Master's Final Project

MASTER'S DEGREE IN ADVANCED MATHEMATICS

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ON A THEOREM OF WARING FOR PLANE ALGEBRAIC CURVES

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Abstract. This project consists in a revision and extension of a classic result, Waring's theorem, about the barycenter of the intersection points of two plane algebraic curves. The theorem arises from the study of the parts with highest degree of the equation of a curve, which are completely determined by the barycentric parallel lines of the groups of asymptotes.

Among other consequences of Waring's theorem we study a result, due to Chasles, about the barycenter of the contact points of parallel tangent lines to a plane curve.

Keywords: Waring's theorem, barycenter, barycentric parallel line, asymptotes of plane algebraic curves, Chasles theorem.

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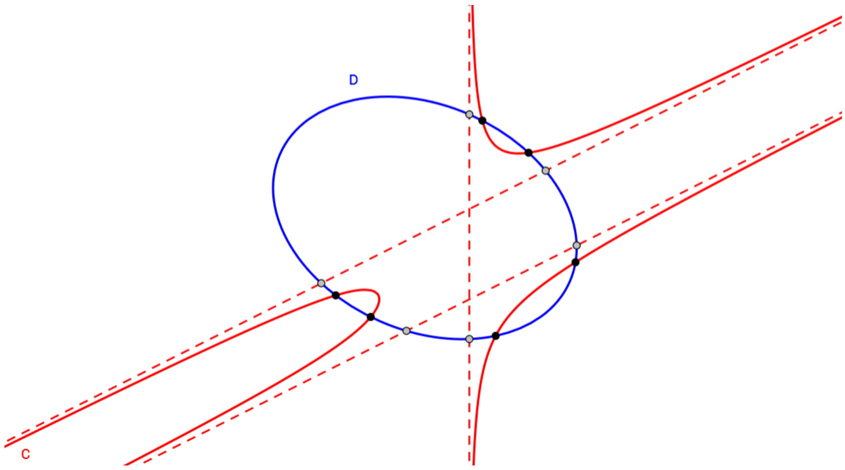
Introduction

Historical review of Waring's theorem

A classic result about the intersection of plane algebraic curves, due to Edward Waring, asserts:

Waring's theorem. *Let C and D be two plane algebraic curves, with no common points in the line at infinity. Then, the barycenter of the points of $C \cap D$ (counted according to intersection multiplicities) is the barycenter of the intersection points when C is replaced by the union of its asymptotes.*

For example, in the following figure, if the cubic curve C is replaced by the union of its three asymptotes, the grey intersection points with the conic D have the same barycenter as the black intersection points of C and D:



Waring's theorem appeared for the first time in the treatise *Proprietates algebricarum curvarum*, whose first edition was published in 1762.

Liouville's article [8], published in 1841, contains a proof of Waring's theorem applying elimination to the variables involved in the equations of the curves.

By means of these elimination methods, Liouville also gave a proof for a nice theorem about the parallel tangent lines to a plane curve. This theorem, together with an analog version for algebraic surfaces and parallel tangent planes to them, is due to Chasles and had been published in *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (1837). The original proof of Chasles consisted in modifying Newton's theorem on the diameters of curves and surfaces. Some years later, Chasles gave a different proof in his *Traité de géométrie supérieure* (1851).

During the rest of 19th century, several articles appeared with simplifications of Liouville's arguments, such as [5] and [10]. All these proofs were done under the (non-explicit) hypothesis of the curve C intersecting transversely the line at infinity, that is, in the same number of distinct points as its degree.

In 1931, Coolidge published the book [4], a very complete recopilation of results on plane algebraic curves. The chapter devoted to their metric properties contains a proof of Waring's theorem following the ideas in [10], this time with the explicit mention of the curve C intersecting transversely the line at infinity. In the same chapter, we may also find Chasles theorem and lots of properties on distances, angles and foci of algebraic curves, most of them forgotten nowadays.

Finally, the paper [7], published in 1995, gave a revision of Waring's theorem that covers the case of curves C with no parabolic branches[1], but restricts D to be a line. The goal of the article was proving a generalization of Waring's theorem for the intersection of algebraic curves with hyperplanes in affine spaces of higher dimension.

The project

The aim of this project is doing a revision, including a proof written according to modern standards and a specification of the hypothesis, of Waring's theorem for its different versions appearing in the literature. As an application, it includes a proof of the classic Chasles theorem for curves.

Since the publication of Liouville's article [8], the most common argument for proving Waring's theorem is divided into two parts:

- On the one hand, the barycenter of the intersection points of two affine curves only depends on the two highest homogeneous parts of (the equations of) the curves. This result appears in the text as theorem 2.4, and it can be proved by using elimination theory.
- On the other hand, the two highest homogeneous parts of a curve are determined by its asymptotes.

In the study of this second part, we present a geometric equivalence (theorem 2.1) to the fact of two curves having identical two highest homogeneous parts, which has not been found in the literature. As a consequence, we deduce the coincidence of asymptotes of two curves as a sufficient condition for the coincidence of their two highest homogeneous parts. Actually, this is also a necessary condition in the classic case of curves intersecting transversely the line at infinity, but not for the more general case of curves with no parabolic branches.

This geometric equivalence, together with the first part of Liouville's argument, allows us to find an extension of Waring's theorem, where the curves may be replaced by a more general type of curves than an union of asymptotes (theorem 2.5).

The necessary background for this project has been summarized in the first chapter. There, we review basic concepts and results on the theory of plane algebraic curves, which were worked in the Master's course: singular points, tangent cones, intersection multiplicities, Bézout theorem, polar curves, ... It also includes basics on pencils of curves and groups of points, which are the basic tool in one of the proofs of theorem 2.1.

[1] That is, having not the line at infinity as a tangent line.

Chapter 1

Preliminaries

In this chapter, we explain all the concepts on plane algebraic curves that are necessary for the project, in order to fix ideas and notations.

Most of the contents (with the exception of groups of points and pencils) were part of the Master's course *Algebraic curves*, and because of this reason many proofs have not been included. The reader is referred mainly to [2], even though other detailed expositions may be found in classic books as [6] and [9].

1.1 Basic considerations

Throught these pages, $\mathbb{P}^n$ will denote the projective n-dimensional space over the field of the complex numbers, with an already fixed system of homogeneous coordinates. The point of $\mathbb{P}^n$ with homogeneous coordinates $x_0, \ldots, x_n$ will be denoted by $(x_0 : ... : x_n)$.

We will consider algebraic hypersurfaces in $\mathbb{P}^n$, defined by equations $F = 0$, with F an homogeneous polynomial in the homogeneous coordinates. Since any polynomial $F \in \mathbb{C}[x_0, x_1]$ of degree d decomposes as the product of d linear factors, hypersurfaces in $\mathbb{P}^1$ are called *groups of points*. The *multiplicity* of a point in a group is just the multiplicity of the associated linear polynomial, as a factor of F.

We will refer to the hypersurfaces in $\mathbb{P}^2$ as *algebraic curves*. A curve $C : F = 0$, with $F = F_1 \ldots F_r$ the product of (possibly repeated) r polynomials, is called the *union* of the curves $C_i : F_i = 0$ ($i = 1, \ldots, r$), and is denoted by $C = C_1 + \ldots + C_r$.

In case that the polynomials F_i are irreducible, each of the curves C_i is called an *irreducible component* of C, with *multiplicity* the multiplicity of F_i as a factor of F. The curve C is said to be *reduced* if all its irreducible components have multiplicity 1.

Recall that, since every point in $\mathbb{P}^n$ has a non-zero homogeneous coordinate, the space $\mathbb{P}^n$ can be covered by the subsets $U_i = \{(x_0 : ... : x_n) \in \mathbb{P}^n : x_i \neq 0\}$ ($i = 0, \ldots, n$). Identifying the point $(x_1, \ldots, x_n)$ (in affine coordinates) with the point $(x_1 : \ldots : x_{i-1} : \underset{i)}{1} : x_i : \ldots : x_n)$ embeds the (complex) affine n-dimensional space $\mathbb{A}^n$ in $\mathbb{P}^n$: the subspace U_i is called the *i-th affine chart*.

Our frame of work will be a complex affine plane $\mathbb{A}^2$, which we consider embedded in $\mathbb{P}^2$ by means of the 0-th affine chart. In other words, we identify the projective point $(x_0 : x_1 : x_2) \in U_0$ with $(x, y) = (\frac{x_1}{x_0}, \frac{x_2}{x_0})$. Recall that the complementary of $\mathbb{A}^2$, when regarded as a subset of $\mathbb{P}^2$, is the *line at infinity* $l_\infty : x_0 = 0$ (also called the *improper line*).

We will deal with algebraic curves in $\mathbb{A}^2$, defined by equations $f = 0$, with f a polynomial (not necessarily homogeneous) in the affine coordinates. Similar considerations relative to irreducible components hold for these curves in $\mathbb{A}^2$.

If $f(x,y) = 0$ is the equation of a curve C in $\mathbb{A}^2$ with degree d, the homogeneous polynomial $F(x_0,x_1,x_2) = x_0^d \cdot f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$ of degree d defines an algebraic curve in $\mathbb{P}^2$ (*projective closure* of C).

Conversely, If $C : F(x_0,x_1,x_2) = 0$ is an algebraic curve in $\mathbb{P}^2$ with some irreducible component different of l_∞, the polynomial $f(x,y) = F(1,x,y)$ is not constant and defines an algebraic curve in $\mathbb{A}^2$, called the *0-th affine part* of C. Similarly, one can define affine parts of C in the remaining affine charts. If no confusion arises, the 0-th affine part will be called simply the *affine part* of the curve.

When dealing with curves C in $\mathbb{P}^2$ having not l_∞ as an irreducible component, the maps "affine part" and "projective closure" are inverse to each other. Furthermore, the irreducible components of the affine part of C are the affine parts of the irreducible components of C, with the multiplicity of each irreducible component of C and that of its affine part coinciding.

Therefore, we usually make no distinction between a plane curve in $\mathbb{A}^2$ and its projective closure in $\mathbb{P}^2$.

1.2 Singular points of an algebraic curve

Let $C : F(x_0,x_1,x_2) = 0$ be an algebraic curve of degree d, and l a line in $\mathbb{P}^2$. The line l may be parametrized by means of a map $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ defined as

$$\varphi((t_0 : t_1)) = (c_0^0 t_0 + c_0^1 t_1 : c_1^0 t_0 + c_1^1 t_1 : c_2^0 t_0 + c_2^1 t_1) ,$$

with the c_i^j ($i \in \{0,1,2\}$, $j \in \{0,1\}$) denoting complex numbers such that the matrix $\begin{pmatrix} c_0^0 & c_1^0 & c_2^0 \\ c_0^1 & c_1^1 & c_2^1 \end{pmatrix}$ has rank 2.

If the polynomial $\overline{F}(t_0,t_1) = F(c_0^0 t_0 + c_0^1 t_1, c_1^0 t_0 + c_1^1 t_1, c_2^0 t_0 + c_2^1 t_1)$ is identically zero, every point in the line l satisfies the equation of C, and hence $l \subset C$.

Otherwise, $\overline{F} \in \mathbb{C}[t_0,t_1]$ is homogeneous of degree d and may be regarded as the equation of a group of points in l (the *intersection* of C and l, denoted by $C \cdot l$).

Definition. The *intersection multiplicity* of C and l at a point $p \in \mathbb{P}^2$, denoted by $[C \cdot l]_p$, is the multiplicity of p in the group of points $C \cdot l$.

Example. The line $l_\infty : x_0 = 0$ admits the parametrization $\varphi((t_0 : t_1)) = (0 : t_0 : t_1)$. If $C : F(x_0,x_1,x_2) = 0$ is a curve having not l_∞ as an irreducible component (for example, if C is the projective closure of an affine algebraic curve), the group of points $C \cdot l_\infty$ determined by $\overline{F}(t_0,t_1) = F(0,t_0,t_1)$ is called the *improper section* of C. Its points are called the *improper points* of C.

Definition. For any point $p \in \mathbb{P}^2$, we define the *multiplicity of p on C*, denoted by $e_p(C)$, as the minimal value of the intersection multiplicities of C and all the lines through p, at the point p.

From this definition, it immediately follows that $e_p(C) = 0$ if, and only if, $p \notin C$. Otherwise:

- If $e_p(C) = 1$, p is called a *simple* (or *smooth*, or *non-singular*) point of C.
- If $e_p(C) > 1$, p is called a *singular* point (or a *singularity*) of C.

An important property is that, for any two algebraic curves $C_1, C_2 \subset \mathbb{P}^2$, $e_p(C_1 + C_2) = e_p(C_1) + e_p(C_2)$. Hence, the singularities of $C_1 + C_2$ consist of the singularities of C_1, the singularities of C_2 and the points of $C_1 \cap C_2$.

The most common characterization of the multiplicity of points on curves is provided by:

Theorem 1.1. *A point p has multiplicity e on a curve $C : F(x_0, x_1, x_2) = 0$ if, and only if, all derivatives of F of order $e-1$ have value 0 at p, and at least one of the e-th derivatives has not. In particular, p is singular if and only if all the first order derivatives of F vanish at p.*

Definition. A line $l \subset \mathbb{P}^2$ is said to be *tangent* to C at a point p if $[C \cdot l]_p > e_p(C)$.

Example. The tangent lines to a curve at some of its improper points are called *asymptotes*.

Another way of finding the multiplicity of a point p on C, as well as the tangent lines to C at p, is the next one: if the polynomial F has degree d, express it as a sum

$$F(x_0, x_1, x_2) = f_d(x_1, x_2) + f_{d-1}(x_1, x_2) \cdot x_0 + \ldots + f_{e+1}(x_1, x_2) \cdot x_0^{d-e-1} + f_e(x_1, x_2) \cdot x_0^{d-e} \qquad (*)$$

of homogeneous polynomials $f_i \in \mathbb{C}[x_1, x_2]$ of degree i ($e \leq i \leq d$), with $f_e \neq 0$. Then, if $p = (1:0:0)$, we have $e_p(C) = e$ and the algebraic curve

$$TC_p(C) : f_e(x_1, x_2) = 0$$

(*tangent cone of C at p*) is the union of the e (possibly repeated) tangent lines to C at p. These notions are generalized to arbitrary points $p \in \mathbb{P}^2$ by means of changes of coordinates.

Remarks.

1. The *multiplicity* of a tangent line l to C at p is just the multiplicity of l as an irreducible component of $TC_p(C)$.
2. In case that p is a smooth point of the curve C (that is, $e_p(C) = 1$), the tangent cone of C at p consists of the single line
$$\frac{\partial F}{\partial x_0}(p) \cdot x_0 + \frac{\partial F}{\partial x_1}(p) \cdot x_1 + \frac{\partial F}{\partial x_2}(p) \cdot x_2 = 0\,,$$
which is called the *tangent line to C at p*.

Definition. The homogeneous polynomials $f_d, f_{d-1} \in \mathbb{C}[x_1, x_2]$ appearing in the expression $(*)$, will be called the *two highest homogeneous parts* (in the equation) of the curve $C : F(x_0, x_1, x_2) = 0$.

We finish this part reviewing some of the most elementary sort of singularities of a curve:

Definition. Let p be a singular point of a curve C, and $e = e_p(C) > 1$. Then:

1. p is an *ordinary singularity* of C if the tangent cone $T_p(C)$ is the union of e distinct lines.
2. p is a *node* if it's an ordinary singularity and $e = 2$.
3. p is an *ordinary cusp* if $e = 2$ and $T_p(C)$ consists of a line l counted twice, such that $[C \cdot l]_p = 3$.

1.3 Branches of a plane curve and intersection multiplicity

Let's start by recalling two basic concepts about fractionary power series:

- If $s \in \mathbb{C}\left[\!\left[x^{1/n}\right]\!\right]$ is a fractionary power series, its *polydromy order* $\nu(s)$ is the minimal common denominator of all the exponents effectively appearing in s. That is, the least number m such that $s \in \mathbb{C}\left[\!\left[x^{1/m}\right]\!\right]$.
- If ε is an n-th root of unity, we have an automorphism of rings $\sigma_\varepsilon : \mathbb{C}\left[\!\left[x^{1/n}\right]\!\right] \longrightarrow \mathbb{C}\left[\!\left[x^{1/n}\right]\!\right]$ (*conjugation*) given by $\sigma_\varepsilon\left(\sum_{i\geq 0} a_i x^{i/n}\right) = \sum_{i\geq 0} a_i \varepsilon^i x^{i/n}$.

Now, let $C : F(x_0, x_1, x_2) = 0$ be a curve in $\mathbb{P}^2$ passing through $O = (1:0:0)$. In order to study C "near" the point O, we take the affine equation

$$C : f(x,y) = 0$$

with $f(x,y) = F(1,x,y)$. We also assume that the coordinates are chosen in such a way that the y-axis $x = 0$ is not tangent to C at $O = (0,0)$.

Theorem 1.2 (Puiseux). *There exist convergent fractionary power series $s_i(x)$ ($1 \leq i \leq k$) with polydromy order $\nu(s_i) = \nu_i$, such that $s_i(0,0) = 0$ and f factorizes in $\mathbb{C}\{x\}[y]$ as*

$$f = g_{s_1} \cdot \ldots \cdot g_{s_k} \cdot \widetilde{f}$$

with $\widetilde{f} \in \mathbb{C}\{x\}[y]$ such that $\widetilde{f}(0,0) \neq 0$ and $g_{s_i} = \prod_{\varepsilon^{\nu_i}=1} (y - \sigma_\varepsilon(s_i)) \in \mathbb{C}\{x\}[y]$.

Definition. Every set $\left\{(x,y) \in \mathbb{A}^2 : g_{s_i}(x,y) = 0\right\}$ is called a *branch* of C at the point O.

The *multiplicity* of the branch $g_{s_i}(x,y) = 0$ is the multiplicity of g_{s_i} as a factor of f. Any of the conjugate series $\sigma_\varepsilon(s_i)$ corresponding to that branch is called a *Puiseux series* of the branch. By a *Puiseux series* of the curve C at the point O, we will simply mean a Puiseux series of a branch of C at O.

Note that the equation $g_{s_i}(x,y) = 0$ defines an analytic curve (not necessarily algebraic) in a neigbourhood of O, admitting the parametrization $x = t^{\nu_i}$, $y = s_i(t^{\nu_i})$. Moreover, it has a single tangent line at O, given by

$$y = a + bx$$

if the Puiseux series s_i is $s_i(x) = a + bx + \ldots$, with $\ldots$ denoting terms of degree strictly higher than 1 in x.

Tangent lines to branches of C at the point O are the irreducible components of the tangent cone of C at O.

The same considerations hold for any projective point $p \in C$, just by taking the appropiate affine part and an appropiate affine change of reference. This allows us to decompose completely the polynomial f, as it may be found in proposition 3.3.3 of [2]:

Proposition 1.3. *Let $C : F(x_0,x_1,x_2) = 0$ be an algebraic curve passing not through $(0:0:1)$. Then, the y-roots of the polynomial $f(x,y) = F(1,x,y)$ are the Puiseux series of C at the points of C on the y-axis: that is,*

$$f(x,y) = a \cdot \prod_{s_i} (y - s_i(x))$$

where $a \in \mathbb{C}$ and the product is taken over all the Puiseux series s_i of C at the points of C on the y-axis.

Definition. Let $p \in C \cap l_\infty$ be an improper point of C. A branch of C at p is said to be *parabolic* if its tangent line at p is l_∞.

Using this notion of branch, we are going to define the intersection multiplicity of two curves. Let $C : f(x,y) = 0$ be an affine algebraic curve, $p \in \mathbb{A}^2$ and C' another affine curve, with a branch γ at p. Assume that γ is given by a Puiseux series s, with polydromy order $\nu = \nu(s)$.

Definition. The *intersection multiplicity* of the curve C and the branch γ is $[C \cdot \gamma] = \nu \cdot o_x f(x, s(x))$.

Definition. The *intersection multiplicity* of two curves C and C' at a point p is $[C \cdot C']_p = \sum_\gamma [C \cdot \gamma]$, with the sum over all the branches γ of C' at p, each of them counted as many times as its multiplicity.

Remarks.

- Naturally, this local notion extends to projective curves C, C' and points $p \in \mathbb{P}^2$, just by considering the affine parts of C and C' corresponding to an affine chart containing p.
- In case that C' is a line, the definition of intersection multiplicity coincides with the one given in the previous section.

The most important properties of the intersection multiplicity are summarized in the following proposition:

Proposition 1.4. *Let C, C', C_1 and C_2 be plane affine curves, and $p \in \mathbb{A}^2$ a point. Then:*

1. $[(C_1 + C_2) \cdot C']_p = [C_1 \cdot C']_p + [C_2 \cdot C']_p$.
2. $[C \cdot C']_p = 0 \iff p \notin C$ *or* $p \notin C'$.
3. $[C \cdot C']_p = \infty \iff C$ *and* C' *have a common irreducible component passing through* p.
4. $[C \cdot C']_p = [C' \cdot C]_p$.
5. $[C \cdot C']_p \geq e_p(C) \cdot e_p(C')$, *with equality if and only if* C *and* C' *have no common tangent line at* p.
6. *If* $C_1 : f = 0$, $C_2 : g = 0$ *and* $C' : af + g = 0$ *for some* $a \in \mathbb{C}[x,y]$, *then* $[C_1 \cdot C_2]_p = [C_1 \cdot C']_p$.

Example. Two algebraic curves C and C' are said to have a *transverse intersection* at a point p if $[C \cdot C']_p = 1$. Equivalently, by properties 1 and 5 of the proposition above, if p is a smooth point of both C and C' where the curves have different tangent lines.

Bézout theorem is probably the most known result on the intersection of plane curves. We take the same statement as in [2] (theorem 3.3.5):

Theorem 1.5 (Bézout). *Let $C, C' \subset \mathbb{P}^2$ be two algebraic curves of respective degrees d and e, with no common irreducible components. Then,*

$$\sum_{p \in \mathbb{P}^2} [C \cdot C']_p = d \cdot e$$

Definition. The *intersection group* of two algebraic curves $C, C' \subset \mathbb{P}^2$ with no common irreducible components is the formal sum

$$\sum_{p \in \mathbb{P}^2} [C \cdot C']_p \cdot p$$

(which is finite according to Bézout theorem).

1.4 Polar curves and polar groups

In this section, we are going to review the basic concepts on polar curves, and introduce the polar groups. Actually, the notion of polarity is general for hypersurfaces in any projective space $\mathbb{P}^n$, but for our purposes we may assume that $n = 1, 2$:

Definition. Let $V : F(x_0, \ldots, x_n) = 0$ and $q = (a_0 : \ldots : a_n)$ be, respectively, an hypersurface of degree $d > 1$ and a point in $\mathbb{P}^n$. We define the *polar of V with respect to q* as the hypersurface

$$\mathscr{P}_q(V) : a_0 \cdot \frac{\partial F}{\partial x_0} + \ldots + a_n \cdot \frac{\partial F}{\partial x_n} = 0$$

when the polynomial on the left-hand side of the equation is not identically 0. Otherwise, we say that the polar of V with respect to q is *undetermined* (or *undefined*).

Remark. The polar of V with respect to q is undetermined if, and only if, q is a point with multiplicity d on V, that is, V is a union of hyperplanes passing through q.

$\mathscr{P}_q(V)$ is an hypersurface of degree $d-1$, whose definition does not depend on the choice of coordinates. Taking successive polars (if possible), we obtain hypersurfaces $\mathscr{P}_q^2(V), \ldots, \mathscr{P}_q^{d-2}(V), \mathscr{P}_q^{d-1}(V)$ with respective degrees $d-2, \ldots, 2, 1$.

Each hypersurface $\mathscr{P}_q^r(V)$ is called the *r-th polar of V with respect to q*, has degree $d-r$ (if determined) and admits the equation

$$\mathscr{P}_q^r(V) : \left(a_0 \cdot \frac{\partial}{\partial x_0} + \ldots + a_n \cdot \frac{\partial}{\partial x_n}\right)^r F(x_0, \ldots, x_n) = 0$$

This equation can also be written as

$$\mathscr{P}_q^r(V):\ \left[\left(x_0\cdot\frac{\partial}{\partial y_0}+\ldots+x_n\cdot\frac{\partial}{\partial y_n}\right)^{d-r} F(y_0,\ldots,y_n)\right]_{(y_0,\ldots,y_n)=(a_0,\ldots,a_n)}=0\,,$$

from which immediately follows the *reciprocity of polars*: for every $p,q\in\mathbb{P}^n$,

$$p\in\mathscr{P}_q^r(V)\iff q\in\mathscr{P}_p^{d-r}(V)\,.$$

Remarks.

- For the case $n=2$, the curves $\mathscr{P}_q^{d-2}(V)$ and $\mathscr{P}_q^{d-1}(V)$ with respective degrees 2 and 1, if determined, are called the *polar conic* and the *polar line* of V with respect to q.

  Looking at the equations above, one deduces that a point $q\in V$ determines a polar line if, and only if, q is a smooth point of V. In such a case, the polar line of V with respect to q is the tangent line to V at q.

- For $n=1$, the group $\mathscr{P}_q^{d-1}(V)$ has degree 1 (consists of a single point) and is called the *point polar* of V with respect to q.

Proposition 1.6. *Let $C\subset\mathbb{P}^2$ be an algebraic curve with degree $d>1$, and $q\in\mathbb{P}^2$ a point such that $\mathscr{P}_q(C)$ is determined. Then, $C\cap\mathscr{P}_q(C)$ consists of the singularities of C, and the simple points of C whose tangent line passes through q.*

The intersection multiplicities of the curves C and $\mathscr{P}_q(C)$ at their intersection points, for curves C with very particular types of singularities, can be deduced from the following lemmas, which appear in section 3.4 of [2]:

Lemma 1.7. *Under the hypothesis of proposition 1.6, if p is a simple point of C, $p\neq q$ and the line $l=p\vee q$ is not contained in C, then $[C\cdot\mathscr{P}_q(C)]_p=[C\cdot l]_p-1$.*

Lemma 1.8. *Under the hypothesis of proposition 1.6, if p is an ordinary singularity with multiplicity e on C and q is contained in no tangent line at p, then $[C\cdot\mathscr{P}_q(C)]_p=e(e-1)$.*

Lemma 1.9. *Under the hypothesis of proposition 1.6, if p is an ordinary cusp of C and the tangent line to C at p does not pass through q, then $[C\cdot\mathscr{P}_q(C)]_p=3$.*

By applying Bézout theorem with the curves C and $\mathscr{P}_q(C)$, and controlling their intersection multiplicities using these results, we obtain:

Theorem 1.10 (Plücker's first formula). *Let $C\subset\mathbb{P}^2$ be an irreducible algebraic curve, whose only singularities are δ nodes and τ ordinary cusps. If a point $q\in\mathbb{P}^2$ does not belong to C, and no tangent line to C at a singular point cointains q, then the number of tangent lines to C passing through q is*

$$d(d-1)-2\delta-3\tau\,,$$

with each tangent line l counted $\sum_{p\in C\cap l}\left([C\cdot l]_p-1\right)$ times.

We finish this section presenting two useful properties of polars of groups of points. The first one is an interpretation of the barycenter of aligned points as a point polar, while the second one states that polarity commutes with the section of a curve by a line.

Proposition 1.11. *Let $p_1,\dots,p_d$ be (possibly repeated) d affine points in $\mathbb{P}^1$. Then, the barycenter of $p_1,\dots,p_d$ is the point polar of the group $G = p_1+\ldots+p_d$ with respect to the point at infinity $p_\infty = (0:1)$.*

Proof. If the points p_i have homogeneous coordinates $p_i = (1:\alpha_i)$ in $\mathbb{P}^1$, the group $G = p_1+\ldots+p_d$ admits the equation $G: F(x_0,x_1) = 0$, with

$$F(x_0,x_1) = \prod_{i=1}^{d}(x_1-\alpha_i x_0) = x_1^d - (\alpha_1+\ldots+\alpha_d)x_1^{d-1}x_0 + \ldots$$

and the last $\ldots$ denoting terms with degree in x_1 strictly smaller than $d-1$. Since

$$\frac{\partial^{d-1}F}{\partial x_1^{d-1}} = d!\cdot x_1 - (\alpha_1+\ldots+\alpha_d)(d-1)!\cdot x_0\ ,$$

the point polar of G with respect to p_∞ is determined, and has equation

$$\mathscr{P}^{d-1}_{p_\infty}(G):\ d\cdot x_1 - (\alpha_1+\ldots+\alpha_d)\cdot x_0 = 0\ .$$

In other words, $\mathscr{P}^{d-1}_{p_\infty}(G)$ consists of the single point $(1:\dfrac{\alpha_1+\ldots+\alpha_d}{d})$, which are the homogeneous coordinates of the barycenter of $p_1,\dots,p_d$. □

Proposition 1.12. *Let q, C and l be, respectively, a point, a curve and a line in $\mathbb{P}^2$, such that $q \in l\setminus C$. Then,*

$$\mathscr{P}_q(C)\cdot l = \mathscr{P}_q(C\cdot l)$$

where the left-hand side is the group of points corresponding to the intersection of $\mathscr{P}_q(C)$ with l, and the right-hand side is the polar of the group of points $C\cdot l$ with respect to q.

Proof. We take coordinates in $\mathbb{P}^2$ in such a way that $q = (0:0:1)$, $C: F(x_0,x_1,x_2)$ and $l: x_1 = 0$.

The intersection group $C\cdot l$ is given by the equation $\overline{F}(x_0,x_2) = F(x_0,0,x_2)$, regarding x_0,x_2 as homogeneous coordinates in l, and hence

$$\mathscr{P}_q(C\cdot l):\ 0 = \frac{\partial\overline{F}}{\partial x_2} = \frac{\partial F(x_0,0,x_2)}{\partial x_2}$$

On the other hand, an equation for $\mathscr{P}_q(C)$ is $\mathscr{P}_q(C):\ \dfrac{\partial F}{\partial x_2} = 0$, which gives

$$\mathscr{P}_q(C)\cdot l:\ \frac{\partial F}{\partial x_2}(x_0,0,x_2) = 0$$

The result becomes a consequence of the equality $\dfrac{\partial F(x_0,0,x_2)}{\partial x_2} = \dfrac{\partial F}{\partial x_2}(x_0,0,x_2)$. □

1.5 Pencils of curves and pencils of groups of points

We want to study when the equations of two affine curves share their two highest homogeneous parts. Our main tool will be pencils, that is, the 1-dimensional linear varieties in the projective space of hypersurfaces of a fixed degree.

In particular, we will make use of pencils of curves (hypersurfaces in $\mathbb{P}^2$) and pencils of groups of points (hypersurfaces in $\mathbb{P}^1$).

Definition. Let $C_1, C_2 \subset \mathbb{P}^2$ be two different plane projective curves of the same degree n, given by equations $C_1 : F(x_0, x_1, x_2) = 0$ and $C_2 : G(x_0, x_1, x_2) = 0$. The *pencil* spanned (or generated) by C_1 and C_2 is the set

$$\Lambda = \left\{ D : \lambda F + \mu G = 0 \,|\, (\lambda : \mu) \in \mathbb{P}^1 \right\}$$

of curves whose equation is a linear combination of F and G.

Remark. By means of the identification of the curve $\lambda F + \mu G = 0$ with the point $(\lambda : \mu) \in \mathbb{P}^1$, the pencil Λ has a natural structure of projective line. In the same way that a line is determined by two different points, any two different curves of Λ span the same pencil.

As it is clear, a point $p \in C_1 \cap C_2$ belongs to all curves of the pencil. Therefore, we define the *locus of base points* of the pencil as the set

$$B_\Lambda = \left\{ p \in \mathbb{P}^2 \,|\, p \in D, \forall D \in \Lambda \right\} = C_1 \cap C_2$$

According to Bézout theorem, if C_1 and C_2 have no common irreducible components, there are at most n^2 distinct base points. And it's easy to see that, in the case of C_1 and C_2 sharing irreducible components, the locus of base points consists in the curve with equation $\gcd(F, G) = 0$, plus finitely many points.

Notice also that, for any non-base point $p \in \mathbb{P}^2 \setminus B_\Lambda$, the condition $\lambda \cdot F(p) + \mu \cdot G(p) = 0$ determines a unique curve of the pencil Λ passing through p.

Lemma 1.13. *Let Λ be the pencil generated by two algebraic curves $C_1, C_2 \subset \mathbb{P}^2$ of the same degree n, and $p \in C_1 \cap C_2$ a base point of Λ. Then,*

$$[C_1 \cdot C_2]_p = [D \cdot D']_p$$

for any two different curves $D, D' \in \Lambda$.

Proof. This result can be easily checked using the property 6 in proposition 1.4. □

Moreover, under certain conditions, we can control the multiplicity of a base point on the curves of the pencil:

Lemma 1.14. *Let Λ be a pencil of algebraic curves of degree n, and let $p \in \mathbb{P}^2$ be a base point for this pencil. The following are equivalent:*

1. *There exist two different curves of Λ having the same tangent cone at p.*
2. *All curves of Λ but one have the same tangent cone at p.*
3. *There are two curves of Λ having different multiplicities at p.*

Proof. During this proof, we take a projective reference such that $p = (1 : 0 : 0)$.

$1) \Longrightarrow 3)$: Assume that $C_1, C_2 \in \Lambda$ are two different curves with the same tangent cone at p, defined by a polynomial $f \in \mathbb{C}[x_1, x_2]$ of degree $e \geq 1$. Then, these curves admit projective equations $C_1 : F(x_0, x_1, x_2) = 0$ and $C_2 : G(x_0, x_1, x_2) = 0$, where

$$F(x_0, x_1, x_2) = f_n(x_1, x_2) + f_{n-1}(x_1, x_2) \cdot x_0 + \ldots + f_{e+1}(x_1, x_2) \cdot x_0^{n-e-1} + f(x_1, x_2) \cdot x_0^{n-e}$$
$$G(x_0, x_1, x_2) = g_n(x_1, x_2) + g_{n-1}(x_1, x_2) \cdot x_0 + \ldots + g_{e+1}(x_1, x_2) \cdot x_0^{n-e-1} + f(x_1, x_2) \cdot x_0^{n-e}$$

and $f_i, g_i \in \mathbb{C}[x_1,x_2]$ are homogeneous polynomials of degree i ($e+1 \le i \le n$).

Thinking of C_1 and C_2 as generators of the pencil, any curve $D \in \Lambda$ has equation $D : H(x_0,x_1,x_2) = 0$, with

$$H = \lambda F + \mu G = (\lambda f_n(x_1,x_2) + \mu g_n(x_1,x_2)) + (\lambda f_{n-1}(x_1,x_2) + \mu g_{n-1}(x_1,x_2)) \cdot x_0 + \ldots +$$
$$+(\lambda f_{e+1}(x_1,x_2) + \mu g_{e+1}(x_1,x_2)) \cdot x_0^{n-e-1} + (\lambda + \mu) f(x_1,x_2) \cdot x_0^{n-e}$$

for some $(\lambda : \mu) \in \mathbb{P}^1$. From this equation, it clearly follows that $e_p(D) \ge e$, and

$$e_p(D) > e \Longleftrightarrow (\lambda + \mu) f_e(x_1,x_2) = 0 \Longleftrightarrow \lambda + \mu = 0 \Longleftrightarrow (\lambda : \mu) = (1 : -1)$$

That is, the point p has exact multiplicity e for all the curves of Λ, with the exception of $F - G = 0$.

$3) \Longrightarrow 2)$: Let $C_1, C_2 \in \Lambda$ be curves with respective multiplicities e_1, e_2 at p, such that $e_1 \neq e_2$. We will assume, without loss of generality, that $e_1 > e_2$. Take equations $C_1 : F(x_0,x_1,x_2) = 0$ and $C_2 : G(x_0,x_1,x_2) = 0$, where

$$F(x_0,x_1,x_2) = f_n(x_1,x_2) + f_{n-1}(x_1,x_2) \cdot x_0 + \ldots + f_{e_1}(x_1,x_2) \cdot x_0^{n-e_1}$$
$$G(x_0,x_1,x_2) = g_n(x_1,x_2) + g_{n-1}(x_1,x_2) \cdot x_0 + \ldots + g_{e_2}(x_1,x_2) \cdot x_0^{n-e_2}$$

and the $f_i, g_i \in \mathbb{C}[x_1,x_2]$ are homogeneous polynomials of degree i. Observe that the tangent cones of C_1 and C_2 at p are given, respectively, by $f_{e_1}(x_1,x_2) = 0$ and $g_{e_2}(x_1,x_2) = 0$.

Spanning Λ by means of C_1 and C_2, each curve $D \in \Lambda$ has equation $D : H(x_0,x_1,x_2) = 0$, with

$$H = \lambda F + \mu G = (\lambda f_n(x_1,x_2) + \mu g_n(x_1,x_2)) + \ldots + (\lambda f_{e_1}(x_1,x_2) + \mu g_{e_1}(x_1,x_2)) \cdot x_0^{n-e_1} +$$
$$+\mu g_{e_1-1}(x_1,x_2) \cdot x_0^{n-e_1+1} + \ldots + \mu g_{e_2}(x_1,x_2) \cdot x_0^{n-e_2}$$

for some $(\lambda : \mu) \in \mathbb{P}^1$. Therefore:

- If $\mu \neq 0$, we have $TC_p(D) : \mu g_{e_2}(x_1,x_2) = 0$, which is the tangent cone $TC_p(C_2)$ of C_2 at p.
- If $\mu = 0$, $D = C_1$ has tangent cone $f_{e_1}(x_1,x_2) = 0$ at p.

In other words, all the curves of Λ, with the exception of C_1, have the same tangent cone at p as the curve C_2.

$2) \Longrightarrow 1)$: Trivial. $\square$

Remark. It follows from the proof of lemma 1.14 that the curve in the condition 2 with different tangent cone has strictly higher multiplicity at p than the other curves of Λ. It will be referred to as the *exceptional curve* of the pencil.

For the case of pencils of groups of points, the main notions are quite similar:

Definition. Let $G_1, G_2 \subset \mathbb{P}^1$ be two different groups of points of the same degree n, with respective equations $F(x_0,x_1) = 0$ and $G(x_0,x_1) = 0$. The *pencil* spanned (or generated) by G_1 and G_2 is the set

$$\widetilde{\Lambda} = \left\{ H : \lambda F + \mu G = 0 \,|\, (\lambda : \mu) \in \mathbb{P}^1 \right\}$$

of groups of points whose equation is a linear combination of F and G.

Again, we have an identification of $\widetilde{\Lambda}$ with $\mathbb{P}^1$, which allows us to generate the pencil by means of any two different groups of points of Λ.

And as before, a point $p \in G_1 \cap G_2$ is contained in all the groups of points of $\widetilde{\Lambda}$. Therefore, we define the *locus of fixed points* of the pencil as the set

$$B_{\widetilde{\Lambda}} = \left\{ p \in \mathbb{P}^1 \,|\, p \in H, \forall H \in \widetilde{\Lambda} \right\} = G_1 \cap G_2$$

(notice that this set may be empty, in contradistinction to the case of pencils of curves).

Any non-fixed point $p \in \mathbb{P}^1 \setminus B_{\widetilde{\Lambda}}$ will be contained in a unique group of the pencil $\widetilde{\Lambda}$, determined by the condition $\lambda \cdot F(p) + \mu \cdot G(p) = 0$.

By means of the two following results, we determine the points in $\mathbb{P}^1$ which are double (in the sense that have multiplicity ≥ 2) for some group of the pencil $\widetilde{\Lambda}$:

Proposition 1.15. *A group of points $G \subset \mathbb{P}^1$ has undetermined point polar with respect to a point $p \in \mathbb{P}^1$ if, and only if, p is a double point for G.*

Proof. Take a reference of $\mathbb{P}^1$ such that $p = (1:0)$ and G has equation $G: F = 0$, with $F(x_0, x_1) = \sum_{i=0}^{n} \alpha_i x_1^i x_0^{n-i}$. The point polar of G with respect to p is

$$\mathscr{P}_p^{n-1}(G) : 0 = \frac{\partial^{n-1} F}{\partial x_0^{n-1}} = \alpha_1 (n-1)! \cdot x_1 + \alpha_0 n! \cdot x_0$$

and therefore

$$\mathscr{P}_p^{n-1}(G) \text{ is not defined} \iff \alpha_1 (n-1)! = 0 = \alpha_0 n! \iff \alpha_1 = 0 = \alpha_0 \iff$$

$$\iff F = x_1^2 \cdot F', \text{ for some } F' \in \mathbb{C}[x_0, x_1] \text{ of degree } n-2 \iff p \text{ is double for } G. \ \square$$

Lemma 1.16. *Let $\widetilde{\Lambda}$ be a pencil of groups of points, and $p \in \mathbb{P}^1$ not a fixed point for this pencil. Then, $\widetilde{\Lambda}$ has a group containing p with multiplicity (at least) 2 if, and only if, p has undetermined point polar for a group of $\widetilde{\Lambda}$ and constant point polar with regard to the other groups of $\widetilde{\Lambda}$.*

Proof. Take a group G_1 of $\widetilde{\Lambda}$, in such a way that $p \notin G_1$ (this is possible because p is not a fixed point), and take G_2 another element of $\widetilde{\Lambda}$ different from G_1.

We fix a reference of $\mathbb{P}^1$, in such a way that $p = (1:0)$ and the homogeneous polyonomials

$$F(x_0, x_1) = \sum_{i=0}^{n} \alpha_i x_1^i x_0^{n-i}, \quad G(x_0, x_1) = \sum_{i=0}^{n} \beta_i x_1^i x_0^{n-i}$$

determine equations $G_1 : F = 0$, $G_2 : G = 0$ for the generators G_1, G_2 of $\widetilde{\Lambda}$. Using the notation $G_{\lambda,\mu} : \lambda F + \mu G = 0$ for the elements of $\widetilde{\Lambda}$, observe that

$$\mathscr{P}_p^{n-1}(G_{\lambda,\mu}) : 0 = \frac{\partial^{n-1}(\lambda F + \mu G)}{\partial x_0^{n-1}} = (\lambda \alpha_1 + \mu \beta_1)(n-1)! \cdot x_1 + (\lambda \alpha_0 + \mu \beta_0) n! \cdot x_0$$

First, suppose that p is double in a group G_{λ_0,μ_0}, for some $(\lambda_0 : \mu_0) \in \mathbb{P}^1$. We may assume that $\mu_0 = 1$, since $p \notin G_1 = G_{1,0}$. By proposition 1.15,

$$p \text{ is double in } G_{\lambda_0,1} \implies \mathscr{P}_p^{n-1}(G_{\lambda_0,1}) \text{ is undetermined} \implies \lambda_0 \alpha_1 + \beta_1 = 0 = \lambda_0 \alpha_0 + \beta_0$$

Hence, for any $(\lambda : \mu) \in \mathbb{P}^1$, the point polar $\mathscr{P}_p^{n-1}(G_{\lambda,\mu})$ is given by the polynomial

$$(\lambda \alpha_1 + \mu \beta_1)(n-1)! \cdot x_1 + (\lambda \alpha_0 + \mu \beta_0) n! \cdot x_0 = (\lambda \alpha_1 - \mu \lambda_0 \alpha_1)(n-1)! \cdot x_1 + (\lambda \alpha_0 - \mu \lambda_0 \alpha_0) n! \cdot x_0 =$$

$$= (\lambda - \mu \lambda_0) \alpha_1 (n-1)! \cdot x_1 + (\lambda - \mu \lambda_0) \alpha_0 n! \cdot x_0$$

If $\lambda - \mu\lambda_0 \neq 0$ (that is, if $(\lambda : \mu) \neq (\lambda_0 : \mu_0)$), we may cancel this factor and thus the point polar

$$\mathscr{P}_p^{n-1}(G_{\lambda,\mu}) :\ \alpha_1(n-1)! \cdot x_1 + \alpha_0 n! \cdot x_0 = 0$$

coincides for all the groups $G_{\lambda,\mu} \in \widetilde{\Lambda}$ with $(\lambda : \mu) \neq (\lambda_0 : \mu_0)$. Notice that these polars are defined, since $p \notin G_{1,0}$.

Conversely, if G_{λ_0,μ_0} is the group of $\widetilde{\Lambda}$ having undetermined point polar with respect to p, it follows from proposition 1.15 that p is contained (at least) twice in G_{λ_0,μ_0}. This finishes the proof. □

Chapter 2

Waring's theorem and consequences

2.1 Characterization of the two highest homogeneous parts of a curve

Let $l_1, \dots, l_r$ be parallel lines in the affine plane $\mathbb{A}^2$. Regard them as lines in $\mathbb{P}^2$, with all of them passing through a certain improper point $p \in l_\infty$.

As we already know from basics on projective geometry, the pencil Λ of lines passing through p has a natural structure of $\mathbb{P}^1$. Furthermore, considering l_∞ as a distinguished element of Λ, the set Λ' of affine lines parallel to the direction determined by p acquires a structure of affine line. Therefore, $l_1 + \ldots + l_r$ is a group of points in Λ'.

Definition. The *barycentric parallel* of $l_1, \dots, l_r$, denoted by $BP(l_1, \dots, l_r)$, is the barycenter of $l_1, \dots, l_r$ as points of Λ'. That is,

$$BP(l_1, \dots, l_r) = \mathscr{P}_{l_\infty}^{r-1}(l_1 + \ldots + l_r) \in \Lambda$$

If the point p is given by the homogeneous coordinates $p = (0 : a : b)$ in $\mathbb{P}^2$, each element of Λ' admits an equation of the form $l : -bx + ay = c$. The number c may be seen as the affine coordinate of l in Λ'.

Hence, if $l_i : -bx + ay = c_i \ (1 \le i \le r)$ are affine equations for $l_1, \dots, l_r$, their barycentric parallel line is

$$BP(l_1, \dots, l_r) : -bx + ay = \frac{c_1 + \ldots + c_r}{r}.$$

Remark. Any affine line $l \subset \mathbb{A}^2$, non-parallel to $l_1, \dots, l_r$, intersects $BP(l_1, \dots, l_r)$ at the barycenter of the intersection points of $l_1, \dots, l_r$ with l.

Example. The affine lines $l_1 : x + 2y = 2,\ l_2 : x + 2y = -2$ and $l_3 : x + 2y = 3$ have barycentric parallel $BP(l_1, l_2, l_3) : x + 2y = 1$.

Consider a plane algebraic curve C of degree n, with projective equation $C : F(x_0, x_1, x_2) = 0$. Recall that, in section 1.2, we took a decomposition

$$F(x_0, x_1, x_2) = f_n(x_1, x_2) + f_{n-1}(x_1, x_2) \cdot x_0 + \ldots + f_1(x_1, x_2) \cdot x_0^{n-1} + f_0 \cdot x_0^n$$

with the $f_i \in \mathbb{C}[x_1,x_2]$ homogeneous polynomials of degree i ($0 \le i \le n$). The two highest homogeneous parts of C were defined as the polynomials f_n and f_{n-1}.

Equivalently, the affine part of C has equation $F(1,x,y)=0$, with

$$F(1,x,y) = f_n(x,y) + f_{n-1}(x,y) + \ldots + f_1(x,y) + f_0$$

and we may regard the two highest homogeneous parts of C as the polynomials of degree n and $n-1$ appearing in the decomposition of the affine equation into a sum of homogeneous polynomials.

In this section, we are going to explore the geometric information that underlies the two highest homogeneous parts. Notice that, with the previous notations, the polynomial f_n is completely determined by the improper section: in fact, $C \cdot l_\infty$ is the group of points in l_∞ given by the equation

$$0 = F(0,x_1,x_2) = f_n(x_1,x_2) \ ,$$

where we consider x_1, x_2 also as homogeneous coordinates in l_∞.

The notion of barycentric parallel line, when applied to groups of parallel asymptotes, provides a geometric criterion for the simultaneous coincidence of the two highest homogeneous parts f_n and f_{n-1}, in case that C has no parabolic branches.

This criterion, not found in the literature, may be formally stated as:

Theorem 2.1. *Let $C_1 : f(x,y)=0$ and $C_2 : g(x,y)=0$ be two affine algebraic curves of degree n, where*

$$f(x,y) = f_n(x,y) + f_{n-1}(x,y) + \ldots + f_1(x,y) + f_0 \ , \quad g(x,y) = g_n(x,y) + g_{n-1}(x,y) + \ldots + g_1(x,y) + g_0$$

and $f_i, g_i \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree i ($0 \le i \le n$). Assume that neither C_1 nor C_2 has parabolic branches.

There exists $k \in \mathbb{C}^*$ such that $g_n = k \cdot f_n$ and $g_{n-1} = k \cdot f_{n-1}$ if, and only if, for any direction δ the groups of asymptotes of C_1 and C_2 that are parallel to δ have the same number of elements (counted with multiplicities) and the same barycentric parallel.

We are going to prove this theorem in two different ways. In the first proof, probably the simplest one, the main tool are pencils of curves and pencils of groups of points:

First proof of theorem 2.1.

We work with the projective equations $C_1 : F(x_0,x_1,x_2)=0$ and $C_2 : G(x_0,x_1,x_2)=0$, where

$$F(x_0,x_1,x_2) = x_0^n \cdot f(\tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}) = f_n(x_1,x_2) + f_{n-1}(x_1,x_2) \cdot x_0 + \ldots + f_1(x_1,x_2) \cdot x_0^{n-1} + f_0 \cdot x_0^n$$

$$G(x_0,x_1,x_2) = x_0^n \cdot g(\tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}) = g_n(x_1,x_2) + g_{n-1}(x_1,x_2) \cdot x_0 + \ldots + g_1(x_1,x_2) \cdot x_0^{n-1} + g_0 \cdot x_0^n$$

We will denote by Λ the pencil of curves spanned by C_1 and C_2.

Assume that $g_n = k \cdot f_n$ and $g_{n-1} = k \cdot f_{n-1}$, for some $k \in \mathbb{C}^*$. Then, the curve $D : kF - G = 0$ of Λ has l_∞ as an irreducible component of multiplicity (at least) 2.

Since g_n and f_n are proportional, they determine the same group of points in l_∞, so that C_1 and C_2 have the same improper section: $C_1 \cdot l_\infty = C_2 \cdot l_\infty$. If $p \in l_\infty$ is any of the improper points of C_1 and C_2, then

$$e_p(C_1) = e_p(C_1) \cdot e_p(l_\infty) = [C_1 \cdot l_\infty]_p = [C_2 \cdot l_\infty]_p = e_p(C_2) \cdot e_p(l_\infty) = e_p(C_2) ,$$

with the second and the fourth equalities following from the curves having no parabolic branches. This proves that the number of asymptotes of C_1 and C_2 in the direction determined by p, counted with multiplicites, is exactly the same.

Write $e = e_p(C_1) = e_p(C_2)$ and, changing coordinates if necessary, assume that $p = (0:0:1)$. Then,

$$F(x_0,x_1,x_2) = \widetilde{f_n}(x_0,x_1) + \widetilde{f_{n-1}}(x_0,x_1) \cdot x_2 + \ldots + \widetilde{f_e}(x_0,x_1) \cdot x_2^{n-e}$$

$$G(x_0,x_1,x_2) = \widetilde{g_n}(x_0,x_1) + \widetilde{g_{n-1}}(x_0,x_1) \cdot x_2 + \ldots + \widetilde{g_e}(x_0,x_1) \cdot x_2^{n-e}$$

with the $\widetilde{f_i}, \widetilde{g_i} \in \mathbb{C}[x_0,x_1]$ homogeneous of degree i.

If the tangent cones $TC_p(C_1) : \widetilde{f_e}(x_0,x_1) = 0$ and $TC_p(C_2) : \widetilde{g_e}(x_0,x_1) = 0$ are equal, the groups of asymptotes of C_1 and C_2 passing through p are the same, and obviously will have the same barycentric parallel.

Otherwise, since every curve of the pencil Λ admits an equation

$$(\lambda \cdot \widetilde{f_n} + \mu \cdot \widetilde{g_n}) + (\lambda \cdot \widetilde{f_{n-1}} + \mu \cdot \widetilde{g_{n-1}}) \cdot x_2 + \ldots + (\lambda \cdot \widetilde{f_e} + \mu \cdot \widetilde{g_e}) \cdot x_2^{n-e} = 0$$

for some $(\lambda : \mu) \in \mathbb{P}^1$, it follows that the pencil $\widetilde{\Lambda}$ spanned by $TC_p(C_1)$ and $TC_p(C_2)$ is the set of tangent cones at p for all the curves of Λ. By the usual identification of the set of lines through p and $\mathbb{P}^1$, we may regard $\widetilde{\Lambda}$ as a pencil of groups of points in $\mathbb{P}^1$.

This pencil $\widetilde{\Lambda}$ has a group where l_∞ is counted (at least) twice: the group corresponding to the tangent cone at p of the curve D. And obviously, l_∞ is not fixed for $\widetilde{\Lambda}$, since l_∞ is not in the groups $TC_p(C_1), TC_p(C_2) \in \widetilde{\Lambda}$ (recall that C_1 and C_2 have no parabolic branches).

According to lemma 1.16, all the elements of $\widetilde{\Lambda}$ have constant point polar with respect to l_∞. That is, all the groups of lines forming a tangent cone $TC_p(C)$, for $C \in \Lambda$, have the same barycentric parallel.

In particular, the barycentric parallels of the respective groups of lines $TC_p(C_1)$ and $TC_p(C_2)$ coincide. This finishes the proof of the first implication.

Conversely: translate the hypothesis of the coincidence of number of asymptotes in the same direction, as $e_p(C_1) = e_p(C_2)$ for every $p \in l_\infty$. Since C_1 and C_2 have no parabolic branches, we deduce that

$$[C_1 \cdot l_\infty]_p = e_p(C_1) \cdot e_p(l_\infty) = e_p(C_1) = e_p(C_2) = e_p(C_2) \cdot e_p(l_\infty) = [C_2 \cdot l_\infty]_p$$

and thus C_1 and C_2 have the same improper section: this gives $g_n = k \cdot f_n$, for some $k \in \mathbb{C}^*$.

Observe that the curve $D : kF - G = 0$ of Λ has l_∞ as an irreducible component, i.e., we can write

$$kF - G = x_0 \cdot H$$

for some homogeneous polynomial $H \in \mathbb{C}[x_0,x_1,x_2]$ of degree $n-1$. In order to finish, it suffices to see that the multiplicity of l_∞ as an irreducible component of D is, at least, 2. Or equivalently, that $D' : H = 0$ has l_∞ as a component.

Denote by $p_1, \ldots, p_s$ ($s \leq n$) the distinct improper points of C_1 and C_2, and write $e_i = e_{p_i}(C_1) = e_{p_i}(C_2)$ for $i \in \{1, \ldots, s\}$. For each point p_i, we distinguish two cases:

- Case 1: Suppose that the tangent cones $T_{p_i}(C_1)$, $T_{p_i}(C_2)$ coincide. Note that l_∞ is a component of $T_{p_i}(D)$ (because it's a component of D), but not of $T_{p_i}(C_1)$ or $T_{p_i}(C_2)$ (C_1 and C_2 have no parabolic branches).

Hence, D must be the exceptional curve of Λ satisfying $e_{p_i}(D) > e_i$ (see lemma 1.14 and the remark following it). Then,

$$e_i + 1 \leq e_{p_i}(D) = e_{p_i}(l_\infty) + e_{p_i}(D') = 1 + e_{p_i}(D') \Longrightarrow e_{p_i}(D') \geq e_i$$

and we deduce that

$$[D' \cdot l_\infty]_{p_i} \geq e_{p_i}(D') \cdot e_{p_i}(l_\infty) = e_{p_i}(D') \geq e_i$$

- Case 2: If the tangent cones $T_{p_i}(C_1)$, $T_{p_i}(C_2)$ do not coincide, the pencil $\widetilde{\Lambda}$ that generate is the set of tangent cones at p_i for all the curves of Λ. As before, we see $\widetilde{\Lambda}$ as a pencil of groups of points in $\mathbb{P}^1$.

  By hypothesis, the groups of lines forming $TC_{p_i}(C_1)$ and $TC_{p_i}(C_2)$ have the same barycentric parallel. Hence, all groups in $\widetilde{\Lambda}$ have the same barycentric parallel, that is, the same point polar with respect to l_∞.

  Notice that l_∞ is not fixed for $\widetilde{\Lambda}$, and the unique group of $\widetilde{\Lambda}$ containing it must be $TC_{p_i}(D)$, since l_∞ is a component of D. According to lemma 1.16, this group $TC_{p_i}(D)$ contains l_∞ twice: hence, D' has a branch tangent to l_∞ at p_i, which gives

$$[D' \cdot l_\infty]_{p_i} > e_{p_i}(D') \cdot e_{p_i}(l_\infty) = e_{p_i}(D') = e_{p_i}(D) - e_{p_i}(l_\infty) = e_i - 1 \Longrightarrow [D' \cdot l_\infty]_{p_i} \geq e_i$$

All together, we have $[D' \cdot l_\infty]_{p_i} \geq e_i$ for every $i \in \{1, \ldots, s\}$, and then

$$\sum_{i=1}^{s} [D' \cdot l_\infty]_{p_i} \geq \sum_{i=1}^{s} e_i = n$$

But D' is a curve of degree $n-1$: by Bézout theorem, l_∞ must be an irreducible component of D', which concludes the proof. $\square$

In the second proof, we factorize the polynomial F by making use of Puiseux parametrizations of branches. This provides a formula that also appears in [7]:

Second proof of theorem 2.1.

In this second proof, we will give an expression of $\dfrac{f_{n-1}(x,y)}{f_n(x,y)}$ as a sum of partial fractions. Take the projective equation $C_1 : F(x_0, x_1, x_2) = 0$, where

$$F(x_0,x_1,x_2) = x_0^n \cdot f\left(\tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}\right) = f_n(x_1,x_2) + f_{n-1}(x_1,x_2) \cdot x_0 + \ldots + f_1(x_1,x_2) \cdot x_0^{n-1} + f_0 \cdot x_0^n$$

Changing coordinates if necesssary, we will suppose that $(0:1:0) \notin C_1$. Consider coordinates (z,t) in the affine chart of $\mathbb{P}^2$ with $x_2 \neq 0$: that is, (z,t) denotes the point with homogeneous coordinates $(z:t:1)$ in the projective frame. The affine part of C_1 for this chart has equation

$$0 = F(z,t,1) = f_n(t,1) + f_{n-1}(t,1) \cdot z + \ldots + f_1(t,1) \cdot z^{n-1} + f_0 \cdot z^n.$$

Assume that C_1 has $s \leq n$ distinct points of intersection with $l_\infty : z = 0$, denoted by $p_i = (0, a_i)$, $1 \leq i \leq s$. Let ν_i be the number of Puiseux series of C_1 at the point p_i.

Since the t-axis l_∞ is not tangent to C_1, the Puiseux parametrizations of the ν_i branches of C_1 at p_i have the form

$$t = a_i + b_{i,j}z + \ldots \quad (j = 1, \ldots, \nu_i),$$

where $\ldots$ denotes terms with degree in z strictly higher than 1. The tangents to these branches will be $l_{i,j} : t = a_i + b_{i,j}z$.

By the hypothesis $(0:1:0) \notin C_1$, the polynomial $F(z,t,1)$ decomposes as

$$F(z,t,1) = \prod_{i=1}^{s} \prod_{j=1}^{\nu_i} (t - a_i - b_{i,j}z - \dots)$$

according to proposition 1.3. Then, the intersection of C_1 with $l_\infty : z = 0$ is given by

$$f_n(t,1) = F(0,t,1) = \prod_{i=1}^{s} \prod_{j=1}^{\nu_i} (t - a_i) = \prod_{i=1}^{s} (t - a_i)^{\nu_i}$$

In order to find an expression for f_{n-1}, we must compute all the terms of $F(z,t,1)$ having degree 1 in z. From a simple observation at the factorization of $F(z,t,1)$, we deduce that such terms are

$$-b_{i,j}z \cdot (t - a_i)^{\nu_i - 1} \cdot \prod_{k \neq i} (t - a_k)^{\nu_k}$$

for $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, \nu_i\}$. Therefore,

$$f_{n-1}(t,1) \cdot z = \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} \left[-b_{i,j}z \cdot (t - a_i)^{\nu_i - 1} \cdot \prod_{k \neq i} (t - a_k)^{\nu_k} \right] = \sum_{i=1}^{s} \left[\left(\sum_{j=1}^{\nu_i} -b_{i,j} \right) \cdot (t - a_i)^{\nu_i - 1} \cdot \prod_{k \neq i} (t - a_k)^{\nu_k} \right] \cdot z$$

which gives

$$f_{n-1}(t,1) = \sum_{i=1}^{s} \left[\left(\sum_{j=1}^{\nu_i} -b_{i,j} \right) \cdot (t - a_i)^{\nu_i - 1} \cdot \prod_{k \neq i} (t - a_k)^{\nu_k} \right]$$

Using these expressions for $f_{n-1}(t,1)$ and $f_n(t,1)$, we deduce that

$$\frac{f_{n-1}(t,1)}{f_n(t,1)} = \sum_{i=1}^{s} \frac{\sum_{j=1}^{\nu_i} -b_{i,j}}{t - a_i} \implies \frac{f_{n-1}(x,y)}{f_n(x,y)} = \sum_{i=1}^{s} \frac{\sum_{j=1}^{\nu_i} -b_{i,j}}{x - a_i y} \qquad (**)$$

And if we come back to the original affine chart ($x_0 \neq 0$) with coordinates (x,y), the asymptotes through each point $(0 : a_i : 1) \in C_1$ are given by

$$l_{i,j} : x = a_i y + b_{i,j} \qquad (j = 1, \dots, \nu_i)$$

whose barycentric parallel is $BP(l_{i,1}, \dots, l_{i,\nu_i}) : x = a_i y + \frac{1}{\nu_i} \cdot \sum_{j=1}^{\nu_i} b_{i,j}$. Hence, in the expression $(**)$:

- The denominators and the improper points of C_1 (that is, the directions of the asymptotes) determine each other.
- The numerators determine and are determined by the barycentric parallels of the groups of parallel asymptotes.

By joining this with the uniqueness of a decomposition into a sum of partial fractions with coprime denominators, the result follows. □

The main advantage of theorem 2.1 is giving a complete geometric determination of the two highest homogeneous parts of curves with no parabolic branches. This provides a generalization of the criterions found in the literature, which are based on the strict coincidence of asymptotes, rather than considering barycentric parallels.

For instance, in [4], as well as all the mathematicians of the 19th century, Coolidge restricts the problem to the curves intersecting transversely l_∞ at each of their improper points:

Corollary 2.2. *Let $C_1 : f(x,y) = 0$ and $C_2 : g(x,y) = 0$ be two affine algebraic curves of degree n, where*

$$f(x,y) = f_n(x,y) + f_{n-1}(x,y) + \ldots + f_1(x,y) + f_0\,, \quad g(x,y) = g_n(x,y) + g_{n-1}(x,y) + \ldots + g_1(x,y) + g_0$$

and $f_i, g_i \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree i ($0 \leq i \leq n$). Assume that each of the curves meets the line at infinity in exactly n distinct points. Then:

$$C_1 \text{ and } C_2 \text{ have the same asymptotes} \iff g_n = k \cdot f_n \text{ and } g_{n-1} = k \cdot f_{n-1}, \text{ for some } k \in \mathbb{C}^*$$

Proof. By hypothesis, the improper section of C_1 and C_2 consists on n distinct (and thus non-singular) points.

Due to this, C_1 and C_2 have a single asymptote associated to each asymptotic direction, and the coincidence of asymptotes becomes an equivalent condition to the coincidence of barycentric parallels of groups of asymptotes with the same direction. Using theorem 2.1 concludes the proof. $\square$

Remark. Coolidge proves this result giving the formula

$$\frac{f_{n-1}(x,y)}{f_n(x,y)} = \sum_{i=1}^{n} \frac{c_i}{a_i x + b_i y}$$

if $a_i x + b_i y + c_i = 0$ ($i = 1, \ldots, n$) are the affine equations for the n different asymptotes of C_1, which is nothing but the formula $(**)$ applied to this particular case.

In [7], Kunz and Waldi prove the formula $(**)$ for curves with no parabolic brances, in a different way. The criterion presented there is again an immediate consequence of theorem 2.1, since considers the strict coincidence of asymptotes as a sufficient condition for the coincidence of two highest homogeneous parts:

Corollary 2.3. *Let $C_1 : f(x,y) = 0$ and $C_2 : g(x,y) = 0$ be two affine algebraic curves of degree n, where*

$$f(x,y) = f_n(x,y) + f_{n-1}(x,y) + \ldots + f_1(x,y) + f_0\,, \quad g(x,y) = g_n(x,y) + g_{n-1}(x,y) + \ldots + g_1(x,y) + g_0$$

and $f_i, g_i \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree i ($0 \leq i \leq n$). Assume that neither C_1 nor C_2 has parabolic branches.

Then, if C_1 and C_2 have the same asymptotes (counted with multiplicity), there is $k \in \mathbb{C}^*$ such that $g_n = k \cdot f_n$ and $g_{n-1} = k \cdot f_{n-1}$.

Clearly, the converse of corollary 2.3 does not hold: two curves may have the same number of asymptotes in any direction determining identical barycentric parallels, but different asymptotes.

For example, consider the cubics

$$C_1 : xy^2 - 2x^2y + x^3 - 4x^2 + 4xy - x - 1 = 0, \quad C_2 : 2xy^2 - 4x^2y + 2x^3 - 8x^2 + 8xy + 2x + 3y - 4 = 0$$

which satisfy $2f_3 = g_3$ and $2f_2 = g_2$, with the previous notations. However, at the improper point $p = (0:1:1)$, the tangent cones

$$TC_p(C_1) : 0 = -x_0^2 + x_1^2 + x_2^2 - 4x_0x_1 + 4x_0x_2 - 2x_1x_2 = ((2-\sqrt{5})x_0 - x_1 + x_2)((2+\sqrt{5})x_0 - x_1 + x_2),$$

$$TC_p(C_2) : 0 = \tfrac{5}{2}x_0^2 + x_1^2 + x_2^2 - 4x_0x_1 + 4x_0x_2 - 2x_1x_2 = (-\tfrac{4+\sqrt{6}}{2}x_0 + x_1 - x_2)(-\tfrac{4-\sqrt{6}}{2}x_0 - x_1 + x_2)$$

give different asymptotes for each of the curves.

The following picture, made with Geogebra, is a representation of C_1 (red) and C_2 (blue) in the real affine plane, together with their asymptotes (dashed lines) and the coincident barycentric parallels for the groups of parallel asymptotes (black):

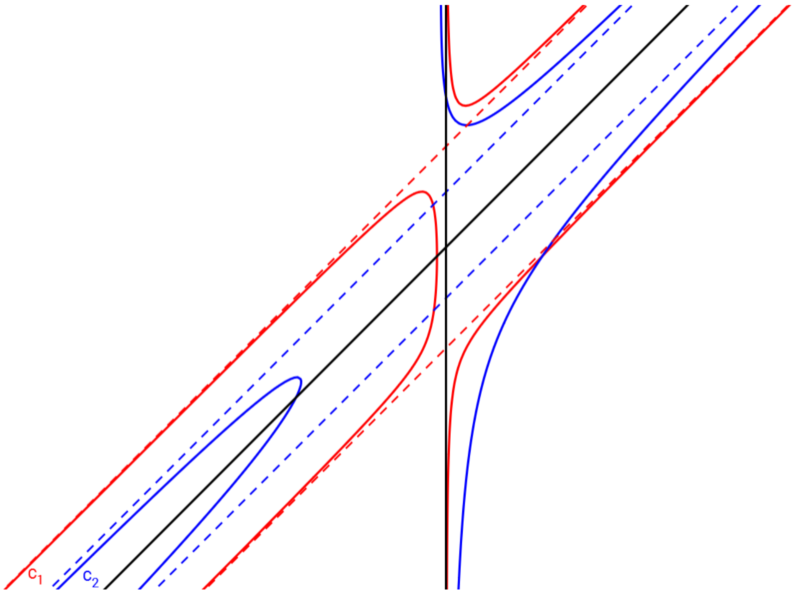


Figure 1

To finish this section, we must point out that these results are far from being true for curves C_1 and C_2 with parabolic branches. For example, for the affine curves

$$C_1 : y^3 + 2xy + 2y^2 + x + y = 0, \;\; C_2 : y^3 + 2xy + x = 0$$

the parts f_2 and g_2 are not proportional, in spite of C_1 and C_2 having the same asymptotes. In fact, both curves have a unique improper point, $p = (0:1:0)$, where the tangent cones coincide:

$$TC_p(C_1) = TC_p(C_2) : x_0(2x_2 + x_0) = 0$$

2.2 An extension of Waring's theorem

As we said in the introduction, the linking between asymptotes and barycenters of intersection groups comes through the two highest homogeneous parts of curves.

The first part of this linking has been widely detailed in the previous section. For the second one, we find explicitly the coordinates of the barycenter, by means of elimination arguments:

Theorem 2.4. *Let C and D be two plane algebraic curves with disjoint improper sections. Then, the barycenter of $C \cdot D$ coincides with the barycenter of the intersection group when any of the curves is replaced by another with the same two highest homogeneous parts.*

Proof. First of all, note that C and D do not share any irreducible component (the intersection of a common component with l_∞ would give common improper points). So, according to Bézout theorem, the intersection $C \cap D$ must be a finite set of points, and the intersection group $C \cdot D$ makes sense.

We take a projective reference such that neither $(0:0:1)$ nor $(0:1:0)$ is an improper point of C or D. In this reference, consider equations $C : F(x_0, x_1, x_2) = 0$ and $D : G(x_0, x_1, x_2) = 0$, in such a way that

$$F(x_0,x_1,x_2) = f_n(x_1,x_2) + f_{n-1}(x_1,x_2)\cdot x_0 + \ldots + f_1(x_1,x_2)\cdot x_0^{n-1} + f_0 \cdot x_0^n$$
$$G(x_0,x_1,x_2) = g_m(x_1,x_2) + g_{m-1}(x_1,x_2)\cdot x_0 + \ldots + g_1(x_1,x_2)\cdot x_0^{m-1} + g_0 \cdot x_0^m$$

with $f_i, g_i \in \mathbb{C}[x_1,x_2]$ homogeneous polynomials of degree i. Denote also by $f(x,y) = F(1,x,y)$ and $g(x,y) = G(1,x,y)$ the equations of their affine parts.

We are going to study the abcissae of the points in the group $C \cdot D$ by eliminating the variable x_2. We will use that the roots of the polynomial $A(x_1) = \text{Res}_{x_2}(F,G)\,|_{x_0=1} = \text{Res}_y(f,g)\,|_{x=x_1}$ in x_1 are exactly the abcissae of the points in $C \cap D$. Furthermore, the multiplicity of each root α of A is the sum of all the intersection multiplicities $[C\cdot D]_p$, where p is a point of abcissa α (see [2], proposition 3.3.4).

Rewrite the polynomials F and G as

$$F(x_0,x_1,x_2) = \widetilde{f_n}(x_0,x_1) + \widetilde{f_{n-1}}(x_0,x_1)\cdot x_2 + \ldots + \widetilde{f_1}(x_0,x_1)\cdot x_2^{n-1} + \widetilde{f_0}\cdot x_2^n$$
$$G(x_0,x_1,x_2) = \widetilde{g_m}(x_0,x_1) + \widetilde{g_{m-1}}(x_0,x_1)\cdot x_2 + \ldots + \widetilde{g_1}(x_0,x_1)\cdot x_2^{m-1} + \widetilde{g_0}\cdot x_2^m$$

with the $\widetilde{f_i}, \widetilde{g_i} \in \mathbb{C}[x_0,x_1]$ homogeneous of degree i.

Since the resultant $\text{Res}_{x_2}(F,G)$ is isobaric of weight nm as a polynomial in $\widetilde{f_0},\ldots,\widetilde{f_n},\widetilde{g_0},\ldots,\widetilde{g_m}$ (see [2], proposition 3.1.5), it follows that $\text{Res}_{x_2}(F,G)$ is homogeneous of degree nm (as a polynomial in x_0,x_1):

$$\text{Res}_{x_2}(F,G) = a_0 x_1^{nm} + a_1 x_1^{nm-1}x_0 + \ldots + a_{nm-1}x_1 x_0^{nm-1} + a_{nm}x_0^{nm} \quad (\text{with the } a_i \in \mathbb{C})$$

Now, observe that:

1. The coefficient a_0 only depends on the polynomials f_n and g_m: in fact, take the expressions

   $$\widetilde{f_i}(x_0,x_1) = \sum_{j=0}^{i} \alpha_{i,j} x_0^j x_1^{i-j} \text{ (for } 0 \le i \le n), \quad \widetilde{g_i}(x_0,x_1) = \sum_{j=0}^{i} \beta_{i,j} x_0^j x_1^{i-j} \text{ (for } 0 \le i \le m)$$

   with $\alpha_{i,j}, \beta_{i,j} \in \mathbb{C}$, and write $z_{i,j} = \alpha_{i,j}x_0^j x_1^{i-j}$ and $z'_{i,j} = \beta_{i,j}x_0^j x_1^{i-j}$. The terms of F and G will have the form $z_{i,j}\cdot x_2^{n-i}$ and $z'_{i,j}\cdot x_2^{m-i}$, respectively.

   Since $\text{Res}_{x_2}(F,G)$ is a polynomial in $\widetilde{f_0},\ldots,\widetilde{f_n},\widetilde{g_0},\ldots,\widetilde{g_m}$, in particular it must be a polynomial Q in the variables $z_{i,j}$ and $z'_{i,j}$:

   $$\text{Res}_{x_2}(F,G) = Q(z_{i,j}, z'_{i,j})$$

   (notice that Q has no independent term, because $\text{Res}_{x_2}(F,G)$ is homogeneous in x_0,x_1). Now,

   $$a_0 x_1^{nm} = \text{Res}_{x_2}(F,G)\,|_{x_0=0} = Q(\alpha_{i,j}x_0^j x_1^{i-j}, \beta_{i,j}x_0^j x_1^{i-j})\,|_{x_0=0},$$

   with this evaluation at $x_0 = 0$ depending only on the variables $z_{i,0} = \alpha_{i,0}x_1^i$ and $z'_{i,0} = \beta_{i,0}x_1^i$ (the remaining variables are identically 0 for this evaluation).

   Therefore, a_0 is univocally determined by the numbers $\alpha_{i,0}, \beta_{i,0} \in \mathbb{C}$, which are precisely the coefficients of the polynomials f_n and g_m.

2. The number a_1 only depends on the polynomials f_n, f_{n-1}, g_m and g_{m-1}:

   With the previous notations, and using the chain rule, we deduce that

   $$a_1 x_1^{nm-1} = \frac{\partial(\text{Res}_{x_2}(F,G))}{\partial x_0}\,|_{x_0=0} = \left[\sum_{i=0}^{n}\sum_{j=1}^{i} \frac{\partial Q}{\partial z_{i,j}} \cdot \frac{\partial z_{i,j}}{\partial x_0} + \sum_{i=0}^{m}\sum_{j=1}^{i} \frac{\partial Q}{\partial z'_{i,j}} \cdot \frac{\partial z'_{i,j}}{\partial x_0}\right]|_{x_0=0} =$$
   $$= \left[\sum_{i=0}^{n}\sum_{j=1}^{i} \frac{\partial Q}{\partial z_{i,j}} \cdot j\alpha_{i,j}x_0^{j-1}x_1^{i-j} + \sum_{i=0}^{m}\sum_{j=1}^{i} \frac{\partial Q}{\partial z'_{i,j}} \cdot j\beta_{i,j}x_0^{j-1}x_1^{i-j}\right]|_{x_0=0}$$

The evaluation at $x_0 = 0$ cancels all the summands, with the exception of those with $j = 1$.

And by the same argument as in 1., $\frac{\partial Q}{\partial z_{i,1}}\,|_{x_0=0}$ and $\frac{\partial Q}{\partial z'_{i,1}}\,|_{x_0=0}$ only depend on the variables of the form $z_{k,0} = \alpha_{k,0} x_1^k$ and $z'_{k,0} = \beta_{k,0} x_1^k$.

Due to this, a_1 only depends on the numbers $\alpha_{i,0}, \beta_{i,0}, \alpha_{i,1}, \beta_{i,1} \in \mathbb{C}$, that is, the coefficients of the polynomials f_n, g_m, f_{n-1} and g_{m-1}.

3. The polynomial $A(x_1) = \mathrm{Res}_{x_2}(F,G)\,|_{x_0=1}$ has exact degree nm, that is, $a_0 \neq 0$. Indeed, as you can see in [2] (proof of theorem 3.3.5), a_0 is the homogeneous resultant of the equations for the groups $C \cdot l_\infty$ and $D \cdot l_\infty$, which share no point by hypothesis.

If Σ denotes the sum of all the roots of A (counted according to multiplicities), we deduce from 3. and Vieta's formulas that the abcissa of the barycenter of $C \cdot D$ is

$$\frac{\Sigma}{nm} = \frac{-a_1}{a_0 \cdot nm}.$$

By 1. and 2., this abcissa only depends on the parts f_n, f_{n-1} and g_m, g_{m-1} in the polynomials F and G, and thus remains unaltered when any of the curves is replaced by another one with identical two highest homogeneous parts.

A similar reasoning with $\mathrm{Res}_{x_1}(F,G)$ proves that the ordinate of the barycenter remains unaltered too. □

Remark. One could ask whether a similar argument still holds without the assumption of the curves having disjoint improper sections. But notice that if C and D have a common improper point, the equations for $C \cdot l_\infty$ and $D \cdot l_\infty$ have a common root and thus the leading coefficient a_0 of A vanishes.

Because of this reason, the sum of the roots of A is expressed as $\Sigma = \frac{-a_k}{a_{k-1}}$, for some $k \geq 2$. And a_k may depend on the homogeneous parts f_{n-k} and g_{m-k} of F and G.

From theorem 2.4 and our characterization of the two highest homogeneous parts of curves provided by theorem 2.1, we obtain the following result:

Theorem 2.5. *Let C_1 and C_2 be two plane algebraic curves with no parabolic branches and such that, for any direction δ, the groups of asymptotes of C_1 and C_2 that are parallel to δ have the same number of elements (counted with multiplicities) and the same barycentric parallel.*

If D is an algebraic curve sharing no improper points with C_1, then the barycenter of $C_1 \cdot D$ coincides with the barycenter of $C_2 \cdot D$.

Proof. The curves C_1 and C_2 having (according to theorem 2.1) the same two highest homogeneous parts, theorem 2.4 establishes that the intersection groups $C_1 \cdot D$ and $C_2 \cdot D$ have identical barycenters. □

Observe that theorem 2.5 is an extension of Waring's theorem, which only considers the case of replacing a curve by the union of its asymptotes:

Corollary 2.6 (Waring's theorem). *Let C and D be two plane algebraic curves, sharing no improper points and such that C has no parabolic branches. Then, the barycenter of $C \cdot D$ coincides with the barycenter of the intersection group when C is replaced by the union of its asymptotes.*

Example. Consider the conics $C: 3x^2 - 32xy + 9y^2 + 3x + 9y - 18 = 0$ and $D: x^2 + 7xy + 3y^2 + x + 3y - 6 = 0$. One can easily see that their improper sections are disjoint, and consist of two different points, so neither C nor D has parabolic branches.

The curves C and D intersect transversely at the four points $(-3,0)$, $(2,0)$, $(0,1)$ and $(0,-2)$, whose barycenter is $(\frac{-1}{4}, \frac{-1}{4})$.

On the other hand, by applying Waring's theorem twice, we should obtain the same barycenter when replacing each of the conics by the union of its asymptotes. A computation of the asymptotes of C gives the lines

$$l_1: 6\sqrt{229}x - (458 + 32\sqrt{229})y = -75 - 3\sqrt{229},$$

$$l_2: -6\sqrt{229}x - (458 - 32\sqrt{229})y = -75 + 3\sqrt{229}$$

while the asymptotes of D are

$$l_3: 2\sqrt{37}x + (7\sqrt{37} - 37)y = 1 - \sqrt{37},$$

$$l_4: -2\sqrt{37}x - (7\sqrt{37} + 37)y = 1 + \sqrt{37}.$$

The points of $(l_1 + l_2) \cap (l_3 + l_4)$ come from the intersection of four pairs of lines: $(-0.456239, 0.0838138)$, $(-0.869784, 0.0439639)$, $(0.0384386, -0.994811)$ and $(0.287585, -0.132966)$. Direct computation shows that the barycenter of these four points is also $(\frac{-1}{4}, \frac{-1}{4})$.

The following figure, made with Geogebra, illustrates the situation in the real affine plane:

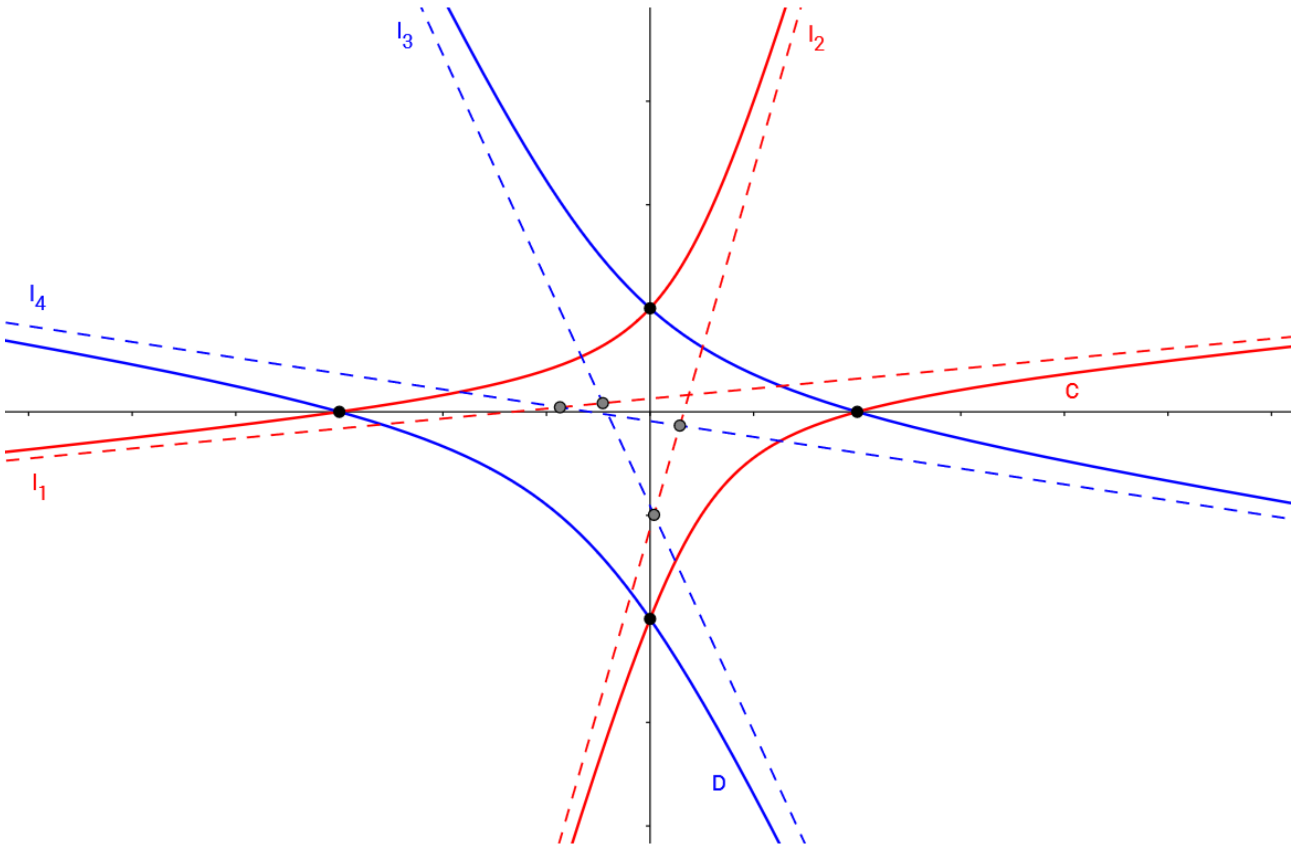


Figure 2

Corollary 2.7. *Let C be a plane algebraic curve, passing through no cyclic point of l_∞. Then, the barycenter of the intersections of C with a circle depends on the position of its centre, but not on its radius.*

Proof. Consider the family of all the circles centered at a fixed point of the affine plane, which we assume having homogeneous coordinates $(1:a:b)$.

That is, we are working with the family of curves $C_r:\ (x_1-ax_0)^2+(x_2-bx_0)^2=r^2x_0^2$ (where r is the radius of the circle C_r). Observe that:

- For any r, the improper section of C_r consists of the cyclic points $I=(0:1:i)$, $J=(0:1:-i)$.
- The asymptotes of C_r are the lines $l_1: x_1+ix_2=(a+bi)x_0$ and $l_2: x_1-ix_2=(a-bi)x_0$.

Since C_r has obviously no parabolic branches, by Waring's theorem we may replace C_r by the union of its asymptotes: the barycenter of the points of $C\cap C_r$ is the barycenter of the points of $C\cap(l_1+l_2)$, which will be the same for every radius r. □

Remark. This result is a nice consequence of Waring's theorem, often mentioned in the classic literature. For instance, it may be found in [4], but restricted to curves C intersecting transversely the line at infinity. It is also quoted in more modern books as [1] (section V.15).

2.3 Chasles theorem

In this section, we are going to prove Chasles theorem on the barycenter of the contact points of the parallel tangent lines to a plane curve. A modern version of this theorem, in terms of polar curves, is:

Theorem 2.8. *Let C be a plane algebraic curve of degree n, meeting the line at infinity in n distinct points. Then, the barycenter of $C\cdot\mathscr{P}_q(C)$ is the same, for any $q\in l_\infty\setminus C$.*

Proof. Take a projective reference in such a way that $q=(0:a_0:a_1)\in l_\infty\setminus C$ and $C: F(x_0,x_1,x_2)=0$, where

$$F(x_0,x_1,x_2)=f_n(x_1,x_2)+f_{n-1}(x_1,x_2)\cdot x_0+\ldots+f_1(x_1,x_2)\cdot x_0^{n-1}+f_0\cdot x_0^n$$

and $f_i\in\mathbb{C}[x_1,x_2]$ is an homogeneous polynomial of degree i, for each $i\in\{0,\ldots,n\}$.

By the hypothesis of C intersecting l_∞ in exactly n distinct points, it is clear that C has not singularities in l_∞ and l_∞ is not tangent to C.

And recall that, by proposition 1.6, $C\cap\mathscr{P}_q(C)$ consists of the singularities of C and the contact points of tangent lines to C passing through q: therefore, C and $\mathscr{P}_q(C)$ have no common improper points.

Now, we are going to apply the results of the previous section, in order to compute the barycenter of $C\cdot\mathscr{P}_q(C)$:

1. By corollary 2.6, we may replace C by the union of its asymptotes (in the sequel, denoted by $A(C)$).
2. By corollary 2.2, f_n and f_{n-1} are the two highest homogeneous parts of the equation of $A(C)$. And observe that the two highest homogeneous parts of the equation of $\mathscr{P}_q(C)$ only depend on f_n and f_{n-1}:

$$\mathscr{P}_q(C):\sum_{j=1}^{n}\left(a_0\frac{\partial f_j}{\partial x_1}+a_1\frac{\partial f_j}{\partial x_2}\right)\cdot x_0^{n-j}=0$$

It follows that $\mathscr{P}_q(C)$ and $\mathscr{P}_q(A(C))$ share their two highest homogeneous parts: according to theorem 2.4, the curve $\mathscr{P}_q(C)$ may be replaced by $\mathscr{P}_q(A(C))$.

So we are studying the barycenter of the group $A(C) \cdot \mathscr{P}_q(A(C))$, whose points are exactly:

- The points of contact of tangents to $A(C)$ through q. But notice that $A(C)$ has a finite number of tangent lines (its own irreducible components, the asymptotes of C), none of which is passing through q, since $q \in l_\infty \setminus C$.
- The singularities of $A(C)$, that is, the points where two (or more) asymptotes intersect.

  These points are ordinary singularities of $A(C)$ (since C has n distinct asymptotes), and hence by lemma 1.8 their multiplicity in the group $A(C) \cdot \mathscr{P}_q(A(C))$ is independent of q.

Consequently, the barycenter of $C \cdot \mathscr{P}_q(C)$ is exactly the barycenter of the intersections of asymptotes of C, which does not depend on the choice of q. □

From theorem 2.8, the classic version of Chasles theorem (in terms of the contact points of tangent lines with a given direction) can be deduced. We prove it for curves whose only singular points are either ordinary singularities or ordinary cusps:

Corollary 2.9. *Let C be a plane algebraic curve of degree n, meeting l_∞ in n distinct points, whose only singular points are either ordinary singularities or ordinary cusps. Then, the locus of the barycenters of the contact points of a complete system of parallel tangents to C (with any direction being non-asymptotic and non-parallel to the tangents of C at its singular points) is a fixed point.*

Proof. Fix a direction, non-asymptotic and non-parallel to the tangents of C at its singular points, and assume it's determined by a point $q \in l_\infty \setminus C$. According to proposition 1.6, $C \cap \mathscr{P}_q(C)$ consists of:

1. The ordinary singularities p of C, where C and $\mathscr{P}_q(C)$ intersect with multiplicity $e_p(C) \cdot (e_p(C) - 1)$, by lemma 1.8.
2. The cusps of C, where C and $\mathscr{P}_q(C)$ intersect with multiplicity 3 (see lemma 1.9).
3. The points of contact of tangent lines to C passing through q. That is, the points of contact of tangent lines which are parallel to that direction.

By Chasles theorem, the barycenter of these points (counted with the proper multiplicities) does not depend on q. But notice that the singularities of C (and the times they are counted) are also independent of q: therefore, the barycenter of the points of type 3 does not depend on q, as desired. □

Remarks.

1. If p is a contact point of a tangent l through q, lemma 1.7 says that the curves C and $\mathscr{P}_q(C)$ intersect at p with multiplicity $[C \cdot l]_p - 1$. So, in order to compute barycenters, p must be counted with weight $[C \cdot l]_p - 1$.

2. Corollary 2.9 is also true with no hypothesis on the singularities of C. A similar proof holds, if one uses that for every point q (contained in no tangent of C at the singular points), the intersection multiplicity of C and $\mathscr{P}_q(C)$ at each singularity of C does not depend on q.

   This result, which may be found (in a local form) as corollary 6.3.2 in [3], requires an advanced study of singularities of curves.

Examples.

1. Let's check Chasles theorem for the smooth cubic $C: -8x^3+10y^3+8x-10y=0$, whose projective closure intersects l_∞ in three distinct points.

   The polar of C with respect to an improper point $q=(0:a:b)\in l_\infty\setminus C$ has affine equation

   $$\mathscr{P}_q(C): a(-24x^2+8)+b(30y^2-10)=0$$

   Notice that the equations of $\mathscr{P}_q(C)$ and C have, respectively, only even and odd exponents. It follows that if a point (x,y) belongs to $C\cdot\mathscr{P}_q(C)$, its opposite point $(-x,-y)$ belongs with the same multiplicity too.

   Therefore, the barycenter of $C\cdot\mathscr{P}_q(C)$ (which consists of the contact points of tangents with the affine direction given by q, by the smoothness of C) is the point $(0,0)$.

   In the following figure, you may find a representation of C in the real affine plane, together with the six tangent lines parallel to $x-2y=0$ (in blue) and the six tangent lines parallel to $x+4y=0$ (in green).

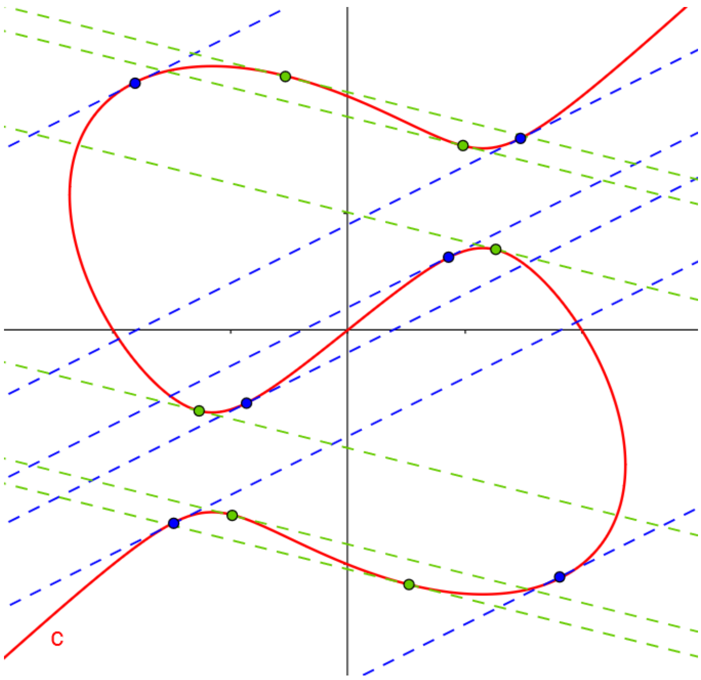


Figure 3

   According to our computations, the barycenter of the six green contact points and the barycenter of the six blue contact points is the same, $(0,0)$.

2. Corollary 2.9 is a generalization of a well-known property for non-degenerate conics. In fact: if we have a smooth conic (not a parabola), and we consider the two tangent lines in any non-asymptotic direction, the midpoint of the contact points coincides with the pole of l_∞ (which is called the *center* of the conic).

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