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The set of undominated imputations and the core: an axiomatic approach

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Abstract

This paper provides an axiomatic framework to compare the D-core (the set of undominated imputations) and the core of a cooperative game with transferable utility. Theorem 1 states that the D-core is the only solution satisfying projection consistency, reasonableness (from above), (∗)-antimonotonicity, and modularity. Theorem 2 characterizes the core replacing (∗)-antimonotonicity by antimonotonicity. Moreover, these axioms also characterize the core on the domain of convex games, totally balanced games, balanced games, and superadditive games.

Resum

En aquest treball es caracteritza axiomàticament el conjunt d'imputacions no dominades (*el D-core*) i se'l compare amb el *core*. El teorema 1 estableix que el *D-core* és l'única solució que satisfà *Projection consistency*, *(*)-antimonotonicity*, *Reasonableness (from above)* i *modularity*. En el teorema 2 es caracteritza el *core* canviant *(*)-antimonotonicity* per *antimonotonicity*. Aquest últim resultat és robust en el sentit que també caracteritza el *core* en el domini dels jocs convexes, totalment equilibrats, equilibrats i superadditius.

Key words: Cooperative TU-game, core, undominated imputations, reasonable outcome.

JEL Classification: C71

1 Introduction

The core and the D-core of a transferable utility coalitional game (TU-game, for short) were introduced by Gillies (1959) by means of a domination relation. The core is the set of undominated preimputations, and it can be rewritten as the solution of a well-known system of linear inequalities. The D-core coincides with the set of undominated imputations. In general, the D-core is a proper extension of the core, but for a large class of games both sets coincide. Moreover, Rafels and Tijs (1997) and Chang (2000) prove that the D-core of a game can be expressed in terms of the core of a new associated game.

The above results show that both concepts are closely related. However, the core has been intensely studied and axiomatized in game theory, but, as far as we know, there is not a proper characterization result for the D-core. This fact opens a natural question: which is the difference, from an axiomatic point of view, between the core and the D-core? In this paper, we axiomatize the D-core on the space of all TU-games. As a byproduct, and only changing one axiom, we obtain a new axiomatic approach for the core. This last result is interesting by itself since it also characterizes the core on the class of convex games, totally balanced games, balanced games, and superadditive games.

The paper is organized as follows. Section 2 contains notation and some definitions. In Section 3 we present the main results: Theorem 1 states that the D-core is the only solution on the space of all TU-games satisfying projection consistency, reasonableness (from above), $(*)$-antimonotonicity, and modularity. Theorem 2 characterizes the core by replacing $(*)$-antimonotonicity by antimonotonicity.

2 Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. By $N \subseteq \mathbb{N}$ we denote a finite set of players, in general $N = \{1, \ldots, n\}$. A *transferable utility coalitional game (a game)* is a pair (N, v) where $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$ and 2^N denotes the set of all subsets (coalitions) of N. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by Γ. Given a coalition $S \subset N, S \neq \emptyset$ and $(N, v) \in \Gamma$, we define the subgame (S, v_S) by $v_S(Q) := v(Q)$, for all $Q \subseteq S$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$.

For the game (N, v), the set of *feasible payoff vectors* is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A *solution* on a set Γ of games is a mapping σ which associates with any game (N, v) a subset $\sigma(N, v)$ of the set $X^*(N, v)$. Notice that the solution set $\sigma(N, v)$ is allowed to be empty. The *pre-imputation set* of a game (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of *imputations* by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq v(i),\ \forall\ i \in N\}$. A game with a non-empty set of imputations is called *essential.* We say that a solution σ is *Pareto optimal* if $\sigma(N, v) \subseteq X(N, v)$ for $(N, v) \in \Gamma$.

Given two pre-imputations $x, y \in X(N, v)$, we say that x dominates y, in short $x \text{ dom}^v\ y$, if there exists a coalition $S \subseteq N$ such that $x_i > y_i$, for all $i \in S$, and $x(S) \leq v(S)$. For a game (N, v) the *set of undominated pre-imputations* is the *core* of the game (Gillies, 1959). The core of a game (N, v) can be rewritten as the set of those imputations where each coalition gets at least its worth, that is $C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. The *D-core* is formed by those imputations which are not dominated by any other imputation. Formally, $DC(N,v) := \{x \in I(N,v) \mid \nexists\ y \in I(N,v)$ such that $y\ \mathrm{dom}^v\, x\}$, while $C(N,v) := \{x \in X(N,v) \mid \nexists\ y \in X(N,v)$ such that $y\ \mathrm{dom}^v\, x\}$. The core is always included in the D-core, $C(N,v) \subseteq DC(N,v)$. Nevertheless, there are games where both concepts are different (see Example 1). A game with a non-empty core is called *balanced* and, if all its subgames have non-empty cores, the game is said to be *totally balanced.*

For our purposes we need to recall the following result (Rafels and Tijs, 1997): for any game (N,v) with $DC(N,v) \neq \varnothing$, $DC(N,v) = C(N,v')$, where (N,v') is defined by

$$v'(S) := \min \left\{ v(S), v(N) - \sum_{i \in N \setminus S} v(i) \right\}, \text{ for all } S \subseteq N, \tag{1}$$

This result can be extended to any essential game (Chang, 2000): for any game (N,v) with $I(N,v) \neq \varnothing$, $DC(N,v) = C(N,v')$. From this result, it is straightforward to see that, for any game (N,v), $DC(N,v) = C(N,v^*)$, where (N,v^*) is defined by

$$v^* := \begin{cases} v' & \text{if } I(N,v) \neq \varnothing, \\ v & \text{if } I(N,v) = \varnothing. \end{cases} \tag{2}$$

A game (N,v) is *convex* (Shapley, 1971) if, for every $S,T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. A game (N,v) is *superadditive* if, for every $S,T \subseteq N$, $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. A game (N,v) is said to be *modular* if there exists a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ such that for every $S \subseteq N$, $v(S) = \sum_{i \in S} x_i$. To indicate the modular game generated by $x \in \mathbb{R}^N$ we will use (N, v_x). A game (N,v) is N-monotonic if $v(S) + \sum_{i \in N \setminus S} v(\{i\}) \leq v(N)$, for all $S \subseteq N$. By Z^N we denote the class of N-monotonic games. For any $(N,v) \in Z^N$, $C(N,v) = DC(N,v)$ (Rafels and Tijs, 1997). Notice that for any $(N,v) \in \Gamma$, $(N,v') \in Z^N$, where (N,v') is defined by (1).

3 An axiomatic characterization of the core and the D-core

This section introduces an axiomatic framework to axiomatize and to compare the D-core and the core of a game. Both characterizations use the same type of axioms and differ only in one of them, which is slightly changed. First, in Theorem 1 we provide an axiomatization of the D-core using projection consistency, reasonableness (from above), $(*)$-antimonotonicity, and modularity. In Theorem 2, by replacing $(*)$-antimonotonicity by antimonotonicity, a new axiomatic characterization of the core is given. We start by defining the above properties.

To introduce consistency first we need to define reduced games.

Definition 1 *Let* $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$ *and* $\varnothing \neq T \subset N$. *The* **projected reduced game relative to** T **at** x *is the game* $(T, r_x^T(v))$ *defined by*

$$r_x^T(v)(S) := \begin{cases} v(S) & \text{if } S \subset T, \\ v(N) - x(N \backslash T) & \text{if } S = T. \end{cases}$$

For interpretation of the notion of the projected reduced game see, for instance, Thomson (1998).

Let σ be a solution on Γ. Then, σ satisfies

- **projection consistency (P-CONS)** if for any $(N, v) \in \Gamma$, all $T \subset N$, $T \neq \varnothing$, and all $x \in \sigma(N, v)$, then $(T, r_x^T(v)) \in \Gamma$ and $x_T \in \sigma(T, r_x^T(v))$.

- **reasonableness (from above) (REAB)** if, for all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$ and all $i \in N$, then $x_i \leq \max_{S \subseteq N \backslash \{i\}} \{v(S \cup \{i\}) - v(S)\}$.

- **antimonotonicity (AMON)** if for any pair $(N, v), (N, w) \in \Gamma$ such that $v(S) \geq w(S)$, for all $S \subset N$, and $v(N) = w(N)$, then $\sigma(N, v) \subseteq \sigma(N, w)$.

- $(*)$**-antimonotonicity** $((*)$**-AMON)** if for any pair $(N, v), (N, w) \in \Gamma$ such that $v^*(S) \geq w^*(S)$, for all $S \subset N$, and $v^*(N) = w^*(N)$, where (N, v^*) and (N, w^*) are defined by (2), then $\sigma(N, v) \subseteq \sigma(N, w)$.

- **modularity (MOD)** if for any modular game (N, v_x), then $\sigma(N, v) = \{x\}$.

Consistency (or reduced game property) is, perhaps, the most fundamental property used in this field. Roughly speaking, this principle says that there is no difference in what the players of the reduced game will get in both the original game and the reduced game (see Thomson, 1998, and Driessen, 1991 for surveys of consistency). Projection consistency has been used by Funaki and Yamato (2001) to characterize the core on the class of balanced games. Here it is important to point out that the D-core does not satisfy the reduced game properties used in the well-known axiomatizations of the core (see, among others, Peleg, 1986, Tadenuma, 1992, Winter and Wooders, 1994, Voorneveld and van den Nouweland, 1998, and Hwang and Sudhölter, 2001). The reason is that the non-emptiness of the imputation set may not be preserved in this kind of reduced games when we take a point of the D-core. Indeed, consider the following example given by Rafels and Tijs (1997):

Example 1 *Let (N, v) be a 3-person game, where $N = \{1, 2, 3\}$ and $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 1$, $v(\{1, 2\}) = 2$, $v(\{1, 3\}) = v(\{2, 3\}) = 1$, and $v(\{1, 2, 3\}) = 2$. The core of this games is empty, but the D-core is $DC(N, v) = [(1, 0, 1), (0, 1, 1)]$. As the reader can easily check, the Davis-Maschler reduced game (Davis and Maschler, 1965) relative to $S = \{2, 3\}$ and $x = \left(\frac{1}{2}, \frac{1}{2}, 1\right) \in DC(N, v)$ is inessential. The same problem appears for* *the other reduce games used in the axiomatizations of the core just commented before.*

Milnor (1952) introduced reasonableness (from above) as a necessary condition to decide whether a payoff vector is a "plausible" outcome for a game. Sudhölter and Peleg (2000) use this principle to characterize the *positive prekernel*.

Antimonotonicity was introduced by Keiding (1986) to axiomatize the core. The intuition is that if the coalitions, except the grand coalition, get impoverished, then any payoff vector in the solution of the original game remains in the solution of the new game. $(*)$-Antimonotonicity is a technical modification of the antimonotonicity in which the worth of the coalitions is compared not in the original game, but in an associated game (N, v^*).

A modular game can be considered as one where no conflict is present: every coalition can get exactly what its members can get for themselves. So, modularity forces the solution to be the "natural" one for these games. It is important to point out that this axiom is satisfied by the main solution concepts.

Now we axiomatize the D-core using projection consistency, reasonableness (from above), $(*)$-antimonotonicity, and modularity.

Theorem 1 *The D-core is the only solution on* Γ *satisfying projection consistency, reasonableness (from above),* $(*)$*-antimonotonicity, and modularity.*

Notice that for games where the set of players is a singleton, any solution satisfying modularity coincides with the D-core. So, from now on, we consider games with at least two players. Theorem 1 is proved with the help of the following lemmata.

Lemma 1 *The D-core satisfies projection consistency, reasonableness (from above),* $(*)$*-antimonotonicity, and modularity.*

PROOF: First we show that the D-core satisfies projection consistency. Let (N, v) be a game, $x \in DC(N, v)$ and $\varnothing \neq S \subset N$. Then, $x \in I(N, v)$, and from the definition of the projected reduced game $x_S \in I(S, r_x^S(v))$. Assume that $x_S \notin DC(S, r_x^S(v))$. Then, there is an imputation $y \in I(S, r_x^S(v))$ and a coalition $\varnothing \neq R \subset S$, $1 < |R| < |S|$, such that $y \text{ dom}^{r_x^S(v)} x_S$ via R. Define $z \in I(N, v)$ as follow: $z_i := y_i$, for all $i \in S$, and $z_i := x_i$, for all $i \in N \setminus S$. Clearly $z \text{ dom}^v x$ via R, which is a contradiction with the fact that $x \in DC(N, v)$. Hence, we can conclude that $x_S \in DC(S, r_x^S(v))$.

To prove Reasonableness (from above), let $x \in DC(N, v)$ and assume, on the contrary, that there is $i \in N$ such that $x_i > \max_{S \subseteq N \setminus \{i\}} \{v(S \cup \{i\}) - v(S)\}$. Now, let $\varepsilon = \min \left\{ \frac{v(N \setminus \{i\}) - x(N \setminus \{i\})}{n-1}, \frac{x_i - v(\{i\})}{n-1} \right\} > 0$, and define the vector $y \in \mathbb{R}^N$ as follows: $y_j := x_j + \varepsilon$, for any $j \in N \setminus \{i\}$, and $y_i := x_i - (n-1)\,\varepsilon$. Then, $y \in I(N, v)$ and $y \text{ dom}^v x$ via $N \setminus \{i\}$, which contradicts the fact that $x \in DC(N, v)$.

To show $(*)$-antimonotonicity it is enough to take into account the antimonotonicity of the core and the fact that, for any game (N, v), $DC(N, v) = C(N, v^*)$, where (N, v^*) is defined by (2). Modularity follows straightforward from the coincidence between the core and the D-core for modular games. □

Lemma 2 *Let σ be a solution on Γ satisfying projection consistency, and modularity. Then, σ is Pareto optimal.*

PROOF: Let σ be a solution on Γ satisfying **P-CONS** and **MOD**, $(N, v) \in \Gamma$, and $x \in \sigma(N, v)$. For $i \in N$, by **P-CONS**, $x_i \in \sigma(\{i\}, r_x^{\{i\}}(v))$. Since $(\{i\}, r_x^{\{i\}}(v))$ is the modular game generated by $y = r_x^{\{i\}}(v)(\{i\}) \in \mathbb{R}$, by **MOD** $x_i = r_x^{\{i\}}(v)(\{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} x_j$, and thus $x(N) = v(N)$. □

Lemma 3 *Let σ be a solution on Γ satisfying $(*)$-antimonotonicity, and modularity. Then, for any $(N, v) \in \Gamma$, $DC(N, v) \subseteq \sigma(N, v)$.*

Proof: Let σ be a solution on Γ satisfying the above properties, $(N,v) \in \Gamma$ and $x \in DC(N,v)$. Then $x \in C(N,v^*)$, where (N,v^*) is defined by (2). Now define the modular game (N,v_x). Clearly $v_x^* = v_x$. But then, $v_x^* \geq v^*$, and by $(*)$-**AMON** we obtain $\sigma(N,v_x) \subseteq \sigma(N,v)$. Finally, by **MOD**, $x \in \sigma(N,v_x)$, and then $x \in \sigma(N,v)$. □

Lemma 4 *Let σ be a solution on Γ satisfying projection consistency, reasonableness (from above), $(*)$-antimonotonicity, and modularity. Then, for any $(N,v) \in Z^N$, $\sigma(N,v) \subseteq C(N,v)$.*

Proof: Let σ be a solution on Γ satisfying **P-CONS**, **REAB**, $(*)$-**AMON** and **MOD**, and $(N,v) \in Z^N$. From Llerena and Rafels (2005) we know that there is a finite collection of convex games $(N,v_1),\ldots,(N,v_k)$ such that

$$v = \max\{v_1,\ldots,v_k\}, \text{ with } v(N) = v_1(N) = \ldots = v_k(N). \tag{3}$$

By N-monotonicity, $v^* = v$, and by convexity, $v_l^* = v_l$, for all $l \in \{1,\ldots,k\}$. Since $v^* = v \geq v_l = v_l^*$, for all $l \in \{1,\ldots,k\}$, by $(*)$-**AMON**,

$$\sigma(N,v) \subseteq \bigcap_{l=1}^{k} \sigma(N,v_l). \tag{4}$$

Let $x \in \sigma(N,v)$ and for any $l \in \{1,\ldots,k\}$ consider the convex game (N,v_l). By (4), $x \in \sigma(N,v_l)$, and by **REAB**,

$$x_i \leq \max_{S \subseteq N\setminus\{i\}} \{v_l(S\cup\{i\}) - v_l(S)\} = v_l(N) - v_l(N\setminus\{i\}), \forall\, i \in N, \tag{5}$$

where the equality follows from the convexity of the game (N,v_l). Now, by Pareto optimality (Lemma 2), $x(N\setminus\{i\}) \geq v_l(N\setminus\{i\})$. Or, equivalently, $x(S) \geq v_l(S)$ for any coalition $S \subset N$ with $|S| = n-1$.

From the convexity of the game (N,v_l), and taking into account that $x_i \leq v_l(N) - v_l(N\setminus$

$\{i\})$, it is straightforward to check that the projected reduced game $(N\setminus\{i\}, r_x^{N\setminus\{i\}}(v_l))$ is also a convex game. By **P-CONS**, $x_{N\setminus\{i\}} \in \sigma(N\setminus\{i\}, r_x^{N\setminus\{i\}}(v_l))$. **REAB** together with the convexity of the projected reduced game implies that, for any player $j \in N\setminus\{i\}$,

$$x_j \leq r_x^{N\setminus\{i\}}(v_l)(N\setminus\{i\}) - r_x^{N\setminus\{i\}}(v_l)(N\setminus\{i,j\}).$$

From the definition of the projected reduced game we have that,

$$x_j \leq v_l(N) - x_i - v_l(N\setminus\{i,j\}), \text{ for any } j \in N\setminus\{i\}.$$

Thus, by Pareto optimality we can conclude that, for any coalition $S \subset N$ with $|S| = n-2$, $x(S) \geq v_l(S)$.

By repeating the same argument, and taking into account that the projected reduction operation is transitive (i.e. for any $(N,v) \in \Gamma$, all $x \in \mathbb{R}^N$ and all $\varnothing \neq S \subset T \subseteq N$, $r_{x_T}^S(r_x^T(v)) = r_x^S(v)$), we can conclude that, for any coalition $S \subseteq N$, $x(S) \geq v_l(S)$. Hence, for any $l \in \{1,\ldots,k\}$,

$$\sigma(N,v_l) \subseteq C(N,v_l). \tag{6}$$

Combining expressions (3), (4) and (6), and taking into account that

$$C(N,v) = C(N,\max\{v_1,\ldots,v_k\}) = \bigcap_{l=1}^{k} C(N,v_l),$$

we obtain

$$\sigma(N,v) \subseteq \bigcap_{l=1}^{k} \sigma(N,v_l) \subseteq \bigcap_{l=1}^{k} C(N,v_l) = C(N,v).$$

□

Lemma 5 *Let σ be a solution on Γ satisfying projection consistency, reasonableness (from above), $(*)$-antimonotonicity, and modularity. Then, for any $(N,v) \in \Gamma$, $\sigma(N,v) \subseteq DC(N,v)$.*

PROOF: Let σ be a solution on Γ satisfying the above properties, $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$. Consider the games (N, v') and (N, v^*) defined by (1) and (2), respectively. Since $(N, v') \in Z^N$, $(v')^* = v'$. Moreover, by definition $v^* \geq v'$. Hence, by $(*)$-AMON and Lemma 4, $\sigma(N, v) \subseteq \sigma(N, v') \subseteq C(N, v')$. Now we can distinguish two cases. First, if $I(N, v) \neq \varnothing$, then $DC(N, v) = C(N, v')$ (Rafels and Tijs, 1997, and Chang, 2000), and so $\sigma(N, v) \subseteq DC(N, v)$. Otherwise, if $I(N, v) = \varnothing$, from Llerena and Rafels (2005) we can express $v = \max\{v_1, \ldots, v_k\}$, where $(N, v_1), \ldots, (N, v_k)$ are convex games with $v(N) = v_1(N) = \ldots = v_k(N)$. Since $I(N, v) = \varnothing$, $v^* = v$, and by convexity $v_l^* = v_l$, for any $l = 1, \ldots, k$. Then, by $(*)$-AMON and Lemma 4:

$$\sigma(N, v) \subseteq \bigcap_{l=1}^{k} \sigma(N, v_l) \subseteq \bigcap_{l=1}^{k} C(N, v_l) = C(N, \max\{v_1, \ldots, v_k\}) = C(N, v).$$

But $C(N, v) = \varnothing$ since $I(N, v) = \varnothing$, which implies $\sigma(N, v) = \varnothing$ and $\sigma(N, v) \subseteq DC(N, v)$.

□

This completes the proof of the Theorem 1.

The following examples show that the above axioms are independent:

- Let σ^1 be the empty set: $\sigma^1(N, v) := \varnothing$, for each $(N, v) \in \Gamma$. Then, σ^1 satisfies **P-CONS**, **REAB**, $(*)$**-AMON**, but not **MOD**.

- Let σ^2 be the set of imputations of a game. Then, σ^2 satisfies **P-CONS**, $(*)$**-AMON**, **MOD**, but not **REAB**.

- Let σ^3 be the core of a game. Then, σ^3 satisfies **P-CONS**, **REAB**, **MOD**, but not $(*)$**-AMON**.

- Let it be $\sigma^4(N, v) := \{x \in X^*(N, v) \mid v^*(\{i\}) \leq x_i \leq b_i^{v^*},\ \text{for all } i \in N\}$, where (N, v^*) is defined by (2). Then, σ^4 satisfies **REAB**, $(*)$**-AMON**, **MOD**, but not **P-CONS**.

Now we introduce a new axiomatic characterization of the core where $(*)$-antimonotonicity is replaced by antimonotonicity.

Theorem 2 *The core is the only solution on Γ satisfying projection consistency, reasonableness (from above), antimonotonicity, and modularity.*

PROOF: Clearly, the core satisfies **P-CONS**, **REAB**, **AMON**, and **MOD**. Let σ be a solution satisfying these properties and $(N, v) \in \Gamma$.

To show the inclusion $C(N, v) \subseteq \sigma(N, v)$, let $x \in C(N, v)$ and consider the modular game (N, v_x) generated by x. Then, by **AMON** and **MOD** we have that $x \in \sigma(N, v)$. To prove the reverse inclusion it is enough to follow the proof of Lemma 4 considering an arbitrary game (N, v) and applying **AMON** instead of $(*)$-**AMON**. $\square$

Remark 1 *: Since the projected reduction operation w.r.t. a core element is closed for convex games, totally balanced games, balanced games, and superadditive games, Theorem 2 also characterizes the core in all these domains. Notice that the max-convex decomposition result (Llerena and Rafels, 2005) used in the proof of Theorem 2 can be applied to extend the axiomatization to the above domains because the class of convex games is included in the other classes. Moreover, it is important to point out that Theorem 2 is valid on the universal domain, that is, no constraints on the number of players are needed.*

The following examples show that the above axioms are independent:

- Let σ^1 be the empty set: $\sigma^1(N, v) := \varnothing$, for each $(N, v) \in \Gamma$. Then, σ^1 satisfies **P-CONS**, **REAB**, **AMON**, but not **MOD**.

- Let σ^2 be the set of imputations of a game. Then, σ^2 satisfies **P-CONS**, **AMON**, **MOD**, but not **REAB**.

- Let σ^5 be the D-core of a game. Then, σ^5 satisfies **P-CONS**, **REAB**, **MOD**, but not **AMON**.
- Let it be $\sigma^6(N, v) := \{x \in X^*(N, v) \mid v(\{i\}) \leq x_i \leq b_i^v, \text{ for all } i \in N\}$, where $b_i^v = v(N) - v(N \backslash \{i\})$, for all $i \in N$. Then, σ^6 satisfies **REAB**, **AMON**, **MOD**, but not **P-CONS**.

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