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Objectes Cel·lulars
en
Categories de Models

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*” **L’instrument més precís de l’esser humà continua essent la paraula. Primer la paraula i just al seu costat la paraula al meu cervell i el plaer del retrobament amb ella. Perquè una paraula no és tan sols la idea: és el só, la resonància que assoleix en la meva ment en ser pronunciada i tot allò que evoca en ella i amb ella. És com si, en emergir de la memòria, aquella paraula arrossegués també amb ella fragments de la meva vida i de la dels altres, i d’un moment i de tots els moments, per d’aquesta manera, per d’alguna manera, tornar a reviure’ls. Inventa mons nous i cuida la teva paraula.** ”*

Per a Xavier Muñoz Igual.

Contents

CELLULAR OBJECTS IN MODEL CATEGORIES

ABSTRACT.

Whitehead's Theorem is a classical result in algebraic topology which states that any continuous map between CW complexes which is both inducing a bijection of path connected components and isomorphisms in homotopy groups for any choice of base point is an homotopy equivalence.

CW complexes are topological spaces built through an interative process of cell attachment. In the 1990s a more general notion of cellular object in the framework of model categories was given and it started a really productive work on cellular objects in many other areas like commutative algebra, group theory or algebraic geometry.

The first aim of this work is to write down the proof of Whitehead's Theorem in pointed model categories which states that an A-equivalence between A-cellular fibrant objects is an homotopy equivalence for any cofibrant object A. In order to reach this objective, the following steps will be completed,

(•) Fully understand the proof of the classical Whitehead's Theorem for the category of topological spaces.

(•) Understand model categories language.

(•) Demonstrate that any weak equivalence between fibrant and cofibrant objects in a model category is an homotopy equivalence.

(•) Characterize the class of A-cellular objects in a pointed model category.

(•) Prove that any A-equivalence between fibrant A-cellular objects is an homotopy equivalence for a given cofibrant object A in a pointed model category.

The second objective in this work is to deeply analize the conditions in those theorem's offering examples and some results involved in the framework of arbitrary model categories.

Part 0. Introduction

Part 0.

Cellularization is the central topic of this work and it is well known that cellularization techniques play a fundamental role in modern homotopy theory. Many applications of homotopy theory naturally reflect how it is often useful to approximate a given topological space by simpler ones and, as a main example, the classical CW approximation functor associates to each pointed topological space a CW complex in the sense of Whitehead through a process of attaching cells.

This initially simple starting point is motivated in one side by the persistent aim in the present work: the conscious choice of minimized prerequisites needed for understanding the whole developments and on the other hand the aim of constructing a ladder to climb and deeply generalize the concept of cellularization using the powerful machinery of model categories. Therefore, it should be enough to have some familiarity with CW complexes, together with a nice kit of topological tools, both equipped with the basic terminology associated with categories (and this is not least at all).

CW complexes appear as especially suitable spaces in classical homotopy theory and a first question is immediately suggested:

Why is it important and useful to work with CW complexes?

Topological spaces are, in general, tough to work with. Instead, CW complexes have many properties that make them nice to work with in homotopy theory, such as

(•) being amenable to study by means of homotopy groups,
(•) and enabling to define maps inductively.

Moreover, these properties are still possessed by spaces with the homotopy type of a CW complex.

Indeed, CW complexes are easier to handle than general topological spaces because of their inductive definition using cells. Thus, a CW complex X is a colimit of its n-skeleta X^n. This makes it much easier to compute things for CW complexes, in particular cellular (co)homology and homotopy. As an instance, Moore spaces and Eilenberg - Mac Lane spaces are CW complexes and their constructions are not too hard. Also, every space X can be constructed as an inverse limit of Eilenberg - Mac Lane spaces via a Postnikov tower, so again CW complexes provide a way to get hands on a general space X. Furthermore, an arbitrary continuous map between CW complexes X and Y is homotopic to a cellular map (which is much nicer), and hence a map which takes the n-skeleton of X to the n-skeleton of Y.

Because of this friendly structure, in order to prove something for all topological spaces, it is often easier to first prove it for CW complexes and then apply the CW approximation theorem to get it for all spaces. The CW approximation theorem states that for every topological space X there is a CW complex Y and a map $f\colon Y \longrightarrow X$ inducing isomorphisms on homotopy, homology, and cohomology. In particular, this expresses X, up to homotopy, as a colimit of a sequence of cellular inclusions $Y^n \hookrightarrow Y$. Thus, the homotopy groups of X are colimits of the homotopy groups of the Y^n and $\pi_n(Y^n) \twoheadrightarrow \pi_n(X)$ is an epimorphism.

However, CW complexes are also nice on the point-set level. They are compactly generated, locally contractible, and every compact subset is contained in a finite CW subcomplex, and what is more, finite CW complexes have almost every space-level regularity property that one could enumerate.

However, this is not a perfect world to live in, since sometimes these point-set considerations can be troublesome: the smash product is not strictly associative, or the cone on a pathological space produces something with the homotopy type of a nice CW complex but terrible point-set behaviour.

A variety of additional nice features can be argued by means of CW complexes while acting as a solid applicant to become the right category to do homotopy theory: the category of CW complexes contains the category of graphs, and geometric realizations of locally finite simplicial sets are in the category of CW complexes as well. They also appear as the first motivating actor to climb to the next step in the ladder: if one wants to find the right category to develop homotopy theory, one should first go to compactly-generated Hausdorff spaces, then put the Quillen model structure on this and recover the category of CW complexes as the fibrant and cofibrant objects. In addition, in this category one has Brown representability, and hence necessary and sufficient conditions for a functor $F\colon Ho(\mathcal{CW})^{op} \longrightarrow Set$ to be representable (where $\mathcal{CW}$ denotes the category of CW complexes). So this allows to understand cohomology theories by their representing objects, which is another nice feature that helps to understand the extensive literature devoted to CW complexes.

In the personal context of the present work, the single most important ingredient is Whitehead's Theorem. It states that if X and Y are connected CW complexes and $f\colon X \longrightarrow Y$ is a weak homotopy equivalence, that is, it induces an isomorphism on π_n for all n, then it is a homotopy equivalence. It seems that, in fact, this was the original justification for CW complexes when John H. C. Whitehead introduced them.

Hence, I will take the Whitehead Theorem as my choice of guidance in this project. It will act on the whole work as the key carrier agent, by following its footsteps along every step in this ladder. So in each step the statement of the Theorem will be recovered and it will be proved.

This is not a random choice, nor hazardous, since CW complexes are, as it was already said, one of the main building blocks of classical homotopy theory. Moreover there are powerful and strong consequences of the Theorem, for instance establishing relations between homotopy and homology groups and by extension with cohomology groups.

Conceptually, the following two theorems (both due to Whitehead) are Eckmann-Hilton duals:

- (•) (Theorem 1)
  A weak homotopy equivalence between CW complexes is a homotopy equivalence.
- (•) (Theorem 2)
  A homology isomorphism between simple spaces is a weak homotopy equivalence.

They do not look dual, but they are (see John Peter May [10]).

The point is that the second statement is really about cohomology, and the standard cellular proof of the first statement dualizes word-for-word to a cocellular proof of the second. Cocellular constructions appear in Postnikov towers, and they can be used more systematically than can be found in the literature. One key point is the Universal Coefficient Theorem, whose details are not completely obvious unless one makes some finiteness assumptions. Another key point that leads to the same result is that, for simply-connected spaces, one can determine the connectivity of a map by looking at the connectivity of the cofiber instead of the connectivity of the fiber.

Part 1 of the work acts as an opportunity to present the classical model case to introduce and develop cellularity, that is, topological spaces and CW complexes, together with some tools and the statement of Whitehead's Theorem in that framework.

The next step will be to raise those concepts to a more general environment, so it will be necessary:

- (•) First, to briefly refresh some category theory in Part 2.
- (•) Secondly, to introduce the powerful machinery of model categories in Part 3.

After having reviewed some facts of category theory in Part 2, the next subject of interest will be homotopy theory in a number of categories, and I would like to establish suitable techniques to compare these.

There is an efficient machinery due to Daniel Quillen, which encodes this structure. As a matter of fact, anyone who properly approaches the theory of model categories must asume that almost all of the material is in some way or another already present in [13] or [14], both works due to Daniel Quillen.

I want to emphasize the new horizon that Daniel Quillen definitively created with those works.

Two main facts are relevant:

- (•) Birth of rational homotopy theory (Daniel Quillen, Dennis Sullivan, etc.).
- (•) Development of derived category theory, along the lines previously devised by Grothendieck.

Homotopical algebra (and in particular the theory of model categories) was mainly developed in the late 60s in seminal works of Daniel Quillen, Daniel Kan, Albrecht Dold, Dieter Puppe and others. It appears as a key theory since it allows to build a general and common abstract framework (of homotopic nature) for,

- (•) classical algebraic topology,
- (•) algebraic models for homotopy types,
- (•) and categories of chain complexes.

More specifically:

- (•) Classical algebraic topology:
    - (◦) Fibrations (Witold Hurewicz, Albrecht Dold, Jean-Pierre Serre, etc.).
    - (◦) Cellular approximations.
    - (◦) Eckmann-Hilton duality ($[\Sigma X, Y] \cong [X, \Omega Y]$).
    - (◦) etc.
- (•) Algebraic models for homotopy types:
    - (a)
        - (◦) Kan simplicial sets as an abstraction of triangulations and polyhedra (combinatorial homotopy).
        - (◦) Simplicial groups (models for loop spaces ΩX).
    - (b)
        - (◦) Sullivan models with Lie algebras for rational homotopy.
        - (◦) Quillen models with differential graded algebras for rational homotopy.
- (•) Categories of chain complexes.

    The derived category of a ring R (or more generally a scheme) is the quotient of the category of chain complexes of R-modules by the homotopy relation (formally analogous to the homotopy of topological spaces).

I want to isolate and remark two of the previous facts:

• The derived category of a ring R (or a scheme) is the quotient of the category of chain complexes of R-modules by the homotopy relation (formally analogous to the homotopy of topological spaces).

• Quillen models with differential graded algebras for rational homotopy.

Daniel Quillen established in a brilliant way the deep relation between these two facts, by developing the theory of model categories (and more generally by developing homotopical algebra). He realized the existence of analogies between homotopy of topological spaces and homotopy of chain complexes.

Model categories surely form a solid foundation of homotopy theory and, in fact, they were originally developed by Daniel Quilen as an abstraction of homotopy theory. The following main problem resides in the deep reason of being of model categories.

Given a category, it often happens that although there are certain maps (weak equivalences) that are not isomorphisms, it would be desirable to consider them as if they were isomorphisms. An always available option is to formally invert the weak equivalences, but this action automatically leads to losing control of the morphisms in the quotient category. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient category from X to Y are merely homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y.

Because this idea of inverting weak equivalences is so central, model categories are extremely important. However, for a long time their natural habitat has been in areas historically associated with algebraic topology, such as homological algebra, algebraic K-theory, and algebraic topology itself, but recently this list has been expanded to cover other areas of mathematics. A remarkable example of this fact is Voevodksy's work that has lifted model categories to the category of a real must in every algebraic geometer's toolkit. Homotopical algebra has received much attention in recent years due to the foundational work of Vladimir Voevodsky, John Friedlander, Andrei Suslin, and others, resulting in the $\mathbb{A}^1$ homotopy theory for quasiprojective varieties over a field. Voevodsky used that new algebraic homotopy theory to prove the Milnor conjecture and later, in collaboration with Markus Rost, the full Bloch-Kato conjecture.

These examples should make it clear that, model categories are really fundamental. There are excellent books and articles totally devoted or partially involving model categories; but all of them take at some point the choice of developing particular cases or examples of application in the framework of the basic examples (topological spaces, simplicial sets, model category structures on chain complexes, simplicial model categories, and many others), but it is surprisingly hard to come across a comprehensive text focused on arbitrary model category.

As it was previously said, the basic material presented in Part 3 is due to Daniel Quillen, but I have replaced his treatment of suspension functors and loop functors by a general construction of homotopy pushouts and homotopy pullbacks in a model category which perfectly suits my general objective of creating constructive developments. Moreover, it is not in any sense a survey of everything that could be found in the literature about model categories, mainly because of the different approach offered (a general framework for arbitrary model categories).

Part 3 in this work is devoted to developing the theory of model categories never leaving the general framework of arbitrary model categories and using particular constructions or examples when a clarification is needed, or the process to ascend for the next rung on the ladder is asked for a new construction.

At this point, one will realise how having each definition and statement in the most general posible framework, always for arbitrary model categories, immediately will show how demanding model category theory is, not a moon, but a universe.

Strong definitions that could be skillfully avoided when dealing with particular cases of model categories (topological spaces, simplicial sets, chain complexes of R-modules over a ring and others... very common in the existing literature). Now in the framework of arbitrary model categories, they will claim and demand to show their real nature.

In order not to get lost within the volume and complexity of this theory, the personal mantra of guidance in this work will be invoked, the Whitehead Theorem, also invoked in order to recover the initial subject and aim of the study, namely cellularization techniques, now lifted to a more general environment, namely model category theory. Because this lifting implies generalization, the general statement of the Theorem will not be word for word equal to that stated in the case of topological spaces. In fact, the proof of the Theorem itself constitutes an extremely beautiful example (as a sort of toy), which includes the main actors, a practical exercise of use of the axioms and most fundamental tools and techniques inherited from algebra and reinterpreted in this new environment).

CW complexes are fundamental in classical homotopy theory and, more generally, one would like to have a similar approximation of a space built out of copies of any given fixed space A. This general concept of cellularization was developed systematicaly for the category of topological spaces and simplicial sets by Emmanuel Dror Farjoun in works as [2] building upon the general foundational work on homotopy localization of Aldridge Bousfield. These localization and cellularization techiques were then extended under some technical conditions to arbitrary model categories by Alexander Nofech in [12] and Philip Hirschhorn in [7].

A cellular homotopy theory is given in general as a model structure derived from a pointed simplicial model category with respect to a fixed cofibrant object.

The last step in this ladder will be to explore in Part 4 the class of A-cellular objects. It will be established again a kind of game because this class of objects will be formally built with an absolutely constructive process using transfinite induction by use of pushouts and telescopes. Thanks to the fact that model category theory has been carefully presented in Part 3, always for arbitrary model categories, complete characterizations will be obtained, in which almost every result reveals subtleties, bridges and developments that deeply show how the involved actors really work.

Indeed, every result in this section is presented with a particular constructive approach:

(•) Characterization of A-cellular objects. Using transfinite induction, those objects will be formally constructed using pushouts and telescopes.
(•) Theorem of CW_A approximation. Using Quillen's small object argument the functor CW_A will be formally constructed together with a study of its universality.
(•) Characterization of A-cellular classes using again a constructive argument by means of pushouts and pullbacks.
(•) Never missed out, a last meeting with the guide and guard, the A-cellular Whitehead Theorem, restated in this new framework, offering a complete proof of it.

Several considerations and consequences will be included, due in particular to the delicate definitions of homotopy colimits and mapping spaces (together with the presentations of cofibrantly generated model categories, proper model categories, combinatorial model categories, locally presentable model categories and others) developed in Part 3.

The widespread availability of those powerful results allow to establish a really compelling, reach and fruitful brainstorming session with oneself. This will result in a more solid knowledge for those students who try to first introduce themselves and go ahead into such a rich universe.

Part 1. CW Complexes

Part 1.

Overview

.

In this part I introduce basic framework, features and tools. The category of topological spaces serves as a reference category: it is here the notion of homotopy appears, and many results and constructions are most naturally understood in this context.

Topology essentially discuss the connectedness of those geometrical objects called topological spaces; however, strictly speaking, one considers topological spaces and two types of continuous maps between them, homeomorphisms and homotopy equivalences. One might classify topological spaces up to homeomorphism or one might do so up to homotopy equivalence. That choice only depends on how strong one wants this classification to be. The classification according to homotopy equivalences is weaker (there are many spaces not homeomorphic to each other that are of the same homotopy type), but it is the one that plays the more important role in algebraic topology, because geometrical properties of homotoypy equivalences translate themselves most successfully into algebra.

Intuitively two spaces are of the same homotopy type if one can be continuously deformed into the other; that is, without losing any holes or introducing any cuts. Two maps are homotopic if the graph of one can be continuously deformed into that of the other. So, one can immediately notice that the homotopy relationship trascends dimension, compactness and cardinality for spaces.

Thus, for spaces and maps, the classification up to homotopy equivalence precisely captures their qualitative features. Homotopy yields algebraic invariants for a topological space, the homotopy groups, which consist of homotopy classes of maps. Continuous maps between spaces induce group isomorphisms between their homotopy groups; moreover, homotopic spaces have isomorphic groups and homotopy maps induce the same group homomorphisms.
Quite intrincate spaces can be synthesized from simplier buildinng pieces. There is fairly natural choice for this pieces, namely homeomorphs of interiors of the disks $\{\ \mathbb{E}^n \mid n \geq 0\}$. It is rather less obvious how to perform the synthesis so as to gain advantage from the cellular substructure for investigating homotopy properties even when infinitely many cells might be involved. This is a fact achieved by *CW* complexes , for given any space (not ust those in *Top* or Top_* it is possible to construct a CW complex having the same homotopy groups. Moreover, maps can be replaced, up to homotopy, by celluar maps wich respect the internal skeletons of *CW* complexes.

The cellular structure of *CW* complexes is ideal for constructing successive approximations to maps, by extending from cell boundaries to interiors. The existence of such extensions is sensitive to the homotopy properties of the space in which the cell sits.

Paths and Homotopy

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Definition 1. *(Homotopic Maps between Topological Spaces)*
Two maps from a topological space X to a topological space Y,

$$f_i : X \to Y \qquad i = 0, 1$$

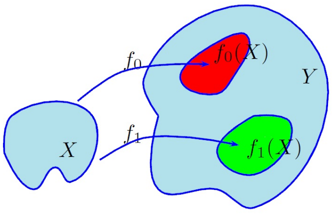
are homotopic if there exists a family of continuous maps,

$$f_t : X \to Y \qquad t \in [0, 1]$$

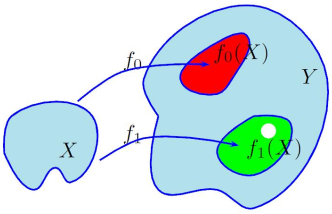
varying continuously from f_0 to f_1 .
This situation is denoted by $f_0 \simeq f_1$ and says that $f_t \quad t \in [0, 1]$ is a homotopy between them.

Intuitivelly speaking, homotopy means that given two continuous mappings f_0 , $f_1 : X \longrightarrow Y$, one can deform the image of f_0 into the image of f_1 within the space Y.

I can deform the image of f_0 into the image of f_1 within the space Y.



Instead, I cannot deform the image of f_0 into the image of f_1 within the space Y, because of the "hole".



If I consider the set S of the continuous maps from a topological space X to a topological space Y.

The relation of being homotopic is an equivalence relation on S that breaks S into equivalence classes called homotopy classes. I denote by $[X, Y]$ the set of homotopy classes of maps from X to Y, which ones calls the homotopy set of X to Y. In other words, one regards all homotopic maps from X to Y as identical and place them in the same homotopy class. Therefore, even if a homotopy class has a large number of continuous maps, one needs to look at only one of them. This is an algebraic simplification.

Let X and Y be topological spaces. A continuous map $f : X \to Y$ is a homotopy equivalence of X and Y if for some continuous map $g : Y \to X$, the composites

$$g \circ f : X \to X$$

and

$$f \circ g : Y \to Y$$

are homotopic to Id_X and Id_Y the identity maps of X and Y, respectively.

One says that X and Y have the same homotopy type if there exists a homotopy equivalence between them.

In general a homotopy equivalence is neither injective nor surjective. I write $X \simeq Y$ when X and Y have the same homotopy type. (Observe that the same symbol is been used here to denote homotopic maps, but this should not cause any confusion here since both sides are topological spaces).

From the definitions it is clear that two topological spaces that are homeomorphic have the same homotopy type; homotopy equivalences are a less strict way of classifying topological spaces.

The intuitive idea of searching spaces with the same homotopy type is clear: one associates to the space X some groups, that are invariants under homotopy, this means that spaces with the same homotopy type had also the same groups, consequently, in order to study a particular space one searchs other one, simplier, but with the same homotopy type and then one proceeds to study only this simplier space, instead.

In topology frequently one considers a pair of topological spaces (X, A) rather than a single space X. Passing form single spaces to pairs of spaces as objects of study was a great breakthrough in algebraic topology in the past.

By a topological pair (X, A), one means a topological space X and a subspace A of X. Given two pairs (X, A) and (Y, B), by a map of pairs $f : (X, A) \to (Y, B)$, one means a continuous map $f \colon X \to Y$ such that $f(A) \subset B$.

One says that two continuous maps of pairs,

$$f_i \colon (X, A) \to (Y, B) \qquad i = 0, 1$$

are homotopic if there exists a family of continuous maps of pairs,

$$f_t : (X, A) \to (Y, B) \qquad t \in [0, 1]$$

varying continuously from f_0 to f_1.
One can partition the continuous maps from a pair (X, A) to another pair (Y, B) into homotopy classes; that is, one can look at the set denoted by,

$$[(X, A), (Y, B)]$$

in which each element is a homotopy class consisting of all homotopic maps from (X, A) to (Y, B). One says that $[(X, A), (Y, B)]$ is the homotopy set of maps from (X, A) to (Y, B). In particular, if $A = B = \emptyset$, then one writes X and Y in place of $(X, \emptyset)$ and $(Y, \emptyset)$. Then one has,

$$[X, Y] = [(X, \emptyset), (Y, \emptyset)]$$

as the right-hand side of the equality is the homotopy set in which an element is a set of homotopic maps from X to Y.

The fundamental group is defined in terms of loops and deformations of loops.

Definition 2. *(Path)*
A path in a space X is a continuous map $f\colon I \longrightarrow X$ where I is the unit interval $[0, 1]$.

The idea of continuously deforming a path, keeping its endpoints fixed, is formalized by the definition of homotopy of paths.

Definition 3. *(Homotopy of paths)*
A homotopy of paths in a space X is a family $f_t\colon I \longrightarrow X \quad 0 \leq t \leq 1$, such that

(1) *The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.*
(2) *The associated map $F\colon I \times I \longrightarrow X$ defined by $F(s, t) = f_t(s)$ is continuous.*

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, thet are said to be homotopic. The notation for this is $f_0 \simeq f_1$.

The relation of homotopy paths with fixed endpoints in any space is an equivalent relation.

The equivalence class of a path f under the equivalence relation of homotopy class of f.

Given two paths $f, g\colon I \longrightarrow X$ such that $f(1) = g(0)$, there is a composition or product path $f \cdot g$ that traverses first f and then g, defined by the formula

$$f \cdot g\ (s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq 1/2 \\ g(2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Thus f and g are traversed twice as fast in order for $f \cdot g$ to be traversed in unit time.

This product operation respects homotopy classes since if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies f_t and g_t, and if $f_0(1) = g_0(0)$ so that $f_0 \cdot g_0$ is defined, then $f_t \cdot g_t$ is defined and provides a homotopy $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

In particular, I can restict attention to paths $f\colon I \longrightarrow X$ with the same starting and ending point $f(0) = f(1) = x_0 \in X$. Such paths are called loops, and the common stating and ending point x_0 is referred to as the basepoint.

The set of all homotopy classes $[f]$ os loops $f\colon I \longrightarrow X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.

This group is called the fundamental group of X at the basepoint x_0.

$\pi_1(X, x_0)$ is the first of a sequence of groups $\pi_n(X, x_0)$ called homotopy groups, which are defined in an entirely analogous way using the n-dimensional cube I^n in place of I.

It is not easy to show that a space has a nontrivial fundamental group since one must somehow demonstrate the nonexistence of homotopies between certain loops.

It is natural to ask about the dependence of $\pi_1(X, x_0)$ on the choice of the base-point x_0. Since $\pi_1(X, x_0)$ involves only the path-component of X containing x_0, it is clear that I can hope to find a relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ for two basepoints x_0 and x_1 only if x_0 and x_1 lie in the same path-component of X.

So, let $h\colon I \longrightarrow X$ be a path form $x_0$to x_1, with the inverse path $\bar{h}(s) = h(1-s)$ from x_1 back to x_0.

I can then associate to each loop f based at x_1 the loop $h \ \cdot \ f \ \cdot \bar{h}$ based at x_0.

Note that strictly speaking, I should choose an order of forming the product $(h \cdot f) \cdot \bar{h}$ or $h \cdot (f \cdot \bar{h})$, but the two choices are homotopic and I am only interested in homotopy classes.

If X is path-connected, the group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of basepoint x_0. In this case the notation $\pi_1(X, x_0)$ is often abbreviated to $\pi_1(X)$.

In a general sense I can define,

Definition 4. *(Simply-Connected Space)*

A space is called simply-connected if it is path-connected and has trivial fundamental group.

Proposition 5. *A space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Proof. Path-connectedness is the existence of paths connecting every pair of points, so I need be concerned only with the uniqueness of connecting paths.

I suppose $\pi_1(X) = 0$.

If f and g are two paths from x_0 to x_1, then $f \simeq f \ \bar{g} \ \cdot \ g \simeq g$ since the loops $\bar{g} \ g$ and $f \ \cdot \ \bar{g}$ are each homotopic to constant loops, using the assumption $\pi_1(X, x_0) = 0$ in the latter case.

Conversely, if there is only one homotopy class of paths connecting a basepoint x_0 to itself, then all loops at x_0 are homotopic to the constant loop and $\pi_1(X, x_0) = 0$ □

Homotopical Tools

.

In this section I present basic constructions and features:

Definition 6. *(Homotopy Extension Property for a topological space)*

Let X be a topological space, and let $A \subset X$. One says that the pair (X, A) has the homotopy extension property if, given a homotopy $f_t\colon A \to Y$ and a map $F_0\colon X \to Y$ such that $F_0|_A = f_0$, there exists an extension of F_0 to a homotopy $F_t\colon X \to Y$ such that $F_t|_A = f_t$.

That is, the pair (X, A) has the homotopy extension property if any map $G\colon ((X \times \{0\}) \cup (A \times I)) \to Y$ can be extended to a map $G'\colon X \times I \to Y$ (that is, G and G' agree on their common domain).

If the pair has this property only for a certain codomain Y, one says that (X, A) has the homotopy extension property with respect to Y.

The homotopy extension property can be depicted as,

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}_0"] \arrow[dr, dashed, "\tilde{f}"] & Y \\
A \arrow[u, "i"] \arrow[r, "f"'] & Y^I \arrow[u, two heads, "p_0"']
\end{tikzcd}$$

If the above diagram (without the dashed map) commutes, which is equivalent to the conditions above, then there exists a map $\tilde{f}$ which makes the diagram commute. Note that a map $\tilde{f}\colon X \to Y^I$ is the same as a map $\tilde{f}\colon X \times I \to Y$.

Remark 7.

(•) *If X is a cell complex and A is a subcomplex of X, then the pair (X, A), has the homotopy extension property.*

(•) *A pair (X, A) has the homotopy extension property if and only if $(X \times \{0\} \cup A \times I)$ is a retract of $X \times I$.*

- **(•)** *If (X, A) has the homotopy extension property, then the simple inclusion map $i : A \longrightarrow X$ is a cofibration.*

  In fact, if one considers any cofibration $i : Y \longrightarrow Z$, then one has that Y is homeomorphic to its image under i.

  This implies that any cofibration can be treated as an inclusion map, and therefore it can be treated as having the homotopy extension property.

Lemma 8. *The following facts hold,*

- **(a)** *The composition of homotopy equivalences $X \longrightarrow Y$ and $Y \longrightarrow Z$ is a homotopy equivalence $X \longrightarrow Z$.*

  It can be deduced that homotopy equivalence is an equivalence relation.
- **(b)** *The relation of homotopy among maps $X \longrightarrow Y$ is an equivalence relation.*
- **(c)** *Any map homotopic to a homotopy equivalence is a homotopy equivalence.*

Proof. I know that a map $f \colon X \longrightarrow Y$ is called a homotopy equivalence if there exists a map $g \colon Y \longrightarrow X$ such that $f \circ g \simeq 1$ and $g \circ f \simeq 1$.

In this case, g will be referred to as the homotopy inverse of f.

Also, I say that two maps f_0,$f_1 \colon X \longrightarrow Y$ are homotopic if there exists a homotopy f_t connecting f_0 to f_1. Here, I will write $f_0 \simeq f_1$.

- **(a)** I can observe that the main claim here is that homotopy equivalence is a transitive relation, the proof of which will verify that homotopy equivalence is an equivalence relation due to the fact that $1_X \colon X \longrightarrow X$ is a homotopy equivalence (where 1_X is its own homotopy inverse), and that X is homotopy equivalent to Y if and only if Y is homotopy equivalent to X.

  To that end, I suppose that X, Y, Z are topological spaces and that $f_1 \colon X \longrightarrow Y$, $f_2 \colon Y \longrightarrow Z$ are homotopy equivalences with homotopy inverses g_1,g_2, respectively. By definition, then, $f_1 \circ g_1 \simeq 1_X$, $g_1 \circ f_1 \simeq 1_Y$ and $f_2 \circ g_2 \simeq 1_Y$, $g_2 \circ f_2 \simeq 1_Z$.

  It suffices to show that the map $f_2 \circ f_1 \colon X \longrightarrow Z$ is a homotopy equivalence, which is immediate due to the fact that it is continuous (and the composition of continuous maps is also continuous) and that $g_1 \circ g_2 \colon Z \longrightarrow X$ is a map for which

  $$(f_2 \circ f_1) \circ (g_1 \circ g_2) = f_2 \circ (f_1 \circ g_1) \circ g_2 \simeq f_2 \circ 1_Y \circ g_2 = f_2 g_2 \simeq 1_Z$$

  and

  $$(g_1 \circ g_2) \circ (f_2 \circ f_1) = g_1 \circ (g_2 \circ f_2) \circ f_1 \simeq g_1 \circ 1_Z \circ f_1 = g_1 \circ f_1 \simeq 1_X.$$

  Hence, I obtain the result.
- **(b)** To show that the relationship of homotopy among maps $X \longrightarrow Y$ is an equivalence relation, I can note again that the reflexive and symmetric properties are free.

  Indeed, if f,$g \colon X \longrightarrow Y$ are maps which are homotopic by way of a homotopy φ_t, then $f \simeq f$ by way of the identity homotopy $F(x, t) = x$ and $g \simeq f$ by way of the homotopy φ_t^{-1}. So, again, it suffices to prove that the relation is transitive.

  To that end, I suppose that $F \colon X \times I \longrightarrow Y$ is a homotopy connecting f to g and that $G : X \times I \longrightarrow Y$ is a homotopy connecting g to h, where f , g , $h \colon X \longrightarrow Y$.

  I define, then, a map $H \colon X \times I \longrightarrow Y$ by,

  $$H(x, t) = \begin{cases} F(x, t) & \text{if } 0 \leq t \leq 1/2 \\ G(x, t) & \text{if } 1/2 < t \leq 1 \end{cases}$$

  Continuity of H is immediate due to the fact that H is made continuous at $t = 1/2$ (the only point of concern), whereby it follows that $f \simeq h$.
- **(c)** Finally, I suppose that $f \colon X \longrightarrow Y$ is a homotopy equivalence with homotopy inverse $g \colon Y \longrightarrow X$ and suppose that $h \colon X \longrightarrow Y$ is homotopic to f, i.e. that there exists a homotopy $F \colon X \times I \longrightarrow Y$ connecting h to f.

  Said in a different way, $h \simeq f$ says that there exists a family $\{f_t : X \longrightarrow Y\}_{t \in I}$ connecting h to f, a fact which immediately implies that the family $\{g \circ f_t\}_{t \in I}$ connects $g \circ h$ to $g \circ f \simeq 1_X$.

Thus, because $1_Y \simeq f \circ g$ by the homotopy inverse property, and because $f \circ g \simeq h \circ g$ because $h \simeq f \iff f \simeq h$ by (b), it follows that $h\colon X \longrightarrow Y$ is also a homotopy equivalence.

$\square$

The description of the next basic constructionss is developed in *Top* the category of topological spaces X and continuous maps $f\colon X \longrightarrow Y$.

I will use for all these constructions, the usual notation.

I denote by I the interval $[0,1] \subseteq \mathbb{R}$ with the usual topology.

I will use the symbol $\star$ referred to the topological space with a single point, and the symbol $\star_c$ referred to the constant map $\star_c\colon X \longrightarrow Y$ such that $\star c(x) = c \quad \forall x \in X$.

I will denote by $\mathbb{E}^n$ the n-dimensional disk,

$$\mathbb{E}^n = \{ \ x \in \mathbb{R}^n \ | \ \parallel x \parallel \leq 1 \ \} \subseteq \mathbb{R}^n$$

and $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere,

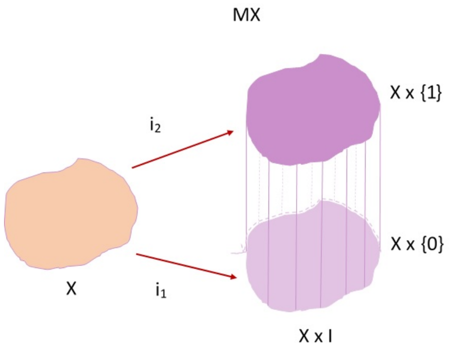
$$\mathbb{S}^{n-1} = \partial \mathbb{E}^n = \{ \ x \in \mathbb{R}^n \ | \ \parallel x \parallel = 1 \ \} \subseteq \mathbb{R}^n$$

both with the usual topology.

I note that $\mathbb{S}^0$ consists on two isolated points and $\mathbb{E}^0 = \star$.
Recall that $f\colon X \longrightarrow Y$ in *Top* is said to by nullhomotopic if $f \cong \star_u\colon X \longrightarrow Y$, for any constant $y \in Y$.

Mapping Cylinder.

.



Let X be a topological space.

The cylinder of X is the space,

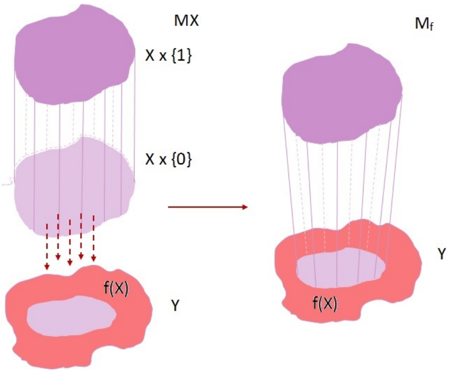
$$MX = X \times I$$

I note that the cylinder defines a functor $M\colon Top \longrightarrow Top$ which maps for each X in *Top* an object MX, and for each $f\colon X \longrightarrow Y$ in *Top* the morphism $Mf\colon MX \longrightarrow MY$ defined as $Mf(x,t) = (f(x),t)$

The mapping cylinder of a function f between topological spaces X and Y is the quotient space,

$$M_f := \frac{(X \times I) \coprod Y}{(x,1) \sim f(x)}$$

where $\sim$ is the equivalence relation generated by $(x,1) \sim f(x) \qquad \forall x \in X$.



So that, the mapping cylinder M_f is obtained by gluing one end of $X \times [0,1]$ to Y via the map f. Notice that the top of the cylinder $\{1\} \times X$ is homeomorphic to X, while the bottom is the space $f(X) \subset Y$.

I observe that M_f can be seen as the following pushout in Top,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d, "i"] \\
MX \arrow[r, "q"'] & M_f
\end{tikzcd}$$

where $q(x,t) = [(x,t)]$ and $i(y) = [y]$

I can observe that the bottom Y, is a deformation retract of M_f.

The projection $M_f \longrightarrow Y$ splits (via $Y \ni y \longrightarrow y \in Y \subset M_f$), and a deformation retraction α is given by,

$$\alpha : M_f \times I \longrightarrow M_f$$
$$([t,x], s) \mapsto [s \cdot t, x]$$

Observe that points in $Y \subset M_f$ stay fixed, because $[0,x] = [s \cdot 0, x] \quad \forall s$

Later I will generalize the following fact but I can already introduce,

The map $f\colon X \longrightarrow Y$ is a homotopy equivalence if and only if the top $1 \times X$ is a strong deformation retract of M_f

I prove first the implication $\Longrightarrow$

The inclusion $i\colon X \hookrightarrow M_f$ is homotopic to the composition $j \circ f$ where j is the inclusion $Y \hookrightarrow M_f$, a homotopy equivalence. By lemma 8, i is a homotopy equivalence if and only if f is a homotopy equivalence.

I prove now the implication $\Longleftarrow$.

Recall that I defined in Definition 6 the homotopy extension property but, I can characterize, by saying that in general, a pair (X, A) has the homotopy extension property if A has a mappping cylinder neighborhood, in the following sense:

There is a map $f\colon Z \longrightarrow A$ and a homeomorphisms h form M_f onto a closed neighborhood N of A in X, with $h|A = 1$ and with $h(M_f - Z)$ an open neighborhood of A. To verify the homotopy extension property, notice first that $I \times I$ retracts onto $I \times \{0\} \cup \partial I \times I$, hence $Z \times I \times I$ retracts onto $Z \times I \times \{0\} \cup Z \times \partial I \times I$, and this retraction induces a retraction on $M_f \times I$ onto $M_f \times \{0\} \cup (Z \amalg A) \times I$.

Thus $(M_f, Z \amalg A)$ has the homotopy extenion property, which implies that (X, A) does also since a map $X \longrightarrow Y$ and a homotopy f its restriction to A, one can take the constant homotopy on the closure of $X - N$ and the apply the homotopy extension property for $(M_f, Z \amalg A)$ to extend the homotopy over N.

So that, by this general fact, the pair (M_f, X) satisfies the homotopy extension property.

But now $M_f \hookrightarrow X$ is a homotopy equivalence, then M_f is a deformation retract of X.

Intuitivelly, the mapping cylinder may be viewed as a way to replace an arbitrary map by an equivalent cofibration, in the following sense:

Given a map $f\colon X \longrightarrow Y$, the mapping cylinder is a space M_f together with a cofibration,

$$\widetilde{f}\colon X \longrightarrow M_f$$

and a surjective homotopy equivalence

$$M_f \longrightarrow Y$$

(indeed, Y is a deformation retract a M_f), such that the composition $X \longrightarrow M_f \longrightarrow Y$ equals f. Thus the space Y gets replaced with a homotopy equivalent space M_f , and the map f with a lifted $\widetilde{f}$.

Equivalently,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\widetilde{f}"'] & & Y \\
& M_f \arrow[ur] &
\end{tikzcd}$$

in the diagram,

$$f\colon X \longrightarrow Y$$

gets replaced with a diagram

$$\widetilde{f}\colon X \longrightarrow M_f$$

together with a homotopy equivalence between them.

The construction serves to replace any map of topological spaces by a homotopy equivalent cofibration.

Note that pointwise, a cofibration is a closed inclusion.

Mapping cylinders are quite common homotopical tools. One uses of mapping cylinders is to apply theorems concerning inclusions of spaces to general maps, which might not be injective.

Consequently, theorems or techniques (such as homology, cohomology or homotopy theory) which are only dependent on the homotopy class of spaces and maps involved may be applied to f : $X \longrightarrow Y$ with the assumption that $X \subset Y$ and that f is actually the inclusion of a subspace.

Another, more intuitive appeal of the construction is that it accords with the usual mental image of a function as "sending" points of X to points of Y, and hence of embedding X within Y, despite the fact that the function needs not be one-to-one.

Proposition 9. *Let*

$$i\colon A \hookrightarrow X$$

be a cofibration

Then i is a homotopy equivalence if and only if is a strong deformation retract of X

Proof. I first prove the $\Longrightarrow$ implication:

Let $j\colon X \longrightarrow A$ be a homotopy inverse of i and let $J\colon Id_A \simeq j \circ i$ and $K\colon Id_X \simeq i \circ j$ be the homotopies.

Because (A, i, X) is a cofibration, there exists a homotopy

$$F\colon X \times I \longrightarrow A$$

such that $F(\bullet, 0) = j$ and $F(i \times Id_I) = J$.

The j is homotopic to a retraction of X onto A.

So I can assume from the begining that j is a retraction.

I define now the homotopy,

$$G\colon ((A \times I) \cup (X \times \partial I) \times I \longrightarrow X$$

by the following conditions:

$$G(x, 0, t) = x \; G(x, 1, t) = K(j(x), 1 - t) \; G(a, s, t) = K(a, (1 - t)s) \; G(x, s, 0) = K(x, s)$$

for all $x \in X$, $a \in A$ and $s, t \in I$.

Now I use the fact that

$$((A \times I) \cup (X \times \partial I, \imath, X \times I)$$

is a cofibration by (potser fer la demostraci d'aixo)... to extend G to a homotopy,

$$G' \colon (X \times I) \times I \longrightarrow X$$

whose restricitions to $((A \times I) \cup (X \times \partial I) \times I$ and $(X \times I) \times \{0\}$ are, respectively, F and K.

The homotopy,

$$H \colon X \times I \longrightarrow X$$

defined by,

$$H(x, s) = G'(x, s, 1)$$

is a strong deformation retraction of X onto A.

I secondly prove the $\Longleftarrow$ implication:

I suppose that

$$H \colon X \times I \longrightarrow X$$

is a strong deformation retraction of X onto A.

The map $j = H(\bullet, 1) \colon X \longrightarrow A$ is a homotopy inverse of i. □

Corollary 10. *A map $f \colon A \longrightarrow B$ is a homotopy equivalence if and only if A is a strong deformation retract of M_f*

Proof. I know from the construction of the mapping cylinder that the map f factors out as $f = r \circ i$ where r is a homotopy equivalence and i is a cofibration.

Using this statements and the previous Proposition 9, I see that f is a homotopy equivalence if and only if i is a homotopy equivalence if and only if A is a strong deformation retract of M_f. □

Mapping Cone.

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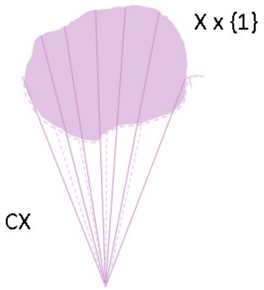


Let X be a topological space.

The cone of X is the quotient space,

$$CX = \frac{MX}{\sim}$$

where $\sim$ is the equivalence relation defined by

$$(x, t) \sim (x', t') \quad \text{if} \quad t = t' = 0 \quad \text{or} \quad (t = t' \text{ and } x = x').$$

I denote by $[(x, t)]$ the class of (x, t) in the quotient, and by c the class of $(x, 0)$

I observe that X is a subspace in the cone of X via the map $j \colon X \longrightarrow CX$ defined by $j(x) = [(x, 1)]$.

Moreover I note that que cone can be defined by the following pushout in *Top*:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i_0"'] & \star \arrow[d] \\
MX \arrow[r, "p"'] & CX
\end{tikzcd}$$

The cone defines a functor $C\colon Top \longrightarrow Top$ which makes correspond each X in Top with the object CX, and each $f\colon X \longrightarrow Y$ in Top the morphism $Cf\colon CX \longrightarrow CY$ defined as $Cf\ [(x,t)] = [(f(x),t)]$

Proposition 11. *CX is contractive.*

Proof. I must prove that $Id_{CX} \simeq \star_c\colon CX \longrightarrow CX$.
I define the homotopy $H\colon CX \times I \longrightarrow CX$ as,

$$H([(x,t)],s) = [(x,ts)]$$

Clearly H is well defined and verifies,

$$H([(x,t)],0) = [(x,0)] = c \quad \text{and} \quad H([(x,t)],1) = [(x,t)]$$

□

Example 12. *In particular, $C\mathbb{S}^n = \mathbb{E}^{n+1} \quad \forall n \geq 0$*
I consider the map, $f\colon C\mathbb{S}^n \longrightarrow \mathbb{E}^{n+1}$
defined by $f([x,t]) = tx$.
f is continuous and bijective and since the spaces are compact and Hausdorff, f results a homeomorphism.

Proposition 13. *Let $f\colon X \longrightarrow Y$ in Top and X contractive.*
Then f is nullhomotopic.

Proof. Since X is contractive then I have that $Id_X \simeq \star_x\colon X \longrightarrow X$ for some (all) x in X. By left composing with the map $f\colon X \longrightarrow Y$ I obtain,

$$f = f \circ Id_x \simeq f \circ \star_x = \star_y$$

Hence f is nullhomotopic. □

Proposition 14. *Let $f\colon X \longrightarrow Y$ be a continuous map between two topological spaces X, Y.*
Then f is nullhomotopic (homotopic to a constant map) if and only if it can be continuously extended to CX, that is, there exists $\widetilde{f}\colon CX \longrightarrow Y$ such that $\widetilde{f} \circ j = f$, where $j\colon X \longrightarrow CX$ is the inclusion into the top of the cylinder (the basis for the cone).

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \\
CX \arrow[ur, dashed, "\widetilde{f}"'] &
\end{tikzcd}$$

Proof. I suppose that $f \simeq \star_{y_0}\colon X \longrightarrow Y$.
That is, there exists a homotopy, $H\colon MX \longrightarrow Y$ such that,

$$H(x,0) = y_0 \qquad \text{and} \qquad H(x,1) = f(x) \qquad \forall x \in X$$

Then, the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i_0"'] & * \arrow[d] \arrow[ddr, dashed, bend left] & \\
MX \arrow[r, "p"'] \arrow[drr, dashed, bend right, "H"'] & CX & \\
 & & Y
\end{tikzcd}$$

Now, by the universal property of the pushout there exists a unique map $\widetilde{f}\colon CX \longrightarrow Y$ making the diagram commute.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i_0"'] & \star \arrow[d] \arrow[ddr, dashed, bend left] & \\
MX \arrow[r, "p"'] \arrow[drr, dashed, bend right, "H"'] & CX \arrow[dr, dashed, "\widetilde{f}"] & \\
 & & Y
\end{tikzcd}$$

and so, $\widetilde{f} \circ j = \widetilde{f} \circ (p \circ i_1) = (\widetilde{f} \circ p) \circ i_1 = H \circ i_1 = f$.
Therefore $\widetilde{f}$ is the extension of f.
Conversely, if there is an extension $\widetilde{f}\colon CX \longrightarrow Y$, then I can build a homotopy $H\colon MX \longrightarrow Y$ in the following way,

$$H(x,t) = \widetilde{f} \circ p(x,t)$$

Finally, I obtain $f \simeq \star_{y_0}$, since,

$$H(x,0) = \widetilde{f} \circ p(x,0) = \widetilde{f}(c) = y_0$$
and
$$H(x,1) = \widetilde{f} \circ p(x,1) = \widetilde{f} \circ (p \circ i_1)(x) = \widetilde{f} \circ j(x) = f(x) \quad \forall x \in X$$

□

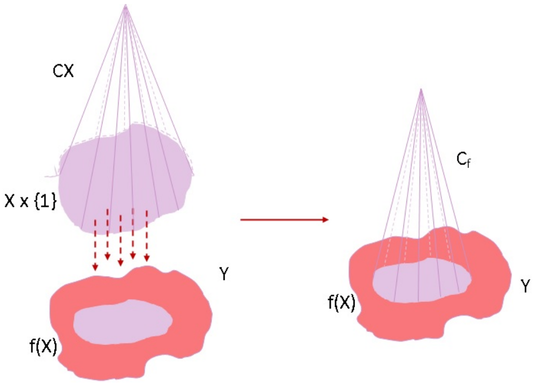
As in immediate consequence,

Corollary 15. *Let $f\colon \mathbb{S}^n \longrightarrow X$ be a continuous map.*
Then f is nullhomotopic if and only if f continuoulsy extends to $\mathbb{E}^{n+1}$

I note that the cone CX is a particular case of the mapping cylinder M_f with $Y = \star$.
The mapping cone of a function f between topological spaces X and Y is the quotient,

$$C_f := \frac{(CX \coprod Y)}{\sim}$$

where $\sim$ is the equivalence relation generated by $j(x) \sim f(x) \qquad \forall x \in X$.
and $j\colon CX$ is the natural inclusion $j(x) = [(x,1)]$



In a categorical language, I can observe that C_f can be described as the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i"] \\
CX \arrow[r, "q"'] & C_f
\end{tikzcd}$$

where q and i are the mappings at the quotient.

Proposition 16. *Let* $f\colon X \longrightarrow Y$ *and* $g\colon Y \longrightarrow Z$ *be two continuous maps.*

Then $g \circ f$ *is null-homotopic (homotopic to a constant map) if and only if it can be continuously extended to* C_f*, that is, there exists* $\widetilde{g}\colon C_f \longrightarrow Z$ *such that* $\widetilde{g} \circ i = g$*.*

Proof. I suppose $g \circ f \simeq *_z\colon X \longrightarrow Z$.

Then by Proposition 14, $g \circ f\colon X \longrightarrow Z$ it extends to CX, that is there exists $\widetilde{g \circ f}\colon CX \longrightarrow Z$ such that $\widetilde{g \circ f} \circ j = g \circ f$.

Therefore, I have the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i"] \arrow[ddr, bend left, "g"] & \\
CX \arrow[r, "p"'] \arrow[drr, dashed, bend right, "\widetilde{g\circ f}"'] & C_f & \\
& & Z
\end{tikzcd}$$

Now, by the universal property of the pushout there exists a unique maping $\widetilde{g}\colon C_f \longrightarrow Z$ making the diagram commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i"] \arrow[ddr, bend left, "g"] & \\
CX \arrow[r, "p"'] \arrow[drr, bend right, "\widetilde{g\circ f}"'] & C_f \arrow[dr, dashed, "\widetilde{g}"'] & \\
& & Z
\end{tikzcd}$$

Therefore, $\widetilde{g} \circ i = g$ and then $\widetilde{g}$ extends g .

Conversely, if g extends to $\widetilde{g}\colon C_f \longrightarrow Z$ I consider the mapping $\widetilde{g} \circ p\colon CX \longrightarrow Z$. This continuous mapping extends to $g \circ f\colon X \longrightarrow Z$ since $\widetilde{g} \circ p \circ j = \widetilde{g} \circ i \circ f = g \circ f$.

Hence, by Proposition 14 $g \circ f\colon X \longrightarrow Z$ is nullhomotopic. ☐

As a direct consequence I obtain

Corollary 17. *A map* $f\colon \mathbb{S} \longrightarrow X$ *is null homotopic if and only if it can be continuously extended to a map* $f\colon \mathbb{E}^{n+1} \longrightarrow X$ *that agrees with* f *on the boundary.*

Proof. I know that $C\mathbb{S}^{n-1} \cong \mathbb{E}^n$ for all n.
Then I have the diagram,

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, hook] \arrow[drr, bend right, "f"'] & \mathbb{E}^{n+1} \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

The result then follows directly from Proposition 16 . ☐

Suspension.

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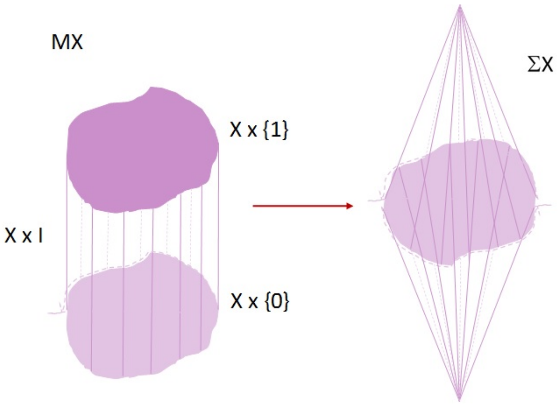

Let X be a topological space.

The suspension of X is the quotient space,

$$\Sigma X = \frac{MX}{\sim}$$

where $\sim$ is the equivalence relation defined by

$$(x,t) \sim (x',t') \qquad \text{if } t = t' = 0 \quad \text{or} \quad t = t' = 1 \quad \text{or} \quad (t = t' \text{ and } x = x').$$

I can observe that $\Sigma X = CX/\sim'$ where $\sim'$ is the equivalence relation generated by $j(x) \sim' j(x')$.

I observe that $\Sigma X x$ can be seen as the following pushout in Top,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & * \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

I note that the suspension defines a functor $\Sigma \colon Top \longrightarrow Top$ which maps for each X in Top an object ΣX, and for each $f \colon X \longrightarrow Y$ in Top the morphism : $\Sigma X \longrightarrow \Sigma Y$ defined as $[(x,t)] = [(f(x),t)]$

Proposition 18. $\Sigma \mathbb{S}^{n-1} \cong S^n$ *for all* $n \geq 1$

Proof. Set $\varphi \colon \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^n$ defined by

$$\varphi([x,t]) = 2\sqrt{t(1-t)}x, 2t-1) \ \mathbb{R}^n \times \mathbb{R}$$

φ is weed defined since $\varphi([x,0]) = (0,0,\cdots,0,-1) = \varphi([x',0])$ and $\varphi([x,1]) = (0,0,\cdots,0,1) = \varphi([x',1])$ and in addition $\varphi([x,t]) \ \mathbb{S}^n$.

φ is trivially bijective and continuous and since $\Sigma \mathbb{S}^{n-1}$ is compact and $\mathbb{S}^n$ is Hausdorff, then φ yields an homeomorphism. ☐

Playing CW Complexes

.

Among the methods of investigating geometrical properties of a given space, one has homology theory and homotopy theory. Homology theory was begun by Poincaré around 1900 while homotopy theory was initiated by Hurewicz in the 1930s.

Briefly speaking, in homology theory one descomposes a given figure into components like points, segments, triangles and in general k-dimensional simplices (the triangulation), and then one extracts a topologic invariant called the homology group out of the way they are connected to each other.

There is another way of decomposing figures, namely by means of CW complexes which are more flexible than triangulations.

This class of spaces is broader and has some better categorical properties than simplicial complexes, but still retains a combinatorial nature that allows for computation (often with a much smaller complex).

A CW complex is a nice topological space which is or can be built up inductively, by a process of attaching n-disks ^{n} along their boundary $(n-1)$-spheres $\mathbb{S}^{(n-1)} \quad \forall n \in \mathbb{N}$: so that is a cell complex built from the basic topological cells $\mathbb{S}^{(n-1)} \hookrightarrow \mathbb{D}^n$.

Being, therefore, essentially combinatorial objects, CW complexes are a main objects of interest in algebraic topology; in fact, most spaces of interest to algebraic topologists are homotopy equivalent to CW-complexes. Notably the geometric realization of every simplicial set, hence also of every groupoid is a CW complex.

Also, CW complexes are the cofibrant objects in the classical model structure on topological spaces. This means in particular that every topological space is weakly homotopy equivalent to a CW complex (but need not be strongly homotopy equivalent to one). Since every topological space is a fibrant object in this model category structure, this means that the full subcategory of Top on the CW complexes is a category of "homotopically very good representatives" of homotopy types.

The terminology "CW-complex" goes back to John H. C. Whitehead. It stands for the following two properties shared by any CW complex:

$C =$ "closure finiteness": a compact subset of a CW complex intersects the interior of only finitely many cells, hence in particular so does the closure of any cell.

$W =$ "weak topology": Since a CW complex is a colimit in Top over its cells, and as such equipped with the final topology of the cell inclusion maps, a subset of a CW complex is open or closed precisely if its restriction to (the closure of) each cell is open or closed, respectively.

In fact, Whitehead called the interior of the n-disks the "cells", so that their closure of each cell is the corresponding n-disk.)

Roughly speaking, a CW complex is made of basic building pieces called cells. The precise definition prescribes how the cells may be topologically glued together.

An n-dimensional closed cell is the image of an n-dimensional closed ball under an attaching map.

For example, a simplex is a closed cell, and more generally, a convex polytope is a closed cell. An n-dimensional open cell is a topological space that is homeomorphic to the n-dimensional open ball. A 0-dimensional open (and closed) cell is a singleton space. Closure-finite means that each closed cell is covered by a finite union of open cells.

Afterwards I will offer a formal definition but I could already introduce that a CW complex is a Hausdorff space X together with a partition of X into open cells (of perhaps varying dimension) that satisfies two additional properties:

- (●) For each n-dimensional open cell C in the partition of X, there exists a continuous map f from the n-dimensional closed ball to X such that,
  - (●) the restriction of f to the interior of the closed ball is a homeomorphism onto the cell C, and
  - (●) the image of the boundary of the closed ball is contained in the union of a finite number of elements of the partition, each having cell dimension less than n.
- (●) A subset of X is closed if and only if it meets the closure of each cell in a closed set.

If the largest dimension of any of the cells is n, then the CW complex is said to have dimension n. If there is no bound to the cell dimensions then it is said to be infinite-dimensional. The n-skeleton of a CW complex is the union of the cells whose dimension is at most n. If the union of a set of cells is closed, then this union is itself a CW complex, called a subcomplex. Thus the n-skeleton is the largest subcomplex of dimension n or less.

A CW complex is often constructed by defining its skeleta inductively. Begin by taking the 0-skeleton to be a discrete space. Next, attach 1-cells to the 0-skeleton. Here, each 1-cell begins as a closed 1-ball and is attached to the 0-skeleton via some (continuous) map from the boundary of the 1-ball, that is, from the 0-sphere $\mathbb{S}^0$. Each point of $\mathbb{S}^0$ can be identified with its image in the 0-skeleton under the aforementioned map; this is an equivalence relation. The 1-skeleton is then defined to be the identification space obtained from the union of the 0-skeleton and 1-cells under this equivalence relation.

In general, given the $(n-1)$-skeleton, the n-skeleton is formed by attaching n-cells to it. Each n-cell begins as a closed n-ball and is attached to the $(n-1)$-skeleton via some continuous map from the boundary of the n-ball, that is, from the $(n-1)$-sphere $\mathbb{S}^n$. Each point of $\mathbb{S}^{(n-1)}$ can be identified with its image in the $(n1)$-skeleton under the previously introduced map; this is again an equivalence relation. The restriction of the attaching map to the inside of the ball, that is to the open n-disk, is required to be a homeomorphism onto its image. The n-skeleton is then defined to be the identification space obtained from the union of the $(n-1)$-skeleton and n-cells under this equivalence relation.

Up to isomorphism every n-dimensional complex can be obtained from its $(n-1)$-skeleton in this sense, and thus every finite-dimensional CW complex can be built up by the process above. This is true even for infinite-dimensional complexes, with the understanding that the result of the infinite process is the direct limit of the skeleta: a set is closed in X if and only if it meets each skeleton in a closed set.

Defining CW Complexes.

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Following Allen Hatcher in [6], the previous introduction leads in the formal inductive definition of a CW complex.

Definition 19. *(CW Complex. Constructive Definition)*

A CW complex is a topological space X constructed in the following way:

(1) *Start with a discrete set X^0 , whose points are regarded as 0-cells.*

(2) *Inductively, form the n-skeleton X^n from $X^{(n-1)}$ by attaching n-cells $\mathfrak{e}^n_\alpha$ via maps $\varphi_\alpha\colon \mathbb{S}^{(n-1)} \longrightarrow X^{(n-1)}$. This means that X^n is the quotient space of the disjoint union $X \amalg_\alpha \mathbb{D}^n_\alpha$ of $X^{(n-1)}$ with a collection of n disks $\mathbb{D}^n_\alpha$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial\mathbb{D}^n_\alpha$. Thus as a set, $X^n = X^{(n-1)} \amalg_\alpha \mathfrak{e}^n_\alpha$ where each $\mathfrak{e}^n_\alpha$ is an open n disk.*

(3) *This inductive process can be either stopped at a finite stage, setting $X = X^n$ for some $n < \infty$, or continued indefinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \bigcap X^n$ is open (or closed) in X^n for each n.*

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the dimension of X, the maximum dimension of cells in X.

Observe that condition (3) is superfluous when X is finite-dimension, with $X = X^n$ for some n. For if A is open in $X = X^{(n-1)}$, and then by the same reasoning $A \bigcap X^{(n-2)}$ is open in $X^{(n-2)}$, and similarly for the skeleta $X^{(n-i)}$.

Let now formally $\mathbb{D}^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ be the n-dimensional disk and let $\mathbb{S}^{n-1} = \partial\mathbb{D}^n = \{x \in \mathbb{R}^n; \|x\| = 1\}$ be its boundary, namely the $(n-1)$-dimensional sphere.

Let X be a topological space and let $f\colon S^{n-1} \longrightarrow X$ be a continuous map.

One denotes by $X \cup_\varphi \mathbb{D}^n$ the space obtained from the disjoint union of X and $\mathbb{D}^n$ by identifying each point $x \in \mathbb{S}^{n-1}$ with $f(x) \in X$. It is called the space obtained by attaching an n-cell $\mathfrak{e}^n = \mathbb{D}^n \backslash \mathbb{S}^{n-1}$ to X by f or simply the attaching space. The map φ is called the attaching map.

A Hausdorff space X is called a cell complex if it is expressed as a disjoint union of cells $E_\lambda (\lambda \in \Lambda)$ in such a way that the image of the attaching map of any n-cell is contained in the union of cells whose dimensions are less than or wqual to $(n-1)$. A subset Y of X, which is a cell complex itself, is called a subcomplex.

Each cell $\mathfrak{e}^n_\alpha$ has its characteristic map ϕ_α , which is by definition the composition $\mathbb{D}^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha \mathbb{D}^n_\alpha \longrightarrow X^n \hookrightarrow X$. This map is continuous since it is a composition of continuous maps, where the inclusion $X^n \hookrightarrow X$ is continuous by (3). The restriction of ϕ_α to the interior of $\mathbb{D}^n_\alpha$ is a homeomorphism onto each cell $\mathfrak{e}^n_\alpha$.

With this presentation, I can offer an alternative definition of a CW complex.

Definition 20. *(CW Complex. Alternative Definition*

A cell complex X is called a CW complex if it satisfies two conditions,

(i) *The closure $\bar{e}$ of any cell e is contained in a finite subcomplex of X.*

(ii) *A subset $U \subset X$ is open if and only, for any cell e, $\bar{e} \cap U$ is open in $\bar{e}$.*

The conditions (i) and (ii) above are called closure finite and weak topology, respectively.

Both definitions are clearly equivalents.

If X is a CW complex, then the set of all cells of dimension less than or equal to n, denoted by $X^{(n)}$, becomes a subcomplex. It is called the n-skeleton of X. In homology theory of CW complexes, the following fact plays a fundamental role. Namely for any n-cell e^n of X, one has

$$H_k(X, X\backslash e^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

In contrast, homotopy theory concerns the set of homotopy classes of maps from the sphere $\mathbb{S}^n$ of each dimension to the given space. Let p_0 be the base point of S^n.

Definition 21. *(Homotopy Groups)*

Let X be a topological space with base point x_0. The set $[\mathbb{S}^n, X]_0$ of homotopy classes, relative to the base points of maps from $\mathbb{S}^n$ to X is denoted by

$$\pi_n(X, x_0)$$

It can be shown that $\pi_n(X, x_0)$ has a natural structure of a group and is called the n-th homotopy group of X with respect to x_0.

$\pi_1(X, x_0)$ is the fundamental group of X and for any $n > 1$, $\pi_n(X, x_0)$ is an abelian group, $\pi_1(X, x_0)$ acts naturally on $\pi_n(X, x_0)$. The group structure of $\pi_n(X, x_0)$ does not depend on the choice of base point so that it is frequently denoted by $\pi_n(X)$. The definition of homotopy groups is simpler than that of homology groups. But this fact is balanced by the fact that,in general, computation of homotopy groups is much harder than that computation of homology groups. For the product $X \times Y$ of two spaces X,Y, however, it follows immediatly from the definition that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$, which is simpler than the K*ü*nneth theorem in homology theory.

One of the basics in developing homology theory is the homology exact sequence for pairs of spaces. There exists a similar exact sequence for pairs of spaces in homotopy theory.

However, what is more important is the homotopy exact sequence for fibrations.

Definition 22. *(Fibration)*

Let E,B be topological spaces. A continuous surjection $\pi\colon E \longrightarrow B$ is called a fibration (or fibering) if π has the covering homotopy property with respect to the n-dimensional cube I^n for all $n \geq 0$ in the following sense:

For any continuous map $f\colon I^n \longrightarrow E$ and any homotopy $\bar{f}\colon I^n \times I \longrightarrow B$ of $\bar{f} = \pi \circ f$, there exists a homotopy $f_t\colon I^n \times I \longrightarrow E$ of f such that $\pi \circ f_t = \bar{f}_t$ for any $t \in I = [0, 1]$.

In this case, one calls $\pi^{-1}(b) \quad (b \in B)$ the fiver over b.

Example 23.

(i) *Fiber bundles are important examples of fibrations.*

(ii) *Let X be an arcwise connected topological space and let x_0 be its base point, Then the set*

$$X = \{\ell : [0, 1] \longrightarrow X \mid \ell(0) = x0\}$$

equipped with the compact open topology is called the path space of X with initial point x_0.

If I define $\pi : \mathcal{P}X \longrightarrow X$ by setting $\pi(\ell) = \ell(1)$, te it becomes a fibration. The fiber over x_0 is the space consisting of all closed paths based there, namely the loop space by ΩX. X is contractible, and it is known that if X has the homotopytype of a CW complex, then so does Ω.

(iii) *Let X,Y be topological spaces and assum that Y is arcwise connected. I show that any continuous map $f\colon X \longrightarrow Y$ can be considered as a fibration in the sense of homotopy. To see this, let $map(I, Y)$ denote the mapping space consisting of all continuous maps from the unit interval I to Y and set*

$$\tilde{X} = \{(x, \ell) \in X \times map(I, Y) | \ell(0) = f(x)\} \subset X \times map(I, Y)$$

If I define $i\colon X \longrightarrow \tilde{X}$ by $i(x) = (x, \ell_{f(x)})$ denotes the constant path at $f(x)$.

If I define $\pi\colon \tilde{X} \longrightarrow Y$ by setting $\pi(x, \ell) = \ell(1)$, then it can be shown that it is a fibration. Clearly $\pi \circ i = f$ so that this fibration is homotopy theoretically equivalent to the original map $f\colon X \longrightarrow Y$. The fiber of this fibration is called the homotopy fiber of f.

Theorem 1. *(Homotopy Exact Sequence for Fibrations) For each fibration* $f \longrightarrow E \longrightarrow B$*, there exists a long exact sequence*

$$\ldots \longrightarrow \pi_{n+1}(X) \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \ldots$$

Definition 24. *Let* n *be a positive integer and let* π *be a group.*

In case $n > 1$ *then I assume that* π *is an abelian group.*

A topological space $K(\pi, n)$ *is called an Eilenberg-Mac Lane space of type* (π, n) *if*

$$\pi_k(K(\pi, n)) \cong \begin{cases} \pi & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

For example S^1 *is a* $K(\mathbb{Z}, 1)$ *and* $\mathbb{CP}^\infty$ *is a* $K(\mathbb{Z}, 2)$*.*

In general, a manifold M *which is a* $K(\pi, 1)$*, namely* $\pi_n(M) = 0$ *for any* > 1*, is called a* $K(\pi, 1)$ *manifold.*

Equivalently I may characterize a $K(\pi, 1)$ *manifold by the property that its universal covering space is contractible.*

Among such manifolds, closed $K(\pi, 1)$ *manifolds are particularly important.*

It is known that any closed Riemannian manifold with negative sectional curvature is an example of such manifolds.

Theorem 2. *For any* (π, n) *given in the above definition, there exists an Eilenberg-MacLane space* $K(\pi, n)$ *which is a CW complex.*

Moreover it is uniquely defined up to homotopy equivalence.

Proof. (Sketch)
In case $n = 1$ I express the group π in terms of a system of generators and relations.

I fix one 0-cell and attach a 1-cell to it corresponding to each generator.

I then realise each relation by attaching a 2-cell.

If I kill higher homotopy groups π_2,π_3,... of the resultant 2-dimensional complex by attaching k-cells for $k = 3, 4, \ldots$, then I obtain a $K(\pi, 1)$.

The cases of $n > 1$ can be treated basically in the same way.

I first express the abelian group π in terms of generators and relations.

I prepare one copy of the n-sphere $\mathbb{S}^n$ corresponding to each generator and attach them to a point.

Next I realise the relations by attaching $(n + 1)$-cells.

Finally I kill higher homotopy groups π_{n+1},π_{n+2}, ... and I am finished.

Uniqueness can be proved by a simple application of the theorem of J.H.C. Whitehead concerning homotopy equivalences which I will prove afterwards. □

If $n > 1$, $K(\pi, n)$ is $(n - 1)$-connected by definition so that I have an isomorphism $H_n(K(\pi, n)) \cong \pi_n(K(\pi, n)) \cong \pi$ by the Hurewicz theorem.

Hence the universal coefficients theorem implies,

$$H^n(K(\pi, n); \pi) \cong hom(\pi, \pi)$$

Let $\imath \in H^n(K(\pi, n); \pi)$ be the element which corresponds to $id \in Hom(\pi, \pi)$ under the above isomorphism.

I call it the fundamental cohomology class of $K(\pi, n)$.

The following theorem shows that the Eilenberg-MacLane spaces can serve as the classifying spaces for the cohomology theory.

Theorem 3. *Let* X *be a CW complex.*

I assume that either $n > 1$ *or* $n = 1$ *and* π *is an abelian group.*

Then the correspondence

$$[X, K(\pi, n)] \ni f \longmapsto f^*(\imath) \in H^n(X; \pi)$$

is a bijection

Proof. (Sketch)

Let $\alpha \in H^n(X;\pi)$ be an arbitrary element and choose a cocycle which represents it.

Construct a continuous map $f\colon X \longrightarrow K(\pi,n)$ as follows.

I send the $(n-1)$-skeleton $X^{(n-1)}$ of X to the base point of $K(\pi,n)$, and over each n-cell e^n of X, I set f to represent the element $\alpha(e^n) \in \pi = \pi_n(K(\pi,n))$.

Since α is a cocycle, f can be extended to $X^{(n+1)}$.

Moreover since the target of the map f is $K(\pi,n)$, I see that it can be extended to the whole of X by an easy application of the obstruction theory.

Clearly $f^*(\imath) = \alpha$ by the construction.

Thus I have proved that the correspondence of theorem is surjective.

Next suppose that I am given two maps $f_i\colon X \longrightarrow K(\pi,n)(i=0,1)$ such that $f_0^*(\imath) = f_1^*(\imath)$.

I may assume that the restrictions of f_i to $X^{(n-1)}$ are both constant maps.

By the assumption, I have $f_0|_{X^{(n)}} \simeq f_1|_{X^{(n)}}$.

Since the target of the map is $K(\pi,n)$, again the obstruction theory implies that the homotopy can be extended to the whole of $X \times I$. ☐

Theorem 4. *The topology of a CW complex is determined by the family of its closed cells.*

Proof. Let X be a CW complex and let $U \subset X$ be a set whose intersection with any closed cell of X is closed.

I want to prove that $U \cap X^n$ is closed, for every integer $n \geq 0$.

Because X^0 is discrete, $U \cap X^0$ is closed.

I assume that $U \cap X^{n-1}$ is closed in X^{n-1}.

Recall that the skeleton X^n is determined by a pushout as,

$$\begin{tikzcd}
\cong_\lambda \mathbb{S}_\lambda^{n-1} \arrow[r, "f"] \arrow[d, "i"] & X^{n-1} \arrow[d, "\bar{i}"] \\
\cong_\lambda B_\lambda^n \arrow[r, "\bar{f}"'] & X^n
\end{tikzcd}$$

and therefore, I must prove that $\bar{f}^{-1}(U \cap X^n)$ is closed in $\bigsqcup_\lambda B_\lambda^n$.

The map $\bar{f}$ induces a set $\{\bar{f}_\lambda \mid \lambda \in \Lambda\}$ of characteristic maps for the n-cells of X.

By the hypothesis

$$\bar{f}_\lambda^{-1}(U \cap X^n) = \bar{f}_\lambda^{-1}(U \cap \bar{e}_\lambda)$$

is closed in B_λ^n, for every $\lambda \in \Lambda$.

Hence,

$$\bar{f}_\lambda^{-1}(U \cap X^n) = \bigcup_{\lambda\in\Lambda} \bar{f}_\lambda^{-1}(U \cap X^n)$$

is closed in B_λ^n.

☐

Theorem 5. *Let K be a compact subset of a CW complex X.*

Then K is contained in a finite union of open cells of X.

Proof. Let $S \subset K$ obtained by taking a point $x_e \in e \cap K$ from each open cell e which intersects K.

I want to prove that S is finite.

I begin by observing that $S \cap X^0 = K \cap X^0$ is a discrete, closed subset of K and thus, $S \cap X^0$ is finite.

I assume, by induction, that $S \cap X^{n-1}$ is finite.

For every closed n-cell $\bar{e}$, $S \cap \bar{e}$ consists of at most x_e and the finitely many elements $S \cap X^{n-1}$ and therefore, $S \cap \bar{e}$ is either empty or is a finte set, in any case, a closed subset $\bar{e}$.

But X^n is itself a CW complex and thus, according to Theorem 4, its topology us determined by the family of its closed cells; thus $S \cap X^n$ is a closed subset of X^n which is discrete and contained in the compact space K and therefore, is a finite set.

I have shown that, for every $n \geq 0$, $S \cap X^n$ is a finite set and so, S is a discrete, closed subset of X and of K.

But a discrete, closed subset of a compact space is finite and so, S is finite. □

As a consequence of the previous results, I have,

Corollary 25. *CW complexes are compactly generated spaces.*

Recall that a space X is said to be compactly generated if X is Haussdorff and its topology is determined by the set of al its compact subsets, i.e. $U \subset X$ is closed if and only if, for every $C \subset X$ compact, $U \cap C$ is closed in C.

I can add that the category of compactly generated spaces is a full subcategory of Top.

Definition 26. *(Cellular Map)*

A map $f\colon X \longrightarrow Y$ of CW complexes is called cellular if $f(X^n) \subset Y^n \quad \forall n$.

That is, a map $f\colon X \longrightarrow Y$ between two CW complexes is said to be cellular if it takes the n-skeleton X^n of X into the n-skeleton Y^n of Y, for every n.

Adjuntions of CW complexes are CW complexes as long as the attaching map is cellular.

Theorem 6. *Let A be a subcomplex of a CW complex Y.*

Let $f\colon A \longrightarrow W$ be a cellular map.

Then $X = W \cup_f Y$ is a CW complex containing W as a subcomplex.

Proof. For every $n \geq 0$, I construct the space

$$X^n = W^n \cup_{f_n} Y^n$$

where $f_n\colon A^n \longrightarrow W^n$ is the restriction of f to A^n.

I can note that X^0 is a discrete space.

I will prove that, for every $n \geq 1$ the pair (X^n, X^{n-1}) is an adjunction of n-cells and that X is a union space of $X^0 \subset \cdots X^n \subset \cdots$.

The first of these assertions will be proved by constructuring an intermediate space $X^{n-1} \subset Z_n \subset X^n$ such that (X^n, Z_n) and (Z_n, X^{n-1}) are adjunctions of n-cells with the attaching map of (X^n, Z_n) factoring through X^{n-1}.

I assume that I succeeded in constructing Z_n with the aforementioned properties.

Let,

$$g\colon S_\Lambda = \textstyle\bigsqcup_{\lambda \in \Lambda} S_\lambda^{n-1} \longrightarrow X^{n-1}$$

$$h\colon S_\Upsilon = \textstyle\bigsqcup_{\upsilon \in \Upsilon} S_\upsilon^{n-1} \longrightarrow Z_n$$

be the attaching maps for (Z_n, X^{n-1}) and (X^n, Z_n), respectively, with h decomposing as

$$S_\upsilon \xrightarrow{h_1} X^{n-1} \xrightarrow{i} Z_n$$

where i is the inclusion.

Let $j\colon Z_n \longrightarrow X^n$ be the inclusion map.

I claim that the following commutative diagram

$$\begin{tikzcd}
S_\Lambda \cup S_\Upsilon \arrow[r, "g \cup h_1"] \arrow[d] & X^{n-1} \arrow[d] \\
B_\Lambda \cup B_\Upsilon \arrow[r, "j\bar{g} \cup \bar{h}"'] & X^n
\end{tikzcd}$$

B_Υ are the topological sums of n-balls corresponding to the topological sums of $(n-1)$-spheres S_Λ and S_Υ, respectively, and the vertical arrows are inclusions) is a pushout. For this, take maps

$l\colon B_\Lambda \cup B_\Upsilon \longrightarrow Z$ and $m\colon X^{n-1} \longrightarrow Z$ giving rise to a commutative diagram when composed with the appropriate maps; then use the universal property of the pushout for (Z_n, X^{n-1}) relative to the maps $l|_{B_\Lambda}$ and m to obtain a map $k\colon Z_n \longrightarrow Z$ which will be used in the pushout diagram of (X^n, Z_n) to generate a map $r\colon X^n \longrightarrow Z$ such that

$$r|X^{n-1} = m \text{ and } r(j\bar{g} \cup \bar{h}) = l$$

I define the space Z_n by

$$Z_n = X^{n-1} \cup W^n$$

Since (W^n, W^{n-1}) is an adjunction of n-cells, the law of horizontal compositions implies that (Z_n, X^{n-1}) is and adjunction of n-cells.

The same law also implies that

$$Z_n \cong W^n \cup_{f_n} (A^n \cup Y^{n-1})$$

The space $W^n \cup_{f_n} (A^n \cup Y^{n-1})$ is a pushout space for the diagram determined by f_n, and the inclusion $A^n \subset A^n \cup Y^{n-1}$.

Let

$$\bar{f} : A^n \cup Y^{n-1} \longrightarrow W^n \cup_{f_n} (A^n \cup Y^{n-1}) \cong Z_n$$

be a characteristic map.

Taking the inclusion

$$A^n \cup Y^{n-1} \subset Y^n$$

viewing $\bar{f}$ as an attaching map and using the law of vertical compositions I conclude that

$$X^n \cong Z_n \cup_{\bar{f}} Y^n$$

since $(Y^n, A^n \cup Y^{n-1})$ is an adjunction of n-cells, it follows that (X^n, Z_n) is an adjunction of n-cells.

Clearly, the attaching map for the n-cells of this pair factors trough $A^n \cup Y^{n-1}$ and thus, through Y^{n-1}.

But the induced map $Y^{n-1} \longrightarrow Z_n$ factors through X^{n-1}, which completes this part of the proof.

It remains to prove that X is a union space for the spaces X^n.

Let $j_n\colon X^n \longrightarrow X$ be the canonical maps and let $g\colon X \longrightarrow Z$ be a map such that, for every $n \geq 0$, $g \circ j_n$ is continuous.

These maps give rise to two sequences of maps

$$\{h_n\colon W^n \longrightarrow Z \mid n \geq 0\}$$

$$\{k_n\colon Y^n \longrightarrow Z \mid n \geq 0\}$$

which, by the universal property of adjunction spaces, produce a continuous function $X \longrightarrow Z$ that coincides with g.

□

As an immediate consequence of the previous theorem,

Corollary 27. *Let X,Y be CW complexes.*

Let $f\colon X \longrightarrow Y$ be a cellular map.

Then the mapping cylinder M_f is a CW complex.

Proof. I consider the mapping cylinder,

$$M_f := \frac{(X \times I) \cup Y}{(x,1) \sim f(x)} = (X \times Y) \cup_f Y.$$

It is a general and well-known fact that the product of two CW complexes is a CW complex.

So $X \times I$ is a CW complex.

And from the construction $X = X \times \{0\}$ is a subcomplex of $X \times I$.

Now, $f\colon X = X \times \{0\} \longrightarrow Y$ is a cellular map.

So that, by Theorem 6 $(X \times I) \cup_f Y = M_f$ is a CW complex.

(Recall that in the proof of Theorem 6 applied to the case of the cylinder, I take as the cells those cells of $Y = X \times I$ not in $A = X \times \{0\}$ and also the cells of $W = Y$. This can be did it, because $(X \times I)\backslash(X \times \{0\})$ is the union of various cells, $X = X \times \{0\}$ being a subcomplex of $X \times I$. The reason I have to require $f : X = X \times \{0\} \longrightarrow Y$ to be cellular, though, is that the boundary of each n-cell has to be contained in a union of $(n-1)$-cells).

□

Remark 28. *I can also note that the product of two CW complexes should not be given the product topology, but the "compactly generated" topology. This is because a CW complex has the W for weak: the topology is the weak topology of the skeleta.*

Theorem 7. *Let (X, A) be an adjunction of n-cells with $n \geq 1$.*

Then (X, A) is $(n-1)$-connected

Corollary 29. *Let (X, A) be a relative CW complex and let $0 \leq n <$ dim X.*
Then (X, X^n) is n-connected.

Proof. I first prove that for every $m > n$, (X^m, X^n) is n-connected.

The previous theorem shows that (X^{n+1}, X^n) is n-connected.

I suppose that $m - 1 > n$ and, by induction, that (X^{m-1}, X^n) is n-connected.

I observe that (X^m, X^{m-1}) is $(m-1)$-connected again by the previous theorem and that the path-components of X^m intersect X^n. In fact, the $(m-1)$-connectivity of (X^m, X^{m-1}) implies that the path-components of X^m intersect X^{m-1} and the n-connectivity of (X^{m-1}, X^n) implies that the path-components of X^{m-1} intersect X^n or, in other words, the following two functions induced by the inclusion maps are onto:

$$\pi_0(X^{m-1}) \longrightarrow \pi_0(X^m) \quad , \quad \pi_0(X^n) \longrightarrow \pi_0(X^{m-1})$$

then,

$$\pi_0(X^n) \longrightarrow \pi_0(X^m)$$

is onto and so, the path-componenets of X^m intersect X^n.

Now, for any $x_0 \in X^n$, the exact sequence of the spaces $X^n \subset X^{m-1} \subset X^m$ shows that

$$\pi_r(X^m, X^n, x_0) = 0 \quad , \quad 1 \leq r \leq n$$

and hence, (X^m, X^n) is n-connected.

To prove that (X, X^n) is n-connected I proceed as follows.

For any $m > n$, let $i_{n,m}\colon X^n \longrightarrow X^m$ be the inclusion map; also denote by i the inclusion of X^n into X.

Now, for any $1 \leq r \leq n$, take the inclusion $i_{r-1}\colon \mathbb{S}^{r-1} \longrightarrow B^r$ and an arrow-map

$$(a, b)\colon i_{r-1} \longrightarrow i$$

Since $b(B^r)$ is compact, there is an $m > n$ such that $b(B^r) \subset X^m$.

There exist a map $b'\colon B^r \longrightarrow X^n$ extending a and a homotopy relative to $\mathbb{S}^{r-1}$ of $i_{n,m}b'$ to b; but then ib' is homotopic rel. $\mathbb{S}^{r-1}$ to b.

Now (X, X^n) is n-connected. □

Theorem 8. *(Cellular Approximation Theorem)*

Let $f\colon X \longrightarrow Y$ be a continuous map of CW complexes.
Then f is homotopic to a cellular map.
Moreover, if there is a subcomplex $L \subset X$ on which $f|_L$ is cellular, then the homotopy can be taken stationary on L.

This theorem is the CW complex version of the simplicial approximation theorem. There are some differences. First, one does not need to do any of that subdivision business in the simplicial's one. Second, a cellular map, unlike a simplicial map, can still be extremely complex. Nonetheless, if f is both a cellular map and an inclusion, then f can be viewed as the inclusion of a subcomplex.

Despite these differences, one can actually prove the cellular approximation theorem by essentially the simplicial theorem. The basic idea is that if a map is not cellular, then it sends an n-cell into an $(n+1)$-cell. But an n-cell is, intuitively, too small to fill up an $(n+1)$-cell. This is not true if one allows space-filling curves, but it is if one allows only smooth maps (by Sards theorem, actually). So one can push it off to the boundary.

Some main ideas in order to understand the proof:

The point is that given $X \longrightarrow Y$, one looks at cells of X which are not mapped into cells of the appropriate dimension in Y. So, one considers a cell e_n^α which is mapped into some union $e_{m_1}^{\beta_1} \cup \ldots \cup e_{m_k}^{\beta_k}$ where some of the $m_i > n$. Then one homotopes f to make it, as a collection of maps between subsets of euclidean space, piecewise-linear on a substantial portion of the domain. The piecewise linearity shows that the image of f has to miss a point. Then one can deformation retract the homotoped version of f onto the boundaries of the things out of e_n^α's league. One must use the fact that the inclusion of a subcomplex is a cofibration to extend these homotopies to the whole complex.

With this intuitions, one can now consider formally the proof.

Proof. For every integer n such that $0 \leq n \leq dim\ X$, take $K^n = X^n \cup L$ and define the map

$$F\colon X \times \{0\} \cup L \times I \longrightarrow Y$$

by

$$F(x,t) = \begin{cases} f(x) & \text{if } x \in X \text{ and } t = 0 \\ f(x) & \text{if } (x,t) \in L \times I \end{cases}$$

Now, for each $x \in X^0 \backslash L$, choose a path $\lambda_x\colon I \longrightarrow Y$ such that $\lambda_x(0) = f(x)$ and $\lambda_x(1) \in Y^0$ (this can be done because, either $f(x)$ is a 0-cell, or $f(x)$ is connected to a 0-cell of Y by a path since every path-component of a CW-complex contains at least a 0-cell).

Next, define

$$F_0\colon K^0 \times I \longrightarrow Y$$

by

$$F_0(x,t) = \begin{cases} F(x,t) & \text{if } (x,t) \in L \times I \\ \lambda_x(t) & \text{if } x \in X^0 \backslash L \end{cases}$$

I can note that

$$F_0|_{(X^0 \times \{0\} \cup L \times I)} = F|_{(X^0 \times \{0\} \cup L \times I)}$$

and $F_0|_{(K^0 \times \{1\})}$ is cellular; in particular, $F_0|_{(X^0 \times \{1\})} \subset Y^0$.

I suppose that I have defined a map

$$F_{n-1}\colon K^{n-1} \times I \longrightarrow Y$$

such that

$$F_{n-1}|_{(X^{n-1} \times \{0\} \cup L \times I)} = F|_{(X^{n-1} \times \{0\} \cup L \times I)}$$

and

$$F_{n-1}(X^{n-1} \times \{1\}) \subset Y^{n-1}.$$

Let e be an n-cell of $X\backslash L$ with characteristic map

$$\bar{c}_e\colon B^n \longrightarrow X$$

and attaching map

$$c_e\colon S^{n-1} \longrightarrow X^{n-1}$$

Observe that there is a retraction

$$r_n\colon B^n \times I \longrightarrow B^n \times \{0\} \cup S^{n-1} \times I$$

given by

$$r_n(x,t) = \begin{cases} \frac{2}{2-t}(x,0) & \text{if } 0 \leq t \leq 2(1- \parallel x \parallel) \\ \frac{1}{\|x\|}(x, 2 \parallel x \parallel +t-2) & \text{if } 2(1- \parallel x \parallel) \leq t \leq 1, \parallel x \parallel \neq 0 \end{cases}$$

(this is the same retraction found while proving that the arrow formed by the inclusion of a sphere into the corresponding ball is a cofibration).
Now I define the map,

$$b_\epsilon\colon B^n \times I \longrightarrow Y$$

as the composition of r_n, with the maps

$$\bar{c}_\epsilon \times 1_I\colon B^n \times \{0\} \cup S^{n-1} \times I \longrightarrow X \times \{0\} \cup K^{n-1} \times I$$

and

$$F|_{(X\times\{0\})} \cup F_{n-1}\colon X \times \{0\} \cup K^{n-1} \times I \longrightarrow Y$$

Because the restriction of b_ϵ to $S^{n-1} \times \{1\}$ maps that space into $Y^{n-1} \subset Y^n$ and the pair (Y, Y^n) is n-connected by Corollary 29.

$$[b_e|_{(B^n\times\{1\})}, b_e|_{(S^{n-1}\times\{1\})}] = 0$$

thus, $b_e|_{(\mathbb{S}^{n-1}\times\{1\})}$ extends to a map

$$\bar{b}_e\colon B^n \times I \longrightarrow Y^n$$

and, denoting the inclusion of Y^n into Y by j_n, there is a homotopy

$$H^e_n\colon B^n \times I \longrightarrow Y$$

relative to $\mathbb{S}^{n-1} \times \{1\}$ between $j_n\bar{b}_e$ and $b_e|_{(B^n\times\{1\})}$.
Now I can construct a commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times I \arrow[r, "c_e\times 1_I"] \arrow[d, "i_{n-1}\times 1_I"'] & X \times \{0\} \cup K^{n-1} \times I \arrow[d] \\
B^n \times I \arrow[r, "\bar{c}_e\times 1_I"] & X \times \{0\} \cup (K^{n-1} \cup e) \times I
\end{tikzcd}$$

whose square is a pushout and thus, giving rise to a map

$$F^e_n\colon X \times \{0\} \cup ((K^{n-1} \cup e) \times I \longrightarrow Y$$

Now,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times I \arrow[r, "c_e\times 1_I"] \arrow[d, "i_{n-1}\times 1_I"'] & X \times \{0\} \cup K^{n-1} \times I \arrow[d] \arrow[rdd, bend left, "F'"] & \\
B^n \times I \arrow[r, "\bar{c}_e\times 1_I"] \arrow[rrd, bend right, "H^e_n"] & X \times \{0\} \cup (K^{n-1} \cup e) \times I \arrow[rd, "F^e_n"] & \\
 & & Y
\end{tikzcd}$$

I can observe that the map F' is the restriction $F|_{(X\times\{0\})} \cup F_{n-1}$.

The crucial property of the restriction $F_n^e|_{(X^{n-1}\cup e)} \times \{1\}$ is that such a map is cellular. Iterating this process for all n-cells of $X\backslash L$, I obtain an extension of F_{n-1} to a map

$$F_n \colon K^n \times I \longrightarrow Y$$

such that $F_n(X^n \times \{1\}) \subset Y^n$ and

$$F_n|_{(X^n \times \{0\} \cup L \times I)} = F|_{(X^n \times \{0\} \cup L \times I)}$$

The union space of the sequence $\{K^n \mid n \geq 0\}$ coincides with X; the maps F_n produce a function

$$G \colon X \times I \longrightarrow Y$$

which is continuous and is a homotopy rel. L between f and $g = G(\bullet, 1)$, a cellular map. ☐

Whitehead's Theorem

.

Theorem 9. *(Whitehead's Theorem)*
Let X and Y be path connected CW complexes.
Let $f \colon X \longrightarrow Y$ be a continuous map.
I suppose that f induces isomorphisms on all homotopy groups (that is, f is a weak homotopy equivalence) and so,

$$f_{ast} \colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an isomorphism for any choice of $x_0 \in X$.
Then f is a homotopy equivalence.

Previously to the formal proof of the Theorem, I want to rremark some general facts.
I know that I can view the homotopy groups as homotopy classes

$$\pi_n(X, x_0) = [(\mathbb{S}^n, p), (X, x_0)]$$

where $\mathbb{S}^n \xrightarrow{f} X$ and $f(p) = x_0$.
In an analogous way, I can view the homotopy groups of pairs

$$\pi_n(X, A, x_0) = [(\mathbb{E}^n, S^{n-1}, p), (X, A, x_0)]$$

A homotopy is now an uniparametric family mapping of pairs in such a way that the following diagram is commutative,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "j"] \arrow[d, "f"'] & E^n \arrow[d, "F"] \\
A \arrow[r, hook, "i"] & X
\end{tikzcd}$$

where $f(p) = x_0 \in A$
Recall that $\mathbb{E}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$

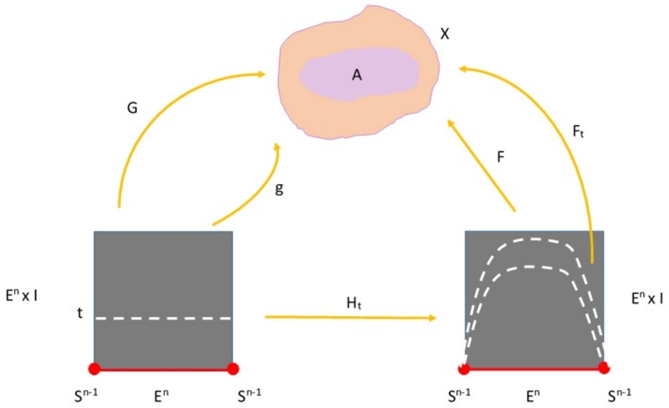
Lemma 30. *Given a homotopy*

$$(F_t, f_t) \colon (\mathbb{E}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A) \quad , 0 \leq t \leq 1$$

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "j"] \arrow[d, "f_t"'] & \mathbb{E}^n \arrow[d, "F_t"] \\
A \arrow[r, hook, "i"] & X
\end{tikzcd}$$

where f_t are not constant but they all always send $\mathbb{S}^{n-1}$ to A
Then there exists another homotopy, $\{(G_t, g_t)\}_{0 \leq t \leq 1}$ where $\{g_t\}_{0 \leq t \leq 1}$

Proof. I construct such a homotopy



g is now the restriction to the sphere and is constant.
Thus $G_t = F_t \circ H_t$ ☐

Proof. I will use a simple cell-by-cell construction to prove this theorem.
First of all, I can assume f is a cellular map. This follows because I can homotope f to be cellular by Theorem 8, the Cellular Approximation Theorem.
Secondly, I observe that the proof of the Theorem can be also reduced to a special case:

the case in which X is a subcomplex of Y and f is the inclusion,

(reduction that follows using the usual trick of considering the mapping cylinder as a way of turning any map into an inclusion).
Indeed, I define the mapping cylinder of f,

$$M_f\colon = \frac{(X\times I)\ \cup\ Y}{(x,1)\sim f(x)} = (X \times I)\ \cup_f\ Y.$$

I know that if $f : X \longrightarrow Y$ is a cellular map between CW complexes, then the mapping cylinder $M_f = (X \times I) \cup_f Y$ is a CW complex by Corollary 27
Then f is the composition of two maps,

$$X \longrightarrow M_f \longrightarrow Y$$

That is, I can factorize f into the previous composition where the first map is an inclusion,

$$i\colon X \longrightarrow M_f$$

defined by sending

$$x \mapsto [(x,0)]$$

In fact, i is an inclusion of a subcomplex.
The second map r is a homotopy equivalence,

$$r\colon M_f \longrightarrow Y$$

defined by sending

$$[(x,t)] \mapsto f(x) \quad \text{if } \ (x,t) \in (X \times I)$$

and

$$[y] \mapsto y \quad \text{if } \ y \in Y$$

Indeed, the map

$$r\colon M_f \longrightarrow Y$$

is a homotopy equivalence since it comes from a deformation retraction of M_f onto Y. Moreover Y is a subcomplex of M_f.
Is defined the inclusion

$$j\colon Y \longrightarrow M_f$$

defined by sending

$$y \longmapsto [y].$$

Clearly j is also a homotopy equivalence (the inverse homotopy for i).
Indeed, by simply observing that $r \circ j = Id_Y$, since

$$(r \circ j)(y) = r(j(y)) = r([y]) = y$$

and there is a homotopy,

$$G\colon M_f \times I \longrightarrow M_f$$

$$G\colon\ = j \circ r \simeq Id_{M_f}$$

given by

$$G([y], t') = [y]$$

and

$$G([(x,t)], t') = [(x, 1 - t'(1-t))]$$

I can observe that $j \circ r$ is sending,

$$[(x,t)] \longmapsto f(x) \longmapsto [f(x)] = [(x,1)]$$

$$[y] \longmapsto y \longmapsto [y]$$

So, in fact,

$$H_0 \cong j \circ r$$

$$H_1 \cong Id_{M_f}$$

To sumarize, in order to prove Whitehead's theorem, I may assume that,

- (•) X is a subcomplex of Y , $X \subset Y$
- (•) and f is a cellular map
- (•) and moreover f is an inclusion, $X \overset{f}{\hookrightarrow} Y$, that induces isomorphisms on all homotopy groups.

That is, it holds,

- (a) $f_*\colon \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, x_0) \qquad \forall n \ \ \forall x_0 \in X$
- (b) Y is obtained by attaching cells.

Case 1: $Y = X \bigcup_\varphi e^n$

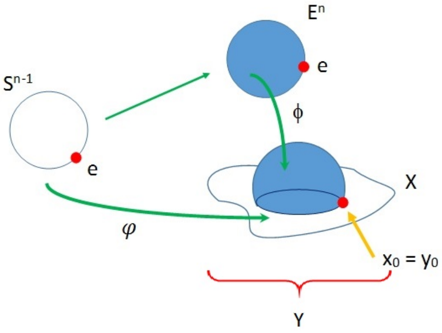
where φ is the attaching map,

$$\mathbb{S}^{n-1} \ \overset{\varphi}{\to}\ X$$

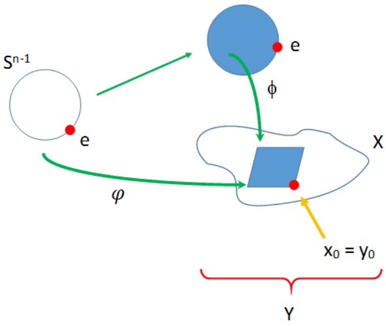
and ϕ is the characteristic map,

$$\mathbb{E}^n \overset{\phi}{\to} Y$$

So that, Y is obtained attaching a single n-cell to X.



Or using a simplier picture which I will use in the subsequents constructive diagrams,



(After the conclusion of this initial case I will need to generalize this construction to the case that Y is obtained attaching arbitrary cells (possibly infinite) of different dimensions in **(Case 2)** and **(Case 3)**.

It is clear that I have the following commutative diagram,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathbb{E}^n \arrow[d, "\phi"] \\
X \arrow[r, hook, "f"'] & Y
\end{tikzcd}$$

Now by the statement and in particular, I suppose that $\pi_n(Y,X) = 0$. This is equivalent to suppose that

$$f_*\colon \pi_n(X) \twoheadrightarrow \pi_n(Y) \text{ is exhaustive,}$$

and

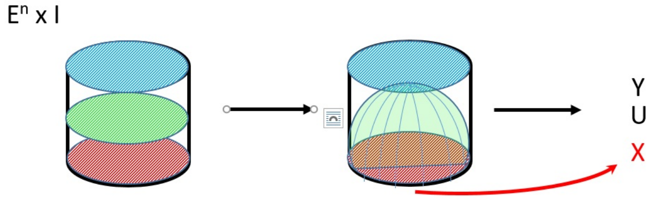
$$f_*\colon \pi_{n-1}(X) \longrightarrow \pi_{n-1}(Y) \text{ is injective.}$$

I shall prove that under this hypothesis, f admits an homotopic inverse. In fact, I will obtain that X is a strong deformation retract of Y.

Recall that $\pi_n(Y,X) = [(\mathbb{E},\mathbb{S}^{n-1}),(Y,X)]$, where all the mappings and homotopies are base point preservant (so I will not specify it although they are being considered).

The hypothesis $\pi_n(Y,X) = 0$ in particular implies that the mappng $\phi\colon \mathbb{E} \longrightarrow Y$ is homotopic to a certain mapping $\mathbb{E}^n \longrightarrow Y$ such that its image is contained in X, and what is more the whole homotopy mantains $\mathbb{S}^{n-1}$ contained in X.

Moreover, I can modify the homotopy in such a way that $\mathbb{S}^{n-1}$ remains fixed, by Lemma 30 as the following picture shows.



So, there exists a homotopy,

$$G\colon \mathbb{E}\times I \longrightarrow Y \quad \text{with} \quad G_0=\pi \quad \text{and such that} \quad G_1 \quad \text{applys in } X\text{:}$$

$$\begin{tikzcd}
\mathbb{E}^n \arrow[r, "\widetilde{\varphi}"] \arrow[rr, bend right, "G_1"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Moreover G is relative to $\mathbb{S}^{n-1}$, that is, $G(z,t)$ not depens on t if $z\in\mathbb{S}^{n-1}$.
Thus,

$$G(z,t)=G(z,0)=G_0(z)=\phi(z)=f(\phi(z)) \qquad \forall t \quad z\in\mathbb{S}^{n-1}$$

At this point, I need to construct a retraction r.
That is, I want to construct,

$$\begin{tikzcd}
X \arrow[r, hook, "f"] \arrow[d, equal] & Y \arrow[d, equal] \arrow[dl, dashed, "\exists\ r\ ?"] \\
X \arrow[r, hook, "f"'] & Y
\end{tikzcd}$$

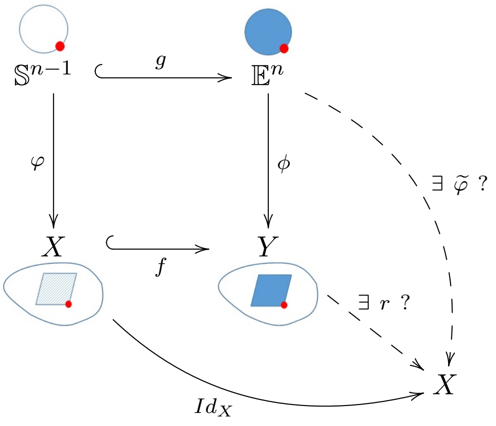
such that,

$$r\circ f = Id_X$$
$$f\circ r \simeq Id_Y \ \ \text{relative to } X$$

I start now with the construction of the retraction r, using the commutative diagram,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathbb{E}^n \arrow[d, "\phi"] \\
X \arrow[r, hook, "f"'] & Y
\end{tikzcd}$$

I want to obtain,



$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathbb{E}^n \arrow[d, "\phi"] \arrow[ddr, dashed, bend left, "\exists\ \widetilde{\varphi}\ ?"] & \\
X \arrow[r, hook, "f"'] \arrow[drr, bend right, "Id_X"'] & Y \arrow[dr, dashed, "\exists\ r\ ?"] & \\
& & X
\end{tikzcd}$$

Remark 31. *I can define* $\widetilde{\varphi}$ *which, in fact is not unique.*
I have,

$$[Id_X \circ \varphi] = [\varphi] \in \pi_{n-1}(X, x_0) \quad \textit{where} \quad x_0 = \varphi(p)$$

Then,

In one hand,

(1)
$f_*([\varphi]) = [f \circ \varphi] = [\phi \circ g] = 0$
Since $\phi \circ g \colon \mathbb{S}^{n-1} \longrightarrow Y$ *is a map that can be extended to* $\mathbb{E}^n$ *and also due to the fact that* $\mathbb{E}^n$ *is contractive,* $\mathbb{E}^n \simeq \{*\}$

But, on the other hand,

(2)
$f_* \colon \pi_{n-1}(X, x_0) \overset{\cong}{\longrightarrow} \pi_{n-1}(Y, x_0)$
So that, $f_*([\varphi]) \in \pi_{n-1}(Y, x_0)$
So, by (1) + (2) I obtain, $[\varphi] = 0$
Hence,

$$[\varphi] = 0 \implies \varphi \ \textit{ is homotopic to constant} \implies \varphi \ \textit{ extends to } \mathbb{E}^n.$$

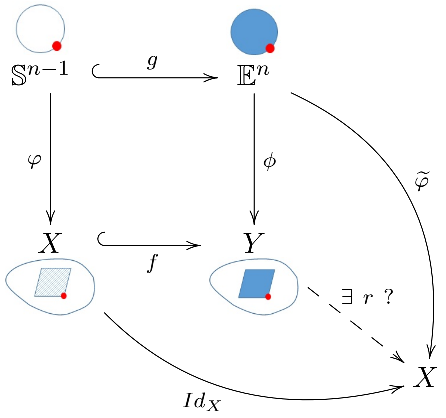
So that, there exists,

$$\widetilde{\varphi} \colon \mathbb{E}^n \longrightarrow X \quad \textit{(which is not unique), such that, } \widetilde{\varphi} \circ g = \varphi$$

Thus, I choose a suitable $\widetilde{\varphi}$ which holds,

$$f(\widetilde{\varphi}(z)) = G_1(z) = G(z, 1) = f(\varphi(z)) \qquad \text{if} \quad z \in \mathbb{S}^{n-1}$$

and so, $\widetilde{\varphi}(z) = \varphi(z) \qquad \text{if} \quad z \in \mathbb{S}^{n-1}$
Up to now, I have,



In fact, I have carefully chosen $\widetilde{\varphi}$ (holding some conditions) in order to built the retraction r in a suitable way
(which is such that $r \circ f = Id_X$ and $r \circ \phi = \widetilde{\varphi}$, but moreover and mainly holds $f \circ r \simeq Id_Y$ relative to X (which is the hardest condition to prove)).
Now, $(\widetilde{\varphi}, Id_X)$ define a map

$$r \colon Y \longrightarrow X$$

such that

$$r \circ f = Id_X$$
and
$$r \circ \phi = \tilde{\varphi}$$

Indeed, note that,

$$r(\phi(e)) = \widetilde{\varphi}(e) \qquad \forall e \in \mathbb{E} \qquad \text{and} \qquad r(f(x)) = x \qquad \forall x \in X.$$

In particular since $r \circ f = Id_X$ then r is a retraction of Y on X.
However, the tough part is to prove that $f \circ r \simeq Id_Y$ relative to X.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathbb{E}^n \arrow[d, "\phi"] & \\
X \arrow[r, hook, "f"'] & Y \arrow[dr, bend left, "f \circ r"] \arrow[dr, bend right, "Id_Y"'] & \\
 & & Y
\end{tikzcd}$$

I need a homotopy, $Y \times I \xrightarrow{H} Y$ such that $H_0 = Id_Y$ and $H_1 = f \circ r$ and furthermore holding that H is a homotopy from $f \circ r$ to Id_Y relative to X.
I know that the diagram

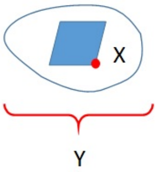
$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\varphi"'] & \mathbb{E}^n \arrow[d, "\phi"] \\
X \arrow[r, hook, "f"'] & Y
\end{tikzcd}$$

is a pushout.
I can also consider the diagram

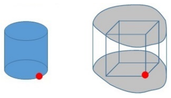
$$\begin{tikzcd}
\mathbb{S}^{n-1} \times I \arrow[r, hook, "g \times Id"] \arrow[d, "\varphi \times Id"'] & \mathbb{E}^n \times I \arrow[d, "\phi \times Id"] \\
X \times I \arrow[r, hook, "f \times Id"'] & Y \times I
\end{tikzcd}$$

which it is also a pushout.
Indeed, (Y, f, ϕ) is a pushout.
and I know that Y is obtained by a process of cell attachment,

$$Y = \frac{\mathbb{E}^n \amalg X}{\sim} \qquad \text{where } g(z) \sim \varphi(z) \quad \forall z \in \mathbb{S}^{n-1}$$



So, clearly now,



$$Y \times I \cong \frac{(\mathbb{E}^n \times I) \amalg (X \times I)}{\sim} \qquad \text{where } (g(z), t) \sim (\varphi(z), t) \qquad \forall t \ \ \forall z$$

therefore, $(Y \times I, f \times Id, \phi \times Id)$ is a pushout as well.
Now I would want to find,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times I \arrow[r, hook, "g\times Id"] \arrow[d, "\varphi\times Id"'] & \mathbb{E}^n \times I \arrow[d, "\phi\times Id"] \arrow[rdd, dashed, bend left, "\exists\ G\ ?"] & \\
X \times I \arrow[r, hook, "f\times Id"'] \arrow[rrd, dashed, bend right, "\exists\ F\ ?"'] & Y \times I \arrow[rd, dashed, "\exists\ H\ ?"] & \\
 & & Y
\end{tikzcd}$$

such that

$$H_0 = Id_Y$$
$$H_1 = f \circ r$$

I note that in fact, I take as the map $G\colon \mathbb{E} \times I \longrightarrow Y$ the which one I earlier obtained, and I can also define,

$$X \times I \xrightarrow{F} Y \qquad \text{as} \qquad F(x,t) = f(x) \qquad \forall\ t$$

Now I have,

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times I \arrow[r, hook, "g\times Id"] \arrow[d, "\varphi\times Id"'] & \mathbb{E}^n \times I \arrow[d, "\phi\times Id"] \arrow[rdd, bend left, "G"] & \\
X \times I \arrow[r, hook, "f\times Id"'] \arrow[rrd, bend right, "F"'] & Y \times I \arrow[rd, dashed, "\exists\ H\ ?"] & \\
 & & Y
\end{tikzcd}$$

Immediately now, there is a d map H because if $z \in \mathbb{S}^{n-1}$, then,

$$G(z,t) = f(\varphi(z)) = F(\varphi(z),t) = F((\varphi \times Id)(z,t))$$

It remains to show that H is a homotopy from f

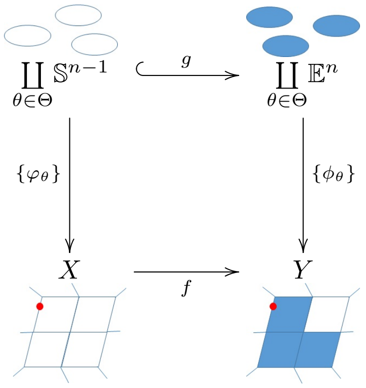
(•) $H(y,0) = H(\phi(e),0) = G(e,0) = G_0(e) = \phi(e) = y$ if $y = \phi(e)$;
(•) $H(y,0) = H(f(x),0) = F(x,0) = f(x) = y$ if $y = f(x)$;
(•) $H(y,1) = H(\phi(e),1) = G(e,1) = G_1(e) = f(\widetilde{\varphi}(e)) = f(r(\phi(e))) = f(r(y))$ if $y = \phi(e)$;
(•) $H(y,1) = H(f(x),1) = F(x,1) = f(x) = f(r(f(x))) = f(r(y))$ if $y = f(x)$;
(•) $H(f(x),t) = F(x,t) = f(x)$ if $\forall x \in X \quad \forall t$

Hence, $f \circ r \simeq Id_Y$ relative to X.
Proof for **Case 1:** $Y = X \cup_\varphi e^n, \quad n \geq 1$ is concluded.
I consider now,

Case 2: $Y = X \cup_{\varphi_\theta} e^n_\theta, \quad n \geq 1, \quad \theta \in \Theta$
where Θ is an arbitrary index set.
This case is absolutely derived from the previous one.
Again, I want to find a retraction,

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "Id_X"'] & Y \arrow[dl, dashed, "r'"] \\ X & \end{tikzcd}$$

I have that the corresponding diagram



$$\begin{tikzcd} \coprod\limits_{\theta\in\Theta} \mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\{\varphi_\theta\}"'] & \coprod\limits_{\theta\in\Theta} \mathbb{E}^n \arrow[d, "\{\phi_\theta\}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is pushout.

Remark 32. *As in the previous* ***(Case 1)****, I can define* $\widetilde{\varphi_\theta} \quad \forall \theta \in \Theta$ *which, in fact are not unique.*
I consider,

$$\forall \theta \in \Theta \ : \qquad\qquad \mathbb{S}^{n-1} \xrightarrow{\varphi_\theta} X \overset{f}{\hookrightarrow} Y$$

Now,

$$\begin{array}{l} [f \circ \varphi_\theta] \in \pi_{n-1}(Y, x_\theta) \quad \textit{where } x_\theta = \varphi_\theta(p) \quad \textit{and } p \textit{ is the base point in } \mathbb{S}^{n-1} \\ \quad \| \\ f_*([\varphi_\theta]) \end{array}$$

Since, $f \circ \varphi_\theta = \phi_\theta \circ g \quad \forall \theta \quad \Longrightarrow \quad [f \circ \varphi_\theta] = 0 \quad \forall \theta$
Now I have,

$$\forall \theta \in \Theta\colon \quad f_*\colon \pi_{n-1}(X, x_\theta) \xrightarrow{\cong} \phi_{n-1}(Y, y_\theta) \quad \textit{where} \quad y_\theta = f(x_\theta) \quad \textit{with } [\varphi_\theta] \in \pi_{n-1}(X, x_\theta)$$

I can observe that, $f_*([\varphi_\theta]) = 0 \;\Rightarrow\; [\varphi_\theta] = 0 \;\Rightarrow\; \exists\, \widetilde{\varphi}_\theta \colon \mathbb{E}^n \longrightarrow Y$ *such that* $\widetilde{\varphi}_\theta \circ g = \varphi_\theta \quad \forall \theta$

But recall that these $\widetilde{\varphi_\theta}$ are not unique.
Thus, following the argument in **Case 1** I can choose the right $\widetilde{\varphi_\theta}$ for all $\theta \in \Theta$.
Now, $(\{\widetilde{\varphi_\theta}\}, Id_X)$ define a map

$$r' \colon Y \longrightarrow X$$

such that

$$r' \circ f = Id_X$$

and

$$r' \circ \{\widetilde{\phi_\theta}\} = \{\widetilde{\varphi_\theta}\}$$

Therefore, r' is defined.

$$\begin{tikzcd}
\coprod\limits_{\theta\in\Theta} \mathbb{S}^{n-1} \arrow[r, hook, "g"] \arrow[d, "\{\varphi_\theta\}"'] & \coprod\limits_{\theta\in\Theta} \mathbb{E}^{n} \arrow[d, "\{\phi_\theta\}"] \arrow[ddr, dashed, bend left, "\{\widetilde{\varphi_\theta}\}"] & \\
X \arrow[r, "f"'] \arrow[drr, bend right, "Id_X"'] & Y \arrow[dr, "r'"] & \\
& & X
\end{tikzcd}$$

And also the same argumet than the previous **(Case 1)**, it shows that

$$f \circ r' \simeq Id_Y$$

and that the homotopy H' which would be constructed to that end, it results in a homotopy from $f \circ r'$ to Id_Y relative to X.

Case 2: $Y = X \cup_{\varphi_\theta} e^n_\theta, \quad n \geq 1, \quad \theta \in \Theta$ is solved.

Finally, I consider

Case 3: $Y = X \cup_{\varphi_0} (\bigvee_{\theta_0} e^0) \cup_{\varphi_1} (\bigvee_{\theta_1} e^1) \cup \cdots \cup_{\varphi_{\theta_m}} e^m \quad , \quad \theta \in \Theta$

Again, I want to find a retraction,

$$\begin{tikzcd}
X \arrow[r, hook, "f"] \arrow[d, "Id_X"'] & Y \arrow[dl, dashed, "r''"] \\
X &
\end{tikzcd}$$

I can observe that $f\colon X \overset{f}{\hookrightarrow} Y$ induces isomorphisms on all homotopy groups, that is,

$$f_*\colon \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, y_0) \qquad \forall n \ \ \forall x_0 \in X \qquad y_0 = f(x_0)$$

I denote by $Y^{(k)}$ the k-skeleton of Y, $\forall k$

Then, $Y^{(k)} \hookrightarrow Y$ induces isomorphisms in $\pi_1, \pi_2, \ldots \pi_{k-1}$.

I consider,

$$Y = \underbrace{\underbrace{X \cup_{\varphi_0} (\bigvee_{\theta_0} e^0)}_{Y^0} \cup_{\varphi_1} (\bigvee_{\theta_1} e^1)}_{Y^1} \cup \cdots \cup_{\varphi_m} e^m \quad , \quad \theta \in \Theta$$

I suppose $m = 0$

Then $Y = X \cup_{\varphi_0} (\bigvee_{\theta_0} e^0)$

This case has been already solved in **(Case 2)**.

I consider now recursively,

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[rrrr, dashed, bend left, "f_n"] & Y^{(n-1)} \arrow[rrr, hook, "i_n"] \arrow[drr, dashed, "Id"'] & & & Y^{(n)} \arrow[dl, "s_n"'] \arrow[ddll, dashed, bend left, "r_n"] \\
& & & Y^{(n-1)} \arrow[dl, "t_n"] & \\
& & X & &
\end{tikzcd}$$

holding $r_n \circ f_n = Id_X$ for all n and $f_n \circ r_n \simeq Id_{Y^{(n)}}$ relative to X for all n .
I have

$$\begin{tikzcd}
X \arrow[r, hook, "f_n"] & Y^{(n)} \arrow[dl, "r_n"] & & X \arrow[r, hook, "f"] & Y \arrow[dl, "r"] & & Y \arrow[r, phantom, "r"] & {} \\
X & & & X & & & &
\end{tikzcd}$$

I know that $Y = \bigcup\limits_{n=0}^{\infty} Y^{(n)}$
Then, r is determined by $\{r_n\}_{n=0}^{\infty}$ because of the following commutative diagrams

$$\begin{tikzcd}
Y^{(n-1)} \arrow[dd, hook, "i_n"'] \arrow[dr, "r_{n-1} = t_n"] & \\
& X \\
Y^{(n)} \arrow[ur, "r_n"'] &
\end{tikzcd}$$

Note that it holds

$$r_n \circ i_n = t_n \circ s_n \circ i_n = t_n \circ Id = t_n = r_{n-1}$$

It remains to show that $f \circ r \overset{?}{\simeq} Id_Y$ relative to X .
I have homotopies, $Y^{(n)} \times I \xrightarrow{H_n} X$ holding,

$$\begin{gathered}
(H_n)_0 = f_n \circ r_n \quad \text{for all } n \\
\text{and} \\
(H_n)_1 = Id_{Y^{(n)}} \quad \text{for all } n
\end{gathered}$$

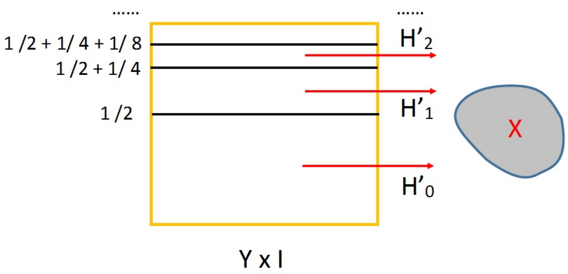
I would want to find a homotopy $Y \times I \xrightarrow{H} X$ holding,

$$\begin{gathered}
H_0 = f \circ r \\
\text{and} \\
H_1 = Id_Y
\end{gathered}$$

But H is not determined by $\{H_n\}_{n=0}^{\infty}$ since the following are noncommutative diagrams,

$$\begin{tikzcd}
Y^{(n-1)} \times I \arrow[dd, hook] \arrow[dr, "H_{n-1}"] & \\
& X \\
Y^{(n)} \times I \arrow[ur, "H_n"'] &
\end{tikzcd}$$

The solution is given by the fact that each homotopy $Y^{(n)} \times I \xrightarrow{H_n} X$ extends to some homotopy $Y \times I \xrightarrow{H'_n} X$ by the property of extension of homotopies, since $Y^{(n)} \hookrightarrow Y$ is a cofibration between topological spaces.
Thus, I define $H \colon Y \times I \longrightarrow X$ "correctly linking" the sequence $\{H'_n\}_{n=0}^{\infty}$.



With this last fact, the retraction r'' is well constructed.

Case 3: $Y = X \cup_{\varphi_0} (\bigvee_{\theta_0} e^0) \cup_{\varphi_1} (\bigvee_{\theta_1} e^1) \cup \ \cdots \ \cup_{\varphi_{\theta_m}} e^m \quad , \quad \theta \in \Theta$ is solved.

The proof is complete.

$\square$

Remark 33. *If all I know is that X and Y have isomorphic homotopy groups, then X and Y need not to be homotopy equivalent (which by Whitehead's theorem means that the isomorphisms on homotopy groups might not be induced by a map $f : X \longrightarrow Y$).*
Indeed, it is essential to the theorem that isomorphisms between $\pi_k(X)$ and $\pi_k(Y)$ for all k are induced by a map $f : X \longrightarrow Y$. If an isomorphism exists which is not induced by a map, it need not be the case that the spaces are homotopy equivalent.

For instance,
I consider,
$\mathbb{P}^n_{\mathbb{R}}$, the n-dimensional real projective space.
$\mathbb{S}^n$, the n-sphere.
Let now,
$X = \mathbb{P}^m_{\mathbb{R}} \times \mathbb{S}^n$
$Y = \mathbb{P}^n_{\mathbb{R}} \times \mathbb{S}^m$
The two spaces X and Y have isomorphic homotopy groups because they both have a universal covering space homeomorphic to $\mathbb{S}^m \times \mathbb{S}^n$, and it is a double covering in both cases.
However, I can also observe that for $m < n$, X and Y are not homotopy equivalent, since for instance using homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$:

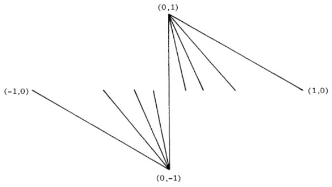
$$H_m(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

but instead of this,

$$H_m(Y; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Remark 34. *I can also observe that the assumption that X and Y are CW complexes is essential in the theorem, since every weak homotopy equivalence not need to be a homotopy equivalence.*

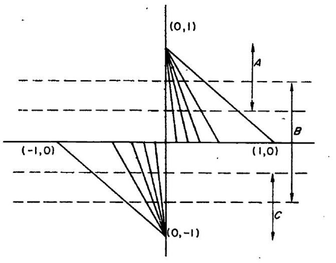
For instance,
Consider the double comb space.



Let X be the subspace of the plane $\mathbb{R}^2$, $X \subset \mathbb{R}^2$, consisting of straight line segments joining $(0, 1)$ to the points $(0, 0)$ and $(1/n, 0)$, for all positive integers n, and $(0, -1)$ to all the points $(0, 0)$ and $(-1/n, 0)$.
This subspace is the union of,

- (●) for each positive integer n, the staight line joining $(0, 1)$ to $(1/n, 0)$;

(•) for each positive integer n, the straight line joining $(0, 1)$ to $(-1/n, 0)$;
(•) the straight line joining $(0, 1)$ and $(0, -1)$



That is, X is the set of all line segments: from $(0, 1)$ to both $(0, 0)$ and $(1/n, 0)$ and from $(0, -1)$ to both $(0, 0)$ and $(-1/n, 0)$, for natural n.

This space holds that $\pi_n(X) = 0$ for all $n \geq 0$, but that X is not contractible. Thus the map that sends all of X to $(0, 0)$,

$$\begin{aligned} f\colon X &\longrightarrow X \\ (x, y) &\longmapsto (0, 0) \end{aligned}$$

is a weak homotopy equivalence that is not a homotopy equivalence (because X is not a CW complex.

To prove that $\pi_n(X) = 0$ for all $n \geq 0$, I take an open covering of X by three open sets A, B, C, defined by $x_2 > 1/3$, $2/3 > x_2 > -2/3$, $-1/3 > x_2$ respectively.

Then if $f\colon \mathbb{S}^n \longrightarrow X$ is any map, the sets $f^{-1}(A)$, $f^{-1}(B)$, $f^{-1}(C)$ form an open covering of S^n, with Lebesgue number δ, say.

If $\mathbb{S}^n$ is triangulated so that the mesh is less that δ, only a finite number of simplexes are mapped into B, and since the image of each is path connected, it follows that $f(S^n) \cap B$ is contained in a finite number of "rays" from $(0, 1)$ or $(0, -1)$.

That is, $f(S^n)$ is contained in Y, the union of A and C with a finite number of rays.

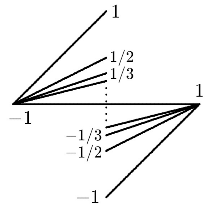
Clearly Y is contractible, and so this means that f is homotopic to the constant map in Y, so certainly in X.

Hence, $\pi_n(X) = 0$.

On the other hand X is not contractible.

Intuitively,

The central point $(0, 0)$ is the limit of the sequences $\{(0, 1/k)\}$ and $\{(0, -1/k)\}$ and a continuous deformation of this space into one single point pushes this central point to the left extreme $(-1, 0)$ as well as to the right extreme $(1, 0)$.



Formally,

If X were contractible, there would be a map $f\colon X \times I \longrightarrow X$ starting with the identity map and ending with the constant map to some point $x_0 \in X$.

Since I is compact, the continuity of f implies that, given $x \in X$ and $\epsilon > 0$, there exists δ such that $d(x, y) < \delta \implies d(f(x, t), f(y, t)) < -\epsilon$ for all $t \in I$.

But for each integer $n > 0$, the homotopy f defines paths u^+ and u^- from $(1/n, 0)$, $(-1/n, 0)$ to x_0 respectively.

Subdivide I (considered as a 1-simplex) so that each simplex of the subdivision is mapped by each of u^+ and u^- into just one of the sets A,B or C.

Since $(1/n, 0)$ and $(-1/n, 0)$ are in different path components of B, there is a first vertex t such that either $u^+(t) \in A$ or $u^-(t) \in C$; if say, $u^+(t) \in A$, then $u^-(t)$ lies in the region $x_2 \leq 0$.

Hence $d(u^+(t), u^-(t)) < 1/3$, which contradicts the continuity of f, since if I take $x = (0, 0)$ and $\epsilon = 1/3$, there is always an n such that $2/n < \delta$, for any δ.

I can add another simplier example,

Let X be the subspace of $\mathbb{R}^1$ consisting of the points 0 and $1/n$, for all integers $n \geq 1$.

Now the path components of X are just the single points (since each point $1/n$ is both open and closed); so if X were homotopy-equivalent to a CW-complex K, then K would have to have an infinite number of path components.

But if $f \colon X \longrightarrow K$ were a homotopy equivalence, $f(X)$ would be compact, since X is, and so would have to be contained in a finite subcomplex of K.

Thus $f(X)$ would be contained in the union of a finite number of path components, and this contradicts the assumption that f is a homotopy equivalence.

Hence X is not homotopy-equivalent to a CW-complex.

In general, Whitehead's theorem can be hard to apply, because if may be hard to check that a map induces isomorphisms on all homotopy groups.

Fortunately, it also holds,

Theorem 10. *Let X and Y be two simply connected CW complexes.*
Let $f : X \longrightarrow Y$ be a continuos map.
I suposse that f induces isomorphisms on all homology groups.
That is,

$$f_* \colon H_n(X) \longrightarrow H_n(Y)$$

is an isomorphism for each n.
Then f is a homotopy equivalence.

Proof. Again, using the mapping cylinder construction M_f for f then I can suppose

$$f \colon X \longrightarrow Y$$

to be an inclusion. Recall that then $Y \hookrightarrow M_f$ is a homotopy deformation retract, and using this, I make the identifications,

$$\pi_i(Y) \cong \pi_i(M_f)$$
$$H_i(Y) \cong H_i(M_f)$$

So, recall too that the inclusion $X \hookrightarrow Y$ gives exact sequences in homotopy and homology;
That is, I have both,

for the induced map f^* in homotopy I have the long exact sequence in homotopy,

$$\cdots \longrightarrow \pi_q(X, x_0) \xrightarrow{f^*} \pi_q(Y, x_0) \longrightarrow \pi_q(Y, X, x_0) \xrightarrow{\partial^*} \pi_{q-1}(X, x_0) \longrightarrow \cdots$$

where ∂^* is the connecting morphism in homotopy;

and for the induced map f_* in homology, I have the long exact sequence in homology,

$$\cdots \longrightarrow H_q(X) \xrightarrow{f_*} H_q(Y) \longrightarrow H_q(Y, X) \xrightarrow{\partial_*} H_{q-1}(X) \longrightarrow \cdots$$

where ∂_* is the connecting morphism in homology;

and using that the absolute Hurewicz morphism h fits with the relative one to form a commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_q(X, x_0) \arrow[r, "f^*"] \arrow[d, "h"] & \pi_q(Y, x_0) \arrow[r] \arrow[d, "h"] & \pi_q(Y, X, x_0) \arrow[r, "\partial^*"] \arrow[d, "h"] & \pi_{q-1}(X, x_0) \arrow[r, "f^*"] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_q(X) \arrow[r, "f_*"] & H_q(Y) \arrow[r] & H_q(Y, X) \arrow[r, "\partial_*"] & H_{q-1}(X) \arrow[r, "f_*"] & \cdots
\end{tikzcd}$$

Because X,Y are both simply connected, then

$$\pi_1(Y, X, x_0) = 0$$

Thus, the pair (Y, X) is 1-connected.
And so $H_1(Y, X) = 0$.
Now I look at a part of the long exact sequence on homology,

$$H_2(X) \longrightarrow H_2(Y) \longrightarrow H_2(Y, X) \longrightarrow H_1(X) \longrightarrow H_1(Y)$$

I can observe that $H_2(Y, X)$ is trapped between isomorphisms, an then it is zero,

$$H_2(Y, X) = 0$$

and so

$$\pi_2(Y, X, x_0) = 0$$

since by the relative Hurewicz theorem, I have the isomorphism,

$$\pi_2(Y, X, x_0) \xrightarrow{\cong} H_2(Y, X) = 0$$

Thus the pair (Y, X) is 2-connected.
Repeating this argument recursively I will obtain,

$$H_n(Y, X) = 0 \quad \forall n$$

and so

$$\pi_n(Y, X, x_0) = 0 \quad \forall n$$

So I have,

$$\cdots \to 0 \to \pi_{q+1}(X, x_0) \xrightarrow{f^*} \pi_{q+1}(Y, x_0) \to 0 \to \pi_q(X, x_0) \xrightarrow{f^*} \pi_q(Y, x_0) \to 0 \to \pi_{q-1}(X, x_0) \to \pi_{q-1}(Y, x_0) \to 0 \to \cdots$$

and so I get that,

$$f_n \colon \pi_n(X) \longrightarrow \pi_n(Y)$$

is an isomorphism for all n.
Therefore, using the Whitehead's Theorem, I can conclude that f is a homotopy equivalence. ☐

Corollary 35. *Let X be a closed, oriented, simply connected n-dimensional manifold with $H_i(X) = 0$ for $0 < i < n$.*
Then X is homotopy equivalent to S^n.

Proof. I can find and embedding of the closed ball D^n into X.
Define a map $f : X \longrightarrow ?$ by sending the interior of D^n homeomorphically to the complement of the north pole, and the rest of X to the north pole. Then Whitehead's theorem applies to show that f is a homotopy equivalence. ☐

Corollary 36. *If X has the homotopy type of an n-dimensional CW complex and if $\pi_i(X) = 0$ for $i \leq n$, then X is contractible.*

Proof. Since $\pi_i(X) = 0$ for $i \leq n$, then $H_i(X) = 0$ for $i \leq n$.
On the other hand, I have $H_i(X) = 0$ for $i > n$ by dimension.
Thus, $\tilde{H}_*(X) = 0$ and so $\pi_*(X) = 0$ by the Hurewicz theorem.
Hence, by the Whitehead's theorem, X is contractible. ☐

Corollary 37. *If X has the homotopy type of an n-dimensional CW complex and if $\pi_i(X) = 0$ for $i \leq n - 1$, then*

$$X \sim \vee \mathbb{S}^n$$

In particular, if $H_n(X) \cong \mathbb{Z}$, then X has the homotopy tipe of S^n.
Thus, a simply connected homology sphere (a homology sphere is a space with the same homology as a sphere) is homotopy equivalent to a sphere).

I can add a nice application for the Whitehead Theorem.

Example 38. *I have three topological spaces, X, Y, Z, two of them, said X and Y are CW-complexes but not the third one, said Z.*
I want to study two cases:

(a) *Assume given the following diagram of arrows,*

$$X \xleftarrow[\sim]{g} Z \xrightarrow[\sim]{h} Y$$

where the maps g and h are weak equivalences.
I want to show that there exists a map

$$f\colon X \longrightarrow Y$$

which is a homotopy equivalence.

(b) *Assume given the following diagram of arrows,*

$$X \xrightarrow[\sim]{g} Z \xleftarrow[\sim]{h} Y$$

where the mamps g and h are weak equivalences.
I want to show that there exists a map

$$f\colon X \longrightarrow Y$$

which is a homotopy equivalence.

I first study part (a).
I know that any topological space has a CW approximation and so I can consider a CW approximation for the topological space Z, which is both a CW complex $\tilde{Z}$ and a weak equivalence,

$$k\colon \tilde{Z} \xrightarrow{\sim} Z$$

So that, I obtain the diagram,

$$\begin{tikzcd} X & Z \arrow[l, "g", "\sim"'] \arrow[r, "h", "\sim"'] & Y \\ & \tilde{Z} \arrow[u, "k", "\sim"'] & \end{tikzcd}$$

Then, I can consider the two compositions,

$$\begin{tikzcd} \tilde{Z} \arrow[r, "k", "\sim"'] \arrow[rr, bend right, "\sim", "g\circ k"'] & Z \arrow[r, "g", "\sim"'] & X \end{tikzcd}$$

$$\begin{tikzcd} \tilde{Z} \arrow[r, "k", "\sim"'] \arrow[rr, bend right, "\sim", "h\circ k"'] & Z \arrow[r, "h", "\sim"'] & Y \end{tikzcd}$$

both of them are compositions of weak equivalences, and so they are weak equivalences too. Indeed, composition of weak equivalences is a weak equivalence.

$$g \text{ and } k \text{ are weak equivalences} \Longrightarrow g\circ k \text{ is a weak equivalence.}$$
$$h \text{ and } k \text{ are weak equivalences} \Longrightarrow h\circ k \text{ is a weak equivalence.}$$

and therefore I obtain the diagram,

$$\begin{tikzcd} X & Z \arrow[l, "g", "\sim"'] \arrow[r, "h", "\sim"'] & Y \\ & \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[ul, bend left, "\sim", "g\circ k"'] \arrow[ur, bend right, "\sim"', "h\circ k"] & \end{tikzcd}$$

where the morphisms $g\circ k$ and $h\circ k$ are weak equivalences but now they are defined between CW complexes and so I can apply the Whitehead Theorem to obtain that the two composition morphisms $g\circ k$ and $h\circ k$ are both of them homotopy equivalences.
Since $g\circ k$ is a homotopy equivalence, I can consider a homotopy inverse which I denote by t.
Now I can define a homotopy equivalence f between the CW complexes X and Y,

$$f\colon X \longrightarrow Y$$

as the composition of the homotopic inverse t defined, followed by the composition $h\circ k$.

$$f = (h \circ k) \circ t$$

Recall that the composition of homotopic equivalences is a homotopic equivalence too.
Finally, I have obtained the following diagram,

$$\begin{tikzcd}
X \arrow[rr, bend left=50, "f"] \arrow[dr, bend right=60, "t"'] & Z \arrow[l, "g", "\sim"'] \arrow[r, "h", "\sim"'] & Y \\
& \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[ul, bend left, "\sim"', "g\circ k"] \arrow[ur, bend right, "\sim", "h\circ k"'] &
\end{tikzcd}$$

I will study next part (b).
Again, I know that any topological space has a CW approximation and so I can consider a CW approximation for the topological space Z, which is both a CW complex $\tilde{Z}$ and a weak equivalence,

$$k\colon \tilde{Z} \xrightarrow{\sim} Z$$

Therefore, I have now the diagram,

$$\begin{tikzcd}
X \arrow[r, "g", "\sim"'] & Z & Y \arrow[l, "h", "\sim"] \\
& \tilde{Z} \arrow[u, "k", "\sim"'] &
\end{tikzcd}$$

In this case, the most delicate part is to determine whether there exist two suitable maps,

$$\tilde{g}\colon X \longrightarrow \tilde{Z}$$

$$\tilde{h}\colon Y \longrightarrow \tilde{Z}$$

such that,

$$k \circ \tilde{g} \simeq g \text{ and } k \circ \tilde{h} \simeq h$$

In order to obtain those $\tilde{g}$ and $\tilde{h}$, recall that, I can consider the so called Theorem of Approximation by CW complexes whose general statement is the following,
For any topological space A, there exists a CW complex $\tilde{A}$ and a weak equivalence

$$\gamma\colon \tilde{A} \longrightarrow A.$$

Moreover, it holds that for a map

$$\varphi\colon A \longrightarrow B$$

and another such CW approximation for the space B, that is both a CW complex $\tilde{B}$ and a weak equivalence,

$$\rho\colon \tilde{B} \longrightarrow B,$$

there exists a map,

$$\tilde{\varphi}\colon \tilde{A} \longrightarrow \tilde{B},$$

which is unique up to homotopy, such that the following diagram is homotopy commutative,

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \\
\tilde{A} \arrow[u, "\gamma"] \arrow[r, "\tilde{\varphi}"'] & \tilde{B} \arrow[u, "\rho"']
\end{tikzcd}$$

This Theorem yields a map $\tilde{g}$

$$\tilde{g}\colon X \longrightarrow \tilde{Z}$$

such that the diagram,

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & Z \\
X \arrow[r, "\tilde{g}"'] & \tilde{Z} \arrow[u, "k"']
\end{tikzcd}$$

is homotopy commutative, that is, such that $k \circ \tilde{g} \simeq g$.
In an analogous way, I have a homotopic commutative diagram,

$$\begin{tikzcd}
Y \arrow[r, "h"] \arrow[d, equal] & Z \\
Y \arrow[r, "\tilde{h}"'] & \tilde{Z} \arrow[u, "k"']
\end{tikzcd}$$

yielding a map $\tilde{h}$,

$$\tilde{h}\colon Y \longrightarrow \tilde{Z}$$

such that $k \circ \tilde{h} \simeq h$.
Therefore I have obtained,

$$k \circ \tilde{g} \simeq g$$

where k *and* g *are weak equivalences and so I can deduce that* $\tilde{g}$ *is a weak equivalence too.*
In an analogous way I have obtained,

$$k \circ \tilde{h} \simeq h$$

where k *and* h *are weak equivalences and so I can deduce that* $\tilde{h}$ *is a weak equivalence too.*
Then I obtain the diagram,

$$\begin{tikzcd}
X \arrow[rr, "g", "\sim"'] \arrow[drr, bend right, "\sim", "\tilde{g}"'] & & Z & & Y \arrow[ll, "h"', "\sim"] \arrow[dll, bend left, "\sim"', "\tilde{h}"] \\
& & \tilde{Z} \arrow[u, "k", "\sim"'] & &
\end{tikzcd}$$

Since $\tilde{g}$ *and* $\tilde{h}$ *are weak equivalences but now both defined between CW complexes, then I can apply the Whitehead Theorem to deduce that* $\tilde{g}$ *and* $\tilde{h}$ *are homotopy equivalences as well.*
Now, since $\tilde{h}$ *is a homotopy equivalence I can consider a homotopic inverse, which I denote by* l.
Now I can define a homotopy equivalence f *between the CW complexes* X *and* Y,

$$f\colon X \longrightarrow Y$$

as the composition of the morphism $\tilde{g}$ *defined, followed by the homotopic inverse* $l =$ *previously defined.*

$$f = l \circ \tilde{g}$$

Recall that the composition of homotopic equivalences is a homotopy equivalence too.
Finally, I have obtained the diagram,

$$\begin{tikzcd}
X \arrow[rrrr, bend left, "f"] \arrow[rr, "g", "\sim"'] \arrow[drr, bend right, "\sim", "\tilde{g}"'] & & Z & & Y \arrow[ll, "h"', "\sim"] \arrow[dll, bend left, "\sim"', "\tilde{h}"] \\
& & \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[urr, bend right, "l"'] & &
\end{tikzcd}$$

Part 2. Review about Category Theory

Part 2.

Notes about Category Theory

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A category $\mathcal{C}$ consists of

(1) A class of objects $Obj(\mathcal{C})$.
(2) For each pair of objects $X,Y \in Obj(\mathcal{C})$ a set of morphisms $hom_{\mathcal{C}}(X,Y)$.
(3) For each $X \in Obj(\mathcal{C})$ and identity morphism $Id_X \in hom_{\mathcal{C}}(X,X)$.
(4) A composition mapping $\circ \colon hom_{\mathcal{C}}(X,Y) \times hom_{\mathcal{C}}(Y,Z) \longrightarrow hom_{\mathcal{C}}(X,Z)$ satisfying
  (a) $f \circ Id_X = Id_Y \circ f = f$
  (b) $f \circ (g \circ h) = (f \circ g) \circ h$

I will sometimes abuse notation and write $X \in \mathcal{C}$ for $X \in Obj(\mathcal{C})$.

In a morphism $f \colon X \longrightarrow Y$ I will call X the source or the domain of f and Y the target or the codomain of f.

I will work with small categories. Recall that a category $\mathcal{C}$ is called small if both $Obj(\mathcal{C})$ and $Hom(\mathcal{C})$ are actually sets rather than proper classes, and large otherwise. A locallly small category is a category such that for all objects X and Y, the hom-class $Hom(X,Y)$ is a set, called a $HomSet$. Many important categories, such as the category of sets, although not small, are at least locally small. An standind assumption in this document is that whenever a category is used to index a product, a coproduct or a diagram, it is assumed to be small.

A discrete category is a category with no non-identity morphisms. Any set naturally gives rise to a discrete category with the elements of a that set as the objects.

A category is finite if both class of objects and its class of morphisms are finite.

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a category consisting of subclasses of the classes of objects and morphisms of a category $\mathcal{C}$.

A subcategory $\mathcal{C}'$ is full if $\forall X,\, Y \in Obj(\mathcal{C}')$, $Hom_{\mathcal{C}}(X,Y)$.

The category Set is a full subcategory of Top where sets are regarded as spaces with discrete topology. Because of this, the constructions I will perfom as examples in Top, such as pushouts and pullbacks as examples of colimits and limits, will automatically hold for Set. The important property that both categories fortunatelly posses is that the limits of small diagrams always exist.

Given a category $\mathcal{C}$, the opposite category of $\mathcal{C}$ is denoted as $\mathcal{C}^{op}$ an defined as the category whose objects are those of $\mathcal{C}$ and whose morphism sets are defined by,

$$Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$$

Intuitively, $\mathcal{C}^{op}$ is $\mathcal{C}$ with the arrows "reversed".

A covariant functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a function which,

(1) Associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$, and
(2) Associates to each morphism $f \in hom_{\mathcal{C}}(X,Y)$ a morphism $F(f) \in hom_{\mathcal{D}}(F(X),F(Y))$ such that
  (a) $F(1_X) = 1_{F(X)} \quad \forall X \in Obj(\mathcal{C})$ and
  (b) $F(g \circ f) = F(g) \circ F(f) \quad \forall f \in hom_{\mathcal{C}}(X,Y)$ and $g \in hom_{\mathcal{C}}(Y,Z)$

A contravariant functor is a covariant functor whose domain is $\mathcal{C}^{op}$. I can think of contravariant functors as those that "reverse arrows" and so, that satisfies the corresponding alterations to the above axioms.

Functors can be thought of as functions from one category to another that respect the categorical structure. When I will speak in general of a functor without mentioning its variance of some other context, it is understood to be covariant.

Functors $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G \colon \mathcal{D} \longrightarrow \mathcal{C}$ are adjoint if there is a natural bijection,

$$hom_{\mathcal{D}}(G(X),Y) \cong hom_{\mathcal{C}}(X,F(Y))$$

As an important example, the functor loop, denoted by Ω that takes a based space to its loop space and the functor suspension, denoted by Σ that takes a based space to its suspension are adjoint functors on Top_*.

A few lines later I will properly and deeply talk about diagrams, but I need to briefly introduce now, that a diagram in a category $\mathcal{C}$ is a functor F from a small category $\mathcal{I}$ to $\mathcal{C}$. A diagram emphasizes the "picture" of the functor, and hence the shape of the indexing category, which plays an important role in defining homotopy colimits.

In order to talk about the homotopy invariance of the homotopy limit of a punctured square, I will require the notion of a "map of diagrams":

A natural transformation (or map of diagrams) $N\colon F \longrightarrow G$ between functors F,$G\colon \mathcal{C} \longrightarrow \mathcal{D}$ associates to each $X \in Obj(\mathcal{C})$ a morphism $N_X\colon F(X) \longrightarrow G(X)$ such that for every morphism $f\colon X \longrightarrow Y$ the following diagram commutes

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "N_X"'] & F(Y) \arrow[d, "N_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

If F and G are both contravariant, the horizontal arrows in the above square are reversed.

Time and again in this work, I will need to speak of a natural transformation $N\colon F \longrightarrow G$ between functors F, $G\colon \mathcal{C} \longrightarrow Top$ or Top_* which is a fibration, cofibration, weak equivalence, or some other property a map of spaces may posses. All this means that for each $c \in \mathcal{C}$, tha map $F(c) \longrightarrow G(c)$ induced by N is a fibration, cofibration, weak equivalence, etc.

In Category Theory, the functors between two given categories $\mathcal{C}$ and $\mathcal{D}$ form a category, called functor category and denoted by $Funct(\mathcal{D}, \mathcal{C})$ or $\mathcal{C}^{\mathcal{D}}$, where the objects are the functors F from $\mathcal{D}$ to $\mathcal{C}$ and the morphisms are natural transformations between the functors.

Time and again in this document I will use this construction supposing $\mathcal{C}$ as an arbitrary category and $\mathcal{I}$ as a small category. So, the category of functors from $\mathcal{I}$ to $\mathcal{C}$, written as $Funct(\mathcal{I}, \mathcal{C})$ or $\mathcal{C}^{\mathcal{I}}$, has as objects the covariant functors from $\mathcal{I}$ to $\mathcal{C}$, and as morphisms the natural transformations between such functors.

Many commonly occurring categories are disguised functor categories, so any statement proved for general functor categories is widely applicable.

Every category embeds in a functor category via the Yoneda embedding; the functor category often has nicer properties than the original category, allowing certain operations that were not available in the original setting.

I can observe that natural transformations can be composed. If,

$$\mu(X)\colon F(X) \longrightarrow G(X)$$

is a natural transformation

$$\text{from the functor} \qquad F\colon \mathcal{I} \longrightarrow \mathcal{C} \qquad \text{to the functor} \qquad G\colon \mathcal{I} \longrightarrow \mathcal{C}$$

and

$$\eta(X)\colon G(X) \longrightarrow H(X)$$

is a natural transformation

$$\text{from the functor} \qquad G\colon \mathcal{I} \longrightarrow \mathcal{C} \qquad \text{to the functor} \qquad H\colon \mathcal{I} \longrightarrow \mathcal{C},$$

then the collection

$$\eta(X) \circ \mu(X)\colon F(X) \longrightarrow H(X)$$

defines a natural transformation from F to H.

With this composition of natural transformations $\mathcal{C}^{\mathcal{I}}$ satisfies the axioms of a category.

In a completely analogous way, one can also consider the category of all contravariant functors from $\mathcal{I}$ to $\mathcal{C}$ and I will write then $Funct(\mathcal{I}^{op}, \mathcal{C})$ or $\mathcal{C}^{\mathcal{I}^{op}}$.

If $\mathcal{I}$ is a small discrete category and so its only morphisms are the identity morphisms, then functor form $\mathcal{I}$ to $\mathcal{C}$ essentially consists of a family of objects of $\mathcal{C}$, indexed by $\mathcal{I}$ and then the functor category $\mathcal{C}^{\mathcal{I}}$ can be identified with the corresponding product category, where its elements are families of objects in $\mathcal{C}$ and its morphisms are families of morphisms in $\mathcal{C}$.

An arrow category $\mathcal{C}^{\rightarrow}$ whose objects are the morphisms of $\mathcal{C}$ and whose morphisms are commuting squares in $\mathcal{C}$ is just $\mathcal{C}^2$, where 2 is the category with two objects and their identity morphisms as well as an arrow from one object to the other but not another arrow back the other way.

A directed graph consists of a set of arrows and a set of vertices, and two functions from the arrow set to the vertex set, specifying each arrow's star and end vertex. The category of all directed graphs is thus nothin but the functor category $Set^{\mathcal{C}}$, where $\mathcal{C}$ is the category with two objects connected by two morphisms, and Set denotes the category of Sets.

I can finally add that the functor category $\mathcal{C}^{\mathcal{S}}$ shares most of the 'nice' properties of $\mathcal{C}$.

More precisely,

If $\mathcal{C}$ is complete or cocomplete, then so is $\mathcal{C}^{\mathcal{S}}$.

If $\mathcal{C}$ is a abelian category, then so is $\mathcal{C}^{\mathcal{S}}$.

Intuitively, in Category Theory, a diagram is the categorical analogue of an indexed family in Set Theory. The primary difference is that in the categorical framework I have morphisms that also need indexing. An indexed family of sets is a collection of sets, indexed by a fixed set; equivalently, a function from a fixed index set to the class of sets. A diagram is a collection of objects and morphisms, indexed by a fixed category; equivalently, a functor from a fixed index category to some other category.

From the previous intuitive idea it leads that there are two natural ways to give the notion of "diagram" a formal definition.

(1) a diagram is a functor, usually one whose domain is a (very) small category. This level of generality is sometimes convenient.
(2) On the other hand, a more direct representation is that when I "draw a diagram", only involves labeling the vertices and edges of a directd graph (or quiver) by objects and morphisms of the category. This sort of diagram can be identified with a functor whose domain is a free category, and this is the most common context when I talk about diagrams "commuting".

I want to briefly develop the first notion, that is, I define diagrams shaped like categories.

Definition 1. *(Diagram (shaped like a category))*

Let $\mathcal{C}$ and $\mathcal{J}$ be two categories.

Then a diagram in $\mathcal{C}$ of shape $\mathcal{J}$, or also said a diagram of tipe $\mathcal{J}$ is a category $\mathcal{C}$ is simply a covariant functor $F\colon \mathcal{J} \longrightarrow \mathcal{C}$.

The category $\mathcal{J}$ is called the index category or the scheme of the diagram D; the functor is sometimes called a $\mathcal{J}$-shaped diagram. The objects and morphism in $\mathcal{J}$ are irrelevant, only the way in which they are interrelated matters. The diagram D is thought of as indexing a collection of objects and morphisms in $\mathcal{C}$ patterned on $\mathcal{J}$.

I can observe that this terminology is often used when speaking about limits and colimits; that is, I speak then about the limit or colimit of a diagram. Similarly, it is common to call the functor category $\mathcal{C}^{\mathcal{J}}$ the category of diagrams in $\mathcal{C}$ of shape $\mathcal{J}$.

Although, technically, there is no difference between an individual diagram and a functor or between a scheme and a category, the change in terminology reflects a change in perspective, just as in the set theoretic case: I fix the index category and I allow the functor to vary and secondarily, do the target category.

I will be most often interested in the case where the scheme $\mathcal{J}$ is a small or even finite category. A diagram is said to be small of finite whenever $\mathcal{J}$ is.

I also want to add some remarks.

(1) For either sort of diagram, $\mathcal{J}$ may be called the shape, scheme, or index category or graph.

(2) Given a diagram $p\colon \mathcal{J} \longrightarrow C$, the image of the shape $\mathcal{J}$ is not necessarily a subcategory of $\mathcal{C}$, even if $\mathcal{J}$ is itself taken to be a category. This is because the functor D could identify objects of $\mathcal{J}$, thereby producing new potential composites which do not exist in $\mathcal{J}$. I note that sometimes I can talk about the image of a functor as a subcategory, but with this I really mean the subcategory generated by the image in the literal sense of objects and morphisms.

(3) $\mathcal{C}$ must be a strict category to make sense of $U(\mathcal{C})$. Instead of this, $F(\mathcal{J})$ always make sense.

Recall that a strict category is a category together with the structure of a set (or class) on its collection of objects; in particular, the objects can be compared for equality (not merely isomorphism). In contrast, a weak category is a category without such structure. Similarly, a strict functor is one which preserves equality of objects and so $F(X) = F(Y)$ if $X = Y$ (this is not inherently size-related, since both large and small categories can be either strict or weak).

A morphism of diagrams of type $\mathcal{J}$ in a category $\mathcal{C}$ is a natural transformation between functors. I can interpret the category of diagrams of type $\mathcal{J}$ in $\mathcal{C}$ as the functor category $\mathcal{C}^{\mathcal{J}}$, and a diagram is the an object in this category.

I can consider some examples of diagrams:

Example 2. *Given any object X in $\mathcal{C}$ I have the constant diagram, usually denoted by $\underline{X}$ which is the diagram that maps all objects in $\mathcal{J}$ to X, and that also maps all morphisms of $\mathcal{J}$ to the identity morphisms on X.*

So that, for every two categories $\mathcal{C}$ and $\mathcal{D}$ there exists a constant functor

$$\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{D}}$$

called the diagonal functor mapping an object $C \in \mathcal{C}$ to te constant diagram of shape $\mathcal{D} \in \mathcal{C}$ where all objects are copies of C and all arrows are copies of $Id_C\colon C \longrightarrow C$. Thus:

$$(\Delta C)(D) \overset{def}{=} C \qquad\qquad (\Delta C)(h) = Id_C$$

for every object B and every arrow h of $\mathcal{D}$.

Similarly, Δ maps an arrow $f\colon C \longrightarrow C'$ to the constant natural transformation,

$$(\Delta f)_D \overset{def}{=} f\colon (\Delta C)(D) \longrightarrow (\Delta C')(D)$$

for every $D \in \mathcal{D}$

Example 3. *If $\mathcal{J}$ is a small discrete category, then a diagram of type $\mathcal{J}$ is essentially just an indexed family of objects in $\mathcal{C}$, family indexed by $\mathcal{J}$. When used in the construction of the limit, the result is the product and when used in the construction of the colimit, the result is the coproduct. So, for instance, when $\mathcal{J}$ is the discrete category with two objects, the resulting limit is just the binary product.*

Example 4. *If $\mathcal{J} = -1 \longleftarrow 0 \longrightarrow +1$, then a diagram of type $\mathcal{J}$, $Y \longleftarrow X \longrightarrow Z$ is a span, and its colimit is a pushout.*

In this example I observe the following,

If I were to "forget" that the diagram had object X and the two arrows $X \longrightarrow Y$, $X \longrightarrow Z$, the resulting diagram would simply be the discrete category with the two object X and Z, and the colimit would simply be the binary coproduct.

Thus, this example shows an important way in which the idea of the diagram generalizes that of the index set in set theory, by including the morphisms $X \longrightarrow Y$, $X \longrightarrow Z$, I can discover additional structure in constructions built from the diagram, structure that would not be evident if one only had an index set with no relations between the objects in the index.

Example 5. *If $J = -1 \longrightarrow 0 \longleftarrow +1$, then a diagram of type J, $Y \longrightarrow 0 \longleftarrow +1$, then a diagram of type $\mathcal{J}$, $Y \longrightarrow X \longleftarrow Z$ is a cospan, and its limit is a pullback.*

Example 6. *Recovering some previous concepts, the index $J = 0 \rightrightarrows 1$ is called "two parallels morphisms" or the free quiver or the walkin quiver. A diagram of type $\mathcal{J}$, $f, g\colon X \longrightarrow Y$, is then a quiver, its limit is an equalizer, and its colimit is a coequalizer.*

Some facts about commutative diagrams can be remarked.

A commutative diagram is a diagram in which composition is path-independent.

If $\mathcal{J}$ is a category, then a diagram $J \longrightarrow C$ is commutative if it factors through a thin category. Equivalently, a diagram of shape $\mathcal{J}$ commutes if and only if any two morphisms in $\mathcal{C}$ that are assigned to any pair of parallel morphisms in $\mathcal{J}$ (that is, with same source and target in $\mathcal{J}$) are equal.

If $\mathcal{J}$ is a quiver, as is more common when I speak about "commutative diagrams, then a diagram of shape $\mathcal{J}$ commutes if the functor $F(\mathcal{J}) \longrightarrow C$ factors through a thin category, Equivalently, this means that given any two parallel paths of finite length (including zero) in $\mathcal{J}$, their images in $\mathcal{C}$ have equal composites.

Recall that a thin category is a category in which, given any two objects x and y and any two morphisms f and g from x to y the morphisms f and g are equal.

Also recall that two morphisms in a category $\mathcal{C}$ are parallel if they have the same source and target. Equivalently a pair of parallel morphisms in $\mathcal{C}$ consists of and object x, and object y, and two morphisms $f, g \colon x \longrightarrow y$.

This can be extended to a family of any number of morphisms, but the morphisms are always compared pairwise to see if they are parallel.

I can remark the existence of some degenerate cases: a family of one parallel morphism is simply a morphism and a family of zero parallel morphisms is simply a pair of objects.

The limit of a pair (or family) of morphisms is called their equalizer; the colimit is their coequalizer. Of course, these do not always exist.

I can start giving a first approach to the definitions of cone, cocone, limit and colimit of diagrams. Altough I offer it in form of a general definition I will better complete later on.

Definition 7. *(Cone and Cocone of a Diagram. Limit and Colimit of a Diagram)*

A cone with vertex N of a diagram $D \colon \mathcal{J} \longrightarrow \mathcal{C}$ is a morphism from the constant diagram $\Delta(N)$ to D. The constant diagram is the diagram which sends every object of $\mathcal{J}$ to an object N of $\mathcal{C}$ and every morphism to the identity morphism of N.

The limit of a diagram D is a universal cone to D. That is, a cone through which all other cones uniquely factor. If the limit exists in a category $\mathcal{C}$ for all diagrams of type $\mathcal{J}$ I obtain a functor,

$$lim \colon \mathcal{C}^{\mathcal{J}} \longrightarrow \mathcal{C}$$

which sends each diagram to its colimit.

Let $D \colon \mathcal{J} \longrightarrow \mathcal{C}$ be a diagram in $\mathcal{C}$. I know that a diagram is a functor from $\mathcal{J}$ to $\mathcal{C}$ and so I think of D as indexing a family of objects and morphisms in $\mathcal{C}$. The category $\mathcal{J}$ is thought of as an index category that of course, it may be the empty category.

Let N be an object of $\mathcal{C}$. A cone from N to D is a family of morphisms,

$$\psi_X \colon N \longrightarrow D(X)$$

for each object X of $\mathcal{J}$ such that for every morphism $f \colon X \longrightarrow Y$ in $\mathcal{J}$ the following diagram commutes,

$$\begin{tikzcd}
 & N \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] \arrow[d, phantom, "\equiv"] & \\
D(X) \arrow[rr, "D(f)"'] & {} & D(Y)
\end{tikzcd}$$

The usually infinite collection of all these triangles can be partially depicted in the shape of a cone with the apex N. The cone ψ is sometimes said to have vertex N and base the diagram D.

I can also define the dual notion of a cone from D to N, called a co-cone, by reversing all the arrows above. Explicitly, a co-cone from D to N is a family of morphisms,

$$\psi_X \colon D(X) \longrightarrow N$$

for each object X of $\mathcal{J}$ such that for every morphism $f\colon X \longrightarrow Y$ in $\mathcal{J}$ the following diagram commutes,

$$\begin{tikzcd}
D(X) \arrow[rr, "D(f)"] \arrow[dr, "\psi_X"'] & & D(Y) \arrow[dl, "\psi_Y"] \\
& N &
\end{tikzcd}$$

I can consider this facts with an equivalent formulations which is usually very useful. Cones are maps from an object to a functor (or vice versa). I would like to define them as morphisms or objects in some suitable category and in fact I can do both.

Indeed, let $\mathcal{J}$ be a small category and let $\mathcal{C}^{\mathcal{J}}$ be the category of diagrams of type $\mathcal{J}$ in$\mathcal{C}$ (this is nothing more than a functor category). I define the diagonal functor,

$$\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{J}}$$

as follows:

$$\Delta(N)\colon \mathcal{J} \longrightarrow \mathcal{C}$$

is the constant functor to N for all N in $\mathcal{C}$.

If D is a diagram of type $\mathcal{J}$ in $\mathcal{C}$, the following statements are equivalent.

- (•) ψ is a cone from N to D.
- (•) ψ is a natural transformation from $\Delta(N)$ to D.
- (•) (N, ψ) is an object in the Category of Cones, $Cone(D)$.

The dual statements are also equivalent:

- (•) ψ is a co-cone from D to N.
- (•) ψ is a natural transformation from D to $\Delta(N)$.
- (•) (N, ψ) is an object in the Category of Cocones, $Cocone(D)$.

These all statements can be verified by a straightforward application of the definitions. Thinking of cones as natural transformations I see that they are just morphisms in $\mathcal{C}^{\mathcal{J}}$ with source or target a constant functor.

Definition 8. *(Category of Cones and Category of Cocones)*

I can define the category of cones $Cone(D)$, where morphisms of cones are then just morphisms in this category, since I can observe that a natural map between constant functors $\Delta(N)$, $\Delta(M)$ corresponds to a morphism between N and M. In this sense, the diagonal functor acts trivially on arrows.

In a similar way, writing down the definition of a natural map from a constant functor $\Delta(N)$ to D yields the same diagram as the above. As I might expect, a morphism from a cone (N, ψ) to a cone (L, φ) is just a morphism $N \longrightarrow L$ such that all the "obvious'" diagrams commute,

$$\begin{tikzcd}
& N \arrow[d, dashed] \arrow[dl, bend right, "\psi_X"'] \arrow[dr, bend left, "\psi_Y"] & \\
& L \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\
D(X) \arrow[rr, "D(f)"'] & & D(Y)
\end{tikzcd}$$

I can define in an analogous way the category of cocones $Co - Cone(D)$, where morphisms of cocones are then just morphisms in this category and also as I might expect, a morphism from a cocone (L, φ) to a co-cone (N, ψ) is just a morphism $L \longrightarrow N$ such that all the "obvious" diagrams commute,

$$\begin{tikzcd}
D(X) \arrow[rr, "D(f)"] \arrow[dr, "\psi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & D(Y) \arrow[dl, "\psi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed] & \\
& N &
\end{tikzcd}$$

Limits and colimits are defined as universal cones, that is, cones through which all other cones factor. I can be more precise.

Definition 9. *(Universal Cone)*

A cone φ from L to D is a universal cone if for any other cone ψ from N to D there is a unique morphism from ψ to φ.

$$\begin{tikzcd}
& N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
& L \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\
D(X) \arrow[rr, "D(f)"'] & & D(Y)
\end{tikzcd}$$

I say that the cone (N, ψ) factors through the cone (L, φ) with the unique factorization u. The morphism u is sometimes called the mediating morphism.

Equivalently, a universal cone to D is a universal morphism from Δ to D (thought of as an object in $\mathcal{C}^{\mathcal{J}}$), or a terminal object in the category of cones, namely $Cone(D)$.

Dually, a cone φ from D to L is a universal cone if for any other cone ψ from D to N there is a unique morphism from φ to ψ.

$$\begin{tikzcd}
D(X) \arrow[rr, "D(f)"] \arrow[dr, "\psi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & D(Y) \arrow[dl, "\psi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}$$

Equivalently, a universal cone from D is a universal morphism from D to Δ, or an initial object in the category of cocones, namely $Cocone(D)$.

The limit of D is a universal cone to D, and the colimit is a universal cone from D.

I can sumarize all these ideas about diagrams, cones and universal cones in the formal definition of limit.

Let $D\colon \mathcal{J} \longrightarrow \mathcal{C}$ be a diagram of type $\mathcal{J}$ in a category $\mathcal{C}$. A cone to D is an object N of $\mathcal{C}$ toghether with a family $\psi_X\colon N \longrightarrow D(X)$ of morphisms indexed by the objects X of $\mathcal{J}$, such that for every morphism $f\colon X \longrightarrow Y$ in $\mathcal{J}$, I have $D(f) \circ \psi_X = \psi_Y$.

A limit of the diagram $D\colon \mathcal{J} \longrightarrow \mathcal{C}$ is a cone (L, φ) to D such that for any other cone (N, ψ) to D there exists a unique morphism $u\colon N \longrightarrow L$ such that $\varphi_X \circ u = \psi_X$ for all X in $\mathcal{J}$.

$$\begin{tikzcd}
& N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
& L \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\
D(X) \arrow[rr, "D(f)"'] & & D(Y)
\end{tikzcd}$$

Limits are referred to as universal cones, since they are characterized by a universal property. As with every universal property, the definition of limit describes a balanced state of generality: The limit object L has to be general enough to allow any other cone to factor through it and on the other hand, L has to be sufficiently specific, so that only one such factoriztion is possible for every cone.

Limits may also be characterized as terminal objects in the category of cones to D, $Cone(D)$.

Indeed, I consider the limit (L, φ) of a diagram $D\colon \mathcal{J} \longrightarrow \mathcal{C}$

If $\mathcal{J}$ is the empty category there is only one diagram of type $\mathcal{J}$, that is the empty diagram (similar to the empty function in Set Theory). A cone to the empty diagram is essentially just an object of $\mathcal{C}$. The limit of D is any object that is uniquely factored through by every other object, but this is just the definition of a terminal object.

I can note that the abstract notion of a limit captures the essential properties of universal constructions as products, pullbacks and inverse limits.

- (•) Products.

  If $\mathcal{J}$ is a discrete category then a diagram D is essentialy nothing but a family of objects of $\mathcal{C}$, indexed by $\mathcal{J}$. The limit L of D is called the product of these objects. The cone φ consists of a family of morphisms $\varphi_X\colon L \longrightarrow D(X)$ called the projections of the product. In the category of sets, for instance, the products are given by cartesian products and the projections are just the natural projections onto the various factors.

  - (◦) Powers.

    A especial case of a product is when the diagram D is a constant functor to an object X of $\mathcal{C}$. The limit of this diagram is called the J-th power of X and denoted by $X^{\mathcal{J}}$.

- (•) Equalizers.

  If $\mathcal{J}$ is a category with two objects A and B and two parallel morphisms from A to B then a diagram of type $\mathcal{J}$ is a pair of parallel morphisms in $\mathcal{C}$. The liit L of such a diagram is called an equalizer of those morphisms.

  - (◦) Kernels.

    A kernel is a special case of an equalizer where one of the morphisms is a zero morphism.

- (◦) Pullbacks.

  Let D be a diagram that picks out three objects X, Y, and Z in $\mathcal{C}$, where the only non-identity morphisms are $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$. The limit L of F is called a pullback of a fiber product. I can nicely be visualized as a commutative square:

$$\begin{tikzcd}
L \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"'] \arrow[dr, "\varphi_Z"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

  That is limits over the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ are pullbacks.

- (•) Inverse limits.

  Let $\mathcal{J}$ be a directed poset (considered as a small category by adding arrows $i \longrightarrow j$ if and only if $i \leq j$ and let $D\colon \mathcal{J}^{op} \longrightarrow \mathcal{C}$ be a diagram. The limit of D is called (confusingly) an inverse limit or a projective limit.

As with all universal constructions, universal cones are not guaranteed to exist for all diagrams D (since it is possible that a diagram does not have a limit al all), but if they do exist they are unique up to a unique isomorphism. For this reason, it is often talked about of "the limit of D".

The dual notions of limits and cones are colimits and cocones. Although it is straightforward to obtain the definitions of these by invertin all morphisms in the above definitions, I will explicitly state them here:

A cocone of a diagram $D\colon \mathcal{J} \longrightarrow \mathcal{C}$ is an object N of $\mathcal{C}$ together with a family of morphisms

$$\psi_X : D(X) \longrightarrow N$$

for every object X of $\mathcal{J}$, such that for every morphisms $f\colon X \longrightarrow Y$ in $\mathcal{J}$, I have $\psi_Y \circ D(f) = \psi_X$.

A colimit of a diagram $D\colon \mathcal{J} \longrightarrow \mathcal{C}$ is a co-cone (L, φ) of D such that for any other coccone (N, ψ) of D there exists a unique morphism $u\colon L \longrightarrow N$ such that $u \circ \varphi_X = \psi_X$ for all X in $\mathcal{J}$.

$$\begin{tikzcd}
D(X) \arrow[rr, "D(f)"] \arrow[dr, "\psi_X"] \arrow[ddr, bend right, "\psi_X"'] & & D(Y) \arrow[dl, "\psi_Y"'] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}$$

Colimits are also referred to as universal co-cones. They can be characterized as initial objects in the category of cocones from D.

Indeed, I consider the colimit (L, φ) of a diagram $D\colon \mathcal{J} \longrightarrow \mathcal{C}$

If $\mathcal{J}$ is the empty category there is only one diagram of type $\mathcal{J}$, that is the empty diagram (similar to the empty function in Set Theory). A co-cone from the empty diagram is essentially just an object of $\mathcal{C}$. The colimit of D is any object that is uniquely factored through by every other object, but this is just the definition of an initial object.

The dual notion of a colimit generalizes constructions such as disjoint unions, direct sums, coproducts, pushouts and direct limits.

- (•) Coproducts are colimits of diagrams indexed by discrete categories.
  - (○) Copowers are colimits of constant diagrams from discrete categories.
- (•) Coequalizers are colimits of a parallel pair of morphisms.
  - (○) Cokernels are coequalizers of a morphisms and a parallel zero morphism.
- (•) Pushouts are colimits of a pair of morphisms with common domain, that is colimits over the category $\bullet \longleftarrow \bullet \longrightarrow \bullet$.
- (•) Direct limits are colimits of diagrams indexed by direct sets.

As with limits, if a diagram D has a colimit then this colimit is unique up to a unique isomorphism.

Equalizers

Let $\mathcal{C}$ be a category.

An equalizer is a limit

$$eq \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

over a parallel pair, equivalently of the diagram of the shape

$$\left\{ X \underset{g}{\overset{f}{\rightrightarrows}} Y \right\}.$$

This means that for $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ two parallel morphisms in a category $\mathcal{C}$, their equalizer is, if it exists

1. (1) an object $eq(f, g) \in \mathcal{C}$,
2. (2) a morphism $eq(f, g) \longrightarrow X$ such that,
   - (a) pulled back to $eq(f, g)$ both morphisms become equal:

$$(eq(f, g) \longrightarrow X \xrightarrow{f} Y) = (eq(f, g) \longrightarrow X \xrightarrow{g} Y)$$

(b) and $eq(f,g)$ is the universal object with this property.

As a property, I can add,

Proposition 10. *A category has equalizers if it has products and pullbacks.*

Proof. For $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ the given diagram, I first form the pullback

$$\begin{tikzcd}
X \times_{f,g} X \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

This gives a morphism $X \times_{f,g} X \longrightarrow X \times X$ into the product.

I define $eq(f,g)$ to be the further pullback

$$\begin{tikzcd}
eq(f,q) \arrow[r] \arrow[d] & X \times_{f,g} X \arrow[d] \\
X \arrow[r, "{Id,Id}"] & X \times X
\end{tikzcd}$$

The vertical morphism $eq(f,g) \longrightarrow X$ equalizes f and g and that it does so universally. □

It also holds that, if a category has products and equalizers, then it has limits.

Proposition 11. *Given $\mathcal{C}$ any category. If $\mathcal{C}$ has all smalls products and equalizers then $\mathcal{C}$ has all smalls limits.*

Proof. I can observe that since $\mathcal{C}$ has all smalls products in particular has the empty products and so a terminal object.

Given a set I and an I-indexed family of objects $\{A_i \mid i \in I\}$ of $\mathcal{C}$, I denote the product by $\prod\limits_{i\in I} A_i$ and projections by $p_i \colon \prod\limits_{i\in I} A_i \longrightarrow A_i$.

An arrow $f\colon X \longrightarrow \prod\limits_{i\in I} A_i$ which is determined by the compositions $f_i = p_i \circ f \circ X \longrightarrow A_i$, is in fact also a collection $\{f_i \mid i \in I\}$.

Now given $\varepsilon \longrightarrow \mathcal{C}$ with ε_0 and ε_1 sets, I construct

$$E \xrightarrow{e} \underset{i\in\varepsilon_0}{prodF(i)} \underset{(F(u)p_{dom(u)}|u\in\varepsilon_1)}{\overset{(p_{dom(u)}|u\in\varepsilon_1)}{\rightrightarrows}} \underset{u\in\varepsilon_1}{prodF(cod(u))}$$

in $\mathcal{C}$ as an equalizer diagram.

Now, the family $\{ \ \mu_i = p_i \circ e\colon E \longrightarrow F(i) \mid i \in \varepsilon_0 \ \}$ is a natural transformation $\Delta \Rightarrow F$ because, given an arrow $u \in \varepsilon_1$, say $u : i \longrightarrow j$, I have that

$$\begin{tikzcd}
 & E \arrow[dl, "p_i \circ e"'] \arrow[dr, "p_j \circ e"] & \\
F(i) \arrow[rr, "F(u)"'] & & F(j)
\end{tikzcd}$$

commutes since $F(u) \circ p_i \circ e = F(u) \circ p_{dom(u)} \circ e = p_{cod(u)} \circ e = p_j \circ e$.

So (E, μ) is a cone for F, but every other cone (D, ν) for F gives a map $d\colon D \longrightarrow \underset{i\in\varepsilon_0}{prodF(i)}$ equalizing the two horizontal arrows. Hence factors uniquely through E. □

The dual concept is that of coequalizer.

The concept of coequalizer in a general category is the generalization of the construction where for two functions f,g between sets X and Y,

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

I can construct the set $Y/_{\sim}$ of equivalence classes induced by the equivalence relation generated by the relation

In this form this may be phrased generally in any category,

In some category $\mathcal{C}$, the coequalizer $coeq(f,g)$ of two parallel morphisms f and g between two objects X and Y is (if it exists), the colimit under the diagram formed by these two morphisms

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl, "p"'] \\ & coeq(f,q) & \end{tikzcd}$$

Equivalently,

In a category $\mathcal{C}$ a diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{p}{\to} Z$$

is called a coequalizer diagram if

(1) $p \circ f = p \circ g$,
(2) p is universal for this property. This means, if $q\colon Y \longrightarrow W$ is a morphism of $\mathcal{C}$ such that $q \circ f = q \circ g$, then there is a unique morphism $q'\colon Z \longrightarrow W$ such that $q' \circ p = q$ as I show in the following diagram,

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[d, "q"', "\exists ! q'"] & Z \arrow[dl] \\ & W & \end{tikzcd}$$

As I said before, by formal duality, a coequalizer in $\mathcal{C}$ is equivalently an equalizer in the opposite category $\mathcal{C}^{op}$.

Proposition 12. *If $\mathcal{C}$ has all coproducts and coequalizers, then it has all colimits.*

Proof. Let $F\colon I \longrightarrow \mathcal{C}$ be a functor, where I is a small category. I need to obtain an object X with morphisms,

$$F_i \longrightarrow X \qquad i \in I$$

such that for each $f\colon i \longrightarrow i'$, the diagram below commutes:

$$\begin{tikzcd} F_i \arrow[r] \arrow[d] & F_{i'} \arrow[dl] \\ X \arrow[r] & Z \end{tikzcd}$$

and such that X is universal among such diagrams.

To give such a diagram, however, is equivalent to giving a collection of maps

$$F_i \longrightarrow X \qquad i \in I$$

that satisfy some conditions.

So X should be thought of as a quotient of the coproduct $\coprod_{i \in I} F_i$. Now, I consider the coproduct $\coprod_{i \in I, f} F_i$, where f ranges over all morphisms in the category I that start from i. I construct two maps,

$$X \coprod_{i \in I, f} F_i \rightrightarrows \coprod_{i \in I, f} F_i$$

whose coequalizer will be that of F. The first map is the identity. The second map sends a factor. □

Coequalizers are closely related to pushouts.

Proposition 13. *A diagram*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Z$$

is a coequalizer diagram precisely if

$$\begin{tikzcd}
X \coprod X \arrow[r, "f \coprod g"] \arrow[d] & Y \arrow[d, "p"] \\
X \arrow[r] & Z
\end{tikzcd}$$

is a pushout diagram.

Conversely,

Proposition 14. *A diagram*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B \arrow[d, "p_1"] \\
C \arrow[r, "p_2"'] & D
\end{tikzcd}$$

is a pushout square, precisely if

$$X \underset{i_2 \circ f_2}{\overset{i_1 \circ f_1}{\rightrightarrows}} B \coprod C \xrightarrow{(p_1, p_2)} D$$

is a coequalizer diagram.

Part 3. Model Categories

Part 3.

Once I have reviewed in the previous part general facts about category theory, next a natural subject of interest is to be able to deal with homotopy theory in any variety of categories, and it may be interestig to stablish good techniques to compare these. There is an efficient machinery due to Daniel Quillen, which encodes this structure. In addition to weak equivalences (which is all that is needed to form the homotopy category) I have fibrations and cofibrations satisfying certain axioms. This structure ensures that the homotopy category actually exists, but more importantly it encodes the deeper homotopical structures, making a large class of arguments formal. It also makes comparison between different homotopical structures more transparent.

Introducing Model Categories

.

In [13] Daniel Quillen introduced the notion of a model category. Following his own first introductory words, a model category is just an ordinary category with three distinguished classes of maps (called weak equivalences, cofibrations and fibrations) satisfying a few simple axioms and he also observed that in such a model category one can "do homotopy theory".

There is a great sort of mess in those classical sources in literature about model categories in many senses but first of all in order to rigorously and right define a model category since in fact after the standard reference [13] was published, Daniel Quillen himself changed the definition later in [14]. After him, Quillen's definitions have been modified over the years again by William G. Dwyer, Philip S. Hirschhorn, and Daniel M. Kan and Jeffrey H. Smith in [4].

Therefore, in dealing with model categories, the first question is to decide what is the "right" generality in which to work. Any reader may object reading all the previous works that there is now more than one different definition of a model category. That is true, but the differences are slight: in practice, a structure that satisfies one definition satisfies them all. Daniel Quillen himself in [13] already noticed that "closed" model categories (that is model categories in which any two of the three distinguished classes of maps (weak equivalences, fibrations and cofibrations) determine the third) can be characterized by five particularly nice axioms and that moreover the requirement that a model category be closed is not a serious one. In fact he showed that a model category is closed iff all three of its distinguished classes of maps are closed under retracts and from this it readily follows that one can turn any model category in which (as always seems to be the case) the class of the weak equivalences is closed under retracts, into a closed model category just by closing the other two classes under retracts. However the first and the last of these five axioms are weaker than one would expect; the first axiom assumes the existence of finite limits and colimits, but not of arbitrary small ones and the last axiom assumes the existence of certain factorizations of maps, but does not insist on their functoriality.

In the work I present today, I will therefore throughout use the term model category for a closed model category which satisfies the above suggested stronger versions of Quillen's first and fifth axioms.

I want to remark that this strengthening of the axioms simplifies many statements and arguments and in particular the closure implies that,

(i) any two of the three distinguished classes of maps (weak equivalences, cofibrations and fibrations) determine the third, and
(ii) the cofibrations and the trivial fibrations (those fibrations which are also weak equivalences) determine each other and dually, so do the fibrations and the trivial cofibrations.

In contrast, although I will work with the strengthened fith axiom, having functorial factorizations, I cannot refrain from saying and still keeping in mind that it could be interesting the study of those model categories having no functorial factorizations in particular I will said something else in Remark 34.

I want to also remark that although I am describing the axioms used to describe what is called a "closed" model category; since no other kind of model category comes up in this paper, I have decided to leave out the word "closed". In [13] Daniel Quillen uses the terms "trivial cofibration" and "trivial fibration" instead of "acyclic cofibration" and "acyclic fibration". This conflicts with the ordinary homotopy theoretic use of "trivial fibration" to mean a fibration in which the total space is equivalent to the product of the base and fibre; in some geometric examples in literature about model categories, the "acyclic fibrations" turn out to be fibrations with a trivial fibre, so that the total space is equivalent to the base. Keeping in mind this fact I will indiscriminately use all those terms in the present work.

Those few simple axioms introduced by Quillen were deliberately reminiscent of properties of topological spaces, because Quillen in fact introduced model categories as an abstraction of the usual situation in topological spaces. This is where the terminology came from as well. Surprisingly enough, these axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory.

In fact, model categories, form the foundation of homotopy theory. The basic problem that model categories solve is the following:

Given a category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. For instance, these maps could be homology isomorphisms of some kind, or homotopy equivalences, or birational equivalences of algebraic varieties. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in the quotient category. In fact, there is a foundational problem inverting the weak equivalences formally, since the class of maps between two objects in the localized category may not be a set. Also, it could be very difficult to understand the maps in the resulting localized category. In a model category, there are weak equivalences, but there are also other classes of maps called cofibrations and fibrations. This extra structure allows one to get precise control of the maps in the category obtained by formally inverting the weak equivalences. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient category from X to Y are simply homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y.

In different and plain words, I suppose I have a category $\mathcal{M}$ and some class of morphisms W which behave somewhat like isomorphisms (for instance: Chain complexes and Quasi-isomorphisms, or topological spaces and homotopy equivalences, or simplicial sets and weak homotopy equivalences ...). I will call this class "weak equivalences". Then I can look at the localized category $[W^{-1}]\mathcal{M}$, where the morphisms in W are made invertible. If I am lucky, not all objects are isomorphic to each other, and if I am really lucky, I can effectively compute something.

The standard procedure to say something about $\mathcal{M} \longrightarrow [W^{-1}]\mathcal{M}$ is to show that $\mathcal{M}$ is complete and co-complete (if necessary, enlarge $\mathcal{M}$ to a presheaf or sheaf category to get these properties) and endow $\mathcal{M}$ with a model structure that incorporates the class W as weak equivalences. Then the model structure allows to compute from $[W^{-1}]\mathcal{M}$ in $\mathcal{M}$ via replacing objects and morphisms by more convenient ones in the same class in $[W^{-1}]\mathcal{M}$. These replacements are similar to injective/projective resolutions that I know from homological algebra, and they are similar to that cellular approximation that I know from topology. Also, mapping cones have their place in the theory.

In some sense, a model category behaves to $[W^{-1}]\mathcal{M}$ as a group presentation to a group. It is not unique, but useful to calculate.

Therefore, the relationship between a model category and its homotopy category are fundamental in the whole theory, and so that model categories are eqquipped with (more than) enough structure to do homotopy theory. Both the model category and its cousins, weak factorization systems, that I will also breafly introduce together with model structures, provide a suitable framework to compute in $[W^{-1}]\mathcal{M}$.

Because this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important and its machinery can then be used immediately in a large number of different settings, as long as the axioms are checked in each case. Although many of these settings are geometric (spaces, fibrewise spaces, G-spaces, spectra, diagrams of spaces ...), some of them are not (chain complexes, simplicial commutative rings, simplicial groups...). Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language. However, so far their utility has been mostly confined to areas historically associated with algebraic topology, such as homological algebra, algebraic K-theory, and algebraic topology itself. Actually this list could be expanded to cover other areas of mathematics, for instance, Voevodksy's work has certain to make model categories a fundamental tool for algebraic geometers.

I would certainly like to introduce some interesting examples of model categories. However, I will have to wait because of the fact that the axioms for a model category are very powerful, and this means that I could be able to prove many theorems about model categories, but it also means that it is hard to check that any particular category is a model category. I will need to develop some theory first, before I could construct any example. In fact, as this work has the aim to be as general as possible, the examples will be only breafly introduced. These examples should make it clear that model categories are really fundamental as a tool.

Proving that a particular category has a model structure is always difficult. There is, however, a standard method, introduced by Quillen in [13] but formalized in William G. Dwyer, Philip S. Hirschhorn, and Daniel M. Kan and Jeffrey H. Smith in[4]. This method is an elaboration of the small object argument and is known as the theory of cofibrantly generated model categories.

Following, in particular, I am interested in functor categories. If I is a small category and $\mathcal{M}$ is a category "in which I know how to do homotopy theory", how can I do homotopy theory in the category of functors from I to $\mathcal{M}$?. This question does not have a unique answer (wich is a good thing since that different answers are serviceable in different situations), but in this sense, I can add:

As with the classical homotopy theory of spaces, having a model structure enables me to make various constructions, such as homotopy limits, unambiguously defined. A more modern viewpoint might suggest regarding model categories themselves as objects of study. In this way, I could seek to understand relationships between different model categories. Left and right Quillen functors provide the correct kinds of maps, with Quillen equivalences the standard means of considering two model categories sufficiently alike. In this framework, one could ask questions such as what a homotopy limit or homotopy colimit of a diagram of model categories would be. Unfortunately, there are no immediate answers to these questions because at present there is no known model structure on the category of model categories.

So that, I will want to define an appropriate kind of "morphisms between model categories" called Quillen functors and corresponding "equivalences between model categories" called Quillen equivalences. It turns out that the useful notion of "morphism between model categories" is not, as one would expect, a functor which is compatible with the model category structures in the sense that it preserves weak equivalences, cofibrations and fibrations, but a functor which is one of a pair of adjoint functors (called Quillen functors), each of which is compatible with one half of the model category structures in the sense that the left adjoint (the left Quillen functor) preserves cofibrations and trivial cofibrations (that is, cofibrations which are also weak equivalences) and the right adjoint (the right Quillen functor) preserves fibrations and trivial fibrations. There is a corresponding notion of "equivalences between model categories" (called Quillen equivalences). These are Quillen functors which induce "equivalences of homotopy theories".

Again and more formally, I will study Quillen functors and their derived functors. The most obvious requirement to make on a functor between model categories is that it preserve cofibrations, fibrations, and weak equivalences. This requirement is too demnding however. Instead, I only require that a Quillen functor preserve half of the model structure: either cofibrations and trivial cofibrations, or fibrations and trivial fibrations, where a trivial cofibration is both a cofibration and a weak equivalence, and similarly for trivial fibrations. This gives me left and right Quillen functors, and could give me two different categories of model categories. However, in practice functors of model categories come in adjoint pairs. I therefore define a morphism of model categories to be an adjoint pair, where the left adjoint is a left Quillen functor and the right adjoint is a right Quillen functor.

A Quillen functor will induce a functor on the homotopy categories, called its total (left or right) derived functor. This operation of taking the derived functor does not preserve identities or compositions, but it does do so up to coherent natural isomorphism.

This observation leads naturally to 2-categories and pseudo-2-functors, which is not a subject of study in this papers but which is well introduced by Mark Hovey in [8]. There he developes how the category of model categories is not really a category at all, but a 2-category. The operation of taking the homotopy category and the total derived functor is not a functor, but instead is a pseudo-2-functor. The 2-morphisms of model categories are just natural transformations, and therefore he just points out that there is a convenient language to talk about these kind of phenomena, rather than introducing any deep mathematics.

The Axioms

.

In my chosen model structure definition, I generally follow the modern axioms of a ”closed Quillen's model category”, but slightly different from Quillen's original definition of a closed model category (introduced in both [13] and [14]) in the sense that I require the existence of all colimits and limits (not just the finite ones), and I require the two factorizations to be functorial.

Definition 1. *A category $\mathcal{M}$ together with three distinguished classes of morphisms W (the weak equivalences), F (the fibrations or 'nice surjections'), C (the cofibrations or 'nice injections') is called a model category if the following axioms are satisfied:*

- **(•)** ***M1: (Limit axiom).***

  The category is complete and cocomplete (that is, it has all small limits and all small colimits).

  Recall that a category $\mathcal{C}$ is called small if both $Obj(\mathcal{C})$ and $Hom(\mathcal{C})$ are actually sets not proper classes, (and it is called large, otherwise).

  Also recall that a category $\mathcal{C}$ is complete if it is closed under small limits, that is, if $lim_{\mathcal{D}}\ F$ exists for every small category $\mathcal{D}$ and every functor $F\colon \mathcal{D} \longrightarrow \mathcal{C}$.

  In a analogous sense, also recall that a category $\mathcal{C}$ is cocomplete if it is closed under small colimits, i.e., if $colim_{\mathcal{D}}\ F$ exists for every small category $\mathcal{D}$ and every functor $F\colon \mathcal{D} \longrightarrow \mathcal{C}$.

- **(•)** ***M2: (Two out of three axiom).***

  The class W satisfies 2 out of 3 property.

  That is, if f and g are morphisms in $\mathcal{M}$ such that $f \circ g$ is well defined,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "f\circ g"] & Y \arrow[r, "f"] & Z & \in \mathcal{M} \end{tikzcd}$$

  and if two of f, g and $f \circ g$ are weak equivalences, then so is the third.

  Then the following diagram commutes,

$$\begin{tikzcd} X \arrow[rr, "h=f\circ g"] \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

- **(•)** ***M3: (Retract axiom).***

  Retracts of morphisms in W (resp. F, C) are again in W (resp. F, C).

  If the morphism $g \in \mathcal{M}$ is a retract of $f \in \mathcal{M}$ and f is a morphism belonging to one of the distinguished classes, (that is f is a fibration a cofibration or a weak equivalence), then so is g.

  That is, in the diagram of retracts,

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "g"'] \arrow[rr, bend left, "Id_X"] & Y \arrow[r, "r"] \arrow[d, "f"] & X \arrow[d, "g"] \\
X' \arrow[r, "j"'] \arrow[rr, bend right, "Id_Y"'] & Y' \arrow[r, "s"'] & X'
\end{tikzcd}$$

if f is a weak equivalence/cofibration/fibration, then the same holds for g.

Recall that, explicitly, the requirement that g is a retract of f means that there exists i, j, r and s such that the previous diagram commutes.

(•) ***M4: (Lifting axiom)***.

Morphisms in F satisfy the right lifting property with respect to the morphisms in $C \cap W$ (the trivial cofibrations also called acyclic cofibrations).

Morphisms in C satisfy the left lifting property with respect to the morphisms in $F \cap W$ (the trivial fibrations also called acyclic fibrations).

Explicity, if the outer square of the diagram commutes, where i is a cofibration and p is a fibration, and moreover i or p is acyclic, then there exists a dotted arrow h completing the following diagram,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

(•) ***M5: (Factorization axiom)***.

There are two functorial factorizations of morphisms into a morphism in C followed by one in F .

For the first one, the C-morphism is also in W.

For the second one, the F-morphism is also in W .

If $g \colon X \longrightarrow Y \in \mathcal{M}$ there exist functorial factorizations,

$$\begin{tikzcd}
& Z_g \arrow[dr, two heads, "f_g"] & \\
X \arrow[ur, tail, "i_g", "\sim"'] \arrow[rr, "g"] \arrow[dr, tail, "j_g"'] & & Y \\
& W_g \arrow[ur, two heads, "p_g"', "\sim"] &
\end{tikzcd}$$

where i_g is a trivial cofibration, f_g is a fibration, j_g is a cofibration and p_g is a trivial fibration.

I can add:

M1 axiom (Limit axiom), essentially means, for every index set, that products and coproducts indexed over this set exist, and that equalizers and coequalizers indexed over this set exist (together these give all small limits and colimits).

M2 axiom (two out of three axiom) means, for f, g morphisms such that $f \circ g$ exists, if two out of $\{\ f,\ g,\ f \circ g\ \}$ are in W , then all three are in W . It follows that W is a subcategory, so that closed under composition, but moreover M2 axiom is stronger than that.

In relation with M3 axiom (Retract axiom). Retracts are to be considered in the arrow category. First, let me remind you of retracts in any category: a morphism $i \colon A \longrightarrow B$ exhibits A as a retract if there is a morphism $r \colon B \longrightarrow A$ which is a retraction, that is $r \circ i = Id_A$. If I apply this to the arrow category, this means that a morphism $f : X \longrightarrow Y$ is a retract of $g \colon X' \longrightarrow Y'$

if there are morphisms $X \longrightarrow X'$ and $Y \longrightarrow Y'$ that commute with f, g and furthermore exhibit X and Y as retracts.

In relation with M4 axiom (Lifting axiom). The lifting properties amount to the following: suppose I have a commutative square with g at the left column, f at the right column, anything as rows. Suppose furthermore that g is a cofibration and f a fibration. The axioms says, if either f or g is, in addition, a weak equivalence, then there exists a "lift" in the diagram, that is a morphism from the lower left corner to the upper right corner that commutes with the other four morphisms in the diagram.

It follows from this axiom that F, C are also subcategories, that is closed under composition.

M5 axiom (Factorization axiom) merely states that there are functorial factorizations. Functoriality means functoriality on the arrow category, that is if I have a commutative square, I can factorize rows or columns simultaneously and get two new commutative squares inside the original one. Later on in this work in some constructions in which the factorizations are built in by use of the small object argument this axiom is fundamental.

I can consider a first auxiliary fact,

Lemma 2. *Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be the empty category (so that, the category with no objects), and $F\colon \mathcal{D} \longrightarrow \mathcal{C}$ the unique functor.*

Then $colim(F)$, if it exists, is an initial object of $\mathcal{C}$

and $lim(F)$, if it exists, is a terminal object of $\mathcal{C}$.

Proof. The proof is immediate since just involves unravelling the definitions.

Indeed, an object $\emptyset$ of a category $\mathcal{C}$ is said to be an initial object if there is exactly one map from $\emptyset$ to any object X of $\mathcal{C}$.

Dually, an object $*$ of $\mathcal{C}$ is said to be a terminal object if there is exactly one map $X \longrightarrow *$ for any object X of $\mathcal{C}$.

And clearly initial and terminal objects of $\mathcal{C}$ are unique up to canonical isomorphism.

Now, in the case of the Lemma $colim(F)$, if it exists, is necessarily that initial object of $\mathcal{C}$ and equivalently, $lim(F)$, if it exists, is necessarily that terminal object of $\mathcal{C}$.

□

I note that,

An object X is called fibrant if the terminal morphism $X \longrightarrow *$ is a fibration. That is, an object for which the unique map to the final object is a fibration is said to be fibrant. In particular, the terminal object $*$ in a model category LM is always fibrant.

Dually,

An object X is called cofibrant if the initial morphism $\emptyset \longrightarrow X$ is a cofibration. That is, an object for which the unique map from the initial object is a cofibration is said to be cofibrant. In particular the initial object $\emptyset$ of a model category LM is always cofibrant.

In a model category $\mathcal{M}$, the initial and terminal morphisms always exist by both the previous Lemma 2 and the fact that a model category is complete and cocomplete by M1 axiom (limit axiom).

Remark 3. *Later on, when I will define the homotopy category $Ho(\mathcal{M})$ of a model category $\mathcal{M}$, I will see that $Hom_{Ho\ (\mathcal{M})}(A, B)$ is in general a quotient of $Hom_{\mathcal{M}}(A, B)$ only if A is cofibrant and B is fibrant. If A is not cofibrant or B is not fibrant, then there are not in general a sufficient number of maps $A \longrightarrow B$ in $\mathcal{M}$ to represent every map in the homotopy category.*

A morphism which is both a fibration (respectively cofibration) and a weak equivalence will be called a trivial fibration or equivalently an acyclic fibration (respectively trivial cofibration or equivalently an acyclic cofibration).

The axioms for a model category are self duals.

Remark 4. *(Duality in Model Categories)*

If $\mathcal{M}$ is a model category, then $\mathcal{M}^{op}$ is a model category such that,

- *the weak equivalences in $\mathcal{M}^{op}$ are the opposites of the weak equivalences in $\mathcal{M}$,*

- (●) *the cofibrations in $\mathcal{M}^{op}$ are the opposites of the fibrations in $\mathcal{M}$, and,*
- (●) *the fibrations in $\mathcal{M}^{op}$ are the opposites of the cofibrations in $\mathcal{M}$.*

All these facts follow directly from the definitions.

This implies that any statement that is proved true for all model categories implies a dual statement in which cofibrations are replaced by fibrations, fibrations are replaced by cofibrations, colimits are replaced by limits and limits are replaced by colimits.

I call a model category (or any category with an initial and terminal object) pointed if the map from the initial object to the terminal object is an isomorphism.

Given a model category $\mathcal{M}$, I defin $\mathcal{M}_*$ to be the category under the terminal object $*$. That is, an object of $\mathcal{M}_*$ is a map $* \xrightarrow{v} X$ of $\mathcal{M}$, often written (X, v).

I think of (X, v) as an object X together with a basepoint v. A morphism from (X, v) to (Y, w) is a morphism $X \longrightarrow Y$ of $\mathcal{M}$ that takes v to w.

I can observe that $\mathcal{M}$ has arbitrary limits and colimits. Indeed, if $F\colon \mathcal{I} \longrightarrow \mathcal{M}_*$ is a functor from a small category $\mathcal{I}$ to $\mathcal{M}_*$, the limit of F as a functor to $\mathcal{M}$ is naturally an element of $\mathcal{M}_*$ and is the limit there. The colimit is a little trickier. For that, I let $\mathcal{J}$ denote $\mathcal{I}$ with an extra initial object $*$. Then F defines a functor $G\colon \mathcal{J} \longrightarrow \mathcal{M}$C, where $G(*) = *$, and G of the map $* \longrightarrow i$ is the basepoint of $F(i)$. The colimit of G in $\mathcal{M}$ then has a canonical basepoint, and this defines the colimit in $\mathcal{M}_*$ of F. For instance, the initial object, the colimit of the empty diagram, in $\mathcal{M}_*$ is $*$, and the coproduct of X and Y is $X \wedge Y$, the quotient of $X \amalg Y$ obtained by identifying the basepoints. In particular, $\mathcal{M}_*$ is a pointed category.

There is an obvious functor $\mathcal{M} \longrightarrow \mathcal{M}_*$ that takes X to $X_+ = X \amalg *$, with basepoint $*$. This operation of adding a disjoint basepoint is left adjoint to the forgetful functor $G\colon \mathcal{M}_* \longrightarrow \mathcal{M}$, and defines a faithful (but not full) embedding of $\mathcal{M}$ into the pointed category $\mathcal{M}_*$. If $\mathcal{M}$ is already pointed, these functors define an equivalence of categories between $\mathcal{M}$ and $\mathcal{M}_*$.

Proposition 5. *Let $\mathcal{M}$ be a model category. I define a map $f \in \mathcal{M}_*$ to be a cofibration (fibration, weak equivalence) if and only if $G(f)$ is a cofibration (fibration, weak equivalence) in $\mathcal{M}$.*

Then $\mathcal{M}_*$ is a model category.

Proof. It is clear that weak equivalences in $\mathcal{M}_*$ satisfy the two out of three property, and that cofibrations, fibrations, and weak equivalences are closed under retracts. Suppose i is a cofibration in $\mathcal{M}_*$ and p is a trivial fibration. Then $G(i)$ has the left lifting property with respect to $G(p)$, it follows that i has the left lifting property with respect to p, since any lift must automatically preserve the basepoint.

Similarly, trivial cofibrations have the left lifting property with respect to fibrations.

If $f = \beta(f) \circ \alpha(f)$ is a functorial factorization in $\mathcal{M}$, then it is also a functorial factorization in $\mathcal{M}_*$; I give the codomain of $\alpha(f)$ the basepoint inherited from α, and then $\beta(f)$ is forced to preserve the basepoint. Thus the factorization axiom also holds and so $\mathcal{M}_*$ is a model category. ☐

I remark that I could replace the terminal object $*$ by any object A of $\mathcal{M}$, to obtain the model category of objects under A. In fact, I could also consider the category of objects over A, whose objects consist of pairs (X, f), where $f\colon X \longrightarrow A$ is a map in $\mathcal{M}$. A similar proof as in Proposition 5 shows that this also forms a model category. Finally, I could iterate these constructions to form the model category of objects under A and over B.

The Axioms at play

.

I can recover the characterizations and some consequences for fibrations and cofibrations living in category theory, newly "remastered" in the new environment of model categories.

Proposition 6. *Let $\mathcal{M}$ be a model category.*

(1) *The map $i\colon A \longrightarrow B$ is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*

(2) *The map $i\colon A \longrightarrow B$ is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations.*
(3) *The map $p\colon X \longrightarrow Y$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.*
(4) *The map $p\colon X \longrightarrow Y$ is a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.*

Proof. The four proofs are rather identical.

(1) Implication $\Rightarrow$), is part of the M4 axiom (Lifting axiom).

Implication $\Leftarrow$).

I can factor i as $i = p \circ j$ where p is a trivial fibration and j is a cofibration, by M5 axiom (Factorization axiom).

Now, the Retract Argument detailed in Section 3 implies that i is a retract of j, and so i is moreover a cofibration by aplying M3 axiom (Retract axiom).

(2) Implication $\Rightarrow$), is part of the M4 axiom (Lifting axiom).

Implication $\Leftarrow$).

Again, I can factor i as $i = p \circ j$ where p is a trivial fibration and j is a cofibration, by M5 axiom (Factorization axiom).

Now, the Retract Argument detailed in Section 3 implies that i is a retract of j, and so i is moreover a trivial cofibration by aplying M3 axiom (Retract axiom).

(3) Implication $\Rightarrow$), is part of the M4 axiom (Lifting axiom).

Implication $\Leftarrow$).

I can factor p as $p = i \circ q$ where i is a cofibration and q is a trivial fibration, by M5 axiom (Factorization axiom).

Now, the Retract Argument detailed in Section 3 implies that p is a retract of q, and so p is moreover a fibration by aplying M3 axiom (Retract axiom).

(4) Implication $\Rightarrow$), is part of the M4 axiom (Lifting axiom).

Implication $\Leftarrow$).

I can factor p as $p = i \circ q$ where i is a cofibration and q is a trivial fibration, by M5 axiom (Factorization axiom).

Now, the Retract Argument detailed in Section 3 implies that p is a retract of q, and so p is moreover a trivial fibration by aplying M3 axiom (Retract axiom).

□

The previous Proposition 6 has some immediate consequences,

In a model category $\mathcal{M}$,

(1) Both, the class C of cofibrations and the class F of fibrations are closed under compositions.
(2) The classes C of cofibrations, F of fibrations, $C \cap W$ of trivial cofibrations and $F \cap W$ of trivial fibrations are closed under products and coproducts.

Proposition 7. *Let $\mathcal{M}$ be a model category.*

A map $f\colon X \xrightarrow{\sim} Y$ is a weak equivalence if and only if it can be factored as a trivial cofibration followed by a trivial fibration.

Proof.

Implication $\Rightarrow$).

By M5 axiom (factorization axiom) I can factor the weak equivalence $f\colon X \xrightarrow{\sim} Y$ as $X \overset{g}{\underset{\sim}{\rightarrowtail}} Z \overset{h}{\twoheadrightarrow} Y$, with g a trivial cofibration and h a fibration.

Now, f and g are weak equivalences, then by M2 axiom (two out of three axiom) so is h.

Hence h is a trivial fibration.

Implication $\Leftarrow$).

Any map that can be factored as a trivial cofibration followed by a trivial cofibration is a composition of weak equivalences and is thus by M2 axiom (two out of three axiom) is a weak equivalence.

□

Proposition 8. *Let $\mathcal{M}$ be a model category.*

Then any two of the classes C of cofibrations, F of fibrations and W of weak equivalences determine the third.

Proof. Directly by Proposition 6, I obtain,

- (•) fibrations and weak equivalences determine cofibrations,
- (•) cofibrations and weak equivalences determine fibrations.

But now, since the classes of cofibrations and of fibrations are closed under compositions, then I obtain that trivial cofibrations and trivial fibrations detemine the weak equivalences.

But now, again by Proposition 6 I obtain,

- (•) fibrations determine determine trivial cofibrations,
- (•) cofibrations determine trivial fibrations.

Hence, I can conclude that fibrations and cofibrations determine the weak equivalences. □

Proposition 9. *Let $\mathcal{M}$ be a model category, and let $p\colon X \longrightarrow Y$ be a morphism in $\mathcal{M}$.*

(1) *The class of morphisms having the left lifting property with respect to p is closed under retracts.*
(2) *The class of morphisms having the right lifting property with respect to p is closed under retracts.*

Proof. I will prove the first part (by dual argument the second part holds).

I suppose that $f\colon A \longrightarrow B$ is a retract of $g\colon C \longrightarrow D$, and I also suppose that g has the left lifting property with respect to p.

I want to show that the dotted arrow ϕ exists in any diagram of the form,

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "f"'] \arrow[rr, bend left, "Id_A"] & C \arrow[r, "q_A"] \arrow[d, "g"] & A \arrow[r, "r"] \arrow[d, "f"'] & X \arrow[d, "p"] \\
B \arrow[r, "i_B"'] \arrow[rr, bend right, "Id_B"'] & D \arrow[r, "q_B"'] & B \arrow[r, "s"'] \arrow[ur, dashed, "?", "\phi"'] & Y
\end{tikzcd}$$

Since g has the left lifting property with respect to p, there exists a map $\varphi\colon D \longrightarrow X$ such that $\varphi \circ g = r \circ q_A$ and $p \circ \varphi = s \circ q_B$.

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "f"'] \arrow[rr, bend left, "Id_A"] & C \arrow[r, "q_A"] \arrow[d, "g"] & A \arrow[r, "r"] \arrow[d, "f"] & X \arrow[d, "p"] \\
B \arrow[r, "i_B"'] \arrow[rr, bend right, "Id_B"'] & D \arrow[r, "q_B"'] \arrow[urr, dashed, "\varphi"] & B \arrow[r, "s"'] & Y
\end{tikzcd}$$

Now, I can define $\phi\colon B \longrightarrow X$ by letting $\phi = \varphi \circ i_B$.

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "f"'] \arrow[rr, bend left, "Id_A"] & C \arrow[r, "q_A"] \arrow[d, "g"] & A \arrow[r, "r"] \arrow[d, "f"] & X \arrow[d, "p"] \\
B \arrow[r, "i_B"'] \arrow[rr, bend right, "Id_B"'] & D \arrow[r, "q_B"'] \arrow[urr, dashed, "\varphi"] & B \arrow[r, "s"'] \arrow[ur, dashed, "\phi"'] & Y
\end{tikzcd}$$

holding, $\phi \circ f = \varphi \circ i_B \circ f = \varphi \circ g \circ i_A = r \circ q_A \circ i_A = r \circ Id_A = r$
and also holding, $p \circ \phi = p \circ \varphi \circ i_B = s \circ q_B \circ i_B = s \circ Id_B = s$.
as desired.

□

I can also put into play pushouts and pullbacks in the framework of model categories.

Proposition 10. *Let $\mathcal{M}$ be a model category, and let $p\colon X \longrightarrow Y$ be a morphism in $\mathcal{M}$.*

1. *The class of morphisms having the left lifting property with respect to p is closed under pushouts.*
2. *The class of morphisms having the right lifting property with respect to p is closed under pullbacks.*

Proof. I will prove the first part (by dual argument the second part holds).
I want to show that if $i\colon A \longrightarrow B$ has the left lifting property with respect to p and if I consider the following diagram,

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "i"'] & C \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "p"] \\
B \arrow[r, "s"'] & D \arrow[r, "v"'] \arrow[ur, dashed, "?", "\phi"'] & Y
\end{tikzcd}$$

in which the square on the left is a pushout, then the dotted arrow ϕ exists.
Since i has the left lifting property with respect to p, then there exists a map $\varphi\colon B \longrightarrow X$ such that $\varphi \circ i = u \circ r$ and $p \circ \varphi = v \circ s$.

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "i"'] & C \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "p"] \\
B \arrow[r, "s"'] \arrow[urr, dashed, "\varphi"] & D \arrow[r, "v"'] & Y
\end{tikzcd}$$

Now since D is the pushout $B \coprod_A C$, this induces a map $\phi\colon D \longrightarrow X$ such that $\phi \circ j = u$ and $\phi \circ s = \varphi$.

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "i"'] & C \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "p"] \\
B \arrow[r, "s"'] \arrow[urr, dashed, "\varphi"] & D \arrow[r, "v"'] \arrow[ur, dashed, "\phi"'] & Y
\end{tikzcd}$$

I then have,
$p \circ \phi \circ s = p \circ \varphi = v \circ s$ and
$p \circ \phi \circ j = p \circ u = v \circ j$
and hence, the universal mapping property of the pushout implies that $p \circ \phi = v$.

□

Proposition 11. *Let $\mathcal{M}$ be a model category.*

1. *The class of cofibrations is closed under pushouts.*
2. *The class of trivial cofibrations is closed under pushouts.*
3. *The class of fibrations is closed under pullbacks.*
4. *The class of trivial fibrations is closed under pullbacks.*

Proof.

Any cofibration has the left lifting property with respect to all trivial fibrations by Proposition 6. But now by Proposition 10 the class of morphisms with the left lifting property with respect a given morphism is closed under pushouts. Hence cofibrations are closed under pushouts.

- **(2)** Any trivial cofibration has the left lifting property with respect to all fibrations by Proposition 6. But now by Proposition 10 the class of morphisms with the left lifting property with respect a given morphisms is closed under pushouts. Hence trivial cofibrations are closed under pushouts.
- **(3)** Any fibration has the right lifting property with respect to all trivial cofibrations by Proposition 6. But now by Proposition 10 the class of morphisms with the right lifting property with respect a given morphism is closed under pullbacks. Hence fibrations are closed under pullbacks.
- **(4)** Any trivial fibration has the right lifting property with respect to all cofibrations by Proposition 6. But now by Proposition 10 the class of morphisms with the right lifting property with respect a given morphism is closed under pullbacks. Hence trivial fibrations are closed under pullbacks.

□

Recall that, given a square,

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "k"'] & D
\end{tikzcd}$$

- **(1)** If the square is a pushout, then I can also call g as the pushout of f along h.
- **(2)** If the square is a pullback, then I can also call f as the pullback of g along k.

Proposition 12. *If $g\colon C \longrightarrow D$ is a pushout of $f\colon A \longrightarrow B$ and $h\colon E \longrightarrow F$ is a pushout of f, then h is a pushout of f.*

Proof. I consider the diagram,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & E \arrow[d, "h"] \\
B \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

Clearly, if every square is a pushout, the rectangle is a pushout too.

□

Proposition 13. *Given any diagram of the form,*

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & E \arrow[d, "h"] \\
B \arrow[r, "u"'] & D \arrow[r, "v"'] & F
\end{tikzcd}$$

- **(1)** *If F is the pushout $E \amalg_A B$ and D is the pushout $B \amalg_A C$, then F is the pushout $D \amalg_C E$*
- **(2)** *If A is the pullback $B \times_F E$ and C is the pullback $D \times_F E$, then A is the pullback $B \times_D C$.*

Proof. I will prove the first part (by dual argument the second part holds).
If X is an object in $\mathcal{M}$ and $\varphi : D \longrightarrow X$ and $\phi\colon E \longrightarrow X$ are morphisms such that $\varphi \circ g = \phi \circ s$,

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & E \arrow[d, "h"] \arrow[ddr, bend left, "\phi"] & \\
B \arrow[r, "u"'] & D \arrow[r, "v"'] \arrow[drr, bend right, "\varphi"'] & F & \\
 & & & X
\end{tikzcd}$$

then $\phi \circ s \circ r = \varphi \circ g \circ r = \varphi \circ u \circ f$.

Since F is the pushout $E \coprod_A B$, then there exists a unique map $k\colon F \longrightarrow X$ such that $k \circ v \circ u = \varphi \circ u$ and $k \circ h = \phi$.

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & E \arrow[d, "h"] \arrow[ddr, bend left, "\phi"] & \\
B \arrow[r, "u"'] & D \arrow[r, "v"'] \arrow[drr, bend right, "\varphi"'] & F \arrow[dr, dashed, "k"] & \\
 & & & X
\end{tikzcd}$$

Since D is the pushout $B \amalg_A C$ and the morphisms φ and $k \circ v$ satisfy both,
$(k \circ v) \circ u = (\varphi) \circ u$ and
$(\varphi) \circ g = \phi \circ s = k \circ h \circ s = (k \circ v) \circ g$,
then I have $\varphi = k \circ v$.
Thus, the morphism k satisfies $k \circ h = \phi$ and $k \circ v = \varphi$.
Now, I prove that k is unique.
If k' is another morphism satisfying both, $k' \circ h = \phi$ and $k' \circ v = \varphi$, then $k' \circ v \circ u = \varphi \circ u$, and then by the universal property of the coproduct $B \coprod_A C$, then I obtain that $k' = k$.

$\square$

Now, also I can consider the cylinder object and the path object in the framework of model categories.

Definition 14. *Let $\mathcal{M}$ be a model category, and let $f, g\colon X \longrightarrow Y$ be maps in $\mathcal{M}$.*
A cylinder object for X is a factorization

$$X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$$

of the fold map

$$Id_X \amalg Id_X\colon X \amalg X \longrightarrow X$$

such that $i_0 \amalg i_1$ is a cofibration and p is a weak equivalence

Definition 15. *A path object for Y is a factorization*

$$Y \xrightarrow{s} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

of the diagonal map,

$$Id_Y \times Id_Y colonY \longrightarrow Y \times Y$$

such that s is a weak equivalence and $p_0 \times p_1$ is a fibration.

Lemma 16. *Let $\mathcal{M}$ be a model category.*

(1) *Every object X of $\mathcal{M}$ has a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ in which p is a trivial fibration.*
(2) *Every object X of $\mathcal{M}$ has a path object $X \xrightarrow{i} Path(X) \xrightarrow{p_0 \times p_1} X \times X$ in which i is a trivial cofibration.*

Proof. I first prove (1).
Using M5 axiom (factorization axiom), I can factor the morphism $Id_X \amalg Id_X\colon X \amalg X \longrightarrow X$ into a cofibration followed by a trivial fibration.
I prove now (2).
Again, using M5 axiom (factorization axiom), I can factor the morphism $Id_X \times Id_X\colon X \longrightarrow X \times X$ into a trivial cofibration followed by a fibration. $\square$

Proposition 17. *Let $\mathcal{M}$ be a model category and let X be an object of LM.*

(1) *If X is cofibrant, then the injections $i_0, i_1\colon X \longrightarrow X \amalg X$ are cofibrations.*
(2) *If X is fibrant, then the projections $p_0, p_1\colon X \times X \longrightarrow X$ are fibrations.*

Proof. I will prove the first part (by dual argument the second part holds).

Since the diagram,

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "i_1"] \\
X \arrow[r, "i_0"'] & X \amalg X
\end{tikzcd}$$

where $\emptyset$ is the initial object in $\mathcal{M}$, is a pushout and the class of cofibrations is closed under pushouts, then I can conclude that i_0 and i_1 are cofibrations.

□

Proposition 18. *Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$.*

(1) *If $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ is a cylinder object for X, then the injections $i_0, i_1 \colon X \longrightarrow Cyl(X)$ are weak equivalences.*

If X is cofibrant, then they are trivial cofibrations.

(2) *If $X \xrightarrow{s} Path(X) \xrightarrow{p_0 \times p_1} X \amalg X$ is a path object for X, then the projections $p_0, p_1 \colon Path(X) \longrightarrow X$ are weak equivalences.*

If X is fibrant, then they are trivial fibrations.

Proof. I will prove the first part (by dual argument the second part holds).

Using the M2 axiom (two out of three axiom), i_0 and i_1 are weak equivalences.

If X is cofibrant, by Proposition 17 then i_0 and i_1 in addition are cofibrations, so they are trivial cofibrations.

□

Keynote Examples

Top. The Model Category of Topological Spaces.

The category of topologogical spaces, denoted by *Top*, is the category with all topological spaces as objects, and the continuous functions between topological spaces as arrows.

The category *Top* of topological spaces can be given the structure of a model category by defining $f \colon X \longrightarrow Y$ to be,

(i) a weak equivalence if f is a weak homotopy equivalence.

(ii) a cofibration if f is a retract of a map $X \longrightarrow Y'$ in which Y' is obtained from X by attaching cells, and

(iii) a fibration if f is a Serre fibration.

With respect to this model category structure, the homotopy category $Ho(Top)$ is equivalent to the usual homotopy category of CW-complexes.

The above model category structure appears to be the one which comes up most frequently in an usual an daily work with algebraic topology. It puts an emphasis on CW structures.

I can note that every object is fibrant, and the cofibrant objects are exactly the spaces which are retracts of generalized CW-complexes (where a "generalized CW-complex" is a space built up from cells, without the requirement that the cells be attached in order by dimension).

Indeed, If X is a topological space and I have a diagram with solid arrows,

$$\begin{tikzcd}
D^n \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d] \\
D^n \times I \arrow[r] \arrow[ur, dashed, "\exists\, ?"] & *
\end{tikzcd}$$

Then the dotted arrow exists defining $h(x,t) = f(x)$. Then $X \longrightarrow *$ is a Serre fibration, and hence, X is fibrant.

On the other hand, is X is a CW complex, taking $A = \emptyset$ then, $\emptyset \longrightarrow X$ has the right lifting property with respect to any mapping p, being both a fibration and a weak equivalence, and so $\emptyset \longrightarrow X$ it results a cofibration and hence X is cofibrant.

In particular, in this model category, is a classical result the following,

(1) a map is a fibration if and only if it has the right lifiting property with respect to the maps $|\Lambda[n,k]| \longrightarrow |\Delta[n]|, \qquad \forall n > 0$ and $0 \leq k \leq n$,

and

(2) a map is a trivial fibration if and only if it has the right lifiting property with respect to the maps $|\partial\Delta[n]| \longrightarrow |\Delta[n]| \quad , \quad \forall n \geq 0$.

If I consider the category Top_* of pointed topological spaces then the model category structure is closely related to de unpointed one, in the sense that for the following category,

Definition 19. *If $f\colon X \longrightarrow Y$ is a map of pointed topological spaces, then*

(1) *f is a weak equivalence if it is a weak equivalence of unpointed topological spaces when I forget about the base points,*

(2) *f is a fibration of unpointed topological spaces when I forget about the base points, and*

(3) *g is a cofibration if it has the left lifting property with rspect to all maps that are both fibrations and weak equivalences.*

the category Top_* of pointed topological spaces with weak equivalences, fibrations and cofibrations as in Definition 19 is a model category.

In particular, this pointed model category hold

(1) a map is a fibration if and only if it has the right lifiting property with respect to the maps $|\Lambda[n,k]|^+ \longrightarrow |\Delta[n]|^+, \qquad \forall n > 0$ and $0 \leq k \leq n$,

and

(2) a map is a trivial fibration if and only if it has the right lifiting property with respect to the maps $|\partial\Delta[n]|^+ \longrightarrow |\Delta[n]|^+ \quad , \quad \forall n \geq 0$.

In some topological situations, though, weak homotopy equivalences are not the correct maps to focus on.

It is natural to ask whether there is another model category structure on Top with respect to which the "weak equivalences" are the ordinary homotopy equivalences.

The paper of Arne Str∅m [15] shows a model structure for Top in this sense. If B is a topological space, I call a subspace inclusion $i\colon A \longrightarrow B$ a closed Hurewicz cofibration if A is a closed subspace of B and i has the homotopy extension property, and so that, for every space B a lift exits in every commutative diagram,

$$\begin{tikzcd}
(B \times 0) \cup (A \times [0,1]) \arrow[r] \arrow[d] & Y \arrow[d] \\
B \times [0,1] \arrow[r] & *
\end{tikzcd}$$

Similarly, I call a map $p\colon X \longrightarrow Y$ a Hurewicz fibration if p has the homotopy lifting property, and so that, for every space A a lift exists in every commutative diagram

$$\begin{tikzcd}
A \times 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
A \times [0,1] \arrow[r] & Y
\end{tikzcd}$$

Then the category Top of topological spaces can be given the structure of a model category by defining a map $f\colon X \longrightarrow Y$ to be

(i) a weak equivalence if f is a homotopy equivalence,

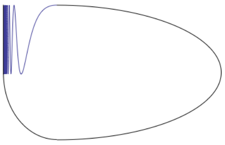
(ii) a cofibration if f is a closed Hurewicz cofibration, and

(iii) a fibration if f is a Hurewicz fibration.

With respect to this model category structure, the homotopy category $Ho(Top)$ is equivalent to the usual homotopy category of topological spaces.

Remark 20. *The last defined model category structure is quite different from the previous one.*

For instance, let W the so called 'Warsaw circle'; this is the compact subspace of the plane, $\mathbb{R}^2$, given by the union of the interval $[-1, 1]$ on the y-axis, the graph of $y = \sin(frac1x)$ for $0 < x \leq 1$, and an arc joining $(1, \sin(1))$ to $(0, -1)$.



Then the map from W to a point is a weak equivalence with respect to the first model category but not a weak equivalence with respect to the second model category structure.

For the present example, the Warsaw circle, a subset of the plane, has all homotopy groups zero, but the map from the Warsaw circle to a single point is not a homotopy equivalence.

In particular, Whitehead's theorem does not apply to the Warsaw Circle because it is not a CW complex. Recall that the Whitehead theorem does not hold for general topological spaces or even for all subspaces of $\mathbb{R}^n$.

sSet. The Model Category of Simplicial Sets.

The category of simplicial sets, denoted by $sSet$, is the category with all simplicial sets as objects, and the simplicial maps as arrows.

The category $sSet$ of simplicial sets can be given the structure of a model category by defining $f\colon X \longrightarrow Y$ to be,

1. a weak equivalence if its geometric realization $|f|\colon |X| \longrightarrow |Y|$ is a weak equivalence of topological spaces,
2. a fibration if it is a Kan fibration, that is if it has the right lifting property with respect to the map $\Lambda[n,k] \longrightarrow \Delta[n]$, $\forall n > 0$ *and* $0 \leq k \leq n$, and
3. a cofibration if it has the left lifting property with respect to all maps that are both fibrations and weak equivalences.

If I consider the category $sSet_*$ of pointed simplicial sets then its model category structure associated is closely related to de unpointed one, in the sense that for the following category,

Definition 21. *If $f\colon X \longrightarrow Y$ is a map of pointed simplicial sets, then*

1. *f is a weak equivalence if it is a weak equivalence of unpointed simplicial sets when I forget about the base points,*
2. *f is a fibration of unpointed simplicial sets when I forget about the base points, and*
3. *g is a cofibration if it has the left lifting property with rspect to all maps that are both fibrations and weak equivalences.*

the category $sSet_*$ of pointed simplicial sets with weak equivalences, fibrations and cofibrations as in Definition 21 is a model category.

In particular, this pointed model category hold

1. a map is a fibration if and only if it has the right lifting property with respect to the maps $\Lambda[n,k]^+ \longrightarrow \Delta[n]^+, \qquad \forall n > 0$ and $0 \leq k \leq n$,

   and
2. a map is a trivial fibration if and only if it has the right lifiting property with respect to the maps $\partial\Delta[n]^+ \longrightarrow \Delta[n]^+ \quad , \quad \forall n \geq 0$.

Remark 22. *The proof for the existence of the standard model category structures for the category of topological spaces and simplicial sets (both pointed and unpointed) and in addition their corresponding characterizations for fibrations and trivial fibrations early stated are rather long and plenty of details, and I will not present them here. The original proofs are due to Daniel Quillen* *and developed in* [13] *(Chapter II, Section* 3*), but an alternative and also beautiful, detailed and readable version can be found in the work by Hovey in* [8] *(Section* 2.4. *and Chapter* 3*).*

In order to well-understand any kind of work which involve simplicial sets I will need to briefly introduce their generalities. In particular, I extensively use these notions in sections below.

The category of Simplicial Sets.

The notion of simplicial sets was arisen from establishing combinatorial models for spaces. Recall that given any topological space X, there is a singular simplicial set $S_*(X)$, where $S_n(X)$ is the set of all continuous maps from the n-simplex to X. Let $C_n(X)$ be the free abelian group generated by S_n. Then I get the chain complex $C_*(X)$ with its homology the singular homology $H_*(X;\mathbb{Z})$. In other words, the singular homology of X is obtained from the free abelian groups generated by $S_*(X)$. I may ask whether the homotopy groups $\pi_*(X)$ can be obtained in a similar way. And the answer is that, in fact, $\pi_*(X)$ can be obtained directly from S_*.

I enunciate the basics about simplicial objects, presenting Δ-set and briefly introducing the relations between Δ-sets and simplicial sets.

In fact the simplicial and singular homology can be directly obtained as the derived functors of Δ-sets while, introducing fibrant simplicial sets, the homotopy theory on the category of simlicial sets can be set up. In particular, the homotopy groups can be combinatorially defined using simplicial sets. All those facts are not discussed in these papers but all of those are motivating facts to start studying simplicial objects.

Definition 23. *(Δ-Set)*
A Δ-set means a sequence of sets $X = \{X_n\}_{n\geq 0}$ with faces,

$$d_i\colon X_n \longrightarrow X_{n-1}, \quad 0 \leq i \leq n,$$

such that,

$$d_i \circ d_j = d_j \circ d_{i+1} \quad \textit{for } i \geq j,$$

which is called the Δ-identity.

Remark 24. *In fact, I can use coordinate projections for catching Δ-identity:*

$$d_i\colon (x_0, \cdots, x_n) \longrightarrow (x_0, \cdots, x_{i-1}, x_{x+1}, \cdots, x_n)$$

Let $\mathcal{O}^+$ be the category whose objects are finite ordered sets and whose morphisms are functions,

$$f\colon X \longrightarrow Y \quad \text{such that} \quad f(x) < f(y) \quad \text{if } x < y.$$

Note that the objects in $\mathcal{O}^+$ are given by $[n] = \{0, 1, \cdots, n\}$ for $n \geq 0$ and the morphisms in $\mathcal{O}^+$ are generated by $d^i\colon [n-1] \longrightarrow [n]$ with

$$d^i(j) = \begin{cases} j & \textit{if } j < i \\ \\ j+1 & \textit{if } j \geq i \end{cases}$$

for $0 \leq i \leq n$, that is d^i is the ordered embedding missing i.

I may write the function d^i in matrix form:

$$d^i = \begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n-1 \\ \\ 0 & 1 & \cdots & i-1 & i+1 & i+2 & \cdots & n \end{pmatrix}$$

The morphisms d^i satisfy the identity:

$$d^j \circ d^i = d^{i+1} \circ d^j$$

for $i \geq j$.

More abstractly, for any category $\mathcal{C}$, a Δ-object over $\mathcal{C}$ means a contravariant functor from $\mathcal{O}^+$ to $\mathcal{C}$. In other words, an Δ-object over $\mathcal{C}$ means a sequence of objects over $\mathcal{C}$, $X = \{X_n\}_{n\geq 0}$ with faces $d_i \colon X_n \longrightarrow X_{n-1}$ as morphisms in $\mathcal{C}$.

Definition 25. *(n-Simplex)*
The n-simplex $\Delta^+[n]$, as a Δ-set, is defined as,

$$\begin{cases} \Delta^+[n]_k = \{\ (i_0, i_1, \cdots, i_k) \quad | \quad 0 \leq i_0 < i_1 < \cdots < i_k \leq n\ \} & \textit{for } k \leq n \\ \\ \Delta^+[n]_k = \emptyset & \textit{for } k > n \end{cases}$$

The face $d_j \colon \Delta^+[n]_k \longrightarrow \Delta^+[n]_{k-1}$ is given by,

$$d_j(i_0, i_1, \cdots, i_k) = (i_0, i_1, \cdots, \widehat{i_j}, \cdots, i_k)$$

that is, deleting i_j.

Let now, $\sigma_n = (0, 1, \cdots, n)$

Then, $(i_0, i_1, \cdots, i_k) = d_{j_1} \circ d_{j_2} \circ d_{j_{n-k}} \circ \sigma_n$,

where $j_1 < j_2 < \cdots < j_{n-k}$ with $\{j_1, \cdots, j_k\} = \{0, 1, \cdots, n\} \setminus \{i_0, i_1, \cdots, i_k\}$.

In other words, any element in $\Delta^{[n]}$ can be written an iterated face of σ_n.

Definition 26. *(Δ-Map)*
A Δ-map $f \colon X \longrightarrow Y$ means a sequence of mappings, $f_n \colon X_n \longrightarrow Y_n$, for each $n \geq 0$ such that $f_{n-1} \circ d_i = d_i \circ f_n$.
That is, the following diagram commutes,

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "d_i"'] & Y_n \arrow[d, "d_i"] \\ X_{n-1} \arrow[r, "f_{n-1}"'] & Y_{n-1} \end{tikzcd}$$

A Δ-subset A of a Δ-set X means a sequence of subsets $A_n \subseteq X_n$ such that $d_i(A_n) \subseteq A_{n-1}$ for all $0 \leq i \leq n < \infty$.

A Δ-set is said to be isomorphic to a Δ-set Y, and denoted by $X \cong Y$, if there is a bijective Δ-map $f \colon X \longrightarrow Y$.

Let X be a Δ-set and let A be a Δ-subset. Clearly the inclusion $A \subseteq X$, that is, $A_n \subseteq X_n$ for each $n \geq 0$, is Δ-map.

I can also note that given X be a Δ-set and given any element $x \in X_n$, then there exists a unique Δ-map,

$$f_x \colon \Delta^+[n] \longrightarrow X$$

such that $f_x(\sigma_n) = x$

Indeed, I can justify this fact.

By the assumption $f_x(\sigma_n) = x$, I have

$$f_x(i_0, i_1, \cdots, i_k) = f_x(d_{j_1} \circ d_{j_2} \circ \cdots \circ d_{j_{n-k}} \circ \sigma_n) = d_{j_i} \circ d_{j_2} \circ \cdots \circ d_{j_{n-k}} \circ f_x(\sigma_n) = d_{j_1} \circ d_{j_2} \circ d_{j_{n-k}} x$$

This defines a Δ-map f_x such that $f_x(\sigma_n) = x$ as desired.

The simplicial map $f_x \colon \Delta^+[n] \longrightarrow X$ is called the representing map of x.

Definition 27. *(Standard Geometric n-Simplex)*
The standard geometric n-simplex Δ^n is defined by,

$$\Delta^n = \left\{\ (t_0, t_1, \cdots, t_n \quad | \quad t_i \geq 0 \textit{ and } \sum_{i=0}^{n} t_i = 1\ \right\}$$

I define, $d^i \colon \Delta^{n-1} \longrightarrow \Delta^n$ by setting,

$$d^i(t_0, t_1, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$$

The maps d^i satisfy the identity:

$$d^j \circ d^i = d^{i+1} \circ d^j \text{ for } i \geq j.$$

The boundary,

$$\partial\Delta^n = \bigcup_{i=0}^{n} d^i(\Delta[n-1])$$

is the union of all faces of Δ^n.

Let $Int(\Delta^n) = \Delta^n \backslash \partial\Delta^n$ be the interior of Δ^n, called open simplex.

Definition 28. *(Δ-Complex Structure)*
A Δ-complex structure on a space X is a collection of maps,

$$C(X) = \left\{ \ \sigma_\alpha \colon \Delta^n \longrightarrow X \quad | \quad \alpha \in J_n \ , \ n \geq 0 \ , \ \textit{with } J \textit{ an index set} \ \right\}$$

such that,

- (•) *the map,*

$$\sigma_{\alpha|Int(\Delta^n)} \colon Int(\Delta^n) \longrightarrow X$$

is injective, and each point of x is in the image of exactly one such restriction $\sigma_{\alpha|Int(\Delta^n)}$.

- (•) *For each $\sigma_\alpha \in C(X)$, each face $\sigma_\alpha \circ d^i \in C(X)$.*
- (•) *A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each $\sigma_\alpha \in C(X)$.*

I define,

$$C_n^\Delta(X) = \left\{ \ \sigma_\alpha \colon \Delta^n \longrightarrow X \quad | \quad \alpha \in J_n \ \right\} \subseteq C(X)$$

with $d_i \colon C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$ given by,

$$d_i(\sigma_\alpha) = \sigma_\alpha \circ d^i$$

for $0 \leq i \leq n$

It holds that $C^\Delta(X) = \{ \ C_n^\Delta(X) \ \}_{n \geq 0}$ is a $\Delta - set$

Now,

Definition 29. *(Geometric Realization)*
Let K be a Δ-set.
The geometric realization $|K|$ of K is defined to be,

$$|K| = \coprod_{\substack{x \in K_n \\ n \geq 0}} (\Delta^n, x) \ / \sim \ = \ \coprod_{n=0}^{\infty} \Delta^n \times K_n \ / \ \sim$$

where (Δ^n, x) is Δ^n labeled by $x \in K_n$ and $\sim$ is generated by,

$$(z, d_i x) \sim (d^i z, x)$$

for any $x \in K_n$ and $z \in \Delta^{n-1}$ labeled by $d_i x$ and

$$(z, s_i x) \sim (s^i z, x)$$

for any $x \in K_n$ and $z \in \Delta^{n+1}$ labeled by $s_i x$.

Note that the points in $(\Delta^{n+1}, s_i x)$ and $(\Delta^{n-1}, d_i x)$ are identified with the points in (Δ^n, x).

I can note that $|\Delta[n]| \cong \Delta^n$.

Now, for any $x \in K_n$, let $\sigma_x \colon \Delta^n = (\Delta^n, x) \longrightarrow |K|$ be the canonical characteristic map.

The topology on K is defined by the fact that $A \subseteq |K|$ is open if and only if the pre-image $\sigma_x^{-1}(A)$ is open in Δ^n for any $x \in K_n$ and $n \geq 0$.

Of course, given K a Δ-set, it holds that $|K|$ is a Δ-complex.

With all those previous definitions, now, I can define,

Definition 30. *(Simplicial Set)*

A simplicial set means a Δ-set X together with a collection of degeneracies $s_i \colon X_n \longrightarrow X_{n+1}$, $0 \leq i \leq n$, such that,

$$\begin{aligned} d_j \circ d_i &= d_{i-1} \circ d_j \qquad for\ j < i \\ s_j \circ s_i &= s_{i+1} \circ s_j \qquad for\ j \leq i \end{aligned}$$

and

$$d_j \circ s_i \quad = \begin{cases} s_{i-1} \circ d_j & if\ j < i \\ \\ Id & if\ j = i, i+1 \\ \\ s_i \circ d_{j-1} & if\ j > i+1 \end{cases} \tag{30.1}$$

The three identities for $d_i \circ d_j$, $s_j \circ s_i$ and $d_i \circ s_j$ detailed in identities(30.1) are called the simplicial identities.

I can use deleting-doubling for catching simplicial identities:

$$\begin{aligned} d_i &\colon (x_0, x_1, \cdots, x_n) \longrightarrow (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \\ s_i &\colon (x_0, x_1, \cdots, x_n) \longrightarrow (x_0, x_1, \cdots, x_{i-1}, x_i, x_i, x_{i+1}, \cdots, x_n) \end{aligned}$$

Let $\mathcal{O}$ be the category whose objects are finite ordered sets and whose morphisms are functions $f \colon X \longrightarrow Y$ such that $f(x) \leq f(y)$ if $x < y$.

The objects in $\mathcal{O}$ are given by $[n] = \{0, \cdots, n\}$ for $n \geq 0$, which are the same as the objects in $\mathcal{O}^+$.

So, the morphisms in $\mathcal{O}$ are generated by d^i, which is defined in $\mathcal{O}^+$, and the following morphism,

$$s^i \colon [n+1] \longrightarrow [n]$$

$$s^i = \begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n+1 \\ \\ 0 & 1 & \cdots & i-1 & i & i & i+1 & \cdots & n \end{pmatrix}$$

for $0 \leq i \leq n$, that is, s^i hits i twice.

More abstractly I have the definition of simplicial objects over any category.

Definition 31. *(Simplicial Object over a Category)*

For any category $\mathcal{C}$, a simplicial object over $\mathcal{C}$ means a contravariant functor from $\mathcal{O}$ to $\mathcal{C}$.

In other words, a simplicial object over $\mathcal{C}$ means a sequence of objects over $\mathcal{C}$,

$$X = \{X_n\}_{n \geq 0}$$

with face morphisms

$$d_i \colon X_n \longrightarrow X_{n-1}$$

and degeneracy morphisms

$$s_i \colon X_n \longrightarrow X_{n+1}, \quad 0 \leq i \leq n,$$

such that the three simplicial identities in identities(30.1) hold.

Definition 32. *(Simplicial Map)*

A simplicial map (simplicial morphisms) $f \colon X \longrightarrow Y$ means a sequence of morphisms $f_n \colon X_n \longrightarrow Y_n$ for each $n \geq 0$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$.

That is, the following diagram commutes.

$$\begin{tikzcd}
X_{n+1} \arrow[d, "f_{n+1}"'] & X_n \arrow[l, "s_i"'] \arrow[r, "d_i"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
Y_{n+1} & Y_n \arrow[l, "s_i"] \arrow[r, "d_i"'] & Y_{n-1}
\end{tikzcd}$$

Moreover, if each X_n is a subset of Y_n such that the inclusions $X_n \subseteq Y_n$ form a simplicial map, then X is called a simplicial subset of Y.

Definition 33. *(Geometric Realization of a Simplicial Map)*
Let $f\colon X \longrightarrow Y$ be a simplicial map.
Then its geometric realization $|f|$ is defined by,

$$|f|(z, x) = (z, f(x))$$

for any $x \in X_n$ and $z \in \Delta^n$ labeled by x.
Clearly $|f|$ is continuous.

A simplicial set X is said to be isomorphic to a simplicial set Y, and denoted $X \cong Y$, if there is a bijective simplicial map $f\colon X \longrightarrow Y$.
The n-simplex $\Delta[n]$, as a simplicial set, is as follows,

$$\Delta[n]_k = \{\ (i_0, i_1, \cdots, i_k) \quad | \quad 0 \leq i_0 < i_1 < \cdots < i_k \leq n\ \} \qquad \textit{for } k \leq n$$

The face $d_j\colon \Delta[n]_k \longrightarrow \Delta[n]_{k-1}$ is given by,

$$d_j(i_0, i_1, \cdots, i_k) = (i_0, i_1, \cdots, \widehat{i_j}, \cdots, i_k)$$

that is, deleting i_j.
The degeneracy $s_j\colon \Delta[n]_k \longrightarrow \Delta[n]_{k-1}$ is defined by,

$$s_j(i_0, i_1, \cdots, d_k) = (i_0, i_1, \cdots, i_j, i_j, \cdots, i_k)$$

that is doubling i_j.
Let $\sigma_n = (0, 1, \cdots, n) \in \Delta[n]_n$.
Then any element in $\Delta[n]$ can be written as iterated compositions of faces and degeneracies of σ_n.
Again, it holds that given X be a simplicial set and given any element $x \in X_n$, then there exists a unique simplicial map,

$$f_x\colon \Delta[n] \longrightarrow X$$

such that $f_x(\sigma_n) = x$.
Let X be a simplicial set ad $A = \{A_n\}_{n \geq 0}$ with $A_n \subseteq X_n$. The simplicial subset of X generated by A is defined by

$$\langle A \rangle = \bigcap \{\ A \subseteq Y \subseteq X \quad | \quad Y \textit{ is a simplicial subset of } X\ \},$$

namely $\langle A \rangle$ consits of elements in X that can be written as iterated compositions of faces and degeneracies of the elements in A.
In particular, the simplicial n-sphere $\mathbb{S}^n$ is defined by,

$$\mathbb{S}^n = \Delta[n]/\partial(\Delta[n]),$$

where $\partial(\Delta[n])$ is the simplicial subset of $\Delta[n]$ generated by $\Delta[n]_k$ for $k < n$.
I can write explicitly for the elements in the simplicial circle $\mathbb{S}^1$.

$$\Delta[1]_k = \{\ (i_0, \cdots, i_k) \mid 0 \leq i_0 \leq \cdots \leq i_k \leq 1\ \} = \{\ (\overbrace{0, \cdots, 0}^{i}, 1, \cdots, 1) \mid 0 \leq i \leq k+1\ \}$$

has $k + 2$ elements.
Now,

$$\partial(\Delta[1])_k = \{\ (0, \cdots, 0), (1, \cdots, 1)\ \}$$

By definition, $\mathbb{S}^1 = \Delta[1]/\partial(\Delta[1])$. Thus,

$$\mathbb{S}^1_k = \{\ *, (i_0, \cdots, i_k) \quad | \quad 0 \le i_0 \le \cdots \le i_k \le 1\ \} = \{\ (\overbrace{0, \cdots, 0}^{i}, 1, \cdots, 1) \quad | \quad 1 \le i \le k\ \}$$

has $k+1$ elements including the basepoint $* = (0, \cdots, 0) \sim (1, \cdots, 1)$.

For a general simplicial n-sphere $\mathbb{S}^n$, I have $\mathbb{S}^n_k = \{*\}$ for $k < n$ and

$$\mathbb{S}^n_k = \{\ *, (i_0, \cdots, i_k) \quad | \quad 0 \le i_0 \le \cdots \le i_k \le n \quad \mathit{with}\ \{i_0, \cdots, i_k\} \ = \ \{0, 1, \cdots, n\}\ \}$$

for $k \ge n$.

I can also define the cartesian product.

Let X and Y be simplicial sets.

I define $X \times Y$ by setting,

$$(X \times Y)_n = X_n \times Y_n$$

with $d_i^{X \times Y} = (d_i^X, d_i^Y)$ and $s_i^{X \times Y} = (s_i^X, s_i^Y)$.

$X \times Y$ is a simplicial set, too.

Let $f\colon X \times Y$ be a simplicial map. Then,

$$Im(f\colon X \longrightarrow Y) = \{\ Im(f\colon X_n \longrightarrow Y_n)\ \}_{n \ge 0}$$

is a simplicial subset of Y.

For a sequence of nonnegative integers $I = (i_1, i_2, \cdots, i_k)$ of lenght $k = l(I)$, I denote $d_I = d_{i_1} \cdots d_{i_k}$ and $s_I = s_{i_1} \cdots s_{i_k}$.

In order to properly work with a pointed simplicial set, I can briefly add the following facts.

Let X be a simplicial set and let $x_0 \in X_0$.

Then the image of the representing map,

$$f_{x_0}\colon \Delta[0] \longrightarrow X$$

is a simplicial subset of X consisting of only one element $f_{x_0}(0, \cdots, 0) = s_I(x_0)$ in each dimension. Thus the base point $*$ of X means a sequence of elements,

$$\{\ f_{x_0}(\overbrace{0, \cdots, 0}^{n+1})\ \}_{n \ge 0}$$

corresponds to the elements $x_0 \in X_0$.

A pointed simplicial set means a simplicial set with a given base point. A pointed simplicial map means a simplicial map that preserves the base points.

A simplicial set is called reduced if X_0 has only one element. For a reduced simplicial set, there is a unique choice of base points. Moreover any simplicial map between reduced simplicial sets is pointed.

Proposition 34. *Let X be a simplicial set. Then $|X|$ is a CW complex.*

Thus the geometric realization gives a functor from the category of simplicial sets to the category of CW complexes.

Proof. From the push-out diagram,

$$\begin{tikzcd}
\coprod\limits_{\substack{x \in X_n \\ nondegenerate}} \partial\Delta[n] \arrow[r, "\coprod f_x|_{\partial\Delta[n]}"] \arrow[d, hook] & sk_{n-1}X \arrow[d, hook] \\
\coprod\limits_{\substack{x \in X_n \\ nondegenerate}} \Delta[n] \arrow[r, "\coprod f_x"] & sk_n X
\end{tikzcd}$$

there is a push-out diagram,

$$\begin{tikzcd}
\coprod\limits_{\substack{x\in X_n \\ nondegenerate}} |\partial\Delta[n]| = \partial|\Delta^n| \arrow[r, "\coprod |f_x|_{\partial\Delta[n]}|"] \arrow[d, hook] & |sk_{n-1}X| \arrow[d, hook] \\
\coprod\limits_{\substack{x\in X_n \\ nondegenerate}} |\Delta[n]| = \Delta^n \arrow[r, "\coprod |f_x|"] & |sk_n X|
\end{tikzcd}$$

Thus $|X|$ is obtained by an attaching cell by cell process and so $|X|$ is a CW complex. □

A notion of homotopy can be defined for all simplicial sets. However, it does not behave well in general. To avoid this problem, I can define a certain type of simplicial sets, called Kan complexes, for which homotopy is well-behaved.

Recall that the category of simplicial sets $sSet$ is the functor category $Fun(\Delta^{op}, Set)$. Then, I can view Δ^n as $\Delta^n = Fun(\bullet, [n])$ the simplicial set represented by $[n]$.

I have previously defined $\partial\Delta^n$, the boundary of Δ^n, as the simplicial subset of Δ^n generated by $d_i(Id_{[n]})$ for $1 \leq i \leq n$.

That is, I have $\partial\Delta^n \subseteq \Delta^n$ is the simplicial subset obtained by removing the interior, namely the n-simplex defined by $Id_{[n]} \colon [n] \longrightarrow [n]$. In particular, $\partial\Delta^0 = \emptyset$.

Definition 35. *(Simplicial Subset Generated by)*

Let $A \subset X$ be a subset of a simplicial set X. The simplicial subset of X generated by A is defined to be the smallest simplicial subset of X containing A. Explicitly, it is the simplicial set consisting of all combinations of faces and degeneracies of elements of A.

Definition 36. *(The k-th Horn)*

The k-th horn Λ^n_k is the simplicial subset of Δ^n generated by $d_i(Id_[n])$ for $i \neq k$.

Therefore, Λ^n_k is obtained from Δ^n by removing the interior and the k-th face.

or equivalently,

For each $0 \leq k \leq n$, $\Lambda^n_k \subseteq \partial\Delta^n$ is the simplicial subset obtained by removing the face opposite to the k-th vertex, namely the $(n-1)$-simplex defined by $d^n_k \colon [n-1] \longrightarrow [n]$.

Definition 37. *(Kan Complex)*

A Kan complex is a simplicial set C such that $C \longrightarrow \Delta^0$ has the right lifting property with respect to all inclusions $\Lambda^n_k \subseteq \Delta^n$ with $0 \leq k \leq n$.

In other words, a simplicial set C is a Kan complex if and only if every map $\Lambda^n_k \longrightarrow C$ with $0 \leq k \leq n$ can be extended to a map $\Delta^n \longrightarrow C$.

The lifting property defining a Kan complex can be adapted to the relative case. More precisely,

Definition 38. *(Kan Fibration)*

A map $f \colon : X \longrightarrow Y$ of simplicial sets is called a Kan fibration if it has the right lifting property with respect to all inclusions $\Lambda^n_k \subseteq \Delta^n$ with $0 \leq k \leq n$.

That is in the following commutative diagram the dotted arrow exists.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists\ \tilde{f}"] & Y
\end{tikzcd}$$

Simplicial Model Categories.

.

It is often the case that the set of maps assemble to a function space (or simplicial set) and that the model structure conforms nicely with the homotopy theory of these function spaces. In these cases many things become somewhat more transparent.

Definition 39. *(S-Category)*
A S-category $\mathcal{C}$ is a class of "objects" Ob $\mathcal{C}$ together with a simplicial set

$$\underline{\mathcal{C}}(A,X) \in S$$

for each pair of objects $A, X \in Ob\ \mathcal{C}$ with a unital and associative composition (the usual axioms for a category, just allowing morphism sets to be simplicial sets).

I can be more precise,
A S-category $\mathcal{C}$ consists of a class O (the objects) and a function that assigns to each ordered pair X , $Y \in O$ a simplicial set $hom(X,Y)$ plus simplicial maps

$$C_{X,Y,Z} \colon hom(X,Y) \times hom(Y,Z) \longrightarrow hom(X,Z)$$

and

$$I_X \colon \Delta[0] \longrightarrow hom(X,X)$$

satisfying the following conditions,
The diagram,

$$\begin{tikzcd}
hom(X,Y) \times (hom(Y,Z) \times hom(Z,W)) \arrow[r, "Id\times C"] \arrow[d, "A"'] & hom(X,Y) \times hom(Y,W) \arrow[dd, "C"] \\
(hom(X,Y) \times hom(Y,Z)) \times hom(Z,W) \arrow[d, "C\times Id"'] & \\
hom(X,Z) \times hom(Z,W) \arrow[r, "C"'] & hom(X,W)
\end{tikzcd}$$

commutes.
and the diagram,

$$\begin{tikzcd}
\Delta[0] \times hom(X,Y) \arrow[r, "L"] \arrow[d, "I\times Id"'] & hom(X,Y) \arrow[d, equal] & hom(X,Y) \times \Delta[0] \arrow[l, "R"'] \arrow[d, "Id\times I"] \\
hom(X,X) \times hom(X,Y) \arrow[r, "C"'] & hom(X,Y) & hom(X,Y) \times hom(Y,Y) \arrow[l, "C"]
\end{tikzcd}$$

commutes.
I can note that,
An S-category $\mathcal{C}$ has an underlying category $\mathcal{C}$ by letting

$$\mathcal{C}(A,X) = \underline{\mathcal{C}}(A,X)_0$$

I note that it is possible to replace S by similar structures like S_* or A, giving rise to parallel theories.

Definition 40. *(S-Functor)*
I suppose that $\mathcal{C}$ and $\mathcal{C}'$ are S-categories with object classes O and O'.
A S-functor $F \colon \mathcal{C} \longrightarrow \mathcal{C}'$ is the specification of a rule that assigns to each object $X \in O$ an object $FX \in O'$ and the specification of a rule that assigns to each ordered pair X, $Y \in O$ a morphism,

$$F_{X,Y} \colon hom(X,Y) \longrightarrow hom(FX,FY)$$

of simplicial sets such that the diagram,

$$\begin{tikzcd}
hom(X,Y) \times hom(Y,Z) \arrow[r, "C"] \arrow[d, "F_{X,Y}\times F_{Y,Z}"'] & hom(X,Z) \arrow[d, "F_{X,Z}"] \\
hom(FX,FY) \times hom(FY,FZ) \arrow[r, "C"'] & hom(FX,FZ)
\end{tikzcd}$$

commutes and the equality $F_{X,X} \circ I_X = I_{FX}$ obtains.

I can note that the underlying functor $UF\colon U\mathcal{C} \longrightarrow U\mathcal{C}'$ sends X to FX and $f\colon \Delta[0] \longrightarrow hom(X, Y)$ to $F_{X,Y} \circ f_\bullet$

I can also note that the opposite of a S-functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor $F^{op}\colon \mathcal{C}^{op} \longrightarrow \mathcal{C}'^{op}$.

Definition 41. *(Tensored and Cotensored S-Category)*
A S-category is tensored and cotensored if for all $S \in \mathcal{C}$ the functor

$$Y \longmapsto \underline{\mathcal{C}}(X, Y)$$

has a left adjoint $K \longmapsto X \otimes X$, and if $K \longmapsto X \otimes K$ has a right adjoint (everything natural).

Definition 42. *(Simplicial Model Category)*
A simplicial model category is a tensored and cotensored S-category $\mathcal{M}$ with a model category structure on the underlying category, such that for all cofibrations $i\colon A \rightarrowtail B$ and fibrations $p\colon X \twoheadrightarrow Y$, the canonical map

$$(i,p)_*\colon \underline{\mathcal{M}}(B, X) \longrightarrow \underline{\mathcal{M}}(B, Y) \times_{\underline{\mathcal{M}}(A,Y)} \underline{\mathcal{M}}(A, X) \in S$$

is a fibration, and that furthermore, if in addition either i or p are weak equivalences, then so is $(i, p)_*$

Remark 43. *Daniel Quillen referred to the "tensored and cotensored" part as axiom SM0 and to the condition on the map $(i, p)_*$ as axiom SM7. The terminology "simplicial model category" is maybe not quite proper, as I would think that it referred to a functor from Δ^{op} to some category of model categories, but the terminology is well established.*

Remark 44. *In simplicial model categories I have the notion of (simplicial) homotopy. This means that I have means of detecting weak equivalences at the function space level.*

Proposition 45. *Let $\mathcal{M}$ be a simplicial model category. A map $f\colon Y \longrightarrow Z$ is a weak equivalence if either of the following conditions are satisfied:*

(1) *For every fibrant object $X \in \mathcal{M}$ the induced map*

$$\underline{\mathcal{M}}(Z, X) \longrightarrow \underline{\mathcal{M}}(Y, X) \in S$$

is a weak equivalence.

(2) *For every cofibrant object $A \in \mathcal{M}$ the induced map*

$$\underline{\mathcal{M}}(A, Y) \longrightarrow \underline{\mathcal{M}}(A, Z) \in S$$

is a weak equivalence.

In the case where both Y and Z are cofibrant the first condition is necessary and sufficient. Likewise with fibrant versus the latter condition.

Whitehead's Theorem in Model Categories

.

I start with a extremly useful result,

From the functorial factorization I get two functors that replace each object by a fibrant (resp. cofibrant) one, and I get a morphism of functors from the identity to each of these functors, which consist of weak equivalences. These are called fibrant and cofibrant replacements.

Theorem 11. *(Theorem of Approximation by Fibrant and Cofibrant Objects)*
Let $\mathcal{M}$ be a model category.

(1) *For any object $X \in \mathcal{M}$, there is a cofibrant object $QX \in \mathcal{M}$ (with $QX = X$ whether X cofibrant) and a weak equivalence*

$$\gamma\colon QX \longrightarrow X.$$

which in fact is a trivial fibration.

Moreover, it holds that for a map

$$\varphi\colon X \longrightarrow Y$$

and another such cofibrant approximation for the object Y, that is a cofibrant object QY and a weak equivalence,

$$\rho\colon QY \longrightarrow Y,$$

there is a morphism,

$$\widetilde{\varphi}\colon \widetilde{QX} \longrightarrow \tilde{QY},$$

which is unique up to homotopy, such that the following diagram is homotopic commutative,

$$\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \\
QX \arrow[u, "\gamma"] \arrow[r, "\widetilde{\varphi}"'] & QY \arrow[u, "\rho"']
\end{tikzcd}$$

QX is called de cofibrant approximation of X.

Dually,

(2) *For any object $X \in \mathcal{M}$, there is a fibrant object $RX \in \mathcal{M}$ (with $RX = X$ whether X fibrant) and a weak equivalence*

$$\lambda\colon X \longrightarrow RX.$$

which in fact is a trivial cofibration.

Moreover, it holds that for a map

$$\phi\colon X \longrightarrow Y$$

and another such fibrant approximation for the object Y, that is a fibrant object RY and a weak equivalence,

$$\mu\colon Y \longrightarrow RY,$$

there is a morphism,

$$\widetilde{\phi}\colon RX \longrightarrow RY,$$

which is unique up to homotopy, such that the following diagram is homotopic commutative,

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\lambda"'] & Y \arrow[d, "\mu"] \\
RX \arrow[r, "\widetilde{\phi}"'] & RY
\end{tikzcd}$$

RX is called de fibrant approximation of X.
RQX is called de fibrant + cofibrant approximation of X.
In addition, RX, QX and RQX endow functors $R\colon \mathcal{M} \longrightarrow \mathcal{M}_f$, $Q\colon \mathcal{M} \longrightarrow \mathcal{M}_c$ and $RQX\colon \mathcal{M} \longrightarrow \mathcal{M}_{fc}$ respectively.

Proof. For each object $X \in \mathcal{M}$, I choose both,

a trivial fibration $\gamma\colon QX \longrightarrow X$ with QX a cofibrant object, just applying the M5 axiom (factorization axiom) to the map $\emptyset \longrightarrow X$ (the initial object $\emptyset$ exists by M1 axiom (limit axiom)),

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr] & & X \\
& QX \arrow[ur, "\sim", "\gamma"'] &
\end{tikzcd}$$

I take $QX = X$ and $\gamma = Id_X$ whether X is cofibrant,

and a trivial cofibration $\lambda\colon X \longrightarrow RX$ with RX a fibrant object, just applying the M5 axiom (factorization axiom) to the map $X \longrightarrow *$ (the final object $*$ exists by M1 axiom (limit axiom)),

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, tail, "\sim", "\lambda"'] & & * \\
& RX \arrow[ur, two heads] &
\end{tikzcd}$$

and I take $RX = X$ and $\lambda = Id_X$ whether X is fibrant.

Now, for each morphism $\varphi\colon X \longrightarrow Y$, I can choose a morphism $\widetilde{\varphi}\colon QX \longrightarrow QY$ such that $\rho \circ \widetilde{\varphi} = \varphi \circ \gamma$.

Indeed, since $\emptyset \rightarrowtail QX$ is a cofibration and γ is a trivial fibration, by the M4 axiom (lifting axiom) there exists $\widetilde{\varphi}$ in the diagram,

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & QY \arrow[d, "\rho"] \\
QX \arrow[r, "\varphi\circ\gamma"'] \arrow[ur, dashed, "\widetilde{\varphi}"] & Y
\end{tikzcd}$$

Now, I know that $\widetilde{\varphi}$ is unique out of a left homotopy by Proposition 58.

Similarly, for each morphism $\phi\colon X \longrightarrow Y$ I can choose a morphism $\widetilde{\phi}\colon RX \longrightarrow RY$ such that $\widetilde{\phi} \circ \lambda = \mu \circ \phi$.

Indeed, since $RX \longrightarrow *$ is a fibration and λ is a trivial cofibration, by the M4 axiom (lifting axiom) there exists $\widetilde{\phi}$ in the diagram,

$$\begin{tikzcd}
X \arrow[r, "\mu\circ\phi"] \arrow[d, tail, "\lambda"'] & RY \arrow[d] \\
RX \arrow[r, two heads] \arrow[ur, dashed, "\widetilde{\phi}"] & *
\end{tikzcd}$$

Now, $\widetilde{\phi}$ is unique out of a right homotopy by Proposition 58.

I want to show that RX, QX and RQX endow functors $R\colon \mathcal{M} \longrightarrow \mathcal{M}_f$, $Q\colon \mathcal{M} \longrightarrow \mathcal{M}_c$ and $RQ\colon \mathcal{M} \longrightarrow \mathcal{M}_{fc}$ respectively.

If $\psi\colon Y \longrightarrow Z$, then I can consider the following commutative diagram,

$$\begin{tikzcd}
Y \arrow[r, "\psi"] & Z \\
QY \arrow[u, "\nu"] \arrow[r, "\widetilde{\psi}"'] & QZ \arrow[u, "\eta"']
\end{tikzcd}$$

I can so consider the commutative diagram too,

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi\circ\varphi"] & Y \arrow[r, "\psi"] & Z \\
QX \arrow[u, "\gamma"'] \arrow[r, "\widetilde{\varphi}"'] \arrow[rr, bend right, "\widetilde{\psi\circ\varphi}"'] & QY \arrow[u, "\rho"'] \arrow[r, "\widetilde{\psi}"'] & QZ \arrow[u, "\eta"']
\end{tikzcd}$$

Since it holds, $\eta \circ \widetilde{\psi} \circ \widetilde{\varphi} = \psi \circ \rho \circ \widetilde{\varphi} = \psi \circ \varphi \circ \gamma$, then it follows that $\widetilde{\psi \circ \varphi} \overset{l}{\simeq} \widetilde{\psi} \circ \widetilde{\varphi}$ and since $\gamma \circ Id_{QX} = Id_X \circ \gamma$ I obtain that $QId_X \overset{l}{\simeq} Id_{QX}$.

Then, $\widetilde{\psi \circ \varphi} \overset{r}{\simeq} \widetilde{\psi} \circ \widetilde{\varphi}$ and $QId_X \overset{r}{\simeq} Id_{QX}$ by Proposition 61.

Hence, I have constructed a well defined functor $Q\colon \mathcal{M} \longrightarrow \mathcal{M}_c$.

with an analogous reasoning,

If $\varsigma\colon Y \longrightarrow Z$, then I can consider the following commutative diagram,

$$\begin{tikzcd}
Y \arrow[r, "\varsigma"] \arrow[d, "\mu"'] & Z \arrow[d, "\delta"] \\
RY \arrow[r, "\widetilde{\varsigma}"'] & RZ
\end{tikzcd}$$

I can so consider the commutative diagram too,

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\lambda"'] \arrow[rr, bend left, "\varsigma\circ\phi"] & Y \arrow[r, "\varsigma"] \arrow[d, "\mu"'] & Z \arrow[d, "\delta"] \\
RX \arrow[r, "\widetilde{\phi}"'] \arrow[rr, bend right, "\widetilde{\varsigma\circ\phi}"'] & RY \arrow[r, "\widetilde{\varsigma}"'] & RZ
\end{tikzcd}$$

Since it holds, $\widetilde{\varsigma} \circ \widetilde{\phi} \circ \lambda = \widetilde{\varsigma} \circ \mu \circ \phi = \delta \circ \varsigma \circ \phi$, then it follows that $\widetilde{\varsigma \circ \phi} \overset{l}{\simeq} \widetilde{\varsigma} \circ \widetilde{\phi}$ and since $\lambda \circ Id_{RX} = Id_X \circ \lambda$, I obtain that $RId_X \overset{l}{\simeq} Id_{RX}$.

Then, $\widetilde{\varsigma \circ \phi} \overset{r}{\simeq} \widetilde{\varsigma} \circ \widetilde{\phi}$ and $RId_X \overset{r}{\simeq} Id_{RX}$ also by Proposition 61.

Hence and again, I have constructed a well defined functor $R\colon \mathcal{M} \longrightarrow \mathcal{M}_f$.

Finally, is X is cofibrant then $\varphi, \phi \in Hom(X, Y)$ and $\varphi \overset{r}{\simeq} \phi$, and by Lemma 65 $\mu\circ\varphi \overset{r}{\simeq} \mu\circ\phi$ and then by Proposition 67 , I obtain $\widetilde{\varphi} \overset{r}{\sim} \widetilde{\phi}$. Therefore, the functor R restricted to $\mathcal{M}_C$ endows a functor $\mathcal{M}_c \longrightarrow \mathcal{M}_{fc}$. (I note that whether X cofibrant, so RX is cofibrant too since $\emptyset \longrightarrow X \longrightarrow RX$ is a cofibration).

Hence, I can conclude that there exists a well defined functor,

$$RQ\colon \mathcal{M} \longrightarrow \mathcal{M}_{fc}$$

given by,

$$X \mapsto RQX$$

$$\varphi \mapsto \widetilde{RQ}$$

□

The classical Whitehead Theorem asserts that any weak equivalence $f\colon X \longrightarrow Y$ between CW complexes is a homotopy equivalence.

CW complexes are spaces in which all their objects are both fibrant and cofibrant objects.

The analogue statement for the Whitehead Theorem in an arbitrary closed model category is the following:

Theorem 12. *(Whitehead)*

I suppose that $f\colon X \longrightarrow Y$ is a morphism defined in a closed model category $\mathcal{M}$ such that the objects X and Y are both fibrant and cofibrant.

I also suppose that f is a weak equivalence.

Then the map f is a homotopy equivalence.

Proof. Every weak equivalence between fibrant + cofibrant objects admits by axiom M5 (factorization axiom) a functorial factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd}$$

in which i is a cofibration and p is a trivial fibration (and so is both a fibration and a weak equivalence as well).

But now, if Y is fibrant and p is a fibration then Z is fibrant.

In an analogous way, if X is cofibrant and i is a cofibration then Z is cofibrant.

Thus, the object Z is also both fibrant and cofibrant.

Now, according the statement, f and p are weak equivalences and so, by M2 axiom (two out of three axiom), I obtain that also i will be a weak equivalence and therefore, since i is moreover a cofibration, then i is an acyclic cofibration.

I know that the composition of homotopy equivalences between fibrant + cofibrant objects is also a homotopy equivalence by Proposition 69, then if now I could be able to prove that any trivial cofibration i (or a trivial fibration p) betwen cofibrant + fibrant objects is a homotopy equivalence, then I could directly deduce the desired proof for the theorem.

I will prove first for the trivial cofibration i and secondly for the trivial fibration p.

Previously I want to add two remarks.

For the first remark, I recall that in a general model category $\mathcal{M}$ the coproduct of any objects A, B in $\mathcal{M}$ denoted by $A \amalg B$ is characterized (up to isomorphism) by the universal property,

There exists morphisms,

$$\begin{tikzcd}
A \arrow[rd, "k_0"] & \\
& A \amalg B \\
B \arrow[ru, "k_1"'] &
\end{tikzcd}$$

such that, for any morphisms, φ_0, φ_1, there exists a unique morphism φ (which could be also denoted by $\varphi = \varphi_0 \amalg \varphi_1$) from the coproduct $A \amalg B$ to any other object C in the category $\mathcal{M}$, as I show in the following diagram,

$$\begin{tikzcd}
A \arrow[rd, "k_0"] \arrow[rrd, bend left, "\varphi_0"] & & \\
& A \amalg B \arrow[r, "\varphi=\varphi_0 \amalg \varphi_1"] & C \\
B \arrow[ru, "k_1"'] \arrow[rru, bend right, "\varphi_1"'] & &
\end{tikzcd}$$

such that

$$\varphi \circ k_0 = \varphi_0 \text{ equivalently } (\varphi_0 \amalg \varphi_1) \circ k_0 = \varphi_0$$

$$\varphi \circ k_1 = \varphi_1 \text{ equivalently } (\varphi_0 \amalg \varphi_1) \circ k_1 = \varphi_1$$

and therefore, all the triangles commute.

So that, in particular, in a general model category $\mathcal{M}$, for a fibrant + cofibrant object B, the coproduct $B \amalg B$ is a colimit for the diagram,

$$\begin{tikzcd}
& B \\
\emptyset \arrow[ru, "j_B"] \arrow[rd, "j_B"'] & \\
& B
\end{tikzcd}$$

where $\emptyset$ is an initial object.

So that, by the universal property of the coproduct I can consider the diagram,

$$\begin{tikzcd}
& B \arrow[dr, "j_0"] \arrow[drr, bend left, "Id_B"] & & \\
\emptyset \arrow[ur, "j_B"] \arrow[dr, "j_B"'] & & B \amalg B \arrow[r, "Id_B \amalg Id_B"] & B \\
& B \arrow[ur, "j_1"'] \arrow[urr, bend right, "Id_B"'] & &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

Also I want to recall that in a general model category $\mathcal{M}$, I define a cylinder object for $B \in \mathcal{M}$ to be the factorization for the map,

$$B \amalg B \xrightarrow{Id_B \amalg Id_B} B$$

as

$$B \amalg B \rightarrowtail Cyl(B) \xrightarrow{\sim} B$$

where $B \amalg B \rightarrowtail Cyl(B)$ is a cofibration and $Cyl(B) \xrightarrow{\sim} B$ is a trivial fibration.

So that I have chosen the commutative diagram,

$$\begin{tikzcd}
B \amalg B \arrow[r, "Id_B \amalg Id_B"] \arrow[d] & B \\
Cyl(B) \arrow[ur] &
\end{tikzcd}$$

I can observe that a cylinder object is not an unique object, not yet well defined, it is merely a choice (which I make functorial for each B) of a factorization of $Id_B \amalg Id_B$ into a cofibration and a trivial fibration.

By the universal property of the coproduct I get two maps $i_0, i_1\colon B \longrightarrow Cyl(B)$, so that I define the cofibration in the initial factorization as $i_0 \amalg i_1$

So, I have,

$$\begin{tikzcd}
B \amalg B \arrow[r, "Id_B \amalg Id_B"] \arrow[d, "i_0 \amalg i_1"'] & B \\
Cyl(B) \arrow[ur] &
\end{tikzcd}$$

and I have,

$$\begin{tikzcd}
& B \arrow[dr, "j_0"] \arrow[drr, bend left, "i_0"] & & \\
\emptyset \arrow[ur, "j_B"] \arrow[dr, "j_B"'] & & B \amalg B \arrow[r, "i_0 \amalg i_1"] & Cyl(B) \\
& B \arrow[ur, "j_1"'] \arrow[urr, bend right, "i_1"'] & &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

Then, I define a (left) homotopy between

$$f, g\colon B \longrightarrow C$$

to be a map,

$$h\colon Cyl(B) \longrightarrow C$$

such that

$$f = h \circ i_0$$

and

$$g = h \circ i_1.$$

Equivalently, h is such that $h \circ (i_0 \amalg i_1) = f \amalg g$.

For the second remark and in an analogous way, I want to recall that in a general model category $\mathcal{M}$ the product of any objects A, B in $\mathcal{M}$ denoted by $A \times B$ is characterized (up to isomorphism) by the universal property,

There exists morphisms,

$$\begin{tikzcd}
 & A \\
A \times B \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & B
\end{tikzcd}$$

such that, for any morphisms, ϕ_0, ϕ_1, there exists a unique morphism ϕ (which could be also denoted by $\varphi = \varphi_0 \times \varphi_1$) from any other object C in the category $\mathcal{M}$ to the product $A \times B$, as I show in the following diagram,

$$\begin{tikzcd}
 & & A \\
C \arrow[r, "\phi=\phi_0\times\phi_1"] \arrow[urr, bend left, "\phi_0"] \arrow[drr, bend right, "\phi_1"'] & A \times B \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & & B
\end{tikzcd}$$

such that

$$l_0 \circ \phi = \phi_0 \text{ equivalently } l_0 \circ (\phi_0 \times \phi_1) = \phi_0$$

$$l_1 \circ \phi = \phi_1 \text{ equivalently } l_1 \circ (\phi_0 \times \phi_1) = \phi_1$$

and therefore, all the triangles commute.

So that, in particular, in a general model category $\mathcal{M}$, for a fibrant + cofibrant object B, the product $B \times B$ is a limit for the diagram,

$$\begin{tikzcd}
B \arrow[dr, "q_B"] & \\
 & * \\
B \arrow[ur, "q_B"'] &
\end{tikzcd}$$

where $*$ is a terminal object.

So that, by the universal property of the product I can consider the diagram,

$$\begin{tikzcd}
 & & B \arrow[dr, "q_B"] & \\
B \arrow[r, "Id_B \times Id_B"] \arrow[urr, bend left, "Id_B"] \arrow[drr, bend right, "Id_B"'] & B \times B \arrow[ur, "q_0"] \arrow[dr, "q_1"'] & & * \\
 & & B \arrow[ur, "q_B"'] &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

Also I want to recall that in a general model category $\mathcal{M}$, I define a path object for $B \in \mathcal{M}$ to be the factorization for the map

$$B \xrightarrow{Id_B \times Id_B} B \times B$$

as

$$B \overset{\sim}{\rightarrowtail} Path(B) \twoheadrightarrow B \times B$$

where $B \overset{\sim}{\rightarrowtail} Path(B)$ is a trivial cofibration and $Path(B) \twoheadrightarrow B \times B$ is a fibration.

So that I have chosen the commutative diagram,

$$\begin{tikzcd}
 & Path(B) \arrow[d] \\
B \arrow[ur] \arrow[r, "Id_B \times Id_B"'] & B \times B
\end{tikzcd}$$

I can observe that the path object is not an unique object, not yet well defined, it is merely a choice (which I make functorial for each B) of a factorization of $Id_B \times Id_B$ into a trivial cofibration followed by a fibration.

By the universal property of the product I get two maps $p_0, p_1 \colon Path(B) \longrightarrow B$, so that I define the fibration in the initial factorization as $p_0 \times p_1$

So, I have,

$$\begin{tikzcd}
 & Path(B) \arrow[d, "p_0 \times p_1"] \\
B \arrow[ur, "\sigma"] \arrow[r, "Id_B \times Id_B"'] & B \times B
\end{tikzcd}$$

and I have,

$$\begin{tikzcd}
 & & B \arrow[dr, "q_B"] & \\
Path(B) \arrow[r, "p_0 \times p_1"] \arrow[urr, bend left, "p_0"] \arrow[drr, bend right, "p_1"'] & B \times B \arrow[ur, "q_0"] \arrow[dr, "q_1"'] & & * \\
 & & B \arrow[ur, "q_B"'] &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

Then, I define a (right) homotopy between

$$f, g \colon A \longrightarrow B$$

to be a map

$$h\colon A \longrightarrow Path(B)$$

such that

$$f = p_0 \circ h$$

and

$$g = p_1 \circ h.$$

Equivalently, such that $(p_0 \times p_1) \circ h = f \times g$.

As a final remark previous to continue the proof of the Whitehead Therorem I want to add that a homotopy equivalence, for generals objects E, $F \in \mathcal{M}$ is defined as usual, that is, considering two maps

$$h : E \longrightarrow F$$

and

$$h' : F \longrightarrow E$$

such that $h' \circ h \simeq Id_E$ and $h \circ h' \simeq Id_F$.

The maps h, h' are homotopy equivalences and h' is in fact the inverse homotopy of h.

Now, going on with my previously desired proof, and using the above introduced remarks I will prove first that any trivial cofibration i betweeen fibrant + cofibrant objects is a homotopy equivalence.

Since X and Z are fibrant objects I have a unique morphism δ (which is a fibration) from those objects to the terminal object of the category (denoted by $*$).

$$X \xrightarrow{\delta} *$$

and

$$Z \xrightarrow{\delta} *.$$

So, I can consider the commutative diagram,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "\delta"] \\
Z \arrow[r, "\delta"'] & *
\end{tikzcd}$$

in which $*$ denotes the terminal object, i is a trivial cofibration and δ is a fibration.

In fact, the fibrant objects of a closed model category are characterized by having the right lifting property with respect to any trivial cofibration in the category (in particular this property makes fibrant objects the 'correct' objects on which to define homotopy groups) and so by M4 axiom (lifting axiom), there exists a dotted arrow θ,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "\delta"] \\
Z \arrow[r, "\delta"'] \arrow[ur, dashed, "\theta"] & *
\end{tikzcd}$$

such that by construction $\theta \circ i = Id_X$.

I want to show that $i \circ \theta \simeq Id_Z$

I consider a path object for Z, that is such a factorization

$$\begin{tikzcd}
 & Path(Z) \arrow[d, "p_0 \times p_1"] \\
Z \arrow[ur, "\sigma"] \arrow[r, "Id_Z \times Id_Z"'] & Z \times Z
\end{tikzcd}$$

where $p_0 \times p_1$ is a fibration and σ is a trivial cofibration.
The diagram commutes by construction,

$$(p_0 \times p_1) \circ \sigma = Id_Z \times Id_Z$$

I consider now the diagram,

$$\begin{tikzcd}
X \arrow[r, "\sigma \circ i"] \arrow[d, "i"'] & Path(Z) \arrow[d, "p_0 \times p_1"] \\
Z \arrow[r, "Id_Z \times (i \circ \theta)"'] & Z \times Z
\end{tikzcd}$$

in which $p_0 \times p_1$ is a fibration and i is a trivial cofibration.
I want to show that this diagram commutes in order to deduce that there exists a dotted arrow h,

$$\begin{tikzcd}
X \arrow[r, "\sigma \circ i"] \arrow[d, "i"'] & Path(Z) \arrow[d, "p_0 \times p_1"] \\
Z \arrow[r, "Id_Z \times (i \circ \theta)"'] \arrow[ur, dashed, "h"] & Z \times Z
\end{tikzcd}$$

and so deduce that $i \circ \theta \simeq Id_Z$
Therefore I want to prove that,

$$(p_0 \times p_1) \circ (\sigma \circ i) = (Id_Z \times (i \circ \theta)) \circ i$$

Indeed, first of all I can observe,

$$(p_0 \times p_1) \circ (\sigma \circ i) = (((p_0 \times p_1) \circ \sigma) \circ i = (Id_Z \times Id_Z) \circ i$$

So that in order to show that the diagram commutes I will want to prove that

$$(Id_Z \times (i \circ \theta)) \circ i = (Id_Z \times Id_Z) \circ i$$

I will use the universal property that characterizes the product; I know that applying the general universal property to my particular case I have two morphisms,

$$\begin{tikzcd}
 & Z \\
Z \times Z \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & Z
\end{tikzcd}$$

such that, for any morphisms, ϕ_0, ϕ_1, there exists a unique morphism $\phi = \phi_0 \times \phi_1$ from any other object X in $\mathcal{M}$ to the product $Z \times Z$ in the diagram,

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, "\phi=\phi_0\times\phi_1"] \arrow[urr, bend left, "\phi_0"] \arrow[drr, bend right, "\phi_1"'] & Z\times Z \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & & Z
\end{tikzcd}$$

such that,

$$l_0 \circ \phi = \phi_0 \text{ equivalently } l_0 \circ (\phi_0 \times \phi_1) = \phi_0$$
$$l_1 \circ \phi = \phi_1 \text{ equivalently } l_1 \circ (\phi_0 \times \phi_1) = \phi_1$$

and therefore, all the triangles commute.
In my case I have the morphisms and such a diagram,

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, bend left, "(Id_Z\times(i\circ\theta))\circ i"] \arrow[r, bend right, "(Id_Z\times Id_Z)\circ i"'] & Z\times Z \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & & Z
\end{tikzcd}$$

In order to prove that

$$(Id_Z \times (i \circ \theta)) \circ i = (Id_Z \times Id_Z) \circ i$$

a necessary and sufficient condition to prove is the following,

$$l_0 \circ (Id_Z \times (i \circ \theta)) \circ i = l_0 \circ (Id_Z \times Id_Z) \circ i$$
$$l_1 \circ (Id_Z \times (i \circ \theta)) \circ i = l_1 \circ (Id_Z \times Id_Z) \circ i$$

I have considered before that in a general model category $\mathcal{M}$ the product of any object C in $\mathcal{M}$ denoted by $C \times C$ is characterized (up to isomorphism) by the universal property,
There exists morphisms,

$$\begin{tikzcd}
 & C \\
C\times C \arrow[ur, "m_0"] \arrow[dr, "m_1"'] & \\
 & C
\end{tikzcd}$$

such that, for any morphisms, ϕ_0,ϕ_1, there exists a unique morphism $\phi = \phi_0 \times \phi_1$ from any other object B in $\mathcal{M}$ to the product $C \times C$ in the diagram,

$$\begin{tikzcd}
 & & C \\
B \arrow[r, dashed, "\phi=\phi_0\times\phi_1"] \arrow[urr, bend left, "\phi_0"] \arrow[drr, bend right, "\phi_1"'] & C\times C \arrow[ur, "m_0"] \arrow[dr, "m_1"'] & \\
 & & C
\end{tikzcd}$$

such that

$$m_0 \circ (\phi_0 \times \phi_1) = \phi_0$$
$$m_1 \circ (\phi_0 \times \phi_1) = \phi_1$$

and therefore, all the triangles commute.
If I consider now any other morphism,

$$\psi \colon A \longrightarrow B$$

again using the universal property of the product,

$$\begin{tikzcd}
 & & & C \\
A \arrow[r, "\psi"] \arrow[urrr, bend left, "\phi_0 \circ \psi"] \arrow[drrr, bend right, "\phi_1 \circ \psi"'] & B \arrow[r, "\phi_0 \times \phi_1"] \arrow[urr, bend left, "\phi_0"] \arrow[drr, bend right, "\phi_1"'] & C \times C \arrow[ur, "m_0"] \arrow[dr, "m_1"'] & \\
 & & & C
\end{tikzcd}$$

Clearly it holds,

$$(\phi_0 \circ \psi) \times (\phi_1 \circ \psi) = (\phi_0 \times \phi_1) \circ \psi$$

And I also know,

$$\psi \circ (\phi_0 \times \phi_1) = (\phi_0 \circ \psi) \times (\phi_1 \circ \psi)$$

Now if I apply these facts to my particular case,

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, bend left, "(Id_Z \times (i \circ \theta)) \circ i"] \arrow[r, bend right, "(Id_Z \times Id_Z) \circ i"'] & Z \times Z \arrow[ur, "l_0"] \arrow[dr, "l_1"'] & \\
 & & Z
\end{tikzcd}$$

I want to prove,

$$l_0 \circ ((Id_Z \times (i \circ \theta)) \circ i) \overset{?}{=} l_0 \circ ((Id_Z \times Id_Z) \circ i)$$
$$l_1 \circ ((Id_Z \times (i \circ \theta)) \circ i) \overset{?}{=} l_1 \circ ((Id_Z \times Id_Z) \circ i)$$

Indeed, on one hand,

$$l_0 \circ (((Id_Z \times (i \circ \theta)) \circ i) \overset{?}{=} l_0 \circ ((Id_Z \times Id_Z) \circ i)$$
$$\Updownarrow$$
$$l_0 \circ ((Id_Z \circ i) \times (i \circ \theta \circ i)) \overset{?}{=} l_0 \circ ((Id_Z \circ i) \times (Id_Z \circ i))$$
$$\Updownarrow$$
$$Id_Z \circ i \overset{?}{=} Id_Z \circ i$$

which is clearly true.
And in the other hand,

$$l_1 \circ ((Id_Z \times (i \circ \theta)) \circ i) \overset{?}{=} l_1 \circ ((Id_Z \times Id_Z) \circ i)$$

$$\Updownarrow$$

$$l_1 \circ ((Id_Z \circ i) \times (i \circ \theta \circ i)) \overset{?}{=} l_1 \circ ((Id_Z \circ i) \times (Id_Z \circ i))$$

$$\Updownarrow$$

$$l_1 \circ ((Id_Z \circ i) \times ((i \circ \theta) \circ i)) \overset{?}{=} l_1 \circ (i \times i)$$

$$\Updownarrow$$

$$(i \circ \theta) \circ i \overset{?}{=} i$$

But this is also true since I can observe that,

$$(i \circ \theta) \circ i = i \circ (\theta \circ i) = i \circ Id_X = i.$$

Then, I have obtained the diagram commutes and so, by definition, h is a right homotopy from $i \circ \theta$ to Id_Z.

Indeed, I get a lifting, since Z is fibrant, the left vertical map is an injection and a weak equivalence, the right vertical map is a fibration and the diagram is commutative, and then the desired homotopy is established.

This implies that $i \circ \theta \simeq Id_Z$.

Thus, θ is a homotopy inverse for i by Proposition 64 and then I can conclude that i is a homotopy equivalence.

Now, I will prove that any trivial fibration p betweeen fibrant + cofibrant objects is a homotopy equivalence.

Since Y and Z are cofibrant objects I have a unique morphism (which is a cofibration) from the initial object of the category (denoted by $\emptyset$) to those objects.

$$\emptyset \xrightarrow{\delta'} Y$$

and

$$\emptyset \xrightarrow{\delta'} Z.$$

So, I can consider the commutative diagram,

$$\begin{tikzcd}
\emptyset \arrow[r, "\delta'"] \arrow[d, "\delta'"'] & Z \arrow[d, "p"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

in which $\emptyset$ denotes the initial object, δ' is a cofibration and p is a trivial fibration.

In fact, the cofibrant objects of a closed model category are characterized by having the left lifting property with respect to any trivial fibration in the category and so by M4 axiom (lifting axiom) there exists a dotted arrow θ',

$$\begin{tikzcd}
\emptyset \arrow[r, "\delta'"] \arrow[d, "\delta'"'] & Z \arrow[d, "p"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "\theta'"] & Y
\end{tikzcd}$$

such that by construction $p \circ \theta' = Id_Y$.

I want to show that $\theta' \circ p \simeq Id_Z$

I consider a cylinder object for Z, that is such a factorization

$$\begin{tikzcd}
Z \amalg Z \arrow[r, "Id_Z \amalg Id_Z"] \arrow[d, "i_0 \amalg i_1"'] & Z \\
Cyl(Z) \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

in which $i_0 \amalg i_1$ is a cofibration and σ' is a trivial fibration.
The diagram commutes by construction,

$$\sigma' \circ (i_0 \amalg i_1) = Id_Z \amalg Id_Z$$

I consider now the diagram,

$$\begin{tikzcd}
Z \amalg Z \arrow[r, "(\theta' \circ p) \amalg Id_Z"] \arrow[d, "i_0 \amalg i_1"'] & Z \arrow[d, "p"] \\
Cyl(Z) \arrow[r, "p \circ \sigma'"'] & Y
\end{tikzcd}$$

in which $i_0 \amalg i_1$ is a cofibration and p is a trivial fibration.
I want to show that this diagram commutes in order to deduce that there exists a dotted arrow h'

$$\begin{tikzcd}
Z \amalg Z \arrow[r, "(\theta' \circ p) \amalg Id_Z"] \arrow[d, "i_0 \amalg i_1"'] & Z \arrow[d, "p"] \\
Cyl(Z) \arrow[r, "p \circ \sigma'"'] \arrow[ur, dashed, "h'"] & Y
\end{tikzcd}$$

and conclude that $\theta' \circ p \simeq Id_Z$.
Therefore I want to prove that,

$$(p \circ \sigma') \circ (i_0 \amalg i_1) = p \circ ((\theta' \circ p) \amalg Id_Z)$$

Indeed, first of all I can observe,

$$(p \circ \sigma') \circ (i_0 \amalg i_1) = p \circ (\sigma' \circ (i_0 \amalg i_1)) = p \circ (Id_Z \amalg Id_Z)$$

So that in order to show that the diagram commutes I will want to prove that,

$$p \circ ((\theta' \circ p) \amalg Id_Z) = p \circ (Id_Z \amalg Id_Z)$$

I will use the universal property that characterizes the coproduct; I know that applying the general universal property to my particular case I have two morphisms,

$$\begin{tikzcd}
Z \arrow[dr, "j_0"] & \\
& Z \amalg Z \\
Z \arrow[ur, "j_1"'] &
\end{tikzcd}$$

such that, for any morphisms, φ_0, φ_1, there exists a unique morphism $\varphi = \varphi_0 \amalg \varphi_1$ defined from the coproduct $Z \amalg Z$ to the object Y, in the diagram,

$$\begin{tikzcd}
Z \arrow[dr, "j_0"] \arrow[drr, bend left, "\varphi_0"] & & \\
& Z \amalg Z \arrow[r, "\varphi=\varphi_0 \amalg \varphi_1"] & Y \\
Z \arrow[ur, "j_1"'] \arrow[urr, bend right, "\varphi_1"'] & &
\end{tikzcd}$$

such that

$$\varphi \circ j_0 = \varphi_0 \text{ equivalently } (\varphi_0 \amalg \varphi_1) \circ j_0 = \varphi_0$$

$$\varphi \circ j_1 = \varphi_1 \text{ equivalently } (\varphi_0 \amalg \varphi_1) \circ j_1 = \varphi_1$$

and therefore, all the triangles commute.

In my case I have the morphisms and such a diagram,

$$\begin{tikzcd}
Z \arrow[dr, "j_0"] & & \\
& Z \amalg Z \arrow[r, bend left, "p\circ((\theta'\circ p)\amalg Id_Z)"] \arrow[r, bend right, "p\circ(Id_Z \amalg Id_Z)"'] & Y \\
Z \arrow[ur, "j_1"'] & &
\end{tikzcd}$$

In order to prove that

$$p \circ ((\theta' \circ p) \amalg Id_Z) = p \circ (Id_Z \amalg Id_Z)$$

a necessary and suficient condition to prove is the following,

$$p \circ ((\theta' \circ p) \amalg Id_Z) \circ j_0 = p \circ (Id_Z \amalg Id_Z) \circ j_0$$

$$p \circ ((\theta' \circ p) \amalg Id_Z) \circ j_1 = p \circ (Id_Z \amalg Id_Z) \circ j_1$$

I have considered before that in a general model category $\mathcal{M}$ the coproduct of any object A in $\mathcal{M}$ denoted by $A \amalg A$ is characterized (up to isomorphsm) by the universal property,

There exists morphisms,

$$\begin{tikzcd}
A \arrow[dr, "k_0"] & \\
& A \amalg A \\
A \arrow[ur, "k_1"'] &
\end{tikzcd}$$

such that, for any morphisms, φ_0,φ_1, there exists a unique morphism $\varphi = \varphi_0 \amalg \varphi_1$ to any other object B, in the diagram,

$$\begin{tikzcd}
A \arrow[dr, "k_0"] \arrow[drr, bend left, "\varphi_0"] & & \\
& A \amalg A \arrow[r, dashed, "\varphi=\varphi_0 \amalg \varphi_1"] & B \\
A \arrow[ur, "k_1"'] \arrow[urr, bend right, "\varphi_1"'] & &
\end{tikzcd}$$

such that

$$(\varphi_0 \amalg \varphi_1) \circ k_0 = \varphi_0$$

$$(\varphi_0 \amalg \varphi_1) \circ k_1 = \varphi_1$$

and therefore, all the triangles commute.
If I consider now any other morphism

$$\phi \colon B \longrightarrow C$$

again using the universal property of the coproduct

$$\begin{tikzcd}
A \arrow[dr, "k_0"] \arrow[drr, bend left, "\varphi_0"] \arrow[drrr, bend left, "\phi\circ\varphi_0"] & & & \\
& A \amalg A \arrow[r, "\varphi_0 \amalg \varphi_1"] & B \arrow[r, "\phi"] & C \\
A \arrow[ur, "k_1"'] \arrow[urr, bend right, "\varphi_1"'] \arrow[urrr, bend right, "\phi\circ\varphi_1"'] & & &
\end{tikzcd}$$

Clearly it holds,

$$(\phi \circ \varphi_0) \amalg (\phi \circ \varphi_1) = \phi \circ (\varphi_0 \amalg \varphi_1)$$

Now if I apply this facts to my particular case,

$$\begin{tikzcd}
Z \arrow[dr, "j_0"] & & \\
& Z \amalg Z \arrow[r, bend left, "p\circ((\theta'\circ p)\amalg Id_Z)"] \arrow[r, bend right, "p\circ(Id_Z \amalg Id_Z)"'] & Y \\
Z \arrow[ur, "j_1"'] & &
\end{tikzcd}$$

I want to prove,

$$p \circ ((\theta' \circ p) \amalg Id_Z) \circ j_0 \overset{?}{=} p \circ (Id_Z \amalg Id_Z) \circ j_0$$

$$p \circ ((\theta' \circ p) \amalg Id_Z) \circ j_1 \overset{?}{=} p \circ (Id_Z \amalg Id_Z) \circ j_1$$

Indeed, in one hand,

$$(p \circ ((\theta' \circ p) \amalg Id_Z))) \circ j_1 \overset{?}{=} (p \circ (Id_Z \amalg Id_Z)) \circ j_1$$

$$\Updownarrow$$

$$((p \circ (\theta' \circ p)) \amalg (p \circ Id_Z)) \circ j_1 \overset{?}{=} ((p \circ Id_Z) \amalg (p \circ Id_Z)) \circ j_1$$

$$\Updownarrow$$

$$p \circ Id_Z \overset{?}{=} p \circ Id_Z$$

which is clearly true.

And on the other hand,

$$(p \circ ((\theta' \circ p) \amalg Id_Z))) \circ j_0 \overset{?}{=} (p \circ (Id_Z \amalg Id_Z)) \circ j_0$$

$$\Updownarrow$$

$$((p \circ (\theta' \circ p)) \amalg (p \circ Id_Z)) \circ j_0 \overset{?}{=} ((p \circ Id_Z) \amalg (p \circ Id_Z)) \circ j_0$$

$$\Updownarrow$$

$$((p \circ (\theta' \circ p)) \amalg (p \circ Id_Z)) \circ j_0 \overset{?}{=} (p \amalg p) \circ j_0$$

$$\Updownarrow$$

$$p \circ (\theta' \circ p) \overset{?}{=} p$$

But this is also true since I can observe that,

$$p \circ (\theta' \circ p) = (p \circ \theta') \circ p = Id_Y \circ p = p.$$

Then, I have obtained the diagram commutes and so, by definition, h' is a left homotopy from $\theta' \circ p$ to Id_Z.

Indeed, I get a lifting, since Z is cofibrant, the left vertical map is an injection, the right vertical map is a fibration and a weak equivalence and the diagram is commutative, and then the desired homotopy is established.

This implies that $\theta' \circ p \simeq Id_Z$.

Thus, θ' is a homotopy inverse for p by Proposition 64 and then I can conclude that p is a homotopy equivalence. □

Next I will show that the converse for the Whitehead Theorem, also holds, and so a homotopy equivalence in a model category is a weak equivalence.

Theorem 13. *(Whitehead)*

Suppose that $f\colon X \longrightarrow Y$ is a morphism of a closed model category $\mathcal{M}$ such that the objects X and Y are both fibrant and cofibrant.

Suppose also that f is a homotopy equivalence.

Then the map f is a weak equivalence.

Proof. First of all I claim that,

If $f\colon X \longrightarrow X$ is a morphism in a model category which is left-homotopic to the identity, Id_X, then f is a weak equivalence.

I prove this claim.

Indeed, let $Cyl(X)$ be a cylinder object of X.

So that, I have the diagram,

$$\begin{tikzcd}
X \amalg X \arrow[r, "Id_x \amalg Id_X"] \arrow[d] & X \\
Cyl(X) \arrow[ur, "\sigma"] &
\end{tikzcd}$$

where σ is a weak equivalence.

Recall that given two morphisms $g, k\colon A \longrightarrow B$ I say that g is left homotopic to k (denoted by $g \overset{l}{\sim} k$) if there is a diagram of the form

$$\begin{tikzcd}
A & A \amalg A \arrow[l, "Id_A \amalg Id_A"'] \arrow[r, "g \amalg k"] \arrow[d, "i_0 \amalg i_1"] & B \\
& Cyl(A) \arrow[ul, "\sigma"] \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

where σ is a weak equivalence.

That is if there exists h such that, $h \circ (i_0 \amalg i_1) = g \amalg k$

So that, since f is left homotopic to the identity Id_X, I have a diagram,

$$\begin{tikzcd}
X & X \amalg X \arrow[l, "Id_X \amalg Id_X"'] \arrow[r, "f \amalg Id_X"] \arrow[d, "i_0 \amalg i_1"] & X \\
& Cyl(X) \arrow[ul, "\sigma"] \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

where σ is a weak equivalence.

So there is a morphism,

$$h\colon Cyl(X) \longrightarrow X$$

which is a homotopy between the morphism f and the identity Id_X.

There are inclusions $i_0, i_1\colon X \hookrightarrow Cyl(X)$ (which are cofibrations if X is cofibrant) such that $h \circ i_0 = f$ and $h \circ i_1 = Id_X$.

Now, both i_0, i_1 are weak equivalences.

Indeed, the inclusion,

$j_0\colon X \hookrightarrow X \amalg X$ is a cofibration by a cobase change (recall that cofibrations are stable under cobase change).

Hence $i_0 = (i_0 \amalg i_1) \circ j_0$ is a cofibration.

$$\begin{tikzcd}
X \arrow[dr, "j_0"] \arrow[drr, bend left, "i_0"] & & \\
& X \amalg X \arrow[r, "i_0 \amalg i_1"] & Cyl(X)
\end{tikzcd}$$

Now $\sigma \circ i_0 = Id_X$ and the M2 axiom (two out of three axiom) implies that i_0 is also a weak equivalence.

Similarly i_1 is a trivial cofibration.

$$\begin{tikzcd}
& X \amalg X \arrow[r, "i_0 \amalg i_1"] & Cyl(X) \\
X \arrow[ur, "j_1"] \arrow[urr, bend right, "i_1"'] & &
\end{tikzcd}$$

So that, I have showed that both i_0, i_1 are weak equivalences.

But I also know that h holds $h \circ i_0 = f$ and $h \circ i_1 = Id_X$.

Applying the M2 axiom (two out of three axiom) to the first identity, $h \circ i_1 = Id_X$, I obtain that h is a weak equivalence.

Now again applying the M2 axiom (two out of three axiom) to the second identity, $h \circ i_0 = f$ I obtain that f is a weak equivalence.

I have concluded the proof for the claim.

I also note, that

In any model category $\mathcal{M}$, a morphism $f\colon X \longrightarrow Y$ that induces an isomorphism in the homotopy category is a weak equivalence.

If I could prove this fact then I will have concluded the proof for the theorem,

Without loss of generality, I can assume that X, Y are cofibrant and fibrant, by replacing them with better objects.

Indeed,

Recall that for any space A I can apply the M5 axiom (factorization axiom) to the unique map:

$$A \xrightarrow{\delta} *$$

to obtain

$$A \overset{\sim}{\rightarrowtail} FA \twoheadrightarrow A.$$

If $A \xrightarrow{\delta} *$ is a fibration, then A is called fibrant.

Similarly,

$$\emptyset \xrightarrow{\delta'} A$$

factorizes to

$$\emptyset \rightarrowtail CA \overset{\sim}{\twoheadrightarrow} A.$$

If $\emptyset \xrightarrow{\delta'} A$ is a cofibration, then A is called cofibrant.

Then I denote by FA the fibrant replacement of A and by CA the cofibrant replacement of A.

Therefore, I consider X, Y as fibrant + cofibrant objects in the category (by replacing them with better objects if necessary, as I have just explained).

Now, I know that the morphisms in the homotopy category between X,Y are just the homotopy classes of morphisms from $X \longrightarrow Y$.

So I have a homotopy equivalence

$$f\colon X \longrightarrow Y$$

between fibrant + cofibrant objects.

I need to show that it is a weak equivalence.

By M5 axiom (factorization axiom) I can factor f as

$$X \underset{\sim}{\overset{i}{\rightarrowtail}} Z \overset{p}{\twoheadrightarrow} Y$$

where i is an acyclic cofibration and p is a fibration.

Clearly Z is cofibrant + fibrant as well.

Now using the previous Theorem 12, Whitehead Theorem, i is a homotopy equivalence, so by composition rule p must be a homotopy equivalence too.

Now I am reduced to showing that p is a weak equivalence.

In particular, I need only prove the result for fibrations (for cofibrations I use duality).

Since p is a homotopy equivalence, there is a morphism $q\colon Y \longrightarrow Z$ which is a homotopy inverse to p.

Therefore it holds, $q \circ p \simeq Id_Z$ and $p \circ q \simeq Id_Y$.

I want a refinement of this map q that will also be a section of p, that is I am searching for a morphism

$$\tilde{q}\colon Y \longrightarrow Z$$

that it stronger holds $p \circ \tilde{q} = Id_Y$.

Recall that, $p \circ q \simeq Id_Y$

So there is also a homotopy $h\colon Cyl(Y) \longrightarrow Y$ between $p \circ q$ and the identity Id_Y.

where I am considering a cylinder object for Y in the usual sense of such a factorization,

$$\begin{tikzcd}
Y \amalg Y \arrow[r, "Id_Y \amalg Id_Y"] \arrow[d, "i_0 \amalg i_1"'] & Y \\
Cyl(Y) \arrow[ur, "\sigma"] &
\end{tikzcd}$$

where $i_0 \amalg i_1$ is a cofibration and σ is a trivial fibration.

The diagram commutes,

$$\sigma \circ (i_0 \amalg i_1) = Id_Y \amalg Id_Y$$

So I am considering the two inclusions $i_0, i_1 \colon Y \hookrightarrow Cyl(Y)$.

Let i_1 be the one corresponding to the identity, Id_Y.

Recall that if h is a homotopy between $p \circ q$ and Id_Y then it holds that $h \circ i_0 = p \circ q$ and $h \circ i_1 = Id_Y$.

So, I get a diagram

$$\begin{tikzcd}
Y \arrow[r, "q"] \arrow[d, "i_0"'] & Z \arrow[d, "p"] \\
Cyl(Y) \arrow[r, "h"] & Y
\end{tikzcd}$$

where i_0 is a cofibration and by the claim is a weak equivalence, p is a fibration and the diagram commutes trivially since by the construction of the homotopy $p \circ q = h \circ i_0$.

Then, by M4 axiom (lifting axiom) there is a lift $g\colon Cyl(Y) \longrightarrow Z$,

$$\begin{tikzcd}
Y \arrow[r, "q"] \arrow[d, "i_0"'] & Z \arrow[d, "p"] \\
Cyl(Y) \arrow[r, "h"] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

Let $\tilde{q} = g \circ i_1$.

Then $\tilde{q}\colon Y \longrightarrow Z$ is homotopic to q, and is another homotopy inverse for p.

But $\tilde{q}$ has the property that I am looking for (and that q needs not have), I have now that

$$p \circ \tilde{q} = p \circ (g \circ i_1) = (p \circ g) \circ i_1 = h \circ i_1 = Id_Y.$$

So, I obtain that I have a fibration $p\colon Z \longrightarrow Y$ between fibrant + cofibrant objects with a homotopy inverse $\tilde{q}\colon Y \longrightarrow Z$.

I also have that $p \circ \tilde{q} = Id_Y$.

I want to prove that p is a weak equivalence.

Now $\tilde{q} \circ p$ is a weak equivalence by my previous claim, as it is homotopic to the identity.

Finally I can consider the following diagram,

$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d, "p"'] & Z \arrow[r, equal] \arrow[d, "\tilde{q} \circ p"] & Z \arrow[d, "p"] \\
Y \arrow[r, "\tilde{q}"] & Z \arrow[r, "p"] & Y
\end{tikzcd}$$

This diagram commutes because $p \circ \tilde{q} = Id_Y$.

This is a retract diagram, and I find that p is a retract of the weak equivalence $\tilde{q} \circ p$.

Hence it is a weak equivalence using the M3 axiom (retract axiom) and with this fact I am concluded the proof for the converse of Whitehead Theorem. □

I can consider the application for the Whitehead Theorem that I showed in the Example 38; I rewrite now this example in terms of model categories.

Example 46. *Suppose three objects, X,Y,Z in a model category $\mathcal{M}$, two of them, said X and Y are fibrant + cofibrant objects and no condition is established on the third object Z.*
I want to study two cases:

(a) *Suppose given the following diagram of arrows,*

$$X \xleftarrow[\sim]{g} Z \xrightarrow[\sim]{h} Y$$

where the morphisms g and h are weak equivalences.
I want to show that there exists a morphism

$$f\colon X \longrightarrow Y$$

which is a homotopy equivalence.

(b) *Assume given the following diagram of arrows,*

$$X \xrightarrow[\sim]{g} Z \xleftarrow[\sim]{h} Y$$

where the morphisms g and h are weak equivalences.
I want to show that there exists a morphism

$$f\colon X \longrightarrow Y$$

which is a homotopy equivalence.

*I begin studing part **(a)**.*
I know that any object in a model category has a cofibrant approximation and so I can consider the cofibrant approximation for the fibrant object Z, which I denote by $\tilde{Z}$.
I know that the cofibrant aproximation for an object Z in the model catagory is both a cofibrant object $\tilde{Z}$ and a weak equivalence,

$$k\colon \tilde{Z} \overset{\sim}{\longrightarrow} Z$$

So that I obtain the diagram,

$$\begin{tikzcd}
X & Z \arrow[l, "g"', "\sim"] \arrow[r, "h", "\sim"'] & Y \\
 & \tilde{Z} \arrow[u, "k", "\sim"'] &
\end{tikzcd}$$

Then, I can consider the two compositions,

$$\begin{tikzcd}
\tilde{Z} \arrow[r, "k", "\sim"'] \arrow[rr, bend right, "\sim", "g\circ k"'] & Z \arrow[r, "g", "\sim"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{Z} \arrow[r, "k", "\sim"'] \arrow[rr, bend right, "\sim", "h\circ k"'] & Z \arrow[r, "h", "\sim"'] & Y
\end{tikzcd}$$

both of them are compositions of weak equivalences, and so they are weak equivalences too. Indeed, composition of weak equivalences is a weak equivalence.

$$g \text{ and } k \text{ are weak equivalences} \Longrightarrow k \circ g \text{ is a weak equivalence.}$$
$$h \text{ and } k \text{ are weak equivalences} \Longrightarrow k \circ h \text{ is a weak equivalence.}$$

and therefore I obtain the diagram,

$$\begin{tikzcd}
X & & Z \arrow[ll, "g"', "\sim"] \arrow[rr, "h", "\sim"'] & & Y \\
 & & \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[ull, bend left, "\sim"', "g\circ k"] \arrow[urr, bend right, "\sim", "h\circ k"'] & &
\end{tikzcd}$$

where the morphisms $g \circ k$ and $h \circ k$ are weak equivalences but now defined between cofibrant objects and so I can apply the Theorem 12, Whitehead Theorem, to obtain that the two composition morphisms $g \circ k$ and $h \circ k$ are both of them homotopy equivalences.

Since $g \circ k$ is a homotopy equivalence, I can consider a homotopic inverse which I denote by $t = (g \circ k)^{-1}$.
I know that now I can define a homotopic equivalence f between the cofibrant objects X and Y,

$$f\colon X \longrightarrow Y$$

as the composition of the homotopic inverse t defined followed by the composition $h \circ k$.

$$f = (h \circ k) \circ t$$

Recall that the composition of homotopic equivalences is a homotopic equivalence too.

Finally, I have obtained the diagram,

$$\begin{tikzcd}
X \arrow[rr, bend left, "f"] \arrow[dr, bend right, "t"'] & Z \arrow[l, "g", "\sim"'] \arrow[r, "h", "\sim"'] & Y \\
& \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[ul, bend left, "\sim", "g\circ k"'] \arrow[ur, bend right, "\sim"', "h\circ k"] &
\end{tikzcd}$$

I will study now part ***(b)****.*

Again, I know that any object in a model category has a cofibrant approximation and so I can consider the cofibrant approximation for the fibrant object Z, which I denote by $\tilde{Z}$.

I know that the cofibrant aproximation for an object Z in the model catagory is both a cofibrant object $\tilde{Z}$ and a weak equivalence,

$$k\colon \tilde{Z} \xrightarrow{\sim} Z$$

Therefore, I obtain now the diagram,

$$\begin{tikzcd}
X \arrow[r, "g", "\sim"'] & Z & Y \arrow[l, "h"', "\sim"] \\
& \tilde{Z} \arrow[u, "k", "\sim"'] &
\end{tikzcd}$$

In this case, the most delicate part is to know whether there exist two proper morphisms,

$$\tilde{g}\colon X \longrightarrow \tilde{Z}$$
$$\tilde{h}\colon Y \longrightarrow \tilde{Z}$$

Recall that, in general, I can use the Theorem 11, Theorem of Approximation by cofibrant objects which statement is the following,

For any object A in a model category, there is a cofibrant object $\tilde{A}$ and a weak equivalence

$$\gamma\colon \tilde{A} \longrightarrow A.$$

Moreover, it holds that for a map

$$\varphi\colon A \longrightarrow B$$

and another such cofibrant approximation for the object B, that is a cofibrant object $\tilde{B}$ and a weak equivalences,

$$\rho\colon \tilde{B} \longrightarrow B,$$

there is a morphism,

$$\tilde{\varphi}\colon \tilde{A} \longrightarrow \tilde{B},$$

which is unique up to homotopy, such that the following diagram is homotopic commutative,

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \\
\tilde{A} \arrow[u, "\gamma"] \arrow[r, "\tilde{\varphi}"'] & \tilde{B} \arrow[u, "\rho"']
\end{tikzcd}$$

So, since I have the homotopic commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & Z \\
X \arrow[r, "\tilde{g}"'] & \tilde{Z} \arrow[u, "k"']
\end{tikzcd}$$

then I can build the morphism,

$$\tilde{g}\colon X \longrightarrow \tilde{Z}$$

such that $k \circ \tilde{g} \sim g$.

In an analogous way, since I have the homotopic commutative diagram,

$$\begin{tikzcd}
Y \arrow[r, "h"] \arrow[d, equal] & Z \\
Y \arrow[r, "\tilde{h}"'] & \tilde{Z} \arrow[u, "k"']
\end{tikzcd}$$

I can build the morphism,

$$\tilde{h}\colon Y \longrightarrow \tilde{Z}$$

such that $k \circ \tilde{h} \sim h$.

Therefore I have obtained,

$$k \circ \tilde{g} \sim g$$

where k *and* g *are weak equivalences and so I can deduce that* $\tilde{g}$ *is a weak equivalence too.*

In an analogous way I have obtained,

$$k \circ \tilde{h} \sim h$$

where k *and* h *are weak equivalences and so I can deduce that* $\tilde{h}$ *is a weak equivalence too.*

Then I obtain the diagram,

$$\begin{tikzcd}
X \arrow[r, "g", "\sim"'] \arrow[dr, bend right, "\sim", "\tilde{g}"'] & Z & Y \arrow[l, "h"', "\sim"] \arrow[dl, bend left, "\sim"', "\tilde{h}"] \\
& \tilde{Z} \arrow[u, "k", "\sim"'] &
\end{tikzcd}$$

Since $\tilde{g}$ *and* $\tilde{h}$ *are weak equivalences but now defined between fibrant + cofibrant objects, then I can apply the Whitehead theorem to deduce that* $\tilde{g}$ *and* $\tilde{h}$ *are both homotopy equivalences.*

Since $\tilde{h}$ *is a homotopy equivalence I can consider a homotopic inverse, which I denote by* $l = (\tilde{h})^{-1}$

I know that now I can define a homotopic equivalence f *between the fibrant + cofibrant objects* X *and* Y*,*

$$f\colon X \longrightarrow Y$$

as the composition of the morphism $\tilde{g}$ *defined followed by the homotopic inverse* l *defined.*

$$f = l \circ \tilde{g}$$

Recall that the composition of homotopic equivalences is a homotopy equivalence too.

Finally, I have obtained the diagram,

$$\begin{tikzcd}
X \arrow[rr, bend left, "f"] \arrow[r, "g", "\sim"'] \arrow[dr, bend right, "\sim", "\tilde{g}"'] & Z & Y \arrow[l, "h"', "\sim"] \arrow[dl, bend left, "\sim"', "\tilde{h}"] \\
 & \tilde{Z} \arrow[u, "k", "\sim"'] \arrow[ur, bend right, "l"'] &
\end{tikzcd}$$

The Homotopy Category of a Model Category

.

In this section, I will follow the standard approach to define and study the homotopy category of a model category $\mathcal{M}$, the original approach of Daniel Quillen in [13].

The basic result is that the localization $Ho\ \mathcal{M}$ of a model category $\mathcal{M}$ obtained by inverting the weak equivalences is equivalent to the quotient category $\mathcal{M}_{fc}/\sim$ of the fibrant + cofibrant objects by the homotopy relation.

I wish to emphasize that in fact, $Ho\ \mathcal{M}$ is not the same category as $\mathcal{M}_fc/\sim$, merely equivalent to it.

Defining a Homotopy Category.

.

As I introduced in the previous section, model categories are used to give an effective construction for the localization of categories, where the problem is to convert a class of morphisms, the weak equivalences, into isomorphisms. The main strategy of model categories is to distinguish subclasses of morphisms, endowed with good homotopical properties in order to obtain a operable and handy representation of morphisms in homotopy categories.

In the case of a model category, the localized category is identified with a homotopy category which results in a category whose morphisms sets consist of equivalence classes of morphisms under a suitable homotopy relation which is determined by the model structure.

More precisely,

Definition 47. *(Homotopy Category of a Model Category)*

I suppose $\mathcal{C}$ is a category with a subcategory of weak equivalences W. I define the homotopy "category" $Ho\ \mathcal{C}$ in the following way,

I form the free category $F(\mathcal{C}, W^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of the arrows of W, then an object of $F(\mathcal{C}, W^{-1})$ is an object of $\mathcal{C}$,

and a morphism is a finite string of composable arrows $(f_1, f_2, \cdots, f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of W.

The empty string at a particular object is the identity at that object,

and composition is defined by concatenation of strings.

Now, I define $Ho\ \mathcal{C}$ to be the quotient category of $F(\mathcal{C}, W^{-1})$ by the relations,

$Id_A = (Id_A)$ *for all objects A,*

$(f, g) = (g \circ f)$ *for all composable arrows f, g of $\mathcal{C}$,*

$Id_{dom\ w} = (w, w^{-1})$ *and* $Id_{codom\ w} = (w^{-1}, w)$ $\forall w \in W$.

where dom w is the domain of w and codom w is the codomain of w.

I will use the notation $Ho\ \mathcal{C}$ to refer the homotopy category of $\mathcal{C}$, that is the localization of $\mathcal{C}$ with respect to the class W of weak equivalences. The notation $Ho\ \mathcal{C}$ is certainly not ideal for this "category". The right notation should be $\mathcal{C}[W^{-1}]$. There are several reasons for not adopting the right notation:

In one hand I will usually refer as $\mathcal{C}$ to be a category or a subcategory of a model category $\mathcal{M}$ and, in addition I take W to be the weak equivalences in $\mathcal{C}$.

On the other hand, the notation $[\mathcal{W}^{-1}]\mathcal{C}$ referred to the localized category directly comes from the classical algebra, but in the present work I will mainly deal with the localization of model categories and so, in this case I will have an identity between the morphisms in the localized category and homotopy classes of morphisms in the original category.

Those are some reasons why I will write $Ho\,\mathcal{C} = [\mathcal{W}^{-1}]\mathcal{C}$ with both meanings:

on one hand for the localization of a category in general,

and on the other hand I will also refer to this category $Ho\,\mathcal{C}$ as the homotopy category associated to $\mathcal{C}$.

I can note that the previous definition makes it clear that $Ho\,\mathcal{C} = (Ho\,\mathcal{C})^{op}$ if $\mathcal{C}$ is a model category.

I introduce now but I will develope afterwards that, in general, the problem of defining a homotopy category is that in many situations I have a category $\mathcal{C}$ together with a class of morphisms $\mathcal{W}$, the weak equivalences which I would like to view as isomorphisms in a localized category $[\mathcal{W}^{-1}]\mathcal{C}$ associated to $\mathcal{C}$.

The reason for the quotes around "category" is that $Ho\,\mathcal{C}(A, B)$ may not be a set in general. So $Ho\,\mathcal{C}$ may not exist until I pass to the higher environment. I will make this passage implicitly until I prove that it is in fact not necessary if $\mathcal{C}$ is a model category.

Note that there is a functor $\mathcal{C} \xrightarrow{\gamma} Ho\,\mathcal{C}$ which is the identity on objects and takes morphisms of W to isomorphisms. The category $Ho\,\mathcal{C}$ is characterized by a universal property.

Therefore, as a general problem, I want to define a category

$$Ho\,\mathcal{C}$$

together with a functor

$$\gamma\colon \mathcal{C} \longrightarrow Ho\,\mathcal{C}$$

which will map weak equivalences to isomorphisms and which will result as universal in the sense that any functor,

$$F\colon \mathcal{C} \longrightarrow \mathcal{D}$$

which maps weak equivalences to isomorphisms will admit a unique factorization,

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathcal{D} \\
& Ho\,\mathcal{C} \arrow[ur, dashed, "\bar{F}"'] &
\end{tikzcd}$$

such that $F = \bar{F} \circ \gamma$.

The basic idea introduced in the Definition 47 of such a category $Ho\,\mathcal{C}$ consists in formally inverting the morphisms in the class of weak equivalences $\mathcal{W}$ (since $Ho\,\mathcal{C} = [\mathcal{W}^{-1}]\mathcal{C}$).

As in the previous section a will continue using the notation $\xrightarrow{\sim}$ to distinguish the morphisms that belong to the class of weak equivalence $\mathcal{W}$ in $\mathcal{C}$

In this setting a morphism of $Ho\,\mathcal{C}$ is represented by a sort of chain,

$$A \xleftarrow{\sim} \cdot \rightarrow \cdot \;\; \ldots \;\; \cdot \xleftarrow{\sim} B$$

where the arrows going in the left direction $\xleftarrow{\sim}$ represent formal inverses of weak equivalences.

But this intuitive approach seems to present deep difficulties and it could become unuseful in practice because I can't even ensure that this construction returns sets and not proper classes of morphisms in general because the intermediate objects of a chain that defines such a morphism may range over a proper class of objects in $\mathcal{C}$ and in addition it appears difficult to compute morphisms sets using that construction.

However, I have that the initial problem of defining a suitable homotopy category (with a proper functor mapping weak equivalence to isomorphism and constructed as universal) has always a solution, but the homotopy category $Ho\,\mathcal{C}$ may not be locally small in the sense that the morphisms between two objects in $Ho\,\mathcal{C}$ define a class but do not form a set in general.

I early said that one of the main purposes of model categories is to give an axiomatic setting satisfied in many practical situations, which enables me to effectively compute morphisms in homotopy categories. In the context of a model category $\mathcal{M}$, I can also ensure that the morphisms between two objects form a set.

The main idea that I have developed in the earlier and initial presentation of the machinery of model categories is to distinguish two additional classes of morphisms besides the weak equivalences ($\xrightarrow{\sim}$) characterized by good homotopical properties: the fibrations ($\twoheadrightarrow$) and the cofibrations ($\rightarrowtail$). I also introduced that the morphisms which are both a weak equivalence and a fibration (respectively a cofibration) are called acyclic fibrations or trivial fibrations and respectively acyclic cofibrations or trivial cofibrations.

In a model category $\mathcal{M}$, I also early introduced a useful tool, the notion of cylinder object, determined by the choice of the class of cofibrations and which enables me to define a homotopy relation $\simeq$ on the morphism sets of the category $Mor_{\mathcal{M}}(X, Y)$ for all X,Y $\in \mathcal{M}$.

In classical examples like topological spaces and simplicial sets, the cylinder object $Cyl(X) \in \mathcal{C}$ which I associated to any object of the category, $X \in \mathcal{M}$ is given by an explicit construction.

So, I have,

$$\begin{tikzcd}
X \amalg X \arrow[r, "Id_X \amalg Id_X"] \arrow[d, tail, "i_0 \amalg i_1"'] & X \\
Cyl(X) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

and I have,

$$\begin{tikzcd}
& X \arrow[dr, "j_0"] \arrow[drr, bend left, "i_0"] & & \\
\emptyset \arrow[ur, "j_X"] \arrow[dr, "j_X"'] & & X \amalg X \arrow[r, tail, "i_0 \amalg i_1"] & Cyl(X) \\
& X \arrow[ur, "j_1"'] \arrow[urr, bend right, "i_1"'] & &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

In the setting of model categories the idea is to take these observations as an abstract definition of the notion of a good cylinder object.

Thus cylinder objects may not be unique but the properties of cofibrations, which are formalized by the model category axioms enable me to correctly manage that choices occurring in the definition.

I also previously defined a (left) homotopy between

$$f, g \colon X \longrightarrow Y$$

to be a map,

$$h \colon Cyl(X) \longrightarrow Y$$

such that

$$f = h \circ i_0$$

and

$$g = h \circ i_1.$$

Equivalently, h is such that $h \circ (i_0 \amalg i_1) = f \amalg g$.

I can proceed dually, and define a homotopy relation by considering path objects, determined by the choice of the class of fibrations instead of cofibrations and cylinder objects. This second approach is equivalent to the approach using cofibrations and cylinder objects.

So I am considering,

a class of cofibrant objects $X \in \mathcal{M}_c$ characterized by the requirement that the initial morphism $\emptyset \to X$ is a cofibration

and a class of fibrant objects $X \in \mathcal{M}_f$ characterized by the dual requirement that the terminal object $X \to *$ is a fibration.

I both know and previously introduced that,

every object $X \in \mathcal{M}$ has a cofibrant resolution also called cofibrant approximation or cofibrant replacement, consisting of a cofibrant object $QX \in \mathcal{M}_c$ equipped with a weak equivalence $QX \xrightarrow{\sim} X$,

(If the weak equivalence is also a fibration, the cofibrant resolution is then called "a good cofibrant resolution").

as well as a fibrant resolution, consisting of a fibrant object $RX \in \mathcal{M}_f$ equipped with a weak equivalence $X \xrightarrow{\sim} RX$.

(If the weak equivalence is also a cofibration, the fibrant resolution is then called "a good fibrant resolution").

I proved in the previous section that the factorization axioms of a model category ensure that such approximations always exist.

But it also holds that for a cofibrant object $QX \in \mathcal{M}_c$, and a fibrant bject $RY \in \mathcal{M}_f$ I have an identity $Mor_{Ho\ \mathcal{M}}(QX, RY) = Mor_{\mathcal{M}}(QX, RY)/ \simeq$, where I consider the quotient of the morphism set associated to the pair (QX, RY) in $\mathcal{M}$ under the homotopy relation $\simeq$.

As soon as I can check this assertion, I have and identity

$$Mor_{Ho\ \mathcal{M}}(X, Y) \colon = Mor_{Ho\ \mathcal{M}}(QX, RY) = Mor_{\mathcal{M}}(QX, RY)/ \simeq$$

for every pair of objects X,$Y \in \mathcal{M}$, where I take a cofibrant resolution of the source of the morphisms $QX \xrightarrow{\sim} X$, and a fibrant resolution of the target $Y \xrightarrow{\sim} RY$.

Hence, I obtain with these facts that the previous problem (about defining a suitable homotopy category) has an effective solution given by a locally small category $Ho\ \mathcal{M}$ which assigns a set of morphisms $Mor_{Ho\ \mathcal{M}}(X, Y)$ (and not a proper class) to every pair of objects X,$Y \in \mathcal{M}$.

The notion of "homotopy relation" is directly loaned from classical topology. The category of topological spaces $\mathcal{M} = Top$ inherits a model structure and in particular, I can work out in this framework the problems of the definition of a homotopy category. In the case of topological spaces, I can take the cartesian product with the interval $X \times [0, 1]$ as a model for a cylinder object associated to any objet $X \in Top$, and the homotopy relation is identified with the classical (unpointed) homotopy relation for (continuous) maps between (unpointed) topological spaces.

It holds,

$$[X, Y]_{\mathcal{M}} = Mor_{\mathcal{M}}(X, Y)/ \simeq,$$

for the set of homotopy classes of morphisms in any model category $\mathcal{M}$, for any pair of objects X,$Y \in \mathcal{M}$. The axioms of model categories ensure that this homotopy class set determines the morphism set of the homotopy category $Mor_{Ho\ \mathcal{M}}(X, Y)$ when I assume $X \in \mathcal{M}_c$ and $Y \in \mathcal{M}_f$ in order to compute this morphism set $Mor_{Ho\ \mathcal{M}}(X, Y)$ in terms of homotopy classes.

The terms fibration and cofibration are also loaned from classical topology, and the axioms of model categories actually reflect the classical properties of fibrations and cofibrations on topological spaces. Note that the homotopy category of a model category only depends on the definition of the class of weak equivalences. In particular, in a category $\mathcal{C}$ equipped with a class of weak equivalences $\mathcal{W}$, I may have several choices of fibrations and cofibrations, leading to non-equivalent model structures, but this does not change the homotopy category $Ho\ \mathcal{C}$.

So, I want to remark that the class of fibrations and of cofibrations in a model category essentially play an auxiliary role for the computation of morphisms sets in the homotopy category; an auxiliary role but that allow me to get a precise control of the maps in the homotopy category. Cofibrations and fibrations will enable me to do homotopy theory, because while many of the homotopy notions involved can be defined in terms of the weak equivalences, the verification of many of their properties requires the fibrations and/or the cofibrations.

Also, recall that in general the localization of a category $\mathcal{C}$ with respect to the class of weak equivalences $\mathcal{W}$ is the universal category under $\mathcal{C}$ where the weak equivalences $w \in \mathcal{W}$ become isomorphisms.

In the case of a model category $\mathcal{M}$ the M5 axiom (factorization axiom) implies that any object $X \in \mathcal{M}$ has a cofibrant resolution consisting of a cofibrant object $QX \in \mathcal{M}$ equipped with a weak equivalence $QX \xrightarrow{\sim} X$ in $\mathcal{M}$.

In the same way, also the M5 axiom (factorization axiom) implies that X has a fibrant resolution consisting of a fibrant object $RX \in \mathcal{M}$ equipped with a weak equivalence $X \xrightarrow{\sim} RX$ in $\mathcal{M}$.

I have yet introduced that I will use the notation $\mathcal{M}_f$ and respectively, $\mathcal{M}_c$ for the full subcategory of a model category $\mathcal{M}$ generated by the class of fibrant objects, respectively cofibrant objects in $\mathcal{M}$.

In addition, I will also consider the category $\mathcal{M}_{fc} = \mathcal{M}_f \cap \mathcal{M}_c$ generated by the objects which are both fibrant and cofibrant.

In fact I will use the existence of those resolutions to prove that the category $Ho\ \mathcal{M}$ is equivalent to the localization of the subcategory $\mathcal{M}_c$ and respectively $\mathcal{M}_f$, $\mathcal{M}_{fc}$ with respect to the class of weak equivalences which this subcategory inherits from the model category $\mathcal{M}$.

The universal property of localizations assures that defining a functor on the homotopy category of a model category $F\colon Ho\ \mathcal{M} \longrightarrow \mathcal{N}$ is equivalent to check that a functor on $\mathcal{M}$ carries weak equivalences to isomorphisms.

The equivalence of the homotopy category $Ho\ \mathcal{M}$ with the localization of the full subcategory $\mathcal{M}_f$ and respectively $\mathcal{M}_c$, implies that it suffices verifying this invariance property for the weak equivalences between fibrant objects and respectively cofibrant objects.

I can introduce now a sequence of general facts, main facts too, all of them relatives to the homotopy category starting with a beautiful introducing result that in particular provides a further reduction of that previous verification, which is used in most applications and which constitutes a nice summation of techniques involving model categories. It is a well known classical result, the so called Ken Brown's Lemma,

Facts about the Homotopy Category of a Model Category.

Lemma 48. *(Ken Brown's Lemma)*

Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a functor defined on a model category $\mathcal{M}$ and with values in a category $\mathcal{N}$ equipped with a class of weak equivalences which satisfies the M2 axiom (two out of three axiom).

(1) *If the functor F carries the acyclic cofibrations between cofibrant objects to weak equivalences then it maps all weak equivalences between cofibrant objects to weak equivalences in the category $\mathcal{N}$.*

(2) *Dually, if the functor F carries the acyclic fibrations between fibrant objects to weak equivalences then it maps all weak-equivalences between fibrant objects to weak equivalences in the category $\mathcal{N}$.*

Proof. I will prove the first part (by dual argument the second part holds).

I assume that $f\colon A \xrightarrow{\sim} B$ is a weak equivalence between cofibrant objects A and B.

Using the M5 axiom (factorization axiom), I factor the map

$$f \amalg Id_B\colon A \amalg B \longrightarrow B$$

into a cofibration followed by a trivial fibration,

$$\begin{tikzcd} A \amalg B \arrow[r, tail, "k"] & Z \arrow[r, two heads, "p", "\sim"'] & B \end{tikzcd}$$

Since A and B are cofibrant, and cofibrations are closed under pushouts by Proposition 11, both the canonical maps $A \xrightarrow{i} A \amalg B \xleftarrow{j} B$ are cofibrations.

Those two canonical maps can be viewed as a pushout diagram,

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, tail, "i"] \\
B \arrow[r, tail, "j"'] & A \amalg B
\end{tikzcd}$$

Both, the composites, $u = k \circ i$ and $v = k \circ j$ are cofibrations because are composition of cofibrations and cofibrations are closed under composition.

$$\begin{tikzcd}
A \arrow[d, tail, "i"'] \arrow[dr, tail, "u"] \arrow[drr, bend left, "f"] & & \\
A \amalg B \arrow[r, tail, "k"] & Z \arrow[r, two heads, "p", "\sim"'] & B \\
B \arrow[u, tail, "j"] \arrow[ur, tail, "v"] \arrow[urr, bend right, "Id_B"'] & &
\end{tikzcd}$$

Since I have $p \circ u = p \circ k \circ i = f$, then by M2 axiom (two out of three axiom), I can deduce that u is also a weak equivalence, and hence and acyclic cofibration (of cofibrant objects).

Similarly, the identity $p \circ v = p \circ k \circ j = Id_B$ implies that v is a weak equivalence by M2 axiom (two out of three axiom), and hence, and acyclic cofibration (of cofibrant objects) as well.

Thus, by assumption in the statement for the functor F,

$F(k \circ i) = F(u)$ and $F(k \circ j) = F(v)$

are weak equivalences.

Since $F(p) \circ F(v) = F(p \circ v) = Id_B$, then applying the M2 axiom (two out of three axiom) (which holds in the category $\mathcal{N}$ by assumption in the statement), I obtain that $F(p)$ is a weak equivalence.

Similarly, since $F(p) \circ F(u) = F(p \circ u) = F(f)$, then applying the M2 axiom (two out of three axiom) (which holds in the category $\mathcal{N}$ by assumption in the statement), I obtain that $F(f)$ is a weak equivalence too, as required.

□

The cylinder object and path object presented in the environment of model categories at the end of the previous part in Definition 14 and Definition 15 respectively are main tools and used in particular to define left and right homotopy.

Definition 49. *(Left Homotopy)*
A left homotopy from f to g consists of a cylinder object,

$$X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$$

for X and a map,

$$H \colon Cyl(X) \longrightarrow Y$$

such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

If there exists a left homotopy from f to g, then I say that f is left homotopic to g and I will write $f \overset{l}{\simeq} g$.

Definition 50. *(Right Homotopy)*
A right homotopy from f to g consists of a path object,

$$Y \xrightarrow{s} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

and a map

$$H \colon X \longrightarrow Path(Y)$$

such that $p_0 \circ H = f$ and $p_1 \circ H = g$.

If there exists a right homotopy from f to g, then I say that f is right homotopic to g and I will write $f \overset{r}{\simeq} g$.

If f is both left homotopic and right homotopic to g, then I say that f is homotopic to g and I will write $f \simeq g$.

Proposition 51. *Let $\mathcal{M}$ be a model category.*

1. *If $f, g\colon X \longrightarrow Y$ are left homotopic and Y is fibrant, then there is a cylinder object $X \amalg X \longrightarrow Cyl(X) \xrightarrow{p} X$ in which p is a trivial fibration and a left homotopy $H\colon Cyl(X) \longrightarrow Y$ from f to g.*
2. *If $f, g\colon X \longrightarrow Y$ are right homotopic and Y is cofibrant, then there is a path object $Y \xrightarrow{s} Path(Y) \longrightarrow Y \times Y$ in which p is a trivial cofibration and a right homotopy $H'\colon X \longrightarrow Path(Y)$ from f to g.*

Proof. I will prove the first part (by dual argument the second part holds).

If $X \amalg X \longrightarrow Cyl(X) \xrightarrow{p} X$ is a cylinder object for X such that there is a left homotopy $K\colon Cyl'(X) \longrightarrow Y$ from f to g, then I factor p as $Cyl'(X) \xrightarrow{j} Cyl(X) \xrightarrow{p} X$ where j is a cofibration and p is a trivial fibration. Now by M2 axiom (two out of three axiom), I obtain that j is a trivial cofibration, and so in the following diagram,

$$\begin{tikzcd}
Cyl'(X) \arrow[r, "K"] \arrow[d, "j"'] & Y \arrow[d] \\
Cyl(X) \arrow[r] \arrow[ur, dashed, "H"'] & *
\end{tikzcd}$$

the dotted arrow exists and so it defines the searched left homotopy H. □

In this section I will use the following notation,

Let $\mathcal{M}$ be a model category and let X and Y be objects of $\mathcal{M}$.

If X is cofibrant, I let $[X, Y]^l$ denote the set of left homotopy classes of maps from X to Y (if X is cofibrant, then left homotopy is an equivalence relation on the set of morphisms from X to Y).

If Y is fibrant, I let $[X, Y]^r$ denote the set of right homotopy classes of maps from X to Y (if Y is fibrant, then right homotopy is an equivalence relation on the set of morphisms from X to Y).

If X is cofibrant and Y is fibrant, I let $[X, Y]$ denote the set of homotopy classes of morphisms from X to Y.

Proposition 52. *Let $\mathcal{M}$ be a model category and let $f, g\colon X \longrightarrow Y$ be morphisms in $\mathcal{M}$.*

1. *The maps f and g are left homotopic if and only if there is a factorization $X \amalg X \xrightarrow{i_0 \amalg i_1} Z \xrightarrow{p} X$ for the map $Id_X \amalg Id_X\colon X \amalg X \longrightarrow X$ such that p is a weak equivalence and a map $H\colon Z \longrightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.*
2. *The maps f and g are right homotopic if and only if there is a factorization $Y \xrightarrow{s} W \xrightarrow{p_0 \times p_1} Y \times Y$ for the map $Id_Y \times Id_Y\colon Y \longrightarrow Y \times Y$ such that s is a weak equivalence and a map $H'\colon X \longrightarrow W$ such that $p_0 \circ H' = f$ and $p_1 \circ H' = g$.*

Proof. I will prove the first part (by dual argument the second part holds).

The necessity of the condition directly follows from the definition.

Conversely, I assume the condition is satisfied. If then I factor $i_0 \amalg i_1$ as $X \amalg X \xrightarrow{i'_0 \amalg i'_1} Z' \xrightarrow{q} Z$ where $i'_0 \amalg i'_1$ is a cofibration and q is a trivial fibration, then $X \amalg X \xrightarrow{i'_0 \amalg i'_1} Z' \xrightarrow{p \circ q} Z$ is a cylinder object for X and $H \circ q\colon Z' \longrightarrow Y$ is a left homotopy from f to g.

□

I can add a result in order to show the behaviour of that notions of left homotopy and right homotopy with functors.

Proposition 53. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $\phi\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a functor.*

1. *$\phi(f)$ is left homotopic to $\phi(g)$ provided the following three conditions hold,*
    - **(a)** *ϕ takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$,*

  - **(b)** *$f, g\colon X \longrightarrow Y$ are left homotopic morphisms in $\mathcal{M}$,*
  - **(c)** *X is cofibrant.*
- **(2)** *$\phi(f)$ is right homotopic to $\phi(g)$ provided the following three conditions hold,*
  - **(a)** *ϕ takes trivial fibrations between fibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$,*
  - **(b)** *$f, g\colon X \longrightarrow Y$ are right homotopic morphisms in $\mathcal{M}$,*
  - **(c)** *Y is fibrant.*

Proof. I will prove the first part (by dual argument the second part holds).

Since f and g are left homotopic, there is a cuylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ for X and a map $H\colon Cyl(X) \longrightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Since $p \circ i_0 = Id_X$, I have $\phi(p) \circ \phi(i_0) = Id_{\phi(X)}$, and since i_0 is a trivial cofibration by Proposition 18, then using the M2 axiom (two out of three axiom) then I obtain that $\phi(p)$ is a weak equivalence. Now, just applying Proposition 52 I obtain that $\phi(f)$ and $\phi(g)$ are left homotopic.

□

Proposition 54. *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a category and let $\phi\colon \mathcal{M} \longrightarrow \mathcal{C}$ be a functor.*

- **(1)** *$\phi(f) = \phi(g)$ provided the following three conditions hold,*
  - **(a)** *ϕ takes trivial cofibrations between cofibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{C}$,*
  - **(b)** *$f, g\colon X \longrightarrow Y$ are left homotopic morphisms in $\mathcal{M}$,*
  - **(c)** *X is cofibrant,*
- **(2)** *$\phi(f) = \phi(g)$ provided the following three conditions hold,*
  - **(a)** *ϕ takes trivial fibrations between fibrant objects in $\mathcal{M}$ to isoorphisms in $\mathcal{C}$,*
  - **(b)** *$f, g\colon X \longrightarrow Y$ are right homotopic morphisms in $\mathcal{M}$,*
  - **(c)** *Y is fibrant,*

Proof. I will prove the first part (by dual argument the second part holds).

Since f and g are left homotopic, there is a cylinder obect $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ for X and a morphism $H\colon Cyl(X) \longrightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Now, since $p \circ i_0 = Id_X$ I have $\phi(p) \circ \phi(i_0) = Id_{\phi(X)}$, and since i_0 is a trivial cofibration by Proposition 18 then $\phi(i_0)$ is n isomorphism, and so $\phi(p)$ is an isomorphism.

Now,

$$\left.\begin{aligned} p \circ i_0 = Id_X = p \circ i_1 \\ \phi(i_0) = (\phi(p))^{-1} = \phi(i_1) \end{aligned}\right\} \quad \Rightarrow \quad \phi(f) = \phi(H) \circ \phi(i_0) = \phi(H) \circ \phi(i_1) = \phi(g)$$

□

I want to add two main remarks in this environment of model categories,

Remark 55. *(Homotopy Extension Property of Cofibrations)*

Let $\mathcal{M}$ be a model category, let X be fibrant, and let $k\colon A \longrightarrow B$ be a cofibration.

If $f\colon A \longrightarrow X$ is a map, $\widetilde{f}\colon B \longrightarrow X$ is an extension of f, $X \xrightarrow{s} Path(X) \xrightarrow{p_0 \times p_1} X \times X$ is a path object for X, and $H\colon A \longrightarrow Path(X)$ is a right homotopy of f, so that is a map H such that $p_0 \circ H = f$, then there exists a map $\widetilde{H}\colon B \longrightarrow Path(X)$ such that $p_0 \circ \widetilde{H} = \widetilde{f}$ and $\widetilde{H} \circ k = H$.

I can observe that I have the solid arrow diagram,

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "k"'] & Path(X) \arrow[d, "p_0"] \\ B \arrow[r, "\widetilde{f}"'] \arrow[ur, dashed, "\widetilde{H}"] & X \end{tikzcd}$$

where Proposition 18 ensures that p_0 is a trivial fibration.

Remark 56. *(Homotopy Lifting Property of Fibrations)*
Let $\mathcal{M}$ be a model category, let A be cofibrant, and let $k\colon X \longrightarrow Y$ be a fibration.

If $f\colon A \longrightarrow Y$ is a map, $\widetilde{f}\colon A \longrightarrow X$ is a lift of f, $A \amalg A \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} A$ is a cylinder object for A, and $H\colon Cyl(A) \longrightarrow Y$ is a left homotopy of f, so that is a map H such that $H \circ i_0 = f$, then there exists a map $\widetilde{H}\colon Cyl(A) \longrightarrow X$ such that $\widetilde{H} \circ i_0 = \widetilde{f}$ and $k \circ \widetilde{H} = H$.
I can observe that I have the solid arrow diagram,

$$\begin{tikzcd}
A \arrow[r, "\widetilde{f}"] \arrow[d, "i_0"'] & X \arrow[d, "k"] \\
Cyl(A) \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"'] & Y
\end{tikzcd}$$

where Proposition 18 again ensures that i_0 is a trivial cofibration.

These properties have some implications,

Proposition 57. *Let $\mathcal{M}$ be a model category.*

- **(1)** *Let X be fibrant and let $k\colon A \longrightarrow B$ be a cofibration.*
  If $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow X$ are maps such that $g \circ k \underset{r}{\simeq} f$,
  then there is a map $g'\colon B \longrightarrow X$ such that $g' \underset{r}{\simeq} g$ and $g' \circ k = f$.
- **(2)** *Let A be cofibrant and let $k\colon X \longrightarrow Y$ be a fibration.*
  If $f\colon A \longrightarrow X$ and $g\colon A \longrightarrow Y$ are maps such that $k \circ f \underset{l}{\simeq} g$,
  then there is a map $f'\colon A \longrightarrow X$ such that $f' \underset{l}{\simeq} f$ and $k \circ f' = g$.

Proof. The first assertion is a direct consequence of the homotopy extension property of cofibrations, Remark 55, while the second assertion follows from the homotopy lifting property of fibrations, Remark 56 □

Proposition 58. *Let $\mathcal{M}$ be a model category.*

- **(1)** *With the following assumptions,*
  - **(a)** *$i\colon A \longrightarrow B$ is a cofibration,*
  - **(b)** *X is fibrant,*
  - **(c)** *i induces an isomorphism $i^*\colon [B, X]^r \cong [A, X]^r$*

  Then for every map $f\colon A \longrightarrow X$ there is a map $g\colon B \longrightarrow X$, unique up to right homotopy, such that $g \circ i = f$.
- **(2)** *With the following conditions,*
  - **(a)** *$p\colon X \longrightarrow Y$ is a fibration,*
  - **(b)** *A is cofibrant,*
  - **(c)** *p induces an isomorphism $p_*\colon [A, X]^l \cong [A, Y]^l$.*

  Then for every map $f\colon A \longrightarrow Y$ there is a map $g\colon A \longrightarrow X$, unique up to left homotopy, such that $p \circ g = f$.

Proof. I will prove the first part (by dual argument the second part holds).
Since $i^*\colon [B, X]^r \longrightarrow [A, X]^r$ is surjective there is a map $h\colon B \longrightarrow X$ such that $h \circ i \underset{r}{\simeq} f$.

Now by Proposition 57, there exists a mapping $g\colon B \longrightarrow X$ such that $g \circ i = f$.
The fact that this mapping is unique up to right homotopy follows from the fact that $i^*\colon [B, X]^r \longrightarrow [A, X]^r$ is and injective mapping.

□

I enunciate now two definitions,

Definition 59. *(Composition of Homotopies)*
Let $\mathcal{M}$ be a model category and let X and Y be objects in $\mathcal{M}$.

- **(1)** *If X is cofibrant,*

$$X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$$

and

$$X \amalg X \xrightarrow{i'_0 \amalg i'_1} Cyl'(X) \xrightarrow{p'} X$$

are cylinder objects for X,
$H\colon Cyl(X) \longrightarrow Y$ is a left homotopy from $f\colon X \longrightarrow Y$ to $g\colon X \longrightarrow Y$,
and $H'\colon Cyl'(X) \longrightarrow Y$ is a left homotopy from g to $h\colon X \longrightarrow Y$,
then the composition of the left homotopies H and H' is the left homotopy $H \cdot H'\colon Cyl''(X) \longrightarrow Y$ from f to h,

(where $Cyl''(X)$ is the pushout of the diagram $Cyl(X) \xleftarrow{i_1} X \xrightarrow{i'_0} Cyl'(X)$) defined by H and H'.
That pushout is,

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i'_0"] & Cyl(X) \arrow[d] \\
Cyl(X)' \arrow[r] & Cyl(X)''
\end{tikzcd}$$

(2) If Y is fibrant,

$$Y \xrightarrow{s} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$$

and

$$Y \xrightarrow{s'} Path'(Y) \xrightarrow{p'_0 \times p'_1} Y \times Y$$

are path objects for Y,
$H\colon X \longrightarrow Path(Y)$ is a right homotopy from $f\colon X \longrightarrow Y$ to $g\colon X \longrightarrow Y$,
and $H'\colon X \longrightarrow Path(Y)'$ is a right homotopy from g to $h\colon X \longrightarrow Y$,
then the composition of the right homotopies H and H' is the right homotopy $H \cdot H'\colon X \longrightarrow Path''(Y)$ from f to h,

(where $Path''(Y)$ is the pullback of the diagram $Path(Y) \xrightarrow{p_1} Y \xleftarrow{p'_0} Path(Y)'$) defined by H and H'.
That pullback is,

$$\begin{tikzcd}
Path''(Y) \arrow[r] \arrow[d] & Path'(Y) \arrow[d, "p'_0"] \\
Path(Y) \arrow[r, "p_1"'] & Y
\end{tikzcd}$$

Definition 60. (Inverse of Homotopies)
Let $\mathcal{M}$ be a model category and let X and Y be objects in $\mathcal{M}$.

(1) If $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ is a cylinder object for X,
and $H\colon Cyl(X) \longrightarrow Y$ is a left homotopy from $f\colon X \longrightarrow Y$ to $g\colon X \longrightarrow Y$,
then the inverse of H is the left homotopy $H^{-1}\colon Cyl(X)^{-1} \longrightarrow Y$ from g to f
where $X \amalg X \xrightarrow{i_0^{-1} \amalg i_1^{-1}} Cyl(X)^{-1} \xrightarrow{p^{-1}} X$ is the cylinder object for X defined by,
$Cyl(X)^{-1} = Cyl(X)$, $i_0^{-1} = i_1$, $i_1^{-1} = i_0$, and $p^{-1} = p$,
and the map H^{-1} equals the map H.

(2) If $Y \xrightarrow{s} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ is a path object for Y
and $H\colon X \longrightarrow Path(Y)$ is a right homotopy from $f\colon X \longrightarrow Y$ to $g\colon X \longrightarrow Y$,
then the inverse of H is the right homotopy $H^{-1}\colon X \longrightarrow Path(Y)^{-1}$ from g to f,
where $Y \xrightarrow{s^{-1}} Path(Y)^{-1} \xrightarrow{p_0^{-1} \times p_1^{-1}} Y \times Y$ is the path object for Y defined by,
$Path(Y)^{-1} = Path(Y)$, $p_0^{-1} = p_1$, $p_1^{-1} = p_0$, and $s^{-1} = s$,

and the map H^{-1} *equals the map* H*.*

The notion of a left and of a right homotopy associated to the standard cylinder and path objects of the category of topological spaces are the same, because I have the adjunction relation,

$$mor_{Top}(A \times [0,1], X) = mor_{Top}(A, mor_{Top}([0,1], X))$$

Furthermore and again I retrieve the classical notion of a homotopy used in topology. In general however the notion of a left and of a right homotopy are not equivalent.

Nevertheless, I have the following assertion

Lemma 61. *If parallel morphisms* $f, g\colon A \longrightarrow X$ *are defined on a cofibrant domain* A*, then I have* $f \underset{l}{\simeq} g \Rightarrow f \underset{r}{\simeq} g$*.*

Dually, if f *and* g *have a fibrant codomain* X*, then I have* $f \underset{r}{\simeq} g \Rightarrow f \underset{l}{\simeq} g$

Proof. This follows directly from Lemma 16 and the following Lemma 62 ☐

Lemma 62. *Let* $\mathcal{M}$ *be a model category.*

Let $f, g\colon X \longrightarrow Y$ *be maps in* $\mathcal{M}$*.*

(1) *If* X *is cofibrant,* f *is left homotopic to* g*, and* $Y \xrightarrow{s} Path(Y) \xrightarrow{p_0 \times p_1} Y \times Y$ *is a path object for* Y*, then there is a right homotopy* $H\colon X \longrightarrow Path(Y)$ *from* f *to* g*.*

(2) *If* Y *is fibrant,* f *is right homotopic to* g*, and* $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ *is a cylinder object for* X*, then there is a left homotopy* $H : Cyl(X) \longrightarrow Y$ *from* f *to* g*.*

Proof. I will prove the first part (by dual argument the second part holds).

Since f is left homotopic to g, there is a cylinder object $X \amalg X \xrightarrow{i_0 \amalg i_1} Cyl(X) \xrightarrow{p} X$ for X and a left homotopy $G\colon Cyl(X) \longrightarrow Y$ from f to g.

Thus, I have the solid arrow diagram

$$\begin{tikzcd}
X \arrow[r, "s \circ f"] \arrow[d, "i_0"'] & Path(Y) \arrow[d, "p_0 \times p_1"] \\
Cyl(X) \arrow[r, "(f \circ p) \times G"'] \arrow[ur, dashed, "h"] & Y \times Y
\end{tikzcd}$$

in which $p_0 \times p_1$ is a fibration.

Since X is cofibrant, Proposition 18 implies that i_0 is a trivial cofibration, and so the dotted arrow h exists.

If I let $H = h \circ i_1$, then H is the right homotopy I require. ☐

I early said that in general morphisms $f, g\colon A \longrightarrow X$ are homotopic when they are both left homotopic, that I denote by $f \underset{l}{\simeq} g$ and right homotopic, that I denote by $f \underset{r}{\simeq} g$. I then write $f \simeq g$. The set of homotopy classes of morphisms $[A, X]_{\mathcal{M}}$, which I associate to any pair of objects A,$X \in \mathcal{M}$, is the quotient of the set of morphisms $mor_{\mathcal{M}}(A, X)$ under the transitive closure of this homotopy relation $\simeq$.

Now the Lemma 61 implies that I have,

$$f \simeq g \Leftrightarrow f \simeq_l g,$$

and hence I have,

$$[A, X]_{\mathcal{M}} = mor_{\mathcal{M}}(A, X)/ \simeq_l$$

when A is cofibrant,

whereas I have,

$$[A, X]_{\mathcal{M}} = mor_{\mathcal{M}}(A, X)/ \simeq_r$$

when A is fibrant.

In the case of topological spaces, I just retrieve the standard set of homotopy classes of topology with this definition.

But the homotopy relation may not be transitive in general. Nevertheless

Proposition 63. *Let $\mathcal{M}$ be a model category, and let X and Y be objects in $\mathcal{M}$.*

(1) *If A is a cofibrant object in $\mathcal{M}$, then left homotopy is an equivalence relation on the set of morphisms from A to X, that is $mor_{\mathcal{M}}(A, X)$.*
(2) *If X is a fibrant object in $\mathcal{M}$, then right homotopy is an equivalence relation on the set of morphisms from A to X, that is $mor_{\mathcal{M}}(A, X)$.*
(3) *If A is cofibrant and X is fibrant, then homotopy is an equivalence relation on the set on morphisms from A to X.*

Proof. I will prove the first part (then by dual argument the second part holds and clearly thrid part follows from the preevious parts).

Since there is a cylinder object for A in which $Cyl(A) = A$, left homotopy is reflexive.

The inverse of a left homotopy implies that left homotopy is symmetric by Definition 60.

The composition of left homotopies implies that left homotopy is transitive by Definition 59.

□

Proposition 64. *Let $\mathcal{M}$ be a model category.*

If X is cofibrant and Y is fibrant, then the left homotopy, right homotopy, and homotopy relations coincide and are equivalence relations on the set of morphisms from X to Y

Proof. This follows directly from Proposition 61 and Proposition 63 □

The definition of composition operation on homotopy classes of morphisms leads to the following simple and more precise observations:

Lemma 65. *Let $\mathcal{M}$ be a model category and let $f, g\colon X \longrightarrow Y$ be maps in $\mathcal{M}$*

(1) *If $f \overset{l}{\simeq} g$ and $h\colon Y \longrightarrow Z$ is a morphism, then $h \circ f \overset{l}{\simeq} h \circ g$.*
(2) *If $f \overset{r}{\simeq} g$ and $k\colon W \longrightarrow X$ is a morphism, then $f \circ k \overset{r}{\simeq} g \circ k$.*

Proof. I will first prove (1). The proof of (2) is dual.

If $X \amalg X \longrightarrow Cyl(X) \longrightarrow X$ is a cylinder object for X and $F\colon Cyl(X) \longrightarrow Y$ is a left homotopy from f to g, then $h \circ F$ is a left homotopy $h \circ f$ to $h \circ g$.

□

As a direct consequence I obtain,

Corollary 66.

(1) *If $f \overset{l}{\simeq} g$ and $h\colon Y \longrightarrow Z$ is a morphism, then composition with h induces a well defined $h_*\colon [X, Y]^l \longrightarrow [X, Z]^l$.*
(2) *If $f \overset{r}{\simeq} g$ and $k\colon W \longrightarrow X$ is a morphism, then composition with h induces a well defined $k^*\colon [X, Y]^r \longrightarrow [W, Y]^r$.*

In order to get the identity of the homotopy category with a localization, I will also use the following two results,

Proposition 67. *Let $\mathcal{M}$ be a model category.*

(1) *If A is cofibrant and $p\colon X \longrightarrow Y$ is a trivial fibration, then p induces an isomorphism of the sets of left homotopy classes of maps $p_*\colon [A, X]^l \longrightarrow [A, Y]^l$.*
(2) *If X is fibrant and $i\colon A \longrightarrow B$ is a trivial cofibration, then i induces an isomorphism of the set of right homotopy classes of maps $i^*\colon [B, X]^r \longrightarrow [A, X]^r$.*

Proof. I will first prove (1). The proof of (2) is dual.

I know that p_* is well defined by Lemma 66.

If $g\colon A \longrightarrow Y$ is a map and $\emptyset$ is the initial object of $\mathcal{M}$, then M4 axiom (lifting axiom) implies that in the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
A \arrow[r, "g"'] \arrow[ur, dashed, "f"] & Y
\end{tikzcd}$$

there exists the dotted arrow and so p_* is surjective.

To see that p_* is injective, let $f, g\colon A \longrightarrow X$ be maps such that $p \circ f \overset{l}{\simeq} p \circ g$.

Then, there exists a cylinder object $A \amalg A \longrightarrow Cyl(A) \longrightarrow A$ for A and a left homotopy $F\colon Cyl(A) \longrightarrow Y$ from $p \circ f$ to $p \circ g$, and so I have the solid arrow diagram,

$$\begin{tikzcd}
A \amalg A \arrow[r, "f \amalg g"] \arrow[d] & X \arrow[d, "p"] \\
Cyl(A) \arrow[r, "F"'] \arrow[ur, dashed, "G"] & Y
\end{tikzcd}$$

Now M4 axiom (lifting axiom) implies that the dotted arrow G exists, and G is a left homotopy from f to g.

□

As a direct consequence of the previous one, I obtain,

Proposition 68. *If A is cofibrant and $h\colon X \longrightarrow Y$ is a weak-equivalence between fibrant objects, then the mapping,*

$$h_*\colon [A, X]_{\mathcal{M}} \xrightarrow{\sim} [A, Y]_{\mathcal{M}}$$

associated to h is one to one.

Dually, if X is fibrant and $k\colon A \longrightarrow B$ is a weak equivalence between cofibrant objects, then the mapping,

$$k^*\colon [B, X]_{\mathcal{M}} \longrightarrow [A, X]_{\mathcal{M}}$$

associated to k is one to one.

Proposition 69. *Let $\mathcal{M}$ be a model category.*

Let X, Y, Z be fibrant + cofibrant objects of $\mathcal{M}$.

Let $f, g\colon X \longrightarrow Y$ and $h, k\colon Y \longrightarrow Z$ be morphisms.

If $f \simeq g$ and $h \simeq k$, then $h \circ f \simeq k \circ g$.

(So composition is well defined on homotopy classes of maps between fibrant + cofibrant objects).

Proof. It follows directly from Lemma 65

□

As an immediate consequence of Proposition 67 when $X = A$ and $Y = B$, I obtain again the Whitehead Theorem (and its converse) for model categories that I formally proved in the previous section.

Recall that I write $\mathcal{M}_{fc}$ for the full subcategory generated by these objects wich are both fibrant and cofibrant in $\mathcal{M}$.

As an immediate consequence from both the previous Proposition 67 and Theorem 12, the Whitehead Theorem for model categories I obtain the following statement.

Proposition 70. *The sets of homotopy classes $[A, B]_{\mathcal{M}}$ define the morphism sets of a category $Ho\mathcal{M}_{fc}$ with the fibrant + cofibrant objects of $\mathcal{M}$ as objects, a composition structure loaned from the composition of morphisms in $\mathcal{M}$, and where the weak equivalences become isomorphisms.*

Proposition 71. *Let $\mathcal{M}$ be a model category.*

If X and Y are fibrant + cofibrant objects in $\mathcal{M}$, then a map $g\colon X \longrightarrow Y$ is a homotopy equivalence if either of the following two conditions hold:

(1) *The map g induces isomorphisms of the sets of homotopy classes of maps $g_*\colon [X, X] \cong [X, Y]$ and $g_*\colon [Y, X] \cong [Y, Y]$.*

(2) *The map g induces isomorphisms of the sets of homotopy classes of maps $g^*\colon [Y, X] \cong [X, X]$ and $g^*\colon [Y, Y] \cong [X, Y]$.*

Proof. I will prove the result using the first condition (1) (the proof using the second condition is analogous).

The isomorphisms $g_*\colon [Y,X] \cong [Y,Y]$ implies that there is a map $h\colon Y \longrightarrow X$ such that $g \circ h \simeq Id_Y$.

But now using Proposition 69 and the isomorphism $g_*\colon [X,X] \cong [X,Y]$ which is satisfied by the statement, both of them implies that h induces and isomorphism $h_*\colon [X,Y] \cong [X,X]$, and therefore there exists a map $k\colon X \longrightarrow Y$ such that $h \circ k \simeq Id_X$.

Thus, h is a homotopy equivalence and g is its inverse, and hence g is a homotopy equivalence as well.

□

It still remains as my initial aim to prove that the localization of the model category $\mathcal{M}$ with respect to the class of weak equivalences is identified with a homotopy category associated to $\mathcal{M}$.

I establish the following intermediate statement about the subcategory $\mathcal{M}_{fc}$ of fibrant + cofibrant objects.

Theorem 14. *The homotopy category* $Ho\ \mathcal{M}_{fc}$ *in the previous Proposition 70 also represents the localization of the category* $\mathcal{M}_{fc}$ *of fibrant + cofibrant objects with respect to the class of morphisms formed by the weak equivalences* $f\colon A \xrightarrow{\sim} B$ *such that* $A,\ B \in \mathcal{M}_{fc}$.

In fact I can precise much more: the canonical functor $\gamma_{fc}\colon \mathcal{M}_{fc} \longrightarrow Ho\ \mathcal{M}_{fc}$*, which is the identity on objects and is yielded by the quotient map on morphism sets, satisfies the universal property for the localization of a category with respect to a class of morphisms, that is, any functor* $F\colon \mathcal{M} \longrightarrow \mathcal{A}$ *with the established property for its morphism admits a unique factorization,*

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma_{fc}"'] & & \mathcal{A} \\
& Ho\ \mathcal{M}_{fc} \arrow[ur, dashed, "\bar{F}"'] &
\end{tikzcd}$$

such that $F = \bar{F} \circ \gamma_{fc}$

Proof. Let $F\colon \mathcal{M}_{fc} \longrightarrow \mathcal{A}$ be any functor mapping weak equivalences in $\mathcal{M}_{fc}$ into isomorphisms. I want to prove that this functor has a factorization $F = \bar{F} \circ \gamma_{fc}$ for a uniquely determined functor $\bar{F}$ on $Ho\ \mathcal{M}_{fc}$.

In order to establish the existence of that factorization, I will check that,

for any pair of parallel morphisms $f,\, g\colon A \longrightarrow B$ in $\mathcal{M}_{fc}$, I have $f \simeq g \overset{?}{\Rightarrow} F(f) = F(g)$.

Let $h\colon Cyl(A) \longrightarrow B$ be a left homotopy between f and g, such that,

$$f = h \circ i_0 \qquad\qquad \text{and} \qquad\qquad g = h \circ i_1.$$

Recall that I am using here the definition of a cylinder object:

So, I have,

$$\begin{tikzcd}
A \amalg A \arrow[r, "Id_A \amalg Id_A"] \arrow[d, "i_0 \amalg i_1"'] & A \\
Cyl(A) \arrow[ur, "\sim"'] &
\end{tikzcd}$$

and I have,

$$\begin{tikzcd}
& A \arrow[dr, "j_0"] \arrow[drr, bend left, "i_0"] & & \\
\emptyset \arrow[ur, "j_A"] \arrow[dr, "j_A"'] & & A \amalg A \arrow[r, "i_0 \amalg i_1"] & Cyl(A) \\
& A \arrow[ur, "j_1"'] \arrow[urr, bend right, "i_1"'] & &
\end{tikzcd}$$

where all the squares and the diagram triangles commute.

I denote by l, the weak equivalence (in fact a trivial fibration) in the definition of cylinder object

$$l\colon Cyl(A) \xrightarrow{\sim} A$$

It holds the identities, $l \circ i_0 = Id_A = l \circ i_1$,

Indeed simply by considering the complete diagram,

$$\begin{tikzcd}
 & A \arrow[dr, "j_0"] \arrow[drr, bend left, "i_0"] \arrow[drrr, bend left, "l\circ i_0"] & & & \\
\emptyset \arrow[ur, "j_A"] \arrow[dr, "j_A"'] & & A \amalg A \arrow[r, "i_0 \amalg i_1"] & Cyl(A) \arrow[r, "l", "\sim"'] & A \\
 & A \arrow[ur, "j_1"'] \arrow[urr, bend right, "i_1"'] \arrow[urrr, bend right, "l\circ i_1"'] & & &
\end{tikzcd}$$

Now, the previous identities

$$l \circ i_0 = Id_A = l \circ i_1$$

and the requirement that l is a weak equivalence in the definition of a cylinder object implies that I have

$$F(i_0) = F(l)^{-1} = F(i_1) \text{ in } \mathcal{A}.$$

So, I obtain

$$F(f) = F(h) \circ F(i_0) = F(h) \circ F(i_1) = F(g)$$

and the construction immediately follows.

The uniqueness of this factorization is immediate since the homotopy category $Ho\, \mathcal{M}_{fc}$ is defined by a quotient construction at the level of morphism sets. □

I use now this category $Ho\, \mathcal{M}_{fc}$ to identify the localization of the category $\mathcal{M}$ with a homotopy category. The main fact in the development relies on the definition of a functor $\gamma'\colon \mathcal{M} \longrightarrow Ho\, \mathcal{M}_{fc}$ which induces an equivalence at the level of this localization.

I will use the concept of a fibrant and a cofibrant resolution which I now remind to define this functor on $\mathcal{M}$

In general,

I call fibrant resolution of an object $X \in \mathcal{M}$ any fibrant object $R \in \mathcal{M}_f$ equipped with a weak equivalence $X \xrightarrow{\sim} R$

and equivalently,

I call cofibrant resolution to any cofibrant object $Q \in \mathcal{M}_c$ equipped with a weak equivalence $Q \xrightarrow{\sim} X$.

The M5 axiom (factorization axiom) implies that any object $X \in \mathcal{M}$ has at least one cofibrant resolution $QX \in \mathcal{M}_c$ fitting in a factorization $\emptyset \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X$ where the weak equivalence towards X is an acyclic fibration.

Both, the M4 axiom (lifting axiom) and the M2 axiom (two out of three axiom) implies that any other cofibrant resolution $Q \xrightarrow{\sim} X$ can be linked to a fixed resolution of this form $QX \overset{\sim}{\twoheadrightarrow} X$ by a weak equivalence $Q \xrightarrow{\sim} QX$ which I obtain by picking a solution of the following lifting problem,

$$\begin{tikzcd}
 & QX \arrow[d, two heads, "\sim"] \\
Q \arrow[ur, dashed] \arrow[r, "\sim"'] & X
\end{tikzcd}$$

Dually, the M5 axiom (factorization axiom) implies that any object $X \in \mathcal{M}$ has at least one fibrant resolution $RX \in \mathcal{M}_f$ fitting in a factorization $X \rightarrowtail RX \overset{\sim}{\twoheadrightarrow} *$ where the weak equivalence to RX is an acyclic cofibration.

Again, both the M4 axiom (lifting axiom) and the M2 axiom (two out of three axiom) also implies that any other fibrant resolution $X \xrightarrow{\sim} R$ can be linked to a fixed resolution of this form $X \overset{\sim}{\rightarrowtail} RX$ by a weak equivalence $RX \xrightarrow{\sim} R$.

Recall that I set $\mathcal{M}_f \subset \mathcal{M}$ and respectively $\mathcal{M}_c \subset \mathcal{M}$ for the full subcategory generated by the class of fibrant and respectively cofibrant objects in the model category $\mathcal{M}$ and I have $\mathcal{M}_{fc} = \mathcal{M}_f \cap \mathcal{M}_c$ for the category of fibrant + cofibrant objects considered in the previous Theorem 14.

Now I consider the following fact,

Proposition 72. *I have a diagram of equivalent categories*

$$\begin{tikzcd}
& Ho\ \mathcal{M}_c \arrow[dr, "\sim"] & \\
Ho\ \mathcal{M}_{fc} \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & Ho\ \mathcal{M} \\
& Ho\ \mathcal{M}_f \arrow[ur, "\sim"'] &
\end{tikzcd}$$

when I consider the localization of the subcategories $\mathcal{M}_{fc}$, $\mathcal{M}_f$, $\mathcal{M}_c \subset \mathcal{M}$ with respect to the class of weak equivalences inherited from $\mathcal{M}$.

I can deduce from the definition of cofibrant resolutions $QX \xrightarrow{\sim} X$ that the functor $Ho\ \mathcal{M}_c \longrightarrow Ho\ \mathcal{M}$ is surjetive, as well as full, but I still have to check that this functor is faithful in order to get the category equivalence that I claimed in the category $Ho\ \mathcal{M}$.

Having such resolutions $QX \xrightarrow{\sim} X$ which define isomorphisms in the category $Ho\ \mathcal{M}$, then proving that the functor $Ho\ \mathcal{M}_c \longrightarrow Ho\ \mathcal{M}$ defines a category equivalence is equivalent to prove that the mapping $Q\colon X \longrightarrow QX$ extends to a functor which forms an inverse category equivalence of the functor $Ho\ \mathcal{M}_c \longrightarrow Ho\ \mathcal{M}$. I establish this claim directly, and I argue similarly for the other functors considered in the statement of the previous proposition.

I precisely deduce the previous Proposition 72 from the following fact:

Proposition 73.

- **(a)** *The mapping $Q\colon X \mapsto QX$, determined by the choice of a cofibrant resolution $QX \xrightarrow{\sim} X$ for every object $X \in \mathcal{M}$, extends to a functor $\bar{Q}\colon Ho\ \mathcal{M} \longrightarrow Ho\ \mathcal{M}_c$ and this functor defines an inverse category equivalence of the functor $Ho\ \mathcal{M}_c \longrightarrow Ho\ \mathcal{M}$ induced by the canonical category embedding $\mathcal{M}_c \hookrightarrow \mathcal{M}$.*
- **(b)** *The mapping $R\colon X \mapsto RX$, determined by the choice of a fibrant resolution $X \xrightarrow{\sim} RX$ for every object $X \in \mathcal{M}$, extends to a functor $\bar{R}\colon Ho\ \mathcal{M} \longrightarrow Ho\ \mathcal{M}_f$ and this functor defines an inverse category equivalence of the functor $Ho\ \mathcal{M}_f \longrightarrow Ho\ \mathcal{M}$ induced by the canonical category embedding $\mathcal{M}_f \hookrightarrow \mathcal{M}$.*
- **(c)** *For every $X \in \mathcal{M}$, I pick a fibrant + cofibrant object RQX that fits in a factorization*

$$\begin{tikzcd}
QX \arrow[r, tail, dashed, "\sim"] & RQX \arrow[r, two heads, dashed, "\sim"] & RX
\end{tikzcd}$$

of the composite morphism $QX \xrightarrow{\sim} X \xrightarrow{\sim} RX$ associated to the choice of a cofibrant resolution QX and of a fibrant resolution RX for that object. The mapping $RQ\colon X \mapsto RQX$ extends to a functor $\bar{RQ}\colon Ho\ \mathcal{M} \longrightarrow Ho\ \mathcal{M}_{fc}$ and this functor defines an inverse category equivalence of the functor $Ho\ \mathcal{M}_{fc} \longrightarrow Ho\ \mathcal{M}$ induced by the canonical category embedding $\mathcal{M}_{fc} \hookrightarrow \mathcal{M}$.

Proof. I will prove the assertion in the first part (a). Assertion (b) and (c) in the statement can be obtained by an analogous argument.

I choose a cofibrant resolution QX such that I have and acyclic fibration $QX \overset{\sim}{\twoheadrightarrow} X$ for every $X \in \mathcal{M}$.

It suffices to complete the prove in this environment, because I early observe that any other choice of resolutions yields an object isomorphic to a resolution of this form in the homotopy category of cofibrant objects $\mathcal{M}_c$.

I define a functor $Q\colon \mathcal{M} \longrightarrow Ho\ \mathcal{M}_c$ and I use the universal property of localization to obtain the category equivalence considered in the proposition.

I take the mapping $Q\colon X \mapsto QX$ to get the value of this functor on objects $X \in \mathcal{M}$.

Let $f\colon X \longrightarrow Y$ be any morphism in $\mathcal{M}$.

Using the M4 axiom (lifting axiom) for model categories, I obtain the morfism $\widetilde{f}$

$$\begin{tikzcd}
X \arrow[d, "f"'] & QX \arrow[l, two heads, "\sim"] \arrow[d, "\exists\ \widetilde{f}"] \\
Y & QY \arrow[l, two heads, "\sim"]
\end{tikzcd}$$

that extends f.

By similar arguments as I used in Theorem 14, I check that this extension $\widetilde{f}$ does not depend on choices when I pass to the homotopy category of cofibrant objects $Ho\ \mathcal{M}_c$.

Let $Cyl(QX)$ be a good cylinder object for QX. I know that cofibrations are closed under composition and this fact inplies that $Cyl(QX)$ is cofibrant, and hence, determines an object in the category $Ho\ \mathcal{M}_c$.

If I have another choice $\widehat{f}$ for this extension, then by completing the diagram

$$\begin{tikzcd}
QX \amalg QX \arrow[rr, "\widetilde{f}\amalg\widehat{f}"] \arrow[d, tail, "i_0\amalg i_1"'] \arrow[dd, bend right=60, "Id_{QX}\amalg Id_{QX}"'] & & QY \arrow[dd, "\sim"] \\
Cyl(QX) \arrow[rru, dashed, "\exists\ h"] \arrow[d, "l"'] & & \\
QX \arrow[r, "\sim"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

I obtain a homotopy between $\widehat{f}$ and $\widetilde{f}$.

In the category $Ho\ \mathcal{M}_c$ I then obtain the identities,

$l \circ i_0 = Id = l \circ i_1 \ \Rightarrow\ i_0 = l^{-1} = i_1 \ \Rightarrow\ h \circ i_0 = h \circ l^{-1} = h \circ i_1 \ \Rightarrow\ \widehat{f} = \widetilde{f}$

So, indeed the uniqueness is proved.

From this result, I deduce that the mapping on morphisms $Q\colon f \mapsto \widetilde{f}$ preserve the composition of morphisms in addition to identities, when I take values in the homotopy category $Ho\ \mathcal{M}_c$.

Therefore, the construction yields a well defined functor $Q\colon \mathcal{M}\ Ho\ \mathcal{M}_c$.

Now by M2 axiom (two out of three axiom), I also deduce that the mapping $Q\colon f \mapsto \widetilde{f}$ assigns weak equivalences to weak equivalences and hence, this functor induces a functor on the homotopy category $\bar{Q}\colon Ho\ \mathcal{M} \longrightarrow Ho\mathcal{M}_c$ as I desired to prove.

Let $\bar{i}\colon Ho\ \mathcal{M}_c \longrightarrow Ho\ \mathcal{M}$ denote the functor induced by the category embedding.

I have $\bar{Q} \circ \bar{i} = Id_{\mathcal{M}_c}$ by construction, and the weak equivalence assocciated to the resolution $QX \xrightarrow{\sim} X$ also gives a natural equivalence between the composite $\bar{i} \circ \bar{Q}$ and the identity in $Ho\ \mathcal{M}$.

□

I finally conclude from the results of Theorem 14 and Proposition 72 that the morphism sets of $Ho\ \mathcal{M}$ are identified with sets of homotopy classes of morphisms in the model category $\mathcal{M}$.

In fact, I have,

$$Mor_{Ho\ \mathcal{M}}(X, Y) = Mor_{Ho\ \mathcal{M}_{fc}}(RQX, RQY) = [RQX, RQY]_{\mathcal{M}}$$

where $RQ\colon X \mapsto RQX$ is the fibrant + cofibrant object construction considered in Proposition 73.

I can use the homotopy invariance properties showed in Proposition 68 to get the following immediate extension of this identity:

Theorem 15. *Let $\mathcal{M}$ be any model category. In the homotopy category $Ho\ \mathcal{M}$, defined as localization with respect to the class of weak equivalences I have the identity,*

$$Mor_{Ho\ \mathcal{M}}(X, Y) = [QX, RY]_{\mathcal{M}}$$

for any pair of objects X, $Y \in \mathcal{M}$, where QX is a cofibrant resolution of the source object X in $\mathcal{M}$, and RY is a fibrant resolution of the target object Y.

I also consider RQX and RQY both fibrant + cofibrant resolutions of X and Y respectively.

To get the identity of this theorem from the previous observation I just use that, according to the result of Proposition 68 the weak equivalences

$$QX \xrightarrow{\sim} RQX$$

and

$$RQY \xrightarrow{\sim} QY$$

in Proposition 73 induce bijections

$$[RQX, RQY] \xrightarrow{\simeq} [QX, RY].$$

In fact, I can be more precise with this statement. Recall that the universal property of localizations characterizes the category $Ho\ \mathcal{M}$ up to isomorphisms only.

I claim that, for any choice of a cofibrant resolution construction $Q \colon X \mapsto QX$ and any choice of fibrant resolution construction $R \colon Y \mapsto RY$ on the objects of the category $\mathcal{M}$, the collection of homotopy class sets $[QX, RY]_{\mathcal{M}}$ considered in the identity of the Theorem define the morphism sets of a category which represents this localization.

The identity of this theorem gives the practical definition of the morphism sets in the homotopy category $Ho\ \mathcal{M}$. In practice, I also use the homotopy invariance result of Proposition 73 to determine the composition operation of morphisms in $Ho\ \mathcal{M}$ from this representation.

Indeed, for any objects X, Y, Z and for any choice of fibrant and cofibrant resolutions in $\mathcal{M}$, I can identify the composition operation,

$$\circ \colon mor_{Ho\ \mathcal{M}(Z,Y)} \times mor_{Ho\ \mathcal{M}(X,Y)} \longrightarrow mor_{Ho\ \mathcal{M}}(X, Y)$$

on the homotopy category $Ho\ \mathcal{M}$ with the composite map:

$$[QZ, RY]_{\mathcal{M}} \times [QX, RZ]_{\mathcal{M}} \xleftarrow{\simeq} [RQZ, RY]_{\mathcal{M}} \times [QX, RQZ]_{\mathcal{M}} \xrightarrow{\circ} [QX, RY]_{\mathcal{M}}$$

where I use the homotopy invariance of homotopy class sets to repalace the cofibrant object defined by $QZ \xrightarrow{\sim} Z$ and the fibrant object defined by $Z \xrightarrow{\sim} RZ$ by a fibrant + cofibrant object RQZ.

Functors between Model Categories

Having introduced the homotopy category associated to a model category $\mathcal{M}$, a first question that arises is to ask when a functor $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ induces a functor from the homotopy category $Ho\ M \longrightarrow \mathcal{N}$.

In some sense, this is equivalent to asking when a functor F is compatible with the internal homotopy relation of $\mathcal{M}$ and I note that this does not depend on whether or not $\mathcal{N}$ is a model category.

In this section I briefly study what might be a good notion of "morphism between model categories".

Definition 74. *(Left and Right Derived Functor)*

Let $\mathcal{M}$ be a model category, $\mathcal{N}$ any category and $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ a functor. A left derived functor of F is a functor $\mathbb{L}F \colon Ho\ \mathcal{M} \longrightarrow \mathcal{N}$ with a natural transformation $\sigma \colon \mathbb{L}F \circ \gamma \implies F$, that is universal among such pairs.

That is, the triangle

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \\
Ho\ \mathcal{M} \arrow[ur, dashed, "\mathbb{L}F"'] &
\end{tikzcd}$$

commute up to the natural transformation σ, and for any other such pair $(\mathbb{L}F', \sigma')$, there is a unique natural transformation $\mathbb{L}F' \overset{\alpha}{\Longrightarrow} \mathbb{L}F$ such that $\sigma \circ (\alpha \circ Id_\gamma) = \sigma'$.
Dually, a right derived functor of F is a functor $\mathbb{R}F \colon Ho\ \mathcal{M} \longrightarrow \mathcal{N}$ with a natural transformation $\sigma \colon F \Longrightarrow \mathbb{R}F \circ \gamma$, that is universal among such pairs.

That is, the triangle

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \\
Ho\ \mathcal{M} \arrow[ur, dashed, "\mathbb{R}F"'] &
\end{tikzcd}$$

commute up to the natural transformation σ, and for any other such pair $(\mathbb{R}F', \sigma')$, there is a unique natural transformation $\mathbb{R}F \overset{\alpha}{\Longrightarrow} \mathbb{R}F'$ such that $(\alpha \circ Id_\gamma) \circ \sigma = \sigma'$

If a functor F from a model category $\mathcal{M}$ sends all weak equivalences to isomorphisms, the universal property of the localized category $Ho\ \mathcal{M}$ says that there is a unique filler

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \\
Ho\ \mathcal{M} \arrow[ur, dashed, "\mathbb{L}F=\mathbb{R}F"'] &
\end{tikzcd}$$

and the triangle strictly commutes (not only up to a natural transformation).

A weaker, and very often used, condition for which a functor $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ admits a left derived functor is when F sends acyclic cofibrations between cofibrant objects to isomorphisms.

Indeed, if F satisfies this property a short argument shows that F sends all weak equivalences between cofibrant objects to isomorphisms (this is Lemma 48, Ken Brown's Lemma).

In this case, a possible derived functor is the composite $\mathbb{L}F = F \circ Q$ for some cofibrant replacement Q. Indeed, any change of cofibrant replacement Q is only seen up to isomorphism in $\mathcal{N}$, thanks to the fact that F sends all weak equivalences between cofibrant objects to isomorphisms in $\mathcal{N}$. A dual condition holds for a right derived functor of F.

If in addition $\mathcal{N}$ is also endowed with a model structure, we can now be interested in when $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ induces a functor between the homotopy categories. A similar condition as in the previous paragraph may now be weakened, because I do not need a "complete" lift $Ho\ \mathcal{M} \longrightarrow \mathcal{N}$, but only a lift in the homotopy category $Ho\ \mathcal{M} \longrightarrow Ho\ \mathcal{N}$.

Definition 75. *(Total Left/Right Derived Functor)*

A total left derived functor of $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ between model categories, is a functor between the homotopy categories $\mathbb{L}F \colon Ho\ \mathcal{M} \longrightarrow Ho\ \mathcal{N}$ that is left derived functor of the composite,

$$\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow Ho\ \mathcal{N}$$

Similarly, a total right derived functor of F is $\mathbb{R}F \colon Ho\ M \longrightarrow Ho\ \mathcal{N}$, that is a right derived functor of the same composite.

However, I am often interested in functors with more structure than just inducing a functor on (from) the homotopy category. In order to be able to compare two homotopy categories $\mathcal{M}$ and $\mathcal{N}$, I would like an adjunction between them, and moreover a Quillen adjunction.

Quillen Functors.

As I have just introduced, given two model categories $\mathcal{M}$ and $\mathcal{N}$, the intuitive notion of a morphism between $\mathcal{M}$ and $\mathcal{N}$ would seem to be a functor $\mathcal{M} \longrightarrow \mathcal{N}$ which is compatible with the model category structures of $\mathcal{M}$ and $\mathcal{N}$, that is, a functor which preserves cofibrations, fibrations and weak equivalences or equivalently a functor which preserves cofibrations, trivial cofibrations, fibrations and trivial fibrations.

However most of the functors between model categories that one usually runs into do not have this property. But many of these are one of a pair of adjoint functors of which the left adjoint is compatible with one half of the model category structures of $\mathcal{M}$ and $\mathcal{N}$ in the sense that it preserves cofibrations and trivial cofibrations while the right adjoint is compatible with the other halves and preserves fibrations and trivial fibrations.
So that, I want to compare model categories betwwen them. I often get the following picture:

Proposition 76. *If I have a pair of adjoint functors between model categories $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$, then anyone of the following assertions implies the others:*

- **(a)** *The functor F maps cofibrations to cofibrations and acyclic cofibrations to acyclic cofibrations.*
- **(b)** *The functor F maps cofibrations to cofibrations and G maps fibrations to fibrations.*
- **(c)** *The functor G maps fibrations to fibrations and acyclic fibrations to acyclic fibrations.*

Proof. I consider $i\colon A \longrightarrow B$ a map in $\mathcal{M}$ and $p\colon X \longrightarrow Y$ a map in $\mathcal{N}$.
The adjointness of F and G implies that there exists a one to one correspondence, that is an equivalence of lifting problems,

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "F(i)"'] & X \arrow[d, "p"] \\ F(B) \arrow[r] \arrow[ur, dashed, "\exists\ h?"] & Y \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & G(X) \arrow[d, "G(p)"] \\ B \arrow[r] \arrow[ur, dashed, "\exists\ h'?"] & G(Y) \end{tikzcd}$$

and so, the dotted arrow h exists if and only if the dotted arrow h' exists.
But now, directly by both this last fact and Proposition 6 it holds,

- **(•)** the left adjoint F preseves cofibrations if and only if the right adjoint G preserves trivial fibrations
- **(•)** the left adjoint F preserves trivial cofibrations if and only if the right adjoint G preserves fibrations.

and so implies the equivalences in the proposition.

□

If the assertions in the previous Proposition 76 are satisfied, then I say that the functors F and G define a Quillen adjunction or equivalently, that they define Quillen adjoint functors.
More precisely, I define,

Definition 77. *(Quillen Functors)*
Given two model categories $\mathcal{M}$ and $\mathcal{N}$, a functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ is called a Left Quillen Functor if,

- **(i)** *F has a right adjoint, and*
- **(ii)** *F preserves cofibrations and trivial cofibrations and hence (by Lemma 48, Ken's Brown Lemma) weak equivalences between cofibrant objects.*

Dually, a functor $G\colon \mathcal{N} \longrightarrow \mathcal{M}$ will be called a Right Quillen Functor if,

- **(a)** *G has a left adjoint, and*
- **(b)** *G preserves fibrations and trivial fibrations and hence (by Lemma 48, Ken's Brown Lemma) weak equivalences between fibrant objects.*

So that, by an adjoint pair of Quillen functors I mean a pair of adjoint functors between model categories for which the left adjoint is a left Quillen functor and, hence, the right adjoint is a right Quillen functor.

The functors of a Quillen adjunction determine a pair of derived adjoint functors on the homotopy categories of model categories.

In the situation of the previous Proposition 76, if I moreover assume that the adjunction bijection $mor_{\mathcal{N}}(F(A), X) \cong mor_{\mathcal{M}}(A, G(X))$ induces a one to one correspondence between the subset of weak-equivalences $f \colon F(A) \xrightarrow{\sim} X$ in $mor_{\mathcal{N}}(F(A), X)$ and the subset of weak equivalences $g \colon A \xrightarrow{\sim} G(X)$ in $mor_{\mathcal{M}}(A, G(X))$ when A is cofibrant and X is a fibrant, then I also say that our functors F and G define a Quillen equivalence.

So that, there is a special kind of Quillen functors which induce "equivalences of homotopy theories" and which I can rigurously define,

Definition 78. *(Quillen Equivalences)*

An adjoint pair of Quillen functors $F \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ will be called an adjoint pair of Quillen equivalences and I call the left adjoint a Left Quillen Equivalence and the right adjoint a Right Quillen Equivalence if

- **(i)** *for every pair of objects $A \in \mathcal{M}_c$ and $X \in \mathcal{N}_f$, a map $A \longrightarrow G(X) \in \mathcal{M}$ is a weak equivalence iff its adjoint $F(A) \longrightarrow X \in \mathcal{N}$ is so,*
  or equivalently if
- **(ii)** *for every pair of objects $B \in \mathcal{M}_{fc}$ and $Y \in \mathcal{N}_{fc}$, a map $B \longrightarrow G(Y) \in \mathcal{M}$ is a weak equivalence iff its adjoint $F(B) \longrightarrow Y \in \mathcal{N}$ is so.*

If F and G both preserve weak equivalences (and not merely the trivial cofibrations or trivial fibrations), then this is also equivalent to requiring that,

- **(iii)** *for every pair of objects $B \in \mathcal{M}_{fc}$ and $Y \in \mathbb{N}_{fc}$, the adjunction maps*

$$B \longrightarrow G(F(B)) \in \mathcal{M} \text{ and } F(G(Y)) \longrightarrow Y \in \mathcal{N}$$

  are weak equivalences.

The functors of a Quillen equivalence determine a pair of derived adjoint equivalences on the homotopy categories of model categories.

I can reduce the verification of the property of a Quillen equivalence to the following statement:

For any object $A \in \mathcal{M}$, I can apply the M5 axiom (factorization axiom) to pick a fibrant object R equipped with a weak-equivalence $r \colon F(A) \longrightarrow R$, where I consider the image of the object A under the functor $F \colon \mathcal{M} \longrightarrow \mathcal{N}$. I say that this object $R = RF(A)$ defines a fibrant resolution of the object $F(A)$ in the category $\mathcal{N}$.

Dually, for any object $X \in \mathcal{N}$, I can pick a cofibrant object Q equipped with a weak equivalence $q \colon Q \xrightarrow{\sim} G(X)$, where I consider the image of the object X under the functor $G \colon \mathcal{N} \longrightarrow \mathcal{M}$. I then say that $Q = QG(X)$ defines a cofibrant resolution of the object $G(X)$ in the category $\mathcal{M}$.

Following, I consider the unit $\alpha : A \longrightarrow G(F(A))$ and respectively, I consider the augmentation $\beta : F(G(X)) \longrightarrow X$ of the adjunction.

Now, the Quillen equivalence property is precisely equivalent to the statement that the composite morphism $G(r) \circ \alpha \colon A \longrightarrow G(RF(A))$, forms a weak equivalence in $\mathcal{M}$ when A is a cofibrant object of $\mathcal{M}$, and for any choice of the fibrant resolution $r \colon F(A) \xrightarrow{\sim} RF(A)$ of the object $F(A)$ in the category $\mathcal{N}$, together with the assumption that the composite morphism $\beta \circ F(q) \colon F(QG(X)) \longrightarrow X$ forms a weak equivalence in $\mathcal{N}$ when X is a fibrant object of $\mathcal{N}$, and for any choice of a cofibrant resolution $q \colon QG(X) \xrightarrow{\sim} G(X)$ of the object $G(X)$ in the category $\mathcal{M}$.

The proof of the equivalence between these properties and my initial definition of a Quillen equivalence is given in Proposition 80.

I want to continue, explaining the definition of derived adjoint functors on homotopy categories associated to Quillen adjoint functors on model categories.

I start with the following simple observation:

Proposition 79. *Let $F \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ be any pair of Quillen adjoint functors between model categories.*

I have $F(\mathcal{M}_c) \subset \mathcal{N}_c$ and $G(\mathcal{N}_f) \subset \mathcal{M}_f$. The functors $F \colon \mathcal{M}_c \longrightarrow \mathcal{N}_c$ and $G \colon \mathcal{N}_f \longrightarrow \mathcal{N}_f$, obtained by taking this restriction of $F \colon \mathcal{M} \longrightarrow \mathcal{N}$, fits in an adjunction relation at the homotopy level:

$$[F(A),X]_{\mathcal{M}} = [A,G(X)]_{\mathcal{N}},$$

where A ranges over the category of cofibrant objects $\mathcal{M}_c \subset \mathcal{M}$ and X ranges over the category of fibrant objects $\mathcal{N}_f \subset \mathcal{N}$.

Proof. The relations $F(\mathcal{M}_c) \subset \mathcal{N}_c$ and $G(\mathcal{N}_f) \subset \mathcal{M}_f$ immediately follow from the definition of a Quillen adjunction.

I will follow notation introduced in Proposition 14 and Proposition 73.

Let $Cyl(A)$ be a good cylinder object for $A \in \mathcal{M}_c$. Recall that the morphism $i_0 \colon A \longrightarrow Cyl(A)$ is automatically an acyclic cofibration in this case. The morphism $F(i_0)$ is an acylic cofibration too because of the fact that F is supposed to preserve cofibrations and acyclic cofibrations by definition of a Quillen adjunction and, by the M2 axiom (two out of three axiom), this assertion implies that $F(l)$ is a weak equivalence since I have $F(l) \circ F(i_0) = F(Id_A)$. The morphism

$$F(A) \amalg F(A) \xrightarrow{\simeq} F(A \amalg A) \xrightarrow{F(i_0 \amalg i_1)} F(Cyl(A))$$

also forms a cofibration (note that the functor F is supposed to preserves colimits by adjunction). From these observations, I conclude that $F(Cyl(A))$ defines a good cylinder object for $F(Cyl(A))$.

For parallel morphisms $f,\ g \colon F(A) \longrightarrow X$ the adjunction between the functors F and G gives an equivalence,

$$\begin{tikzcd}
F(A) \amalg F(A) \arrow[r, "f \amalg g"] \arrow[d] & X & \Leftrightarrow & A \amalg A \arrow[r, "f_\star \amalg g_\star"] \arrow[d] & G(X) \\
F(Cyl(A)) \arrow[ur, dashed, "\exists\ h?"'] & & & Cyl(A) \arrow[ur, dashed, "\exists\ h_\star?"'] &
\end{tikzcd}$$

where I write $(\bullet)_\star$ for the adjunction relation on morphisms.

Thus, I have $f \sim g \ \Leftrightarrow\ f_\star \sim g_\star$, and I conclude from this equivalence that the adjunction relation $mor_{\mathcal{N}}(F(A),X) = mor_{\mathcal{M}}(A,G(X))$ induces a bijection on homotopy classes of morphisms and, hence, the assertion of the proposition follows.

□

Theorem 16. *Any pair of Quillen adjoint functors $F \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon G$ determine a pair of derived adjoint functors $LF \colon Ho\ \mathcal{M} \rightleftarrows Ho\ \mathcal{N} \colon RG$ such that $LF|_{Ho\ \mathcal{M}_c} = F$ (respectively, $LG|_{Ho\ \mathcal{N}_f} = G$) when I restrict the work to the homotopy category of the category generated by cofibrant (respectively, fibrant) objects in $\mathcal{M}$ (respectively, $\mathcal{N}$).*

Proof. The functor $F \colon \mathcal{M}_c \longrightarrow \mathcal{N}_c$ considered in the proof of Proposition 79 carries acyclic cofibrations to acyclic cofibrations by the definition of a Quillen adjunction. Now, the Lemma 48, the Ken's Brown Lemma, also implies that F carries any weak equivalence between cofibrant objects in $\mathcal{M}_c$ to an isomorphism (a weak equivalence) in the homotopy category $Ho\ \mathcal{N}_c$. Hence, the functor $F \colon \mathcal{M}_c \longrightarrow \mathcal{N}_c$ admits an extension to the homotopy category:

$$\begin{tikzcd}
\mathcal{M}_c \arrow[r, "F"] \arrow[d] & \mathcal{N}_c \arrow[d] & \\
Ho\ \mathcal{M}_c \arrow[r, dashed, "\widetilde{F}"'] & Ho\ \mathcal{N}_c \arrow[r, "\sim"] & Ho\ \mathcal{N}
\end{tikzcd}$$

and this is this extension which I consider in the definition of the left derived functor LF on $Ho\ \mathcal{M}_c$.

In the symmetric case of the functor G, I form a similar diagram, where I just replace the subcategories of cofibrant objects by subcategories of fibrant objects, in order to get the definition of the right derived functor RG on $Ho\ \mathcal{N}_f$.

In the statement of the theorem, I write F (respectively, G) for the functor $\widetilde{F}$ (respectively, $\widetilde{G}$) defined on the homotopy category $Ho\ \mathcal{M}_c$ (respectively, $Ho\ \mathcal{N}_f$).

The value of the derived functor LF on the whole homotopy category $Ho\ \mathcal{M}$ can be determined by composing this functor

$$F\colon Ho\,\mathcal{M}_c \longrightarrow Ho\,\mathcal{N}$$

with the cofibrant resolution functor $Q\colon A \mapsto QA$ of Proposition 73 and similarly in the case of the case of the right derived functor RF (I then consider the fibrant resolution functor $r\colon X \mapsto RX$). The adjunction relation of the theorem is therefore immediate from the result of Proposition 79.

Let $QA \xrightarrow{\sim} A$ be any cofibrant resolution of an object $A \in \mathcal{M}$. In general, the weak equivalence $QA \xrightarrow{\sim} QA$ with a distinguished resolution QA considered in the proof of Proposition 73 gives rise to a weak equivalence at the derived functor level $F(QA) = LF(Q) \xrightarrow{\sim} LF(A)$.

□

I previously said that a pair of Quillen adjoint functors $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ defines a Quillen equivalence if for every cofibrant object A in $\mathcal{M}$ and for every fibrant object X in $\mathcal{N}$ the adjunction bijection $mor_{\mathcal{M}}(F(A), X) = mor_{\mathcal{M}}(A, G(X))$ induces a one to one correspondence between the subset of weak equivalences $f\colon F(A) \xrightarrow{\sim} X$ in $\mathcal{N}$ and the subset of weak equivalences $g\colon A \xrightarrow{\sim} G(X)$ in $\mathcal{M}$. I use the observations of Proposition 79 and Theorem 16 to get the following equivalent characterization early mentioned of this notion:

Proposition 80. *Let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ be any pair of Quillen adjoint functors between model categories. Let α (respectively, β) denotes the unit (respectively, augmentation) morphism of this adjunction. The pair $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ denotes a Quillen equivalence if and only if the following assertions hold:*

- **(a)** *For every cofibrant object $A \in \mathcal{M}$, and for any choice of a fibrant resolution $r\colon F(A) \xrightarrow{\sim} RF(A)$ of the object $F(A) \in \mathcal{N}$, the composite,*

$$A \xrightarrow{\alpha} G(F(A)) \xrightarrow{G(r))} G(RF(A))$$

defines a weak equivalence in $\mathcal{M}$.

- **(b)** *For every fibrant object $X \in \mathcal{N}$, and for any choice of a cofibrant resolution $q\colon QG(X) \xrightarrow{\sim} G(X)$ of the object $G(X) \in \mathcal{M}$, the composite,*

$$F(QG(X)) \xrightarrow{F(q)} F(G(X)) \xrightarrow{\beta} X$$

defines a weak equivalence in $\mathcal{N}$.

Proof. I see that the composite morphism in the first assertion (a) represents the adjoint morphism of the weak equivalence,

$$r\colon F(A) \xrightarrow{\sim} RF(A)$$

while the composite morphism in the second assertion (b) represents the adjoint morphism of the weak equivalence

$$q\colon QG(X) \xrightarrow{\sim} G(X)$$

The ”only if” claim of the statement is therefore immediate.

To get the ”if” part, I just use that these composite morphisms represent the unit and augmentation morphisms of the derived functor adjunction of Theorem 16.

□

Proposition 81. *The derived functors $LF\colon Ho\,\mathcal{M} \rightleftarrows Ho\,\mathcal{N}\colon RG$ define adjoint equivalences of categories when the pair $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ forms a Quillen equivalence.*

Proof. This proposition is an immediate consequence of the observation that the morphisms in the requirements of Proposition 80 represent the unit and augmentation morphisms of the derived functor adjunction of Theorem 16.

□

Example 82. *The adjoint pair of the singular set functor and geometric realization between Top and sSet is an important tool in classical algebraic topology.*

Recall that Sing : Top $\longrightarrow$ sSet is given by sending a topological space X to the simplicial set,

$$n \mapsto Top(|\Delta^n|, X)$$

where $|\Delta^n|$ is the standard topological n-simplex.

Roughly speaking, the geometric realization of a simplicial set K is given by gluing edges along vertices. A concise definition can be given by considering the category $\Delta \downarrow K$ of maps $\sigma \colon \Delta^n \longrightarrow X$ and diagrams

$$\begin{tikzcd}
\Delta^n \arrow[d, "\theta"'] \arrow[dr, "\sigma"] & \\
 & X \\
\Delta^m \arrow[ur, "\tau"] &
\end{tikzcd}$$

Geometric realization is then the functor $| \bullet | \colon sSet \longrightarrow Top$ given by sending a simplicial set K to,

$$|K| = \underset{\sigma \in \Delta\downarrow}{colim} |\Delta^n|$$

with the colimit topology.

The adjointness of Sing and $| \bullet |$ follows from the observation that there is an isomorphism of simplicial sets,

$$K \cong \underset{\sigma \in \Delta\downarrow}{colim} \Delta^n$$

Hence, there are isomorphisms

$$\begin{aligned} Top(|K|, X) &\cong Top(\underset{\sigma \in \Delta\downarrow}{colim} |\Delta^n|, X) \\ &\cong \underset{\sigma \in \Delta\downarrow}{lim}\, Top(|\Delta^n|, X) \\ &\cong \underset{\sigma \in \Delta\downarrow}{lim}\, sSet(|\Delta^n|, Sing(X))) \end{aligned}$$

The proof that $(Sing, | \bullet |)$ is a Quillen pair, and in fact a Quillen equivalence, is due to Daniel Quillen in [13]. *Also Mark Hovey offers a proof in* [8]*, in fact, the results that lead to this is an important part of the proof of the theorem asserting that sSet is model category.*

Model Category Structures

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Beside the usual definition of a model category, there is an equivalent, more compact, definition of a model category defined with the help of weak factorization systems (where the term "weak" is used to remind me that this factorization is not necessarily unique).

Definition 83. *(Weak Factorization System).*

A weak factorization system on a category $\mathcal{C}$ is a pair of classes of maps $(\mathcal{L}, \mathcal{R})$ that are closed under retracts, and satisfying the following two axioms,

(1) *each morphism $f \in Mor(\mathcal{C})$ factorizes as a map from $\mathcal{L}$ followed by a map from $\mathcal{R}$;*
(2) *every pair of morphisms $(l, r) \in \mathcal{L} \times \mathcal{R}$ has the lifting property, that is, any commutative square in $\mathcal{C}$ admits a filler as in the diagram*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "l"'] & X \arrow[d, "r"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Equivalently to my initial definition, a model structure on a complete and cocomplete category $\mathcal{M}$ is a class W of weak equivalences that has the 2 out of 3 property, together with two weak factorization systems $(C \cap W, F)$ and $(C, F \cap W)$. Weak factorizations systems, as well as other types of factorization systems are brilliantly treated by Richard Gardner in [5].

The introduction of weak factorization system and this new point of view for a model category helps to better understand several facts that I previously introduced as propositions or lemmas in the introductory sections about general facts in model categories.

Indeed, for instance, in Proposition 8 I introduced that in any model category $\mathcal{M}$, two of the three classes C, F and W completely determine the third. Indeed, it follows from the definition that all fibrations have the right lifting property with respect to acyclic cofibrations. In fact, the fibrations turn out to be exactly the morphisms that have the right lifting property with respect to all acyclic cofibrations.

Suppose that a morphism $X \xrightarrow{f} Y$ has the right lifting property with respect to all acyclic cofibrations, and choose a factorization,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, tail, "i"', "\sim"] & & Y \\
& Z \arrow[ur, two heads, "p"'] &
\end{tikzcd}$$

Since i is an acyclic cofibration, the following square has a diagonal filler

$$\begin{tikzcd}
X \arrow[r, "Id_X"] \arrow[d, tail, "i"'] & X \arrow[d, "f"] \\
Z \arrow[r] \arrow[ur, dashed, "\exists\ g"'] & Y
\end{tikzcd}$$

and then it follows that f is a retract of the fibration p, and so is itself a fibration. A similar argument shows that the cofibrations are exactly the morphisms that have the left lifting property with respect to all acyclic fibrations. For the last case, if I know all the cofibrations and all the fibrations, this argument shows that I know all the acyclic cofibrations and all the acyclic fibrations. By the 2 out of 3 property, a morphism $X \longrightarrow Y$ is a weak equivalence if and only if it can be written as a composite of an acylic cofibration and an acyclic fibration.

Those all facts had beeen early detailed in Proposition 8 as I early said, but the interesting feature concerning to the fact that two classes determines the third relies on the fact that I am given two weak factorization systems. In a weak factorization system $(\mathcal{L}, \mathcal{R})$, one of the two classes $\mathcal{L}$ or $\mathcal{R}$ determines the other. Moreover, the fact that the cofibrations are exactly the morphisms that have the (left) lifting property with respect to all acyclic fibrations, implies that the class of cofibrations are closed under composition. Similarly, the class of acyclic cofibrations, fibrations and acyclic fibrations are also closed under composition. More precisely, C , F and W are subcategories of $Mor(\mathcal{M})$, containing all the objects as a domain or codomain of an arrow, and containing all isomorphisms.

Given a model category $\mathcal{M}$ with cofibrations C , fibrations F and weak equivalences W , there is a canonical model category on the opposite underlying category $\mathcal{M}^{op}$ where the cofibrations are F^{op}, the fibrations are C^{op} and the weak equivalences are W^{op}.

Given 3 classes of maps in a complete and cocomplete category $\mathcal{M}$, that are closed by retracts and by composition and one of them having the 2 out of 3 property, it is never easy to check that this corresponds to a model structure on $\mathcal{M}$. In this situation, the axioms to be checked are the lifting property and the existence of the two (functorial) factorizations, that is, showing the existence of two weak factorizations systems. Maybe I should emphasize the fact that the class of weak equivalences is the most important of the three classes. Indeed, it is clear that the homotopy category, which is the localization $\mathcal{M}[W^{-1}]$, only depends on the class of the weak equivalences. Therefore, the class of weak equivalences should be the first class to be determined, in order to endow a category with a model structure. Afterwards, there is a balance to be found between the cofibrations and the fibrations. By the lifting properties, more cofibrations implies fewer fibrations, and more fibrations implies fewer cofibrations. Furthermore, there should always remain enough of both cofibrations and fibrations, in order to find functorial factorizations.

The small object argument is a generic tool that provides weak factorization systems, given as input only a set of morphisms where the domain of each morphism is "not too big". This requirement of objects being small enough is very important, since it is one of the only problem that may occur. This machinery outputs two classes of maps $\mathcal{I}-cell$ and $\mathcal{I}-inj$ such that any morphism of $\mathcal{M}$ can be factored by a map from $\mathcal{I}cell$ followed by a map from $\mathcal{I}-inj$. The only missing property of the pair $(\mathcal{I}-cell, \mathcal{I}-inj)$ for being a weak factorization system is that $\mathcal{I}-cell$ is not necessarily closed by retracts. Therefore, if I call $Icof$ the closure by retracts of $\mathcal{I}-cell$, the couple $(\mathcal{I}-cell, \mathcal{I}-inj)$ is a weak factorization system, that is, there is the functorial factorization required and every couple $(i,p) \in (\mathcal{I}-cell, \mathcal{I}-inj)$ satisfies the lifting property. This will be seen as one of the two factorization systems of a model category.

However, this is only half of the model structure, since a model category is defined with the two weak factorization systems $(C, F \cap W)$ and $(C \cap W, F)$. Since these two weak factorization systems are certainly not independent, I could not just give as input two sets of maps $\mathcal{I}$ and $\mathcal{J}$ and hope that the output gives a model structure on $\mathcal{M}$. For example, a relation between them may simply be the fact that every acyclic cofibration (that is, an element of $\mathcal{J}-cof$) is in particular a cofibration (and so, an element of $\mathcal{I}-cof$). The recognition theorem will give a sufficient condition on the sets $\mathcal{I}$ and $\mathcal{J}$ in order to have an induced model structure.

The functorial factorization of a morphism $X \longrightarrow Y$ will eventually be given by successively factoring it trough bigger and bigger objects Z_α for some (infinite) indexing, until such an object Z_β is big enough so that $Z_\beta \longrightarrow Y$ has the right lifting property with respect to all desired maps. I will need first define such infinite compositions, and what is such a notion of "size" for objects in a category.

Definition 84. *(λ-sequence, Transfinite Composition).*

Let λ be an ordinal, seen as a poset category. A λ-sequence in a category $\mathcal{M}$ is a functor $X\colon \lambda \longrightarrow \mathcal{M}$, that is, a λ-diagram,

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots \quad \in \mathcal{M}$$

satisfying the property that the natural maps $\underset{\beta<\gamma}{colim}\ X_\beta \xrightarrow{\cong} X_\gamma$ are isomorphisms for all limit ordinals $\gamma \leq \lambda$.

The transfinite composition of a λ-sequence X is the morphism

$$X_0 \longrightarrow \underset{\beta<\gamma}{colim}\ X_\beta$$

If $\mathcal{D}$ is a class of morphisms in $\mathcal{M}$, a λ-sequence X in $\mathcal{D}$ is a λ-sequence X such that every morphism $X_\beta \longrightarrow X_{\beta+1}$ lies in $\mathcal{D}$.

Intuitively, the condition $\underset{\beta<\gamma}{colim}\ X_\beta \xrightarrow{\cong} X_\gamma$ for all limit ordinal γ is included to ensure that X does not make gaps at these levels γ. Indeed, by their definition, limit ordinals cannot be reached from below, and this condition is necessary in order to have some sort of "continuity" in a λ-sequence. These transfinite compositions allow the construction of very big objects, by glueing objects together by means of pushouts. To control the size of the objects that will arise in the functorial factorizations, I need to impose a condition on the objects of the domains of $\mathcal{I}$.

Definition 85. *(Regular Cardinal)*

A cardinal λ is said to be a regular cardinal if for every set S of cardinality less than λ and every collection of sets $\{S_s\}_{s\in S}$ such that each set S_s is of cardinality less than λ, then the union $\underset{s\in S}{\cup} S_s$ is also of cardinality less than λ.

A regular cardinal can be seen as a limit ordinal that cannot be broken into a smaller collection of smaller parts.

I can now define what small objects are.

Definition 86. *($(k-)$Small Object (with respect to $\mathcal{I}$)*

Let $\mathcal{I}$ be a class of morphisms in $Mor(\mathcal{M})$ and k be an ordinal.

An object $Z \in \mathcal{M}$ is said to be k-small with respect to $\mathcal{I}$ if for every regular cardinal $\lambda \geq k$ and everry λ-sequence

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots \quad \in \mathcal{M}$$

the induced map of $hom-sets$ is a bijection,

$$\underset{\beta<\lambda}{colim}\ \mathcal{M}(Z,X_\beta)\xrightarrow{\cong}\mathcal{M}(Z,\underset{\beta<\lambda}{colim}\ X_\beta)$$

An object Z is said to be small with respect to $\mathcal{I}$ if there exists an ordinal k such that Z is k-small with respect to $\mathcal{I}$.

Moreover, the object Z is said to be small if it is small with respect to the class of all morphisms $Mor(\mathcal{M})$.

To best understand the definition, let's consider a λ-sequence

$$X_0\hookrightarrow X_1\hookrightarrow\cdots\hookrightarrow X_\beta\hookrightarrow\cdots\hookrightarrow colim\ X_\beta\quad\in\mathcal{M}$$

where all the maps are monomorphisms of $\mathcal{M}$. This is in fact not really restrictive, it turns out that it will often be the case that these maps are monomorphisms. In this case, the induced function is injective,

$$colim\ \mathcal{M}(Z,X_\beta)\hookrightarrow\mathcal{M}(Z,colim\ X_\beta),$$

where the colimits are over $\beta<\lambda$.

Indeed, observe that this function is just composition with monomorphisms

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X_1 \arrow[r, hook] & \cdots \arrow[r, hook] & X_\beta \arrow[r, hook] & \cdots \arrow[r, hook] & colim\ X_\beta \\
 & & Z \arrow[ur] \arrow[urrr, dashed] & & &
\end{tikzcd}$$

The bijectivity of this function is therefore equivalent to its surjectivity, which says that any morphism $Z\xrightarrow{f}X_\beta$ admits a preimage, that is exactly to say that there is some X_β trough which f factors

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X_1 \arrow[r, hook] & \cdots \arrow[r, hook] & X_\beta \arrow[r, hook] & \cdots \arrow[r, hook] & colimX_\beta \\
 & & Z \arrow[ur, dashed, "\exists"] \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

This is essentially the meaning of the definition.

For instance, in the category Set of simplicial sets, every set S is card S-small. In the category $sSet$ of simplicial sets, every simplicial set with a finite number of non-degenerate simplices is $\aleph_0$-small with respect to all monomorphisms (cofibrations). Similarly, in the category Top of topological spaces, any finite CW complex is $\aleph_0$-small with respect to the inclusions of CW complexes.

Recall that the aleph numbers are the sequence of numbers used to represent the cardinality or "size" of infinite sets that can be well-ordered. So that $\aleph_0$ (aleph-naught) is the smallest infinite cardinal number and is the cardinality of the natural numers $\mathcal{N}$. A set has cardinality $\aleph_0$ if and only if it is countably infinite, that is, there is a bijection (one to one correspondence) between it and the natural numbers.

Now in order to continue, I need to formally present the general terminology early introduced and to well define the classes of maps that will give the weak factorization systems.

Definition 87. *($rlp(S)$, $llp(S)$)*

Let $\mathcal{M}$ be a cocomplete category and let $S\subset Mor(\mathcal{C})$ a class of morphisms.

I define,

- (•) *$rlp(S)$ for the collection of morphisms with the right lifting property with respect to S.*
- (•) *$llp(S)$ for the collection of morphisms with the left lifting property with respect to S.*

Moreover, I will also write $\mathcal{I}\subset Mor(\mathcal{M})$, $\mathcal{I}$ a set of morphisms in $\mathcal{M}$:

I define five classes of morphisms by letting,

Definition 88. *($\mathcal{I}-cell$, $\mathcal{I}-cof$, $\mathcal{I}-inj$, $\mathcal{I}-proj$ and $\mathcal{I}-fib$)*

(•) *$cell(\mathcal{I})$ for the class of morphisms obtained by transfinite composition of pushouts of coproducts of elements in $\mathcal{I}$.*

Equivalently, it is usual to say,

$\mathcal{I}-cell$ contains all the morphisms that are obtained as transfinite compositions of pushouts of coproducts of morphisms from $\mathcal{I}$, that is, all the transfinite compositions of the λ-sequences X in which each step $X_\beta \longrightarrow X_{\beta+1}$ is obtained as a pushout

$$\begin{tikzcd}
\coprod A_{\beta,x} \arrow[r] \arrow[d, "\coprod f_{\beta,x}"'] & X_\beta \arrow[d, dashed] \\
\coprod B_{\beta,x} \arrow[r] & X_{\beta+1}
\end{tikzcd}$$

where each $f_{\beta,x} \in \mathcal{I}$.

A morphism $A \longrightarrow B$ in $\mathcal{I}-cell$ is called a relative $\mathcal{I}-cell$ complex, and an object $X \in \mathcal{M}$ is called an $I-cell$ complex if the unique morphism $\emptyset \longrightarrow X$ is a relative $\mathcal{I}-cell$ complex.

A picture of such a sequence, would thus look like,

$$\begin{tikzcd}
X_0 \arrow[rr] & & X_1 \arrow[rr] & & X_2 \arrow[r] & \cdots \\
\coprod A_{0,x} \arrow[u] \arrow[r] & \coprod B_{0,x} \arrow[ur] & \coprod A_{1,x} \arrow[u] \arrow[r] & \coprod B_{1,x} \arrow[ur] & \cdots \arrow[u] & \cdots
\end{tikzcd}$$

(•) *$cof(\mathcal{I})$ for the class of retracts (in the arrow category $Arr(\mathcal{M})$) of elements in $cell(\mathcal{I})$.*

So that,

$\mathcal{I}-cof$ contains the morphisms that have the left lifting property with respect to all morphisms in $\mathcal{I}-inj$.

(•) *$inj(\mathcal{I}) := rlp(I)$ for the class of morphisms having the right lifting property with respect to all morphisms in $\mathcal{I}$, the $\mathcal{I}$-injective morphisms.*

Equivalently, it is usual to say,

$\mathcal{I}-inj$ contains the morphisms that have the right lifting property with respect to all morphisms in $\mathcal{I}$.

(•) *$proj(\mathcal{I})$ for the class of morphisms having the left lifting property with respect to all morphisms in $\mathcal{I}$.*

Equivalently, it is usual to say, $\mathcal{I}-proj$ or $\mathcal{I}$-projections.

(•) *$fib(\mathcal{I})$ for the class of morphisms having the right lifting property with respect to all morphisms in $\mathcal{I}-proj$.*

Equivalently, it is usual to say, $\mathcal{I}-fib$ to this class of morphisms.

The machinery that produces a weak factorization is the so called Quillen's Small Object Argument that I will after describe.

Definition 89. *A set of morphisms $\mathcal{I}$ in $\mathcal{M}$ is said to permit the small object argument if the domains of each morphism in $\mathcal{I}$ are small with respect to $\mathcal{I}$.*

Since a class of morphisms having the left (or right) lifting property with respect to another class is closed under transfinite compositions, $\mathcal{I}-cell \subseteq \mathcal{I}-cof$. The opposite inclusion is not true, since $\mathcal{I}-cell$ is in general not closed under retracts, while $\mathcal{I}-cof$ is. When $\mathcal{I}$ permits the small object argument, this is the only obstruction and the closure by retracts of $\mathcal{I}-cell$ gives exactly $\mathcal{I}-cof$.

Afterwards I will describe in detail the Small Object Argument but right now I can briefly introduce the statement:

Let $\mathcal{M}$ be a cocomplete category having and let $\mathcal{I}$ a set of morphisms that admits the small object argument. Then there exists a functorial factorization of every morphism of $\mathcal{M}$ by a morphism of $\mathcal{I}-cell$ followed by a morphism of $\mathcal{I}-inj$.

and moreover, I can add a sketch of the proof in order to remark some main details.

Talking about the proof.

Let $X \xrightarrow{f} Y$ be a morphism. The idea is to inductively (transfinitly) factorize it trough bigger and bigger objects, until the "projection" to Y admits the right lifting property in all possible squares. More precisely, I will force this property to be true by factorizing at each step "trough all possible squares".

Graphically, the construction is the following

$$\begin{tikzcd}
X = X_0 \arrow[dd, "f=p_0"'] \arrow[dr, "j_1"] & & & & \\
& X_1 \arrow[r, "j_2"] \arrow[dl, dashed, "p_1"'] & X_2 \arrow[r] \arrow[dll, dashed, bend left, "p_2"] & \cdots \arrow[r] & colimX_\beta \arrow[dllll, dashed, bend left, "p"] \\
Y & & & &
\end{tikzcd}$$

where X_β is a λ-sequence where each morphism is a pushout of coproducts of maps from $\mathcal{I}$, and the morphisms p_β are induced at each step by the universal property of the pushout.

I will use an inductive process, supposing the construction of $X_\beta \xrightarrow{p_\beta} Y$ done, and I will then build $X_{\beta+1}$. I will consider all possible commutative squares,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_\beta \arrow[d, "p_\beta"] \\
B \arrow[r] & Y
\end{tikzcd}$$

where I require that $i \in \mathcal{I}$.

In order to have all the liftings required, I will formally consider the pushout,

$$\begin{tikzcd}
\coprod dom\ i \arrow[r] \arrow[d, "\coprod i"'] & X_\beta \arrow[d] \arrow[ddr, bend left, "p_\beta"] & \\
\coprod cod\ i \arrow[r] \arrow[drr, bend right] & X_{\beta+1} \arrow[dr, dashed, "p_{\beta+1}"'] & \\
& & Y
\end{tikzcd}$$

At first sight, I could proceed that way, and the algorithm may never stop, since at each step there are new squares that I need to take care of. The trick comes from the fact that the *dom i* that appear in these squares are all small. Therefore, each of them has an associated ordinal k such that for $\beta \geq k$, a morphism from *dom i* must factor trough X_β. Furthermore, by taking the supremum (union) of all these k's, I get an ordinal λ which has the property that for any $\beta \geq \lambda$, any map $dom\ i \longrightarrow X_\beta$ must factor trough X_λ, and so, there are no new squares. All this is to say that if I consider the factorization of $X \xrightarrow{f} Y$ at

$$X \xrightarrow{j} \underset{\beta<\lambda}{colim}\ X_\beta \xrightarrow{p} Y$$

the morphism $j \in \mathcal{I} - cell$ since it is a transfinite composition of pushouts of coproducts of maps from $\mathcal{I}$, and the morphism p grew enough to admit the right lifting property with respect to all morphisms from $\mathcal{I}$, as desired. For more details, see the Section 3 devoted to the Quillen's Small Object Argument.

By closing the class $\mathcal{I} - cell$ under retracts, I get $\mathcal{I} - cof$, which are exactly the maps that have the left lifting property with respect to $\mathcal{I}inj$.

Corollary 90. *Let $\mathcal{M}$ be a cocomplete category and $\mathcal{I}$ a set of morphisms that admits the small object argument. Then the pair $(\mathcal{I} - cof, \mathcal{I} - inj)$ is a weak factorization system on $\mathcal{M}$.*

Proof. Since $\mathcal{I}-cof$ are exactly the morphisms that have the left lifting property with respect to $\mathcal{I}-inj$, and since $\mathcal{I}-cell \subseteq \mathcal{I}-cof$, the process previously described (Quillen's Small Object Argument) gives the desired functorial factorization. ☐

This corollary gives a method of creating weak factorisation systems, and thus model structures. A model category that can be obtain with two sets of maps $\mathcal{I}$ and $\mathcal{J}$ is called cofibrantly generated. A few lines below I will properly develope this concept.

For $\mathcal{M}$ a model category and $\mathcal{I}$ any small category there are two ways to put a model category structure on the functor category $[\mathcal{I}, \mathcal{M}]$ or $\mathcal{M}^{\mathcal{I}}$, called the projective and the injective model structures. For completely general $\mathcal{M}$, neither one need exist. The projective model structure exists as long as $\mathcal{M}$ is cofibrantly generated, while the injective model structure exists as long as $\mathcal{M}$ is combinatorial.

More formally,

Definition 91. *For $\mathcal{M}$ a combinatorial model category or, in the projective case, just a cofibrantly generated model category, and $\mathcal{I}$ a small category there exist the following two (combinatorial) model category structures on the functor category $[\mathcal{I}, \mathcal{C}]$ (also denoted by $\mathcal{C}^{\mathcal{I}}$):*

- (•) *The projective structure $[\mathcal{I}, \mathcal{M}]_{proj}$: weak equivalences and fibrations are the natural transformations that are objectwise such morphisms in $\mathcal{M}$.*
- (•) *The injective structure $[\mathcal{I}, \mathcal{C}]_{inj}$: weak equivalences and cofibrations are the natural transformations that are objectwise such morphisms in $\mathcal{C}$.*

Cofibrantly Generated Model Categories.

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Intuitively, a model category $\mathcal{M}$ is cofibrantly generated if there is a set, meaning a small set, not a proper class, of cofibrations and also another one set of trivial cofibrations, such that all other trivial cofibrations are generated from these.

More formally,

Definition 92. *A model category $\mathcal{M}$ is cofibrantly generated if there are small sets of morphisms $\mathcal{I}$, $\mathcal{J} \subset Mor(\mathcal{M})$ such that,*

- (•) *$cof(\mathcal{I})$ (equivalently $\mathcal{I}-cof$) is precisely the collection of cofibrations of $\mathcal{M}$,*
- (•) *$cof(\mathcal{J})$ (equivalently $\mathcal{J}-cof$) is precisely the collection of acyclic cofibrations in $\mathcal{M}$ and*
- (•) *I and J permit the small object argument.*

The set $\mathcal{I}$ is called the set of generating cofibrations and $\mathcal{J}$ is called the set of generating acyclic cofibrations.

Since $\mathcal{I}$ and $\mathcal{J}$ are assumed to admit the small object argument the collection of cofibrations and acyclic cofibrations has the following characterization,

Proposition 93. *In a cofibrantly generated model category $\mathcal{M}$ I have*

- (•) $cof(\mathcal{I}) = llp(rlp(\mathcal{I}))$
- (•) $cof(\mathcal{J}) = llp(rlp(\mathcal{J}))$.

And therefore the fibrations are precisely $rlp(\mathcal{J})$ (and so are $\mathcal{J}-inj$) and the acyclic fibrations precisely $rlp(\mathcal{I})$ (and so are $\mathcal{I}-inj$).

Proof. The argument is analogous for $\mathcal{I}$ and $\mathcal{J}$. So I develope the proof for $\mathcal{I}$.

By definition I have $\mathcal{I} \subset llp(rlp(\mathcal{I}))$ and it is readily checked that collections of morphisms given by a left lifting property are stable under pushouts, transfinite composition and retracts (see here for details).

So $cof(\mathcal{I}) \subset llp(rlp(\mathcal{I}))$.

For the converse inclusion, I use the Quillen's Small Object Argument 3:

Let $f\colon X \longrightarrow Y$ be in $llp(rlp(\mathcal{I}))$. The small object argument produces a factorization,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i \in cof(\mathcal{I})"'] & & Y \\ & Z \arrow[ur, "p \in rlp(\mathcal{I})"'] & \end{tikzcd}$$

Finally I apply the Retract Argument 3:

It follows that f has the left lifting property with respect to p which yields a morphism α in

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "\sigma"] & Y \end{tikzcd}$$

which exhibits f as a retract of i

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "\sigma"] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

Therefore $f \in cof(\mathcal{I})$.

□

In a cofibrantly generated model category, the functorial factorization may not be the one given by the small object argument, even this one is always available.

Many usual model categories are cofibrantly generated.

Examples 94.

1. *The Quillen model structure on Top with weak homotopy equivalences and Serre fibrations is cofibrantly generated. The set $\mathcal{I}$ of generating cofibrations can be given by the natural inclusions $\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ for $n \in \mathcal{N}$, while the set $\mathcal{J}$ of generating acyclic cofibrations can be given be the inclusions $\mathbb{D}^n \hookrightarrow \mathbb{D}^n \times I$ for all $n \in \mathbb{N}$.*
2. *The standard model structure on sSet with monomorphisms as cofibrations, Kan fibrations and weak equivalences the maps that are weak homotopy equivalences after realization, is cofibrantly generated by letting the set $\mathcal{J}$ of generating acyclic cofibrations being $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $0 \leq k \leq n$ and the set $\mathcal{I}$ of generating cofibrations $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n \in \mathbb{N}$.*

Remark 95. *Most of the model categories in common use are cogibrantly generated, and are often finitely generated. One (possible) exception is the category of chain complexes of abelian groups, where the weak equivalences are chain homotopy equivalences. Similar model categories, such as the model category of topological spaces with the Hurewicz model category structure, where the weak equivalences are the homotopy equivalences, are also probably not cofibrantly generated (see the paper of Arne Str∅m* [15]*).*

The dual notion of a fibrantly generated model category also makes sense, by letting a model category being fibrantly generated if and only if its opposite model category is cofibrantly generated. However, this notion is usually not relevant since the notion of cosmall object is not very flexible. Indeed, even in the category *Set* of sets, the only cosmall objects are the empty set and the singletons.

Cofibrantly generated model categories are useful for several reasons. First of all, it is easier to endow a category with a model structure that is cofibrantly generated as I will show in the Recognition Principle, Theorem 18, in particular because the Small Object Argument detailed in Theorem 3 gives the functorial factorization. Moreover, carrying only part of the data, by only keeping in mind the two sets $\mathcal{I}$ and $\mathcal{J}$ instead of C , F and W simplifies many arguments. For example, it is now easier to verify when a functor F is a left Quillen functor (Proposition 97), by only verifying it on the generating sets. With the same idea, it is easier to transport a cofibrantly generated model structure to other categories. A main example is the transport of a cofibrantly generated model structure on $\mathcal{M}$ to categories of diagrams $[\mathcal{C}, \mathcal{M}]$.

The following Lemma is often useful,

Lemma 96. *Let $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ be an ajunction between model categories. $\mathcal{I}$ is a class of maps in $\mathcal{M}$ and $\mathcal{J}$ is a class of maps in $\mathcal{N}$. Then*

- **(a)** $G(F\mathcal{I} - inj) \subseteq \mathcal{I} - inj$
- **(b)** $F(\mathcal{I} - cof) \subseteq F\mathcal{I} - cof$
- **(c)** $F(G\mathcal{J} - proj) \subseteq \mathcal{J} - proj$
- **(a)** $G(\mathcal{J} - fib) \subseteq G\mathcal{J} - fib$

Proof. Proof for part **(a)**.

I suppose $g \in F\mathcal{I} - inj$, and $f \in \mathcal{I}$. Then g has the right lifting property with respect to $F(f)$, and so, by adjointness, $G(g)$ has the right lifting property with respect to f. Thus $G(g) \in \mathcal{I} - inj$ as required.
Proof for part **(b)**.

I suppose $f \in \mathcal{I} - cof$, and $g \in F\mathcal{I} - inj$. Then by the previous part (a), $G(g) \in \mathcal{I} - inj$, and so f has the left lifting property with respect to $G(g)$, and so, by adjointness, $F(f)$ has the left lifting property with respect to g. Thus $F(f) \in (F\mathcal{I} - inj) - proj = FI - cof$ as required.

Proofs for parts (c) and (d) are dual from (a) and (b) respectively. ☐

If $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ is an adjunction between model categories, where $\mathcal{M}$ is cofibrantly generated. The fact that F is a left Quillen functor (and thus that $F \dashv G$ is a Quillen adjunction) can be checked on the sets of generating cofibrations $\mathcal{I}$ and the set of generating acyclic cofibrations $\mathcal{J}$ of $\mathcal{M}$.

Proposition 97. *Let $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ be an adjunction between a cofibrantly generated model category model category $\mathcal{M}$, with generating sets $\mathcal{I}$ and $\mathcal{J}$, and any model category $\mathcal{N}$. This adjunction is a Quillen adjunction if and only if $F(f)$ is a cofibration in $\mathcal{N}$ for all generating cofibrations $f \in \mathcal{I}$ and $F(f)$ is an acyclic cofibration in $\mathcal{N}$ for all generating acyclic cofibrations $f \in \mathcal{J}$.*

Proof. The conditions are clearly necessary, then I will show that they are also sufficient.
By Lemma 96 I have $F(\mathcal{I} - cof) \subseteq FI - cof$.

Furthermore, if I denote by $C_{\mathcal{N}}$ the class of cofibrations of $\mathcal{N}$, the hypothesis says that $F\mathcal{I} \subseteq C_{\mathcal{N}}$. It follows that $FI - cof \subseteq (C_{\mathcal{N}}) - cof$, but since $(C_{\mathcal{N}}) - cof = C_{\mathcal{N}}$, then I have,

$$F(\mathcal{I} - cof) \subseteq FI - cof \subseteq (C_{\mathcal{N}}) - cof = C_{\mathcal{N}},$$

and therefore F sends cofibrations to cofibrations. With a similar argument, changing $\mathcal{I}$ by $\mathcal{J}$ gives that F preserves acyclic cofibrations and hence, F is a left Quillen functor.

☐

Another important result is that a cofibrantly generated model structure can be pushed trough an adjunction under natural assumptions.

Theorem 17. *(Kan)*

Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating trivial cofibrations $\mathcal{J}$. Let $\mathcal{N}$ be a complete and cocomplete category and $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ be an adjunction such that,

- **(1)** *$F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument.*
- **(2)** *G takes relative $F\mathcal{J} - cell$ complexes to weak equivalences, that is, $G(F\mathcal{J} - cell) \subseteq W_{\mathcal{M}}$.*

Then $\mathcal{N}$ admits a cofibrantly generated model structure in which, $F\mathcal{I}$ is a set of generating cofibrations, $F\mathcal{J}$ is a set of generating trivial cofibrations and the weak equivalences are the maps that G takes to weak equivalences in $\mathcal{M}$. Furthermore, the adjunction $F \dashv G$ is a Quillen adjunction with respect to these model structures.

Proof. I will use Theorem 18, Recognition Theorem, and so the assumption that $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument is essential.

I define the weak equivalences $W_{\mathcal{N}}$ to be the morphisms that G takes to weak equivalences in $\mathcal{M}$. Since G preserves composition and retracts, the class $W_{\mathcal{N}}$ is closed by retracts and satisfies the 2 out of 3 property.

By hypothesis $F\mathcal{J}-cell \subseteq W_{\mathbb{N}}$ and so by closing under retracts I get $F\mathcal{J}-cof \subseteq W_{\mathcal{N}}$.

Moreover,

$$\mathcal{I}-inj \subseteq \mathcal{J}-inj \;\Rightarrow\; F\mathcal{I}-inj \subseteq F\mathcal{J}-inj \;\Rightarrow\; F\mathcal{J}-cof \subseteq F\mathcal{I}-cof,$$

and so the first hypothesis $F\mathcal{J}-cof \subseteq F\mathcal{I}-cof \cap W_{\mathcal{N}}$ of the Recognition Theorem is satisfied.

Now, since $FI-inj \subseteq F\mathcal{J}-inj$ by adjunction and using the lifting property I get that $G(F\mathcal{I}-inj) \subseteq \mathcal{I}-inj \subseteq W_{\mathcal{M}}$. In particular this gives the second inclusion $F\mathcal{I}-inj \subseteq F\mathcal{J}-inj \cap W_{\mathcal{N}}$.

For a reverse inclusion, I pick a morphism $X \xrightarrow{f} Y \in F\mathcal{J}-inj \cap W_{\mathcal{N}}$.

Again by adjunction I get that $G(f) \in \mathcal{J}-inj \cap W_{\mathcal{M}} = \mathcal{I}-inj$, which gives the desired inclusion $F\mathcal{J}-inj \cap W_{\mathcal{N}} \subseteq F\mathcal{I}-inj$.

By Theorem 18, Recognition Theorem, the sets $F\mathcal{I}$ and $F\mathcal{J}$ define a cofibrantly generated model structure on $\mathcal{N}$.

For the last point, I observe that since F is a left adjoint, it preserves all colimits. In particular $F(\mathcal{I}-cell) \subseteq F\mathcal{I}-cell$ and $F(\mathcal{J}-cell) \subseteq F\mathcal{J}-cell$.

Moreover, since any functor preserves retracts, by closing under retracts these two inclusions, I get that F preserves cofibrations as well as acyclic cofibrations, and so that, F is a left Quillen functor and so the adjunction, $F\colon \mathcal{M} \leftrightarrows \mathcal{N}\colon G$ is a Quillen adjunction.

□

I note that the strentgh of this theorem is that it suffices to prove that G sends the (regular) "acyclic cofibrations" to weak equivalences, and the rest follows for free.

An example of an adjunction that allows such a lifting of a model structure is the adjunction between a cofibrantly generated model category $\mathcal{M}$ and its pointed version.

Given a category $\mathcal{M}$ with finite coproducts and with a terminal object $*$, its associated pointed category, denoted by $\mathcal{M}_*$, is the category $(* \downarrow \mathcal{M})$ under the terminal object.

Moreover, there is an adjunction

$$(\bullet)_+ \colon \mathcal{M} \leftrightarrows \mathcal{M}_* \colon G,$$

where G is the forgetful functor and $(\bullet)_*$ adds a disjoint base point. More generally any above or under category of $\mathcal{M}$ admits a natural model structure from $\mathcal{M}$.

Proposition 98. *Let $\mathcal{M}$ be a model category, and let $X \in \mathcal{M}$ be any object. Then the categories $(X \downarrow \mathcal{M})$ and $(\mathcal{M} \downarrow X)$ admit a model structure in which a morphism is a cofibration, a fibration or a weak equivalence if it is so in $\mathcal{M}$.*

Proof. Everything follows from the definitions. □

In particular, there is a model structure on $\mathcal{M}_*$, where a map is a cofibration, a fibration or a weak equivalence if it is so in $\mathcal{M}$ after applying the forgetful functor G. More precisely, if $\mathcal{M}$ is cofibrantly generated, then so is $\mathcal{M}_*$.

Corollary 99. *Let $\mathcal{M}$ be a cofibrantly generated model category with $\mathcal{I}$ as set of generating cofibrations and $\mathcal{J}$ as generating acyclic cofibrations. There is an induced cofibrantly model structure on M_* with generating cofibrations $\mathcal{I}_+$ and generating acyclic cofibrations $\mathcal{J}_+$, where a morphism f is a cofibration, fibration or weak equivalence if and only if $G(f)$ is respectively a cofibration, a fibration or a weak equivalence.*

Proof. I consider the adjunction

$$(\bullet)_+ \colon \mathcal{M} \leftrightarrows \mathcal{M}_* \colon G,$$

The model structure where the cofibrations, fibrations and weak equivalences of $\mathcal{M}_*$ are the ones that are so after applying the forget functor G turns $\mathcal{M}_*$ into a model category. The lifting property follows by the lifting property in $\mathcal{M}$, which is lifted to $\mathcal{M}_*$ trough the adjunction, and the functorial factorization is similarly given by the one in $\mathcal{M}$.

By adjointness, it follows that $\mathcal{I}_+ - cof$ are the cofibrations, $\mathcal{J}_+ - cof$ the acyclic cofibrations, $\mathcal{J}_+ - inj$ are the fibrations, $\mathcal{I}_+ - inj$ the acyclic fibrations.

It remains to prove that $\mathcal{I}_+$ and $\mathcal{J}_+$ permit the small object argument in $\mathcal{M}_*$. Since the forget functor G commutes with colimits of diagrams of the type

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots \quad \in \mathcal{M}$$

the fact that $\mathcal{I}_+$ (or $\mathcal{J}_+$) permits the small object argument is the same as the fact that $\mathcal{I}$ (or $\mathcal{J}$) permits the small object argument, which is true by assumption. □

Cellular and Combinatorial Model Categories.

A cellular model category is essentially a cofibrantly model category, with two extra conditions, a stronger statement of smallness of objects and a condition on cofibrations. A combinatorial model category requires an even stronger condition of smallness. These conditions are in particular required so that constructions, such as localization of model categories, always exist.

I start by defining the extra condition of smallness, which is a generalization of a small object.

Definition 100. *(Compact Object).*
Let $\mathcal{I}$ be a class of morphisms in a cocomplete category $\mathcal{M}$, and let k be a cardinal. An object $Z \in \mathcal{M}$ is said to be k-compact with respect to $\mathcal{I}$ if for every $\lambda \geq k$ and any λ-sequence in $\mathcal{I}$

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_\beta \hookrightarrow \cdots \hookrightarrow colim\ X_\beta \quad \in \mathcal{M}$$

the induced function of hom-sets is a bijection,

$$\underset{\beta<\lambda}{colim}\ \mathcal{M}(Z, X_\beta) \xrightarrow{\cong} \mathcal{M}(Z, \underset{\beta<\lambda}{colim}\ X_\beta)$$

The object Z is said to be compact with respect to $\mathcal{I}$ if it is k-compact with respect to $\mathcal{I}$ for some k, and it is said to be compact if it is compact with respect to all morphisms $Mor(\mathcal{M})$.

By considering the covariant hom-functor $\mathcal{M}(Z, \bullet)\colon \mathcal{M} \longrightarrow Set$, the fact that Z is k-compact is to say that it preserves all the colimits $\underset{\beta<\lambda}{colim} X_\beta$ for all $\lambda \geq k$. Recall that an object Z was defined to be k-small if it preserves all these colimits for only the regular cardinals $\lambda \geq k$.

I will now define the condition imposed on cofibrations.

Definition 101. *(Effective Monomorphism)*
In a category $\mathcal{M}$, a morphism $K \xrightarrow{i} L$ is said to be and effective monomorphism if the pushout $L \amalg_K L$ exists, and if $K \xrightarrow{i} L$ is the equalizer of $L \rightrightarrows L \coprod_K L$.
If I construct the pushout square,

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "i"'] & L \arrow[d] \\
L \arrow[r] & L \coprod_K L
\end{tikzcd}$$

saying that $K \xrightarrow{i} L$ is the desired equalizer, is the same as requiring a unique filler $A \longrightarrow K$ in,

$$\begin{tikzcd}
A \arrow[rrd, bend left, "j"] \arrow[rdd, bend right, "j"'] \arrow[rd, dashed, "\exists\,!"] & & \\
& K \arrow[r, "i"] \arrow[d, "i"'] & L \arrow[d] \\
& L \arrow[r] & L \coprod_K L
\end{tikzcd}$$

for any map $A \xrightarrow{j} L$ *that gets equalized in* $L \coprod_K L$.

Example 102. *(Effective Monomorphisms in Set and sSet)*

I first show that effective monomorphisms are exactly the injections in the category Set of sets.

If $K \xrightarrow{i} L$ *is an injection, and if* $A \xrightarrow{j} L$ *is equalized by the induced pair* $L \rightrightarrows L \coprod_K L$, *then* $j(A) \subseteq i(K) \subseteq L$.

Therefore such a filler $A \longrightarrow K$ *exists and is forced to send an element* $a \mapsto i^{-1} \circ j(a)$.

More generally, an effective monomorphisms is necessary a monomorphism. Indeed, pick two morphisms $f, g \colon A \rightrightarrows K$ *that are the same after composition with* $K \xrightarrow{i} L$, *that is,* $i \circ f = i \circ g$. *They are therefore the same after composition with* $L \rightrightarrows L \coprod_K L$, *and by the property of* $K \xrightarrow{i} L$ *being an effective monomorphism, there is only one such* $A \longrightarrow K$, *that is,* $f = g$. *This shows that the morphism* $K \xrightarrow{i} L$ *is a monomorphism.*

In the category sSet of simplicial sets, let $K_\bullet \xrightarrow{i} L_\bullet$ *be a monomorphism. Since colimits are computed degree-wise, I can restrict the study to diagrams of sets.*

$$\begin{tikzcd}
A_n \arrow[rrd, bend left, "j_n"] \arrow[rd, dashed, "\exists ?"] \arrow[rdd, bend right, "j_n"'] & & \\
& K_n \arrow[r, "i_n"] \arrow[d, "i_n"'] & L_n \arrow[d] \\
& L_n \arrow[r] & L_n \coprod_{K_n} L_n
\end{tikzcd}$$

Since monomorphisms of simplicial sets are degree-wise injections (of sets), the argument shows that there is a filler, and it is unique. Doing the argument in each degree show that $K_\bullet \xrightarrow{i} L_\bullet$ *is an effective monomorphism.*

Definition 103. *(Cellular Model Category)*

A cofibrantly generated model category $\mathcal{M}$ *is said to be cellular if there is a set* $\mathcal{I}$ *of generating cofibrations and a set* $\mathcal{J}$ *of generating acyclic cofibrations such that*

- (•) *the domains and codomains of morphisms in* $\mathcal{I}$ *are compact objects,*
- (•) *the domains of morphisms in* $\mathcal{J}$ *are small with respect to* $\mathcal{I}$,
- (•) *the cofibrations (given by* $I - cof$*) are effective monomorphisms.*

I suppose now given a model category $\mathcal{M}$ that is cofibrantly generated with set $\mathcal{I}$ of generating cofibrations and set $\mathcal{J}$ of generating acyclic cofibrations.

If the model category $\mathcal{M}$ is cellular, the generating sets $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}}$ of the cellular structure need not be directly related to $\mathcal{I}$ and $\mathcal{J}$ (as sets). Of course, since the cellular structure has the same underlying model structure as the given structure on $\mathcal{M}$, relations such as $\mathcal{I} - cof = \widetilde{\mathcal{I}} - cof$ must hold.

Examples 104.

(1) *The usual structure on sSet, where the set* $\mathcal{I}$ *of generating cofibrations is given by,*

$$\mathcal{I} = \{\ \partial\Delta[n] \hookrightarrow \Delta[n]\ \}_{n \in \mathbb{N}}$$

the set $\mathcal{J}$ *of generating acyclic cofibrations is given by,*

$$\mathcal{J} = \{\ \Lambda^k[n] \hookrightarrow \Delta[n]\ \}_{k \leq n\ ,\ n > 0}$$

is a cellular model structure.

The cofibrations are (exactly all the) effective monomorphisms since they are exactly the monomorphisms. Moreover, all the domains and codomains appearing both in I and J are compact (with respect to all morphisms) since they are finite. In particular, this model structure is cellular.

(2) *Similarly, the category* $sSet_*$ *of pointed simplicial sets is also a cellular model category.*

The second type of model category in this section is a Combinatorial Model Category. As its name indicates, it is combinatorial in the sense that all objects are built up (as colimits) from smaller objects.

Definition 105. *(Locally Presentable Category)*

A cocomplete category $\mathcal{C}$ is called a locally presentable category if

- (•) *all objects are small,*
- (•) *there exists a set S of objects of $\mathcal{C}$ such that each object of $\mathcal{C}$ can be obtained as a colimit of a diagram using only objects of S.*

To emphasize the size of the objects of S, a locally presentable category $\mathcal{C}$ where the set S may be chosen among k-small objects (for a regular cardinal k) is said to be k-locally presentable. If I choose $k = \omega$, the countable infinite, an ω-locally presentable category is called locally finitely presentable.

Recall that $\aleph_0$ (aleph-naught, also aleph zero or the German term Aleph-null) is the cardinality of the set of all natural numbers, and is an infinite cardinal. The set of all finite ordinals, called ω or ω_0 has cardinality $\aleph_0$. A set has cardinality $\aleph_0$ if and only if it is countably infinite, that is, there is a bijection (one-to-one correspondence) between it and the natural numbers.

Examples 106.

(1) *The category Set of sets is locally finitely presentable, since any set $X \in Set$ is the (directed) colimit of the poset (under inclusion $\subseteq$) of its finite subsets. Therefore, the set of its generators contains one set of n elements for any natural number $n \in \mathbb{N}$.*

(2) *The category $sSet$ of simplicial sets is also locally finitely presentable since a simplicial set $K_\bullet$ is a colimit over its category of simplices. More precisely, I can construct the category of simplices of $K_\bullet$, denoted by $\Delta K_\bullet$*
  - (•) *Objects : morphisms $\Delta[n] \longrightarrow K_\bullet \quad \in sSet \quad$, for any $n \in \mathbb{N}$,*
  - (•) *Morphisms : morphisms between the domains $\Delta[n] \xrightarrow{f} \Delta[m]$ over $K_\bullet$, are such that the diagram commutes*

$$\begin{tikzcd}
\Delta[n] \arrow[rr, "f"] \arrow[dr] & & \Delta[m] \arrow[dl] \\
& K_\bullet &
\end{tikzcd}$$

The composite of the projection functor,

$$\pi\colon \Delta K_\bullet \longrightarrow \Delta\colon (\Delta[n] \to K_\bullet) \mapsto [n]$$

with the Yoneda embedding

$$\Delta \hookrightarrow [\Delta^{op}, Set] = sSet\colon [n] \mapsto \Delta[n]$$

gives the chain of functors

$$\Delta K_\bullet \longrightarrow \Delta \hookrightarrow sSet\colon (\Delta[n] \to K_\bullet) \mapsto \Delta[n]$$

The simplicial set $K_\bullet$ is canonically isomorphic to the colimit of this diagram in $sSet$

$$colim_{\Delta K_\bullet} \Delta[n] \cong K_\bullet$$

naturally in the category $sSet$.

Therefore, the category $sSet$ of simplicial sets is locally finitely presentable, generated by the representables $S = \{\Delta[n]\}_{n\in\mathbb{N}}$

Now, intuitively, a combinatorial model category is a particularly tractable model category structure.

Being combinatorial means that there is very strong control over the cofibrations in these model structures: there is a set (meaning small set, not proper class) of generating acyclic cofibrations, and all objects, in particular the domains and codomains of these cofibrations, are small objects.

So as an initial idea I have that a combinatorial model structure is one that is generated from small data: it is generated from a small set of (acyclic) cofibrations between small objects.

In fact, the combinatoriality condition is a bit stronger than that, as it requires even that every object is small and is the colimit over a small set of generating objects.

More formally,

Definition 107. *(Combinatorial Model Category)*

A model category $\mathcal{M}$ is called combinatorial if it is

- *Locally presentable as a category,*

and

- *Cofibrantly generated as a model category.*

Recall from the definition at cofibrantly generated model category that this last fact means that $\mathcal{M}$ has a set (not a proper class) $\mathcal{I}$ of generating cofibrations and a set $\mathcal{J}$ of generating trivial cofibrations such that

$$cof = llp(rlp(\mathcal{I}))$$
$$fib = llp(rlp(\mathcal{J}))$$

where fib, $cof \subset Mor(\mathcal{M})$ is the collection of fibrations and cofibration, respectively, and in general $llp(S)$, $rlp(S)$ is the collection of morphisms satisfying the left or right, respectively, lifting property with respect to a given collection of morphisms S.

Example 108. *The model structure on sSet is combinatorial, since it is cofibrantly generated and the underlying category is locally finitely presentable.*

Locally presentable categories are very useful in homotopy theory because, since all objects are small, I can freely apply the small object argument which helps creating model structures.

Proper Model Categories.

Properness is another useful property that model categories may enjoy. It follows from their lifting properties that cofibrations are preserved by pushouts and fibrations are preserved by pullbacks, as in the diagrams,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
B \arrow[r] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & X \arrow[d, two heads] \\
B \arrow[r] & Y
\end{tikzcd}$$

Definition 109. *(Left/Right Proper Model Category)*

A model category $\mathcal{M}$ is called,

- *Right proper if weak equivalences are preserved by pullback along fibrations.*

  That is, if for a pullback diagram with p a fibration and g a weak equivalence, then f is a weak equivalence,
- *Left proper whether weak equivalece are preserved by pushout along cofibrations.*

  That is, if for a pushout diagram with i a cofibration and f a weak equivalence, then g is a weak equivalence,
- *Proper whether the model category is both left and right proper.*

$$\begin{tikzcd}
A \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"'] & X \arrow[d, tail] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "\sim", "g"'] & Y
\end{tikzcd}$$

The axioms of a model category imply that the pushout along a cofibration of a weak equivalence between cofibrant objects, is necessarily a weak equivalence, and dually for pullbacks along fibrations.

Now, I introduce an auxiliar Lemma which I will use as a sort of characterization of weak equivalences of cofibrants in order to prove the immediately below Proposition 111.

Lemma 110. *If A and B are cofibrant objects in $\mathcal{M}$ then $f\colon A \longrightarrow B$ is a weak equivalence if and only if any lifting problem.*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "p"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}$$

where p is a fibration, can be solved to the extent that there exists a map $l\colon B \longrightarrow X$ and a homotopy $h\colon A \times I \longrightarrow X$, such that

$p \circ l = v$,

and also h is a homotopy from u to $l \circ f$,

and also $p \circ h = v \circ f \circ pr_A$,

where $pr_A\colon A \times I \longrightarrow A$ is the projection.

Proof. If f is a weak equivalence then I factor $f = k \circ j$, where j is a trivial cofibration, k a trivial fibration, $j\colon A \longrightarrow Z$, $k\colon Z \longrightarrow B$.

I find, $s : B \longrightarrow Z$ a section, and $H : Z \times I \longrightarrow Z$ a homotopy from the identity to $s \circ k$, covering the identity of B.

I also find, $\widehat{u}\colon Z \longrightarrow X$ with $p \circ \widehat{u} = v \circ k$ and $\widehat{u} \circ j = u$, by lifting j against p.

Let $l = \widehat{u} \circ s$, and $h = \widehat{u} \circ H \circ i$, where i is the inclusion $A \times I \longrightarrow Z \times I$. Then l and h are the desired maps.

If f has the property, then f induces an epimorphism on left homotopy classes $[B, Y]_l \longrightarrow [A, Y]_l$, and a monomorphism on right homotopy classes $[B, Y]_r \longrightarrow [A, Y]_r$, where Y is a fibrant object.

Since A and B are cofibrant, then f induces and isomorphism $[B, Y] \longrightarrow [A, Y]$, and so f is a weak equivalence, since $\mathcal{M}$ is closed.

□

Proposition 111. *Let $\mathcal{M}$ be a model category.*

(1) *Every pushout of a weak equivalence between cofibrant objects, along a cofibration, is again a weak equivalence.*
(2) *Every pullback of a weak equivalence between fibrant objects, along a fibration, is again a weak equivalence.*

Proof. I will prove the first part (1) (by dual argument the second part (2) holds).
I consider the following starting situation,

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"', "\sim"] & B \arrow[d, "g"] \\
C \arrow[r, "j"'] & D
\end{tikzcd}$$

is a pushout diagram in $\mathcal{M}$. So, $D = B \amalg_A C$, with i a cofibration, f a weak equivalence, and A and C are cofibrant.

I want to prove that then g is a weak equivalence as well.

I consider the following diagram,

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"', "\sim"] & B \arrow[r, "u"] \arrow[d, "g"] & X \arrow[d, two heads, "p"] \\
C \arrow[r, "j"'] & D \arrow[r, "v"'] & Y
\end{tikzcd}$$

with p a fibration, I know by Lemma 110 that there exists a $\widehat{v}\colon C \longrightarrow X$ such that $p \circ l = \widehat{v} \circ j$, and $H\colon A \times I \longrightarrow X$ such that H is a homotopy from $u \circ i$ to $\widehat{v} \circ f$ and $p \circ H = v \circ j \circ f \circ pr_A$.

Since A and B are cofibrant objects then there exists a cylinder object $B \times I$ such that the map $B \amalg_A A \times I \longrightarrow B \times I$ is a cofibration and a weak equivalence (the coproduct is over the zero inclusion $A \longrightarrow A \times I$).

Now I lift in the diagram,

$$\begin{tikzcd}
B \amalg_A (A \times I) \arrow[r, "(u \amalg H)"] \arrow[d] & X \arrow[d, "p"] \\
B \times I \arrow[r, "v \circ g \circ pr_B"] & Y
\end{tikzcd}$$

to obtain h. I define $l = (h_1 \amalg \widehat{v})\colon D = B \amalg_A C \longrightarrow X$, where h_1 is the one end of h.

The conditions in Lemma 110 are now satisfied by l and h and I obtain the result required. □

Main Tools.

Retract Argument.

The Retract Argument is a useful standard tool in discussion of weak factorization systems. It asserts that if a morphism factors as the composition of two factors such that it is has the left or right lifting property against its second or first factor, respectively, then it is a retract (as an object of the arrow category) of the respective other factor.

The retract argument is frequently used in the verification of the axioms of model category structures.

Its general statement is the following,

Consider a composite morphism,

$$f\colon X \xrightarrow{i} A \xrightarrow{p} Y$$

Then, it holds,

(1) If f has the left lifting property against p, then f is a retract of i.
(2) If f has the right lifting property against i, then f is a retract of p.

Recall that by a retract of a morphism $X \xrightarrow{f} Y$ in some category $\mathcal{C}$ is meant a retract of f as an object in the arrow category $\mathcal{C}^{\Delta[1]}$, hence a morphism $A \xrightarrow{g} B$ such that in $\mathcal{C}^{\Delta[1]}$ there is a factorization of the identity on g through f

$$Id_g\colon g \longrightarrow f \longrightarrow g.$$

This means equivalently that in $\mathcal{C}$ there is a commuting diagram of the form,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\
B \arrow[r] & Y \arrow[r] & B
\end{tikzcd}$$

where the upper row is, $Id_A\colon A \longrightarrow X \longrightarrow A$

and the lower row is, $Id_B\colon B \longrightarrow Y \longrightarrow B$

I will discuss the first statement since the second one is formally dual.

I write the factorization of f as a commuting square of the form,

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

It is assumed by the statement that f has the left lifting property against p so there exists g making a commuting diagram of the form

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\
Y \arrow[r, equal] \arrow[ur, "g"] & Y
\end{tikzcd}$$

By a simply rearrangement of the previous diagram, I obtain the equivalent one,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "i"] & \\
Y \arrow[r, "g"'] & A \arrow[r, "p"'] & Y
\end{tikzcd}$$

where the lower row is, $Id_Y \colon Y \xrightarrow{g} A \xrightarrow{p} Y$

Completing this to the right, this yields a diagram showing the desired retract,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & A \arrow[r, "p"'] & Y
\end{tikzcd}$$

Quillen's Small Object Argument.

Quillen's small object argument is a transfinite functorial construction of a weak factorization system (on some category $\mathcal{C}$ that is cofibrantly generated by a set of morphisms $\mathcal{I} \subset Mor(\mathcal{C})$.

This originally categorical construction is notably used in the theory of model categories and in particular cofibrantly generated model categories in order to demonstrate the existence of the required factorization of morphisms into composites of (acyclic) cofibrations following by (acyclic) fibrations, and in order to find such factorization choices functorially.

Of course, recall that a small object (sometimes also called a compact object) S in a category $\mathcal{C}$ is an object such that every morphism into a coproduct $k \colon S \longrightarrow \coprod_{i \in \mathcal{I}} C_i$ factors though $\coprod_{i \in \mathcal{I}} C_i$ for some finite subset Ω of $\mathcal{I}$. That is, any map from S into a coproduct factors through a finite coproduct.

Also, recall that to say that a weak factorization system is cofibrantly generated by $\mathcal{I}$ is to say that the right class $\mathcal{R}$ of the system consists of precisely those maps which have the right lifting property with respect to $\mathcal{I}$ (the $\mathcal{I}$-injective morphisms).

$$\mathcal{R} = rlp(\mathcal{I})$$

The left class $\mathcal{L}$ is then necessarily the class of maps who have the left lifting property with respect to the right class (the $\mathcal{I}$-cofibrations).

$$\mathcal{L} = llp(\mathcal{R}) = llp(rlp(\mathcal{I}))$$

Provided the classes of cofibrantly generated weak factorzation system are determined by lifting properties, the content of the small object argument is to produce the required factorizations. With care, this construction is functorial, so the result is a functorial weak factorization system.

The conditions established on the category $\mathcal{C}$ are fundamental. If the category $\mathcal{C}$ is just assumed to have all colimits then the domains of the maps in $\mathcal{I}$ are required to satisfy a smallness condition that says that any morphism from these objects to sufficiently-large-directed colimit will factor through the base of the colimiting diagram. If the category is required to be a locally presentable category then no further condition is required.

The general statement for the small object argument is the following,

Let $\mathcal{I} \subset Mor(\mathcal{C})$ be a set of morphisms in a category $\mathcal{C}$.

Let $\mathcal{C}$ be, a locally presentable category, (or more generally a category having all colimits and with each domain of morphisms in $\mathcal{I}$ being a small object, or what is more general a category having all colimits and with each domain of morphisms in $\mathcal{I}$ being small relative to transfinite composites of pushouts of maps in $\mathcal{I}$).

Then every morphisms $f\colon X \longrightarrow Y$ admits a factorization of the form,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "l\in cell(\mathcal{I})"'] & & Y \\
& Z \arrow[ur, "r\in rlp(\mathcal{I})"'] &
\end{tikzcd}$$

where,

- (•) $rlp(\mathcal{I})$ is the set of morphisms with the right lifting property with respect to $\mathcal{I}$.
- (•) $cell(\mathcal{I})$ is the set of transfinite compositions of pushouts of morphisms in $\mathcal{I}$.

A collection $\mathcal{I}$ of morphisms is said to admit the small object argument if all domains are small relative to transfinite composties of pushotus of elements of $\mathcal{I}$.

I will directy follow the process of construction made by Daniel Quillen in his Homotopical Algebra ([13]).

Given the morphism $f\colon X \longrightarrow Y$, I would like to factor f as $l\colon X \longrightarrow Z$ followed by $r\colon Z \longrightarrow Y$, where r has the right lifting property with respect to all arrows in $\mathcal{I}$ and l will be constructed to be a transfinite composite of pushout of coproducts of maps in $\mathcal{I}$. The left class of a weak factorization system is closed under all these constructions, so l will be in the left class cofibrantly generated by $i \in \mathcal{I}$.

The category $\mathcal{C}$ is locally small. I can then consider the set $\mathcal{D}_1$, of lifting problems between f (on the right) and elements $i \in \mathcal{I}$ (on the left), so the set of commuting diagrams.

$$\mathcal{D}_1 = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd} \quad \middle| \quad i \in \mathcal{I} \right\}$$

So that I am also considering the set $\mathcal{L}_1 = \{(h,g) \in Hom_{\mathcal{M}}(A,X) \times Hom_{\mathcal{M}}(B,Y)\}$ of pairs of morphisms that make every diagram in $\mathcal{D}_1$ commute.

So, in fact $\mathcal{D}_1$ forms the coproduct morphism

$$(A_{(\mathcal{I}/f)} \longrightarrow B_{(\mathcal{I}/f)})\colon = \coprod_{i\in\mathcal{D}_1} (A \xrightarrow{i} B)$$

over $\mathcal{D}_1$ of the corresponding elements of $\mathcal{I}$; the squares of $\mathcal{D}_1$ then specify a canonical morphism

$$A_{(\mathcal{I}/f)} \longrightarrow X =\colon Z_0$$

from the domain of this morphism to $Z_0\colon = X$.

Now the pushout

$$\begin{tikzcd}
A_{(\mathcal{I}/f)} \arrow[r] \arrow[d, "i"'] & Z_0 = X \arrow[d, "l_1"] & \\
B_{(\mathcal{I}/f)} \arrow[r] & Z_0 \coprod_{B_{(\mathcal{I}/f)}} A_{(\mathcal{I}/f)} =\colon Z_1 \arrow[dr, "r_1"] & \\
& & Y
\end{tikzcd}$$

of this diagram defines and object Z_1 and two morphisms,

$$l_1\colon Z_0 \longrightarrow Z_1$$

and

$$r_1\colon Z_1 \longrightarrow Y$$

factoring f.

$$\begin{tikzcd}
X = Z_0 \arrow[rr, "f"] \arrow[dr, "l_1"'] & & Y \\
& Z_1 \arrow[ur, "r_1"'] &
\end{tikzcd}$$

Remark 112. *To clarify this situation, for instance, if I think of the morphism $i \in \mathcal{I}$ as being inclusions of spheres into balls, I have formed Z_1 by sphere attachments for every attaching map from a domain of $\mathcal{I}$ into $X\colon = Z_0$*

Note that I also have,

$$\begin{tikzcd}
A_{(\mathcal{I}/f)} \arrow[r] \arrow[d, "i"'] & Z_0 = X \arrow[d, "l_1"] \arrow[ddr, bend left, "r_0"] & \\
B_{(\mathcal{I}/f)} \arrow[r] \arrow[drr, bend right] & Z_0 \coprod_{B_{(\mathcal{I}/f)}} A_{(\mathcal{I}/f)} =\colon Z_1 \arrow[dr, "r_1"] & \\
& & Y
\end{tikzcd}$$

Now, I will iterate this construction with $r_1\colon Z_1 \longrightarrow Y$ in place of f and taking colimits to construct Z_α for limit ordinals α.

This construction does not converge. So I will choose instead to stop at a sufficiently large ordinal β, $\alpha < \beta$ chosen so that the domains of the maps in $\mathcal{I}$ will satisfy the smallness property assumed in the statement.

I will define,

l to be the transfinite composite of the l_α

and r to be the induced map from the colimit $Z_\beta = \underset{\alpha}{colim} Z_\alpha$ to Y,

so that I am obtaining the factorization for $X \longrightarrow Y$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "l"'] & & Y \\
& Z_\beta = \underset{\alpha}{colim} Z_\alpha \arrow[ur, "r"'] &
\end{tikzcd}$$

where l is a transfinite composite,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\alpha"'] \arrow[dr, "l"] & & Y \\
Z_\alpha \arrow[r, "l'_\alpha"'] & Z_\beta = \underset{\alpha}{colim} Z_\alpha \arrow[ur, "r"] &
\end{tikzcd} \qquad \beta < \alpha$$

(where I am denoting by $l'_\alpha\colon Z_\alpha \longrightarrow Z_\beta\colon = \underset{\alpha}{colim}\ Z_\alpha \quad \alpha < \beta$ the morphisms given by the existence of the colimit $Z_\beta\colon = \underset{\alpha}{colim}$).

and r is he induced map,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\alpha"'] \arrow[dr, "l"] & & Y & \alpha < \beta \\
Z_\alpha \arrow[r, "l'_\alpha"'] \arrow[urr, bend right=60, "r_\alpha"'] & Z_\beta = \underset{\alpha}{colim} Z_\alpha \arrow[ur, "r"'] & &
\end{tikzcd}$$

(where I am denoting by $r_\alpha \colon Z_\alpha \longrightarrow Y$, $\alpha < \beta$ the uniquely determined induced morphisms making the corresponding triangles commute).

I can summarize the construction in the following diagram,

$$\begin{tikzcd}
Z_0 = X \arrow[d, "l_1"'] \arrow[dr, "l"] \arrow[drr, bend left, "f"] & & \\
Z_1 \arrow[r, "l'_1"] \arrow[d, "l_2"'] \arrow[rr, bend right, "r_1"'] & Z_\beta = \underset{\alpha}{colim} Z_\alpha \arrow[r, "r"] & Y \\
Z_2 \arrow[ur, "l'_2"] \arrow[urr, bend right, "r_2"'] \arrow[d, "l_3"'] & & \\
Z_3 \arrow[uur, "l'_3"] \arrow[uurr, bend right, "r_3"'] \arrow[d] & & \\
\vdots \arrow[d] & & \\
Z_n \arrow[uuuur, "l'_n"] \arrow[uuuurr, bend right, "r_n"'] & &
\end{tikzcd}$$

I can deduce from the construction that l is in the left class of the weak factorization system, so it remains to show that r has the right lifting property with respect to each $i \in \mathcal{I}$.

Given a lifting problem,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & Z_\beta \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

the map from A to Z_β factors through some Z_α, with $\alpha < \beta$, since Z_β is a filtered colimit and using the assumed smallness of A. Because $Z_{\alpha+1}$ was defined to be a pushout over squares including this one, I have a map $B \longrightarrow Z_{\alpha+1} \longrightarrow Z_\beta = \underset{\alpha}{colim} Z_\alpha$, which is the desired lift:

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & Z_\alpha \arrow[r] \arrow[d] & Z_\beta \arrow[dd] \\
& Z_{\alpha+1} \arrow[ur] & \\
B \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

Finally I have obtained a diagram of the form

$$\begin{tikzcd}
X\colon = Z_0 \arrow[r, "l_1"] \arrow[dr, "f\colon = r_0"'] & Z_1 \arrow[r, "l_2"] \arrow[d, "r_1"] & Z_2 \arrow[r] \arrow[dl, "r_2"] & \cdots \\
& Y & &
\end{tikzcd}$$

where I amb defining, $Z_0\colon = X$ and $r_0\colon = f$.

Following in order to clarify the whole process, I will develope the Small Object Argument in the case of topological spaces.

I note that in order to simplify and clarify notation I have slightly move the indexes from the previous development.

Example 113. *Let X and Y be topological spaces and $f\colon X \longrightarrow Y$ be a continuous map.*

I will use the Quillen's small object argument and I will obtain a factorization for f as $f = r \circ l$ where l is a cofibration and r is trivial fibration.

Given $f\colon X \longrightarrow Y$, I will construct a diagram,

$$\begin{tikzcd}
X\colon = Z_{-1} \arrow[r, "l_0"] \arrow[dr, "f\colon = r_{-1}"'] & Z_0 \arrow[r, "l_1"] \arrow[d, "r_0"] & Z_1 \arrow[r] \arrow[dl, "r_1"] & \cdots \\
& Y & &
\end{tikzcd}$$

I will follow an inductive process,

I consider the set D_n of all the commutative diagrams d of the form,

$$\begin{tikzcd}
\mathbb{S}^{q_d-1} \arrow[r, "\rho_d"] \arrow[d, "i_d"'] & Z^{n-1} \arrow[d, "r_{n-1}"] \\
B^{r_d} \arrow[r, "\eta_d"'] & Y
\end{tikzcd}$$

with $q_d \geq 0$ for each diagram, $\forall d \in D_n$.

I note that indeed, is a set since $q_d \in \aleph_0$ and maps between two arbitrary topological spaces became a set.

In the first step,

I call $Z_{-1}\colon = X$ and $r_{-1}\colon = f$.

One of the important conclusions of the small object argument is that it is functorial, hence, it produces functorial factorizations. But since in its ordinary form the process does not converge (in the up to isomorphism sense) but rather is merely stopped when it has gone far enough along, for functoriality I have to take care to terminate the constructions at the same ordinal β for every input.

I suppose now that I have yet built Z_{n-1} , $n \in \aleph_0$

I define Z_n and $l_n\colon Z_{n-1} \longrightarrow Z_n$ as the pushout,

$$\begin{tikzcd}
\coprod\limits_{d\in D_n} \mathbb{S}^{q_d-1} \arrow[r, "\coprod\limits_{d\in D_n} \rho_d"] \arrow[d, "\bigvee\limits_{d\in D_n} i_d"'] & Z_{n-1} \arrow[d, "in_1 = l_n"] \\
\coprod\limits_{d\in D_n} \mathbb{E}^{q_d} \arrow[r, "in_2"'] & Z_n
\end{tikzcd}$$

I define $r_n\colon Z_n \longrightarrow Y$ as the only morphism that holds, $r_n \circ l_n = r_{n-1}$ and $r_n \circ in_2 = \coprod\limits_{d\in D_n} \eta_d$.

I can observe that such a r_n exists, since

$$\Big(\coprod_{d\in D_n} \eta_d\Big) \circ \Big(\bigvee_{d\in D_n} i_d\Big) = \coprod_{d\in D_n} (\eta_d \circ i_d) = \coprod_{d\in D_n} (r_{n-1} \circ \rho_d) = r_{n-1} \circ \Big(\coprod_{d\in D_n} \rho_d\Big)$$

I take Z as the colimit of the diagram built by the Z_n, $n \geq -1$ and by the l_n, $n \in \aleph_0$.

I call $l'_n \colon Z_{n-1} \longrightarrow Z$ to the morphisms given by the existence of the colimit Z and I take $l = l'_{-1}$,

$$\begin{tikzcd}
X \arrow[d, "l_0"'] \arrow[dr, "l"] \arrow[drr, bend left, "f"] & & \\
Z_0 \arrow[r, "l'_0"] \arrow[d, "l_1"'] \arrow[rr, bend right, "r_0"'] & Z \arrow[r, "r"] & Y \\
Z_1 \arrow[ur, "l'_1"] \arrow[urr, bend right, "r_1"'] \arrow[d, "l_2"'] & & \\
Z_2 \arrow[uur, "l'_2"] \arrow[uurr, bend right, "r_2"'] \arrow[d] & & \\
\vdots \arrow[d] & & \\
Z_n \arrow[uuuur, "l'_n"] \arrow[uuuurr, bend right, "r_n"'] & &
\end{tikzcd}$$

And since $r_n \circ l_n = r_{n-1} \quad \forall \in \aleph_0$, then there exists r making the previous diagram commute. In particular, I have that $f = r \circ l$.

Now, $\bigvee_{d \in D_n} i_d$ has the left lifting property with respect to the trivial fibrations $\forall n \in \aleph_0$, and since l_n is the base extension for the previous morphism then I have that also l_n has the left lifting property with respect to the trivial fibrations.

I want to prove that also l holds the same property.

I suppose that I have a commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "i"'] & V \arrow[d, "r'"] \\
Z \arrow[r, "h"'] & W
\end{tikzcd}$$

where r' is a trivial fibration.

I will prove by induction over n that $\forall n \in \aleph_0$ there exists $\phi_n \colon Z_n \longrightarrow V$ such that

$$\phi_0 \circ l_0 = g,$$

$$\phi_n \circ l_n = \phi_{n-1} \quad \forall n \in \mathbb{N}$$

and

$$r \circ \phi_n = h \circ l'_n \qquad \forall n \in \aleph_0.$$

Case $n = 0$.

I consider the commutative diagram with solid arrows,

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "l_0"'] & V \arrow[d, "r'"] \\
Z_0 \arrow[r, "h \circ l'_0"'] \arrow[ur, dashed, "\phi_0"] & W
\end{tikzcd}$$

Since l_0 has the left lifting property with respect to the trivial fibrations and r' is a trivial fibration, there exists $\phi_0 \colon Z_0 \longrightarrow V$ such that $\phi_0 \circ l_0 = g$ and $r' \circ \phi_0 = h \circ l'_0$.

I suppose now that I have built ϕ_{n-1}.

I consider the commutative diagram with solid arrows,

$$\begin{tikzcd}
Z_{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, "l_{n-1}"'] & V \arrow[d, "r'"] \\
Z_n \arrow[r, "h \circ l'_n"'] \arrow[ur, dashed, "\phi_n"] & W
\end{tikzcd}$$

Since l_{n-1} *has the left lifting property with respect to the trivial fibrations, there exists* $\phi_n \colon Z_n \longrightarrow V$ *such that,*

$\phi_n \circ l_{n-1} = \phi_{n-1}$ *and* $r' \circ \phi_n = h \circ l'_n$ *as desired to conclude the inductive step.*

Now, since $\phi_n \circ l_{n-1} = \phi_{n-1}$ *and* Z *is the colimit for the* Z_n*, there exists* $\phi \colon Z \longrightarrow V$ *such that* $\phi \circ l = g$ *and* $\phi \circ l'_n = \phi_n \qquad \forall n \in \aleph_0$*.*

Then, I obtain that $r' \circ \ \phi \circ l'_n = r' \circ \phi_n = h \circ l'_n \qquad \forall n \in \aleph_0$ *and* $r' \circ \phi \circ l = r' \circ g = h \circ l$*,*

and so, $r' \circ \phi$ *and* h *can be used as a punctured arrows to make the following diagram commute.*

$$\begin{tikzcd}
X \arrow[d, "l_0"'] \arrow[dr, "i"] \arrow[drr, bend left, "r' \circ g"] & & \\
Z_0 \arrow[r, "l'_0"] \arrow[d, "l_1"'] \arrow[rr, bend right, "r \circ \phi_0"'] & Z \arrow[r, dashed] & Y \\
Z_1 \arrow[ur, "l'_1"] \arrow[d, "l_2"'] \arrow[urr, bend right, "r \circ \phi_1"'] & & \\
Z_2 \arrow[uur, "l'_2"] \arrow[d] \arrow[uurr, bend right, "r \circ \phi_2"'] & & \\
\vdots \arrow[d] & & \\
Z_n \arrow[uuuur, "l'_n"] \arrow[uuuurr, bend right, "r \circ \phi_n"'] & &
\end{tikzcd}$$

So $r' \circ \phi = h$ *and I can conclude that* ϕ *is the desired lifting. Therefore,* l *has the left lifting property with respect to the trivial fibrations, and hence,* $l \in Cof \colon = \mathcal{I} - cof$ *is a cofibration.*

It still remains to prove that r *is a trivial fibration.*

With this challenge, I first prove that if K *is a compact in* Z *then there exists* $m \in \mathbb{N}$ *such that* $K \subset Z_m$*.*

By reduction to the absurd, I then suppose that $\forall j \in \mathbb{N}$ *there exist* $m_j \in \mathbb{N}$ *with* $m_{j+1} \geq m_j \quad \forall j$ *and* $k_j \in (Z_{m_j} - Z_{m_j - 1}) \cap K$*.*

I call $T = \{k_j \mid j \in \mathbb{N}\}$*.*

T *is not finite. I consider* $T' \subset T$*. Then* $T' \cap Z_n$ *is finite* $\forall n \in \mathbb{N}$*. But now, since the* Z_n*'s were built from the* Z_{n-1}*'s by "gluying" n-cells" and the points* k_j *are interiors with respect to those cells, then* $\{k_j\}$ *is closed in* Z_n *for all* j *(or eventuallly empty).*

I want to remark that is fundamental that those points k_j *are not in* X *since I have no extra hypotesis over* X *(for instance conditions as Hausdorff are not included in the statement).*

Then, $T' \cap Z_n$ *is closed in* Z_n *, for all* $n \in \mathbb{N}$*.*

Now, Z *is the colimit for the* Z_n*, and therefore has the final topology with respect to the inclusions* $Z_n \hookrightarrow Z$ *and then* T' *is closed in* Z*.*

Since this argument is absolutely general for any $T' \subset T$ *then* T *is discrete en* Z*.*

Moreover, T *is closed (so, I can take* $T' = T$*) and* $T \subset K$*.*

Therefore T *is compact, but* T *is dicrete as well, then* T *must be finite!!!. I have obtined a contradiction.*

Now, I need to prove that r is a trivial fibration. It suffices to show that it has the right lifting property with respect to the inclusions $\mathbb{S}^{n-1} \hookrightarrow E^n$.

I consider the commutative diagram,

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\rho"] \arrow[d, hook, "i"'] & Z \arrow[d, "r"] \\ \mathbb{E}^n \arrow[r, "\eta"'] & Y \end{tikzcd}$$

Since $\mathbb{S}^{n-1}$ is compact, then $\rho(\mathbb{S}^{n-1})$ is compact too, then, by the previous fact showed, there exists $m \in \mathbb{N}$ such that $\rho(\mathbb{S}^{n-1}) \subset Z_m$.

Therefore, I obtain the commutative diagram,

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "\rho"] \arrow[d, hook, "i"'] & Z_m \arrow[d, "r"] \\ \mathbb{E}^n \arrow[r, "\eta"'] & Y \end{tikzcd}$$

and then, by the construction of Z_{m+1}, ρ is one of the ρ_d, $d \in D_{m+1}$, more precisely if I call d_0 to the previous diagram then $\rho_{d_0} = \rho$. I take φ the 'coordinate' d_0 of in_2 (seen as $\varphi\colon \mathbb{E}^n \longrightarrow Z$). Then, looking at the coordinate d_0 in the pushout for the definition of Z_{m+1}, I obtain that $\varphi \circ i = \rho$. On the other hand, I have previously obtained that $r_{m+1} \circ in_2 = \coprod_{d \in D_{m+1}} \eta_d$, then looking at the corresponding coordinate at d_0 I obtain $r_{m+1} \circ \varphi = \eta_{d_0} = \eta$, and then, since r_{m+1} is the restriction of r to Z_{m+1}, then I obtain $r \circ \varphi = \eta$. Then φ is the desired lifting and r is a trivial fibration as I wanted to prove.

Recognition Principle.

.

The following theorem allows me to recognize cofibrantly generated categories by checking fewer conditions.

Theorem 18. *(Recognition Principle)*
Let $\mathcal{C}$ be a category with all small limits and colimits and Υ a class of maps satisfying the M2 axiom (2 out of 3 axiom) of a model category.
If I and J are sets of maps in $\mathcal{C}$ such that

- **(1)** *both I and J permit the small object argument.*
- **(2)** *$cof(J) \subset cof(I) \cap \Upsilon$.*
- **(3)** *$inj(I) \subset inj(J) \cap \Upsilon$;*
- **(4)** *one of the following holds*
  - **(a)** *$cof(I) \cap \Upsilon \subset cof(J)$*
  - **(b)** *$inj(J) \cap \Upsilon \subset inj(I)$*

Then there is the stucture of a cofibrantly generated model category on $\mathcal{C}$ with

- **(1)** *Weak equivalences $\mathcal{C}_w\colon = \Upsilon$.*
- **(2)** *Generating cofibrations I, equivalently $\mathcal{C}_c\colon = llp(rlp(I))$*
- **(3)** *Generating acyclic cofibrations J.*

This is originally due to Daniel Kan, reproduced for instance as (Hirschhorn 03, theorem 11.3.1).

Proof. I have to show that with weak equivalences Υ setting $\mathcal{C}_c\colon = cof(I)$ and $\mathcal{C}_f\colon = inj(J)$ defines a model category structure.

The existence of limits, colimits and the 2-out-of-3 property holds by assumption. Closure under retracts of the weak equivalences will hold automatically if I check the rest of the axioms without using it, by an argument of A. Joyal.

Closure under retracts of Fib and Cof follows by the general statement that classes of morphisms defined by a left or right lifting property are closed under retracts.

I consider the set $Cof\colon = I - cof$. I want to prove that is a closed set under retracts.

I assume that given morphisms f and g, f is a retract of g, as I show in the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "Id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "Id_B"'] & D \arrow[r] & B
\end{tikzcd}$$

Moreover, I assume that $g \in I - cof$. I consider the following diagram,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & E \arrow[d, "h"] \\
B \arrow[r] & F
\end{tikzcd}$$

where $h \in I - inj$.

I can plug in the retract to obtain the following one,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[r] & E \arrow[d, "h"] \\
B \arrow[r] & D \arrow[r] \arrow[urr, dashed] & B \arrow[r] & F
\end{tikzcd}$$

In which I formally obtain the dotted lifting indicated using that g is an I-cofibration.

I compose now the maps $B \longrightarrow D$ an $D \longrightarrow E$ to obtain the lift I want,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[r] & E \arrow[d, "h"] \\
B \arrow[r] \arrow[urr] & D \arrow[r] \arrow[urr] & B \arrow[r] & F
\end{tikzcd}$$

Similarly, I consider now the set $Fib\colon = J - inj$. I want to prove that is a closed set under retracts.

I assume that given morphisms f and g, f is a retract of g, as I show in the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "Id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "Id_B"'] & D \arrow[r] & B
\end{tikzcd}$$

Moreover, I assume now that $g \in J - inj$. I consider the following diagram,

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "h"] & A \arrow[d, "f"'] \\
F \arrow[r] & A
\end{tikzcd}$$

where now $h \in J$, (J is the set of generating trivial cofibration).

I can again plug in the retract to obtain the following one,

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "h"'] & A \arrow[r] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
F \arrow[r] \arrow[rru, dashed] & B \arrow[r] & D \arrow[r] & B
\end{tikzcd}$$

In which I formally obtain the dotted lifting indicated using that g is an I-injection.

I compose now the maps $F \longrightarrow C$ an $C \longrightarrow A$ to obtain the lift I want,

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "h"'] & A \arrow[r] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
F \arrow[r] \arrow[rru] \arrow[rrru] & B \arrow[r] & D \arrow[r] & B
\end{tikzcd}$$

The factorization property follows by applying the Small Object Argument 3 to the set I, showing that every morphism $f\colon X \longrightarrow Y$ may be factored as

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rd, "i\in cof(J)"'] & & Y \\
& Z \arrow[ru, "p\in inj(J)=\mathcal{C}_f"'] &
\end{tikzcd}$$

and noting that $inj(I) \subset \Upsilon$ by assumption (3).

Similarly applying the Small Object Argument 3 to J gives factorizations

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rd, "i\in cof(J)"'] & & Y \\
& Z \arrow[ru, "p\in inj(J)=\mathcal{C}_f"'] &
\end{tikzcd}$$

and assumption (2) guarantees that $cof(J) \subset \Upsilon$.

It remains to verify the lifting axiom. This verification depends on which of the two parts of assumption (4) is satisfied. I assume the first one is, and the argument for the second one is analogous.

Then using the assumption $cof(I) \cap \Upsilon \subset cof(J)$ and remembering that I have set $inj(J) = \mathcal{C}_f$ I immediately have the lifting of trivial cofibrations on the left against fibrations on the right.

To get the lifting of cofibrations on the left with acyclic fibrations on the right, I show finally that $inj(J) \cap \Upsilon \subset inj(I)$. To see this, I apply the factorization established before to an acyclic fibration $f\colon X \longrightarrow Y$ to get,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "g\in cof(I)\cap\Upsilon"'] & X \arrow[d, "f\in inj(J)\cap\Upsilon"] \\
Z \arrow[r, "inj(I)"'] & Y
\end{tikzcd}$$

that with assumption (4.a) this is,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "g\in cof(J)"'] & X \arrow[d, "f\in inj(J)\cap\Upsilon"] \\
Z \arrow[r, "inj(I)"'] & Y
\end{tikzcd}$$

so that I have a lift,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "g\in cof(J)"'] & X \arrow[d, "f\in inj(J)\cap\Upsilon"] \\
Y \arrow[r, "inj(I)"'] \arrow[ur, "\sigma"] & Y
\end{tikzcd}$$

which establishes a retract,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Z \arrow[r, "\sigma"] \arrow[d, "g\in\Upsilon"] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Therefore f is a weak equivalence.

□

A cofibrantly generated model category for which the domains of the morphisms in I and J are small objects is a cellular model category.

The category of diagrams indexed by a fixed small category $\mathcal{I}$, taking values in another cofibrantly generated model category $\mathcal{M}$.

Homotopy limits and colimits

In an absolute general sense, a homotopy limit is a way of constructing appropriate sorts of limits in a (weak) higher category, using some presentation of that higher category by a 1-categorical structure. The general study of such presentations is of course the Homotopy Theory.

Recall that also in a very general sense, the Higher Category Theory is the generalization of Category Theory to a context where there are not only morphisms between objects, but generally k-morphisms between $(k-1)$-morphisms, for all $k \in \mathcal{N}$.

In the Classical Homotopy Theory, the presentation is given by a category with weak equivalences, possibly satisfying extra axioms such as those of a Homotopical Category, a Category of Fibrant Objects, or a Model Category.

In particular, recall that, a homotopical category is a structure used in homotopy theory, related to but more flexible than a model category.
More precisellly

Definition 114. *(Homotopical Category)*
A homotopical category is a category with weak equivalences where on top of the two out of 3 property the morphisms satisfy the following and so called two out of six property.
If morphisms $h \circ g$ and $g \circ f$ are weak equivalences, then so are f, g, h and $h \circ g \circ f$.

I can remark that,

- (•) The two out of 6 property implies the 2 out of 3 property, hence every homotopical category is a category with weak equivalences.
- (•) Every model category yields a homotopical category.
- (•) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between homotopical categories which preserves weak equivalences is a homotopical functor.

In Enriched Homotopy Theory, the presentation of that Higher Category is given by an Enriched Model Category or an Enriched Homotopical Category.

Recall that an Enriched Model Category or more generally an Enriched Homotopical Category is an enriched Category with extra information on how it behaves as a model in Higher Category. An enriched model category is an enriched category $\mathcal{C}$ together with the structure of a Model Category on the underlying category $\mathcal{C}_0$ such that both structures are compatible in a reasonable way. In the Enriched Category theory context the appropriate notion of homotopy limit is a weighted homotopy limit, Recall that, the idea is that the weight functor $W\colon K \longrightarrow V$ (where K is a small category and V is a monoidal category so that a category equipped with some notion of "tensor product"), encodes the way in which one generalizes the concept of a cone over a diagram F (that is, something with just a tip from which morphisms are emanating down to F) to a more intricate structure over the diagram F. For instance having $V = sSet$, the weight is such that it ensures that not only 1-morphisms are emanating from the tip, but that any triangle formed by these is filled by a 2-cell, every tetrahedron by a 3-cell, etc.

Now, as for ordinary limits, there are two ways to define homotopy limits:

- (•) with explicit constructions that satisfy a local universal property: the homotopy limit object "represents homotopy-coherent cones up to homotopy".
- (•) as derived functors of a Homotopy Kan Extension that satisfy a global universal property: the homotopy limit functor is Euniversal among homotopical approximations to the strict limit functor".

One of the central Theorems of the subject is that in good cases, the two of them give equivalent results.

Recall the definition of a Kan extension

Definition 115. *(Kan Extension)*

Let A be a category and $p\colon \mathcal{C} \longrightarrow \mathbb{D}$ be a functor between small categories.

I have the functor categories $[\mathcal{C}, A]$ and $[\mathcal{D}, A]$ and composition with p p induces a functor,

$$p^*\colon [\mathcal{D}, A] \longrightarrow [\mathcal{C}, A].$$

If A has all limits and colimits, then this functor has a left adjoint denoted by Lan_p and a right adjoint denoted by Ran_p,

$$Lan_p \dashv p^* \dashv Ran_p)\colon = (\ p_! \dashv f^* \vdash p_*\ )\colon [C, A] \underset{Ran_p}{\overset{\overset{Lan_p}{\longrightarrow}}{\overset{p^*}{\rightleftarrows}}} [\mathcal{D}, A].$$

These are respectively the left and right Kan extension functors.

A First Approach.

.

Limit and colimit are tough operations in the category of spaces: they tend to destroy homotopical information and in addition they are not invariant under homotopies of maps.

For instance, the colimit of the diagram,

$$* \longleftarrow \mathbb{S}^2 \longrightarrow *$$

is a point and has no collection of the homotopy type for the space $\mathbb{S}^2$.

Another example constitute the diagram $* \rightrightarrows [0,1]$, where both maps send the point to 0, is also a point, but becomes empty if I deform one or the two maps away from 0.

I would want to work with homotopy versions of limit and colimit better preserving the homotopical structure of spaces. To this end, I consider two main examples.

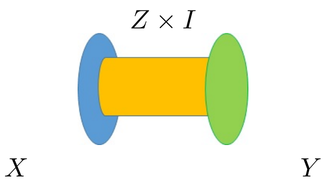
Example 116. *The colimit of the diagram of spaces $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ is,*

$$(X \amalg Y)\ /\ (\ f(z) \sim g(z)\ , \quad z \in Z\ )$$

Now, instead of identify $f(z)$ and $g(z)$, I take a path between them.

This is the double mapping cylinder construction and is an example of a homotopy colimit:

$$hocolim\ (X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y) = colim\ (X \longleftarrow Z \longrightarrow Z \times [0,1] \longleftarrow Z \longrightarrow Y) =$$
$$X \amalg (Z \times [0,1]) \amalg Y)\ /\ \{\ (z,0) \sim f(z)\ ,\ (z,1) \sim g(z)\ \}$$



I can note that I recover the suspension functor as a special case of homotopy colimit, when $X = Y = *$.

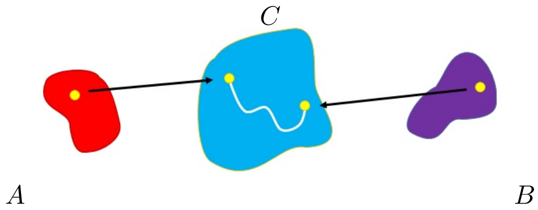
Example 117. *The limit of the diagram of spaces* $A \xrightarrow{f} C \xleftarrow{g} B$ *is,*

$$\{\ (a,b) \in A \times B \mid f(a) = g(b)\ \}$$

Instead of expecting $f(x)$ *and* $g(y)$*, to be equal in this limit, I can merely demand that they be connectted by a chosen path.*

This is the double path space construction and is an example of a homotopy limit:

$$holim\ (A \xrightarrow{f} C \xleftarrow{g} B) = lim\ (A \longrightarrow C \longleftarrow C^{[0,1]} \longrightarrow C \longleftarrow B) = \{\ (a\ ,\ q\colon [0,1] \longrightarrow C\ ,\ b) \mid f(a) = q(0)\ ,\ g(b) = q(1)\ \}$$



I can note that I recover the loop functor as a special case of homotopy limit, when $X = Y = *$ *and* $f = g$.

Local and Global Approaches.

As I have just introduced, there are two natural candidates for a definition of homotopy limit.

The first one is defined for any category enriched over topological spaces, or over simplicial sets.

In this case there are explicit constructions of homotopy limits and colimits. The classical work of Aldridge Bousfield and Daniel Kan in ([1]) develope those case. Also, Philip Hirschhorn in [7] offers a more modern exposition.

These homotopy limits are objects satisfying a "homotopical" sort of the usual universal property. Indeed, instead of representing commuting cones over a diagram, they represent "homotopy coherent" cones. This universal property is local in that it characterizes only a single object, although usually the homotopy limit can be extended to a functor.

The second approach studies categories which are equipped with some notion of weak equivalence. Then, I can invert the weak equivalences homotopically to get a simplicially enriched localization, but it is usually more convenient to deal with the original category and its weak equivalences directly. This is the context where the notion of derived functor, which is a universal homotopical approximation to some given functor.

From this last point of view, it is natural to define a homotopy limit to be a derived functor of the usual limit functor. This kind of homotopy limit has a global universal property that is referred to the whole possible homotopical replacements for the limit functor.

Both approaches have advantages:

- The universal property of the global constructions makes it easier to obtain coherence and preservation results.
- The local construction, is very powerful in computations, since has a natural filtration which gives rise to spectral sequences (when expressed as a bar construction). On the other hand.

There are several works establishing comparisons between those two definitions in order to center whether they agree, up to homotopy, so modern homotopy theory and moreover algebraic topology can both use whichever is most convenient for a given purpose.

From the global point of view, a homotopy limit should be a derived functor of the limit functor. Since the limit is a right adjoint and the colimit is a left adjoint, I expect the first one to have a right derived functor while the second one a left derived functor.

So that, my dreamt environment would be one in which for all model categories $\mathcal{M}$ and all small categories $\mathcal{I}$ the diagram category $M^{\mathcal{I}}$ always would have model structures for which the colimit and limit functors were left and right Quillen, respectively. Sadly, the more common notion of Quillen model category does not have those properties.

However, I can enumerate some main special cases in which the diagram category does have a model structure and limit or colimit functors are both Quillen.

- If $\mathcal{M}$ is cofibrantly generated, then all categories $M^{\mathcal{I}}$ have a projective model structure in which the weak equivalences and fibrations are objectwise. When this model structure exists, the colimit functor is left Quillen on it.
- If $\mathcal{M}$ is "sheafifiable" (which is a stronger hypothesis) then each functor category $\mathcal{M}^{\mathcal{I}}$ have an injective model structure in which the weak equivalences and cofibrations are objectwise. Whether such a model structure exists, then the limit functor is right Quillen on it.
- If $\mathcal{I}$ is a Reedy category, then for any model category $\mathcal{M}$, the category $\mathcal{M}^{\mathcal{I}}$ has a Reedy model structure in which the weak equivalences are objectwise, but the cofibrations and fibrations are generally not. If furthermore $\mathcal{I}$ has fibrant constants defined in some suitable sense, then the colimit functor is left Quillen for this model structure. Dually, if $\mathcal{I}$ has cofibrant constants, then the limit functor is right Quillen.

To sum up, when a suitable model structure exists on $\mathcal{M}^{\mathcal{I}}$, the global definition of homotopy limits or colimits is rather simple. I then apply a fibrant or cofibrant replacement in the appropriate model structure and take the usual limit or colimit.

Using the following example I will show a way of constructing the inverse homotopy colimit and the direct homotopy limit. In addition I will show two very useful constructions the mapping telescope and the mapping microscope.

Example 118. *I suppose* $\mathcal{I} = \{\ 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \ \cdots\ \}$ *and let* $X\colon \mathcal{I} \longrightarrow \mathcal{M}$ *be a functor with* $X(i) = X_i$ *and* $X(i \longrightarrow i-1) = f_i$.

Thus the diagram is an inverse system of objects

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \ \cdots$$

For any X_i, *the matching object* $M_i(X)$ *is equal to* X_{i-1}, *as the object* $i \longrightarrow i-1$ *is initial in the category indexing the limit which defines* $M_i(X)$.

The above tower constitutes a fibrant diagram; to prove this, it suffices that the maps $f_i\colon X_i \longrightarrow X_{i1}$ *are fibrations for each* i.

Hence, a model for the homotopy inverse limit of a tower can be achieved by replacing all of the maps systematically by fibrations.

To do this, I will define a new tower $T_m X$ *and a map of towers* $X \longrightarrow T_m X$ *that will be a homotopy equivalence. More precisely, I will construct a commutative diagram,*

$$\begin{tikzcd}
X_0 \arrow[d] & X_1 \arrow[l, "f_1"'] \arrow[d] & X_2 \arrow[l, "f_2"'] \arrow[d] & \cdots \arrow[l] \\
T_mX(0) & T_mX(1) \arrow[l, "p_1"] & T_mX(2) \arrow[l, "p_2"] & \cdots \arrow[l]
\end{tikzcd}$$

where all the vertical maps are equivalences.

The tower $T_m X$ *is defined inductively:*

Let $T_m X(0) = X_0$, *with the vertical map being the identity.*

Then let $T_m X(1) = T_{m_{f_1}}$, *with the maps* $X_1 \longrightarrow T_m X(1)$ *and* $T_m X(1) \longrightarrow T_m X(0)$ *being the natural maps. The vertical maps are homotopy equivalences, and the map* $T_m X(1) \longrightarrow T_m X(0)$ *is a fibration.*

Provided $T_m X(k1)$ *has been constructed, then I define* $T_m X(k)$, *via the pullback square,*

$$\begin{tikzcd}
T_mX(k) \arrow[r] \arrow[d, "p_k"'] & T_{m_{f_k}} \arrow[d] \\
T_mX(k-1) \arrow[r] & X_{k-1}
\end{tikzcd}$$

where $T_mX(k-1) \longrightarrow X_{k-1}$ *is the composition of the projection,*

$$T_mX(k-1) = lim\ (T_mX(k-2) \longrightarrow X_{k-2} \longleftarrow T_{m_{f_{k-1}}} \longrightarrow T_{m_{f_{k-1}}}$$

with the canonical projection $T_{m_{f_{k-1}}} \longrightarrow X_{k-1}$.

Since the right vertical map in the above square is a fibration, so is the map p_k. *Since the bottom horizontal map is a homotopy equivalence, so is the map* $T_mX(k) \longrightarrow T_{m_{f_k}}$.

To define the map $X_k \longrightarrow T_mX(k)$, *I will do the following. I have a natural map* $X_k \longrightarrow T_{m_{f_k}}$ *and the composition,*

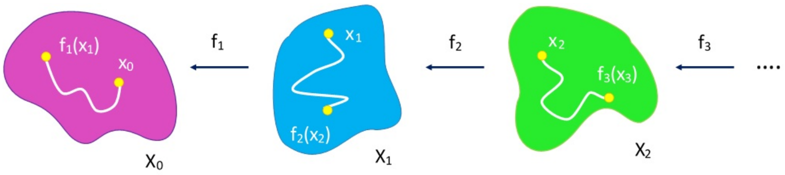
$$X_k \longrightarrow X_{k-1} \longrightarrow T_mX(k-1)$$

and if I compose both of those with maps to X_{k1}, *they agree. Since* $X_k \longrightarrow T_{m_{f_k}}$ *and* $T_mX(k) \longrightarrow T_{m_{f_k}}$ *are both homotopy equivalences, the map* $X_k \longrightarrow T_mX(k)$ *is also a homotopy equivalence.*

Because of the fact that the maps p_i *are fibrations, by the homotopy invariance of the homotopy limit, the induced map* $\underset{\mathcal{I}}{holim}\ T_mX \longrightarrow \underset{\mathcal{I}}{holim}\ X$ *is a homotopy equivalence. Since* T_mX *is fibrant, the natural map* $\underset{\mathcal{I}}{lim}\ T_mX \longrightarrow \underset{\mathcal{I}}{holim}\ T_mX$ *is a weak equivalence.*

It remains to describe $\underset{\mathcal{I}}{lim}\ T_mX$ *explicitly. The map* $p_k\colon T_mX(k) \longrightarrow PX(k-1)$ *sends a pair* $(z_{k-1}, (x_k, \gamma_{k1}))$ *to* z_{k-1}. *A point in* $\underset{\mathcal{I}}{lim}\ T_mX$ *consists of a sequence of points* $(x_0, x_1, \cdots)$ *and paths* $(\gamma_0, \gamma_1, \cdots)$ *where* $x_i \in X_i$ *and* $\gamma_i\colon \mathcal{I} \longrightarrow X_i$ *has the property that* $\gamma_i(0) = f_{i+1}(x_{i+1})$ *and* $\gamma_i(1) = x_i$.

The mapping microscope of an inverse system of objects can be pictured as follows,



$\cdots$

There is a homeomorphism,

$$\underset{\mathcal{I}}{lim}\ X \cong \underset{\mathcal{I}}{lim}\left(\prod_i X_i \xrightarrow{\Theta_m} \prod_i X_i \times \prod_i X_i \xleftarrow{Id_{\amalg_i X_i} \times F)} \prod_i X_i\right)$$

where $F = (f_1, f_2, \cdots)$ *and* Θ_m *is the diagonal.*

And the the object,

$$\underset{\mathcal{I}}{lim}\left(T_{\Theta_m} \longrightarrow \prod_i X_i \times \prod_i X_i \xleftarrow{Id_{\amalg_i X_i} \times F)} \prod_i X_i\right)$$

where T_{Θ_m} *is the mapping path object is the mapping microscope described above.*

To sum up, to create a homotopy invariant limit, I replace a map by a fibration.

Dually, I suppose $\mathcal{I} = \{\ 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots\ \}$ *indexes a directed system of objects*

$$Y\colon \mathcal{I} \longrightarrow \mathcal{M},$$

namely

$$Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots$$

In this case, the latching object $L_i(Y)$ *of* Y_i *is* Y_{i1} *and it thus suffices for all the maps in this system to be cofibrations in order for it to be cofibrant.*

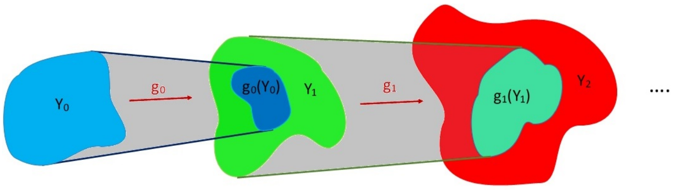
I define a tower T_tY *inductively as follows,*

$T_tY(0) = Y_0$, $T_tY(1) = T_{t_{g_0}}$ *, and*

$$T_tY(k) = colim\ (T_tY(k-1) \longleftarrow Y_{k-1} \longrightarrow T_{t_{g_{k-1}}}.$$

The maps $t_k \colon T_tY(k) \longrightarrow T_tY(k+1)$ *are the canonical inclusions* $T_tY(k) \longrightarrow colim\ (T_tY(k) \longleftarrow Y_k \longrightarrow T_{t_{g_k}})$*. Then by an argument dual to the previous one, I obtain a tower of cofibrations with a map* $Y \longrightarrow T_tY$ *which is a pointwise homotopy equivalence, and so I obtain ,* $\underset{\mathcal{I}}{hocolim}\ Y \simeq \underset{\mathcal{I}}{colim}\ T_tY$*.*

This colimit, $\underset{\mathcal{I}}{colim} T_tY$ *is called the mapping telescope of the diagram* Y *and I can roughly picture it as,*



$\dots$

This is the mapping telescope of a directed system of objects.

There is a homeomorphism,

$$\underset{\mathcal{I}}{colim}\ Y \cong \left(\coprod_i Y_i \xleftarrow{\Theta_t} (\ \coprod_i Y_i\ ) \coprod (\ \coprod_i Y_i\ ) \xrightarrow{Id_{\coprod_i Y_i} \coprod G} \coprod_i G_i \right)$$

where $G = g_i(y_i)$*, for* $y_i \in Y_i$ *and* Θ_t *is the fold map.*

Then, the space,

$$\left(T_{\Theta_t} \longleftarrow (\ \coprod_i Y_i\ ) \coprod (\ \coprod_i Y_i\ ) \xrightarrow{Id_{\coprod_i Y_i} \coprod F} \coprod_i G_i \right)$$

where T_{Θ_t} *is the mapping cylinder object, gives precisely the mapping telescope described above.*

To sum up, to create a homotopy invariant colimit, I replace a map by a cofibration.

I have shown, to summ up, how when a suitable model structure exists on $\mathcal{M}^{\mathcal{I}}$, the global definition of homotopylimits or colimits is rather simple. I then apply a fibrant or cofibrant replacement in the appropriate model structure and take the usual limit or colimit.

However, for projective and injective model structures, the cofibrant and fibrant replacements are constructed using the Quillen's small object arguments (or variation) and so are difficult to get a handle on. Hence, I may desire a more explicit construction.

In a Reedy model structures the fibrant and cofibrant replacements are relatively easy to define and compute. Moreover, many common diagram shapes have Reedy structures. In particular, the category

$$(\ \bullet \longleftarrow \bullet \longrightarrow \bullet\ )$$

which indexes pushout diagrams. So many global homotopy limits and colimits can be computed in this way by simply replacing a few maps by fibrations or cofibrations.

But, in fact, the conditions to be a Reedy category are quite particular, is some sense a too odd property of the diagram category, so in the general case it is far too much to expect.

Most model categories I can think of, are in fact cofibrantly generated, so that projective model structures exist, but equally many (such as those arising from topological spaces) are not sheafifiable, so there is no known model structure for which the limit functor is Quillen. Thus more technical methods are needed to construct global homotopy limit functors at all in this context.

One approach is to use a suitable "homotopical replacement" for the shape category $\mathcal{I}$.

For any functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ and any object $Y \in \mathcal{D}$, the comma category of F over Y has,

as objects the arrows $FX \longrightarrow Y$ in $\mathcal{D}$, and as arrows those in $\mathcal{C}$ whose images under F configure commutative triangles.

I will write $(F \downarrow Y)$ for the comma-category. When F is the identity functor of $\mathcal{C}$, I then write $(\mathcal{C} \downarrow X)$ for the comma-category, but in this case it is also called the over category of X. There exists a dual comma category $(Y \downarrow F)$ which leads me to the under category $(X \downarrow \mathcal{C})$.

More precisely, I develope these facts in the following section.

Constructing Homotopy Limits and Homotopy Colimits.

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Trying to understand the system of coherent homotopies at a given space, I will be interested in the local shape of the indexing category near the object indexing that space. This is captured by the notions of overcategory and undercategory.

Definition 119. *(OverCategory and UnderCategory)*
Suppose $\mathcal{I}$ is a small category and X is and object of $\mathcal{I}$.
The category of objects over X, denoted by $(\mathcal{I} \downarrow X)$, is the category

- (•) *whose objects are morphisms $(A \longrightarrow X)$*
- (•) *and whose morphisms $(A \longrightarrow X) \longrightarrow (B \longrightarrow X)$ are commutative diagrams,*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X & \end{tikzcd}$$

The category of objects under X, denoted by $(X \downarrow \mathcal{I})$, is the category

- (•) *whose objects are morphisms $(X \longrightarrow A)$*
- (•) *and whose morphisms $(X \longrightarrow A) \longrightarrow (X \longrightarrow B)$ are commutative diagrams,*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd}$$

The identity morphism $(X \longrightarrow X)$ is a final object in $(\mathcal{I} \downarrow X)$ and an initial object in $(X \downarrow \mathcal{I})$. I refer to $(\mathcal{I} \downarrow X)$ as an overcategory and $(X \downarrow \mathcal{I})$ as an undercategory.

I can observe that the overcategory and undercategory are dual in the sense that there is a natural isomorphism of categories,

$$(X \downarrow \mathcal{I})^{op} \cong (\mathcal{I}^{op} \downarrow X).$$

Moreover, if $\mathcal{I}$ has a final object X_f , then there is a natural isomorphism $(\mathcal{I}^{op} \downarrow X_f) \cong \mathcal{I}$ given by forgetting the final object, and similarly holds if it has an initial object.

These definitions can be extended to functors between categories in the following way:

Definition 120. *(OverCategory and UnderCategory of a Functor)*
If $F \colon \mathcal{I} \longrightarrow \mathcal{J}$ is a functor, and Y is an object of $\mathcal{J}$, I define the category of objects of $\mathcal{I}$ over Y, denoted $(F \downarrow Y)$, as consisting of pairs (X, ϕ) where $\phi \colon F(X) \longrightarrow Y$ is a morphism. A morphism $(X, \phi) \longrightarrow (X', \phi')$ in this category is a morphism $(X \longrightarrow X')$ such that $\phi' \circ F(X \longrightarrow X') = \phi$. That is, the diagram,

$$\begin{tikzcd} F(X) \arrow[rr, "F(X\longrightarrow X')"] \arrow[dr, "\phi"'] & & F(X') \arrow[dl, "\phi'"] \\ & Y & \end{tikzcd}$$

commutes.

In a similar way, I define $(Y \downarrow F)$, the category of objects of $\mathcal{I}$ under Y. I may refer to $(F \downarrow Y)$ as an overcategory of F and $(Y \downarrow F)$ as an undercategory of F. I observe that in the special case *where $F = Id_{\mathcal{I}}$ is the identity functor in the above definition I recover the notions of overcategory and undercategory.*

Construction of Homotopy Colimits

Let $\mathcal{I}$ be a small category, and let $D\colon \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram. I can construct the homotopy colimit of D using a simplicial replacement.

The simplicial replacement of D is the simplicial space,

$$\coprod_{i_0} D(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} D(i_1) \Lleftarrow \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} D(i_2) \,\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\, \cdots$$

I will denote this $s_{rep}(D)$. So I have,

$$s_{rep}(D)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n)$$

where the coproduct ranges over chains of composable maps in $\mathcal{I}$.

As every simplicial space I must define the face and degeneracy maps,

The faces can be described as follows:

If $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ is a chain and $0 \leq j \leq n$, then I can "cover up" i_j and obtain a chain of $n-1$ composable maps. I denote by $\sigma(j)$ this new chain.

When $j < n$, the map $d_j\colon s_{rep}(D)_n \longrightarrow s_{rep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to σ to the identical copy of $D(i_n)$ in $s_{rep}(D)_{n-1}$ indexed by $\sigma(j)$.

When $j = n$, I must modify this slightly, as covering up i_n now yields a chain that ends with i_{n-1}. So $d_n\colon s_{rep}(D)_n \longrightarrow s_{rep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to the chain σ to the summand $D(i_{n-1})$ corresponding to $\sigma(n)$, and the map which I work is the map $D(i_n) \longrightarrow D(i_{n-1})$ induced by the last map in σ.

The degeneracy maps $s_j\colon s_{rep}(D)_n \longrightarrow s_{rep}(D)_{n+1}, 0 \leq j \leq n$, can be described as follows:

Each s_j sends the summand $D(i_n)$ corresponding to the chain $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots i_n]$ to the identical summand $D(i_n)$ corresponding to the chain $\sigma[j]$ in which I have inserted the identity map $i_j \leftarrow i_j$.

Remark 121. *Note that I have made a choice when defining the simplicial replacement.*

I could have defined the n-th object to be,

$$\coprod_{i_0 \to i_1 \to \cdots \to i_n} D(i_0) \tag{121.1}$$

and again defined the degeneracy d_j to be the map associated to "covering up" i_j.

This is related to the distinction between the nerve of a category $\mathcal{I}$ and the nerve of its opposite that I have introduced in remark 130. The simplicial space from 121.1 is not isomorphic to $s_{rep}(D)$, early defined, although their geometric realizations are homeomorphic.

So there are two natural definitions of the simplicial replacement (as well as for the nerve of a category), and is useful to have both definitions around at the same time to use them at will.

Definition 122. *(Homotopy Colimit of a Diagram)*

The homotopy colimit of a diagram $D\colon \mathcal{I} \longrightarrow \mathcal{C}$ is the geometric realization of its simplicial replacement. That is,

$$\underset{\mathcal{I}}{hocolim}\ D = |s_{rep}(D)|.$$

I write $\underset{\mathcal{I}}{hocolim}\ D$ to remind of the indexing category $\mathcal{I}$.

The homotopy invariance of the homotopy colimit is an interesting fact to be proved.

Proposition 123. *If $D, D'\colon \mathcal{I} \longrightarrow \mathcal{M}$ are two diagrams consisting of cofibrant objects and $\omega\colon D \longrightarrow D'$ is a natural weak equivalence, then the induced map hocolim $D \longrightarrow$ hocolim D' is a weak equivalence.*

Proof. I get a map of simplicial spaces $s_{rep}(D) \longrightarrow s_{rep}(D')$, and this is an objectwise weak equivalence. Since $s_{rep}(D)$ and $s_{rep}(D')$ are both Reedy cofibrant, it follows that the induced map of realizations is also a weak equivalence. □

Remark 124. *Proposition 123 is perhaps weaker than I would hope for, because of the cofibrancy conditions on the objects of D and D', but they are a must. In a general model category, to get the "correct" homotopy colimit of a diagram D, I should first arrange things so that all the objects are cofibrant, for instance, by applying a cofibrant replacement functor to all the objects of D. And, then I can apply specific formulas for the hocolim, such as the one above.*

In some categories, like the category Top, the cofibrancy conditions on the objects are not necessary at all and then, Proposition 123 is true even without these conditions. But it is a main fact in this work to state each result as generalized as possible, in order to deeply understand the environment in which every statement or fact, efectively lives.

I can also note that *colim* D is the coequalizer of the mapping faces d_0 and d_1 in $s_{rep}(D)$, that is, it is the quotient space $[\coprod_i D(i)]/\sim$ where for every map $\sigma\colon i \longrightarrow j$ in $\mathcal{I}$, I identify points $x \in D(i)$ with $\sigma_*(x) \in D(j)$.

The canonical map

$$|s_{rep}(D)| \longrightarrow coeq\left[s_{rep}(D)_1 \rightrightarrows s_{rep}(D)_0\right]$$

therefore can be written as a map *hocolim* $D \longrightarrow$ *colim* D.

This is the natural map from the homotopy colimit to the colimit.

I can add another formula for the homotopy colimit. The space it describes is homeomorphic to that of my previous definition:

$$\underset{\mathcal{I}}{hocolim}\ D = coeq\left[\coprod_{i\to j} D_i \times B(j\downarrow\mathcal{I})^{op} \rightrightarrows \coprod_i D_i \times B(i\downarrow\mathcal{I})^{op}\right] \tag{124.1}$$

Recall that if $\mathcal{C}$ is a category, the geometric realization of its nerve is called its classifying space and it is denoted by $B(\mathcal{C})$ and $\mathcal{C}^{op}$ is the opposite category. The op's are needed in the above formula only to make it conform with the choices I made in defining the simplicial replacement.

Finally, if $i \longrightarrow j$ is a map in $\mathcal{I}$ then there is an evident induced map of categories $(j\downarrow\mathcal{I}) \longrightarrow (i\downarrow\mathcal{I})$. The above formula 124.1 gives a more direct comparison between the homotopy colimit and the ordinary colimit. The colimit is, after all, the coequalizer,

$$\underset{\mathcal{I}}{colim}\ D = coeq\left[\coprod_{i\to j} X_i \rightrightarrows \coprod_i X_i\right]$$

I find a map from the previous coequalizer diagram to this last one simply by collapsing the spaces $B(i\downarrow\mathcal{I})^{op}$ to a point.

Thus, I obtain the map *hocolim* $D \longrightarrow$ *colim* D.

Remark 125. *I can justify the fact that the space defined in formula 124.1 is homeomorphic to the space $|s_{rep}(D)|$.*

Previously, I want to explain the general idea. In constructing $|s_{rep}(D)|$, for every chain $i_n \leftarrow i_{n-1} \leftarrow \cdots \leftarrow i_0$, I have added a copy of $D_{i_0} \times \Delta^n$. So if I fix a particular spot D_i of the diagram, this means that I am adding a copy of $D_i \times \Delta^n$ for every string $i_n \leftarrow i_{n1} \leftarrow \cdots \leftarrow i_1 \leftarrow i$.

Such a string gives an n-simplex in $B(i\downarrow\mathcal{I})^{op}$, corresponding to the chain

$$[i, i_n \leftarrow i] \leftarrow [i, i_{n-1} \leftarrow i] \leftarrow \cdots \leftarrow [i, i_1 \leftarrow i] \leftarrow [i, i \leftarrow i\colon Id]$$

(which is a chain in $(i\downarrow\mathcal{I})$).

In the formula 124.1, I am simply grouping all these $D_i \times \Delta^n$'s together, fixing i and letting n vary, into the space $D_i \times B(i\downarrow\mathcal{I})^{op}$.

In other words, the space $B(i\downarrow\mathcal{I})^{op}$ is parameterizing all the "D_i-homotopies" that are being added into the homotopy colimit.

In order to properly develope the detailed proof that my two formulas for hocolim D are naturally homeomorphic, a couple of observations are needed.

First, if K is a simplicial set then $X \times |K|$ can be identified with the geometric realization of the simplicial space

$$[n] \mapsto X \times K_n = \coprod_{K_n} X$$

I consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \cdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\coprod_{i,k_1 \leftarrow k_0 \leftarrow j \leftarrow i} X_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{i,j_1 \leftarrow j_0 \leftarrow i} X_i \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \coprod_{j_0 \leftarrow j_1} X_{j_i} \arrow[d, shift left] \arrow[d, shift right] \\
\coprod_{i,k_0 \leftarrow j \leftarrow i} X_i \arrow[r, shift left] \arrow[r, shift right] & \coprod_{i,j_0 \leftarrow i} X_i \arrow[r] & \coprod_{j_0} X_{j_0}
\end{tikzcd}$$

Each column is a simplicial space.

The rightmost column is $s_{rep}(X)$, the middle column is $\coprod_i (X_i \times N(\mathcal{I} \downarrow i)^{op})$, and the leftmost column is $\coprod_{i \leftarrow j} (X_i \times N(\mathcal{I} \downarrow j)^{op})$.

I have a map of simplicial spaces from the middle column to the right column. In dimension n this sends the summand X_i corresponding to the string $[j_0 \leftarrow j_1 \leftarrow \cdots \leftarrow j_n \leftarrow i]$ to the summand X_{j_n} corresponding to $[j_0 \leftarrow \cdots \leftarrow j_n]$ via the map $X_i \longrightarrow X_{j_n}$ induced by $i \to j_n$. This is compatible with face and degeneracies.

I have two maps of simplicial spaces from the left column to the middle column. In dimension n, one map sends the summand X_i corresponding to the index $[i, k_n \leftarrow k_{n-1} \leftarrow \cdots \leftarrow k_0 \leftarrow j \leftarrow i]$ to the summand X_i indexed by $[i, k_n \leftarrow \cdots \leftarrow k_0 \leftarrow i]$ (forget about j). The other map sends the summand X_i to the summand X_j indexed by $[j, k_n \leftarrow \cdots \leftarrow k_0 \leftarrow j]$ (forget about i).

Now, each horizontal level in the diagram is a coequalizer diagram; that is to say, the objects in the right column are the coequalizers of the objects in the other two columns. Geometric realization is a left adjoint, and therefore will commute with coequalizers. So this identifies $|srep(D)|$ with the coequalizer of

$$\coprod_{i \to j} |X_i \times N(\mathcal{I} \downarrow j)^{op}| \rightrightarrows \coprod_i |X_i \times N(\mathcal{I} \downarrow i)^{op}|.$$

This is the identification that I was searching for.

The facts about homotopy limits are completely dual to that for homotopy colimits, I can briefly outline the main facts.

A cosimplicial space is a functor $X : \Delta \longrightarrow \mathcal{M}$, pictured as follows:

$$\begin{tikzcd}
X_0 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

(and here I am omitting the codegeneracy maps...). Let Δ^* denote the cosimplicial space corresponding to the standard inclusion $\Delta \hookrightarrow Top$. As a cosimplicial space, Δ^* is

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left] \arrow[r, shift right] & \Delta^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Delta^2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

If X is any cosimplicial space I can talk about the space of maps from Δ^* to X: the points are the natural transformations $\Delta^* \longrightarrow X$, and they are topologized as a subspace of $\prod_n X_n^{\Delta^n}$.

This space of maps is sometimes denoted $map(\Delta^*, X)$, but is more commonly denoted $Tot\ X$. It is called the totalization of X.

I can also describe it as an equalizer:

$$Tot\ X = eq \left[\prod_n X_n^{\Delta^n} \rightrightarrows \prod_{[n] \to [k]} X_n^{\Delta^n} \right].$$

The two maps in the equalizer can be defined as follows, using that any map $\sigma \colon [n] \longrightarrow [k]$ induces a corresponding map $\sigma_* \colon \Delta^n \longrightarrow \Delta^k$.

Given a sequence of elements $s_n \in X_n^{\Delta^n}$, one of the maps sends this sequence to the collection $\sigma \mapsto s_k \circ \sigma_* \in X_k^{\Delta^n}$.

The other map sends the sequence s_n to the collection $\sigma \mapsto X(\sigma) \circ s_n \in X_k^{\Delta^n}$, where $X(\sigma)$ is the induced map $X_n \longrightarrow X_k$.

That is, a point in $Tot\ X$ consists of a point $x_0 \in X_0$, an edge x_1 in X_1, a 2-simplex x_2 in X_2, and so on, which are "compatible" in the following two ways:

(1) The two images of x_0 under $X_0 \rightrightarrows X_1$ are the two endpoints of x_1; the three images of x_1 under the maps d^0, d^1, $d^2 \colon X_1 \longrightarrow X_2$ are the three faces of the 2-simplex x_2 and so on.

(2) The image of x_1 under the codegeneracy $X_1 \longrightarrow X_0$ is the map $\Delta^1 \longrightarrow X_0$ collapsing everything to x_0; the image of x_2 under the two codegeneracies $X_2 \rightrightarrows X_1$ are the two maps $\Delta^2 \rightrightarrows \Delta^1 \xrightarrow{x_1} X_1$, etc.

Intuitively, I can think of a point in $Tot\ X$ as being a point $x_0 \in X_0$ plus an edge connecting its two images in X_1, plus a 2-simplex connecting the three images of this edge in X_2, and so on, with the added fact that all this data must be compatible under the codegeneracies.

I can note that there is a map $eq(X_0 \rightrightarrows X_1) \longrightarrow Tot\ X$ defined as follows. If $x_0 \in X_0$ is equalized by the two maps to X_1, then I can choose the 1-simplex x_1 in X_1 to be constant. Then I can also choose a 2-simplex in X_2 to be constant, and so on down the line. All of these choices are automatically compatible under codegeneracies, so I get a point in $Tot\ X$.

Construction of homotopy limits.

Let $\mathcal{I}$ be a small category and $D \colon \mathcal{I} \longrightarrow \mathcal{M}$ a diagram.

Any diagram of spaces could be turned into a cosimplicial one in such a way that the totalization of the cosimplicial diagram is the homotopy limit of the original diagram. In fact, many authors take the totalization of the cosimplicial replacement of a diagram as the definition of its homotopy limit.

Given a diagram $D \colon \mathcal{I} \longrightarrow \mathcal{M}$, with $\mathcal{I}$ small, I consider the space,

$$\prod\nolimits^n D = \prod_{i_0 \to \cdots \to i_n} D(i_n)$$

In fact, this construction is like the definition of the nerve of $\mathcal{I}$.

The product is taken over all composable morphisms of length n in $\mathcal{I}$. (Composable morphisms of any length always exist in a diagram because of the existence of identity maps.) I note that,

$$\prod\nolimits^0 D = \prod_{i \in \mathcal{I}} D(i)$$

In fact, the coface and codegeneracy maps in the cosimplicial replacement are determined by faces and degeneracies in $N(\mathcal{I})$.

The coface maps d^j are defined as follows. The projection of

$$d^j \colon \prod_{i_0 \to \cdots \to i_{n-1}} D(i_{n-1}) \longrightarrow \prod_{i'_0 \to \cdots \to i'_n} D(i'_n) \qquad 0 \leq j \leq n$$

onto the factor $D(i'_n)$ indexed by $i'_0 \to \cdots \to i'_n$ is the composition of the identity map of $D(i'_n)$ with the projection onto the factor indexed by $i'_0 \to i'_1 \to \cdots \to i'_{j-1} \to i'_{j+1} \to \cdots \to i'_n = d_j(i'_0 \to \cdots \to i'n)$.

The codegeneracies are defined as,

$$s^j \colon \prod_{i_0 \to \cdots \to i_{n+1}} D(i_{n+1}) \longrightarrow \prod_{i'_0 \to \cdots \to i'_n} D(i'_n) \qquad 0 \leq j \leq n,$$

The projection onto the factor $D(i'_n)$ indexed by $i'_0 \to \cdots \to i'_n$ is the composition of the identity map of $D(i'_n)$ with the projection onto the factor indexed by $i'_0 \to i'_1 \to \cdots \to i'_j \to i'_j \to \cdots \to i'_n = s_j(i'_0 \to \cdots \to i'n)$.

These cofaces and codegeneracies satisfy the cosimplicial identities since the maps d_j and s_j in the definition of $N(I)$ satisfy the simplicial identities.

Now,

Definition 126. *The cosimplicial space*

$$\prod\nolimits^\bullet F = \{\prod\nolimits^n D\}_{n=0}^\infty$$

with coface and degeneracy maps as decribed above is called the cosimplicial replacement of the diagram $D\colon \mathcal{I} \longrightarrow \mathcal{M}$*.*

The cosimplicial replacement of D *which I denote by* $c_{rep}(D)$ *and the previous definition is usually expressed in a equivalent way as,*

$$c_{rep}(D)_n = \prod_{i_0 \mapsto i_1 \mapsto i_1 \mapsto \cdots \mapsto i_n} D(i_n)$$

Remark 127. *I could have used F(i0) instead of F(in) in,*

$$\prod\nolimits^n D = \prod_{i_0 \to \cdots \to i_n} D(i_n)$$

This choice, would result in a cosimplicial replacement whose totalization is homeomorphic to the totalization of the replacement defined above.

I can define the homotopy limit of D by,

$$holim\ D = Tot[c_{rep}(D)]$$

This construction is homotopy invariant.

The equalizer of $c_{rep}(D)_0 \rightrightarrows c_{rep}(D)_1$ is just $lim\ D$.

A point in this equalizer consists of a choice of point in each D_i which are compatible as one moves around the diagram. The natural map from this equalizer into $Tot(c_{rep}(D))$ gives me a natural map $lim\ D \longrightarrow holim\ D$.

Just as for homotopy colimits, I can describe $holim\ D$ via another formula, this time an equalizer formula:

$$holim\ D \cong eq\left[\prod_i X_i \times B(i \downarrow \mathcal{I}) \rightrightarrows \prod_{i \to j} X_j \times B(i \downarrow \mathcal{I})\right]$$

I can consider now a useful adjunction formula.

$$map_{\mathcal{M}}(\underset{\mathcal{I}}{colim}\ D, X) \cong \underset{\mathcal{I}}{lim}\ map_{\mathcal{M}}(D(i), X). \tag{127.1}$$

This formula just says that giving a map $colim\ D \longrightarrow X$ is the same as giving a bunch of maps $D(i) \longrightarrow X$ which are compatible as i changes.

There is a similar formula,

$$map_{\mathcal{M}}(X, \underset{\mathcal{I}}{lim}\ D) \cong \underset{\mathcal{I}}{lim}\ map_{\mathcal{M}}(X, D(i))$$

which has an analogous interpretation.

I can generalise adjunction formuli 127.1 to homotopy limits and homotopy colimits. For this I need to assume I am working in 'suitable' model category where the mapping space is a true right adjoint (like the category of compactly generated model category).

Lemma 128. *Let* $\mathcal{M}$ *be a model category and let* $\mathcal{I}$ *be a small category. If* D *is a* $\mathcal{I}$*-diagram in* $\mathcal{M}$ *and* Z *is an object in* $\mathcal{M}$*, then* $map(D, Z)$ *is a* $\mathcal{I}^{op}$*-diagram and I then have a natural map,*

$$map\ (\underset{\mathcal{I}}{hocolim}\ D, Z) \cong \underset{\mathcal{I}^{op}}{holim}\ map(D(i), Z)$$

Proof.

I know that by definition,

$$\underset{\mathcal{I}}{hocolim}\ D = coeq\left[\coprod_{i \to j} D_i \times B(j \downarrow \mathcal{I})^{op} \rightrightarrows \coprod_i D_i \times B(i \downarrow I)^{op}\right]$$

Now I apply $map(\bullet, Z)$ to this diagram.

$$map(\underset{\mathcal{I}}{hocolim}\ D, Z)$$

Now using the facts that: there exists a isomorphisms of categories $(i \downarrow \mathcal{I})^o p \cong (\mathcal{I}^o p \downarrow i)$, also that $map(\bullet, Z)$ takes colimits to limits and that $map(\bullet, Z)$ takes coproducts to products lemma 3, adjunction and the exponential law lemma 131, the fact that I obtain,

$$map\left(\underset{i\in\mathcal{I}}{hocolim}\ D, Z\right)$$

$$\downarrow\cong$$

$$eq\left[map\left(\coprod_i D_i\times B(i\downarrow\mathcal{I})^{op}, Z\right)\rightrightarrows map\left(\coprod_{i\to j} D_i\times B(j\downarrow\mathcal{I})^{op}, Z\right)\right]$$

$$\downarrow\cong$$

$$eq\left[\prod_i map\left(D_i\times B(i\downarrow\mathcal{I})^{op}, Z\right)\rightrightarrows \prod_{i\to j} map\left(D_i\times B(j\downarrow\mathcal{I})^{op}, Z\right)\right]$$

$$\downarrow\cong$$

$$eq\left[\prod_i map\left(D_i, Z\right)^{B(i\downarrow\mathcal{I})^{op}}\rightrightarrows \prod_{i\to j} map\left(D_i, Z\right)^{B(j\downarrow\mathcal{I})^{op}}\right]$$

$$\downarrow\cong$$

$$eq\left[\prod_i map\left(D_i, Z\right)^{B(\mathcal{I}^{op}\downarrow\mathcal{I})}\rightrightarrows \prod_{i\to j} map\left(D_i, Z\right)^{B(\mathcal{I}^{op}\downarrow j)}\right]$$

$$\|$$

$$\underset{\mathcal{I}^{op}}{holim}\ map\Big(D(i), Z\Big)$$

□

It also holds,

$$map\ (Z, \underset{\mathcal{I}}{holim}\ D)\cong \underset{\mathcal{I}}{holim}\ map(Z, D(i))$$

which could be proved with a similar argument to the previous one.

Mapping Spaces

.

As an initial intuitive idea, I can think about the model case, the category Top of topological spaces. Let $[X, Y]$ denote the set of homotopy classes of maps from X to Y . Then the functor $[\bullet, \bullet]$ encodes a lot of homotopical information: for example $\pi_n(X) = [\mathbb{S}^n, X]$ (with a little careless about base points). I note that I can identify homotopies of maps from X to Y with maps from $X\times|\Delta^1|$ to Y.

Setting $X^n = X\times|\Delta^n|$, I see that maps $X^n\longrightarrow Y$ should record "n-th order homotopies" between maps $X\longrightarrow Y$. I note that the X^n together form a cosimplicial object in Top. Moreover, $X^0\cong X$ and $X^n\simeq X$ since $|\Delta^n|$ is contractible.

If I denote now by cX^* the constant simplicial object at X, I have a natural map $X^*\longrightarrow cX^*$ where the maps are objectwise weak equivalences. The existence of this map is equivalent to saying that I have a weak equivalence $X^*\longrightarrow cX^*$ in Top^Δ.

I can generalise this to any model category. Let X be an object of a model category $\mathcal{M}$. Then a cosimplicial resolution of X is an acyclic cofibration $A^*\longrightarrow cX^*$ in the model category $\mathcal{M}^\Delta$. Dually a simplicial resolution of X is an acyclic fibration $cX^*\longrightarrow A^*$ in $\mathcal{M}^{\Delta^{op}}$, (where I denote the category of cosimplicial objects in $\mathcal{M}$ by $\mathcal{M}^\Delta$ and I also denote the category of simplicial objects in $\mathcal{M}$ by $\mathcal{M}^{\Delta^{op}}$).

If $A^*\longrightarrow cX^*$ is a cosimplicial resolution, then $A^0\longrightarrow X$ is an acyclic cofibration and the map $A^0\coprod A^0\longrightarrow A^1\longrightarrow A^0$ is a cylinder object for A^0. This is a good clue that a cosimplicial resolution of X records a lot of the higher homotopical data about X. Loosely I can think of the A^n as higher cylinder objects for X.

I suppose now that in the model category $\mathcal{M}$, I can construct splittings of maps (as in the factorization axiom) functorially. Then I can also construct cosimplicial resolutions functorially and hence I have a cosimplicial resolution functor $r\colon \mathcal{M} \longrightarrow \mathcal{M}^{\Delta}$, along with an analogous simplicial resolution functor $\bar{r}$. Since cosimplicial resolutions are unique up to weak equivalence, choosing a particular functor will not affect what I want to do: I want to use cosimplicial resolutions to construct mapping spaces.

First, I note that if X and Y are objects then $hom(rX, Y)$ is a simplicial set. This suggests that to get a mapping space I should take a cosimplicial resolution for X. But I should probably also replace Y by a simplicial resolution too. Keeping this in mind, if X and Y are objects in a model category, then I can form a bisimplicial set $hom(rX, \bar{r}Y)$. I can take the diagonal of this bisimplicial set to get a fibrant simplicial set $map_{\mathcal{M}}(X, Y)$ which is called the homotopy function complex from X to Y.

In fact, this is a two-sided function complex. I could have chosen just a cosimplicial resolution for X or a simplicial resolution for Y to get a left (respectively right) function complex. In either case I have ended up with something weakly equivalent to my original definition.

If I had not fixed a cosimplicial resolution functor, I have to keep track of the cosimplicial resolutions which I chose to define the function complexes with.

One of my initial intuitions for mapping spaces is that the set $\pi_0\, map_{\mathcal{M}}(X, Y)$ of path components of $map_{\mathcal{M}}(X, Y)$ should be in bijection with the homotopy classes of maps from X to Y. This does indeed happen with this construction.

Finally, I can add that weakly equivalent objects have weakly equivalent mapping spaces: if $f\colon X \xrightarrow{\sim} Y$ is a weak equivalence then f induces weak equivalences $map_{\mathcal{M}}(W, X) \xrightarrow{\sim} map_{\mathcal{M}}(W, Y)$ and $map_{\mathcal{M}}(X, Z) \xrightarrow{\sim} map_{\mathcal{M}}(Y, Z)$, as long as W is cofibrant and Z is fibrant.

In order to well-understand the following section I need to have in mind the generalities about simplicial sets introduced in Section 3.

Defining Mapping Spaces.

.

Definition 129. *(Mapping Space)*
Let $\mathcal{M}$ be a model category and I consider any two objects in the model category, X,$Y \in \mathcal{M}$.

Then, $map(X, Y)\colon = \bigcup_{n=1}^{\infty} N(\mathcal{C}_n^{X,Y})$ where $\mathcal{C}_n^{X,Y}$ is the following category,

- (•) *An object in $\mathcal{C}_n^{X,Y}$ is a chain in $\mathcal{M}$ of the form,*

$$X = X_0 \longleftrightarrow X_1 \longleftrightarrow X_2 \longleftrightarrow \cdots \longleftrightarrow X_n = Y$$

  where every arrow is a morphism in $\mathcal{M}$ that is $\longrightarrow$ or $\longleftarrow$, and every $\longleftarrow$ is a weak equivalence.

- (•) *A morphism in $\mathcal{C}_n^{X,Y}$ is a commutative diagram of the form,*

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X_0 \arrow[r, leftrightarrow] \arrow[d] & X_1 \arrow[r, leftrightarrow] \arrow[d] & X_2 \arrow[r, leftrightarrow] \arrow[d] & \cdots \arrow[r, leftrightarrow] & X_n \arrow[r, equal] \arrow[d] & Y \arrow[d, equal] \\
X \arrow[r, equal] & X'_0 \arrow[r, leftrightarrow] & X'_1 \arrow[r, leftrightarrow] & X'_2 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & X_n \arrow[r, equal] & Y
\end{tikzcd}$$

  where the scheme of arrows $\longrightarrow\!\longleftarrow$ is the same in each row.

Recall that in category theory the nerve $N(\mathcal{I})$ of a small catergory $\mathcal{I}$ is a simplicial set constructed from the objects and morphisms of $\mathcal{I}$. The geometric realization of this simplicial set is a topological space, called the classifying space of the category $\mathcal{I}$. These closely related objects can provide information about some familiar and useful categories using algebraic topology, but most often homotopy theory.

The nerve of a category $\mathcal{C}$ is often used to construct topological versions of moduli spaces. If X is an object of $\mathcal{C}$, its moduli space should somehow encode all objects isomorphic to X and keep track of the various isomorphisms between all of these objects in that category. This can become rather complicated, especially if the objects have many non-identity automorphisms. In particular, the nerve provides a combinatorial way of organizing this data. Since simplicial sets have a good homotopy theory, I can then, in the development of a given problem, ask questions about the meaning of the various homotopy groups $\pi_n(N(\mathcal{C}))$ in the hope that the answers to such questions could provide useful information about the original category $\mathcal{C}$, or about related categories in the development of that given problem.

The construction of the nerve is the following,

Let $\mathcal{I}$ be a small category. I can define the sets $N(\mathcal{I})_k$ for small k, which leads to the general definition. In particular, there is a 0-simplex of $N(\mathcal{I})$ for each object in $\mathcal{I}$ and also there is a 1-simplex for each morphism $f\colon X \longrightarrow Y$ in $\mathcal{I}$.

Now I suppose that $g\colon X \longrightarrow Y$ and $h\colon Y \longrightarrow Z$ are morphisms in $\mathcal{I}$. Then I also have their composition $h \circ g\colon X \longrightarrow Z$.

$$\begin{tikzcd} & Z \\ X \arrow[ur, "h\circ g"] \arrow[r, "g"'] & Y \arrow[u, "h"'] \end{tikzcd}$$

The previous diagram suggests the natural next course of action: I will add a 2-simplex for this commutative triangle. Every 2-simplex of $N(\mathcal{I})$ comes from a pair of composable morphisms in this way. I note that the addition of these 2-simplices does not erase or otherwise disregard morphisms obtained by composition, it merely remembers that that is how they arise.

In general, $N(\mathcal{I})_k$ consists of the k-tuples of composable morphisms

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_{k-1} \longrightarrow A_k$$

of $\mathcal{I}$. To entirely conclude the definition of $N(\mathcal{I})$ as a simplicial set, I must also specify the face and degeneracy maps. These are also provided to me by the structure of $\mathcal{I}$ as a category.

The face maps,

$$d_i\colon N(\mathcal{I})_k \longrightarrow N(\mathcal{I})_{k-1}$$

are given by composition of morphisms as the i-th object (or removing the i-th object from the sequence, when i is 0 or k). This means that d_i sends the k-tuple

$$A_0 \longrightarrow \cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_k$$

to the $(k-1)$-tuple,

$$A_0 \longrightarrow \cdots \longrightarrow A_{i-1} \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_k$$

That is, the map d_i composes the morphisms $A_{i-1} \longrightarrow A_i$ and $A_i \longrightarrow A_{i+1}$ into the morphism $A_{i-1} \longrightarrow A_{i+1}$, yielding a $(k-1)$-tuple for every k-tuple.

Similarly, the degeneracy maps

$$s_i\colon N(\mathcal{I})_k \longrightarrow N(\mathcal{I})_{k+1}$$

are given by inserting an identity morphism at the object A_i.

Recall that simplicial sets may also be regarded as functors $\Delta^{op} \longrightarrow Set$, where Δ is the category of totally ordered finite sets and order-preserving morphisms. Every partially ordered set P yields a (small) category $i(P)$ with objects the elements of P and with a unique morphisms from p to q whenever $p \leq q$ in P. I thus obtain a functor i from the category Δ to the category of small categories. I can now describe the nerve of the category $\mathcal{I}$ as the functor $\Delta^{op} \longrightarrow Set$,

$$N(\mathcal{I})(\bullet) = F(i(\bullet), \mathcal{I})$$

This description of the nerve makes functoriality quite transparent; for example, a functor between small categories $\mathcal{I}$ and $\mathcal{J}$ induces a map of simplicial sets $N(\mathcal{I}) \longrightarrow N(\mathcal{J})$. Moreover a natural transformation between two such functors induces a homotopy between the induced maps. It follows that adjoint functors induce homotopy equivalences. In particular, if $\mathcal{I}$ has an initial or final object, its nerve is contractible.

Remark 130. *To sum up, the nerve of a small category $\mathcal{I}$ is the simplicial set $N(\mathcal{I})$ which in dimension n consists of all strings*

$$[A_0 \to A_1 \to \cdots \to A_n]$$

of n composable arrows.

The face map d_i corresponds to "covering up" the object A_i , as above. The classifying space of $\mathcal{I}$ is the geometric realization of the nerve, denoted by $B(\mathcal{I})$.

The nerve of the opposite category $\mathcal{I}^{op}$ may be identified with the simplicial set which in dimension n consists of all strings $[A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_n]$ of n composable arrows, where the face map d_i again corresponds to covering up the object A_i. This is very similar to the nerve of $\mathcal{I}$, but not identical since the order of the faces and degeneracies have been reversed. These simplicial sets are not isomorphic, but they are naturally weakly equivalent.

Main Examples.

.

Mapping Space in Top.

.

For Y, Z topological spaces, the mapping space $map_{Top}(Y, Z)$ is defined to be the set of all continuous mappings $f\colon Y \longrightarrow Z$ with the compact-open topology defined as follows.

For every compact $K \subseteq Y$ and every open $U \subseteq Z$ I have the subbasic open set,

$$\langle K, U\rangle \colon = \{\ f \in map(Y, Z) \quad | \quad f(K) \subseteq U\ \}$$

The basic open sets are all finite intersections of these. The open sets are arbitrary unions of the basic open sets.

I can add, two useful consequences of using the compact-open topology. The first is that the composition map,

$$map(X, Y) \times map(Y, Z) \longrightarrow map(X, Z)$$

which sends (f, g) to $g \circ f$ is continuous. The same is true for based mapping spaces.

The second is known as the exponential law, most concisely expressed as

$$Y^{X\times Z} \cong (Y^X)^Z.$$

More precisely,

Lemma 131. *(Exponential Law, unbased version)*

For spaces X, Y, Z,

$$map(X \times Z, Y) \longrightarrow map(Z, map(X, Y))$$

which associates $F\colon X \times Z \longrightarrow Y$ with the map $z \mapsto (x \mapsto F(x, z))$ is a homeomorphism.

I note that the spaces X, Y, and Z are compactly generated Hausdorff. This theorem is not true for arbitrary topological spaces.

Proof. By definition, the compact-open topology on the space X^Y of maps $f\colon Y \longrightarrow X$ has a subbasis consisting of the sets $M(K, U)$ of mappings taking a compact set $K \subset Y$ to an open set $U \subset X$. Thus a basis for X^Y consists of sets of maps taking a finite number of compact sets $K_i \subset Y$ to open sets $U_i \subset X$.

First, I show that a subbasis for $X^{Y\times Z}$ is formed by the sets $M(A \times B, U)$ as A and B range over compact sets in Y and Z respectively and U ranges over open sets in X. Given a compact $K \subset Y \times Z$ and $f \in M(K, U)$, let K_Y and K_Z be the projections of K onto Y and respectively ont Z. Then $K_Y \times K_Z$ is compact Hausdorff and hence normal, so for each point $k \in K$ there are compact sets $A_k \subset Y$ and $B_k \subset Z$ such that $A_k \times B_k$ is a compact neighborhood of k in $f^{-1}(U) \cap (K_Y \times K_Z)$. By compactness of K a finite number of the products $A_k \times B_k$ cover K. Discarding the others, I then have $f \in \bigcap_k M(A_k \times B_k, U) \subset M(K, U)$, which shows that the sets $M(A\times, U)$ form a subbasis.

Under the bijection $X^{Y\times Z} \longrightarrow (X^Y)^Z$ these sets $M(A\times B, U)$ correspond to the sets $M(B, M(A,U))$, so it will suffice to show the latter sets form a subbasis for $(X^Y)^Z$. I show more generally that X^Y has as a subbasis the sets $M(K,V)$ as V ranges over a subbasis for X and K ranges over compact sets in Y, assuming that Y is Hausdorff.

Given $f \in M(K,U)$, write U as a union of basic sets U_α with each U_α an intersection of finitely many sets $V_{\alpha,j}$ of the given subbasis. The cover of K by the open sets $f^{-1}(U_\alpha)$ has a finite subcover, say by the open sets $f^{-1}(U_i)$. Since K is compact Hausdorff and hence normal, I can write K as a union of compact subsets K_i with $K_i \subset f^{-1}(U_i)$. Then f lies in $M(K_i, U_i) = M(K_i, \bigcap_j V_{ij}) = \bigcap_j M(K_i, V_{ij})$ for each i.

Hence f lies in $\bigcap_{i,j} M(K_i, V_{ij}) = \bigcap_i M(K_i, U_i) \subset M(K,U)$.

Since $\bigcap_{i,j} M(K_i, V_{ij})$ is a finite intersection, this shows that the sets $M(K,V)$ form a subbasis for X^Y.

□

The based version requires the smash product construction:

Lemma 132. *(Exponential Law, based version)*
If X, Y, and Z are based, the map

$$map_*(X\wedge Z, Y) \longrightarrow map_*(Z, map_*(X,Y))$$

which associates $F\colon X\times Z \longrightarrow Y$ with the map $z \mapsto (x \mapsto F(x,z))$ is a homeomorphism.

One consequence of the above fact is that the suspension operation Σ is dual to the loop space operation $\Omega = map_*(\mathbb{S}^1, \bullet)$ in the following sense,

Proposition 133. *Let Z and Y be based spaces. Then there is a homeomorphism,*

$$map_*(\Sigma Z, Y) \cong map_*(Z, \Omega Y).$$

Proof. it follows from the based exponential law, lemma 124.1, by setting $X = \mathbb{S}^1$. Then, $\mathbb{S}^1 \wedge Z \cong \Sigma Z$ and $map_*(\mathbb{S}^1, Y)$ is by definition ΩY. □

I can add two results about the interaction of pullbacks and pushouts. They are both analogous to the familiar exponential laws from albegra; the first is the analog of $(a^b)^c = a^{bc}$, and the second the analog of $a^{b+c} = a^b a^c$. I have natural homeomorphisms,

$$map(Z, lim(X \longleftarrow W \longrightarrow Y)) \xrightarrow{\cong} lim(map(Z,X) \longrightarrow map(Z,W) \longleftarrow map(Z,Y))$$
$$map(colim(X \longleftarrow W \longrightarrow Y), Z) \xrightarrow{\cong} lim(map(X,Z) \longrightarrow map(W,Z) \longleftarrow map(Y,Z))$$

Mapping Space in sSet.

.

In the category of simplicial sets, $map_{sSet}(X,Y)$ is the simplicial set that at level n has the set $hom_{sSet}(X' \times \Delta^n, Y')$, where X' and Y' are fibrant + cofibrant replacements for X and Y. This agrees with the initial intuition that maps $X \times \Delta^n \longrightarrow Y$ should record "n-th order homotopies" between maps $X \longrightarrow Y$.

In view of the introductory part about simplicial sets in 3 and also in order to remind how simplicial sets works I can add a constructive process to obtain the mapping space as a simplicial set from X to Y.

Let X and Y be simplicial sets. I write $hom_{\mathcal{M}}(X,Y)$ for the set of simplicial maps from X to Y.

Let $\sigma_n = (0, 1, \cdots, n) \in \Delta[n]$ be the nondegenerate element.

Recall that a element $x \in X_n$ is called nondegenerate if $x \neq s_i(y)$ for any $y \in X_{n-1}$

$$d^i = f_{d_i \circ \sigma_n} \colon \Delta[n-1] \longrightarrow \Delta[n]$$
$$s^i = f_{s_i \circ \sigma_n} \colon \Delta[n+1] \longrightarrow \Delta[n]$$

be the representing maps of $d_i \circ \sigma_n$ and $s_i \circ \sigma_n$ for $0 \le i \le n$.

I observe that the composites,

$$\Delta[n-2] \xrightarrow{d^j} \Delta[n-1] \xrightarrow{d^j} \Delta[n]$$

$$\Delta[n+2] \xrightarrow{s^j} \Delta[n+1] \xrightarrow{s^i} \Delta[n]$$
$$\Delta[n] \xrightarrow{d^j} \Delta[n+1] \xrightarrow{s^i} \Delta[n]$$

are representation maps of the elements $d_j \circ d_i \circ \sigma_n$, $s_j \circ s_i \circ \sigma_n$ and $d_j \circ s_i \circ \sigma_n$ respectively.

From the simplicicial identities in (30.1) (also at introductory part about simplicial sets in 3), the sequences of simplicial sets $\{\Delta[n]\}_{n\geq 0}$ with d^i and s^i is a cosimplicial simplicial set, namely the identities,

$$\begin{aligned} d^j \circ d^i &= d^{i+1} \circ d^j \qquad for\ i \geq j \\ s^i \circ s^j &= s^j \circ s^{i+1} \qquad for\ i \geq j \end{aligned}$$

$$s^i \circ d^j \;=\; \begin{cases} d^j \circ s^{i-1} & if\ j < i \\ \\ Id & if\ j = i, i+1 \\ \\ d^{j-1} \circ s^i & if\ j > i+1 \end{cases} \tag{133.1}$$

Let $map(X,Y)_n = hom_S(X \times \Delta[n], Y)$ and let,

$$\begin{aligned} d_i \;&=\; (Id_X \times d^i)^* \colon map(X,Y)_n \;=\; hom_S(X \times \Delta[n], Y) \\ &\qquad\qquad\qquad\qquad\qquad \longrightarrow hom_S(X \times \Delta[n-1], Y) \\ &\qquad\qquad\qquad\qquad\qquad \;=\; map(X,Y)_{n-1} \\ s_i \;&=\; (Id_X \times s^i)^* \colon map(X,Y)_n \;=\; hom_S(X \times \Delta[n], Y) \\ &\qquad\qquad\qquad\qquad\qquad \longrightarrow hom_S(X \times \Delta[n+1], Y) \\ &\qquad\qquad\qquad\qquad\qquad \;=\; map(X,Y)_{n+1} \end{aligned}$$

for $0 \leq i \leq n$.

From identities in (133.1), $map(X,Y) = \{\ map(X,Y)_n\ \}_{n\geq 0}$ with d_i and s_i is a simplicial set, which is called the mapping space from X to Y.

Now, I can scale and generalize into a more general frame of examples.

Simplicial Mapping Spaces.

.

The most usual categories in fact are simplicial model categories.

I will refer to the simplicial mapping sepace between two objects as the function complex between those objects.

More precisely,

Definition 134.

(1) *If X and Y are objects in the categry SS of simplicial sets, then $map(X,Y)$ is the simplicial set winth n-simplices the simplicial maps $X \times \Delta[n] \longrightarrow Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.*

(2) *If X and Y are objects of SS_*, the category of pointed simplicial sets, then $map(X,Y)$ is the simmplicial set with n-simplices the base point preserving simplicial maps $X \wedge \Delta[n]^+ \longrightarrow Y$ and face and degeneracy maps induced by the standard maps between the $\Delta[n]$.*

(3) *If X and Y are objects of Top the category of topological spaces, then $map(X,Y)$ is the simplicial set with n-simplices the constinuous functions $X \times |\Delta[n]| \longrightarrow Y$ and face and degeneacy maps induced by the standard maps between the $\Delta[n]$.*

(4) *If X and Y are objects of Top_* the category of pointed topological spaces, then $map(X,Y)$ is the simplicial set with n-simplices the continuous functions $X \wedge |\Delta[n]|^+ \longrightarrow Y$ and face and degeneracy maps induced by the satandard maps between the $\Delta[n]$.*

I can observe that, in all cases, $map(X,Y)$ is an unpointed simplicial set.

Part 4. A-Cellular Categories

Part 4.

A-Cellular Classes

.

In this last part I am interested in the study of collections of objects called cellular classes. In particular I am interested in cellular classes generated by a given object A, whose elements are called A-cellular objects.

The initial framework where cellular classes were developed was topological spaces and spectra. In fact, cellular classes of topological spaces have been studied over several years, but it is only in the last years that the study of cellular classes has been started in other settings.

The basic way to understand a topological space is by comparing it to more familiar spaces, such as the spheres. By studying maps from spheres I obtain the homotopy groups, the most fundamental of topological invariants.

I want, however, to develop a similar study in the general framework of pointed model categories and using a given object A and its suspensions instead of spheres.

Let $\mathcal{M}$ be a pointed model category, and let A be an object of $\mathcal{M}$.

The intuitive idea is that the A-cellular objects are the ones that can, up to homotopy, be built from A. This definition is precisely the same as the usual notion of cellularity for the category of pointed topological spaces.

In the same way, also the idea of cellular classes is intuitively clear. Suppose that I have a space A. I then study which other spaces can be constructed out of A using only wedges and homotopy push-outs and telescopes. This class is called the class of A-cellular spaces. If X is A-cellular, then I know that any invariant of A that is preserved by wedges and push-outs will also become an invariant of X.

More formally,

Definition 1. *(A-Equivalence)*

Let $\mathcal{M}$ be a pointed model category and let A be an object in $\mathcal{M}$.

A map $f\colon X \longrightarrow Y$ between objects in $\mathcal{M}$ is an A-equivalence if it induces a weak equivalence on pointed mapping spaces,

$$map_*(A,X) \xrightarrow{\sim} map_*(A,Y).$$

I think of A-equivalences as morphisms that from the point of view of A are weak equivalences.

Definition 2. *(A-Cellular Object)*

An object $Z \in \mathcal{M}$ (a pointed model category) is said to be A-cellular if Z is cofibrant and any A-equivalence $f\colon X \longrightarrow Y$ between objects in $\mathcal{M}$ induces a weak equivalence on pointed mapping spaces,

$$map_*(Z,X) \xrightarrow{\sim} map_*(Z,Y).$$

Both definitions lead to a more formal presentation of cellular classes or the class of A-cellular objects.

Theorem 19. *(Characterization of Cellular Classes)*
Let $\mathcal{M}$ be a pointed model category, and let A be a cofibrant object in $\mathcal{M}$.
The class of A-cellular objects which I will denote by $Cell(A)$ is the smallest class of objects of $\mathcal{M}$ such that,

(1) *The object of that collection A is A-cellular.*
That is, every object in A is an element of the class $Cell(A)$.

(2) *If X is weakly equivalent to an A-cellular object, then X is A-cellular as well.*
That is, if $X \in Cell(A)$ and $X' \underset{w}{\simeq} X$, then $X' \in Cell(A)$

(2) *The class is closed under homotopy pushouts and telescopes.*

I have presented the class $Cell(A)$ in the form of a central theorem in this section and my aim is to prove it in detail.
I want to add two main remarks about A-equivalences.

Remark 3. *The class of A-equivalences is closed by composition (whenever it can be defined).*
Indeed, I consider,

$$X \xrightarrow{f} Y \qquad\qquad Y \xrightarrow{g} Z$$

where both f and g are A-equivalences.
Therefore, by Definition 1, they both induce weak equivalences on pointed mapping spaces,

$$\begin{array}{c} f \text{ is an } A\text{-equivalence} \\ \Downarrow \\ map_*(A,X) \xrightarrow{\sim} map_*(A,Y) \\ \Downarrow \\ [A,X] \xrightarrow[\cong]{f_*} [A,Y] \end{array} \qquad\qquad \begin{array}{c} g \text{ is a an } A\text{-equivalence} \\ \Downarrow \\ map_*(A,Y) \xrightarrow{\sim} map_*(A,Z) \\ \Downarrow \\ [A,Y] \xrightarrow[\cong]{g_*} [A,Z] \end{array}$$

Now, composing,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

At the level of homotopy classes I have isomorphisms, then,

$$\begin{tikzcd} {[A,X]} \arrow[r, "f_*", "\cong"'] \arrow[rr, bend right, "(g \circ f)_*"'] & {[A,Y]} \arrow[r, "g_*", "\cong"'] & {[A,Z]} \end{tikzcd}$$

But now, $(g \circ f)_* = g_* \circ f_*$, and composition of isomorphisms is an isomorphism.
Therefore, $g \circ f$ is an A-equivalence.

Moreover,

Remark 4. *The class of A-equivalences has the 2 out of 3 property.*
Indeed, having a composite,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

I have,

$$\begin{array}{c} \text{case } \textbf{(a)} \\ \text{if } g \circ f \text{ is an } A\text{-equivalence} \\ \text{and } f \text{ is an } A\text{-equivalence} \\ \Downarrow \end{array} \qquad\qquad \begin{array}{c} \text{case } \textbf{(b)} \\ \text{if } g \circ f \text{ is an } A\text{-equivalence} \\ \text{and } g \text{ is an } A\text{-equivalence} \\ \Downarrow \end{array}$$

$$\begin{array}{c}
[A,X] \xrightarrow[\cong]{(g\circ f)_*} [A,Z] \\
\textit{and} \;\; [A,X] \xrightarrow[\cong]{f_*} [A,Y] \\
\Downarrow \\
\underbrace{(g\circ f)_*}_{\substack{\uparrow \\ \cong}} = g_* \circ \underset{\substack{\uparrow \\ \cong}}{f_*} \\
\Downarrow \\
g_* \; \textit{isomorphism} \\
\Downarrow \\
g \; A\textit{-equivalence}
\end{array}
\qquad\qquad
\begin{array}{c}
[A,X] \xrightarrow[\cong]{(g\circ f)_*} [A,Z] \\
\textit{and} \;\; [A,Y] \xrightarrow[\cong]{g_*} [A,Z] \\
\Downarrow \\
\underbrace{(g\circ f)_*}_{\substack{\uparrow \\ \cong}} = \underset{\substack{\uparrow \\ \cong}}{g_*} \circ f_* \\
\Downarrow \\
f_* \; \textit{isomorphism} \\
\Downarrow \\
f \; A\textit{-equivalence}
\end{array}$$

Later in this section, I will prove that given a cofibrant object $A \in \mathcal{M}$, there is a functor CW_A which assigns to every object X the universal A-approximation of X in the sense that CW_AX is A-cellular and there is a map $CW_AX \longrightarrow X$ which, from the point of view of A, is a weak equivalence. As the name suggests, I should think of CW_AX as a generalization of the classical construction of a CW approximation using A instead os $\mathbb{S}^0$.

So I can also define,

Definition 5. *(A-Cellular Approximation)*

Let X, $A \in \mathcal{M}$ be objects in a pointed model category.

A pair $(\widetilde{X}, \varphi)$ where $\widetilde{X}$ is a cofibrant object in $\mathcal{M}$ and $\varphi\colon \widetilde{X} \longrightarrow X$ is a morphism is called an A-cellular approximation of X if $\widetilde{X}$ is an A-cellular object in $\mathcal{M}$ and φ is an A-equivalence.

The formal construction of the coapproximation functor CW_A also be a main fact in this section acquiring the form of Theorem 20 that I will later prove in detail.

An A-cellular approximation is a universal construction (as I will later prove as well), that is, it is terminal among maps from A-cellular objects and initial among A-equivalences.

In order to achieve all those objectives I will need some previous facts, most of them purely categorical.

As a first and almost immediate result I observe that being an A-equivalence automatically translates to being a Z-equivalence for all A-cellular objects Z.

Lemma 6. *Let $\mathcal{M}$ be a pointed model category and let $A, X, Y, Z \in \mathcal{M}$.*

If Z is A-cellular and $f\colon X \longrightarrow Y$ is an A-equivalence then $f\colon X \longrightarrow Y$ is a Z-equivalence.

Proof. As an assumption $f : X \longrightarrow Y$ is an A-equivalence so by definition it induces a weak equivalence on pointed mapping spaces,

$$map_*(A,X) \xrightarrow{\sim} map_*(A,Y).$$

But moreover Z is A-cellular and then any A-equivalence (in particular $f\colon X \longrightarrow Y$) induces a weak equivalence on pointed mapping spaces,

$$map_*(Z,X) \xrightarrow{\sim} map_*(Z,Y)$$

But this is precisely the definition for f being a Z-equivalence.

□

Assuming given the construction for an A-cellular approximation for a given object $X \in \mathcal{M}$ (that I will later complete), a relevant fact about an A-approximation is the following,

Lemma 7. *Suppose that $(\widetilde{X}, \varphi)$ is an A-cellular approximation of X and that $\phi\colon Y \longrightarrow X$ is a map from a cofibrant object Y to X.*

If Y is A-cellular then there is a map $\psi\colon Y \longrightarrow \widetilde{X}$ making the following diagram commute up to homotopy:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\varphi"] \\ Y \arrow[ur, "\psi"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

If ϕ is an A-equivalence then there is a map $\psi\colon \widetilde{X} \longrightarrow Y$ making the following diagrama commute up to homotopy:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\varphi"] \arrow[dl, "\psi"'] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

Proof. I suppose first that Y is A-cellular. By Lemma 6, φ is a Y-equivalence so it induces a weak equivalence on pointed mapping spaces,

$$map_*(Y, \widetilde{X}) \xrightarrow{\sim} map_*(Y, X).$$

In particular there exists an isomorphism from the set of homotopy classes of maps from Y to $\widetilde{X}$ to the set of homotopy classes of maps from Y to X, hence I can find the desired ψ.

If ϕ is an A-equivalence then it induces a weak equivalence on pointed mapping spaces,

$$map_*(A, Y) \xrightarrow{\sim} map_*(A, X).$$

In particular there exist an isomorphism from the set of homotopy classes of maps from $\widetilde{X}$ to Y to the set of homotopy classes of maps from $\widetilde{X}$ to X, hence I can find the desired ψ. □

A consequence of this lemma is that A-cellular approximations are unique up to homotopy.

A more relevant fact is that I will need a characterization which allows me to effectively present and construct A-cellular objects.

Characterization 8. *(Characterization for A-Cellular Objects)*

Let $\mathcal{M}$ be a pointed model category.

Given objects A and X in $\mathcal{M}$, X is called A-cellular if and only if it can be built from A by iterating telescopes and homotopy pushouts.

Given this characterization, if X is A-cellular I will also use the notation $A \ll X$ (sometimes called celllular inequality) and I will also say, in that case, that A constructs X.

As I said before, the concept of cellular class of objects generalizes the construction of CW-complexes using spheres as pieces (in that case $A = \mathbb{S}^0$).

It is intended to extract information of the target space X using as input the homotopy structure of the building space A and taking into account the structure of the diagram (or diagrams) which build X from A.

In order to justify the characterization I will use some general categorical operations on morphisms and general facts already introduced.

Review of Categorical Operations on Morphisms and General Facts.

.

(1)

In general, I can think of the pushout of two maps,

$$f\colon A \longrightarrow B \qquad \text{and} \qquad g\colon A \longrightarrow C$$

That is,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

in *Set* as computing the disjoint union of B and C with an identification $f(a) = g(a)$ for each element a of A.

I could imagine forming this as either the quotient by an equivalence relation, or by gluing in a segment joining $f(a)$ to $g(a)$ for each a and taking π_0 of the resulting space. If two elements a,a' of A satisfy $f(a) = f(a')$ and $g(a) = g(a')$, the pushout is unaffected by removing a' from A.

Similarly, the homotopy pushout is formed by gluing in a segment joining $f(a)$ to $g(a)$ for each a and not forgetting the number of ways in which two elements of $B \amalg C$ are identified; instead I take the entire space as the result. It can be understood as the derived version of the pushout.

In general I can think of the homotopy pushout of

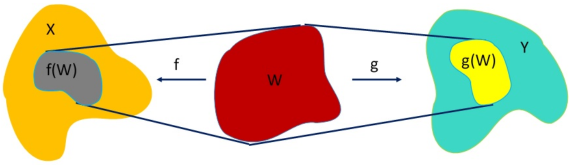
$$f\colon A \longrightarrow B \quad \text{and} \quad g\colon A \longrightarrow C$$

Equivalently,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

as the free thing generated by B and C with relations coming from A. But it is importat that the relations are imposed exactly once, since in the homotopical/derived setting one keeps track of such things (and have relations between relations, etc.)

A picture of a homotopy pushout is given by the following:



In fact I have already showed a picture like this when I described homotopy colimits, because a definition of homotopy pushout can be given in this sense.

Definition 9. *(Homotopy Pushout)*

The homotopy pushout, or homotopy colimit, of a diagram $D = (X \xleftarrow{f} W \xrightarrow{g} Y)$, *denoted by* hocolim D *or* hocolim$(X \xleftarrow{f} W \xrightarrow{g} Y)$, *is the quotient space of* $X \amalg (W \times I) \amalg Y$ *under the equivalence relation generated by* $f(w) \sim (w, 0)$ *and* $g(w) \sim (w, 1)$ *for* $w \in W$. *If* W *is a based space with basepoint* w_0, *I add the relation* $(w_0, t) \sim (w_0, s)$ *for all* s , $t \in I$.

The terminology homotopy colimit and the notation hocolim is used because the homotopy pushout is an example of a homotopy colimit.

Equivalently I can consider a notion of homotopy pullback.

Definition 10. *(Homotopy Pullback)*

In a model category $\mathcal{M}$, *a (commuting) square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback square if for the functorial factorizations

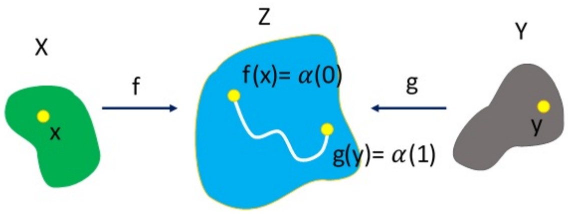
$$\begin{tikzcd} B \arrow[r, tail, "\sim"] & X \arrow[r, two heads] & D \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[r, tail, "\sim"] & Y \arrow[r, two heads] & D \end{tikzcd}$$

(into trivial cofibrations followed by fibrations) the canonical map $A \longrightarrow X \times_D Y$ *is a weak equivalence. The homotopy pullback is the object* $X \times_D Y$*.*

A picture of a homotopy pullback is given by the following:



In fact and again, I have already showed a picture like this when I described homotopy limits, because an alternatively definition of pullback can be given in this sense.

The homotopy pullback, or homotopy limit, or homotopy fiber product of a diagram $D = (X \xrightarrow{f} Z \xleftarrow{g} Y)$ also denoted by *holim* D or $holim(X \xrightarrow{f} Z \xleftarrow{g} Y)$ is the subspace of $X \times map(I, Z) \times Y$ consisting of $\{\ (x, \alpha, y) \colon \alpha(0) = f(x)\ ,\ \alpha(1) = g(y)\ \}$.

As previously, the terminology homotopy limit and the notation holim is used because the homotopy pullback is an example of a homotopy limit.

The good thing about homotopy pullbacks in (right) proper model categories is that they are homotopy invariant, and if the square in the definition were already a (categorical) pullback diagram and $B \twoheadrightarrow D$ a fibration, then it would be homotopy cartesian.

(2)

Also in this section, I will use a main familiar example: in a derived category, the mapping cone of a morphism

$$f \colon A \longrightarrow B$$

is the homotopy pushout of f and the zero map $A \longrightarrow *$.

This certainly depends on A, even when B is the zero object: it is the suspension of A.

So, in a model category, given an object A, I can construct the suspension of A, denoted by ΣA, in the following way,

I consider the factorization of the mapping $A \xrightarrow{t} *$,

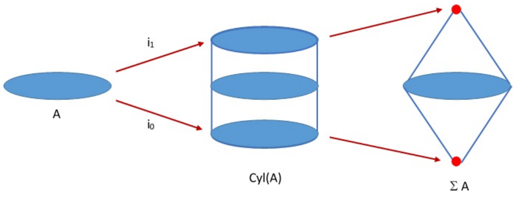
$$\begin{tikzcd} A \arrow[rr, "t"] \arrow[dr, tail, "i"'] & & * \\ & Cone(A) \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

Then, ΣA, the suspension of A, is the pushout

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "t"'] & Cone(A) \arrow[d] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

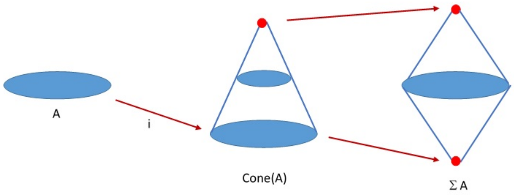
Recall that I both have the diagram,

$$\begin{tikzcd} A \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_0"'] & Cyl\ (A) \arrow[r] & \Sigma\ A \end{tikzcd}$$



and the diagram,

$$\begin{tikzcd} A \arrow[r, tail, "i"] & Cone\ (A) \arrow[r] & \Sigma\ A \end{tikzcd}$$



(3)

For $\mathcal{C}$ a category and $X, Y \in Obj(\mathcal{C})$ two objects, their coproduct is an object $X \amalg Y$ in $\mathcal{C}$ equipped with two morphisms

$$\begin{tikzcd} X \arrow[dr, "i_X"'] & & Y \arrow[dl, "i_Y"] \\ & X \amalg Y & \end{tikzcd}$$

such that this is universal with this property, meaning that for any object Z with maps like this

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

there exists a unique morphism

$$(f \amalg g) \colon X \amalg Y \longrightarrow Z$$

such that I have a commuting diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \amalg Y \arrow[d, "f \amalg g"'] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

This morphism $f \amalg g$ is usually called the copairing of f and g. The morphisms $X \xrightarrow{i_X} X \amalg Y$ and $Y \xrightarrow{i_Y} X \amalg Y$ are called coprojections or sometimes injections or inclusions, although in general they may not be monomorphisms.

A coproduct is thus the colimit over the diagram that consists of just two objects.

More generally, for any set S and $F \colon S \longrightarrow \mathcal{C}$ a collection of objects in $\mathcal{C}$ indexed by S, their coproduct is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps,

$$F(s) \longrightarrow \coprod_{s \in S} F(s)$$

such that this is universal among all objects with maps from the $F(s)$.

The notation $\mathcal{C}^X$ for any set X is commonly used to denote infinite coproducts. Here the set X is regarded as a discrete category, so the functor category $\mathcal{C}^X$ has as its objects the X-indexed families $a = \{a_x \mid x \in X\}$ of objects of $\mathcal{C}$. The corresponding diagonal functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}^X$ sends each c to the constant family (all $c_x = c$). The universal arrow from a to F is an X-fold coproduct diagram; it consists of a coproduct object $\amalg_{x\in X} a_x \in \mathcal{C}$ with the prescribed universal property.

This universal property states that the assignment $f \mapsto \{f \circ i_x \mid x \in X\}$ is a bijection:

$$\mathcal{C}(\amalg_{x\in X} a_x, c) \cong \prod_{x\in X} \mathcal{C}(a_x, c)$$

natural in every c.

The coproduct of any two objects exists in many of the familiar categories, where it has a variety of names. For instance: *Set* (disjoint union of sets), *Top* (disjoint union of spaces, Top_* (wedge product or gluing two spaces at the base points), *Ab* or $R-Mod$ (direct sum $A \oplus B$), *Grp* (free product), *CRing* (tensor product $R \otimes S$).

(4)

If the factors in a coproduct are all equal ($a_x = b$ for all x), the coproduct $\coprod_{x\in X} b$ is called a copower and is written $X \cdot b$, so that

$$\mathcal{C}(X \cdot b, c) \cong \mathcal{C}(b, c)^X$$

natural in each c. For instance, in *Set* with $b = Y$ a set, the copower $X \cdot Y = X \times Y$ is the cartesian product of the sets X and Y.

So, for every set A, the copower $A \cdot X$ denotes the coproduct of $|A|$-copies of X, hence it comes with an injection $i_a\colon X \longrightarrow A \cdot X$ for each a in A; if X is a set, $i_a(x)$ is written as $a \cdot x$. As I early said, in *Set*, the expression $a \cdot x$ can also be understood as the pair (a, x) since the copower $A \cdot X$ is isomorphic to the product $A \times X$.

Usually a copower is also called a tensor and the notation $\otimes$ substitutes the notation $\cdot$

Using this notation I can rewrite all the previous facts as follows,

Every locally small category $\mathcal{C}$ with all coproducts is canonically tensored over *Set*: the copowering functor,

$$\otimes\colon Set \times \mathcal{C} \longrightarrow \mathcal{C}$$

sends (S, b) to $|S|$-many copies of $b \in \mathcal{C}$:

$$S \otimes b\colon = \amalg_{s\in S} b\ .$$

The defining natural isomorphism in this situation is then just the fact that the *Hom* functor sends colimits in its first argument to limits:

$$\mathcal{C}(\amalg_{s\in S} b, c) \cong \prod_{s\in S} \mathcal{C}(b, c) \cong Set(S, \mathcal{C}(b, c))\ .$$

(5)

In fact, in many problems, the definition of lifting morphisms relies on the following categorical operation:

In this section, I will consider sequences of composable morphisms,

$$A_0 \xrightarrow{f_{01}} A_1 \xrightarrow{f_{12}} A_2 \xrightarrow{f_{23}} \cdots \longrightarrow A_\nu \xrightarrow{f_\nu} A_{\nu+1} \longrightarrow \cdots$$

which range over a (maybe transfinite) ordinal λ, and which I formally define as functors

$$F\colon \lambda \longrightarrow \mathcal{M}$$

such that,

(i) $\underset{\nu<\mu}{colim}\ A_\nu = A_\mu$ for every limit ordinal $\mu < \lambda$.

(ii) $f_{ij}\colon A_i \longrightarrow A_j$ is a morphism (a cofibration if homotopy colimits) for all $i < j$.

The composite of such a sequence is then the morphism $f\colon A_0 \longrightarrow \underset{\nu<\lambda}{colim}\, A_\nu$ by taking the colimit of that diagram.

I can note that I retrieve the plain composition operation in the category when I apply this construction to an ordinal which is finite.

Main Theorems.

Characterization for A-Cellular Objects.

With all these constructions I want to justify the claim that I early wrote, my characterization of A-cellular objects in Characterization 8: given pointed objects A and X, X is called A-cellular if and only if it can be built from A by iterating telescopes and homotopy pushouts.

I consider a transfinite sequence of maps, that is the mapping telescope,

$$A_0 \xrightarrow{f_{01}} A_1 \xrightarrow{f_{12}} A_2 \xrightarrow{f_{23}} A_3 \xrightarrow{f_{34}} \cdots$$

in a cofibrantly generated simplicial model category.

I replace now the generating maps by a sequence of cofibrations between cofibrant objects.

I know that in the general framework of model categories, each map $f_{i(i+1)}$ may be replaced by its mapping cylinder. Gluing these together, I precisely obtain the mapping telescope.

Let λ denote the ordinal category,

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

and I write $F\colon \lambda \longrightarrow \mathcal{M}$ for the previous sequence of spaces.

I will construct a projective cofibrant replacement of F by an inductive process.

Recall that I will use the operation that I denote by $\cdot$. The $\cdot$ is called a copower or a tensor. In particular, if S is a set and $b \in B$ then $S \ \cdot \ b$ is the S-indexed coproduct of copies of b.

For the initial step, I take a cofibrant replacement

$$q_0\colon Q_0 \xrightarrow{\sim} A_0$$

of A_0 and so I construct the functor $G^0\colon = \lambda(0, \bullet) \cdot Q_0$.

This functor is projectively cofibrant.

Now, by construction, there is a natural transformation

$$\begin{tikzcd}
G^0 \arrow[dd, equal] & Q_0 \arrow[r, equal] \arrow[dd, "q_0"', "\sim"] & Q_0 \arrow[r, equal] \arrow[dd, dotted] & Q_0 \arrow[r, equal] \arrow[dd, dotted] & Q_0 \arrow[r, equal] \arrow[dd, dotted] & \cdots \\
 & & & & & \\
F & A_0 \arrow[r, "f_{01}"] & A_1 \arrow[r, "f_{12}"] & A_2 \arrow[r, "f_{23}"] & A_3 \arrow[r, "f_{34}"] & \cdots
\end{tikzcd}$$

in which the dotted arrows are defined to be the composites of the arrows to their left. I can note that $\lambda(0, \bullet) \cdot Q_0$ is not yet a cofibrant replacement of F because only its initial component is a weak equivalence.

For the next step, I use the factorization in $\mathcal{M}$ to form a cofibrant replacement of the map $f_{01} \circ q_0$

$$\begin{tikzcd}
Q_0 \arrow[r, tail] \arrow[d, "q_0"', "\sim"] & Q_1 \arrow[d, "\sim"', "q_1"] \\
A_0 \arrow[r, "f_{01}"] & A_1
\end{tikzcd}$$

I then obtain a pushout in $\mathcal{M}^\lambda$, the functor category.

$$\begin{tikzcd}
\lambda(1, \bullet) \cdot Q_0 \arrow[r] \arrow[d, tail, "g_{01}"'] & G^0 = \lambda(0, \bullet) \cdot Q_0 \arrow[d, tail] \\
\lambda(1, \bullet) \cdot Q_1 \arrow[r] & G^1
\end{tikzcd}$$

Here the top horizontal attaching map is adjunct to the identity at Q_0. The vertical maps are projective cofibrations.

The universal property of the pushout defining the functor G^1 leads to a natural transformation

$$\begin{tikzcd}
G^1 \arrow[dd, Rightarrow] & Q_0 \arrow[r, tail, "g_{01}"] \arrow[dd, "q_0"', "\sim"] & Q_1 \arrow[r, equal] \arrow[dd, "\sim"', "q_1"] & Q_1 \arrow[r, equal] \arrow[dd, dashed] & Q_1 \arrow[r, equal] \arrow[dd, dashed] & \cdots \\
& & & & & \\
F & A_0 \arrow[r, "f_{01}"] & A_1 \arrow[r, "f_{12}"] & A_2 \arrow[r, "f_{23}"] & A_3 \arrow[r, "f_{34}"] & \cdots
\end{tikzcd}$$

in which the first two components are weak equivalences.

I continue iteratively, and at step n, I define a functor G^n in an analogous way:

$$\begin{tikzcd}
Q_{n-1} \arrow[r, tail, "g_{(n-1)n}"] \arrow[d, "q_{n-1}"', "\sim"] & Q_n \arrow[d, "\sim"', "q_n"] \\
A_{n-1} \arrow[r, "f_{(n-1)n}"] & A_n
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda(n,\bullet)\cdot Q_{n-1} \arrow[r] \arrow[d, tail, "g_{(n-1)n}"'] & G^{n-1} = \lambda(n-1,\bullet)\cdot Q_{n-1} \arrow[d, tail] \\
\lambda(n,\bullet)\cdot Q_n \arrow[r] & G^n
\end{tikzcd}$$

The transfinite composite,

$$\emptyset \rightarrowtail G^0 \rightarrowtail G^1 \rightarrowtail G^2 \rightarrowtail \cdots \rightarrowtail colim_n G^n =: G$$

is a projectively cofibrant functor with a natural weak equivalence to F.

$$\begin{tikzcd}
G \arrow[d, Rightarrow, "q"', "\sim"] & Q_0 \arrow[r, tail, "g_{01}"] \arrow[d, "q_0"', "\sim"] & Q_1 \arrow[r, tail, "g_{12}"] \arrow[d, "q_1"', "\sim"] & Q_2 \arrow[r, tail, "g_{23}"] \arrow[d, "q_2"', "\sim"] & Q_3 \arrow[r, tail, "g_{34}"] \arrow[d, "q_3"', "\sim"] & \cdots \\
F & A_0 \arrow[r, "f_{01}"] & A_1 \arrow[r, "f_{12}"] & A_2 \arrow[r, "f_{23}"] & A_3 \arrow[r, "f_{34}"] & \cdots
\end{tikzcd}$$

As I would want to end constructing.

A particularly interesting example of an A-cellular object is the suspension ΣA.

Example 11. *Let A be an A-cellular object in $\mathcal{M}$.*

The suspension ΣA is A-cellular.

Indeed, I consider the diagram,

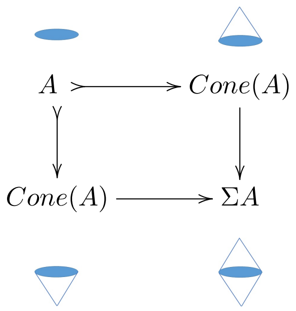
$$\begin{tikzcd}
A \arrow[r] \arrow[d] & * \\
* &
\end{tikzcd}$$

However, this diagram is not cofibrant.

Using the object $Cone(A)$, I know that I can easily turn it into a diagram which is now weakly equivalent to the previous one, but now cofibrant.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & Cone(A) \\
Cone(A) &
\end{tikzcd}$$

The supension ΣA is obtained now as the pushout of the previous diagram,



$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, tail] & Cone(A) \arrow[d] \\
Cone(A) \arrow[r] & \Sigma A
\end{tikzcd}$$

Now simply by definition ΣA is A-cellular since A and $Cone(A)$ are both A-cellular.

Moreover, as a consequence of an induction argument, $\Sigma^n A$ is A-cellular , forall n.

I observe that the use of the suspension ΣA of any object A provides me with a useful tool to construct new A-cellular spaces.

Indeed, the telescope

$$A_0 \xrightarrow{f_{01}} A_1 \xrightarrow{f_{12}} A_2 \xrightarrow{f_{23}} A_3 \xrightarrow{f_{34}} \cdots$$

will be obtained by considering in the first step, $A_0 = A$ and using the construction of the suspension ΣA as a pushout to obtain $A_1 = \Sigma A$ and iteratively $A_n = \Sigma^n A$.

Observe than I can obtain an alternative definition of A-equivalence to the one given in Definition 1.

Definition 12. *(A-Equivalence (alternative definition))*
Let $\mathcal{M}$ be a pointed model category and let A be an object in $\mathcal{M}$.
A morphism $X \xrightarrow{f} Y$ in $\mathcal{M}$ is an A-equivalence if

$$[\Sigma^k A, X] \xrightarrow{f_*} [\Sigma^k A, Y]$$

is an isomorphism for $k \geq 0$.

I observe that $f_*(\alpha) = f \circ \alpha$ for $\Sigma^k A \xrightarrow{\alpha} X \xrightarrow{f} Y$.

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& \Sigma^k A \arrow[ul, dashed, "\exists\,!"] \arrow[ur, dashed] &
\end{tikzcd}$$

In particular, for pointed topological spaces,

Example 13. $\mathcal{M} = Top_*$ *and* $A = \mathbb{S}^0$

$$[\Sigma^k A, X] = \begin{cases} \pi_k(X, x_0) & \text{if } k \geq 1 \\ \pi_0(X) & \text{if } k = 0 \end{cases}$$

The A-equivalences in this case are the classical weak homotopy equivalences.

Now, given an object $A \in \mathcal{M}$, the A-cellularization or A-cellular approximation (or also A-cellular cover) is a canonical way of turning every object $X \in \mathcal{M}$ into an A-cellular object which is identical to X from the point of view of A-equivalences.

I can introduce something else about A-cellularization.

Given an object $A \in \mathcal{M}$, in the notion of A-homotopy I have already introduced how A and its suspensions play the same role as the spheres in classical homotopy of topological spaces. I have also introduced that the A-homotopy groups of an object X are defined to be homotopy classes of pointed maps from $\Sigma^n A$ to X. The idea of CW-complex space (in the model category of topological spaces) is replaced by the one of A-cellular object, equivalently, an object built from A by means of pointed homotopy colimits. The concept of cellular approximation of X is replaced by A-cellular approximation, that is, an A-cellular object $CW_A X$ called the A-cellularization of X, and a natural map from $CW_A X$ to X inducing an equivalence $map_*(A, CW_A X) \simeq map_*(A, X)$.

In other words, there exists an A-cellularization functor CW_A that provides the best possible A-cellular approximation, for any object $X \in \mathcal{M}$, that is, the natural map $CW_AX \longrightarrow X$ induces a weak equivalence between pointed mapping spaces $map_*(A, CW_AX) \simeq map_*(A, X)$.

More formally,

Theorem of CW_A Approximation.

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Theorem 20. *(Theorem of CW_A Approximation)*

Let $\mathcal{M}$ be a pointed model category which is right proper and admitting Quillen's small object argument and let A be a cofibrant object in $\mathcal{M}$.

There exists an augmented endofunctor CW_A in $\mathcal{M}$ and whose image lies in the category of pointed cellular objects, called A-cellularization, such that for every object X the augmentation $CW_AX \longrightarrow X$ induces a weak homotopy equivalence $map_*(A, CW_AX) \simeq map_*(A, X)$.

The A-cellulariation functor is a universal construction, that is, it is terminal among maps from A-cellular objects and initial among all maps $Y \longrightarrow X$ which induce an A-equivalence.

This functor only makes sense in the pointed context.

Recall that a functor $\phi\colon T_1 \longrightarrow T_2$ is called augmented if there is a natural transformation of functors $\varrho\colon \phi \longrightarrow Id$.

Moreover it is called idempotent if $\phi(\phi(X))$ is weakly equivalent to $\phi(X)$ via the augmentation ϱ.

An endofunctor is a functor that maps a category to itself.

A simplicial set is contractible if the unique map to a point is a weak homotopy equivalence.

A functor $F\colon T_1 \longrightarrow T_2$ is homotopy final if the simplicial set $N(t/F)$ is contractible for all $t \in T_2$ and homotopy initial if each $N(F/t)$ is contractible.

The A-cellularization of an object X in a pointed model category must be then interpreted as the closest object analogous to X that can be built of copies of A. I am using the closedness term here meaning in particular that

$$[\Sigma^n A, CW_AX]_* \cong [\Sigma^n A, X]_*$$

for every $n \geq 0$, where ΣA stands for the suspension of A.

In the case $A = \mathbb{S}^0$ this amounts to say that CW-approximation preserves the homotopy groups of the space.

In topology I can construct CW-approximations and in algebra cofibrant replacements (or projective resolutions). These are functorial approximations of a given object by something that can be constructed out of spheres and disks. I will show how a similar approximation can be obtained using other objects than spheres. It turns out that the A-cellular objects are precisely the objects that can be reconstructed, up to weak equivalence, out of A.

I want to construct under suitable conditions for objects A, $X \in \mathcal{M}$ in a pointed model category, an A-cellular approximation of X, that is, an A-cellular object, denoted by $CW_A\ X$, also called the A-cellularization of X, and a natural map $r_X : CW_A(X) \longrightarrow X$ inducing an A-equivalence.

The general idea is to first consider the map $\emptyset \longrightarrow X$ and to obtain a factorization

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr] & & X \\
& CW_A\ X \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

where the object CW_AX is a cofibrant object constructed from A by homotopy pushouts and telescopes and moreover the map $CW_AX \overset{\sim}{\twoheadrightarrow} X$ becomes an A-equivalence.

In this construction I directly use Quillen's small object argument: under suitable conditions, given objects A, $X \in \mathcal{M}$, I factor the map $\emptyset \longrightarrow X$ into an A-cellular map followed by an $\mathcal{I}$-injective map. Following that analogous development to Quillen's small object argument,I will produce a pair $(CW_A X, r_X)$ such that, when A is cofibrant, this pair determines an A-cellular approximation of X.

Indeed, let γ denote some limit ordinal such that the cofinality of γ is bigger than the cardinality of the underlying set of A and so the underlying set of $\amalg A_i$. I also let γ denote the category of all ordinal numbers smaller than γ with a unique map $i \longrightarrow j$ if $i \leq j$. This assumption on γ means that given any functor indexed by γ, $F\colon \gamma \longrightarrow \mathcal{M}$ and a map $A \xrightarrow{h} \underset{\gamma}{colim} F$, then there is an $i < \gamma$ and a factorization of the map h into $A \longrightarrow F_i \longrightarrow \underset{\gamma}{colim} F$.

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr] & & \underset{\gamma}{colim} F \\
& F_i \arrow[ur] &
\end{tikzcd}$$

So, let $F\colon \gamma \longrightarrow \mathcal{M}$ denote the functor defined inductively by:

(1) For $i = 0$, I let:

$$F_0 = \coprod_{\substack{hom(\sum^k A, X)\\ 0 \leq k < \infty}} \sum\nolimits^k A$$

There is an induced map $p_0\colon F_0 \longrightarrow X$.

It is clear that F_0 is A-cellular.

(2) If $i = j + 1$ then let I_{j+1} be the set of all commutative squares:

$$I_{j+1} = \left\{ \begin{tikzcd}
\Sigma^k A \arrow[r] \arrow[d] & F_j \arrow[d] \\
C\Sigma^k A \arrow[r] & X
\end{tikzcd} \right\}$$

Now, F_{j+1} is defined as the push-out of:

$$\coprod_{I_{j+1}} C\Sigma^k A \longleftarrow \coprod_{I_{j+1}} \Sigma^k A \longrightarrow F_j$$

that is,

$$\begin{tikzcd}
\coprod_{I_{j+1}} \Sigma^k A \arrow[r] \arrow[d] & F_j \arrow[d] \\
\coprod_{I_{j+1}} C\Sigma^k A \arrow[r] & F_{j+1}
\end{tikzcd}$$

Moreover there are maps $p_{j+1}\colon F_{j+1} \longrightarrow X$ and $q_j\colon F_j \longrightarrow F_{j+1}$ such that $p_j = p_{j+1} \circ q_j$.

Of course by the construction F_{j+1} is A-cellular.

(3) Finally if i is a limit ordinal then $F_i = \underset{j<i}{colim} F_j$.

I let $p_i\colon F_i \longrightarrow X$ be the induced map.

Cellularity is preserved by directed colimits, hence F_i is A-cellular.

Now $CW_A X = \underset{\gamma}{colim} F$.

and let $r_X\colon CW_A X \longrightarrow X$ be the induced map.

If A is cofibrant then inductively all the maps $q_j\colon F_j \longrightarrow F_{j+1}$ are cofibrations and all F_i are A-cellular. Hence $CW_A X$ is also A-cellular.

From the construction $CW_A X$ and r_X are natural in $X \in \mathcal{M}$. Then in fact I have constructed a functor $CW_A\colon \mathcal{M} \longrightarrow \mathcal{M}$ and a natural transformation $r\colon CW_A \longrightarrow Id$.

The value of this functor depends on the choice of the ordinal γ. However its homotopy type does not.

I claim that if A is cofibrant then the pair (CW_AX, r_X) constructed above is an A-cellular approximation of X.

Indeed, by the construction $CW_AX = \underset{\gamma}{colim}\ F_i$ and each F_i is A-cellular, so CW_AX is A-cellular.

In order to see that r_X is an A-equivalence, I note that it is enough to find a lift in diagramas of the form:

$$\begin{tikzcd}
\Sigma^k A \arrow[r] \arrow[d] & CW_AX \arrow[d] \\
C\Sigma^k A \arrow[r] & X
\end{tikzcd}$$

I assumed that A is γ-small, so there is some j such that the map $\Sigma^k A \longrightarrow CW_AX$ factors as

$$\Sigma^k A \longrightarrow F_j \longrightarrow F_{j+1} \longrightarrow CW_AX\ .$$

The definition of F_{j+1} implies that there is a map $h\colon C\Sigma^k A \longrightarrow F_{j+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma^k A \arrow[r] \arrow[d] & F_{j+1} \arrow[d] \\
C\Sigma^k A \arrow[r] \arrow[ur, "h"] & X
\end{tikzcd}$$

The composition of h with the natural map $F_j \longrightarrow X$ yields the required lift.

A Remark and an Example.

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Remark 14. *I observe that if for any object $X \in \mathcal{M}$, I factor the map $\emptyset \longrightarrow X$ into an A-cellular map followed by an $\mathcal{I}$-injective map, then for a non-fibrant X I will not be able to show that the $\mathcal{I}$-injective map is an A-equivalence.*

Indeed, in this sense I will consider Example 20 but first I recall some generalities about homotopy theory an in particular about the category of simplicial sets, topological spaces, and fibrations and fibre bundles.

First of all, generally speaking, if U and V are objects in the category of simplicial sets $sSet_*$ and K is a simplicial set, then $U \otimes K$ and V^k will denote the objects of $sSet_*$ defined by the simplicial model category structure on $sSet_*$ which are characterized by the natural isomorphisms of sets,

$$sSet_*(U \otimes K, V) \cong sSet(K, map(U, V)) \cong sSet_*(U, V^K).$$

Thus,

$$U \otimes K = U \wedge K^+$$

and

$$U^K = map_*(K^+, U).$$

Lemma 15. *Let K be a simplicial set (either pointed or unpointed).*

If L is a simplicial set (either pointed or unpointed), then there is a natural homeomorphism of (either pointed or unpointed) topological spaces $|L \otimes K| \cong |L| \otimes K$.

Proof. If L is either a pointed or unpointed simplicial set, then there are natural homeomorphisms,

$$|L \otimes K| = |L \wedge K^+| \cong |L| \wedge |K|^+ = |L| \otimes K\ .$$

$\square$

Lemma 16. *If L is a simplicial set and W is a topological space (both either pointed of unpointed), then the standard adjunction of the geometric realization and total singular complex functors extends to a natural isomorphism of simplicial mapping spaces*

$$map(|L|, W) \cong map(L, Sing\ W)\ .$$

Proof. I know that by Lemma 15 there exists a natural homeomorphism,

$$|L \otimes \Delta[n]| \cong |L| \otimes |\Delta[n]| \ .$$

Now,

$$|L \otimes \Delta[n]| \cong |L| \otimes |\Delta[n]| = |L| \wedge |\Delta[n]|^{+}$$

and $|L|^{|\Delta[n]|} = map_*(|\Delta[n]|^{+}, L)$
And it induces the desired natural isomorphism of simplicial mapping spaces,

$$map(|L|, W) \cong map(L, SingW) \ .$$

Also

$$map_*(|L|, W) \cong map_*(L, SingW) \ .$$

□

Lemma 17. *If N and U are objects of $sSet_*$ and U is fibrant, then there is a natural weak equivalence of simplicial sets*

$$map(N, U) \xrightarrow{\sim} map(|N|, |U|) \ .$$

Proof. Since all simplicial sets are cofibrant, the natural map $U \longrightarrow Sing|U|$ induces a weak equivalence $map(N, U) \xrightarrow{\sim} map(N, Sing|U|)$.
Now by Lemma 16 I obtain the desired weak equivalence,

$$map(N, U) \xrightarrow{\sim} map(|N|, |U|) \ .$$

□

Now, generally speaking, if I wish to study topological spaces, one way of doing it is the following:

I may take a cell decomposition (or using cells, I then construct a new space) and next I try to reduce its topological properties to algebraic or combinatorial relationships between the boundaries of the cells. For instance, I can construct simplicial complexes or apply a homology theory.

A second possibility is directly related with fibrations and can be illustrated by the following algebraic situation:

I may take an exact sequence of groups,

$$0 \longrightarrow F \xrightarrow{i} E \longrightarrow B \longrightarrow 0 \ ,$$

and ask what possible values of E can be taken for given F and B (for instance, $E = F \times B$ is always possible).

It is a useful idea to compare this question with the following general setup:

Let $p\colon E \longrightarrow B$ be any continuous map. The inverse images $p^{-1}(b)$ of points b in B constitute a decomposition of E into fibers $p^{-1}(b)$. I get closer to the algebraic situation described above if all fibers were homeomorphic to each other as it will be the case in the two following examples.

The maps $p\colon E \longrightarrow B$ that I shall be dealing with are the fibrations, without any conditions. Later on, according to their particular (lifting) properties, they will be qualified with a special name, such as trivial fibration, Serre fibration, Hurewicz fibration, locally trivial fibration, Kan fibration, and so on.

- **(a)** The topological product defined as follows. Let B and F be topological spaces and take the projection

  $$p = proj_1\colon E = B \times F \longrightarrow B$$

  This is a fibration, called the trivial fibration or the product fibration.
- **(b)** The covering maps. Recall,

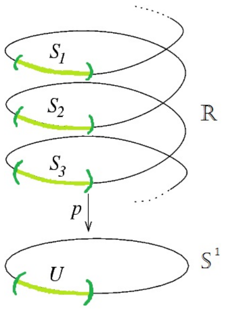
Definition 18. *A map $p\colon X \longrightarrow Y$ is called a covering map (and X is called a covering space of Y) if X and Y are Hausdorff, arcwise connected, and locally arcwise connected, and if each point $y \in Y$ has an arcwise connected neighborhood U such that $p^{-1}(U)$ is a nonempty disjoint union of sets U_α (which are the arc components of $p^{-1}(U)$ on which $p|_{U_\alpha}$ is a homeomorphism $U_\alpha \xrightarrow{\approx} U$. Such sets U will be called elementary, or evenly covered.*

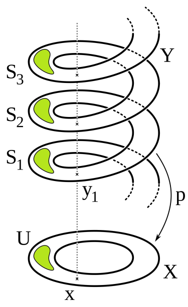
Note that a covering map must be onto, because that is part of homeomorphism. Also, it is not enough for a map to be a local homeomorphism (meaning each point of X has a neighborhood mapping homeomorphically onto a neighborhood of the image point). Consider the map $p\colon (0,2) \longrightarrow \mathbb{S}^1$ defined by $p(t) = (\cos(2\pi t), \sin(2\pi t)))$. That is a local homeomorphism, but for any small neighborhood U of $1 \in \mathbb{S}^1$, some component of $p^{-1}(U)$ does not map onto U.

The number of points in the inverse image of a point, under a covering map, is locally constant, and hence constant since the base space is connected. This number is called the number of sheets of the covering. Covering maps with two sheets are also called double coverings or two fold coverings.

The usual example of covering map is the following,

The map $p\colon \mathbb{R} \longrightarrow \mathbb{S}^1 \subset \mathbb{C}$ taking $t \mapsto \mathfrak{e}^{2\pi i t}$ is a covering map with infinitely many sheets.



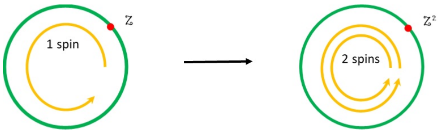


The fibers $p^{-1}(s)$, $\quad s \in \mathbb{S}^1$, are homeomorphic to $\mathbb{Z}$ (as a set with the discrete topology). I also note that $\mathbb{R} \not\approx \mathbb{S}^1 \times \mathbb{Z}$ since $\mathbb{R}$ is connected while $\mathbb{S}^1 \times \mathbb{Z}$ has infinitely many components.

The map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ taking $z \mapsto z^n$ for a fixed positive integer n is a covering with n sheets.

In particular,

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^1$$
$$z \mapsto z^2$$



Recall that a map $p\colon E \longrightarrow B$ is called a fibre bundle if p is onto and if for every map $\bar{b}\colon \Delta[n] \longrightarrow B$, the induced map $p^{\bar{b}}\colon E^{\bar{b}} \longrightarrow \Delta[n]$ is strongly isomorphic to $p^*\colon F \times \Delta[n] \longrightarrow \Delta[n]$, where $p^*(f,k) = k$ and where F is a given complex called the fibre of the bundle.

If F is a Kan complex, then $p\colon E \longrightarrow B$ is called a Kan fibre bundle. (See the concept of Kan complex in 37).

Lemma 19. *Every Kan fibre bundle is a Kan fibration.*

(See the concept of Kan fibration in 38.

Proof. Let $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1}$ be $q+1$ compatible q-simplicies of E and suppose $\partial_i b = x_i$, $i \neq k$, $b \in B_{q+1}$.

Let now $\alpha_b\colon F \times \Delta[q+1] \longrightarrow E^{\bar{b}}$ be a strong isomorphism, say

$$\alpha_b(y_i, \partial_i \Delta_{q+1}) = (x_i, \partial_i \Delta_{q+1}) \ .$$

There exists $y \in F_{q+1}$ such that $\partial_i y = y_i$, $i \neq k$, and if

$$\alpha_b(y_i, \Delta_{q+1}) = (x, \Delta_{q+1})$$

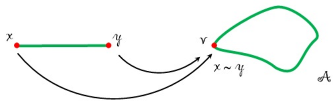
then $p(x) = b$ and $\partial_i x = x_i$, $i \neq k$.

□

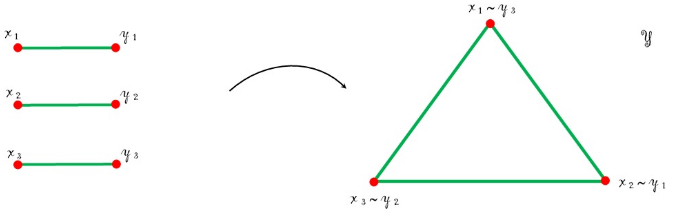
These are main general facts used in the following main example.

Example 20. *Let $\mathcal{M} = sSet_*$ be the category of pointed simplicial sets.*

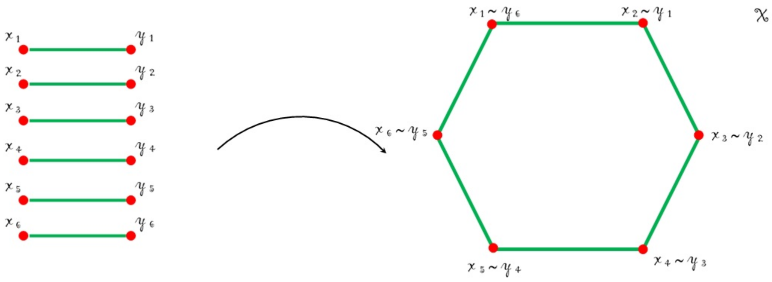
Let A be the quotient of $\Delta[1]$ (1-simplex) obtained by identifying the two vertices of $\Delta[1]$ (denoted by x and y), so that the geometric realization of A is homeomorphic to a loop with a distinguished base point v.



Let Y be $\partial\Delta[2]$, the boundary of a 2-simplex, so that Y consists of three 1-simplices with vertices identified so that its geometric realization is homeomorphic to a circle with a distinguished base point y_0.

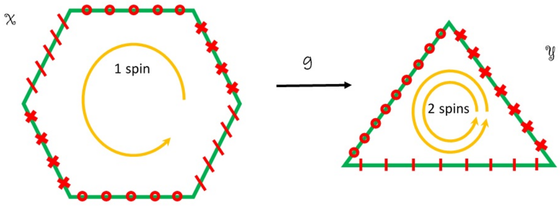


Let X be a simplicial set built from six 1-simplices by identifying vertices so that the geometric realization of X is homeomorphic to a circle with a distinguished base point x_0.



It is clear that all the (nondegenerate) 1-simplices in X have different vertices.

Now, there is a map $g\colon X \longrightarrow Y$ whose geometric realization is the double cover of the circle.



As I early said in the introduction to this example, in Lemma 19 the map g is a fibration since it is a fiber bundle with fiber two discrete points (recall that this concept is equivalent to the concept of a covering space).

Since no nondegenerate 1-simplex of X has its vertices equal, the only pointed map from A to X is the constant map to the base point,

$$A \longrightarrow X$$
$$a \mapsto x_0$$

for all $a \in A$ *.*

Indeed, the identified vertex v *in* A *forms part of a loop but in* X *the only nondegenerate loop is the whole* X *which contains* 6 *different vertices.*

Recall that there is a functor called the geometric realization taking a simplicial set to its corresponding realization in the category of compactly generated Hausdorff topological spaces.

Recall that in general for pointed spaces (C, c_0)*,* (D, d_0)*, the smash product is defined as*

$$C \wedge D = \frac{C \times D}{C \vee D}$$

where $(c, d_0) \sim (c_0, d)$ *and*

$$X \hookrightarrow X \times \{d_0\} \subset C \times D \ .$$
$$Y \hookrightarrow \{c_0\} \times D \subset C \times D \ .$$

That is, inside a product space $C \times D$ *there are copies of* C *and* D*, namely* $C \times \{d_0\}$ *and* $\{c_0\} \times Y$ *for points* $c_0 \in C$ *and* $d_0 \in D$*. These two copies of* C *and* D *in* $C \times D$ *intersect only at the point* (c_0, d_0)*, so their union can be identified with the wedge sum* $C \vee D$*. I can think of* $C \wedge D$ *as a reduced version of* $X \times Y$ *obtained by collapsing away the parts that are not genuinely a product, the separate factors* C *and* D*.*

Moreover, in the category of pointed spaces, the smash product plays the role of a tensor product. In particular I have an adjunction such that for any pointed space (T, t_0)*,*

$$map_{*}hom(C \wedge T, D) \cong map_{*}(C, map_{*}(T, D)) \ .$$

So that applying this adjunction,

$map_{*}(A \wedge \Delta[1]^{+}, X) = map_{*}(A, map_{*}(\Delta[1]^{+}, X)) = map_{*}(\Delta[1]^{+}, map_{*}(A, X)$ *.*

However, $map_{*}(A, X)$ *has a single* n*-simplex for every* n *since the only pointed map* $A \longrightarrow X$ *is constant.*

Hence, the only map from $A \wedge \Delta[1]^{+} \longrightarrow X$ *is the constant map to the base point.*

The same argument, shows that the only map from $A \wedge \Delta[n]^{+} \longrightarrow X$ *is the constant map to the base point.*

(When the target is a single point, there can be only one singular n*-simplex for each* n*, namely, the map sending* Δ^{n} *to the base point).*

Thus, $map_{*}(A, X)$ *has only one simplex in each dimension.*

With an analogous argument, the only map from A *to* Y *is the constant map to the base point and* $map_{*}(A, Y)$ *has only one simplex in each dimension.*

Hence, the induced map $g_{*} \colon map_{*}(A, X) \longrightarrow map_{*}(A, Y)$ *is an isomorphism.*

Hence g *is* $\mathcal{I}$*-injective.*

However, g *is not an* A*-equivalence.*

Indeed, $Sing|g| \colon Sing|X| \longrightarrow Sing|Y|$ *is a fibrant approximation to* g*, and the map*

$$map(A, sing|X|) \longrightarrow map(A, sing|Y|) \text{ is isomorphic to the map}$$

$$map(|A|, |X|) \longrightarrow map(|A|, |Y|).$$

Since the map $|g| \colon |X| \longrightarrow |Y|$ *is homeomorphic to the double covering map of the circle, the induced map*

$$map(|A|, |X|) \longrightarrow map(|A|, |Y|).$$

is not surjective on the set of components, and so g *is not an* A*-equivalence.*

I want to remark that, if X and Y are simplicial sets, I denote by $map(X, Y)$ the simplicial mapping space from X to Y, whose n-simplices are simplicial maps $\Delta[n] \times X \longrightarrow Y$, where $\Delta[n]$ denotes the standard n-simplex.

If the simplicial set Y is fibrant, then the geometric realization $|map(X, Y)|$ has te same weak homotopy type as the topological mapping space $map(|X|, |Y|)$.

Now, if X and Y are simplicial sets with distinguished base points x_0 an y_0 , let $map_{*}(X, Y)$ denote the pointed mapping space, whose n-simplices are simplicial maps $\Delta[n]^{+} \wedge X \longrightarrow Y$, where in fact, $\Delta[n]^{+} \wedge X \longrightarrow X$ is the space obtained by collapsing $\Delta[n]^{+} \wedge \{x_0\}$ inside $\Delta[n]^{+} \wedge X$.

The geometric realization $|map_*(X,Y)|$ has the same weak homotopy type as the space of maps $f\colon |X| \longrightarrow |Y|$ such that $f(x_0) = y_0$, with the compact-open topology, if Y is fibrant. There is a fibration,

$$map_*(X,Y) \longrightarrow map(X,Y) \longrightarrow Y$$

where the second arrow is evaluation at the basepoint, that is, the map induced by the inclusions $\{x_0\} \hookrightarrow X$ and the isomorphism $map(\{x_0\},Y) \cong Y$.

Therefore, the X fibrant condition in the statement is a must. However I can always choose a fibrant approximation $\widetilde{X}$.

The following is the additional process that I need to follow to complete the construction of the functor CW_A if X is a non fibrant object in $\mathcal{M}$.

For any $X \in \mathcal{M}$ I choose a functorial fibrant approximation $j\colon X \rightarrowtail \widetilde{X}$. So that, I have $X \rightarrowtail \widetilde{X} \longrightarrow *$.

Recall that in a model category fibrations are stable under pullback and cofibrations are stable under pushout, but weak equivalences need not to have either property. In a proper model category weak equivalences are also preserved under certain pullbacks and/or certain pushouts.

I apply the generalized Quillen's small object argument to factorize the map $\emptyset \longrightarrow \widetilde{X}$ into a cellular map $r \in cell(\mathcal{I})$ followed by a $\mathcal{I}$-injective map $s \in inj(\mathcal{I})$:

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr, tail, "r\in cell(\mathcal{I})"'] & & \widetilde{X} \\
& \widetilde{W} \arrow[ur, two heads, "s\in inj(\mathcal{I})"'] &
\end{tikzcd}$$

following the process I earlier explained.

Recall that $cell(\mathcal{I})$ is the class obtained by transfinite composition of pushouts of coproducts of elements in $\mathcal{M}$ and that $inj(\mathcal{I})$ is the class of morphisms with the right lifting property with respect to all morphisms in $\mathcal{I}$.

Also, I note that s is a fibration, since $s \in \mathcal{I} - inj$.

Next, I take $W = X \times_{\widetilde{X}} \widetilde{W}$.

$$\begin{tikzcd}
W \arrow[r, "t", "\sim"'] \arrow[d, two heads, "v"'] & \widetilde{W} \arrow[d, two heads, "s"] \\
X \arrow[r, tail, "\sim", "j"'] & \widetilde{X}
\end{tikzcd}$$

Then the natural map $t\colon W \longrightarrow \widetilde{W}$ is a weak equivalence as a pullback of a weak equivalence j along the fibration s.

The natural map $v\colon W \longrightarrow X$ is in $\mathcal{I} - inj$ as a pullback of the $\mathcal{I}$-injective map s.

I can consider now a functorial cofibrant approximation,

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr, tail] & & W \\
& CW_A\ X \arrow[ur, two heads, "u"'] &
\end{tikzcd}$$

$$\emptyset \rightarrowtail CW_A\ X \overset{u}{\twoheadrightarrow} W$$

and this factorization supplies me with an augmented functor CW_A, where the augmentation p_X is given by the composition $p_X = v \circ u$

To sum up, I obtain the commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[rrrd, bend left, "r"] \arrow[dr, hook, bend right, "g"'] & & & & \\
& CW_A\ X \arrow[r, "u", "\sim"'] \arrow[dr, "p_X"'] & W \arrow[r, "t", "\sim"'] \arrow[d, two heads, "v"'] & \widetilde{W} \arrow[d, two heads, "s"] & \\
& & X \arrow[r, tail, "\sim", "j"'] & \widetilde{X} \arrow[r, two heads] & *
\end{tikzcd}$$

The map $p_X\colon CW_A\ X \longrightarrow X$ is an A-equivalence since its fibrant approximation $s\colon \widetilde{W} \longrightarrow \widetilde{X}$ is $\mathcal{I}$-injective and so it has the right lifting property with respect to $\mathcal{I}$.

Moreover, $p_X \in \mathcal{I} - inj$. I note $p_X = v \circ u$ is a composition of two fibrations, hence a fibration and so it has the right lifting property with respect to any morphism in $\mathcal{I}$.

For any other map $C \hookrightarrow D$ in $\mathcal{I}$ and any commutative square,

$$\begin{tikzcd}
C \arrow[r] \arrow[d, hook] & CW_A\ X \arrow[d, two heads, "p_X"] \\
D \arrow[r] & X
\end{tikzcd}$$

I construct first a lift $\widetilde{l}\colon D \longrightarrow \widetilde{W}$, which exists since $s \in \mathcal{I} - inj$.

$$\begin{tikzcd}
\emptyset \arrow[rrrd, bend left, "r"] \arrow[dr, hook, "g"] & & & & \\
C \arrow[r] \arrow[d, hook] & CW_A\ X \arrow[r, "u", "\sim"'] \arrow[dr, "p_X"'] & W \arrow[r, "t", "\sim"'] \arrow[d, two heads, "v"'] & \widetilde{W} \arrow[d, two heads, "s"] & \\
D \arrow[rrruu, "\widetilde{l}"] \arrow[rr] & & X \arrow[r, tail, "\sim", "j"'] & \widetilde{X} \arrow[r, two heads] & *
\end{tikzcd}$$

To clarify, I can call $h\colon D \longrightarrow W$ the natural map into the pullback W. Finally the required lift $l\colon D \longrightarrow CW_A\ X$ exists, since the map u is a trivial fibration and the map $C \hookrightarrow D$ is a cofibration. So,

$$\begin{tikzcd}
\emptyset \arrow[rrrd, bend left, "r"] \arrow[dr, hook, "g"] & & & & \\
C \arrow[r] \arrow[d, hook] & CW_A\ X \arrow[r, "u", "\sim"'] \arrow[dr, "p_X"'] & W \arrow[r, "t", "\sim"'] \arrow[d, two heads, "v"'] & \widetilde{W} \arrow[d, two heads, "s"] & \\
D \arrow[ru, "l"] \arrow[rrruu, "\widetilde{l}"] \arrow[rr] & & X \arrow[r, tail, "\sim", "j"'] & \widetilde{X} \arrow[r, two heads] & *
\end{tikzcd}$$

It remains to show that $CW_A\ X$ is A-cellular for any $X \in \mathcal{M}$. But $CW_A\ X$ is cofibrant and weakly equivalent to $\widetilde{W}$, hence A-cellular by construction.

Remark 21. *I can note that $CW_A X$ is fibrant if X is fibrant.*

Indeed, recall that I have

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr, hook] & & X \\
& CW_A\ X \arrow[ur, two heads] &
\end{tikzcd}$$

and by definition, X is fibrant if $X \twoheadrightarrow *$ is a fibration.

Then, I have

$$\begin{tikzcd} CW_AX \arrow[r, two heads] \arrow[rr, bend left] & X \arrow[r, two heads] & * \end{tikzcd}$$

Composition of fibrations is a fibration too, so $CW_AX \twoheadrightarrow *$ is a fibration.

Hence, by definition CW_AX is fibrant.

Remark 22. *In the construction of the CW_A functor the object A is demanded to be pointed. This condition is also a key requirement.*

Otherwise, CW_A will be trivial.

That is, for any unbased $X \in \mathcal{M}$ (non based), I build CW_AX with the early described process using unbased homotopy colimits.

I claim that $CW_AX \xrightarrow{\sim} X \qquad forall \;\; X \in \mathcal{M}$.

Indeed, let $\mathcal{M}$ be a model category and let A be a cofibrant object in $\mathcal{M}$.

I define an unbased A-equivalence as a morphism $f\colon X \longrightarrow Y$ which induces a weak equivalence of unbased mapping spaces,

$$map(A,X) \xrightarrow[\sim]{f_*} map(A,Y) \ .$$

Assume that $\emptyset = *$ in $\mathcal{M}$.

Thus I can take,

$$\begin{tikzcd} * \arrow[rr] \arrow[dr, tail] & & * & & map(*,X) \arrow[d, phantom, "//"] \arrow[rr] \arrow[dr, tail] & & map(*,X) \arrow[d, phantom, "\backslash\backslash"] \\ & A \arrow[ur] & & \Longrightarrow & X & map(A,X) \arrow[ur] & X \end{tikzcd}$$

X is a retract of $\quad map(A,X) \qquad forall \;\; X$

I know that any retract of a weak equivalence is a weak equivalence.

Set now, $f\colon X \longrightarrow Y$ an A-equivalence.

Then I obtain, $map(A,X) \xrightarrow[f_*]{\sim} map(A,Y)$.

Therefore,

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, equal] & Y \arrow[d, equal] \\ map(*,X) \arrow[r, "f_*"'] & map(*,Y) \end{tikzcd}$$

Any A-equivalence is a weak equivalence.

Hence, $\quad CW_AX \xrightarrow{\sim} X \qquad for\ all \;\; X$.

which is the triviality result that I claimed.

Universality of CW_A.

Finally I want to prove the universality of CW_A, but before doing this previously I consider some generalities.

Universal Property.

The concrete details of a particular given construction may be intrincate, but if the construction satisfies a universal property, then I can forget all those details: everything there is to know about the construct is already contained in the universal property.

Universal properties define objects uniquely up to a unique isomorphism or homeomorphism or homotopy equivalence or weak equivalence... Therefore, one strategy to prove that two objects are isomorphic or homeomorphic or homotopic or weakly equivalent is to show that they satisfy the same universal property.

Universal constructions are functorial in nature: if I can carry out the construction for every object in a general category $\mathcal{C}$ then I obtain a functor on $\mathcal{C}$. Furthermore, this functor is a right or left adjoint to the functor F used in the definition of the universal property.

To sum up, by understanding the abstract universal properties, I can obtain information about all these constructions and moreover I can avoid repeating the same analysis for each individual instance.

Formally,

Definition 23. *(Initial Morphism and Terminal Morphism)*

Suppose that $F\colon \mathcal{D} \longrightarrow \mathcal{C}$ is a functor from a category $\mathcal{D}$ to a category $\mathcal{C}$, and let C be an object of $\mathcal{C}$.

Consider the following dual (opposite) notions:

An initial morphism from C to F is an initial object in the category $(C \downarrow F)$ of morphisms from C to F (the comma category). In other words, it consists of a pair (D, ϕ) where D is an object of $\mathcal{D}$ and $\phi\colon C \longrightarrow F(D)$ is a morphism in $\mathcal{C}$, such that the following initial property is satisfied:

Whenever T is an object of $\mathcal{D}$ and $f\colon C \longrightarrow F(T)$ is a morphism in $\mathcal{C}$, then there exists a unique morphism $g\colon D \longrightarrow T$ such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[dr, "f"'] & F(D) \arrow[d, dashed, "F(g)"] & & D \arrow[d, dashed, "g"] \\
& F(T) & & T
\end{tikzcd}$$

A terminal morphism from F to C is a terminal object in the comma category $(F \downarrow C)$ of morphisms from F to C. In other words, it consists of a pair (D, ϕ) where D is an object of $\mathcal{D}$ and $\phi\colon F(D) \longrightarrow C$ is a morphism in $\mathcal{C}$, such that the following terminal property is satisfied:

Whenever T is an object of $\mathcal{D}$ and $f\colon F(T) \longrightarrow C$ is a morphism in $\mathcal{C}$, then there exists a unique morphism $g\colon T \longrightarrow D$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[d, dashed, "g"] & & F(T) \arrow[d, dashed, "F(g)"] \arrow[dr, "f"] & \\
D & & F(D) \arrow[r, "\phi"] & C
\end{tikzcd}$$

The term universal morphism refers either to an initial morphism or a terminal morphism, and the term universal property refers either to an initial property or a terminal property. In each definition, the existence of the morphism g intuitively expresses the fact that (D, ϕ) is general enough, while the uniqueness of the morphism ensures that (D, ϕ) is not too general.

I can note that since the notions of initial and terminal are dual, it is often enough to discuss only one of them, and simply reverse arrows in $\mathcal{C}$ for the dual discussion. Alternatively, the word universal is often used in place of both words.

CW_A is terminal.

.

I show now that CW_A is terminal. That is, for any map $\alpha\colon Z \longrightarrow X$ from an A-cellular Z there exists a factorization,

$$\begin{tikzcd}
Z \arrow[rr, "\alpha"] \arrow[dr] & & X \\
& CW_A\, X \arrow[ur, "\sim"'] &
\end{tikzcd}$$

which is unique up to homotopy.

Indeed, I know that the natural map $p_X\colon CW_A X \longrightarrow W$ is $\mathcal{I}-inj$.

The map $\emptyset \longrightarrow Z$ is in $\mathcal{I}-cof$, since Z is cofibrant and for every $f\colon A \longrightarrow B$ the induced map $f_*\colon map_*(Z, A) \longrightarrow map_*(Z, B)$ is a weak equivalence.

Then the commutative square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & CW_A\ X \arrow[d, "p_X"] \\
Z \arrow[r, "\alpha"'] \arrow[ur, dashed] & X
\end{tikzcd}$$

admits a lift, which provides the required factorization.

The factorization above is unique since $map_*(Z, CW_AX) \longrightarrow map_*(Z, X)$ is a weak equivalence as desired.

CW_A is initial.

The augmentation map r_X is a fibration for any X, hence the subcategory of fibrant diagrams is stable under localizations.

Now, I denote by CW^f_A the restriction of CW_A to the subcategory of fibrant diagrams.

Then CW^f_A is initial with respect to the A-equivalences. In more detail, I have:

On the subcategory of fibrant diagrams the augmentation map $r_X\colon CW^f_AX \longrightarrow X$ is initial, up to homotopy, among all A-equivalences.

Therefore, I want to prove that, for any A-equivalence of fibrant diagrams $\psi\colon Y \longrightarrow X$, there exists a unique, up to homotopy, mapping $\phi\colon CW^f_A(\psi) = CW_A(\psi)$ such that $r_X \simeq \psi \circ \phi$.

Indeed, I apply the functor CW^f_A on the A-equivalence ψ, then the map $CW^f_A(\psi) = CW_A(\psi)$ is an A-equivalence by the M2 axiom (2 out of 3 axiom). But then $CW_A(\psi)$ is also a weak equivalence, therefore it has a homotopy inverse φ.

If I take $\phi = r_Y \circ \varphi$, then $\psi \circ \phi = \psi \circ r_Y \circ \varphi = r_X \circ CW_A(\psi) \circ \varphi \simeq r_X \circ Id_{CW_AX} = r_X$ as the following diagram shows:

$$\begin{tikzcd}
CW_AY \arrow[r, "CW_A(\psi)", "\sim"'] \arrow[d, "r_Y"'] & CW_AX \arrow[d, "r_X"] \arrow[l, dashed, bend right=60, "\varphi"'] \arrow[dl, dashed, "\phi"'] \\
Y \arrow[r, "\psi"'] & X
\end{tikzcd}$$

I prove now that the map is determined up to homotopy.

I suppose there exists $\phi'\colon CW_AX \longrightarrow Y$, $\phi' \neq \phi$, and such that $\psi \circ \phi' \simeq r_X$.

By the previously proved terminal property of the cellularization functor there exists a map $\varphi'\colon CW_AX \longrightarrow CW_AY$ such that $\phi' = r_Y\varphi'$.

It will suffice to show that $\varphi' \simeq \varphi$, since this implies that $\phi' \simeq \phi$.

The induced map on homotopy classes $[CW_AX, \psi \circ r_Y]$ is an isomorphism, since $\psi \circ r_X$ is an A-equivalence of fibrant diagrams and CW_AX is cofibrant.

But now, $\psi \circ r_X \circ \varphi = \psi \circ \phi \sim r_x \simeq \psi \circ \phi' = \psi \circ r_Y\varphi'$, and hence $\varphi' \simeq \varphi$ as I would want to show.

Remark 24. *Beside the Theorem of CW_A Approximation (Theorem 20), there also exists a Theorem of L_f Localization, which states the following,*

Theorem 21. *(Theorem of L_f Localization)*

Let $\mathcal{M}$ be a pointed model category which is left proper and admits Quillen's small object argument.

For every morphism $f\colon A \longrightarrow B$ there exists a coaugmented endofunctor L_f on $\mathcal{M}$, called f-localization, such that for every object $X \in \mathcal{M}$, the coaugmentation $X \xrightarrow{l} L_fX$ is a cofibration where L_fX is f-local and l is an f-equivalence.

L_fX is f-local if it is fibrant and $map_*(B, L_fX) \xrightarrow{\sim} map_*(A, L_fX)$.

l is an f-equivalence if $map_*(L_fX, Z) \xrightarrow{\sim} map_*(X, Z)$ for every f-local object Z.

The functor L_f has the following universal property:

For every f-local object Z and every object X,

$$\begin{tikzcd} X \arrow[rr, "l"] \arrow[dr] & & L_f X \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

there is a bijection $[L_f X, Z] \cong [X, Z]$ induced by l in the homotopy category $Ho\ \mathcal{M}$.

In the construction of the coapproximation object $CW_A X$, using Quillen's small object argument,

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, tail] & & X \\ & CW_A\ X \arrow[ur, two heads, "p_X"'] & \end{tikzcd}$$

$$\emptyset \rightarrowtail CW_A\ X \overset{p_X}{\twoheadrightarrow} X$$

This results in a cofibrant cellular object $CW_A X$ of $\mathcal{M}$ and the mapping p_X in a fibration forall $X \in \mathcal{M}$.

I can construct the f-localization object $L_f X$, using the Quillen's small argument with an analogous argument, to obtain a functorial factorization of $X \longrightarrow *$,

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, tail, "l_X"'] & & * \\ & L_f\ X \arrow[ur, two heads] & \end{tikzcd}$$

$$X \overset{l_X}{\rightarrowtail} L_f\ X \twoheadrightarrow *$$

This results in a fibrant object $L_f X$ of $\mathcal{M}$ and the mapping l_X in a cofibration forall $X \in \mathcal{M}$.

The existence of the f-localization functor allows me to define new structures in the model category $\mathcal{M}$.

Indeed, with the weak equivalences as the mappings g such that $L_f\ g$ is a weak equivalence and with the same cofibrations of those in $\mathcal{M}$ and fibrations induced, I will obtain a new model structure on $\mathcal{M}$, which I could denote by $L_f\mathcal{M}$, the f-localized model structure.

Equivalently, the existence of the CW_A-approximation functor allows me to define new structures in the model category $\mathcal{M}$.

Indeed, with the weak equivalences as the mappings g such that $CW_A g$ is an A-equivalence and with the same fibrations as those in $\mathcal{M}$ and cofibrations induced, I will obtain a new model structure on $\mathcal{M}$, which I could denote by $CW_A\mathcal{M}$, the A-colocalized model structure.

Given an object X, both with the mappings l_X and p_X, a fibrant approximation of X and respectively a cofibrant approximation of X could be defined in the new categories, the localized category and the colocalized category.

Characterization of A-Cellular Classes.

In order to prove the main Theorem 19 in this section I previously need some auxiliary facts.

Lemma 25. *Let $\mathcal{M}$ be a pointed model category.*

If $f\colon A \xrightarrow{\sim} B$ is a weak equivalence between cofibrant objects A, $B \in \mathcal{M}$ and $i\colon A \rightarrowtail C$ is a cofibration, then, in the following pushout diagram, $\bar{f}$ is a weak equivalence:

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"'] & B \arrow[d] \\ C \arrow[r, "\bar{f}"'] & C \amalg_A B \end{tikzcd}$$

Proof. I consider the diagram,

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A \arrow[r, "f", "\sim"'] \arrow[d, tail, "\bar{i}"'] \arrow[dd, tail, bend left=60, "\bar{f}"'] & B \arrow[d] \arrow[ddr, tail, bend left] & \\
C \arrow[r, tail, "\sim", "\bar{j}"'] \arrow[dr, "Id_C"'] & \bar{C} \arrow[r, tail, "i"] \arrow[d, "w"'] & C \arrow[dr, "\bar{w}"'] & \\
& C \arrow[rr, "\bar{f}"'] & & C \amalg_A B
\end{tikzcd}$$

in which the mapping $j\colon C \longrightarrow A$ is the section of f and the mappings $Id_C = \bar{j} \circ w$ and $i = w \circ \bar{i}$ are the factorizations in $\mathcal{M}$ where $\bar{j}$ is a trivial cofibration and $\bar{i}$ is a cofibration.

Now, since $Id_C = \bar{j} \circ f = w \circ \bar{j}$, I obtain that $\bar{f}$ and w are weak equivalences.

Moreover, w is a weak equivalence between cofibrant objects and so $\bar{w}$ is a weak equivalence, too. ☐

Recall that generally, given a map $h : X \longrightarrow Y$, I say that a map $k\colon Y \longrightarrow X$ is a section of h if $h \circ k = Id_Y$.

Dually, I have the following.

Lemma 26. *Let $\mathcal{M}$ be a pointed model category.*

If $f\colon X \longrightarrow Y$ is a weak equivalence between fibrant objects X, $Y \in \mathcal{M}$ and $p\colon E \longrightarrow Y$ is a fibration, then, in the following pullback diagram, $\bar{f}$ is a weak equivalence:

$$\begin{tikzcd}
E \times_Y X \arrow[r] \arrow[d, "\bar{f}"'] & X \arrow[d, "f"] \\
E \arrow[r, "p"'] & Y
\end{tikzcd}$$

Lemma 27. *(Cube Lemma for Pushouts)*

I consider a cube diagram such that the objects X_0, Y_0, X_2 and Y_2 are cofibrant and both the top face and the bottom face are pushouts and i, j are cofibrations.

$$\begin{tikzcd}
& X_0 \arrow[rr, tail, "i"] \arrow[dl, "w_0"', "\sim"] \arrow[dd, "j"' near end] & & X_1 \arrow[dl, "\sim"', "w_1"] \arrow[dd] \\
Y_0 \arrow[rr, tail] \arrow[dd] & & Y_1 \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dl, "w_2"', "\sim"] & & X \arrow[dl, "w"] \\
Y_2 \arrow[rr] & & Y &
\end{tikzcd}$$

Then if w_0,w_1 and w_2 are weak equivalences then w is a weak equivalence too.

Proof. Considering the top face and the bottom face in the cube, I complete the cube as follows:

$$\begin{tikzcd}
& X_0 \arrow[rr, tail, "i"] \arrow[dl, "w_0"', "\sim"] \arrow[dd] & & X_1 \arrow[dl, "\sim"', "q"] \arrow[dd] \\
Y_0 \arrow[rr, tail, "k"] \arrow[dd] & & Q \arrow[dd] & \\
& X_2 \arrow[rr] \arrow[dl, "w_2"', "\sim"] & & X \arrow[dl, "r"] \\
Y_2 \arrow[rr] & & R &
\end{tikzcd}$$

I know that q and r are weak equivalences.

Then, I have the diagrams

$$\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd] & \\
Y_0 \arrow[ur, "k"] \arrow[rr, tail, "j"] \arrow[dd] & & Y_1 \arrow[dd] \\
& R \arrow[dr, "v"] & \\
Y_2 \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd] & \\
X_1 \arrow[ur, "q", "\sim"'] \arrow[rr, "\sim", "w_1"'] \arrow[dd] & & Y_1 \arrow[dd] \\
& R \arrow[dr, "v"] & \\
X \arrow[ur, "r", "\sim"'] \arrow[rr, "w"'] & & Y
\end{tikzcd}$$

where w_1 is a weak equivalence, so u is also a weak equivalence by the M2 axiom (two out of three axiom) and moreover w is a weak equivalencece if and only if v is a weak equivalence again by the M2 axiom (two out of three axiom).

I can show that the two partial diagrams fit into the complete diagram,

$$\begin{tikzcd}
& X_0 \arrow[rrr, "i"] \arrow[ddl, "w_0"', "\sim"] \arrow[ddd] & & & X_1 \arrow[dll, "\sim"', "w_1"] \arrow[ddl, "\sim"', "q"] \arrow[ddd] \\
& & Y_1 \arrow[ddd, shift right] \arrow[ddd, shift left] & & \\
Y_0 \arrow[urr, "j"] \arrow[rrr, tail, "k"] \arrow[ddd] & & & Q \arrow[ul, shift right] \arrow[ul, shift left, "u"'] \arrow[ddd, shift right] \arrow[ddd, shift left] & \\
& X_2 \arrow[rrr] \arrow[ddl, "w_2"', "\sim"] & & & X \arrow[dll, "\sim"', "w"] \arrow[ddl, "\sim"', "r"] \\
& & Y & & \\
Y_2 \arrow[urr] \arrow[rrr] & & & R \arrow[ul, shift right] \arrow[ul, shift left, "v"'] &
\end{tikzcd}$$

Considering again the two partial diagrams, and in particular the left one (the red coloured) and for there, I consider the case $w_0 = Id$ and $w_2 = Id$. I want to prove that v is a weak equivalence.

I consider $Y_0 \longrightarrow Y_2$ and a factorization $Y_0 \longrightarrow \bar{Y_0} \longrightarrow Y_2$, where $Y_0 \longrightarrow \bar{Y_0}$ is a cofibration and $\bar{Y_0} \longrightarrow Y_2$ is a weak equivalence. Then the pushout, deconstructed in steps, can be seen as

$$\begin{tikzcd}
& Q \arrow[dr, "u"] \arrow[dd, tail] & \\
Y_0 \arrow[ur, "k"] \arrow[rr, "j"] \arrow[dd, tail] & & Y_1 \arrow[dd] \\
& \bar{Q} \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd] & \\
\bar{Y_0} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, tail, shift left] \arrow[rr, tail, shift right] \arrow[dd] & & \bar{Y_1} \arrow[dd] \\
& R \arrow[dr, "v"] & \\
Y_2 \arrow[ur] \arrow[rr] & & Y
\end{tikzcd}$$

where the mappings $Y_0 \longrightarrow \bar{Y_0}$ and $Q \longrightarrow \bar{Q}$ are cofibrations. But cofibrations are closed under pushouts hence I obtain that $\bar{Y_0} \longrightarrow \bar{Y_1}$ is a cofibration too.

I can show that the two partial diagrams fit into the complete diagram

$$\begin{tikzcd}
& X_0 \arrow[rrr, tail, "i"] \arrow[ddl, "w_0"', "\sim"] \arrow[ddd] & & & X_1 \arrow[dll, "\sim", "w_1"'] \arrow[ddl, "\sim"', "q"] \arrow[ddd] \\
& & Y_1 \arrow[ddd, shift right] \arrow[ddd] \arrow[ddd, shift left] & & \\
Y_0 \arrow[urr, "j"] \arrow[rrr, tail, "k"] \arrow[ddd] & & & Q \arrow[ul, shift left, "u"] \arrow[ul, shift right] \arrow[ddd, shift right] \arrow[ddd, shift left] & \\
& X_2 \arrow[rrr] \arrow[ddddl, "w_2"', "\sim"] & & & X \arrow[ddl, "\sim"', "w"] \arrow[ddddl, "\sim"', "r"] \\
& & \bar{Y_1} \arrow[dd, shift right] \arrow[dd] \arrow[dd, shift left] & & \\
\bar{Y_0} \arrow[urr, shift left] \arrow[urr, shift right] \arrow[rrr, shift left] \arrow[rrr, shift right] \arrow[dd] & & & \bar{Q} \arrow[ul, shift left, "v"] \arrow[ul] \arrow[ul, shift right] \arrow[dl] \arrow[dd, shift right] \arrow[dd, shift left] & \\
& & Y & & \\
Y_2 \arrow[urr] \arrow[rrr] & & & R \arrow[ul, shift left, "v"] \arrow[ul, shift right] &
\end{tikzcd}$$

Now I consider the pushout diagrams

$$\begin{tikzcd}
\bar{Y}_0 \arrow[r] \arrow[d] & \bar{Y}_2 \arrow[d] \\
\bar{Q} \arrow[r, "\sim"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{Y}_0 \arrow[r] \arrow[d] & \bar{Y}_2 \arrow[d] \\
\bar{Y}_1 \arrow[r, "\sim"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r] \arrow[d] & Y_1 \arrow[d] \\
\bar{Q} \arrow[r, "\sim"'] & \bar{Y}_1
\end{tikzcd}$$

where the mappings $\bar{Q} \longrightarrow R$, $\bar{Y}_1 \longrightarrow Y$ and $\bar{Q} \longrightarrow \bar{Y}_1$ are weak equivalences and then v is a weak equivalence too.

□

Dually, I have

Lemma 28. *(Cube Lemma for Pullbacks)*
I consider a cube diagram such that the objects X_0, Y_0, X_2 and Y_2 are fibrant and both the front face and the back face are pullbacks and p, q are fibrations.

$$\begin{tikzcd}
& X \arrow[rr] \arrow[dl, "w"'] \arrow[dd] & & X_2 \arrow[dl, "\sim", "w_2"'] \arrow[dd] \\
Y \arrow[rr] \arrow[dd] & & Y_2 \arrow[dd] & \\
& X_1 \arrow[rr, two heads, "p"] \arrow[dl, "w_1"', "\sim"] & & X_0 \arrow[dl, "\sim", "w_0"'] \\
Y_1 \arrow[rr, two heads, "q"] & & Y_0 &
\end{tikzcd}$$

Then if w_0, w_1 and w_2 are weak equivalences then w is a weak equivalence too.

Lemma 29. *If $\mathcal{M}$ has sequential direct limits (and so, $A_0 \longrightarrow A_1 \longrightarrow \cdots$ has a direct limit), then given a commuting diagram of sequences*

$$\begin{tikzcd}
A_0 \arrow[r, tail, "i_0"] \arrow[d, "f_0"', "\sim"] & A_1 \arrow[r, tail, "i_1"] \arrow[d, "f_1", "\sim"'] & \cdots \\
B_0 \arrow[r, tail, "j_0"] & B_1 \arrow[r, tail, "j_1"] & \cdots
\end{tikzcd}$$

where each f_n is a weak equivalence, each i_n and j_n is a cofibration, and A_0 and B_0 are cofibrant, then colim$A_n \longrightarrow$ colim B_n, the direct limit, is a weak equivalence.

Proof. The proof of Lemma 110 can be extended to show that a lifting and homotopy on B_n and $A_n \times I$ can be chosen extending the ones on B_{n-1} and $A_{n-1} \times I$.
Since $colim(A \times I)_n$ will be a cylinder object for $colim\ A_n$ if I choose $A_n \times I$ so that

$$(A_n \amalg A_n) \vee A_{n-1} \times I \longrightarrow A_n \times I$$

is a cofibration, the direct limit of the lifting and homotopy shows that $colim\ A_n \longrightarrow colim\ B_n$ is a weak equivalence by Lemma 110.

□

Remark 30. *(Tower of Fibrations)*
I early introduced the term combinatorial, which means that the model category is cofibrantly generated and the underlying category is locally presentable. Daniel Dugger proved in [3] *that every combinatorial model category is equivalent to a localization of a category of diagrams of simplicial sets. Among many other examples, the model category of simplicial sets is combinatorial.*

Also, recall that, if $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is a small category, then the projective model structure on the category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-indexed diagrams in $\mathcal{M}$ has objectwise weak equivalences and objectwise fibrations, while the injective model structure has objectwise weak *equivalences and objectwise cofibrations. In the projective model structure, if the model category $\mathcal{M}$ is simplicial, then $\mathcal{M}^{\mathcal{C}}$ with the projective model structure is also simplicial.*

Lemma 31. *Let $\mathcal{M}$ be a cofibrantly generated simplicial model category and $\mathcal{I}$ a small category. Suppose that A is a cofibrant diagram in the projective model category structure of $\mathcal{M}^{\mathcal{I}}$ and X is a fibrant object of $\mathcal{M}$. Then $map(A, X)$ is fibrant in the injective model structure on the category of $\mathcal{I}^{op}$-diagrams of simplicial sets.*

Proof. I want to prove that I can find a lift for any commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, tail, "i"', "\sim"] & map(A, X) \arrow[d] \\
Z \arrow[r] & *
\end{tikzcd}$$

where i is an objectwise trivial cofibration of $\mathcal{I}^{op}$-diagrams of simplicial sets.

By adjunction this is equivalent to proving that I can find a lift in the commutative diagram in $\mathcal{M}$,

$$\begin{tikzcd}
A \otimes_{\mathcal{I}} Y \arrow[r] \arrow[d] & X \arrow[d, two heads] \\
A \otimes_{\mathcal{I}} Z \arrow[r] & *
\end{tikzcd}$$

And again by adjunction the previous one is equivalent to finding a lift in the commutative diagram in $\mathcal{M}^{\mathcal{I}}$

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & X^Z \arrow[d, two heads, "p_{X^i}"', "\sim"] \\
A \arrow[r] & X^Y
\end{tikzcd}$$

p_{X^i} is a projective trivial fibration (it is an objectwise trivial fibration) and A is projectively cofibrant. Hence this last lifting exists, and so does each one previously presented. ☐

In a locally presentable model category $\mathcal{M}$ I consider

$$A_0 \rightarrowtail A_1 \rightarrowtail \cdots$$

where each A_n is a cofibration.

I choose a regular cardinal λ such that any member of the set of domains and codomains of maps in this sequence is λ-presentable (such a cardinal exists since $\mathcal{M}$ is locally presentable,

Let $\mathcal{I}$ be any λ-directed partially ordered set, and suppose given a diagram $f \colon \mathcal{I} \longrightarrow Arr\mathcal{M}$ where $Arr\mathcal{M}$ is the category of maps in $\mathcal{M}$.

Recall that a partially ordered set $\mathcal{I}$ is called λ-directed, where λ is a regular cardinal, if every subset of $\mathcal{I}$ of cardinality smaller than λ has an upper bound.

I denote by $X \colon \mathcal{I} \longrightarrow \mathcal{M}$ the domain of f and by $Y \colon \mathcal{I} \longrightarrow \mathcal{M}$ the codomain, so f can also be seen as a map from X to Y in $\mathcal{M}^{\mathcal{I}}$:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, "f_n"'] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & \cdots
\end{tikzcd}$$

Since $\mathcal{M}$ is cocomplete, Arr $\mathcal{M}$ is cocomplete as well, and I may consider the colimit of the diagram f.

I choose a cofibrant approximation $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ to f using the projective model structure on $\mathcal{M}^{\mathcal{I}}$, hence obtaining the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd}
\widetilde{X_0} \arrow[rr, tail] \arrow[dd, "\widetilde{f}_0"', "\sim"] \arrow[dr, "\sim"] & & \widetilde{X_1} \arrow[rr, tail] \arrow[dd, "\widetilde{f}_1"', "\sim"] \arrow[dr, "\sim"] & & \cdots \arrow[rr, tail] & & \widetilde{X_n} \arrow[rr, tail] \arrow[dd, "\widetilde{f}_n"', "\sim"] \arrow[dr, "\sim"] & & \cdots & \\
& X_0 \arrow[rr] \arrow[dd, "f_0"', "\sim"] & & X_1 \arrow[rr] \arrow[dd, "\sim"', "f_1"] & & \cdots \arrow[rr] & & X_n \arrow[rr] \arrow[dd, "f_n"', "\sim"] & & \cdots \\
\widetilde{Y_0} \arrow[rr, tail] \arrow[dr, "\sim"] & & \widetilde{Y_1} \arrow[rr, tail] \arrow[dr, "\sim"] & & \cdots \arrow[rr, tail] & & \widetilde{Y_n} \arrow[rr, tail] \arrow[dr, "\sim"] & & \cdots & \\
& Y_0 \arrow[rr, tail] & & Y_1 \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & Y_n \arrow[rr, tail] & & \cdots
\end{tikzcd}$$

For every $Z \in \mathcal{M}$, let $\widetilde{Z}$ be a fibrant approximation to Z. The induced map

$$map_*(colim\widetilde{f}, \widetilde{Z})\colon map_*(colim\widetilde{Y}, \widetilde{Z}) \longrightarrow map_*(colim\widetilde{X}, \widetilde{Z})$$

can be written as a limit of a diagram,

$$lim\ map_*(\widetilde{f}, \widetilde{Z})\colon lim\ map_*(\widetilde{Y}, \widetilde{Z}) \longrightarrow lim\ map_*(\widetilde{X}, \widetilde{Z})$$

The $\mathcal{I}^{op}$-diagrams $map_*(\widetilde{X}, \widetilde{Z})$ and $map_*(\widetilde{Y}, \widetilde{Z})$ are fibrant in the injective model structure by Lemma 31.

Therefore, their inverse limits are homotopy inverse limits since the constant diagram of points is cofibrant in the injective model structure.

Hence, $map_*(colim\widetilde{f}, \widetilde{Z})$ is a weak equivalence, as a map induced between homotopy inverse limits by levelwise weak equivalences $map_*(\widetilde{f}_i, \widetilde{Z})$.

Now, I will prove the main Theorem 19 in this section, with two assumptions:

(a) Definition of A-cellular object, (Definition 2).

(b) There exists an augmented functor CW_A in the pointed model category $\mathcal{M}$ such that forall $X \in \mathcal{M}$ the natural map

$$CW_A X \longrightarrow X$$

induces a weak equivalence $map_*(A, CW_A X) \xrightarrow{\sim} map_*(A, X)$ and the object $CW_A X$ can be built from A using homotopy pushouts and telescopes (therefore $CW_A X$ is A-cellular).

Proof. (of the Theorem 19, the Theorem of Characterization for Cellular Classes)

Let $Cell(A)$ be the class of A-cellular objects.

I need to prove the three consequences stated in the theorem.

(a) A is A-cellular clearly by Definition 1 and Definition 2.

Remark 32. *In the definitions, A could be considered as a set of objects $\{A_i\}_{i \in I}$.*

In any case, I can always reduce the proof to the case the collection of objects is a single object A.

Indeed, let $\mathcal{A}$ a set of objects $\{A_i\}_{i \in I}$.

Then I can consider $A = \coprod_{i \in I} A_i$, and an object X is A-cellular if and only if it is $\mathcal{A}$-cellular.

I can deduce this claim from the fact that

$$map_*(A, X) = map_*(\coprod_{i \in I} A_i, X) \cong \prod_{i \in I} map_*(A_i, X)\ .$$

Given $f\colon X \longrightarrow Y$,

$$\begin{tikzcd}
map_*(A,X) \arrow[rr, "f_*"] \arrow[d, equal] & & map_*(A,Y) \arrow[d, equal] \\
\prod_{i\in I} map_*(A_i,X) \arrow[rr, dashed, "\prod (f_i)_*"] & & \prod_{i\in I} map_*(A_i,Y) \\
& \upuparrows & \\
& \textit{factor by factor} &
\end{tikzcd}$$

Now,

$$\begin{gathered}
f_* \textit{ is a weak equivalence of simplicial sets} \\
\Updownarrow \\
(f_i)_* \textit{ is a weak equivalence of simplicial sets for all } i \\
\textit{provided that } map_*(A_i,X) \neq \emptyset \quad \textit{for all } i.
\end{gathered}$$

But this fact always holds in a pointed model category $\mathcal{M}$ since

$$A_i \longrightarrow * \longrightarrow X$$

always exists.

(b) I want to prove,

$$\left.\begin{array}{r} X \in Cell(A) \\ X' \underset{w}{\simeq} X \end{array}\right\} \quad \overset{?}{\Longrightarrow} \quad X' \in Cell(A)$$

The fact that, $X' \underset{w}{\simeq} X$ means by definition that there exists a finite sequence of weak equivalences

$$X = X_0 \overset{\sim}{\longleftrightarrow} X_1 \overset{\sim}{\longleftrightarrow} X_2 \overset{\sim}{\longleftrightarrow} \cdots \overset{\sim}{\longleftrightarrow} X_n = X'$$

where all the maps are double-arrowed to mean that the sequence could be left or right-sided.

I can observe that it suffices to prove that $X_1 \in Cell(A)$ since composition of weak equivalences of A-cellular objects is also an A-cellular object.

So I prove this fact considering separately both arrow cases.

(b.1) (namely $X = X_0 \underset{\sim}{\xrightarrow{\varphi}} X_1$).

Let $V \xrightarrow{g} W$ be an A-equivalence. So by Definition 1, it induces a weak equivalence $map_*(X,V) \underset{\sim}{\xrightarrow{g_*}} map_*(X,W)$ since by hypothesis $X \in Cell(A)$.

On the other hand $X = X_0 \underset{\sim}{\xrightarrow{\varphi}} X_1$ is a weak equivalence so it induces weak equivalences,

$$map_*(X_1,V) \underset{\sim}{\xrightarrow{\varphi^*}} map_*(X,V)$$

and

$$map_*(X_1,W) \underset{\sim}{\xrightarrow{\varphi^*}} map_*(X,W)$$

Hence, I have the following diagram:

$$\begin{tikzcd}
map_*(X_1,V) \arrow[r, "g_*"] \arrow[d, "\varphi^*"', "\sim"] & map_*(X_1,W) \arrow[d, "\sim"', "\varphi^*"] \\
map_*(X,V) \arrow[r, "\sim", "g_*"'] & map_*(X,W)
\end{tikzcd}$$

Now, by the M2 axiom (two out of three axiom) for model categories, the upper horizontal map is a weak equivalences too, as desired.

(b.2) (namely $X = X_0 \underset{\sim}{\xleftarrow{\varphi}} X_1$)

In fact, this proof is absolutely analogous to the previous one but with the vertical arrows in the final diagram inverted.

Indeed, let $V \overset{g}{\to} W$ be an A-equivalence. So by Definition 1, it induces a weak equivalence $map_*(X,V) \underset{g_*}{\overset{\sim}{\longrightarrow}} map_*(X,W)$ since by hypothesis $X \in Cell(A)$.

On the other hand $X = X_0 \underset{\psi}{\overset{\sim}{\longleftarrow}} X_1$ is a weak equivalence so it also induces weak equivalences

$$map_*(X,V) \underset{\sim}{\overset{\psi^*}{\longrightarrow}} map_*(X_1,V)$$

and

$$map_*(X,W) \underset{\sim}{\overset{\psi^*}{\longrightarrow}} map_*(X_1,W)$$

Hence, I have the correspondig diagram

$$\begin{tikzcd}
map_*(X_1,V) \arrow[r, "g_*"] & map_*(X_1,W) \\
map_*(X,V) \arrow[u, "\psi^*", "\sim"'] \arrow[r, "\sim", "g_*"'] & map_*(X,W) \arrow[u, "\sim", "\psi^*"']
\end{tikzcd}$$

Now, by the M2 axiom (two out of three axiom) for model categories, the upper horizontal map is a weak equivalence too, as desired.

(c) Again I develop the proof of the closedness property separately for homotopy pushouts and afterwards for telescopes.

(c.1) (namely closedness for the class of A-cellular objects under homotopy pushouts).

Given a homotopy pushout,

$$\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[d, tail] & X_1 \arrow[d] \\
X_2 \arrow[r] & X
\end{tikzcd}$$

I want to prove that

$$X_0,\ X_1,\ X_2 \in Cell(A) \qquad \overset{?}{\Rightarrow} \qquad X \in Cell(A)\ .$$

Let $V \overset{g}{\to} W$ be an A-equivalence. By Definition 1, it induces the following weak equivalences:

$$map_*(X_0,V) \underset{\sim}{\overset{g_*}{\longrightarrow}} map_*(X_0,W) \text{ since by hypothesis } X_0 \in Cell(A).$$

$$map_*(X_1,V) \underset{\sim}{\overset{g_*}{\longrightarrow}} map_*(X_1,W) \text{ since by hypothesis } X_1 \in Cell(A).$$

$map_*(X_2,V) \underset{\sim}{\overset{g_*}{\longrightarrow}} map_*(X_2,W)$ since by hypothesis $X_2 \in Cell(A)$.

On the other hand, the homotopy pushout

$$\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[d, tail] & X_1 \arrow[d] \\
X_2 \arrow[r] & X
\end{tikzcd}$$

induces the following two pullbacks:

$$\begin{tikzcd}
map_*(X_0,V) & map_*(X_1,V) \arrow[l, two heads] \\
map_*(X_2,V) \arrow[u, two heads] & map_*(X,V) \arrow[l] \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
map_*(X_0,W) & map_*(X_1,W) \arrow[l, two heads] \\
map_*(X_2,W) \arrow[u, two heads] & map_*(X,W) \arrow[l] \arrow[u]
\end{tikzcd}$$

Then, in fact I have the diagram

$$\begin{tikzcd}
map_*(X_0,V) \arrow[dr, "\sim", "g_*"'] & & map(X_1,V) \arrow[ll, two heads] \arrow[dr, "g_*", "\sim"'] & \\
& map_*(X_0,W) & & map_*(X_1,W) \arrow[ll, two heads] \\
map_*(X_2,V) \arrow[uu, two heads] \arrow[dr, "\sim", "g_*"'] & & map_*(X,V) \arrow[ll] \arrow[uu] \arrow[dr, red, "g_*", "\sim"'] & \\
& map_*(X_2,W) \arrow[uu, two heads] & & map_*(X,W) \arrow[ll] \arrow[uu]
\end{tikzcd}$$

So, by Lemma 28 (Cube Lemma for pullbacks) also the red arrow $(\searrow^{\sim})$ is a weak equivalence.

Hence $X \in Cell(A)$.

(c.2) (namely closedness for the class of A-cellular objects under telescopes).

Given a telescope

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X = colim X_i$$

I suppose $X_i \in Cell(A)$ forall i , and I want to see that $X \overset{?}{\in} Cell(A)$.

Let $V \xrightarrow{g} W$ be an A-equivalence.

By Definition 1, it induces weak equivalences forall i,
$map_*(X_i,V) \underset{g_*}{\xrightarrow{\sim}} map_*(X_i,W)$ since by hypothesis $X_i \in Cell(A)$ for all X_i .

On the other hand the telescope yields the following towers of fibrations:

$$map_*(X \ = \ colim \ X_i, V) = lim \ map_*(X_i,V) \twoheadrightarrow \cdots \twoheadrightarrow map_*(X_1,V) \twoheadrightarrow map_*(X_0,V)$$

and

$$map_*(X \ = \ colim \ X_i, W) = lim \ map_*(X_i,W) \twoheadrightarrow \cdots \twoheadrightarrow map_*(X_1,W) \twoheadrightarrow map_*(X_0,W)$$

In fact I obtain the following diagram:

$$\begin{tikzcd}
map_*(X \ = \ colim \ X_i, V) = lim \ map_*(X_i,V) \arrow[r, two heads] \arrow[d, "g_*"] & \cdots & map_*(X_1,V) \arrow[r, two heads] \arrow[d, "\sim"', "g_*"] & map_*(X_0,V) \arrow[d, "\sim"', "g_*"] \\
map_*(X \ = \ colim \ X_i, W) = lim \ map_*(X_i,W) \arrow[r, two heads] & \cdots & map_*(X_1,W) \arrow[r, two heads] & map_*(X_0,W)
\end{tikzcd}$$

So that, also the map

$$map_*(X \ = \ colim \ X_i, V) = lim \ map_*(X_i,V) \underset{g_*}{\xrightarrow{\sim}} map_*(X \ = \ colim \ X_i, W) = lim \ map_*(X_i,W),$$

is a weak equivalence as I desired to prove.

Finally, I want to prove the last part in the theorem, namely that $Cell(A)$ is the smallest class of objects in $\mathcal{M}$, holding conditions **(a)** , **(b)** and **(c)**.

Let $\mathcal{C}$ be a class of objects in $\mathcal{M}$ such that conditions **(a)** , **(b)** and **(c)** hold.

I want to prove that $\mathcal{C}$ contains the class $Cell(A)$.

$$Cell(A) \overset{?}{\subset} \mathcal{C}.$$

Let $Z \in Cell(A)$. I want to prove that $Z \overset{?}{\in} \mathcal{C}$.

I know that for every X, there is a morphism $CW_A\ X \longrightarrow X$ where $CW_A\ X \in \mathcal{C}$ such that there exists a weak equivalence

$$map_*(A, CW_A\ X) \xrightarrow{\sim} map_*(A, X)\ .$$

I apply this functor CW_A to $Z \in Cell(A)$

$$CW_A(Z) \xrightarrow{\alpha} Z\ .$$

I claim that this mapping α is a weak equivalence. In fact, I will prove that α is a homotopy equivalence.

I know that,

$$\begin{array}{c} map_*(A, CW_A\ Z) \xrightarrow{\sim} map_*(A, Z) \\ \upuparrows \\ \text{induced by } \alpha \end{array}$$

which means, by Definition 1, that α is an A-equivalence.

Now, since $Z \in Cell(A)$, by Definition 2 I have that

$$\begin{array}{c} map_*(Z, CW_A\ Z) \xrightarrow{\sim} map_*(Z, Z) \\ \upuparrows \\ \text{induced by } \alpha \end{array}$$

is a weak equivalence.

I want to prove

$$\begin{array}{c} \text{induced by } \alpha \\ \downdownarrows \\ map_*(Z, CW_A\ Z) \xrightarrow{\sim} map_*(Z, Z) \\ \beta \overset{?}{\longleftarrow} Id_Z \end{array}$$

that $\beta\colon Z \longrightarrow CW_A\ Z$ is a homotopy inverse of α

$$\left.\begin{array}{ccc} \alpha \circ \beta & \overset{?}{\simeq} & Id_Z \\ \beta \circ \alpha & \overset{?}{\simeq} & Id_{CW_A\ Z} \end{array}\right\}$$

By Definition 1 + Definition 2, I know that

$$\begin{array}{ccc} \begin{array}{c} map_*(Z, CW_A\ Z) \xrightarrow{\sim} map_*(Z, Z) \\ \upuparrows \\ \text{induced by, } \alpha\colon CW_A\ Z \longrightarrow Z \end{array} & \Longrightarrow & \begin{array}{ccc} \pi_0\ map_*(Z, CW_A\ Z) & \xrightarrow{\cong} & map_*(Z, Z) \\ \| & & \| \\ [Z, CW_A\ Z] & & [Z, Z] \end{array} \end{array}$$

Every weak equivalence induces isomorphisms into π_n , for $n \geq 1$

Therefore, I have

$$[Z, CW_A\ Z] \underset{\cong}{\xrightarrow{\alpha_*}} [Z, Z]\ .$$

But $[Id] \in [Z, Z]$ and so,

$$\begin{array}{c} [Z, CW_A\ Z] \xrightarrow{\sim} [Z, Z] \\ [\beta] \overset{\exists}{\longleftarrow} [Id] \end{array}$$

$$\exists\ \beta\colon Z \longrightarrow CW_A\ Z \qquad \text{such that} \qquad \alpha_*([\beta]) = [Id]\ .$$

But now,

$$\alpha_*([\beta]) = [Id] \iff \alpha \circ \beta \simeq Id_{CW_A\ Z}\ .$$

Moreover, β is unique up to homotopy,

Indeed, I know that any object B that can be built from A by pushouts and telescopes, which yields with A-equivalences

$$map_*(B, CW_A\ Z) \xrightarrow{\sim} map_*(B, Z)\ .$$

In particular for $B = CW_A\ Z$,

$$\begin{array}{rcl} map_*(CW_A\ Z, CW_A\ Z) & \xrightarrow{\sim} & map_*(CW_A Z, Z) \\ \beta \circ \alpha & \mapsto & \alpha \circ \beta \circ \alpha \simeq \alpha \\ & & \uparrow \\ & & \alpha \circ \beta \simeq Id_{CW_A Z} \\ Id_{CW_A\ Z} & \mapsto & \alpha \circ Id_{CW_A Z} = \alpha \end{array}$$

Hence, $\beta \circ \alpha \simeq Id_Z$ as desired. □

Remark 33. *Recall that, generally speaking, if $\mathcal{M}$ is a model category and X, Y are objects in $\mathcal{M}$, then an element of $[X, Y]$ can be represented by a morphism $f\colon X \longrightarrow Y$ in $\mathcal{M}$ if X is cofibrant and Y is fibrant.*

Otherwise, an element of $[X, Y]$ is represented by a sort of ig-zag diagram, taking QX a cofibrant approximation for X and RY a fibrant approximation for Y,

$$\begin{tikzcd} X & Y \arrow[d, "\sim"] \\ QX \arrow[u, "\sim"] \arrow[r, "f"'] & RY \end{tikzcd}$$

since this is precisely the way how the homotopy category $Ho\ \mathcal{M}$ is constructed (where $[X, Y]$ is the set of morphisms from X to Y in $Ho\ \mathcal{M}$).

Remark 34. *I observe that in the construction of CW_AX I have used the small object argument. The small object argument allways offers functorial factorizations.*

As I know, in particular in such context the cylinders are not, in general, functors. They are determined from a not necessarily unique factorization of a morphism into a cofibration followed by a weak equivalence and such a factorization need not be canonical.

Of course if I am working in a cofibrantly generated model category, a functorial factorization may not be the one given by the small object argument, even if this one is always available.

Instead of this, the construction of CW_AX will not be available on those model categories which do not admit functorial factorizations.

There are interesting examples of model categories without functorial factorizations. In fact William Gerard Dwyer proved that the category of bounded chain complexes of finitely generated abelian groups with the standard (Quillen) model structure does not have functorial factorizations. It seems that there is no functorial cylinder in this category. On the other hand Daniel C. Isaksen has given in [9] a strict model structure in the category of pro-objects of a proper model category. In such structure the factorizations are not functorial.

A-Cellular Whitehead's Theorem.

Theorem 22. *Let $\mathcal{M}$ be a pointed model category.*

Let A a cofibrant object in $\mathcal{M}$. Let X, Y be two fibrant A-cellular objects in a $\mathcal{M}$.

Let $f\colon X \longrightarrow Y$ be a morphism in $\mathcal{M}$. Then if f is an A-equivalence, f is a homotopy equivalence.

Proof. I want to show that f has an homotopy inverse.
I consider

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& Y \arrow[ul, dashed] \arrow[ur, dashed] &
\end{tikzcd}$$

I have

$$\left.\begin{array}{r} f \text{ is an } A\text{-equivalence} \\ Y \text{ is } A\text{-cellular} \end{array}\right\} \quad \Longrightarrow \qquad map_*(Y,X) \xrightarrow[\sim]{f_*} map_*(Y,Y)$$

But now,

$$\begin{array}{c} \Downarrow \\ \text{induced by } f \\ \downdownarrows \end{array}$$

$$\begin{tikzcd}
\pi_0\ map_*(Y,X) \arrow[r, "\cong"] \arrow[d, equal] & map_*(Y,Y) \arrow[d, equal] \\
{[Y,X]} & {[Y,Y]}
\end{tikzcd}$$

But $[Id] \in [Y,Y]$ and so,

$$\begin{array}{rcl} [Y,X] & \xrightarrow{\sim} & [Y,Y] \\ [g] & \overset{\exists}{\longleftarrow} & [Id] \end{array}$$

$[g]$ is represented by a morphism

$$\begin{tikzcd}
Y \arrow[r, "g"] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

such that $f_*[g] = [Id]$ and so, $f \circ g \simeq Id_X$.
It remains to show that $g \circ f \overset{?}{\simeq} Id_Y$.
I consider now

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& X \arrow[ul, dashed] \arrow[ur, dashed] &
\end{tikzcd}$$

I have

$$\left.\begin{array}{r} f \text{ is an } A\text{-equivalence} \\ X \text{ is } A\text{-cellular} \end{array}\right\} \quad \Longrightarrow \qquad map_*(X,X) \xrightarrow[\sim]{f_*} map_*(X,Y)$$

But now,

$$\begin{array}{c} \Downarrow \\ \text{induced by } f \\ \downdownarrows \end{array}$$

$$\begin{tikzcd}
\pi_0\ map_*(X,X) \arrow[r, "\cong"] \arrow[d, equal] & \pi_0\ map_*(X,Y) \arrow[d, equal] \\
{[X,X]} & {[X,Y]}
\end{tikzcd}$$

Now, by the previous part, I know that there exists $[g]$, represented by a morphism

$$\begin{tikzcd}
Y \arrow[r, "g"] & X \\
X \arrow[u, "f"] &
\end{tikzcd}$$

such that $f_*[g] = [Id]$ and so, $f \circ g \simeq Id_X$

But I know $[X,X] \overset{\cong}{\longrightarrow} [X,Y]$

$$f \circ g \simeq Id_X$$
$$\Downarrow$$

So, in one hand, $[g \circ f] \longrightarrow f_*[g \circ f] = [f \circ g \circ f] = [f]$

But $[Id] \in [X,X]$ and so, on the other hand, $[Id] \longrightarrow f_*[Id] = [f]$

Therefore, $[g \circ f] = [Id]$ and hence, $g \circ f \simeq Id_Y$

□

A-Cellular Model Category

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When studying topological spaces, concern is placed on CW complexes up to homotopy equivalence.

One way to make such considerations is by using the Quillen model category structure on the category of topological spaces. While CW complexes are very useful, I have showed how I could also consider the spaces obtained by building with other A-cellular objects besides the zero sphere (and the associated homotopy colimits). The general approach towards cellularization was developed by Emmanuel Dror Farjoun [2] in the context of pointed simplicial sets and pointed topological spaces. The framework in the context of general pointed closed model categories was developed systematically by Philip Hirschhorn [7]. An independent account along similar lines was also given by Alexander Nofech [12].

In the initial generalization to other A-cellular spaces due to Emmanuel Dror Farjoun in [2], the A-cellular spaces are the cofibrant objects in the model category structure on pointed topological spaces associated to a pointed cofibrant object A (see Definition 35) much like pointed CW complexes are the cofibrant objects in the standard model category structure on pointed topological spaces.

Also highly brilliant and compelling is the work due to Alexander Nofech in [12]. Nofech defines this localized model category structure in more generality and starts with any pointed closed simplicial model category (not necessarily pointed topological spaces). On that work, for a given cofibrant object A of a closed simplicial model category $\mathcal{M}$, a new closed model category structure is defined, in which cofibrant objects are A-cellular.

In particular, if $\mathcal{M}$ is the category of pointed topological spaces (Top_*) or pointed simplicial sets ($sSet_*$), then the cofibrations are relative A-cellular complexes and their retracts, and weak equivalences are maps that induce isomorphisms on A-homotopy. When $A = \mathbb{S}^0$ this specializes to Quillen's model category structure.

Only in this final part and in order to simplify and moreover clarify the definition, I will use the following particular notation (following that of Alexander Nofech):

$hom(X,Y)$ denotes the simplicial function complex in $\mathcal{M}$, $sSet$ is the usual category of simplicial sets and the subscript f denotes fibrant approximation, so that if $\varphi\colon X \longrightarrow Y$ is a morphism in the category $\mathcal{M}$, then $\varphi_f\colon X_f \longrightarrow Y_f$ is the induced morphism between the fibrant approximations X_f and Y_f of the objects X and Y respectively.

Definition 35. *(A-Cellular Model Category)*

Let $\mathcal{M}$ be a pointed closed simplicial model category and A be a cofibrant object of $\mathcal{M}$. Let W_{sSet} denote the class of weak equivalences of simplicial sets, $F_{\mathcal{M}}$ denote the class of fibrations in $\mathcal{M}$ and $C_{\mathcal{M}}$ denote the class of cofibrations in $\mathcal{M}$.

An A-cellular closed model category structure denoted by $\mathcal{M}^A$ on the underlying category of $\mathcal{M}$ is a closed model category structure where the classes $(W_{\mathcal{M}^A}, F_{\mathcal{M}^A}, C_{\mathcal{M}^A})$ of weak equivalences, fibrations and cofibrations of $\mathcal{M}^A$ are defined as follows:

(1) $W_{\mathcal{M}^A} : \; = \{ \; \varphi\colon hom(A, \varphi_f) \in W_{sSet} \; \}$,

(2) $F_{\mathcal{M}^A} = F_{\mathcal{M}}$,

(3) $F_{\mathcal{M}^A} = \{ \; \psi \; : \; \psi$ *has the left lifting property with respect to* $(W_{\mathcal{M}^A} \cap F_{\mathcal{M}^A}) \; \}$.

The cofibrations and weak equivalences of $\mathcal{M}^A$ are my well known A-cofibrations and A-equivalences respectively. Any weak equivalence in $\mathcal{M}$ is an A-equivalence for any cofibrant object A of $\mathcal{M}$.

In the presence of a set of generators of trivial cofibrations in $\mathcal{M}$, that is a set of trivial cofibrations $\{t_j\}$ such that a morphism φ is a fibration if and only if any of the t_j has the Left Lifting Property with respect to φ, also in [12], Nofech proved the following rather general theorem for the existence of A-cellular closed model category structures, which I will just state.

Theorem 23. *Let $\mathcal{M}$ be a pointed proper simplicial closed model category with arbitrary colimits having a set t_j of generators of trivial cofibrations and let A be a cofibrant, s-definite object of $\mathcal{M}$. Then there exists an A-cellular closed model category structure $\mathcal{M}^A$ admitting functorial factorizations.*

There are several questions in various contexts that can be asked about this category. For instance:

What model category structure is obtained from the right intersection of two of these A-cellular model categories?

For this I will consider the most simple case when $\mathcal{M}$ is the category of pointed topological spaces Top_*. Then, I have the following proposition.

Proposition 36. *Let A and A' be two cofibrant pointed topological spaces. Then $Top_*^A \cap Top_*^{A'} = Top_*^{A\vee A'}$.*

Proof. I need to show that the model category $Top_*^{A\vee A'}$ can be described via the model category $Top_*^A \cap Top_*^{A'}$, but this amounts to showing that the weak equivalences agree.

Given a map $\varphi\colon X \longrightarrow Y$, then $\varphi_f\colon X_f \longrightarrow Y_f$ is the induced morphism between the fibrant approximations X_f and Y_f of the objects X and Y respectively.

I would like to show that both

$$\varphi_f^A\colon hom(A, X_f) \longrightarrow hom(A, Y_f)$$

$$\varphi_f^{A'}\colon hom(A', X_f) \longrightarrow hom(A', Y_f)$$

are weak equivalences if and only if $\varphi_f^{A\vee A'}\colon hom(A\vee A', X_f) \longrightarrow hom(A \vee A', Y_f)$ is a weak equivalence.

I see that since $hom(A\vee A', \bullet) \cong hom(A, \bullet) \times hom(A, \bullet)$, and both φ_f^A and $\varphi_f^{A'}$ are weak equivalences, then $\varphi_f^{A\vee A'}$ must also be a weak equivalence.

Now, I have that A and A' are both retracts of $A \vee A'$ and so I obtain that φ_f^A and $\varphi_f^{A'}$ are both retracts of $\varphi_f^{A\vee A'}$. This implies that if $\varphi_f^{A\vee A'}$ is a weak equivalence, and hence φ_f^A and $\varphi_f^{A'}$ must be weak equivalences too. □

Conclusions

Part .

There is a large amount of articles involving model categories, most of them exploring the general theory or application cases based on the keynote examples (topological spaces, simplicial Sets, chain complexes of R-modules and at most simplicial model categories). However, not a single one is entirely developed for arbitrary model categories.

The reason is rather clear: in the framework of arbitrary model categories, definitions and statements show their real nature and this fact adds much complexity.

As an instance, only references to concepts as homotopy colimits (limits) or mapping spaces (both of them essential and unavoidable) may result in a serious fattened bulk of prerequisites. Indeed,

- (•) in one hand homotopy colimits (limits) admit different approaches to be defined, all of them interrelated, and those relations are not intuitive at all.
- (•) and on the other hand the definition of mapping spaces involves the notion of nerve. Nerves of categories are simplicial sets, so a convenient presentation of these is needed. In fact up to now, the author cannot be able to find a single complete definition, or merely an intuitive approach to mapping spaces in the environment of arbitrary model categories.

Depending on the environment structure to deal with, each author will accommodate the definitions which in the basic cases previously referenced admit simplified definitions.

Of course it is not merely a question of including two or three new concepts. The natural habitat, the real jungle of model category theory, then shows an overflowing brilliance. Once the door is opened for both of them, the rest of the animals will show their complete heads (neither any partial ones, nor the accommodated versions) and will be ready to play (weak factorization systems, cofibrantly model categories, proper model categories, locally presentable model category, Reedy categories, adjointness, Quillen equivalences...).

As a result, the gap between the notions and knowledge required for complete developments in particular model categories is enormous when compared with those required to offer statements and proofs for arbitrary model categories.

Consequently, only partial references to results in arbitrary cases are offered by some authors and always in a higher conceptual environment:

- (•) for instance in higher homotopy theory (given that the category of model categories is a 2-category)
- (•) or exploring very special cases as those model categories not admitting functorial factorizations.

On the contrary, this line of work based on a general framework for arbitrary model categories could be offered, and what is more, showing absolutely friendly constructive arguments in many cases. Of course friendly is understood in the fattened framework of model category theory.

Once the initial fear to that wild environment for arbitrary model categories is overcome and beaten, the student will be aware of the benefits, since these are clear and straightforward.

- (•) Right understanding of how the initial conditions in the definitions and theorems are playing and why they are needed.
- (•) Right management of the homotopical techniques (the small object argument, approximation or localization functors, replacements, arguments involving fibrancy and cofibrancy, transfinite induction processes).

- (•) Proper understanding about concepts so relevant as mapping spaces or homotopy colimits (limits).
- (•) A huge amount of knowledge acquired. Moreover this knowledge is qualitatively outstanding, allowing the student to manage a wider kit of tools and powerful arguments that will be useful in future developments.
- (•) And many others.

Of course, all these notions are rather clear for experienced topologists while they remain unaffordable for students. When they try to climb the ladder of this theory, the lack of suitable information is extensive; affordable references are barely present and they return to the comfortable range of topological spaces, simplicial sets, chain complexes of R-modules or simplicial model categories.

This work has been made by a student to help students who want to go ahead and who are looking forward to hearing from deep literature solving the requirements of arbitrary model categories, including complete definitions and statements, besides practical constructions (mainly establishing constructive arguments), examples and full details about why requirements and conditions in the statements and proofs are included.

Part 5. Bibliography

Part 5.

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