DOCUMENTS DE TREBALL

DE LA DIVISIÓ DE CIÈNCIES JURÍDIQUES

ECONÒMIQUES I SOCIALS

Col·lecció d'Economia

The assignment game: core bounds for mixed-pair coalitions

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* This work has been supported by the University of Barcelona (Divisió II), and the second author also by Spanish Research grant DGICYT Project PB98-0867

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Abstract: In the assignment game framework, we try to identify those assignment matrices in which no entry can be increased without changing the core of the game. These games will be called buyer–seller exact games and satisfy the condition that each mixed–pair coalition attains the corresponding matrix entry in the core of the game. For a given assignment game, a unique buyer–seller exact assignment game with the same core is proved to exist. In order to identify this matrix and to provide a characterization of those assignment games which are buyer–seller exact in terms of the assignment matrix, attainable upper and lower core bounds for the mixed–pair coalitions are found. As a consequence, an open question posed in Quint (1991) regarding a canonical representation of a "45^o–lattice" by means of the core of an assignment game can now be answered.
Key words: Assignment games, core, exact games.
JEL: C71, C78

Resum: Aquest treball tracta de determinar aquells jocs d'assignació la matriu associada dels quals té la propietat que cap de les seves entrades pot ser incrementada sense modificar el core del joc. Els anomenarem jocs d'assignació "buyer–seller" exactes ja que tota coalició comprador–venedor assoleix el pagament indicat por la seva corresponent entrada de la matriu en una distribución del core. Donat qualsevol joc d'assignació, provem l'existència d'un únic joc d'assignació amb el seu mateix core i amb la propietat de ser "buyer–seller" exacte. Per tal de trobar la matriu d'aquest nou joc i de caracteritzar, en termes de la matriu, els jocs d'assignació que són "buyer-seller" exactes, fites assolibles per al pagament de les coalicions comprador–venedor dins del core han estat calculades. Com a conseqüència podem tancar una qüestió oberta per Quint (1991) referent a la representació canònica dels reticles de 45^o per mitjà del core d'un joc d'assignació.

1 Introduction

Assignment games associated to different assignment matrices may have the same core (no examples will be found in 2×2 matrices). Nevertheless, among all the matrices leading to the same core, only one has a property we will call buyer–seller exactness. We will be able to determine which this matrix is. This representation result (the core of an assignment game can be represented by means of a buyer–seller exact assignment game) will be one of the aims of this paper.

Roughly speaking, an assignment game is buyer–seller exact when all matrix entries are necessary to determine the core of the game. In other words, for each two–person mixed–pair coalition (a coalition formed by a buyer and a seller), there exists a core allocation which is tight at the corresponding core constraint, that is to say, the addition of both players' payoff in this core allocation coincides with the corresponding matrix entry.

The following example, taken from Shapley and Shubik (1971), will help us to illustrate the above idea. Take the assignment game with set of buyers $M = \{1, 2, 3\}$, set of sellers $M' = \{4, 5, 6\}$, and defined by matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 5 & \textcircled{8} & 2 \\ 7 & 9 & \textcircled{6} \\ \textcircled{2} & 3 & 0 \end{pmatrix} \end{array}$$

This matrix has only one optimal matching which is $\mu = \{(1, 5), (2, 6), (3, 4)\}$ and the core of the game is the convex hull of the following extreme core allocations: (3,5,0;2,5,1), (5,6,1;1,3,0), (3,6,0;2,5,0), (4,6,1;1,4,0), (5,6,0;2,3,0) and (4,5,0;2,4,1).

It is well known that every optimally matched pair achieves the corresponding matrix entry in each core allocation, but what happens in the case of the remaining matrix entries? Take the pair (1,4), with matrix entry $a_{14} = 5$, and notice that this worth is achieved in three of the extreme core allocations. But for pair (1,6) the worth $a_{16} = 2$ is never achieved, as the minimum core payoff for this coalition, which is attained in the third extreme core payoff, is 3. A quick inspection shows that all remaining matrix entries are attained in some extreme core allocation.

This lower core bound for the pair (1,6) could also have been obtained from the matrix entries, without making use of the extreme core allocations. If x is a core allocation, then $x_1 + x_6 \geq a_{16} = 2$, but also, as $x_3 + x_4 = 2$,

$$(x_1 + x_6) + (x_3 + x_4) = (x_1 + x_4) + (x_3 + x_6) \geq a_{14} + a_{36} = 5 + 0$$

which leads to $x_1 + x_6 \geq 3$ and shows that $a_{16} = 2$ will never be reached in the core.

The raising of a_{16} from 2 to 3 does not change the core. In fact, there is a parametric family of assignment matrices leading to the same core as the original matrix and, in this case, the matrices of this family are the only ones with this property. We can describe this family by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 5 & 8 & \alpha \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix} \end{array}$$

where $0 \leq \alpha \leq 3$. However, only one of these matrices, the one in which $\alpha = 3$, satisfies the condition that every entry is attained by some core allocation. This will be said to be *buyer–seller exact*.

As might be expected, when the number of players becomes larger, more inequalities have to be taken into account to determine the minimum core payoff of a mixed–pair coalition; nevertheless, a procedure similar to that used in the example will still be applicable.

The main goal of this paper is to characterize the minimum joint payoff (and also the maximum one) that an arbitrary pair of agents of different sides of the market can achieve in the core of an assignment game. After section 2, which is devoted to definitions and preliminaries, two approaches will be carried out, which will turn out to be connected. The first one (section 3) will make use of a recent characterization of the extreme core allocations of the assignment game (Núñez and Rafels, 2001) while the second one (section 4) works with the matrix entries. The result will be that for any assignment game there exists a unique buyer–seller exact assignment matrix which is a "good" representation of the game from the viewpoint of the core.

Some direct consequences will be deduced. First, an open question posed by Quint (1991) searching for a canonical representation of a "45^o–lattice" by means of the core of an assignment game can now be answered. The second consequence is a characterization of those assignment matrices such that there is no other one leading to a game with the same core.

2 Definitions and preliminaries

A transferable utility cooperative game is a pair (N, v), where the set $N = \{1, 2, \ldots, n\}$ is its finite player set and $v : 2^N \longrightarrow \mathbf{R}$ its characteristic function satisfying $v(\emptyset) = 0$. A payoff vector will be $x \in \mathbf{R}^n$ and, for every coalition $S \subseteq N$ we shall write $x(S) := \sum_{i \in S} x_i$ the payoff to coalition S (where

$x(\emptyset) = 0$). The core of the game (N, v) consists of those payoff vectors which allocate the worth of the grand coalition in such a way that every other coalition receives at least its worth by the characteristic function: $C(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$. The core is a bounded convex polyhedron and thus has a finite number of extreme points, $Ext(C(v))$.

The marginal contribution of player $i \in N$ in the game v, $b_i^v = v(N) - v(N \setminus \{i\})$ is an upper bound for player i's payoff in the core of the game. In general this upper bound may not be attained. However, there are balanced games with the property that all players can attain their marginal contribution in the core. This is the case of assignment games.

The reduction of a game is a well known concept in the general framework of cooperative TU games. Let v be an arbitrary cooperative game with player set N and suppose that some subset of players, $T \subseteq N$, is given. For a fixed vector $x \in \mathbf{R}^{N \setminus T}$, members of coalition T can reconsider their cooperative situation by means of a new game with player set T where the worth of coalitions in T is reevaluated taking into account the worth they could achieve by joining players outside T and paying them according to x. We will consider the special case where $T = N \setminus \{i\}$, for some player $i \in N$ and $x_i = b_i^v = v(N) - v(N \setminus \{i\})$, and the reduction of the game is *à la* Davis and Maschler (1965). This is what we will call the i–marginal game; we denote it by v^i (Núñez and Rafels, 1998)

Given a cooperative game (N, v) and a player $i \in N$ its *i–marginal game* is $(N \setminus \{i\}, v^i)$ where $v^i(\emptyset) = 0$ and for all $\emptyset \neq S \subseteq N \setminus \{i\}$,

$$v^i(S) = \max\{v(S \cup \{i\}) - b_i^v, v(S)\},$$

The game $v^{i_n \cdots i_{k+1} i_k}$ is the i_k–marginal game of $v^{i_n \cdots i_{k+1}}$.

Then, reduced marginal worth vectors are inspired by the marginal worth vectors. For each ordering $\theta = (i_1, i_2, \ldots, i_{n-1}, i_n)$, the reduced marginal worth vector rm_θ^v is a vector in $\mathbf{R}^n$ where each player receives her marginal contribution to her set of predecessors, and a reduction of the game is performed in each step (Núñez and Rafels, 1998): $(rm_\theta^v)_{i_n} = b_{i_n}^v$ and, for all $1 \leq k < n$, $(rm_\theta^v)_{i_k} = b_{i_k}^{v^{i_n i_{n-1} \cdots i_{k+1}}}$. These vectors will play an important role in the core of the assignment game.

Assignment games were introduced by Shapley and Shubik (1971) as a model for a two–sided market with transferable utility. The player set consists of the union of two finite disjoint sets $M \cup M'$, where M is the set of buyers and M' is the set of sellers. We will denote by n the cardinality of $M \cup M'$, $n = m + m'$, where m and m' are, respectively, the cardinalities of M and M'. The worth of any two–player coalition formed by a buyer $i \in M$ and a seller $j \in M'$ is $w(i,j) = a_{ij} \geq 0$. These real numbers can be arranged in a matrix A and determine the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, $\mathcal{M}(S,T)$ being the set of matchings between S and T. We will sometimes denote the game as $(M \cup M', w_A)$. A matching (or assignment) between S and T is a subset μ of $S \times T$ such that each player belongs at most to one pair in μ. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. We say a matching μ is optimal if for all $\mu' \in \mathcal{M}(M,M')$, $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ and will denote by $\mathcal{M}^*(A)$ the set of optimal matchings for the grand coalition. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Shapley and Shubik proved that the core of the assignment game $(M \cup M', w)$ is nonempty and can be represented in terms of an optimal matching in $M \cup M'$. Let μ be one such optimal matching, then

$$C(w) = \left\{ (u,v) \in \mathbf{R}^{M \times M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M\,;\; v_j \geq 0, \text{ for all } j \in M' \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu\,. \end{array} \right\} \tag{1}$$

Moreover, the core has a lattice structure with two special extreme core allocations: the buyers–optimal core allocation, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the sellers–optimal core allocation, $(\underline{u}, \overline{v})$, where each seller does. Notice that when agents on one side of the market obtain their maximum core payoff, the agents on the opposite side obtain their minimum core payoff, as the joint payoff of an optimally matched pair is fixed: $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w)$ if $(i,j) \in \mu$.

From Demange (1982) and Leonard (1983) we know the expression of the maximum and the minimum core payoffs in terms of the characteristic function:

$$\overline{u}_i = w(N) - w(N \setminus \{i\}) \text{ and } \underline{v}_j = w(N \setminus \{i\}) - w(N \setminus \{i,j\}) \text{ if } (i,j) \in \mu\,.$$

Similarly, $\underline{u}_i = w(N \setminus \{j\}) - w(N \setminus \{i,j\})$ and $\overline{v}_j = w(N) - w(N \setminus \{j\})$.

The two extreme core allocations mentioned of the assignment game are not, in general, the only ones. In Núñez and Rafels (2001) the extreme core allocations of the assignment game are proved to coincide with the set of reduced marginal worth vectors.

3 Buyer–seller exactness

A balanced cooperative TU game (N, v) is *exact* (Schmeidler, 1972) if for each coalition $S \subset N$ there exists a core allocation x such that $x(S) = v(S)$. In other words, the core of the game makes use of all information provided by the characteristic function, and whenever the worth of a coalition is increased the core changes.

On the other hand, given (N, v), a non exact balanced game, a unique game (N, v') can be found such that $C(v) = C(v')$ and v' is exact. Schmeidler (1972) proves this result for arbitrary games but for games with a finite player set, as in our case, it is enough to define $v'(S) = \min_{z \in Ext(C(v))} z(S)$.

Assignment games are not exact in general. The example given by Shapley and Shubik (1971), analyzed in the introduction, is an example of a non exact game. Recently, Solymosi and Raghavan (2001) have given a characterization of exact assignment games in terms of the assignment matrix. Moreover, when exactifying an assignment game the resulting game may not be an assignment game. For instance, when computing v' for the example in the introduction of this paper, $v'(1) = 3 \neq 0$.

But the relevant coalitions for the core of an assignment game are the mixed–pair coalitions: in the example in the introduction, one matrix entry could be increased without modifying the core of the assignment game. We will now define when the assignment matrix determines the core of the game, in the sense that any increase of any matrix entry will change the core.

Definition 1 *An assignment game $(M \cup M', w_A)$ is buyer–seller exact if for all $i \in M$ and all $j \in M'$, there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$.*

It is easy to find examples of assignment games which are buyer–seller exact but not exact, as we will show later in this paper.

The following property will be useful to show that some assignment games are not buyer–seller exact, by means of the minimum core payoffs.

Proposition 2 *If $(M \cup M', w_A)$ is a buyer–seller exact assignment game, then $\underline{u}_i + \underline{v}_j \leq a_{ij}$ for all $i \in M$ and all $j \in M'$.*

PROOF: From buyer–seller exactness, for all $i \in M$ and $j \in M'$ there exists $(u, v) \in C(w_A)$ such that $a_{ij} = u_i + v_j \geq \underline{u}_i + \underline{v}_j$. □

When applying this condition to Shapley and Shubik's game, $\underline{u}_1 = 3$ and $\underline{v}_6 = 0$, but $\underline{u}_1 + \underline{v}_6 = 3 > a_{16} = 2$.

However the condition in proposition 2 is necessary but not sufficient for an assignment game to be buyer–seller exact. Take the assignment game defined by the following matrix A

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right) \end{array}$$

and notice that, as $\underline{u}_i = 0$ for all $i \in \{1,2,3\}$ and $\underline{v}_j = 0$, for all $j \in \{3,4,5\}$ the condition of proposition 2 holds. But $u_3 + v_4 > a_{34} = 0$ for all $(u,v) \in C(w_A)$, as $C(w_A) = convex\{(1,1,1,0,0,0),(0,0,0,1,1,1)\}$.

Now, a new assignment game (with the same player set), which will be buyer–seller exact, will be canonically associated to any given assignment game.

Theorem 3 *For all $(M \cup M', w_A)$ there exists a unique assignment game $(M \cup M', w_{A^r})$ such that*

1. *$C(w_A) = C(w_{A^r})$ and*
2. *$(M \cup M', w_{A^r})$ is buyer–seller exact.*

Moreover, $\mathcal{M}^*(A) \subseteq \mathcal{M}^*(A^r)$ and $A \leq A^r$.

Proof: Define $A^r = (a^r_{ij})_{(i,j)\in M \times M'}$ by

$$a^r_{ij} = \min_{(u,v)\in C(w_A)} u_i + v_j = \min_{(u,v)\in Ext(C(w_A))} u_i + v_j = \min_{\theta\in\mathcal{S}_n} rm^{w_A}_\theta(i) + rm^{w_A}_\theta(j),$$

where $rm^{w_A}_\theta$ is the reduced marginal worth vector related to permutation θ and $\mathcal{S}_n$ is the set of all possible orderings in the player set. Notice that $a^r_{ij} = a_{ij}$ if $(i,j) \in \mu$, where $\mu \in \mathcal{M}^*(A)$, and $a^r_{ij} \geq a_{ij}$ otherwise, which leads to $A \leq A^r$.

Take $\mu \in \mathcal{M}^*(A)$ and $\mu' \in \mathcal{M}^*(A^r)$, then, for any $(u,v) \in Ext(C(w_A))$,

$$\sum_{(i,j)\in\mu'} a^r_{ij} \leq \sum_{(i,j)\in M\times M'} u_i + v_j = w(M \cup M') = \sum_{(i,j)\in\mu} a_{ij} = \sum_{(i,j)\in\mu} a^r_{ij}$$

and thus $\mu \in \mathcal{M}^*(A^r)$. We leave to the reader the proof of $C(w_A) = C(w_{A^r})$ and of the buyer–seller exactness of w_{A^r}.

From the definition follows easily that two buyer–seller assignment matrices defining assignment games with the same core, must be the same. □

Notice first that the inclusion $\mathcal{M}^*(A) \subseteq \mathcal{M}^*(A^r)$ cannot be improved. Take for instance the assignment matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which has four optimal matchings and notice that $C(w_A) = convex\{(1,1,1,0,0,0),(0,0,0,1,1,1)\}$. Then its buyer–seller exact associated assigned matrix A^r corresponds to the classical glove market, $A^r = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which has two more optimal matchings.

The proof of theorem 3 shows that what determines matrix A^r are the lower core bounds for the mixed–pair coalitions. These lower core bounds will be analyzed in the next section.

The corollary below states that all matrices between A and A^r (with the usual order: $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $(i,j) \in M \times M'$) have the same core and at least one optimal matching in common. As a consequence (and this will be useful later on in this paper) if a matrix A is not buyer–seller exact, then there is an infinity of assignment matrices such that the corresponding assignment games have the same core as w_A. The proof of this corollary follows easily from theorem 3 and is left to the reader.

Corollary 4 *Given a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$, for all $B = (b_{ij})_{(i,j)\in M\times M'}$ such that $A \leq B \leq A^r$ it holds that they have at least one optimal matching in common and the same core. In fact $\mathcal{M}^*(A) \subseteq \mathcal{M}^*(B) \subseteq \mathcal{M}^*(A^r)$.*

Another immediate consequence of theorem 3 is that two assignment games $(M \cup M', w_A)$ and $(M \cup M', w_B)$ have the same core if and only if $A^r = B^r$, that is to say, the corresponding buyer–seller exact matrices coincide. Moreover, for a fixed assignment game $(M \cup M', w_A)$, the set of assignment matrices B defining assignment games w_B with the same core as w_A is closed by taking the maximum, with maximal element A^r. It is easy to see that the same class is not closed by taking the minimum.

Corollary 5 *Let $(M \cup M', w_A)$ and $(M \cup M', w_B)$ be two assignment games such that $C(w_A) = C(w_B)$ and take the game $(M \cup M', w_{A\vee B})$ where $A \vee B = (\max\{a_{ij}, b_{ij}\})_{(i,j)\in M\times M'}$. Then,*

1. *$C(w_{A\vee B}) = C(w_A) = C(w_B)$ and*
2. *$\mathcal{M}^*(A) \cup \mathcal{M}^*(B) \subseteq \mathcal{M}^*(A \vee B)$.*

Although, by the characterization of the extreme core allocations of the assignment game mentioned in the introduction of this paper, the matrix A^r can be computed from the reduced marginal worth vectors of the assignment game $(M \cup M', w_A)$, a direct computation in terms of the matrix entries would be desirable. It will be provided in next section.

4 Core bounds for mixed–pair coalitions

In this section a characterization of those assignment games which are buyer–seller exact, in terms of a property of the assignment matrix, will be given. This property, which will provide a practical method to compute the buyer–seller exact matrix A^r related to the assignment game $(M \cup M', w_A)$, will turn out to be a generalization of another property introduced by Solymosi and Raghavan (2001) and known as doubly dominant diagonal.

From now on the following notation will be used. By adding dummy players (that is to say, zero rows or columns in the assignment matrix) we can assume without loss of generality that the number of sellers equals the number of buyers and thus the assignment matrix is square. We will now denote $M = \{1, 2, \dots, m\}$ and $M' = \{1', 2', \dots, m'\}$ to distinguish the j–th seller j' from the j–th buyer j. Moreover, we will assume that rows and columns in the matrix are arranged in such a way that $\mu = \{(i, i') \mid i \in M\}$ is an optimal matching.

As we want to find out which assignment games are buyer–seller exact, we first need to know the lower bound for any mixed–pair coalition payoff in the core. In fact, these lower bounds can be obtained from the upper ones, which are easier to determine.

Given any assignment game $(M \cup M', w_A)$, for all $i \in M$ and $j' \in M'$, we define

$$\begin{aligned} K_{ij'} &= \max_{(u,v)\in C(w_A)} u_i + v_{j'} \qquad \text{and} \\ k_{ij'} &= \min_{(u,v)\in C(w_A)} u_i + v_{j'}\,. \end{aligned} \tag{2}$$

Theorem 6 *Let* $(M \cup M', w_A)$ *be an assignment game. Then*

$$\begin{aligned} K_{ij'} &= w_A(M \cup M') - w_A(M \cup M' \setminus \{i, j'\}) \ \textit{and} \\ k_{ij'} &= a_{ii'} + a_{jj'} + w_A(M \cup M' \setminus \{j, i'\}) - w_A(M \cup M')\,. \end{aligned} \tag{3}$$

Proof: We will first prove that the expressions (3) are bounds for the core payoff of a mixed pair coalition and we will then show that these bounds ara attainable.

Take $i \in M$ and $j' \in M'$. If $j' = i'$, then from (1) we have that, for any $(u, v) \in C(w_A)$, $u_i + v_{j'} = u_i + v_{i'} = a_{ii'}$. Notice that, on one side, $w_A(M \cup M') - w_A(M \cup M' \setminus \{i, j'\}) = w_A(M \cup M') - w_A(M \cup M' \setminus \{i, i'\}) = a_{ii'}$ and, on the other side, $a_{ii'} + a_{jj'} + w_A(M \cup M' \setminus \{j, i'\}) - w_A(M \cup M') = a_{ii'} + a_{ii'} + w_A(M \cup M' \setminus \{i, i'\}) - w_A(M \cup M') = a_{ii'}$.

Assume now that $j' \neq i'$, then, as $(u, v) \in C(w_A)$,

$$u(M \setminus \{i\}) + v(M' \setminus \{j'\}) \geq w_A(M \cup M' \setminus \{i, j'\})\,,$$

which leads to

$$u_i + v_{j'} \leq w_A(M \cup M') - w_A(M \cup M' \setminus \{i, j'\}) = K_{ij'} .$$

By applying the above result to the mixed–pair coalition $\{j, i'\}$ we obtain $u_j + v_{i'} \leq w_A(M \cup M') - w_A(M \cup M' \setminus \{j, i'\})$. As $u_j = a_{jj'} - v_{j'}$ and $v_{i'} = a_{ii'} - u_i$, the above inequality leads to $a_{ii'} + a_{jj'} - u_i - v_{j'} \leq w_A(M \cup M') - w_A(M \cup M' \setminus \{j, i'\})$ or equivalently

$$k_{ij'} = a_{ii'} + a_{jj'} + w_A(M \cup M' \setminus \{j, i'\}) - w_A(M \cup M') \leq u_i + v_{j'} .$$

Let us now prove that these lower and upper bounds, $k_{ij'}$ and $K_{ij'}$, are always attainable by the mixed pair coalition $\{i, j'\}$ in the core of the assignment game.

If $j' = i'$, then we have already pointed out that $u_i + v_{j'} = a_{ii'} = k_{ij'} = K_{ij'}$.

Assume then $j' \neq i'$ and take the reduced marginal worth vector related to any ordering $\theta = (k_1, k_2, \ldots, k_n)$ such that $k_n = i$ and $k_{n-1} = j'$. Then, i being the last player in θ, $rm_\theta^{w_A}(i) = w_A(M \cup M') - w_A(M \cup M' \setminus \{i\})$. Let us compute $rm_\theta^{w_A}(j') = w_A^i(M \cup M' \setminus \{i\}) - w_A^i(M \cup M' \setminus \{i, j'\})$. From the definition of i–marginal game, $w_A^i(M \cup M' \setminus \{i\}) = w_A(M \cup M' \setminus \{i\})$ and

$$w_A^i(M \cup M' \setminus \{i, j'\}) = \max\{w_A(M \cup M' \setminus \{j'\}) - b_i^{w_A}, w_A(M \cup M' \setminus \{i, j'\})\} ,$$

where $b_i^{w_A} = w_A(M \cup M') - w_A(M \cup M' \setminus \{i\})$. But as players i and j' are from different sides of the market, we know from Shapley (1962) that

$$w_A(M \cup M' \setminus \{j'\}) - w_A(M \cup M' \setminus \{i, j'\}) \leq w_A(M \cup M') - w_A(M \cup M' \setminus \{i\}) ,$$

and thus $w_A^i(M \cup M' \setminus \{i, j'\}) = w_A(M \cup M' \setminus \{i, j'\})$ and $rm_\theta^{w_A}(j') = w_A(M \cup M' \setminus \{i\}) - w_A(M \cup M' \setminus \{i, j'\})$.

Then,

$$rm_\theta^{w_A}(i) + rm_\theta^{w_A}(j') = w_A(M \cup M') - w_A(M \cup M' \setminus \{i, j'\}) = K_{ij'} ,$$

and thus $K_{ij'}$ is attained in $C(w_A)$.

Applying the above reasoning to the ordering θ' of $M \cup M'$ such that $k_n = i'$ and $k_{n-1} = j$ we obtain

$$rm_{\theta'}^{w_A}(i') + rm_{\theta'}^{w_A}(j) = w_A(M \cup M') - w_A(M \cup M' \setminus \{j, i'\}) \tag{4}$$

and from $rm_{\theta'}^{w_A} \in C(w_A)$ it follows that $rm_{\theta'}^{w_A}(i) + rm_{\theta'}^{w_A}(i') + rm_{\theta'}^{w_A}(j) + rm_{\theta'}^{w_A}(j') = a_{ii'} + a_{jj'}$ and from equation (4),

$$rm_{\theta'}^{w_A}(i) + rm_{\theta'}^{w_A}(j') = a_{ii'} + a_{jj'} + w_A(M \cup M' \setminus \{j, i'\}) - w_A(M \cup M') ,$$

and $k_{ij'}$ is attained in $C(w_A)$. □

Once the lower bound for a mixed–pair coalition payoff in the core is determined,

$$k_{ij'} = a_{ii'} + b_{jj'} + w_A(M \cup M' \setminus \{j, i'\}) - w_A(M \cup M'),$$

we would like to express it in terms of the matrix. To this end, we will compute the worth of $w_A(M \cup M' \setminus \{j, i'\})$.

Proposition 7 *Let $(M \cup M', w_A)$ be an assignment game, $i \in M$ and $j' \in M'$ such that $j' \neq i'$. Then*

$$\begin{aligned} w_A(M \cup M' \setminus \{j,i'\}) = \max \Bigg(& a_{ij'} + \sum_{t \in M \setminus \{i,j\}} a_{tt'}, \\ & \max_{\substack{k_1,\dots,k_r \in M \setminus \{i,j\} \\ \text{different}}} \left\{ a_{ik'_1} + a_{k_1k'_2} + \cdots + a_{k_rj'} + \sum_{t \in M \setminus \{i,j,k_1,\dots,k_r\}} a_{tt'} \right\} \Bigg) \end{aligned} \quad (5)$$

Proof: Let μ^* be an optimal assignment for coalition $M \cup M' \setminus \{j, i'\} = \{i, j', k_1, \dots, k_{m-2}, k'_1, \dots, k'_{m-2}\}$.

If $(i, j') \in \mu^*$, as, by the notation we have fixed, the diagonal determines an optimal matching for matrix A, we have $(k_t, k'_t) \in \mu^*$ for all $t \in \{1, 2, \dots, m-2\}$. Otherwise, if players in $M \setminus \{i, j\}$ and $M' \setminus \{i', j'\}$ could be matched in a better way, by adding to this matching the pairs (i, i') and (j, j') we would contradict that $\{(t, t') \mid t \in M\}$ is an optimal matching for $M \cup M'$. Thus the worth achieved by μ^* is

$$a_{ij'} + \sum_{t \in M \setminus \{i,j\}} a_{tt'}. \quad (6)$$

If $(i, k'_{s_1}) \in \mu^*$, where $k'_{s_1} \in M' \setminus \{i', j'\}$, two cases are possible:
Case 1: $(k_{s_1}, j') \in \mu^*$.

By the same reasoning as above, $(t, t') \in \mu^*$ for all $t \in M \setminus \{i, j, k_{s_1}\}$ and the worth achieved by this matching is

$$a_{ik'_{s_1}} + a_{k_{s_1}j'} + \sum_{t \in M \setminus \{i,j,k_{s_1}\}} a_{tt'}. \quad (7)$$

Case 2: $(k_{s_1}, k'_{s_2}) \in \mu^*$ and $k'_{s_2} \neq j'$.

Again two possibilities appear. Either $(k_{s_2}, j') \in \mu^*$ which implies $(t, t') \in \mu^*$ for all $t \in M \setminus \{i, j, k_{s_1}, k_{s_2}\}$ and the worth achieved by this matching is

$$a_{ik'_{s_1}} + a_{k_{s_1}k'_{s_2}} + a_{k_{s_2}j'} + \sum_{t \in M \setminus \{i,j,k_{s_1},k_{s_2}\}} a_{tt'}. \quad (8)$$

or else $(k_{s_2}, j') \notin \mu^*$ and by repeating the same procedure, in a finite number of steps we obtain

$$\mu^* = \{(i, k'_{s_1}), (k_{s_1}, k'_{s_2}), \ldots, (k_{s_r}, j')\} \cup \{(t, t')\}_{t \in M \setminus \{i, j, k_{s_1}, \ldots, k_{s_r}\}}$$

while the worth achieved by this matching is

$$a_{ik'_{s_1}} + a_{k_{s_1} k'_{s_2}} + \cdots + a_{k_{s_r} j'} + \sum_{t \in M \setminus \{i, j, k_{s_1}, \ldots, k_{s_r}\}} a_{tt'} \,. \tag{9}$$

Now, identity (5) follows from equations (6) to (9). □

Recall that Solymosi and Raghavan (2001) define a square matrix A to be doubly dominant diagonal whenever for all $i, j, k \in M$ and different, $a_{ij} + a_{kk} \geq a_{ik} + a_{kj}$, and this property is a necessary, but not sufficient, condition for the game w_A to be exact.

A new property will now be defined.

Definition 8 *Let A be a square matrix with rows $M = \{1, 2, \ldots, m\}$ and columns $M' = \{1', 2', \ldots, m'\}$ and such that $\mu = \{(i, i') \mid i \in M\} \in \mathcal{M}^*(A)$. A is strongly dominant diagonal if and only if for all $i \in M$ and $j' \in M'$,*

$$a_{ij'} + a_{k_1 k'_1} + \cdots + a_{k_r k'_r} \geq a_{ik'_1} + a_{k_1 k'_2} + \cdots + a_{k_r j'}$$

for all $k_1, k_2, \ldots, k_r \in M \setminus \{i, j\}$, all of them different.

For instance, the introductory example has not a strongly dominant diagonal matrix. After reordering the player set so that the diagonal is an optimal matching, the assignment matrix A is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} \textcircled{8} & 2 & 5 \\ 9 & \textcircled{6} & 7 \\ 3 & 0 & \textcircled{2} \end{pmatrix} \end{array} \,. \tag{10}$$

Notice that $a_{12'} + a_{33'} < a_{13'} + a_{32'}$.

Definition 8 can be read as follows: let $A^{ji'}$ be the submatrix obtained by deleting row j and column i', with $i' \neq j'$, from matrix A. If A is strongly dominant diagonal, then all pairs (t, t') with $t \neq i, j$ will still be optimally paired in $A^{ji'}$ and thus, from proposition 7,

$$w_A(M \cup M' \setminus \{j, i'\}) = a_{ij'} + \sum_{t \in M \setminus \{i, j\}} a_{tt'} = a_{ij'} + w_A(M \setminus \{i, j\}, M' \setminus \{i', j'\}) \,.$$

Moreover, if A is a strongly dominant diagonal matrix, any submatrix obtained by deleting some optimally matched pairs will also be strongly dominant diagonal and thus, for all $S \subseteq M \setminus \{i,j\}$ and taking $S' = \{t' \in M' \mid t \in S\}$, we obtain $w_A(S \cup S' \cup \{i,j'\}) = a_{ij'} + \sum_{t \in S} a_{tt'} = a_{ij'} + w_A(S \cup S')$.

This means that given any coalition $S \cup S'$ formed by a set of optimally matched pairs and one additional mixed pair $\{i,j'\}$, all optimally matched pairs will also be matched in an optimal assignment for coalition $S \cup S' \cup \{i,j'\}$. In fact this property characterizes the strongly dominant diagonal matrices in terms of the characteristic function of the assignment game.

Proposition 9 *Let A be a square matrix with rows $M = \{1,2,\ldots,m\}$ and columns $M' = \{1',2',\ldots,m'\}$ and assume $\mu = \{(t,t') \mid t \in M\}$ is an optimal matching for A, then the following statements are equivalent:*

1. *A is strongly dominant diagonal*
2. *For all $i \in M$ and $j' \in M'$ such that $i' \neq j'$ and all $S \subseteq M \setminus \{i,j\}$,*
$$w_A(S \cup S' \cup \{i,j'\}) = a_{ij'} + w_A(S \cup S'),$$
where $S' = \{t' \in M' \mid t \in S\}$.

Proof: It follows easily from definition 8 and the above considerations. □

Notice that a strongly dominant diagonal matrix is always doubly dominant diagonal (just take $r = 1$). The property of being strongly dominant diagonal will characterize those matrices which are buyer–seller exact.

Theorem 10 *Let $(M \cup M', w_A)$ be an assignment game. The following statements are equivalent:*

1. *w_A is buyer–seller exact*
2. *A is strongly dominant diagonal*

Proof: $1 \Rightarrow 2)$ Take $i, j, k_1, \ldots, k_r \in M$, all of them different. As w_A is a buyer–seller exact game, there exists $(u,v) \in C(w_A)$ such that $u_i + v_{j'} = a_{ij'}$. Moreover, from the expression of the core of an assignment game in (1), $u_{k_l} + v'_{k_l} = a_{k_l k'_l}$ holds for all $l \in \{1,2,\ldots,r\}$ while $u_{k_l} + v_{k'_{l+1}} \geq a_{k_l k'_{l+1}}$ for all $l \in \{1,2,\ldots,r-1\}$, $u_i + v_{k'_1} \geq a_{ik'_1}$ and $u_{k_r} + v_{j'} \geq a_{k_r j'}$. Then,

$$u_i + v_{j'} + u_{k_1} + v_{k'_1} + \cdots + u_{k_r} + v_{k'_r} = a_{ij'} + a_{k_1 k'_1} + \cdots + a_{k_r k'_r}$$

while on the other hand

$$u_i + v_{j'} + u_{k_1} + v_{k'_1} + \cdots + u_{k_r} + v_{k'_r} \geq a_{ik'_1} + a_{k_1 k'_2} + \cdots + a_{k_r j'}$$

and thus A is strongly dominant diagonal.
$2 \Rightarrow 1$) If A is strongly dominant diagonal, then we deduce, from proposition 7, that for all $i \in M$ and $j' \in M'$, $w_A(M \cup M' \setminus \{j, i'\}) = a_{ij'} + \sum_{t \in M \setminus \{i,j\}} a_{tt'}$ and

$$k_{ij'} = a_{ii'} + a_{jj'} + a_{ij'} + \sum_{t \in M \setminus \{i,j\}} a_{tt'} - \sum_{t \in M} a_{tt'} = a_{ij'} .$$

By theorem 6, the lower bound $k_{ij'}$ is attained by the mixed–pair coalition $\{i, j'\}$ in the core of the assignment game w_A, and consequently w_A is buyer–seller exact. □

As a by–product of theorem 10, we have a method to compute the buyer–seller exact matrix A^r associated to A. Define $a^r_{ij'} = \max\{a_{ij'}, \tilde{a}_{ij'}\}$, where

$$\tilde{a}_{ij'} = \max_{\substack{k_1, k_2, \ldots, k_r \in M \setminus \{i,j\} \\ different}} \left\{ a_{ik'_1} + a_{k_1 k'_2} + \cdots + a_{k_r j'} - (a_{k_1 k'_1} + \cdots + a_{k_r k'_r}) \right\} .$$

This means that each mixed–pair coalition $\{i, j'\}$ evaluates what it could achieve by cooperating with some optimal matched pairs out of $\{i, j, i', j'\}$, let us say pairs $(k_1, k'_1), (k_2, k'_2), \ldots, (k_r, k'_r)$, on the basis that these pairs will be paid what they obtain in the fixed optimal matching, and then takes the maximum between this worth and $a_{ij'}$.

Let us now compute the matrix A^r related to matrix (10). Notice that $a^r_{23'} = \max\{a_{23'}, a_{21'} + a_{13'} - a_{11'}\} = \max\{7, 6\} = 7 = a_{23'}$ while $a^r_{12'} = \max\{a_{12'}, a_{13'} + a_{32'} - a_{33'}\} = \max\{2, 3\} = 3 > a_{12'} = 2$. Proceeding in the same way we will get $a^r_{ij'} = a_{ij'}$ for any other $(i, j') \in M \times M'$.

There is still another consequence of theorem 10 regarding those assignment games w_A such that there is no other assignment game with the same core. In other words, we are interested in those assignment games such that any change in a matrix entry would produce a change in the core of the game. Of course the original game must be buyer–seller exact, otherwise w_{A^r} and, from corollary 4 also all w_B with $A \leq B \leq A^r$, will have the same core. However this condition is not sufficient, as the assignment game defined by matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is buyer–seller exact, has the same core as the one defined by matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Next corollary provides a necessary and sufficient condition for an assignment game to be the only one with its same core.

Corollary 11 *The game* $(M \cup M', w_A)$ *is the only assignment game with core* $C(w_A)$ *if and only if for all* $i \in M$ *and* $j' \in M'$ *,* $i' \neq j'$ *,*

$$a_{ij'} + a_{k_1 k_1'} + \cdots + a_{k_r k_r'} > a_{ik_1'} + a_{k_1 k_2'} + \cdots + a_{k_r j'}$$

for all $k_1, k_2, \ldots, k_r \in M \setminus \{i, j\}$ *, all of them different.*

PROOF: If there exists $a_{i_* j_*'}$, with $i_*' \neq j_*'$, such that $a_{i_* j_*'} + a_{k_1 k_1'} + \cdots + a_{k_r k_r'} \leq a_{i_* k_1'} + a_{k_1 k_2'} + \cdots + a_{k_r j_*'}$ for some $k_1, k_2, \ldots, k_r \in M \setminus \{i_*, j_*\}$, all of them different, then by taking matrix B such that $b_{i_* j_*'} = 0$ and $b_{ij'} = a_{ij'}$ otherwise, we get $A^r = B^r$ and thus $C(w_A) = C(w_B)$.

On the other side, if $a_{ij'} + a_{k_1 k_1'} + \cdots + a_{k_r k_r'} > a_{ik_1'} + a_{k_1 k_2'} + \cdots + a_{k_r j'}$ for all $(i, j') \in M \times M'$ such that $i' \neq j'$, then A is buyer–seller exact. Assume there exists another matrix B such that $C(w_B) = C(w_A)$. From the uniqueness in theorem 3, $A = B^r$. Then, from corollary 4, all matrices C such that $B \leq C \leq B^r = A$ will define games w_C with the same core as w_A . But, from the assumption on A it is possible to define a matrix C in these conditions and it will be buyer–seller exact. Take a pair $(i_*, j_*') \in M \times M'$ such that $b_{i_* j_*'} < c_{i_* j_*'} < a_{i_* j_*'}$, $c_{ij'} = a_{ij'}$ otherwise and $c_{i_* j_*'} + a_{k_1 k_1'} + \cdots + a_{k_r k_r'} > a_{ik_1'} + a_{k_1 k_2'} + \cdots + a_{k_r j'}$. As C is buyer–seller exact and $C \leq A$, $C(w_C) \neq C(w_A)$, which involves a contradiction. □

To close this section, let us point out what happens with 2×2 assignment games.

Corollary 12 *Let* $(M \cup M', w_A)$ *be a* 2×2 *assignment game. Then*

1. $(M \cup M', w_A)$ *is buyer–seller exact and*
2. *there is no other assignment game* $(M \cup M', w_B)$ *such that* $C(w_A) = C(w_B)$ *.*

PROOF: As $k_{12'} = a_{12'}$ and $k_{21'} = a_{21'}$, the game is buyer–seller exact. Any other 2×2 assignment game $(M \cup M', w_B)$ will then also be buyer–seller exact and if $B \neq A$ they must differ in terms of some matrix entry. If $a_{ii'} \neq b_{ii'}$ for some $i \in \{1, 2\}$, then their cores cannot coincide. If the two matrices differ in terms of another entry, say for instance $a_{12} < b_{12}$ then, as A is buyer–seller exact, there exists $x \in C(w_A)$ such that $x_1 + x_{2'} = a_{12'} < b_{12'}$ and thus the cores will also be different. □

From the above corollary, the remark made after definition 1 can now be improved. Take a 2×2 assignment game which is not dominant diagonal (see definition in subsection 5.2 below). From Solymosi and Raghavan (2001) it will not be exact, but from corollary 12 it will be buyer–seller exact.

5 Concluding remarks

5.1 45^o–lattices

We are now in a position to answer a question posed by Quint (1991). Notice first that the core of an assignment game $(M \cup M', w_A)$ can be expressed just in terms of the payoffs to one side of the market (let us take the buyers without loss of generality) by using the constraints $u_i + v_{j'} = a_{ij'}$ if $(i, j') \in \mu$, where μ is an optimal matching. This is what is called the u–space core of w_A, $C_u(w_A) = \{u \in \mathbf{R}^m \mid \exists v \in \mathbf{R}^{m'} \text{ and } (u,v) \in C(w_A)\}$ and it has a very particular shape: it is a 45^o–lattice in $\mathbf{R}^m$, that is to say, there exist real numbers d_{ik} for all i, $k \in \{1, \ldots, m\}$, and non negative real numbers b_i and e_i, for all $i \in \{1, 2, \ldots, m\}$, such that

$$C_u(w_A) = \left\{ u \in \mathbf{R}^m \;\middle|\; \begin{array}{l} u_i - u_k \geq d_{ik} \text{ for all } i, k \in \{1, \ldots, m\}, i \neq k \\ b_i \leq u_i \leq e_i \text{ for all } i \in \{1, 2, \ldots, m\} \end{array} \right\}.$$

Quint (1991) gives a characterization of the cores of an assignment game by proving that for any 45^o–lattice, that is to say, a non empty set L such that

$$L = \left\{ u \in \mathbf{R}^m \;\middle|\; \begin{array}{l} u_i - u_k \geq d_{ik} \text{ for all } i, k \in \{1, \ldots, m\}, i \neq k \\ b_i \leq u_i \leq e_i \text{ for all } i \in \{1, 2, \ldots, m\} \end{array} \right\},$$

there exists an assignment game $(M \cup M', w_L)$ such that $C_u(w_L) = L$.

Taking $M' = M \cup \{m+1\}$, the game w_L is defined by the assignment matrix A_L:

$$\begin{array}{ll} a_{ii'} = e_i & \text{for all } i \in \{1, 2, \ldots, m\} \\ a_{ij'} = \max\{e_j + d_{ij}, 0\} & \text{for all } i, j \in \{1, 2, \ldots, m\},\ i \neq j \\ a_{i,m+1} = b_i & \text{for all } i \in \{1, 2, \ldots, m\}, \end{array}$$

and $\mu = \{(i, i')\}_{i \in M}$ is an optimal matching for the grand coalition.

This game w_L is not the unique one with u–space core coinciding with L. As Quint points out, on one hand, more sellers could be added by suitably choosing the entries in their columns, without changing the optimal matching and the core. However, an assignment game with minimal number of sellers can be defined such that $C_u(w_L) = L$. This minimal number of sellers will be $m+1$ if $0 \notin L$ and m otherwise. Secondly, rearranging the columns of A_L, the optimal matching may change, but not the core. Finally, the definition of A_L above may produce a_{ij} for which $u_i + v_j > a_{ij}$ for every $(u, v) \in C(w)$. Then we could lift a_{ij} slightly without changing the core.

At that point, Quint (1991, page 419) makes the following conjecture which can now be proved.

Theorem 13 *For a given 45^o–lattice $L \subseteq \mathbf{R}^m$, there exists a unique assignment game $(M \cup M', w_L)$, defined by matrix $A'_L = (a'_{ij'})_{(i,j') \in M \times M'}$, such that:*

1. *$C(w_L) = L$,*

2. *w_L contains $m+1$ sellers if $0 \notin L$ (or m sellers if $0 \in L$),*

3. *μ is an optimal matching for A'_L, where $(i,i') \in \mu$ for all $i \in \{1, 2, \ldots, m\}$,*

4. *the $a'_{ij'}$, for $i \neq j$, are "as high as possible".*

Proof: By Quint's method, given a 45^o–lattice $L \subseteq \mathbf{R}^m$ we know the existence of at least a matrix A_L such that $C_u(w_{A_L}) = L$. Now take matrix A'_L to be the (unique) buyer–seller exactification of A_L, that is to say, $A'_L = A^r_L$. By theorem 3, $C(w_{A^r_L}) = C(w_{A_L})$ and thus $C_u(w_{A^r_L}) = L$. If B is another matrix, with the same dimensions as A_L, such that $C_u(w_B) = L$, then, since $C(w_B) = C(w_{A_L})$, again by theorem 3 we obtain that $B^r = A^r_L$ and thus $B \leq B^r = A^r_L$ which proves that A^r_L is "as high as possible". □

5.2 About the exactification w^e_A

At the beginning of section 3 we remarked that when exactifying an assignment game w_A the resulting game w^e_A, $w^e_A(S) = \min_{z \in Ext(C(w_A))} z(S)$, may not be an assignment game, although in some cases it is. We can now characterize when this does happen.

For this purpose, recall that a square matrix A is dominant diagonal (Solymosi and Raghavan, 2001) when $a_{ii'} \geq a_{ij'}$ for all $j' \in M'$ and $a_{ii'} \geq a_{ji'}$ for all $j \in M$, that is to say, each diagonal element is the largest one in its row and column. Solymosi and Raghavan also prove that an assignment game is exact if and only its assignment matrix is dominant diagonal and doubly dominant diagonal.

Now, it is easy to see that the game w^e_A is an assignment game if and only if A^r is dominant diagonal.

Corollary 14 *Given an arbitrary assignment game $(M \cup M', w_A)$, the following statements are equivalent:*

1. *w^e_A is an assignment game*

2. *$w^e_A = w_{A^r}$*

3. *A^r is a dominant diagonal matrix*

Proof: On one side, if A^r is dominant diagonal, as it is also doubly dominant diagonal, from Solymosi and Raghavan (2001) w_{A^r} is an exact game and thus it coincides with w_A^e, which is the only exact game with the same core as w_A. On the other hand, if w_A^e is an exact assignment game, as it has the same core as w_A, then it is also buyer–seller exact and from theorem 3 must coincide with w_{A^r}. But then A^r is dominant diagonal. □

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