a particular assignment game where there are several optimal matchings and the graph (whose nodes are all mixed–pair coalitions appearing in some optimal matching and two nodes are connected if they have a player in common) contains a spanning tree. They prove that in that particular case only one–player coalitions play a role when computing the nucleolus. Moreover, in the above case, the core allocations are determined by a single parameter. The same authors note that in the Böhm–Bawerk horse market game, which also has a line segment as its core, one–player coalitions are not enough to compute the nucleolus. To justify this, they take the example of horse market game given in Böhm–Bawerk (1923) and also analyzed in Shapley and Shubik (1972); they show that the excesses of some mixed–pair coalitions are to be considered and moreover, for this numerical example, that the nucleolus is the midpoint of the core.

In this section we determine the nucleolus of any Böhm–Bawerk horse market game. Lemmas 5 and 6 tell us which are the essential coalitions for the calculus of this nucleolus, and this set of coalitions does not always coincide with that of the example analyzed in Granot and Granot (1992), although the nucleolus is proved to be always the midpoint of the core. In fact, lemma 6 shows that only four coalitions matter when computing the nucleolus: the individual coalitions formed by the last active buyer or the last active seller, and those mixed pair coalitions formed by the last active agent on one side of the market and the first non active one on the opposite side (if there is one).

To obtain the nucleolus of a Böhm–Bawerk horse market game, we must analyze the excess of each coalition S with respect to any core allocation z, $e(S, z) = w_A(S) - z(S)$. Notice first that if $S \subseteq M$ and $T \subseteq M'$, then the restricted game $w_{A|S \cup T}$ is also a Böhm–Bawerk horse market game. If

$|S| = |T|$ and all agents in these two coalitions are active, then $e(S \cup T, z) = 0$ for all $z \in C(w_A)$. This happens because if A is the matrix defining the above horse market, then in any square submatrix obtained from active players, any possible matching is optimal. When necessary, we will denote the ith seller as i', to avoid confusion with the ith buyer. Recall also that s (or s') denotes the last active agent on each side of the market, that is to say, $s = \max\{i \in \{1, \ldots, r\} \mid h_i - c_i \geq 0\}$, where $r = \min\{m, m'\}$ and, by a convention made above, $h_1 \geq h_2 \geq \cdots \geq h_m \geq 0$ and $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m'}$.

Lemma 5 *Let* $(M \cup M', w_A)$ *be a Böhm–Bawerk horse market game and* $z \in C(w_A)$. *Then*

1. *For all* $i \in M$ *active,* $e(\{i\}, z) \leq e(\{s\}, z)$.
2. *For all* $i \in M'$ *active,* $e(\{i'\}, z) \leq e(\{s'\}, z)$.
3. *If* $s' < m'$, *for all* $i \in M$ *active,* $e(\{i, s'+1\}, z) \leq e(\{s, s'+1\}, z)$.
4. *If* $s < m$, *for all* $i \in M'$ *active,* $e(\{s+1, i'\}, z) \leq e(\{s+1, s'\}, z)$.

PROOF: Let $z = (x, y)$ and take p_z the competitive price corresponding to this core element, that is $x_i = h_i - p_z$ for all $i \in M$ and active and $y_j = p_z - c_j$ for all $j \in M'$ and active. Then,

$$e(\{i\}, z) = -x_i = -(h_i - p_z) \leq -(h_s - p_z) = -x_s = e(\{s\}, z)$$

and

$$e(\{i'\}, z) = -y_i = -(p_z - c_i) \leq -(p_z - c_s) = -y_s = e(\{s'\}, z),$$

which proves statements 1 and 2.

Assume now $s' < m'$ and take $i \in M$ active. Then,

$$e(\{i, s'+1\}, z) = \max\{h_i - c_{s+1}, 0\} - x_i = \max\{h_i - c_{s+1} - x_i, -x_i\}.$$

If $h_i < c_{s+1}$, then also $h_s - c_{s+1} < 0$ and we have

$$e(\{i, s'+1\}, z) = -x_i \leq -x_s = e(\{s, s'+1\}, z).$$

If $h_i - c_{s+1} \geq 0$ and $h_s - c_{s+1} \geq 0$, then

$$\begin{aligned} e(\{i, s'+1\}, z) &= h_i - c_{s+1} - x_i = h_i - c_{s+1} - (h_i - p_z) = -c_{s+1} + p_z = \\ &= h_s - c_{s+1} - (h_s - p_z) = h_s - c_{s+1} - x_s = e(\{s, s'+1\}, z). \end{aligned}$$

If $h_i - c_{s+1} \geq 0$ but $h_s - c_{s+1} < 0$,

$$\begin{aligned} e(\{i, s'+1\}, z) &= h_i - c_{s+1} - x_i = h_i - c_{s+1} - (h_i - p_z) = -c_{s+1} + p_z < \\ &< -h_s + p_z = -x_s = e(\{s, s'+1\}, z). \end{aligned}$$

This proves statement 3, while statement 4 is proved analogously. $\square$

Lemma 6 *Let $(M \cup M', w_A)$ be a Böhm–Bawerk horse market game. For all $S \subseteq M$, $T \subseteq M'$ such that $e(S \cup T, z)$ does not vanish identically in $C(w_A)$, if $z \in C(w_A)$, then*

$$e(S \cup T, z) \leq \max\{e(\{s\}, z), e(\{s'\}, z), e(\{s, s'+1\}, z), e(\{s+1, s'\}, z)\},$$

where the excess $e(\{s, s'+1\}, z)$ ($e(\{s+1, s'\}, z)$) is only considered if $s' < m'$ ($s < m$).

Proof: Notice first that if S and T have the same number of active players (which includes the case where none of them has active players), then $e(S \cup T, z) = 0$ for all $z \in C(w_A)$. Assume then that S has more active players than T. Let $S = \{i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_{k+l}\}$, $l \geq 0$, where $i_1, \ldots, i_k$ are active in w_A and $i_1 \leq i_2 \leq \cdots \leq i_{k+l}$, and $T = \{j_1, j_2, \ldots, j_{k-r}, j_{k-r+1}, \ldots, j_{k-r+l'}\}$, $l' \geq 0$, where $j_1, \ldots, j_{k-r}$ are active in w_A, $0 < r \leq k$, and $j_1 \leq j_2 \leq \cdots \leq j_{k-r+l'}$. Let $k_0 = \min\{k, k-r+l'\}$. Then $\mu = \{(i_t, j_t) \mid 1 \leq t \leq k_0\}$ is an optimal matching in $S \cup T$ and

$$w_A(S \cup T) = \sum_{t=1}^{k_0} a_{i_t j_t} = \sum_{t=1}^{k-r} (h_{i_t} - c_{j_t}) + \sum_{t=k-r+1}^{k_0} \max\{h_{i_t} - c_{j_t}, 0\}, \quad (7)$$

where we assume that the summation over an empty set of indices is zero.

Let $z = (x, y) \in C(w_A)$, then

$$e(S \cup T, z) = \sum_{t=k-r+1}^{k_0} (\max\{h_{i_t} - c_{j_t}, 0\} - x_{i_t}) + \sum_{t=k_0+1}^{k} -x_{i_t} \quad (8)$$

because, for all $1 \leq t \leq k-r$, both i_t and j_t are active in w_A and then $x_{i_t} + y_{j_t} = a_{i_t j_t}$, as there exists an optimal matching μ' in w_A such that $(i_t, j_t) \in \mu'$ and z is a core allocation. Moreover, $y_{j_t} = 0$ for $k-r+1 \leq t \leq k_0$, as j_t is non active in w_A and non active players receive zero payoff in any core allocation.

Notice that if $l' = 0$, then $k_0 = k - r$ and the set of indices of the first summation in (8) is empty. If $l' > 0$ and $k_0 = k$, then the set of indices of the second summation in (8) is empty.

As all summands in (8) are non positive, if $l' = 0$ or $l' > 0$ but $k_0 < k$, then

$$e(S \cup T, z) \leq -x_k \leq -x_s = e(\{s\}, z),$$

where the last inequality follows from lemma 5.

If $l' > 0$ and $k_0 = k$ but there exists $t^* \in \{k-r+1, \ldots k_0\}$ such that $h_{i_{t^*}} - c_{j_{t^*}} \leq 0$, then

$$e(S \cup T, z) \leq -x_{i_{t^*}} \leq -x_s = e(\{s\}, z),$$

where the last inequality follows from lemma 5.

Otherwise, $h_{i_t} - c_{j_t} > 0$ for all $t \in \{k-r+1, \dots k_0\}$, and as j_t is non active, then $s < j_t \leq m'$ and $c_{s+1} \leq c_{j_t}$, which implies that for all $t \in \{k-r+1, \dots k_0\}$ we obtain

$$e(S \cup T, z) \leq h_{i_t} - c_{j_t} - x_{i_t} \leq h_{i_t} - c_{s+1} - x_{i_t} = e(\{i_t, s'+1\}, z) \leq e(\{s, s'+1\}, z)\,,$$

where the last inequality follows from part 3 of lemma 5.

The proof of the case where T has more active players than S is analogous and left to the reader. □

With the help of the above technical lemmas, we can now compute the nucleolus of the Böhm-Bawerk horse market game.

Proposition 7 *Let $(M \cup M', w_A)$ be a Böhm–Bawerk horse market game and $\eta(w_A)$ the nucleolus of w_A, then*

$$\eta(w_A) = \frac{1}{2}(\overline{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \overline{v})\,.$$

Proof: As the horse market game has a non empty core, the nucleolus will be a core allocation, which means that $\eta(w_A) = \lambda(\overline{u}, \underline{v}) + (1-\lambda)(\underline{u}, \overline{v})$ for some $\lambda \in [0,1]$. We will distinguish four cases to prove $\lambda = \frac{1}{2}$ for all Böhm–Bawerk horse market game.

Case 1: ($m > s$ with $h_{s+1} - c_s > 0$) and ($m' > s'$ with $h_s - c_{s+1} > 0$).

As $h_s > c_{s+1}$, it is straightforward to see that

$$e(\{s\}, z) = -x_s < h_s - c_{s+1} - x_s = e(\{s, s'+1\}, z)$$

for all $z \in C(w_A)$. Similarly, taking into account $h_{s+1} > c_s$, we obtain that

$$e(\{s'\}, z) = -y_s < h_{s+1} - c_s - y_s = e(\{s+1, s'\}, z)$$

for all $z \in C(w_A)$.

Then, from the definition of the nucleolus and lemma 6, we deduce $e(\{s, s'+1\}, \eta(w_A)) = e(\{s+1, s'\}, \eta(w_A))$. From (5), (6) and (4), and taking into account the assumptions of case 1, we can write

$$\begin{aligned} x_s &= h_s - (\lambda \underline{p} + (1-\lambda)\overline{p}) = h_s - (\lambda h_{s+1} + (1-\lambda)c_{s+1}) \\ y_s &= (\lambda h_{s+1} + (1-\lambda)c_{s+1}) - c_s\,, \end{aligned}$$

and the above equality between excesses leads to $\lambda(h_{s+1} - c_{s+1}) = (1-\lambda)(h_{s+1} - c_{s+1})$ which is equivalent to $(1-2\lambda)(h_{s+1} - c_{s+1}) = 0$. Then, either $\lambda = \frac{1}{2}$ or $h_{s+1} = c_{s+1}$ but this last case contradicts s being the last active agent on each side of the market, and the claim of the proposition follows trivially.

Case 2: ($m = s$ or $m > s$ with $h_{s+1} - c_s \leq 0$) and ($m' = s'$ or $m' > s'$ with $h_s - c_{s+1} \leq 0$).

If $s' < m'$, as $h_s \leq c_{s+1}$, we obtain $e(\{s, s'+1\}, z) = e(\{s\}, z)$ for all $z \in C(w_A)$. Similarly, if $s < m$, as $h_{s+1} \leq c_s$, $e(\{s+1, s'\}, z) = e(\{s'\}, z)$ for all $z \in C(w_A)$.

Then, from the definition of the nucleolus and lemma 6, we know $e(\{s\}, \eta(w_A)) = e(\{s'\}, \eta(w_A))$. From (5), (6) and (4), and taking into account the assumptions of case 2, we can write

$$\begin{aligned} x_s &= h_s - (\lambda c_s + (1-\lambda)h_s) \\ y_s &= (\lambda c_s + (1-\lambda)h_s) - c_s\,, \end{aligned}$$

and the above equality between excesses leads to $(1-2\lambda)(h_s - c_s) = 0$. Then, either $\lambda = \frac{1}{2}$ or $h_s = c_s$. But in this second case, by proposition 4, the core reduces to only one point and the claim also follows trivially.

Case 3: ($m > s$ with $h_{s+1} - c_s > 0$) and ($m' = s'$ or $m' > s'$ with $h_s - c_{s+1} \leq 0$).

If $s' < m'$, as $h_s \leq c_{s+1}$, we obtain $e(\{s, s'+1\}, z) = e(\{s\}, z)$. Similarly, by using $c_s < h_{s+1}$, we obtain $e(\{s'\}, z) < e(\{s+1, s'\}, z)$ for all $z \in C(w_A)$.

Then, from the definition of the nucleolus and lemma 6, $e(\{s\}, \eta(w_A)) = e(\{s+1, s'\}, \eta(w_A))$. From (5), (6) and (4), and taking into account the assumptions of case 3, we can write

$$\begin{aligned} x_s &= h_s - (\lambda h_{s+1} + (1-\lambda)h_s) \\ y_s &= (\lambda h_{s+1} + (1-\lambda)h_s) - c_s\,, \end{aligned}$$

and the above equality between excesses is equivalent to $(1-2\lambda)(h_s - h_{s+1}) = 0$. Then, either $\lambda = \frac{1}{2}$ or $h_{s+1} = h_s$, but in this second case, from proposition 4, the core reduces to only one point and the claim also follows trivially.

Case 4: ($m = s$ or $m > s$ with $h_{s+1} - c_s \leq 0$) and ($m' > s$ with $h_s - c_{s+1} > 0$).

If $s < m$, as $h_{s+1} \leq c_s$, we obtain $e(\{s+1, s'\}, z) = e(\{s'\}, z)$ for all $z \in C(w_A)$. Similarly, from $c_{s+1} < h_s$, $e(\{s\}, z) < e(\{s, s'+1\}, z)$ for all $z \in C(w_A)$.

Then, from the definition of the nucleolus and lemma 6, $e(\{s'\}, \eta(w_A)) = e(\{s, s'+1\}, \eta(w_A))$. From (5), (6) and (4), and taking into account the assumptions of case 4, we can write

$$\begin{aligned} x_s &= h_s - (\lambda c_s + (1-\lambda)c_{s+1}) \\ y_s &= (\lambda c_s + (1-\lambda)c_{s+1}) - c_s\,, \end{aligned}$$

and the above equality between excesses is equivalent to $(1-2\lambda)(c_{s+1}-c_s) = 0$. Then, either $\lambda = \frac{1}{2}$ or $c_{s+1} = c_s$. But in the latter case, by proposition 4, the core reduces to only one point and the claim also follows trivially. $\square$

To sum up, the following theorem has been proved.

Theorem 8 *In a Böhm–Bawerk horse market game $(M \cup M', w_A)$, the τ–value and the nucleolus coincide with the midpoint of the core.*

To finish this analysis of single–valued solutions for the Böhm–Bawerk horse market game, let us consider again the Shapley value. We have already mentioned that the Shapley value of this game usually lies outside the core, but in those cases where it is a core allocation, can it select a point other than the midpoint of the core?

A particular case of the Böhm–Bawerk horse market is the glove market (Shapley, 1959), where the valuations of all buyers coincide, as do the valuations of all sellers. In that case $a_{ij} = c$ for all $i \in M$ and $j \in M'$. It is well known that, when $m = m'$, the core is the segment with extreme points $(\overline{u}, \underline{v})$, where $\overline{u}_i = c$ for all $i \in M$ and $\underline{v}_j = 0$ for all $j \in M'$, and $(\underline{u}, \overline{v})$, where $\underline{u}_i = 0$ for all $i \in M$ and $\overline{v}_j = c$ for all $j \in M'$. In that case the Shapley value is also the midpoint of the core: $\Phi(w_A)_k = \frac{c}{2}$ for all $k \in M \cup M'$.

On the other hand, in a glove market with different numbers of buyers and sellers the core reduces to one point where the payoff to each agent on the short side of the market is c and the payoff to each agent on the opposite side is zero. In that case, if $c > 0$, the Shapley value is not a core allocation, as all agents on the large side of the market have positive marginal contribution to some coalitions.

We will now see that the square glove markets are the only Böhm–Bawerk horse markets where the Shapley value belongs to the core.

Proposition 9 *Let $(M \cup M', w_A)$ be a Böhm–Bawerk horse market and $\Phi(w_A)$ its Shapley value. Then*

$$\Phi(w_A) \in C(w_A) \Leftrightarrow w_A \textit{ is a square glove market}$$

and in that case $\Phi(w_A) = \frac{1}{2}(\overline{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \overline{v})$.

Proof: If $(M \cup M', w_A)$ is a Böhm–Bawerk horse market such that $\Phi(w_A) \in C(w_A)$, then non active players must receive a zero payoff. This implies that rows and columns of A corresponding to non active players must be null. Otherwise, if $j \in M'$ is non active and $h_i - c_j > 0$ for some $i \in M$, then $w_A(i,j) - w_A(j) = h_i - c_j > 0$ and $\Phi_j(w_A) > 0$.

We can then only consider those Böhm–Bawerk horse markets where all players are active, $m = m' = s$. Let us now decompose A in the following way: $A = A_1 + A_2$ where

$$A_1 = \begin{pmatrix} (h_1 - h_s) - (c_1 - c_s) & (h_1 - h_s) - (c_2 - c_s) & \cdots & h_1 - h_s \\ (h_2 - h_s) - (c_1 - c_s) & (h_2 - h_s) - (c_2 - c_s) & \cdots & h_2 - h_s \\ \cdots & \cdots & \cdots & \cdots \\ c_s - c_1 & c_s - c_2 & \cdots & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} h_s - c_s & h_s - c_s & \cdots & h_s - c_s \\ h_s - c_s & h_s - c_s & \cdots & h_s - c_s \\ \cdots & \cdots & \cdots & \cdots \\ h_s - c_s & h_s - c_s & \cdots & h_s - c_s \end{pmatrix}$$

Notice that, as for every subset of buyers and sellers the corresponding submatrices of A_1 and A_2 have at least one optimal matching in common, $w_A = w_{A_1} + w_{A_2}$. Then, by additivity of the Shapley value, $\Phi(w_A) = \Phi(w_{A_1}) + \Phi(w_{A_2})$.

Moreover, the core also behaves additively for this decomposition. The inclusion $C(w_{A_1}) + C(w_{A_2}) \subseteq C(w_A)$ always holds. To prove the other inclusion, notice first that A_1 is a Böhm–Bawerk horse market and, following the proof of theorem 3, its valuations can be defined as $h'_i = (h_i - h_s) - (c_1 - c_s)$ for all $i \in M$ and $c'_j = c_j - c_1$ for all $j \in M'$. On the other hand, from proposition 4, $C(w_{A_1})$ reduces to only one point which is proved to be (u', v') where $u'_i = h_i - h_s$ for all $i \in M$ and $v'_j = c_s - c_j$ for all $j \in M'$. Notice also that, as the minimum and maximum competitive prices for the original market A are $\underline{p} = c_s$ and $\overline{p} = h_s$, from equations (3) we obtain $\underline{u}_i = h_i - h_s$ and $\underline{v}_j = c_s - c_j$, and thus the vector (u', v') coincides with the vector of minimum core payoffs in $C(w_A)$, which is $(\underline{u}, \underline{v})$.

Now, for all $(u, v) \in C(w_A)$ define (u'', v'') in the following way:

$$u''_i = u_i - (h_i - h_s) \quad \text{for all } i \in M \text{ and } v''_j = v_j - (c_s - c_j) \quad \text{for all } j \in M'.$$

Notice that $u''_i \geq 0$ as $u_i \geq \underline{u}_i$, $v''_j \geq 0$ as $v_j \geq \underline{v}_j$, and moreover $u''_i + v''_j = h_s - c_s$ for all $i \in M$ and all $j \in M'$. Then $(u, v) = (u', v') + (u'', v'')$ where $(u', v') \in C(w_{A_1})$ and $(u'', v'') \in C(w_{A_2})$ which proves that $C(w_A) \subseteq C(w_{A_1}) + C(w_{A_2})$.

As $(M \cup M', w_{A_2})$ is a glove market with as many buyers as sellers, $\Phi(w_{A_2}) \in C(w_{A_2})$ and moreover $\Phi_k(w_{A_2}) = \frac{1}{2}(h_s - c_s)$ for all $k \in M \cup M'$.

By the additivity of the core follows $\Phi(w_{A_1}) \in C(w_{A_1})$. If we denote by a'_{ij} the entries of matrix A_1, as $a'_{ss} = 0$, it holds $\Phi_s(w_{A_1}) = \Phi_{s'}(w_{A_1}) = 0$ where s' denotes the s–th seller, to be distinguished from the s–th buyer. But this implies that the s–th row and column of A_1 must be null, otherwise, if there exists $i \in M$, $i \neq s$, such that $h_i - h_s > 0$, then the marginal contribution $w_{A_1}(i, s') - w_{A_1}(i)$ would be positive, in contradiction with $\Phi_{s'}(w_{A_1}) = 0$. Something similar happens if there exists $j \in M'$, $j \neq s'$ and $c_s - c_j > 0$.

Thus, $h_i = h_s$ for all $i \in M \setminus \{s\}$ and $c_s = c_j$ for all $j \in M' \setminus \{s'\}$ which implies $A_1 = 0$, $w_A = w_{A_2}$ and, consequently, w_A is a square glove market and

$$\Phi(w_A) = \Phi(w_{A_2}) = (\frac{h_s - c_s}{2}, \ldots \frac{h_s - c_s}{2}; \frac{h_s - c_s}{2}, \ldots, \frac{h_s - c_s}{2}).$$

$\square$

From the above proposition, whenever the Shapley value lies in the core of the Böhm–Bawerk horse market, it coincides with all the single–valued solutions analyzed in the previous section.

We can define a Shapley like solution α for any cooperative TU game taking the average of the reduced marginal worth vectors, that is to say, $\alpha(w_A) = \frac{1}{n!} \sum_{\theta \in \mathcal{S}_n} rm_\theta^{w_A}$, where the summation is taken over the set $\mathcal{S}_n$ of all possible permutations over the player set $N = M \cup M'$. See the introduction of the present paper or Núñez and Rafels (1998) for the definition of the reduced marginal worth vectors. In the case of assignment games, this solution α will always lie in the core, as the reduced marginal worth vectors are the extreme core allocations of the assignment game (Núñez and Rafels, 2003a). Thus, in the case of the Böhm–Bawerk horse market game, each reduced marginal worth vector must coincide either with the buyers–optimal or with the sellers–optimal core allocation. It is then easy to prove that the average of the reduced marginal worth vectors also coincides with the midpoint of the core, reinforcing once more the role of this point as a focal cooperative solution for this particular markets (we leave this proof to the reader).

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