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The Aggregate Claims Distribution of a Life Insurance Portfolio with a Pairwise Positive Dependence Structure

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Abstract

The individual life model has always been considered as the one closest to the real situation of the total claims of a life insurance portfolio. It only makes the "nearly inevitable assumption" of independence of the lifelenghts of insured persons in the portfolio. Many clinical studies, however, have demonstrated positive dependence of paired lives such as husband and wife. In our opinion, it won't be unrealistic expecting a considerable number of married couples in any life insurance portfolio (e.g. life insurance contracts formalized at the time of signing a mortatge) and these dependences materially increase the values for the stop-loss premiums associated to the aggregate claims of the portfolio. Since the stop-loss order is the order followed by any risk averse decison maker, the simplifying hypothesis of independence constitute a real financial danger for the company, in the sense that most of their decisions are based on the aggregated claims distribution. In this paper, we will determine approximations for the distribution of the aggregate claims of a life insurance portfolio with some married couples and we will describe how to make safe decisions when we don't know exactly the dependence structure between the risks in each couple. Results in this paper are partly based on results in Dhaene and Goovaerts (1997).

Resum

El model individual de vida ha estat considerat com el més realista a l'hora de modelitzar la sinistralitat total d'una cartera d'assegurances de vida. Aquest model, només estableix com a hipòtesi inicial la independència entre la mortalitat dels assegurats en una mateixa cartera. Alguns estudis clínics han demostrat, no obstant, l'existència de dependència positiva entre la mortalitat d'alguns parells d'individus. Un cas clar n'és la mortalitat entre els cònjuges d'una mateixa parella. Creiem que en qualsevol cartera d'assegurances de vida podríem trobar un considerable nombre de cònjuges assegurats (per exemple, aquells que contracten una assegurança de vida a l'hora de formalitzar una hipoteca). L'existència de dependències de signe positiu incrementa el valor de les primes stop-loss associades a la distribució de cost total. L'ordre stop-loss és l'ordre que estableix qualsevol decisor advers al risc i, així, quan establim en la cartera la simplificativa hipòtesi de independència estem incorrent en un risc financer real ja que, la majoria de les decisions que es prenen sobre aquesta cartera, es basen en la distribució del cost total. En aquest article, determinen aproximacions per a la distribució del cost total d'una cartera d'assegurances de vida amb varies parelles assegurades i argumentem quina és l'estratègia a seguir per tal de no infravalorar el risc de la cartera quan l'estructura de dependència entre els riscos de cada parella no és del tot coneguda. El resultats d'aquest article es basen, en part, en resultats de Dhaene i Goovaerts (1997).

Key words: Individual life model, aggregate claims distribution, De Pril's recursion (1986), stop-loss order, dependent lifetimes, comonotonic risks.

1 Introduction

Since the beginning of the 1990's, several papers have treated dependence between risks in the actuarial field. Within the framework of the individual life model, simple expressions for computing the riskiest and safest distribution for the aggregate claims of a life insurance portfolio with multivariate dependencies are derived in Dhaene and Goovaerts (1997) and Hu and Wu (1999), respectively. The riskiest one follows by assuming that the multivariate distribution corresponding to the dependent risks in the portfolio is given by the Fréchet upper bound (in this case, the risks are said to be mutually comonotonic). On the other hand, the safest one corresponds to the case of dependent risks with the Fréchet lower bound as multivariate distribution when the conditions assuring that the Fréchet lower bound is a proper distribution function are fulfilled (in this case, the risks are said to be mutually exclusive). Hence, in any life insurance portfolio, we can consider all kind of dependencies and easily compute the riskiest and safest distributions in the stop-loss order sense. These bound distributions will help us to give an idea of the degree of underestimation (overestimation) of the real risk of a portfolio with positively (negatively) dependent risks when computing the distribution of the aggregate claims under the traditional hypothesis of independence. Unfortunately, they turn out to be useless as a measure of the risk of the portfolio since the dependencies between the individual risks will be, in most real situations, nearest to the independence than to these extreme dependence relations.

In this paper, we derive results concerning the aggregate claims distribution of a life insurance portfolio with bivariate intermediate positive dependence relations. More precisely, we assume that the portfolio contains a number of married couples with a positive dependence structure. This hypothesis is based on the fact that the husband and wife are more or less exposed to the same risks since they share a common way of life, go together away and, as the saying goes, "birds of a feather flock together". Moreover, from the medical point of view, several clinical studies put the "broken heart syndrome" in a prominent position; the latter may cause an increase of the mortality rate after the death of one's spouse (using a data set consisting of 4.486 55-year-old widowers, Parkers, Benjamin and Fitzgerald (1969) showed that there is a 40% increase in mortality among the widowers during the first few months after the death of their wives; see also Jagger and Sutton (1991)). There is thus strong empirical evidence that supports the dependence of mortality of pairs of individuals, specially, if we are just considering a short reference period for computing the aggregate claims distribution (e.g. one year).

The paper is organized as follows. In Section 2, we describe the model and we extract some results from the actuarial literature which we need for later sections. We prove, in Section 3, that the bivariate probabilities associated to each couple under any intermediate positive dependence hypothesis about their mortality can always be written as a convex linear combination between the independent and the comonotonic ones. Considering such form for the bivariate distributions, and assuming that the bivariate dependence relations are completely known, the aggregate claims distribution of the portfolio can be then easily computed. In case the dependence structure between the members of each couple is not exactly known (as will occur in practice), safe approximations for the aggregate claims distribution follow from results in Section 4. Some numerical results are summarized in Section 5.

2 A pairwise positive dependence structure in the individual life model

Consider the individual life model where the total claims of the portfolio during a certain reference period (e.g. one year) is given by

$$S = \sum_{i=1}^{n} X_i, \tag{1}$$

with X_i having a given two-point distribution in 0 and $\alpha_i > 0$:

$$\Pr(X_i = 0) = p_i \qquad \Pr(X_i = \alpha_i) = q_i = 1 - p_i. \tag{2}$$

The amount α_i is due if the policy holder i ($i = 1, \cdots, n$) dies during the reference period. Hence, the aggregate claim of the portfolio is the sum of all amounts payable during the reference period. Usually, it is assumed that the risks X_i are mutually independent because models without this restriction turn out to be less manageable. In the sequel we will derive results concerning the aggregate claims, S, if the assumption of mutually independence is relaxed. More precisely, we will assume that the portfolio contains a number m ($m \leq n/2$) of married couples with a positive dependence structure. Following Dhaene and Goovaerts (1996) nomenclature, we will rearrange and rewrite (1) as

$$S = \sum_{i=1}^{m} (X_{2i-1} + X_{2i}) + \sum_{i=2m+1}^{n} X_i$$

with m ($m \leq n/2$) the number of coupled risks. For any i and j $(i, j = 1, \cdots, n;\ i \neq j)$, we will assume that risks X_i and X_j are independent risks except if they are on the set $(X_{2k-1}, X_{2k})\,,\ k = 1, \cdots, m.$

In this case, the distribution of aggregate claims is no longer uniquely determined by the survival probabilities p_i of the individual risks. Intuitively, it is clear that the riskiness of the aggregate claims, S, will be strongly dependent on the way of dependency between the mortality of members of couples. This fact has been proved in terms of stop-loss order.

Definition 1. *A risk S_1 is said to precede a risk S_2 in stop-loss order (written $S_1 \leq_{sl} S_2$), or also S_1 is less risky than S_2, if their stop-loss premiums are ordered uniformly:*

$$E\left(S_1 - d\right)_+ \leq E\left(S_2 - d\right)_+$$

for all retentions $d \geq 0$.

Let $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ be two multivariate risks with identically marginal distributions for the individual risks given by (2) and with the pairwise dependency structure as defined above. Denote their respective sums by

$$S_1 = \sum_{i=1}^{m} (X_{2i-1} + X_{2i}) + \sum_{i=2m+1}^{n} X_i \tag{3}$$

and

$$S_2 = \sum_{i=1}^{m} (Y_{2i-1} + Y_{2i}) + \sum_{i=2m+1}^{n} Y_i. \tag{4}$$

As $X_i \stackrel{d}{=} Y_i,\ i = 1, \cdots, n$, where the symbol $\stackrel{d}{=}$ is used to indicate equality in distribution, the only difference between both distributions will be given by the bivariate dependency relations. As pointed out by Dhaene and Goovaerts (1996), finding a partial order $\leq_{ord}$ between bivariate distributed risks such that

$$(X_{2k-1}, X_{2k}) \leq_{ord} (Y_{2k-1}, Y_{2k})\,, \quad k = 1, \cdots, m \tag{5}$$

implies

$$S_1 \leq_{sl} S_2 \tag{6}$$

can be restricted to finding a partial ordering $\leq_{ord}$ between bivariate risks (X_1, X_2) and (Y_1, Y_2) with $X_i \stackrel{d}{=} Y_i,\ i = 1, 2$, from which the following property holds:

$$(X_1, X_2) \leq_{ord} (Y_1, Y_2) \tag{7}$$

implies

$$X_1 + X_2 \leq_{sl} Y_1 + Y_2. \tag{8}$$

As is well known, the stop-loss order is preserved under the convolution of independent risks, see e.g. Kaas et al. (1994). Hence, an ordering $\leq_{ord}$ from which (7) implies (8) will immediately lead to a solution of the problem described by (5) and (6).

Let us denote by $R(p_1, p_2; \alpha_1, \alpha_2) \equiv R_2$ the class of all bivariate distributed risks with two-point marginal distributions defined by (2). The following theorem give bounds for the riskiness of elements of R_2 and can be found in Dhaene and Goovaerts (1997).

Theorem 1. *Let (X_1, X_2) and (Y_1, Y_2) be two elements of R_2 and let r denote the Pearson's correlation coefficient. Then we have that*

$$r(X_1, X_2) \leq r(Y_1, Y_2)$$

implies

$$X_1 + X_2 \leq_{sl} Y_1 + Y_2.$$

In practice, the risks of each couple will be positively correlated. Hence, we define the subclass $R_{2,+} \subseteq R_2$ of bivariate risks with non-negative correlations, i.e., for any $(X_1, X_2) \in R_{2,+}$:

$$r(X_1, X_2) \geq 0,$$

and, in the following, we will restrict ourselves to risks in this subclass. Theorem 2 considers this case and follows immediately from theorem 1.

Theorem 2. *Let $\left(X_1^{\mathrm{I}}, X_2^{\mathrm{I}}\right)$ and (X_1, X_2) be two elements of $R_{2,+}$ with $X_i^{\mathrm{I}}\ (i = 1, 2)$ mutually independent. Then we have that*

$$X_1^{\mathrm{I}} + X_2^{\mathrm{I}} \leq_{sl} X_1 + X_2.$$

Hence, we can conclude that the assumption of mutual independence will underestimate the stop-loss premiums corresponding to S, when the couples (X_{2k-1}, X_{2k}), $k = 1, \cdots, m$, are positively correlated. This means that, in fact, when we assume the traditional assumption of mutually independence we are replacing the real aggregate claims distribution by a less risky one, which is a dangerous strategy. One can think that a prudent choice for approximating the unknown distribution of S could be considering the strongest positive dependency relation that can hold between the mortality of members of each couple. This result is obtained when assuming that the corresponding bivariate distribution function associated to the risks in each couple is given by the Fréchet upper bound.

Definition 2. *For any pair of risks (X_1, X_2) with marginal distribution functions F_1 and F_2, i.e.,*

$$F_i(x) = \Pr(X_i \leq x), \ x \geq 0, \ i = 1, 2,$$

the Fréchet upper bound is defined by

$$\Pr(X_1 \leq x_1, X_2 \leq x_2) = \min\{F_1(x_1), F_2(x_2)\}, \quad x_1, x_2 \geq 0.$$

When the bivariate distribution associated to (X_1, X_2) is given by the Fréchet upper bound, risks X_1 and X_2 are said to be mutually comonotonic. This dependency relation has been frequently considered in the recent actuarial literature.

It is easy to proof that for the individual risks we are considering, i.e., those with individual risk distributions defined by (2), X_1 and X_2 will be mutually comonotonic if and only if

$$\Pr(X_1 = \alpha_1, X_2 = \alpha_2) = \min\{q_1, q_2\}. \tag{9}$$

The relation in (9) means that, for the couple considered, the death of the younger one (the one with the higher survival probability) implies the death of the older one.

Observe that

$$r(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)\ Var(X_2)}} = \frac{\alpha_1 \alpha_2 (\Pr(X_1 = \alpha_1, X_2 = \alpha_2) - q_1 q_2)}{\sqrt{Var(X_1)\ Var(X_2)}}.$$

Since the relation

$$\Pr(X_1 = \alpha_1, X_2 = \alpha_2) \leq \min\{q_1, q_2\}$$

holds for all $(X_1, X_2) \in R_{2,+}$, the following theorem can be immediately obtained from theorem 1.

Theorem 3. *Let* (X_1, X_2) *and* $\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right)$ *be two elements of* $R_{2,+}$ *with* $X_1^{\mathrm{C}}, X_2^{\mathrm{C}}$ *mutually comonotonic. Then we have that*

$$X_1 + X_2 \leq_{sl} X_1^{\mathrm{C}} + X_2^{\mathrm{C}}.$$

Combining the stop-loss preserving property under convolution of independent risks with bounds in theorems 2 and 3, we can conclude that our portfolio is, in terms of risk, between the one obtained when assuming mutually independency between the mortality of the components of each married couple and the resultant when the hypothesis about their mortality is comonotonicity. These bivariate distributions are the only known in advance and can be immediately obtained from the individual risk distributions given in (2). Unfortunately, they turn out to be useless for most practical situations. In contrast to the independency assumption, assuming comonotonicity is an extremely prudent strategy and it only will be realistic in case of duplicates (i.e. the portfolio contains two policies concerning the same live). The dependency relations that we are considering (married couples) are closer to the independency than to the comonotonicity and this last hypothesis can never been accepted for different purposes than giving the best upper bound for the riskiness of the portfolio if the only information available consists of the individual risks distributions. We prove, in next section, that the bivariate distribution function associated to each couple under any intermediate positive dependency hypothesis about their mortality can always be obtained as a convex linear combination between the two extreme dependency relations considered here.

3 Bivariate distribution functions

Consider any couple $(X_1, X_2) \in R_{2,+}$. From results in previous section follows that

$$0 \leq r\,(X_1, X_2) \leq r\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right) \tag{10}$$

where $\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right)$ is the comonotonic pair in $R_{2,+}$.

Relations in (10) indicate that there exist a real value for s in $[0,1]$ such that

$$r\left(X_1, X_2\right) = s \cdot r\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right) \tag{11}$$

or, equivalently,

$$Cov\left(X_1, X_2\right) = s \cdot Cov\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right) \tag{12}$$

since $X_i \overset{d}{=} Y_i,\ i = 1, 2$.

The respectively covariances associated to each pair of risks are defined by

$$\begin{aligned} Cov\left(X_1, X_2\right) &= \alpha_1\alpha_2\left(\Pr\left(X_1 = \alpha_1, X_2 = \alpha_2\right) - q_1 q_2\right) \\ &= \alpha_1\alpha_2\left(\Pr\left(X_1 = \alpha_1, X_2 = \alpha_2\right) - \Pr\left(X_1^{\mathrm{I}} = \alpha_1, X_2^{\mathrm{I}} = \alpha_2\right)\right) \end{aligned}$$

and

$$\begin{aligned} Cov\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right) &= \alpha_1\alpha_2\left(\Pr\left(X_1^{\mathrm{C}} = \alpha_1, X_2^{\mathrm{C}} = \alpha_2\right) - q_1 q_2\right) \\ &= \alpha_1\alpha_2\left(\Pr\left(X_1^{\mathrm{C}} = \alpha_1, X_2^{\mathrm{C}} = \alpha_2\right) - \Pr\left(X_1^{\mathrm{I}} = \alpha_1, X_2^{\mathrm{I}} = \alpha_2\right)\right) \end{aligned}$$

where $\left(X_1^{\mathrm{I}}, X_2^{\mathrm{I}}\right)$ is the independent pair in $R_{2,+}$.

Substituting these expressions in (12), we have

$$\begin{aligned} &\Pr\left(X_1 = \alpha_1, X_2 = \alpha_2\right) - \Pr\left(X_1^{\mathrm{I}} = \alpha_1, X_2^{\mathrm{I}} = \alpha_2\right) \\ = \ & s\left(\Pr\left(X_1^{\mathrm{C}} = \alpha_1, X_2^{\mathrm{C}} = \alpha_2\right) - \Pr\left(X_1^{\mathrm{I}} = \alpha_1, X_2^{\mathrm{I}} = \alpha_2\right)\right) \end{aligned}$$

and the probability that both risks lead to a claim in the reference period is defined by

$$\begin{aligned} \Pr\left(X_1 = \alpha_1, X_2 = \alpha_2\right) = s \cdot \Pr\left(X_1^{\mathrm{C}} = \alpha_1, X_2^{\mathrm{C}} = \alpha_2\right) \\ +\left(1 - s\right) \cdot \Pr\left(X_1^{\mathrm{I}} = \alpha_1, X_2^{\mathrm{I}} = \alpha_2\right), \quad 0 \le s \le 1. \end{aligned} \tag{13}$$

As risks are two point distributed, the following theorem can be immediately obtained from expression (13).

Theorem 4. *Let* $\left(X_1^{\mathrm{I}}, X_2^{\mathrm{I}}\right)$ *and* $\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right)$ *denote, respectively, the independent and comonotonic couples in* $R_{2,+}$*. Then, for any* $(X_1, X_2) \in R_{2,+}$ *with Pearson's correlation coefficient given by*

$$r\left(X_1, X_2\right) = s \cdot r\left(X_1^{\mathrm{C}}, X_2^{\mathrm{C}}\right), \quad 0 \le s \le 1,$$

the following expressions hold for the bivariate probability density and distribution functions respectively:

$$\Pr\left(X_1 = x_1, X_2 = x_2\right) = s \cdot \Pr\left(X_1^{\mathrm{C}} = x_1, X_2^{\mathrm{C}} = x_2\right) \\ + (1-s) \cdot \Pr\left(X_1^{\mathrm{I}} = x_1, X_2^{\mathrm{I}} = x_2\right), \quad \forall x_1, x_2 \geq 0$$

and

$$\Pr\left(X_1 \leq x_1, X_2 \leq x_2\right) = s \cdot \Pr\left(X_1^{\mathrm{C}} \leq x_1, X_2^{\mathrm{C}} \leq x_2\right) \\ + (1-s) \cdot \Pr\left(X_1^{\mathrm{I}} \leq x_1, X_2^{\mathrm{I}} \leq x_2\right), \quad \forall x_1, x_2 \geq 0.$$

Results in theorem 4 follow for all couples in the portfolio. Hence, we can conclude that the values of $s_k,\ s_k \in [0,1]$, such that

$$r\left(X_{2k-1}, X_{2k}\right) = s_k \cdot r\left(X_{2k-1}^{\mathrm{C}}, X_{2k}^{\mathrm{C}}\right), \quad k = 1, \cdots, m, \tag{14}$$

turn out to be the key quantity for the exact knowledge of the bivariate probability functions associated to each couple in the portfolio. Given the values of s_k, the bivariate probabilities result, in each point, as a convex linear combination between the corresponding bivariate probabilities when the hypothesis with respect to the dependency relations of the risks in each pair are independency and comonotonicity, respectively. Notice that these two extreme bivariate probabilities are the only known in advance.

Once the bivariate probability density functions associated to the risks in each couple are given the bivariate sum distributions can be immediately obtained. Then, the distribution of the aggregated claims of the portfolio results by convoluting the bivariate sum distributions with the corresponding to the independent risks in the portfolio, which easily follow by applying, e.g., De Pril's recursion (1986).

4 Approximating the bivariate distribution functions in practice

Results in previous section indicate that the aggregate claims distribution of a life insurance portfolio can be exactly obtained if we can give exact values for the percentages of correlation that correspond to risks in each couple with respect to the maximum they could have (the corresponding comonotonic ones). It is clear that the exact knowledge of these values is only possible in the measure we can know the exact values for the bivariate probabilities associated to each pair and we are precisely searching such quantities. Hence, in any practical situation, we have to restrict ourselves to the problem of finding a good approximation for the aggregate claims distribution of the portfolio, which will be solved once we have given a good approximation for the values s_k in (14).

Let us consider one of the couples in the portfolio: $(X_1, X_2) \in R_{2,+}$ and assume that the individual risk X_1 corresponds to a x_1-year-old man while X_2 is associated to his x_2-year-old wife. We will denote by T_{x_1} and T_{x_2} respectively their remaining life times. Without loss of generality, we can assume that the corresponding reference period is one year. Then, for $i = 1, 2$:

$$\Pr(X_i = \alpha_i) = \Pr(T_{x_i} \leq 1) = q_i$$

and

$$\Pr(X_i = 0) = \Pr(T_{x_i} > 1) = 1 - q_i$$

Let I denote the indicator function and define the random variables $I(T_{x_1} \leq 1)$ and $I(T_{x_2} \leq 1)$, then

$$r(X_1, X_2) = r(I(T_{x_1} \leq 1), I(T_{x_2} \leq 1))$$

and, hence, the correlation associated to the risks (X_1, X_2) depends on the correlation between the bivariate remaining lifetimes for next year. The latter correlation will increase when the probability that husband and wife die during the same year increase. Because of reasons state in Section 1, we can assume that this bivariate death probability is greater than in the independent case and, then, it will be necessary to revise historical results referred to the mortality of husband/wife during the same year in order to quantify the increment we can expect with respect to the greater that can happen, i.e., the corresponding to the comonotonic case. The increment occurred will be strongly dependent on two factors: the possibility that both die at the same time (e.g. due to an accident) and the “broken heart syndrome”. Presumably this last factor will be greater for older couples than for younger ones and then, we can expect different values depending on the ages of the components of each couple.

The analysis state above will lead us to give an approximative value for s in (11). On what concerns to this value, notice that results in Theorem 1 indicate that the coefficient

s is monotonically increasing with respect to the risk of the couple, in the stop-loss order sense. Hence, a safe strategy will be overestimating a little s if we don't know exactly its value, as will occur in practice. This will be also the desirable strategy when estimating the values s_k in (14) that define the bivariate probability distributions associated to each couple in the portfolio. Indeed, combining the stop-loss order preservation property for convolutions of independent risks with results in Theorem 1, we have that for the multivariate risk sequences considered in section 1 with corresponding sum distributions S_1 and S_2 given in (3) and (4), if

$$r\left(X_{2k-1}, X_{2k}\right) = s_k \cdot r\left(X_{2k-1}^{\mathrm{C}}, X_{2k}^{\mathrm{C}}\right) \quad \text{and} \quad r\left(Y_{2k-1}, Y_{2k}\right) = s_k' \cdot r\left(X_{2k-1}^{\mathrm{C}}, X_{2k}^{\mathrm{C}}\right)$$

with

$$s_k \leq s_k', \quad \text{for } k = 1, \cdots, m,$$

then

$$S_1 \leq_{sl} S_2.$$

Finally, we want to remark that the importance of the results in this paper consist in giving improved upper bounds for the riskiness of the portfolio but not in obtaining exact values for the aggregate claims distribution. Indeed, as stated before, exact results are only possible when the exact values for the corresponding correlations are known and this will never occur in practice. Nevertheless, until now, numerical results have only been carried out by assuming comonotonicity for the risks in each couple. The resultant aggregate claims distribution is a supreme in terms of stop-loss order but, except in case of duplicates, the distribution obtained this way is even more unrealistic than the corresponding independent one. Our proposal consist in giving an approximative value for the coefficients s_k in (14) by combining two criteria: they have to be smaller enough to describe reality, but, they also have to be larger enough to guarantee they will never be exceeded. For safety reasons, the latter criteria has to prevail in our approximations and then, the upper bound obtained turns out to be sharper than the resultant when the comonotonicity hypothesis is assumed for risks in each couple.

5 Numerical example

In this section we will illustrate previous results by a numerical example. We will use Gerber's (1979) life insurance portfolio which is represented in the following table.

	Amount at risk				
Claim probability	1	2	3	4	5
0.03	2	3	1	2	-
0.04	-	1	2	2	1
0.05	-	2	4	2	2
0.06	-	2	2	2	1

The portfolio consists of 31 risks. Each risk can either produce no claim or a fixed positive claim amount (the amount at risk) during a certain reference period. The claim probability is, in this case, the probability that the insured dies during the reference period. We label the risks from 1 to 31, row by row. Hence, risks 1 and 2 have claim probability 0.03 and a conditional claim amount equal to 1, risks 3, 4 and 5 have claim probability 0.03 and conditional claim amount 2,...

Dhaene and Goovaerts (1996) assume, among others, the following dependency relations:

Situation	Hypothesis
1	All risks mutually independent
2	Positive dependence in couples: (1,2), (3,4), (5,6), (7,8)
3	Positive dependence in couples: (24,31), (14,23), (29,30), (21,22)

and deduce the maximal stop-loss premiums associated to each situation, which are obtained by assuming comonotonicity for the risks in each couple in situations 2 and 3. These quantities can also be obtained from results in section 2 by giving a value $s = 1$ to each couple in the portfolio. Moreover, results in previous sections allow us to consider intermediate dependency relations. Assuming that the dependent couples in situations 2 and 3 are married couples and, after revising the population historical results referred to the mortality of husband/wife during the same reference period, we could conclude that the same value of s can be used for all couples in the portfolio or, on the contrary, that they have to be different because they depend, for instance, on the age of the components of each couple. For simplicity reasons we will assume here that the same value of s can be applied to all couples in the portfolio.

The ratio (multiplied by 100) of the corresponding stop-loss premiums under different hypothesis for the value of s divided by the stop-loss premium in the independent case is given in next table for the situations considered above.

	Situation - Value of s						
Retention	1-s=0	2-s=0.15	2-s=0.25	2-s=1	3-s=0.15	3-s=0.25	3-s=1
0	100	100.0	100.0	100.0	100.0	100.0	100.0
2	100	100.2	100.4	101.6	100.5	100.9	103.8
4	100	100.6	101.0	103.8	102.3	103.9	116.5
6	100	101.2	102.0	108.0	105.4	109.1	137.6
8	100	101.9	103.2	112.8	110.6	117.7	169.1
10	100	103.1	105.2	120.7	117.6	129.0	206.4
12	100	104.5	107.6	130.1	127.9	145.9	226.4
14	100	106.4	110.6	143.8	139.9	166.0	354.2

Columns 1 (independency hypothesis), 4 and 7 (comonotonicity hypothesis) are also derived in Dhaene and Goovaerts (1996). In addition, we have obtained results for couples in situation 2 and 3 under two intermediate positive dependency hypothesis. Assuming that an accurate statistical study of the historical data lead us to conclude that a value of $s = 0.15$ is larger enough to guarantee, at the same time, safety and exactitude in our results, the riskiness of the aggregated claims distribution with respect to the independent case can be measured by the ratio for the stop-loss premiums obtained in columns 4 (situation 2) and 6 (situation 3). Moreover, if we feel more risk for the couples considered or, simply, if we want to assure a less riskiness distribution for the portfolio, we just have to increase the value for s as has been done for obtaining results in columns 5 (situation 2) and 7 (situation 3), where we have assumed that $s = 0.25$.

Comparing the obtained results we can conclude that under any positive dependency assumption, the relative increase of the stop-loss premium is an increasing function of the retention. Comparing results in columns 2, 3, 4 and in 5, 6, 7, we can conclude that increasing the value of s lead to an increased effect, which will be greater when increasing the claim probabilities and the claim amounts of the couples.

Before concluding this example, it may be noticed that the values of s have been taken with the only purpose of numerically illustrating results in previous sections and they haven't been contrasted in practice.

6 Concluding remarks

Finally, we remark that in this paper we have always considered that the bivariate dependencies in the portfolio arise from married couples. Despite the apparently limitation of this analysis, in our opinion these dependency relations will be the greater part of the dependencies we could find in any real life insurance portfolio. Nevertheless, any other kind of positive bivariate dependency relations can be considered with our proposal.

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