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Uniform-price assignment markets

Marina Núñez and Carles Rafels [1]

Adreça correspondència

Departament de Matemàtica Econòmica, Financera i Actuarial, i CREB

Facultat de Ciències Econòmiques i Empresarials

Universitat de Barcelona

Avda. Diagonal, 690

08034 BARCELONA

Tfn. 93 403 19 91

Fax. 93 403 48 92

e-mail: mnunez@ub.edu, crafels@ub.edu

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Abstract: Uniform–price assignment games are introduced as those assignment markets with the core reduced to a segment. In these games, for all active agents, competitive prices are uniform although products may be non-homogeneous. A characterization in terms of the assignment matrix is given. The only assignment markets where all submarkets are uniform are the Böhm–Bawerk horse markets. We prove that for uniform-price assignment games the kernel, or set of symmetrically-pairwise bargained allocations, either coincides with the core or reduces to the nucleolus.
Key words: assignment game, core, Böhm–Bawerk horse market, kernel
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Resum: Els jocs d'assignació amb preu uniforme són aquells mercats d'assignació on el core es redueix a un segment. En aquests casos, per a tots els agents actius en el mercat, els preus competitius varien de forma uniforme, tot i que els productes poden ser no homogenis. En aquest treball es dona una caracterització dels mercats amb preu uniforme a partir de la matriu d'assignació. Els únics mercats on tots els subjocs són de preu uniforme són els mercats de cavalls de Böhm-Bawerk. Finalment, provem que en aquests mercats de preu uniforme el kernel, o conjunt de pagaments que s'obtenen a partir d'un procés de negociació bilateral i simètric, o bé coincideix amb tot core o es redueix al seu punt mig que és el nucleolus.

1 Introduction

In an assignment market, two disjoint sets of agents exist, let us say buyers and sellers, and when a buyer is paired with a seller an additional value is created. The first problem is to find an optimal matching, that is to say an assignment of buyers to sellers that maximizes the total profit. The second one is to know how the output of each pairing will be divided between the two agents involved in the trade. It is assumed in such markets that the goods traded are indivisible and heterogeneous, and utilities are transferable. The difference between what the object sold is worth to the buyer and the minimum that would be accepted by the seller must be divided between them by determining a price. Thus, one of the most interesting questions regarding the classical assignment market is that of the formation of prices.

Given a matching of buyers to sellers, a vector of incomes that allocates the output of each pairing between the corresponding paired agents is said to be stable if no other mixed-pair formed by a buyer and a seller could produce together more than the sum of their incomes. In 1972 Shapley and Shubik introduce the assignment game as a cooperative model for two–sided markets with transferable utility. They prove that assignment games have a non empty core and it coincides with the set of stable allocations. Moreover, the core turns out to be in one–to–one correspondence with the set of competitive price vectors.

Before that, Böhm–Bawerk had carried out a deep analysis of some very simple markets which are now known as Böhm–Bawerk horse markets. After Shapley and Shubik (1972), Böhm–Bawerk horse markets appear as a particular case of an assignment market when there is no product differentiation. In the Böhm–Bawerk horse market each buyer places the same valuation on each one of the objects and in equilibrium all transactions take place at the same price. There is then no possibility of price discrimination in those assignment markets where there is no product differentiation. In fact, Shapley and Shubik (1972) prove that, in the Böhm–Bawerk horse market, the core is a segment and consequently the set of competitive prices is also one–dimensional. The term *uniform prices* is already used there.

The aim of the present paper is thus to study those markets where although the buyers discriminate between the objects (each buyer may place different values on different objects), the prices still behave uniformly (that is, they move in a segment). More formally, we will analyze those assignment markets where the core reduces to a segment, not necessarily being Böhm–Bawerk markets.

Section 2 presents the definitions and notations for the assignment model that will be needed in the paper. Section 3 defines the uniform–price assignment games as those assignment games with a one–dimensional core, and gives a characterization of these games in terms of their matrix. In Section 4 we analyze when an assignment market is such that every submarket is also uniform–price and prove that this only happens in the case of Böhm–Bawerk horse markets.

Although it reduces to a segment, there are still infinitely many possible allocations in the core of a uniform-price assignment game. A way of selecting some core allocations of an assignment market with additional stability properties is to consider the set of pairwise–bargained allocations introduced by Rochford (1984), which coincides with the kernel of the game. In Section 4 we compute the set of pairwise–bargained allocations of a uniform–price assignment game, and it turns out that this set either coincides with the core or reduces to the midpoint of the core.

2 The assignment model

Let M be a finite set of buyers and M' a finite set of sellers and let us denote by m and m' their cardinalities. We may think of the formal model of assignment games as arising from a situation where each seller $j \in M'$ has an object for sale which he valuates in $c_j \in \mathbf{R}_+$ (reservation price of seller j), being $\mathbf{R}_+$ the set of non negative real numbers, while each buyer $i \in M$ wants exactly one indivisible object and places a value of $h_{ij} \in \mathbf{R}_+$ in the object offered by seller j, $h_i = (h_{ij})_{j \in M'}$. Then, if $h = (h_i)_{i \in M}$ and $c = (c_j)_{j \in M'}$, a matrix $A = A(h, c) = (a_{ij})_{(i,j) \in M \times M'}$ is defined, where $a_{ij} = \max\{h_{ij} - c_j, 0\}$ are the potential gains from the trade between i and j. We will denote by $\mathcal{M}_{m \times m'}(\mathbf{R}_+)$ the set of non negative matrices with m rows and m' columns. An assignment market is then a triple (M, M', A).

A *matching* (or assignment) between M and M' (or a matching for A) is a subset μ of $M \times M'$ such that each $k \in M \cup M'$ belongs to at most one pair in μ. We will denote by $\mathcal{M}(A)$ the set of matchings of A. A matching μ is *optimal* if for all $\mu' \in \mathcal{M}(A)$, $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ and we denote by $\mathcal{M}^*(A)$ the set of optimal matchings.

The above two–sided market can be described by means of a cooperative game where the player set consists of the union $M \cup M'$ of the sets of buyers and sellers. Then, $m+m'$ is the cardinality of the player set. The profits of mixed–pair coalitions, $\{i,j\}$ where $i \in M$ and $j \in M'$, are $w_A(i,j) = a_{ij} \geq 0$, and the matrix A also determines the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, $\mathcal{M}(S,T)$ being the set of matchings between S and T. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. Moreover, we say a buyer $i \in M$ is *not assigned* by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers). We denote by $\mu(i)$ the seller j such that $(i,j) \in \mu$ and then we also write $i = \mu^{-1}(j)$.

Shapley and Shubik (1972) prove that the core, $C(w_A)$, of the assignment game $(M \cup M', w_A)$ is nonempty and coincides with the set of *stable outcomes*. This means that given any optimal matching μ of $M \cup M'$, a payoff vector $(u,v) \in \mathbf{R}_+^M \times \mathbf{R}_+^{M'}$ is in the core if $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$, $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M \times M'$, and the payoff to any agent not matched by μ is zero.

Moreover, the core has a lattice structure with two special extreme core allocations: the *buyers–optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains her maximum core payoff, and the *sellers–optimal core allocation*, $(\underline{u}, \overline{v})$, where each seller does. Notice that, when agents on one side of the market obtain their maximum core payoff, the agents on the opposite side obtain their minimum core payoff, as the joint payoff of an optimally matched pair is fixed.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his marginal contribution:

$$\overline{u}_i = w_A(N) - w_A(N \setminus \{i\}) \text{ and } \overline{v}_j = w_A(N) - w_A(N \setminus \{j\}) \tag{1}$$

for all $i \in M$ and all $j \in M'$. As a consequence, for each optimally matched pair

(i, j), the minimum core payoffs are

$$\begin{aligned} \underline{u}_i &= w_A(M \cup M' \setminus \{j\}) - w_A(M \cup M' \setminus \{i,j\}) \text{ and} \\ \underline{v}_j &= w_A(M \cup M' \setminus \{i\}) - w_A(M \cup M' \setminus \{i,j\}) . \end{aligned} \tag{2}$$

The present paper is devoted to the analysis of those assignment games which have a segment as a core, the segment with extreme points the buyers–optimal and the sellers–optimal core allocations. It is important to point out that different assignment games may have the same core. For instance, if there exists a matrix entry a_{ij} which is not attained by any core allocation, $u_i + v_j > a_{ij}$ for all $(u, v) \in C(w_A)$, then the worth of a_{ij} can be slightly raised without changing the core. It is shown in Núñez and Rafels (2002) that for every assignment matrix A there exists a unique matrix A^r such that $C(w_A) = C(w_{A^r})$ and w_{A^r} is *buyer–seller exact*, which means that for all $(i, j) \in M \times M'$ there exists $(u, v) \in C(w_{A^r})$ with $u_i + v_j = a^r_{ij}$. Thus no entry in A^r can be raised without changing the core.

If we assume that A is square, and this can always been achieved by adding dummy players, then the entries in matrix A^r, once fixed an optimal matching μ, are

$$a^r_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} + w_A(M \cup M' \setminus \{\mu^{-1}(j), \mu(i)\}) - w_A(M \cup M') . \tag{3}$$

Moreover, an assignment game $(M \cup M', w_A)$ is buyer–seller exact ($A = A^r$) if and only if its matrix A is *doubly dominant diagonal*, which means that $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}$ for all $i,\, k \in M$ and $j \in M'$.

A particular case of assignment market is a *glove market*. In a glove market (Shapley, 1959), not only goods are homogeneous but, in addition to that, all buyers have the same valuation for them all, and all sellers have the same reservation price. Then an assignment market $(M,\, M',\, A)$ is a glove market if $a_{ij} = c \geq 0$ for all $i \in M$ and all $j \in M'$. Those glove markets where the number of buyers differs from the number of sellers have only one core point, where each agent on the short side of the market gets c while agents on the large side get zero. If there are as many buyers as sellers, the core is a line segment where each agent can obtain any amount from zero to c. We will now extend this notion by allowing for some dummy agents in the glove market.

Definition 1 *An assignment market* (M, M', A) *is an* extended glove market *if there exists a subset of buyers* $M_1 \subseteq M$ *and a subset of sellers* $M'_1 \subseteq M'$ *such that* $a_{ij} = c \geq 0$ *for all* $(i,j) \in M_1 \times M'_1$ *and* $a_{ij} = 0$ *if* $i \notin M_1$ *or* $j \notin M'_1$.

If $|M_1| = |M'_1|$ *we say the game is an extended square glove market.*

The core of the extended square glove market is also a segment, where all $i \in M \setminus M_1$ and all $j \in M' \setminus M'_1$, receive zero payoff in any core allocation. These games will play an important representative role in the next section.

To end this section, we recall the definitions of competitive prices and competitive equilibrium. As a notational convention, we assume in this definition that M' contains an artificial agent 0 the object of whom has null value, $h_{i0} = 0$ for all $i \in M$. Several buyers may buy the object of seller 0. A *feasible price vector* is $p \in \mathbf{R}^{m'}_{+}$ such that $p_j \geq c_j$ for all $j \in M' \setminus \{0\}$, where c_j is the reservation price of seller j, and $p_0 = 0$. Once fixed a feasible price p, the *demand set* for buyer $i \in M$ at price p is defined by $D_i(p) = \{j \in M' \mid h_{ij} - p_j = \max_{k \in M'}\{h_{ik} - p_k\}\}$. Now, a price vector p is *quasi-competitive* if there is a matching $\mu \in \mathcal{M}(A)$ such that if $\mu(i) = j$ then $j \in D_i(p)$ and if i is not matched by μ then $0 \in D_i(p)$. Then μ is said to be *compatible* with price p. Finally, the pair (p, μ) is a *competitive equilibrium* if p is quasi-competitive, μ is compatible with p and $p_j = c_j$ for all j not matched by μ. We then say that p is an *equilibrium price vector.*

It is easy to check that if (p, μ) is a competitive equilibrium then the corresponding payoffs (u, v) are stable, where $u_i = h_{ij} - p_j$, if $(i,j) \in \mu$, and $v_j = p_j - c_j$ for all $j \in M' \setminus \{0\}$. Conversely, if (u, v) is stable and $p_j = v_j + c_j$ for all $j \in M'$, then p is an equilibrium price vector.

3 Uniform–price assignment markets: definition and characterization

We introduce now those assignment markets which have a segment as a core as *uniform–price assignment markets.*

Definition 2 *An assignment market* $(M,\, M',\, A)$ *is uniform–price if and only if the core is a segment, that is to say*

$$C(w_A) = [(\underline{u}, \overline{v}), (\overline{u}, \underline{v})] = \{\lambda(\underline{u}, \overline{v}) + (1-\lambda)(\overline{u}, \underline{v})\,,\ \lambda \in [0,1]\,\}.$$

Notice that in the above definition those assignment games where the competitive equilibrium price is unique are also included. In these games the core shrinks as much as possible.

We know from Shapley and Shubik (1972) that Böhm–Bawerk horse markets are uniform–price assignment games. We will see in the next section that if we consider 2×2 matrices, all uniform–price markets are Böhm–Bawerk horse markets, but for higher dimensions both classes differ. The assignment game with set of buyers $M = \{1, 2, 3\}$, set of sellers $M' = \{1', 2', 3'\}$ and defined by matrix

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

has a one–dimensional core $C(w_{A_1}) = [(1,1,1;0,0,0),(0,0,0;1,1,1)]$, but is not a Böhm–Bawerk horse market. To see that, notice that from $a_{11'} = a_{21'}$ we deduce that buyers 1 and 2 have the same valuation for the object of seller 1', but this enters in contradiction with $a_{12'} \neq a_{22'}$.

In the above example, we can rise the entry $a_{12'}$ in one unit to obtain a Böhm–Bawerk horse market with the same core. This fact might suggest that the existence of uniform–price assignment games which are not horse markets is only caused by the lack of exactness and that given any such game A, its buyer–seller exact representative A^r (see Núñez and Rafels, 2002) is always a Böhm–Bawerk horse market. However, this is not the case, as it is shown by matrix

$$A_2 = \begin{pmatrix} 8 & 8 & 5 \\ 8 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}. \tag{4}$$

It is easily obtained that $C(w_{A_2}) = [(5,6,0;3,3,0),(6,6,0;2,3,0)]$ and it can then be noticed that every matrix entry is attained in some extreme core allocation, which means that w_{A_2} is buyer–seller exact. But the same argument used for matrix A_1 shows that A_2 does neither define a Böhm-Bawerk horse market.

If $(M,\, M',\, A)$ is an arbitrary assignment market, let us denote by I_0 the subset of buyers with fixed core payoff, and J_0 the subset of sellers with the same property:

$$I_0 = \{i \in M \mid \overline{u}_i = \underline{u}_i\} \text{ and } J_0 = \{j \in M \mid \overline{v}_j = \underline{v}_j\}\,.$$

Recall that $(\overline{u}, \underline{v})$ can be obtained as the solution of a few linear programs, since all buyers achieve their maximum core payoff in the same core allocation, and the same happens with $(\underline{u}, \overline{v})$. Consequently, the sets I_0 and J_0 are easily determined even for assignment games with large number of players.

Agents in I_0 or J_0 can be assumed to be *non-active* as, although they may take part of some transaction, they have no bargaining capability as their core payoff is fixed. Then, agents in $M \setminus I_0$ or $M' \setminus J_0$ are *active agents*. Let μ be any optimal matching for A. Notice first that if i is not matched by μ, then $u_i = 0$ for all $(u, v) \in C(w_A)$, and thus $i \in I_0$. Also $j \in J_0$ if j is not matched by μ. Moreover, if $i \in I_0$ and $(i, j) \in \mu$ for some $\mu \in \mathcal{M}^*(A)$, then from $\underline{u}_i + \overline{v}_j = \overline{u}_i + \underline{v}_j = a_{ij}$ follows $\underline{v}_j = \overline{v}_j$ and thus $j \in J_0$. Similarly, if $j \in J_0$ and $(i, j) \in \mu$, $\mu \in \mathcal{M}^*(A)$, then $i \in I_0$. This implies that the number of active players is always even and we will sometimes refer to any pair (i, j) formed by active agents as an *active pair.*

Since the core of an assignment game is always a 45^o–lattice (Quint, 1991), given a uniform-price assignment market $(M,\, M',\, A)$, its core segment cannot have arbitrary slopes: the core payoffs to the active buyers must be of the type $\overline{u} - \varepsilon \cdot 1$, where $1 \in \mathbf{R}^m$ gives unitary payoff to any active buyer and null payoff to any non-active one, and $0 \le \varepsilon \le K$ for some fixed $K \ge 0$. Thus, the core can be described by

$$C(w_A) = \left\{ (u, v) \in \mathbf{R}_+^{m+m'} \;\middle|\; \begin{array}{l} u_i = \overline{u}_i \text{ for } i \in I_0\,, \\ u_i = \overline{u}_i - \varepsilon \text{ for } i \in M \setminus I_0\,, \\ v_j = \underline{v}_j \text{ for } j \in J_0\,, \\ v_j = \underline{v}_j + \varepsilon \text{ for } j \in M' \setminus J_0\,, \end{array} \text{ for some } \varepsilon \in [0, K] \right\} \tag{5}$$

where $K = \overline{u}_i - \underline{u}_i = \overline{v}_j - \underline{v}_j$ for all $i \in M \setminus I_0$ and $j \in M' \setminus J_0$.

Notice that the prices of those transactions between active agents are described by the single parameter ε: once fixed an arbitrary $j \in M' \setminus J_0$, $p_j = c_j + \underline{v}_j + \varepsilon$. We then say that prices vary uniformly.

The next theorem characterizes the uniform–price assignment markets in terms of the assignment matrix, under the assumption that A is square. Notice that this can always be achieved by adding null rows or columns, and this action does not modify the dimension of the core. Recall also that the minimum core payoff of each agent can be easily computed from the matrix, by using equations (2).

Theorem 3 *Let (M, M', A) be an assignment market with as many buyers as sellers, A^r the unique buyer–seller exact matrix such that $C(w_A) = C(w_{A^r})$ and let it be $\underline{A} = (\underline{a}_{ij})_{(i,j)\in M\times M'}$ where $\underline{a}_{ij} = \underline{u}_i + \underline{v}_j$. Then the following statements are equivalent:*

1. *(M, M', A) is uniform–price.*

2. *$A^c = A^r - \underline{A}$ defines an extended square glove market.*

Proof: $1 \Rightarrow 2$) Since $(M \cup M', w_A)$ is a uniform–price assignment game, A^r defines a buyer–seller exact assignment game such that $C(w_A) = C(w_{A^r})$ is a segment. For all $(i,j) \in M \times M'$, there exists $(u,v) \in C(w_A)$ such that $a^r_{ij} = u_i + v_j \geq \underline{u}_i + \underline{v}_j$ and consequently $A^c \geq 0$. Moreover, if $(i,j) \in M \times M'$ with $i \in I_0$, we get $a^r_{ij} \geq \underline{u}_i + \underline{v}_j = \overline{u}_i + \underline{v}_j$. But, being $(\overline{u}, \underline{v})$ a core allocation, it must hold $a^r_{ij} = \underline{u}_i + \underline{v}_j$, and then $a^c_{ij} = 0$. The same happens if $j \in J_0$. Finally, as $C(w_{A^r})$ is a segment, from (5) follows that, if $i \in M$ and $j \in M'$ are active, $u_i + v_j = u'_i + v'_j$ for any pair of core allocations (u,v) and (u',v') and consequently, from buyer–seller exactness, $u_i + v_j = a^r_{ij}$ for all $(u,v) \in C(w_{A^r})$. In particular, $\underline{u}_i + \overline{v}_j = a^r_{ij}$.

Then, for $i \in M \setminus I_0$ and $j \in M' \setminus J_0$,

$$a^c_{ij} = a^r_{ij} - \underline{u}_i - \underline{v}_j = \underline{u}_i + \overline{v}_j - \underline{u}_i - \underline{v}_j = \overline{v}_j - \underline{v}_j = K > 0\,,$$

which proves A^c is an extended square glove market.

$2 \Rightarrow 1$) Assume now $A^r = \underline{A} + A^c$, where A^c is an extended square glove market. We prove first that any optimal matching $\mu \in \mathcal{M}^*(A^c)$ which is maximal, in the sense that all agents are matched by μ, is also optimal for A^r. To see that, take any $\mu \in \mathcal{M}^*(A^c)$ and maximal. This can always be achieved since A^c is square. Then $\sum_{(i,j)\in\mu} a^c_{ij} = \sum_{(i,j)\in\mu} a^r_{ij} - \sum_{i\in M} \underline{u}_i - \sum_{j\in M'} \underline{v}_j$. Moreover, for any $\mu' \in \mathcal{M}(A^c)$,

$$\sum_{(i,j)\in\mu} a^c_{ij} \geq \sum_{(i,j)\in\mu'} a^c_{ij} = \sum_{(i,j)\in\mu'} a^r_{ij} - \sum_{(i,j)\in\mu'} (\underline{u}_i + \underline{v}_j) \geq \sum_{(i,j)\in\mu'} a^r_{ij} - \sum_{i\in M} \underline{u}_i - \sum_{j\in M'} \underline{v}_j \,,$$

which implies $\sum_{(i,j)\in\mu} a^r_{ij} \geq \sum_{(i,j)\in\mu'} a^r_{ij}$, and $\mu \in \mathcal{M}^*(A^r)$.

Now, we prove that $C(w_{A^r}) \subseteq (\underline{u},\underline{v}) + C(w_{A^c}) = \{(u,v) \in \mathbf{R}^{m+m'} | (u,v) = (\underline{u},\underline{v}) + (u',v') \text{ and } (u',v') \in C(w_{A^c})\}$. To see that, for all $(u,v) \in C(w_{A^r})$ define (u',v') in the following way:

$$u'_i = u_i - \underline{u}_i \quad \text{for all } i \in M \;\text{ and }\; v'_j = v_j - \underline{v}_j \quad \text{for all } j \in M' .$$

Let us see that $(u',v') \in C(w_{A^c})$. Notice first that $u'_i \geq 0$ and $v'_j \geq 0$. Taking an arbitrary maximal optimal matching $\mu \in \mathcal{M}^*(A^c)$, $u'_i + v'_j = a^c_{ij}$ if $(i,j) \in \mu$, since we also have $\mu \in \mathcal{M}^*(A^r)$. Moreover, for all $i \in M$ and all $j \in M'$, $u'_i + v'_j = u_i - \underline{u}_i + v_j - \underline{v}_j \geq a^r_{ij} - \underline{u}_i - \underline{v}_j = a^c_{ij}$. Then $(u,v) = (\underline{u},\underline{v}) + (u',v')$ where $(u',v') \in C(w_{A^c})$, which proves that $C(w_{A^r}) \subseteq (\underline{u},\underline{v}) + C(w_{A^c})$.

As A^c defines an extended square glove market, $C(w_{A^c})$ is a segment and, from the above argument, $C(w_{A^r})$ is included in the translation of $C(w_{A^c})$ by the vector $(\underline{u},\underline{v})$ and thus $C(w_{A^r}) = C(w_A)$ is also a segment. □

Let us remark that, by the proof of the above theorem, we get that given a uniform-price assignment market (M, M', A) and its related extended square glove market A^c, $a^c_{ij} = 0$ if and only if i or j are non-active.

Theorem 3 shows that any uniform–price assignment market, after exactification (A^r) and substraction of the minimum core payoffs ($\underline{A}$), gives an extended square glove market where only active pairs have a positive output. For instance, if we take again matrix A_2 in expression (4), which is buyer–seller exact, the corresponding extended glove market is $A^c_2 = A_2 - \underline{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Let us look into another example. Consider a market with four buyers and four sellers, $M = \{1, 2, 3, 4\}$ and $M' = \{1', 2', 3', 4'\}$, described by matrix

$$A_3 = \begin{pmatrix} 9 & 8 & 0 & 4 \\ 8 & 7 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 3 & 0 & 1 & 0 \end{pmatrix} \tag{6}$$

which is not buyer–seller exact. To see that, just notice that an optimal matching is placed in the diagonal and $a_{13'} + a_{22'} < a_{12'} + a_{23'}$ shows that A_3 is not doubly dominant diagonal. If we compute the marginal contribution of each agent and take into account the aforementioned optimal matching, we get the buyers–optimal core allocation $(\overline{u}, \underline{v}) = (6, 5, 2, 0; 3, 2, 1, 0)$ and the sellers–optimal core allocation $(\underline{u}, \overline{v}) = (4, 3, 2, 0; 5, 4, 1, 0)$. How to know if these are the unique extreme allocations in $C(w_{A_3})$?

The first step is to compute the matrix A_3^r by means of equation (3). Then,

$$\begin{aligned} a^r_{13'} &= a_{11'} + a_{33'} + w_{A_3}(M \cup M' \setminus \{3, 1'\}) - w_{A_3}(M \cup M') = 5 \\ a^r_{23'} &= a_{22'} + a_{33''} + w_{A_3}(M \cup M' \setminus \{3, 2'\}) - w_{A_3}(M \cup M') = 4 \\ a^r_{24'} &= a_{22'} + a_{44'} + w_{A_3}(M \cup M' \setminus \{4, 2'\}) - w_{A_3}(M \cup M') = 3 \,. \end{aligned}$$

Similarly, $a^r_{31'} = 5$, $a^r_{32'} = 4$, $a^r_{42'} = 2$ and $a^r_{ij} = a_{ij}$ otherwise, which leads to

$$A_3^r = \begin{pmatrix} 9 & 8 & 5 & 4 \\ 8 & 7 & 4 & 3 \\ 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix} .$$

On the other hand, as $(\underline{u}, \underline{v}) = (4, 3, 2, 0; 3, 2, 1, 0)$, the matrix $\underline{A_3}$ is

$$\underline{A_3} = \begin{pmatrix} 7 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

and consequently

$$A_3^c = A_3^r - \underline{A_3} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Now, from Theorem 3 follows that A_3 is a uniform–price assignment game where only the two first agents on each side of the market are active.

Finally, as a consequence of Theorem 3 we see that the equilibrium price vector is unique if and only if $A^r = \underline{A}$, which means that the buyer–seller exactification (but maybe not the original matrix) is additively generated by the minimum core payoffs.

4 Totally uniform-price assignment markets

In this section we look for those uniform-price assignment markets where all subgames also have uniform prices. A well known example of that are the Böhm–Bawerk horse markets. We ask whether these are the only assignment markets with this property.

The *Horse Market of Böhm–Bawerk* (1891) is also studied from the viewpoint of game theory in Shapley and Shubik (1972). In this market, each seller has one horse for sale and each buyer wishes to buy one horse and places the same valuation in all the horses available, as they are all alike. Let $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m'}$ be the reservation prices of the sellers and $h_1 \geq h_2 \geq \cdots \geq h_m \geq 0$ the valuations of the buyers. If $h_i < c_j$, no transaction is possible between these two agents but whenever $h_i \geq c_j$, agents i and j can trade and obtain a joint profit of $h_i - c_j$. Thus, the assignment matrix describing this market is defined by $a_{ij} = \max\{h_i - c_j, 0\}$.

It is already known from Shapley and Shubik (1972) that the core of the Böhm–Bawerk horse market game consists of a segment, with extreme points the buyers–optimal and the sellers–optimal core allocations. Moreover, in a core allocation all transactions take place at the same price. This means that there exists an interval of prices $[\underline{p}, \overline{p}]$ such that $(u, v) \in C(w_A)$ if and only if there exists $p \in [\underline{p}, \overline{p}]$ and

$$u_i = h_i - p \text{ and } v_j = p - c_j \tag{7}$$

if buyer i and seller j are involved in some transaction, while the remaining agents receive a zero payoff.

In this section, given an arbitrary assignment matrix, we will need to recognize, merely by inspection of the matrix entries, if it represents such a particular market as the Böhm–Bawerk horse market. To this end we will develop an idea already present in the work of Shapley and Shubik (1972), who point out that a property of the assignment matrix of these particular markets is that *in each* 2×2 *submatrix with nonzero entries, the sums of the diagonals are equal.* This property is not enough to characterize the matrices defining a Böhm–Bawerk horse market (see for instance matrix A_1).

However, it is not difficult to prove that a 2×2 assignment matrix defines a Böhm–Bawerk horse market if and only if either two optimal matchings exist or there is only one optimal matching but one of the optimally matched pairs has a null outcome. Thus, 2×2 matrices defining a Böhm–Bawerk horse market are, up to possible permutations of buyers or sellers, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{11} + a_{22} = a_{12} + a_{21}$, or $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}$ with $a_{11} \geq a_{12} + a_{21}$.

The next corollary, which is a consequence of Theorem 3, states that the class of uniform-price assignment markets coincides with the class of Böhm–Bawerk horse markets if we restrict to 2×2 matrices. We leave the proof for the appendix.

Corollary 4 *Every uniform-price assignment market defined by a* 2×2 *matrix is a Böhm-Bawerk horse market.*

Notice that, unlike what happens with Böhm–Bawerk horse markets, the subgames of a uniform–price assignment market, need not be uniform–price. Take for instance the submatrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ from $A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and notice that it defines an assignment game with a two–dimensional core.

We name *totally uniform–price assignment markets* those assignment markets such that every submarket is also uniform–price. Next theorem states that this property characterizes the Böhm–Bawerk horse markets.

Theorem 5 *Let* $(M,\, M',\, A)$ *be an assignment market. The following statements are equivalent:*

1. $(M,\, M',\, A)$ *is totally uniform–price,*
2. $(M,\, M',\, A)$ *is a Böhm–Bawerk horse market.*

PROOF: If (M, M', A) is a Böhm–Bawerk horse market, then all subgames are also Böhm–Bawerk horse markets, and thus they are all uniform–price.

To prove the converse statement notice first that if one side of the market has only one agent, then trivially the market is a Böhm–Bawerk horse market. So, let us assume that (M, M', A) is a totally uniform–price assignment market with at least two agents on each side. Then every 2×2–subgame is also uniform-price and, by Corollary 4, it is a Böhm–Bawerk horse market. We claim that this property characterizes the Böhm–Bawerk horse markets:

Claim: *If* $A \in \mathcal{M}_{m\times m'}(\mathbf{R}_+)$ *is such that every* 2×2 *submatrix defines a Böhm–Bawerk horse market, then* A *also defines a Böhm–Bawerk horse market.*

The proof of this claim is done in the appendix. □

5 The kernel or symmetrically pairwise–bargained allocations

In this second part of the paper, we analyze which core allocations of a uniform-price assignment game are supported by another cooperative set-solution concept as it is the kernel of the game. The kernel of a cooperative game (Davis and Maschler, 1965) is always nonempty and it always contains the nucleolus, a single-valued solution concept introduced by Schmeidler (1969).

The first analysis of the kernel of an assignment game, $\mathcal{K}(w_A)$, is carried out by Rochford (1984) where the optimally matched players are engaged in bargaining of the sort modelled by Nash, using as their threats the maximum they could receive in an alternative matching. A symmetrically pairwise–bargained (SPB) allocation is a core allocation such that all partners are in bargained equilibrium. Rochford proves that an SPB allocation always exists and that the set of SPB allocations coincides with the intersection of the kernel and the core of the assignment game. Once proved, for assignment games, the inclusion of the kernel in the core (Driessen, 1998), it turns out that the set of SPB allocations is the kernel of the game.

Given an assignment game $(M \cup M', w_A)$, it is easy to see, and it is justified in Rochford (1984), that once fixed an optimal matching $\mu \in \mathcal{M}^*(A)$ and given $(u,v) \in C(w_A)$, we get that $(u,v) \in \mathcal{K}(w_A)$ if and only if $s_{ij}(u,v) = s_{ji}(u,v)$ for all $(i,j) \in \mu$, where $s_{ij}(u,v) = \max\{-u_i, a_{ij'} - u_i - v_{j'}, \forall j' \in M' \setminus \{j\}\}$ and $s_{ji}(u,v) = \max\{-v_j, a_{i'j} - u_{i'} - v_j, \forall i' \in M \setminus \{i\}\}$. In fact, since the kernel of an assignment game is included in its core, the above equalities characterize the kernel of the assignment game:

$$\mathcal{K}(w_A) = \{(u,v) \in C(w_A) \mid s_{ij}(u,v) = s_{ji}(u,v)\,, \text{ for all } (i,j) \in \mu\}\,.$$

In these markets, the kernel can be viewed as those imputations for which any two optimally matched players are equally powerful concerning their mutual threats. We want to remark the fact that, unlike the case of arbitrary coalitional games, to compute the kernel of an assignment game, only the excesses of individual coalitions and mixed-pair coalitions are to be taken into account and, moreover, equilibrium is only required for pairs of agents which are optimally matched.

Our aim in this section is to prove that the kernel of a uniform–price assignment game either coincides with the core or reduces to the nucleolus. This is not true for arbitrary assignment games: an example can be found in Granot and Granot (1992) of an assignment game with a kernel which is not a convex set.

If no active pair exists, then the core of the assignment market reduces to only one point and, since the kernel is always nonempty, it coincides with the kernel (and also with the nucleolus). We can thus assume that at least one active pair exists.

Theorem 6 *Let* $(M \cup M', w_A)$ *be a uniform–price assignment game with at least* *one active pair. Then, the kernel $\mathcal{K}(w_A)$ either coincides with the core or reduces to only one point. This last case happens if and only if a unique active pair exists.*

PROOF: We will assume without loss of generality, that A is buyer–seller exact, $A = A^r$, since by Núñez (2004) we know that $\mathcal{K}(w_A) = \mathcal{K}(w_{A^r})$. Once fixed $\mu \in \mathcal{M}^*(A)$, let us prove first that if $(i, j) \in \mu$ is a non-active pair, then $s_{ij}(u, v) = s_{ji}(u, v)$ for all $(u, v) \in C(w_A)$. To do that, we consider two cases.

Case 1: Assume first that $(i, j) \in \mu$ is a non-active pair but it is not the only one. Since there exists $j' \neq j$ non-active, $a_{ij'} - u_i - v_{j'} = a_{ij'} - \underline{u}_i - \underline{v}_{j'} = 0$ for all $(u, v) \in C(w_A)$, where the second equality follows from the fact that w_A is buyer–seller exact. Thus, $s_{ij}(u, v) = 0$ and, by the same argument, $s_{ji}(u, v) = 0$.

Case 2: Assume now that $(i, j) \in \mu$ is the unique non-active pair. For all $(u, v) \in C(w_A)$, either $u_i = 0$ or if $u_i > 0$ we claim that there exists $j' \in M' \setminus \{j\}$ such that $a_{ij'} - u_i - v_{j'} = 0$. To prove the claim notice that if $u_i > 0$ and $a_{ij'} - u_i - v_{j'} < 0$ for all $j' \in M \setminus \{j\}$, then we could choose $\varepsilon > 0$ small enough so that $(u', v') \in \mathbf{R}^{m+m'}$, defined by $u'_i = u_i - \varepsilon$, $v'_j = v_j + \varepsilon$, $u'_k = u_k$ for $k \in M \setminus \{i\}$, and $v'_l = v_l$ for $l \in M' \setminus \{j\}$, would be a core allocation. Since $u_i \neq u'_i$ and both (u, v) and (u', v') belong to the core, this contradicts that i is a non-active buyer. Once proved the claim, we have that also in this case $s_{ij}(u, v) = 0$, and the same argument applies to obtain $s_{ji}(u, v) = 0$.

At this point we can state that to know if a core allocation (u, v) belongs to the kernel of a buyer-seller exact assignment game you only need to check the constraints $s_{ij}(u, v) = s_{ji}(u, v)$ for those $(i, j) \in \mu$ formed by active agents. We now consider the two cases that appear in the statement of the theorem.

Let us now assume there exists more than one active pair. Take $(i, j) \in \mu$ an active pair and consider $s_{ij}(u, v) = \max\{-u_i, a_{ij'} - u_i - v_{j'}, \forall j' \in M \setminus \{j\}\}$. Notice that if j' is also active then, taking into account the description of the core of uniform–price assignment games given in (5), $a_{ij'} - u_i - v_{j'}$ is constant for all core allocations. This means that, since w_A is buyer–seller exact, $a_{ij'} - u_i - v_{j'} = 0$ for all core allocation (u, v) and all j' active. By assumption, an active seller $j' \neq j$ exists and so $s_{ij}(u, v) = 0$. The same argument proves that $s_{ji}(u, v) = 0$, and thus

$s_{ij}(u,v) = s_{ji}(u,v)$.

To sum up, if more than one active pair exists, we have seen that for all $(u,v) \in C(w_A)$ and all $(i,j) \in \mu$, $s_{ij}(u,v) = s_{ji}(u,v)$ which means that $C(w_A) \subseteq \mathcal{K}(w_A)$. Since the other inclusion always holds, we have obtained in this case the coincidence of the kernel with the core.

Assume now that A has only one active pair $(i_1, j_1) \in \mu$. Since the kernel of an arbitrary coalitional game is always nonempty, take $(x,y) \in \mathcal{K}(w_A)$. We then have

$$s_{i_1 j_1}(x,y) = s_{j_1 i_1}(x,y)\,, \tag{8}$$

where

$$\begin{aligned} s_{i_1 j_1}(x,y) &= \max\{-x_{i_1}, a_{i_1 j} - x_{i_1} - y_j,\ \forall j \in M' \setminus \{j_1\}\} \\ s_{j_1 i_1}(x,y) &= \max\{-y_{j_1}, a_{i j_1} - x_i - y_{j_1},\ \forall i \in M \setminus \{i_1\}\}\,. \end{aligned}$$

We will prove that (x,y) is the unique allocation in the kernel.

If there exist some $(u,v) \in \mathcal{K}(w_A)$, $(x,y) \neq (u,v)$, then, since $\mathcal{K}(w_A) \subseteq C(w_A)$, both (x,y) and (u,v) must be of the form described in (5). Taking this into account and the fact that $I_0 = M \setminus \{i_1\}$ and $J_0 = M' \setminus \{j_1\}$, there exists $\varepsilon' > 0$ such that either $u_{i_1} = x_{i_1} + \varepsilon'$, $u_i = x_i$ for all $i \in M \setminus \{i_1\}$, $v_{j_1} = y_{j_1} - \varepsilon'$ and $v_j = y_j$ for all $j \in M' \setminus \{j_1\}$, or else $u_{i_1} = x_{i_1} - \varepsilon'$, $u_i = x_i$ for all $i \in M \setminus \{i_1\}$, $v_{j_1} = y_{j_1} + \varepsilon'$ and $v_j = y_j$ for all $j \in M' \setminus \{j_1\}$. We will do the proof only in the first case, as the second one is proved analogously.

Since $M' \setminus \{j_1\} = J_0$ and $M \setminus \{i_1\} = I_0$, we get

$$s_{i_1 j_1}(u,v) = \max\{-x_{i_1} - \varepsilon', a_{i_1 j} - (x_{i_1} + \varepsilon') - y_j, \forall j \in M' \setminus \{j_1\}\} = s_{i_1 j_1}(x,y) - \varepsilon'$$

and

$$s_{j_1 i_1}(u,v) = \max\{-y_{j_1} + \varepsilon', a_{i j_1} - x_i - (y_{j_1} - \varepsilon'), \forall i \in M \setminus \{i_1\}\} = s_{j_1 i_1}(x,y) + \varepsilon'$$

Thus, $s_{i_1 j_1}(u,v) = s_{j_1 i_1}(u,v)$ if and only if $\varepsilon' = 0$, and this means that (u,v) coincides with (x,y). □

Notice that when there exists only one active pair and the kernel reduces to only one point, this point is necessarily the nucleolus of the game (Schmeidler, 1969).

Moreover, looking at some examples given before, we realize that both cases in the theorem above can really happen. The game w_{A_2} has only one active pair, while the game w_{A_3} has several active pairs.

A Appendix

Proof of Corollary 1: Recall from Núñez and Rafels (2002) that all 2×2 assignment games are buyer-seller exact. Then, by Theorem 3, $A = A^r = \underline{A} + A^c$ where A^c is an extended square glove market where $a^c_{ij} = 0$ if and only if i or j are non-active in A. This means that, up to permutations of the buyers or the sellers, either $A^c = 0$ and all agents are non-active, or $A^c = \begin{pmatrix} c & c \\ c & c \end{pmatrix}$ with $c > 0$ and all agents are active, or $A^c = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ with $c > 0$ and only the first buyer and the first seller are active. In the first two cases, by adding $\underline{A}$, we obtain a 2×2 matrix with two optimal matchings, and thus a Böhm–Bawerk horse market.

In the last case, $A^r = \begin{pmatrix} c + \underline{u}_1 + \underline{v}_1 & \underline{u}_1 + \underline{v}_2 \\ \underline{u}_2 + \underline{v}_1 & \underline{u}_2 + \underline{v}_2 \end{pmatrix}$, and we claim that $\underline{u}_2 = \underline{v}_2 = 0$. To see that, recall first that in every extreme core allocation of an assignment game there is at least one agent who receives a null payoff (see Balinski and Gale, 1987). Since in this case $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$ are two different extreme core allocations there exists $i \in \{1, 2\}$ such that $\underline{u}_i = 0$ and there exists $j \in \{1, 2\}$ such that $\underline{v}_j = 0$.

If we assume that $\underline{v}_2 > 0$, then it must hold $\underline{v}_1 = 0$. Let us now consider $\overline{u}_2$. By equation (1), $\overline{u}_2$ is the marginal contribution of buyer 2 to the grand coalition but, since this is a non-active agent, we know $\overline{u}_2 = \underline{u}_2$. When computing this marginal contribution we get

$$\overline{u}_2 = c + \underline{u}_1 + \underline{v}_1 + \underline{u}_2 + \underline{v}_2 - \max\{c + \underline{u}_1 + \underline{v}_1, \underline{u}_1 + \underline{v}_2\}.$$

If the maximum is attained in $c + \underline{u}_1 + \underline{v}_1$, we have $\overline{u}_2 = \underline{u}_2 + \underline{v}_2 = \underline{u}_2$ in contradiction with $\underline{v}_2 > 0$, and in the second case we obtain $\overline{u}_2 = c + \underline{u}_2 = \underline{u}_2$, in contradiction with $c > 0$. Thus, $\underline{v}_2 = 0$. The fact that $\underline{u}_2 = 0$ is proved analogously. This means that $A = A^r = \begin{pmatrix} c + \underline{u}_1 + \underline{v}_1 & \underline{u}_1 \\ \underline{v}_1 & 0 \end{pmatrix}$ and thus it is also a Böhm–Bawerk horse market. $\square$

Proof of the Claim in Theorem 6: *If $A \in \mathcal{M}_{m\times m'}(\mathbf{R}_+)$ is such that every 2×2 submatrix defines a Böhm–Bawerk horse market, then A also defines a Böhm–Bawerk horse market.*

To prove the claim let us assume, without loss of generality, that rows and columns have been ordered in such a way that $a_{1j} \geq a_{1j+1}$ for all $j \in \{1,\ldots,m'-1\}$, $a_{i1} \geq a_{i+11}$ for all $i \in \{1,\ldots,m-1\}$ and, moreover, $a_{11} \geq a_{ij}$ for all $i \in M$ and $j \in M'$. Notice that this can always be achieved.

Under the assumption that all 2×2 submatrices define Böhm–Bawerk markets, the above ordering implies that, for all $i \in M$ and $j \in M'$, $a_{ij} \geq a_{ij'}$ for all $j' \geq j$ and $a_{ij} \geq a_{i'j}$ for all $i' \geq i$.

We prove the first inequality of the above statement (the second one is proved analogously). Take $j' > j$ and consider the matrix $A' = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}$. As this matrix defines a Böhm–Bawerk horse market, and because of the given orders in the sets of buyers and sellers, if $a_{1j} = 0$, then $a_{ij} = 0$. But on the other side, as $a_{1j} \geq a_{1j'}$, we obtain $a_{1j'} = 0$ and since matrix $\begin{pmatrix} a_{11} & a_{1j'} \\ a_{i1} & a_{ij'} \end{pmatrix}$ is a Böhm–Bawerk horse market, we deduce that $a_{ij'} = 0$ and thus $a_{ij} \geq a_{ij'}$.

If $a_{1j} > 0$ we will first see that $a_{1j} \geq a_{ij}$. As this is obvious when $a_{ij} = 0$, let us assume $a_{ij} > 0$. Then, since A' is a Böhm–Bawerk horse market, we obtain $a_{11} + a_{ij} = a_{1j} + a_{i1}$, which from $a_{11} \geq a_{i1}$ implies $a_{1j} \geq a_{ij}$.

Now take matrix $A'' = \begin{pmatrix} a_{1j} & a_{1j'} \\ a_{ij} & a_{ij'} \end{pmatrix}$. If $a_{ij'} = 0$, then trivially $a_{ij} \geq a_{ij'}$. If $a_{ij'} > 0$, since A'' is a Böhm–Bawerk horse market, $a_{1j} + a_{ij'} = a_{ij} + a_{1j'}$ which, as $a_{1j} \geq a_{1j'}$, implies $a_{ij} \geq a_{ij'}$.

We now define valuations for buyers and sellers which show that A is a Böhm–Bawerk horse market.

Define $h_i = a_{i1}$ for all $i \in M$ and $c_j = a_{11} - a_{1j}$ for all $j \in M'$. Let us consider the submarket $A' = \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix}$ which, by assumption, is a Böhm–Bawerk horse market. If $a_{ij} > 0$, then $A' > 0$ and

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = \max\{a_{ij}, 0\} = a_{ij} .$$

If $a_{ij} = 0$, then $a_{11} \geq a_{1j} + a_{i1}$, which means

$$\max\{h_i - c_j, 0\} = \max\{a_{i1} - (a_{11} - a_{1j}), 0\} = 0 = a_{ij} .$$

□

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