UNIVERSITAT DE BARCELONA

Master Thesis

Master's Degree in Advanced Mathematics

Weighted Inequalities for the Hardy Operator

Author: Sergi Arias García

Supervisor: Dr. F. Javier Soria de Diego
Department: Matemàtiques i Informàtica

Barcelona, June 2017

Contents

Abstract **iii**

Acknowledgments **v**

1 Introduction **1**

2 Preliminary concepts **3**
- 2.1 Lebesgue Spaces 3
- 2.2 Duality principles and Minkowski's integral inequality 5
- 2.3 Distribution functions, decreasing rearrangements and Lorentz spaces 6

3 Classical Hardy inequalities **13**
- 3.1 The classical Hardy's integral inequality 13
- 3.2 Hardy's inequalitiy with power weights 15

4 Hardy inequalities with weights **17**
- 4.1 First results involving weights 17
- 4.2 Characterization of weighted Hardy inequalities 21
- 4.3 Weighted Hardy inequalities of (p, q) type 26

5 Hardy inequalities for monotone functions **43**
- 5.1 First results for monotone functions 43
- 5.2 Maximal operator and the Hardy inequality for monotone functions . 46
- 5.3 Characterization of the weighted Hardy inequality for monotone functions 50
- 5.4 Applications 61
  - 5.4.1 Normability of Lorentz spaces 61
  - 5.4.2 Normability of weak Lorentz spaces 69
  - 5.4.3 Maximal operator and normability of Lorentz spaces 78

Bibliography **81**

Abstract

This project revolves around Hardy's integral inequality, proved by G. H. Hardy in 1925. This inequality has been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. We study the classical Hardy's integral inequality and its generalizations. We analyse some of the first results including weighted inequalities and prove the key theorem of B. Muckenhoupt, who characterized Hardy's integral inequality with weights for the diagonal case in 1972. After this fundamental result, different authors considered the general context and new characterizations appeared until closing definitely the problem in 2000.

Also we study Hardy's integral inequality in the cone of monotone functions. This point of view is really interesting and has a lot of surprising consequences. For example, M. A. Ariño and B. Muckenhoupt realized in 1990 that Hardy's inequality in the cone of monotone functions is equivalent to the boundedness of the Hardy-Littlewood maximal operator between Lorentz spaces. Just after E. Sawyer proved that the classical Lorentz space $\Lambda^p(w)$ is normable if, and only if, Hardy's integral inequality in the cone of monotone functions is satisfied for w. We study also the normability of both spaces $\Lambda^p(w)$ and $\Lambda^{p,\infty}(w)$ in terms of the boundedness of the maximal operator.

Acknowledgments

To my master thesis advisor, Prof. Javier Soria, for his dedication and patience, and for choosing such an interesting topic.

Chapter 1

Introduction

The project that follows corresponds to the Master Thesis in Mathematics of the Faculty of Mathematics of the University of Barcelona by Sergi Arias.

This Master Thesis is organized in four chapters. In Chapter 2 we give some preliminary concepts and results. Throughout the project we will need to work with some specific spaces such as weighted Lebesgue spaces, weak-type Lebesgue spaces, classical Lorentz spaces or weak-type Lorentz spaces, which are defined in this chapter. It is necessary to introduce some concepts in order to define Lorentz spaces, and therefore the distribution function and nonincreasing rearrangement function are presented, as well as some important properties. It is also presented some duality principles needed in Chapter 5.

This project revolves around Hardy's integral inequality, discovered in 1925 by G. H. Hardy. This inequality has been studied by a large number of authors during the twentieth century and has motivated some important lines of study which are currently active. In Chapter 3 we study the classical Hardy's integral inequality and its first generalization, studied by Hardy himself, including power weights. We also study in both cases if the constants appearing are sharp. In addition, Hardy's inequality can be generalized even more. It can be studied with weights instead of power weights, which at the same time can be considered different in both sides of the inequality, as well as working with different indexes. In Chapter 4 we deal with this kind of problem. We present some of the first results considering weights until reaching the key theorem of B. Muckenhoupt, who characterized Hardy's integral inequality with weights for the diagonal case (when the indexes are the same) in 1972. After this fundamental result, the authors studied the general case and new results appeared until closing definitely the problem in 2000.

Finally, in Chapter 5 we study Hardy's integral inequality in the cone of monotone functions, that is, Hardy's inequality considered just for positive decreasing functions. This point of view is really interesting and has a lot of surprising consequences. M. A. Ariño and B. Muckenhoupt realized in 1990 that the boundedness of Hardy's inequality in the cone of monotone functions is equivalent to the boundedness of the Hardy-Littlewood maximal operator between Lorentz spaces. Thus, the characterization of the weighted Hardy inequality for positive decreasing functions has been widely studied. M. A. Ariño and B. Muckenhoupt proved that the characterization of Hardy's inequality in the cone of monotone functions for the diagonal case is not equivalent to the one given by B. Muckenhoupt in 1972. Here a new class of weights, called B_p, play a crucial role. But this approach of Hardy's inequality has another surprising consequences. For example, E. Sawyer realized that the classical Lorentz space $\Lambda^p(w)$ is normable if, and only if, Hardy's integral inequality in the cone of monotone functions is satisfied for the weight w and the index p. Actually, the normability of the classical Lorentz space $\Lambda^p(w)$ is equivalent to the weak boundedness of the Hardy operator in the cone of monotone functions (when $p \geq 1$) and, similarly, the normability of the weak-type Lorentz space $\Lambda^{p,\infty}(w)$ is equivalent to Hardy's integral inequality for positive decreasing functions. Furthermore, the weak boundedness of the Hardy-Littlewood maximal operator is equivalent to the weak boundedness of the Hardy operator in the cone of monotone functions.

In the execution of this project, the chronological evolution of the study of Hardy's inequality has been extracted from A. Kufner, L. Maligranda and L.-E. Person's book [9] and A. Kufner and L.-E. Person's book [10]. The main results appearing along this Master Thesis have been consulted directly from the original articles. In some cases, the proofs have been extracted from other articles, whose reasoning were easier to understand. Also, some general concepts have been studied in C. Bennett and R. Sharpley's book [3].

Chapter 2

Preliminary concepts

In this chapter we are going to present some definitions and results that we will need in the subsequent chapters.

We start defining the Lebesgue spaces, weighted Lebesgue spaces and weak-type Lebesgue spaces. The weighted Lebesgue spaces will be very important throughout the project and they will be used constantly. The weak-type Lebesgue spaces will appear in Chapter 5. Next we present some duality principles that will be useful in Chapter 5 and the Minkowski's integral inequality. Finally, we define the concepts of distribution function and nonincreasing rearrangement, we give some basic properties and we define both the Lorentz spaces and the weak-type Lorentz spaces. These materials will be used as well in Chapter 5.

2.1 Lebesgue Spaces

We define the classical Lebesgue spaces. We present the general definition for an arbitrary measure space although we will mostly work with $\mathbb{R}^n$.

Definition 2.1.1. Given a measure space (X, μ) and $0 < p \leq \infty$, we define the Lebesgue space L^p as the set of measurable functions on X such that

$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} < \infty,$$

when $0 < p < \infty$, and such that

$$\|f\|_\infty := \operatorname*{ess\,sup}_X f = \inf\{a \in \mathbb{R} : \mu\left(\{x \in X : f(x) > a\}\right) = 0\} < \infty.$$

The next definition states what we will consider by a weight function.

Definition 2.1.2. Consider the interval $(a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq +\infty$. We say that $w : (a, b) \longrightarrow \mathbb{R}$ is a *weight function* if it is measurable, $w(x) \geq 0$ a.e. $x \in (a, b)$ and it is locally integrable on (a, b).

Definition 2.1.3. Given a weight function w and a real number $0 < p < \infty$, we define, for measurable functions $f : (a,b) \longrightarrow \mathbb{R}$,

$$\|f\|_{p,w} := \left(\int_a^b |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

Proposition 2.1.4. *$\|f\|_{p,w}$ is a quasinorm.*

Proof. We observe that $\|f\|_{p,w} \geq 0$ and $\|\lambda f\|_{p,w} = |\lambda| \|f\|_{p,w}$ for all $\lambda \in \mathbb{R}$. In addition, as two real numbers $|x| \geq |y|$ satisfy the inequality

$$|x+y| \leq |x| + |y| \leq 2|x|$$

we have

$$|x+y|^p \leq (2|x|)^p \leq 2^p \left(|x|^p + |y|^p \right). \tag{2.1.1}$$

So using (2.1.1), we conclude

$$\begin{aligned}
\|f+g\|_{p,w} &\leq \left(\int_a^b |f(x)+g(x)|^p w(x) dx \right)^{\frac{1}{p}} \leq \left(\int_a^b 2^p \left(|f(x)|^p + |g(x)|^p \right) w(x) dx \right)^{\frac{1}{p}} \\
&= 2 \left(\int_a^b |f(x)|^p w(x) dx + \int_a^b |g(x)|^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq 2 \cdot 2^{\frac{1}{p}} \left(\left(\int_a^b |f(x)|^p w(x) dx \right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p w(x) dx \right)^{\frac{1}{p}} \right) \\
&= 2^{1+\frac{1}{p}} \left(\|f\|_{p,w} + \|g\|_{p,w} \right).
\end{aligned}$$

Finally, if $f \equiv 0$ then $\|f\|_{p,w} = 0$. Furthermore, if $\|f\|_{p,w} = 0$, we deduce $|f(x)| = 0$ a.e. $x \in (a,b)$.

□

Definition 2.1.5. For a given weight function w and a real number $0 < p < \infty$, we define the weighted Lebesgue space $L^p(a,b;w)$ as (classes of equivalent) functions f on (a,b) such that $\|f\|_{p,w} < \infty$.

Remark 2.1.6. We will usually work with the interval $(0,\infty)$ and then the weighted Lebesgue space will be denoted by $L^p(w)$.

We finally define the weak-type Lebesgue spaces.

Definition 2.1.7. Given $0 < p < \infty$ and a weight w, we define the weak-type Lebesgue space as

$$L^{p,\infty}(w) = \left\{ f \ : \|f\|_{L^{p,\infty}(w)} = \sup_{t>0} \ t^{\frac{1}{p}} \int_{\{s:|f(s)|>t\}} w(s) ds < \infty \right\},$$

where f is a measurable function defined on $\mathbb{R}^n$ or $\mathbb{R}^+$.

Remark 2.1.8. We observe that the $\|.\|_{L^{p,\infty}(v)}$-norm can be also written as

$$\|f\|_{L^{p,\infty}(v)} = \sup_{t>0} |f(t)| \left(\int_0^t w(s) ds \right)^{\frac{1}{p}}.$$

2.2 Duality principles and Minkowski's integral inequality

We present this well-known duality principle for L^p spaces.

Proposition 2.2.1. *Let v and g be measurable functions in $(0,\infty)$ with v positive. Then*

$$\sup_{f\geq 0} \frac{\left|\int_0^\infty f(x)g(x)dx\right|}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx\right)^{\frac{1}{p'}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. First of all we observe that, by Hölder's inequality,

$$\begin{aligned}\left|\int_0^\infty f(x)g(x)dx\right| &\leq \int_0^\infty f(x)v(x)^{\frac{1}{p}}|g(x)|v(x)^{-\frac{1}{p}}dx \\ &\leq \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx\right)^{\frac{1}{p'}},\end{aligned}$$

for any $f\geq 0$. On the other hand, if we consider

$$f(x) = \operatorname{sign}(g(x))v(x)^{1-p'}|g(x)|^{p'-1},$$

then

$$\left|\int_0^\infty f(x)g(x)dx\right| = \int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx$$

and

$$\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} = \left(\int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx\right)^{\frac{1}{p}}.$$

Therefore,

$$\frac{\left|\int_0^\infty f(x)g(x)dx\right|}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx\right)^{1-\frac{1}{p}}.$$

□

Another duality principle, which can be proved in a similar way, is the following.

Proposition 2.2.2. *Given a function $f\in L^p$, $1<p<\infty$, we have that*

$$\|f\|_p = \sup_{g\in L^{p'}, g\neq 0} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|dx}{\|g\|_{p'}}.$$

As a consequence of this last duality principle (Proposition 2.2.2), we can deduce the Minkowski's integral inequality.

Theorem 2.2.3. *(Minkowski's Integral Inequality) For $1 \leq p \leq \infty$,*

$$\left\| \int_{\mathbb{R}^n} F(\cdot, y)dy \right\|_p \leq \int_{\mathbb{R}^n} \|F(\cdot, y)\|_p \, dy.$$

Proof. The case $p = 1$ corresponds to Fubini's Theorem. For the case $p = \infty$ we just notice that

$$\left| \int_{\mathbb{R}^n} F(x, y)dy \right| \leq \int_{\mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |F(x, y)|dy.$$

If $1 < p < \infty$, applying Fubini's Theorem and Proposition 2.2.2, we get

$$\begin{aligned}
\left\| \int_{\mathbb{R}^n} F(\cdot, y)dy \right\|_p &= \sup_{g \in L^{p'}, g \neq 0} \frac{\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x, y)dy \; g(x) \right| dx}{\|g\|_{p'}} \\
&\leq \sup_{g \in L^{p'}, g \neq 0} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x, y)g(x)|dxdy}{\|g\|_{p'}} \\
&\leq \int_{\mathbb{R}^n} \sup_{g \in L^{p'}, g \neq 0} \frac{\int_{\mathbb{R}^n} |F(x, y)g(x)|dx}{\|g\|_{p'}} dy = \int_{\mathbb{R}^n} \|F(\cdot, y)\|_p dy.
\end{aligned}$$

□

2.3 Distribution functions, decreasing rearrangements and Lorentz spaces

We present the notion of distribution function. Throughout this section we will work on a measure space (X, μ).

Definition 2.3.1. Given a measurable function f, we define its distribution function as

$$\lambda_f(t) = \mu(\{x \in X : |f(x)| > t\}),$$

with $t \geq 0$.

Example 2.3.2. ([3, Example I.1.4]) Let us consider a positive simple function

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $a_1 > ... > a_n > 0$ and the sets E_j are pairwise disjoint with finite measure. Then, we have

$$\lambda_f(t) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(t),$$

where

$$m_j = \sum_{i=1}^{j} \mu(E_i).$$

We give some properties of the distribution function (cf. [3, Proposition II.1.3]).

Proposition 2.3.3. *The following properties are satisfied:*

(i) If $|g| \leq |f|$ $\mu-a.e.$*, then* $\lambda_g \leq \lambda_f$.

(ii) If $|f| \leq \liminf_{n\longrightarrow\infty} |f_n|$*, then* $\lambda_f \leq \liminf_{n\longrightarrow\infty} \lambda_{f_n}$.

Proof. For (i) we just notice that $|g(x)| > t \Rightarrow |f(x)| > t$ for almost every $x \in X$. Hence $\lambda_g \leq \lambda_f$. To prove (ii) we fix $t > 0$ and we define the sets

$$E := \{x \in X : |f(x)| > t\} \text{ and } E_n := \{x \in X : |f_n(x)| > t\},\ n \in \mathbb{N}.$$

Notice that $\mu(E) = \lambda_f(t)$ and $\mu(E_n) = \lambda_{f_n}(t)$. Now by hypothesis we deduce that there is an $m \in \mathbb{N}$ such that for all $n > m$ we have $|f(x)| \leq |f_n(x)|$ and, therefore, $E \subseteq \cup_{m\in\mathbb{N}} \cap_{n>m} E_n$. We also notice that

$$\mu\left(\bigcap_{n>m} E_n\right) \leq \inf_{n>m} \mu(E_n) \leq \sup_{m\in\mathbb{N}} \inf_{n>m} \mu(E_n) =: \liminf_{n\longrightarrow\infty} \mu(E_n) \tag{2.3.1}$$

for all $m \in \mathbb{N}$. Finally, as $\bigcap_{n>m} E_n$ increases with m, we conclude, by the Monotone Convergence Theorem and (2.3.1), that

$$\begin{aligned}\mu(E) \leq \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n\right) &= \int_X \chi_{\bigcup_{m=1}^{\infty} \cap_{n>m} E_n}(x)d\mu(x)\\ &= \lim_{m\longrightarrow\infty} \int_X \chi_{\cap_{n>m} E_n}(x)d\mu(x) = \lim_{m\longrightarrow\infty} \mu\left(\bigcap_{n>m} E_n\right) \leq \liminf_{n\longrightarrow\infty} \mu(E_n).\end{aligned}$$

□

A concept related to the distribution function is the nonincreasing rearrangement function.

Definition 2.3.4. Given a measurable function f, we define its nonincreasing rearrangement as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\},$$

with $t \geq 0$.

We present some properties of nonincreasing rearrangement functions (cf. [3, Proposition I.1.7]).

Proposition 2.3.5. *Let* f*,* f_n *and* g *be measurable functions. The following properties are satisfied:*

(i) f^* *is decreasing.*

(ii) *We have*

$$(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$$

for any t_1, $t_2 > 0$.

(iii) *If $|g| \leq |f|$ μ-a.e., then $g^* \leq f^*$.*

(iv) *$\lambda_f(f^*(t)) \leq t$ whenever $f^*(t) < \infty$.*

(v) *If $|f| \leq \liminf_{n\longrightarrow\infty} |f_n|$, then $f^* \leq \liminf_{n\longrightarrow\infty} f_n^*$.*

(vi) *If $|f_n| \nearrow |f|$, then $f_n^* \nearrow f^*$.*

Proof. Properties (i), (ii) and (iv) are immediate consequences of the definition. To see (iii) we notice that, by Proposition 2.3.3 (i) we have $\lambda_g \leq \lambda_f$ and hence

$$\{s > 0 : \lambda_f(s) \leq t\} \subseteq \{s > 0 : \lambda_g(s) \leq t\},$$

from which the property follows.

Now, to prove (v) we observe that $\lambda_f \leq \liminf_{n\longrightarrow\infty} \lambda_{f_n}$ by Proposition 2.3.3 (ii). So there exists an $m \in \mathbb{N}$ such that for all $n > m$ we have $\lambda_f(t) \leq \lambda_{f_n}(t)$ for all $t > 0$ and, therefore,

$$\{s > 0 : \lambda_{f_n}(s) \leq t\} \subseteq \{s > 0 : \lambda_f(s) \leq t\}$$

for all $n > m$, from which the property can be deduced.

Finally we proof (vi). First of all, by (iii) we deduce that $f_n^* \leq f^*$ and hence

$$\liminf_{n\longrightarrow\infty} f_n^* \leq \limsup_{n\longrightarrow\infty} f_n^* \leq f^*. \tag{2.3.2}$$

Furthermore, as $|f| \leq \liminf_{n\longrightarrow\infty} f_n = \lim_{n\longrightarrow\infty} f_n = |f|$, by (v) we deduce that

$$f^* \leq \liminf_{n\longrightarrow\infty} f_n^*. \tag{2.3.3}$$

The result follows by combining (2.3.2) and (2.3.3). □

Example 2.3.6. For the function defined in Example 2.3.2, we have

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}(t),$$

where we define $m_0 = 0$.

The following proposition [3, Proposition II.1.8] is a well known property of the nonincreasing rearrangement functions, which states that f and f^* have the same $\|.\|^p$-norm.

Proposition 2.3.7. *If $0 < p < \infty$ and f is a measurable function, then*

$$\int_X |f(x)|^p d\mu(x) = p\int_0^\infty t^{p-1}\lambda_f(s)ds = \int_0^\infty f^*(t)^p dt.$$

Furthermore, when $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in X} |f(x)| = f^*(0).$$

G. H. Hardy and J. E. Littlewood provided the following inequality ([3, Theorem II.2.2]), which bounds the $\|.\|^1$-norm of the product $f \cdot g$ of two functions by the $\|.\|^1$-norm of the product $f^* \cdot g^*$ of their nonincreasing rearrangement functions.

Proposition 2.3.8. *If f and g are measurable functions, then*

$$\int_X |f(x)g(x)|d\mu(x) \le \int_0^\infty f^*(s)g^*(s)ds.$$

We define the function f^{**}, the average of the nonincreasing rearrangement function f^*.

Definition 2.3.9. For a measurable function f, we define f^{**} as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds,$$

with $t > 0$.

The operator $f \mapsto f^{**}$ is subadditive ([3, Theorem II.3.4]).

Proposition 2.3.10. *If f and g are measurable functions, then*

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$$

for all $t > 0$.

Now we present a theorem [3, Theorem II.6.2] that provides a useful expression for f^{**} in terms of the Peetre's K-functional (cf. [3, Definition V.1.1]) for L^1 and L^∞ , $K(t, f, (L^1, L^\infty))$, which is a very common operator in Interpolation Theory.

Theorem 2.3.11. *If f is a measurable function, then*

$$K(t, f, (L^1, L^\infty)) := \inf_{f=g+h}\{\|g\|_{L^1} + t\|h\|_{L^\infty}\} = \int_0^t f^*(s)ds = tf^{**}(t), \tag{2.3.4}$$

for all $t > 0$.

Proof. The last identity in (2.3.4) follows from the definition of f^{**}. In order to prove the second identity in (2.3.4), we fix a measurable function f and $t > 0$, denoting by a_t the infimum on (2.3.4). We want to prove first that

$$\int_0^t f^*(s)ds \leq a_t. \tag{2.3.5}$$

Let us assume that $f \in L^1 + L^\infty$ since, otherwise, the infimum would be infinite and (2.3.5) would hold trivially. So expressing f as $f = g + h$ with $g \in L^1$ and $h \in L^\infty$, and applying Proposition 2.3.10, we get

$$\int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds + \int_0^t h^*(s)ds.$$

Now, by Proposition 2.3.7, we have

$$\int_0^t g^*(s)ds \leq \int_0^\infty g^*(s)ds = \|g\|_{L^1}$$

and

$$\int_0^t h^*(s)ds \leq th^*(0) = t \operatorname*{ess\,sup}_{x\in X} |h(x)| = t\|h\|_{L^\infty}.$$

Therefore,

$$\int_0^t f^*(s)ds \leq \|g\|_{L^1} + t\|h\|_{L^\infty},$$

and taking the infimum over the representations of f, we get (2.3.5).

In order to prove the reverse inequality,

$$a_t \leq \int_0^t f^*(s)ds,$$

we are going to construct functions $g \in L^1$ and $h \in L^\infty$ such that

$$\|g\|_{L^1} + t\|h\|_{L^\infty} \leq \int_0^t f^*(s)ds. \tag{2.3.6}$$

Assuming that the right hand side on (2.3.6) is finite (otherwise there is nothing to prove), Proposition 2.3.8 and Example 2.3.6 provides that f is integrable over any subset of X of measure at most t. Now if we define $E := \{x \in X : |f(x)| > f^*(t)\}$ and we denote $t_0 = \mu(E)$, by Proposition 2.3.5 (iv) it must be $t_0 \leq t$, concluding that f is integrable over E. As a consequence, the function

$$g(x) := \max\{|f(x)| - f^*(t), 0\} \cdot \operatorname{sign}(x)$$

is integrable, as well as

$$h(x) := \min\{|f(x)|, f^*(t)\} \cdot \operatorname{sign}(x)$$

is a function in L^∞ bounded by $f^*(t)$. We observe that $f = g + h$. Hence, by Proposition 2.3.8,

$$\|g\|_{L^1} = \int_E (|f(x)| - f^*(t))d\mu(x) = \int_E |f(x)|d\mu - \mu(E)f^*(t) \le \int_0^{t_0} f^*(s)ds - t_0 f^*(t),$$

and so

$$\|g\|_{L^1} + t\|h\|_{L^\infty} \le \int_0^{t_0} f^*(s)ds + (t - t_0)f^*(t) = \int_0^t f^*(s)ds,$$

where in the last equality we have used that $f^*(s)$ is constant when $t_0 \le s \le t$ with value $f^*(t)$. □

Finally, we present the classical Lorentz spaces and the weak-type Lorentz spaces.

Definition 2.3.12. Given a weight w in $\mathbb{R}^+$ and $0 < p < \infty$, we define the weighted Lorentz space $\Lambda_p(w)$ as the set of measurable functions satisfying

$$\|f\|_{\Lambda_p(w)} := \|f^*\|_{p,w} = \left(\int_X (f^*(t))^p w(t)d\mu(t)\right)^{\frac{1}{p}} < \infty.$$

Definition 2.3.13. Given a weight w in $\mathbb{R}^+$ and $0 < p < \infty$, we define the weak-type Lorentz space as

$$\Lambda^{p,\infty}(w) = \left\{ f \ : \|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} \ t \left(\int_0^{\lambda_f(t)} w(s)ds\right)^{\frac{1}{p}} < \infty \right\},$$

where f is a measurable function defined on $\mathbb{R}^n$.

Remark 2.3.14. We observe that the $\|.\|_{\Lambda^{p,\infty}(w)}$-norm can be also written as

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} f^*(t) \left(\int_0^t w(s)ds\right)^{\frac{1}{p}}.$$

Chapter 3

Classical Hardy inequalities

In this short chapter we present the classical Hardy inequality, an historical result that G. H. Hardy proved in 1925, and its generalization with power weights, also studied by G. H. Hardy in 1928.

3.1 The classical Hardy's integral inequality

The following theorem is known as the Hardy's integral inequality.

Theorem 3.1.1. *If $f(x) \geq 0$, $p > 1$ and $\int_0^\infty f^p(x)dx$ is convergent, then*

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx.$$

Proof. Changing the variable ($t = xs$) we get

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^1 f(xs)ds \right)^p dx \right)^{\frac{1}{p}}.$$

Using Minkowski's integral inequality (Theorem 2.2.3) and changing again the variable $u = xs$, we conclude

$$\begin{aligned} \left(\int_0^\infty \left(\int_0^1 f(xs)ds \right)^p dx \right)^{\frac{1}{p}} &\leq \int_0^1 \left(\int_0^\infty f(xs)^p dx \right)^{\frac{1}{p}} ds \\ &= \int_0^1 \left(\int_0^\infty f(u)^p \frac{du}{s} \right)^{\frac{1}{p}} ds \\ &= \left(\frac{p}{p-1} \right) \left(\int_0^\infty f(x)^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

□

Definition 3.1.2. We define the *classical Hardy operator* as

$$\mathcal{H}f(x) := \frac{1}{x} \int_0^x f(t)dt.$$

Remark 3.1.3. (See Example 1 in [9]) The constant $\frac{p}{p-1}$ in Theorem 3.1.1 is sharp. Indeed, $\mathcal{H}$ is bounded on $L^p(0,\infty)$ and $\|\mathcal{H}\|_{L^p(0,\infty)\longrightarrow L^p(0,\infty)} \leq \frac{p}{p-1}$ but, actually, we are going to prove that the norm of $\mathcal{H}$ is exactly $\frac{p}{p-1}$. To see this, we can take the functions defined as $f_\epsilon(t) = t^{\frac{-1}{p}+\epsilon}\chi_{(0,a)}(t)$ with $0<\epsilon<\frac{1}{p}$ and $a>0$. Then

$$\|f_\epsilon\|_p = \left(\int_0^a x^{-1+p\epsilon}dx\right)^{\frac{1}{p}} = \left(\left[\frac{x^{p\epsilon}}{p\epsilon}\right]_0^a\right)^{\frac{1}{p}} = \frac{a^\epsilon}{(p\epsilon)^{\frac{1}{p}}}$$

and, changing variables $\left(s=\frac{t}{x}\right)$,

$$\begin{aligned}
\|\mathcal{H}f_\epsilon\|_p &= \left(\int_0^\infty \left(\frac{1}{x}\int_0^x t^{-\frac{1}{p}+\epsilon}\chi_{(0,a)}(t)dt\right)^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(\int_0^1 (sx)^{-\frac{1}{p}+\epsilon}\chi_{\left(0,\frac{a}{x}\right)}(s)ds\right)^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^a \left(\int_0^1 (sx)^{-\frac{1}{p}+\epsilon}ds\right)^p dx + \int_a^\infty \left(\int_0^{\frac{a}{x}} (sx)^{-\frac{1}{p}+\epsilon}ds\right)^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^a \left(\frac{1}{x}\frac{x^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}\right)^p dx + \int_a^\infty \left(\frac{1}{x}\frac{a^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}\right)^p dx\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{(1-\frac{1}{p}+\epsilon)^p}\left[\frac{x^{\epsilon p}}{\epsilon p}\right]_{x=0}^a + \frac{\left(a^{1-\frac{1}{p}+\epsilon}\right)^p}{(1-\frac{1}{p}+\epsilon)^p}\left[\frac{x^{1-p}}{1-p}\right]_{x=a}^\infty\right)^{\frac{1}{p}} \\
&= \frac{a^\epsilon}{1-\frac{1}{p}+\epsilon}\left(\frac{1}{\epsilon p}+\frac{1}{p-1}\right)^{\frac{1}{p}}.
\end{aligned}$$

As a consequence

$$\|\mathcal{H}\|_{L^p(0,\infty)\longrightarrow L^p(0,\infty)} \geq \frac{\|\mathcal{H}f_\epsilon\|_p}{\|f_\epsilon\|_p} = \frac{1}{1-\frac{1}{p}+\epsilon}\left(1+\epsilon\frac{p}{p-1}\right)^{\frac{1}{p}} \underset{\epsilon\to 0}{\longrightarrow} \frac{p}{p-1}$$

and then $\|\mathcal{H}\|_{L^p(0,\infty)\longrightarrow L^p(0,\infty)} = \frac{p}{p-1}$.

Remark 3.1.4. Theorem 3.1.1 is not true for $p=1$. If it would exist a constant C such that

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)dx \leq C\int_0^\infty f(x)dx,$$

we could choose $f(x)=\chi_{(0,1)}(x)$ and we would get a contradiction, since

$$\mathcal{H}f(x) = \begin{cases} 1 & \text{if } 0<x<1, \\ \frac{1}{x} & \text{if } x\geq 1, \end{cases}$$

and

$$\int_0^1 dx + \int_1^\infty \frac{1}{x} dx = \int_0^\infty (\mathcal{H}f)(x)dx \leq C \int_0^\infty f(x)dx = C,$$

where the left hand side of the inequality is not convergent.

3.2 Hardy's inequalitiy with power weights

The following theorem (cf. [9, Theorem 2]) generalizes the classical Hardy's integral inequality by introducing power weights x^α.

Theorem 3.2.1. *If f is a positive function, $p \geq 1$ and $\alpha < p - 1$, then*

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p x^\alpha dx \leq \left(\frac{p}{p - \alpha - 1} \right)^p \int_0^\infty f^p(x) x^\alpha dx.$$

Proof. Changing the variable ($t = xs$) we get

$$\begin{aligned} \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p x^\alpha dx \right)^{\frac{1}{p}} &= \left(\int_0^\infty \left(\int_0^1 f(xs)ds \right)^p x^\alpha dx \right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \left(\int_0^1 f(xs) x^{\frac{\alpha}{p}} ds \right)^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, applying Minkowski's integral inequality (cf. Theorem 2.2.3) and changing the variable ($u = xs$), we conclude

$$\begin{aligned} \left(\int_0^\infty \left(\int_0^1 f(xs) x^{\frac{\alpha}{p}} ds \right)^p dx \right)^{\frac{1}{p}} &\leq \int_0^1 \left(\int_0^\infty f^p(xs) x^\alpha dx \right)^{\frac{1}{p}} ds \\ &= \int_0^1 \left(\int_0^\infty f^p(y) \left(\frac{y}{s} \right)^\alpha \frac{dy}{s} \right)^{\frac{1}{p}} ds \\ &= \left(\int_0^1 s^{-\frac{1}{p}(\alpha+1)} ds \right) \left(\int_0^\infty f^p(y) y^\alpha dy \right)^{\frac{1}{p}} \\ &= \frac{p}{p - \alpha - 1} \left(\int_0^\infty f^p(y) y^\alpha dy \right)^{\frac{1}{p}}. \end{aligned}$$

$\square$

Remark 3.2.2. The constant $\frac{p}{p-\alpha-1}$ in Theorem 3.2.1 is sharp. Indeed, proceeding in the same way as in Remark 3.1.3, we can consider the functions

$$f_\epsilon(t) = t^{-\frac{1}{p} - \frac{1}{\alpha} + \epsilon} \chi_{(0,a)}(t),$$

with $a > 0$. Then

$$\|f_\epsilon\|_{p,x^\alpha} = \left(\int_0^a x^{-1-\alpha+p\epsilon} x^\alpha dx \right)^{\frac{1}{p}} = \left(\left[\frac{x^{p\epsilon}}{p\epsilon} \right]_0^a \right)^{\frac{1}{p}} = \frac{a^\epsilon}{(p\epsilon)^{\frac{1}{p}}}$$

and, changing variables $\left(s = \frac{t}{x}\right)$,

$$
\begin{aligned}
\|\mathcal{H}f_\epsilon\|_{p,x^\alpha} &= \left(\int_0^\infty \left(\frac{1}{x}\int_0^x t^{-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\chi_{(0,a)}(t)dt\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left(\int_0^1 (sx)^{-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\chi_{(0,\frac{a}{x})}(s)ds\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^a \left(\int_0^1 (sx)^{-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}ds\right)^p x^\alpha dx + \int_a^\infty \left(\int_0^{\frac{a}{x}} (sx)^{-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}ds\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^a \left(\frac{1}{x}\frac{x^{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}}{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\right)^p x^\alpha dx + \int_a^\infty \left(\frac{1}{x}\frac{a^{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}}{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\right)^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{(1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon)^p}\left[\frac{x^{\epsilon p}}{\epsilon p}\right]_{x=0}^{a} + \frac{a^{p-1-\alpha+\epsilon p}}{(1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon)^p}\left[\frac{x^{\alpha-p+1}}{\alpha-p+1}\right]_{x=a}^{\infty}\right)^{\frac{1}{p}} \\
&= \frac{a^\epsilon}{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\left(\frac{1}{\epsilon p}+\frac{1}{p-\alpha-1}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|\mathcal{H}\|_{L^p(0,\infty;x^\alpha)\longrightarrow L^p(0,\infty;x^\alpha)} \geq \frac{\|\mathcal{H}f_\epsilon\|_{p,x^\alpha}}{\|f_\epsilon\|_{p,x^\alpha}} &= \frac{1}{1-\frac{1}{p}-\frac{\alpha}{p}+\epsilon}\left(1+\epsilon\frac{p}{p-\alpha-1}\right)^{\frac{1}{p}} \\
&\xrightarrow[\epsilon\to 0]{} \frac{p}{p-\alpha-1}
\end{aligned}
$$

and hence the constant in Theorem 3.2.1 is sharp.

Remark 3.2.3. (cf. [9, Theorem 2]) Conditions $p \geq 1$ and $\alpha < p-1$ are essential in Theorem 3.2.1. Indeed, if we consider the functions $f_a(x) = \chi_{(a,a+1)}(x)$ with $a > 0$, then

$$
\mathcal{H}f(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{x} & a \leq x \leq a+1, \\ \frac{1}{x} & x \geq a+1, \end{cases}
$$

and, if $\alpha \geq p-1$,

$$
\int_0^\infty \mathcal{H}f_a(x)^p x^\alpha dx \geq \int_{a+1}^\infty \mathcal{H}f_a(x)^p x^\alpha dx = \int_{a+1}^\infty x^{\alpha-p}dx,
$$

where this last integral is not convergent since $\alpha \geq p-1$.

On the other hand, if $0 < p < 1$ and $\alpha \leq p-1$, then

$$
\begin{aligned}
\frac{\int_0^\infty \mathcal{H}f_a(x)^p x^\alpha dx}{\int_0^\infty f_a(x)^p x^\alpha dx} \geq \frac{\int_{a+1}^\infty x^{\alpha-p}dx}{\int_a^{a+1} x^\alpha dx} &\geq \frac{\int_{a+1}^\infty x^{\alpha-p}dx}{(a+1)^\alpha} \\
&= \frac{1}{p-\alpha-1}\frac{(a+1)^{\alpha-p+1}}{(a+1)^\alpha} \xrightarrow[a\to\infty]{} \infty,
\end{aligned}
$$

and Theorem 3.2.1 does not hold.

Chapter 4

Hardy inequalities with weights

Hardy's integral inequality can be generalized by considering different weights (instead of just power weights) in both sides of the inequality. In Section 4.1 we present some of the first results considering the Hardy inequality with weights. Several authors considered this kind of inequalities until B. Muckenhoupt gave the first result characterizing completely the Hardy's integral inequality with weights in the diagonal case ($p = q$). In Section 4.2 we study this result. Finally, another large amount of authors studied the general case ($p \neq q$), which is presented in Section 4.3.

4.1 First results involving weights

Definition 4.1.1. A weighted Hardy inequality is an inequality of the form

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}},$$

where u and v are weights, and p, q are positive real numbers.

The first result characterizing completely the weighted Hardy inequality for the case $v(x) = 1$ and $p = q = 2$ appeared in 1958 and it is due to Kac-Krein (cf. [9, Theorem 3]).

Theorem 4.1.2. *The inequality*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^2 u(x)dx \leq C\int_0^\infty f(x)^2 dx \tag{4.1.1}$$

holds for every $f \in L^2(0,\infty)$ if, and only if, the supremum

$$A := \sup_{r>0} r\int_r^\infty \frac{u(x)}{x^2}dx$$

is finite.

Remark 4.1.3. Equivalently, we can write $u(x)$ instead of $\frac{u(x)}{x^2}$ and then (4.1.1) becomes

$$\int_0^\infty \left(\int_0^x f(t)dt\right)^2 u(x)dx \le C\int_0^\infty f(x)^2dx \tag{4.1.2}$$

and A is now

$$A := \sup_{r>0} r\int_r^\infty u(x)dx.$$

Proof. We are going to work with the changes done in Remark 4.1.3. First we assume inequality (4.1.2) holds for every $f \in L^2(0,\infty)$. We consider, for $r, h > 0$, the functions

$$f_h(x) := r^{-\frac{1}{2}}\chi_{(0,r]}(x) - r^{\frac{1}{2}}h^{-1}\chi_{[r+h,r+2h)}(x),$$

which are in $L^2(0,\infty)$. Now, we notice that

$$\int_0^\infty f_h(x)^2dx = \int_0^r r^{-1}dx + \int_{r+h}^{r+2h} rh^{-2}dx = 1 + \frac{r}{h}$$

and

$$\begin{aligned}\int_0^x f_h(t)dt &= \int_0^x r^{-\frac{1}{2}}\chi_{(0,r]}(t)dt - \int_0^x r^{\frac{1}{2}}h^{-1}\chi_{[r+h,r+2h)}(t)dt \\ &= xr^{-\frac{1}{2}}\chi_{(0,r]}(x) + r^{\frac{1}{2}}\chi_{(r,\infty)}(x) \\ &\quad - r^{\frac{1}{2}}h^{-1}(x-r-h)\chi_{(r+h,r+2h]}(x) - r^{\frac{1}{2}}\chi_{(r+2h,\infty)}(x) \\ &= xr^{-\frac{1}{2}}\chi_{(0,r]}(x) + r^{\frac{1}{2}}\chi_{(r,r+h]}(x) \\ &\quad + \left(r^{\frac{1}{2}} - r^{\frac{1}{2}}h^{-1}(x-r-h)\right)\chi_{(r+h,r+2h]}(x).\end{aligned}$$

Then

$$\begin{aligned}&\int_0^\infty \left|\int_0^x f_h(t)dt\right|^2 u(x)dx \\ &\quad = \int_0^r x^2r^{-1}u(x)dx \\ &\qquad + \int_r^{r+h} ru(x)dx \int_{r+h}^{r+2h} \left|\left(r^{\frac{1}{2}} - r^{\frac{1}{2}}h^{-1}(x-r-h)\right)\chi_{(r+h,r+2h]}(x)\right|^2 u(x)dx \\ &\quad \ge \int_r^{r+h} ru(x)dx.\end{aligned}$$

We conclude

$$r\int_r^{r+h} u(x)dx \le C\int_0^\infty f_h(x)^2dx = C\left(1+\frac{r}{h}\right),$$

and hence, making $h \longrightarrow \infty$, we obtain

$$r\int_r^\infty u(x)dx \le C,$$

which implies

$$A = \sup_{r>0} r \int_r^\infty u(x)dx < \infty.$$

Now we assume that $A < \infty$. Given the integral

$$\int_0^\infty \left(\int_0^x f(t)dt \right)^2 u(x)dx,$$

we are going to integrate by parts with $dv = u(x)$. First we notice that, by the Lebesgue Differentiation Theorem, if $v = -\int_x^\infty u(t)dt$ then $dv = u(x)$ a.e. x. So we have

$$\begin{aligned}
\int_0^\infty \left(\int_0^x f(t)dt \right)^2 u(x)dx &= -\left[\left(\int_0^x f(t)dt \right)^2 \left(\int_x^\infty u(t)dt \right) \right]_{x=0}^\infty \\
&\quad + 2\int_0^\infty \left(\int_0^x f(t)dt \right) f(x) \left(\int_x^\infty u(t)dt \right) dx \\
&= 2\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt \right) f(x) \left(x\int_x^\infty u(t)dt \right) dx.
\end{aligned}$$

Now, applying Hölder's inequality and Theorem 3.1.1, we get

$$\begin{aligned}
&2\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt \right) f(x) \left(x\int_x^\infty u(t)dt \right) dx \\
&\leq 2\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt \right)^2 dx \right)^{\frac{1}{2}} \left(\int_0^\infty f(x)^2 \left(x\int_x^\infty u(t)dt \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq 2A\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt \right)^2 dx \right)^{\frac{1}{2}} \left(\int_0^\infty f(x)^2 dx \right)^{\frac{1}{2}} \\
&\leq 4A\left(\int_0^\infty f(x)^2 dx \right)^{\frac{1}{2}} \left(\int_0^\infty f(x)^2 dx \right)^{\frac{1}{2}} = 4A\int_0^\infty f(x)^2 dx
\end{aligned}$$

□

Definition 4.1.4. For a given weight u, we define the modified Hardy operator as

$$H_u f(x) := \frac{1}{xu(x)} \int_0^x f(t)u(t)dt.$$

The following theorem was proved by N. Levinson in 1964 (cf. [13, Theorem 4]).

Theorem 4.1.5. *Let $p > 1$ and $f \geq 0$. Let $r(x)$ be an absolutely continuous function defined for $x > 0$. Assume*

$$\frac{p-1}{p} + \frac{xr'}{r} \geq \frac{1}{\lambda} \tag{4.1.3}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then

$$\int_0^\infty H_r f(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx.$$

Proof. We consider $0 < a < b < \infty$ and let

$$h_{r,a}f(x) := \frac{1}{r(x)} \int_a^x r(t)f(t)dt.$$

Then, defining $H_{r,a}f(x) = \frac{1}{x} h_{r,a}f(x)$ and integrating by parts (with $u = (h_{r,a}f)^p$ and $dv = x^{-p}$), we obtain

$$\int_a^b H_{r,a}f(x)^p dx = \int_a^b \frac{1}{x^p}(h_{r,a}f)^p dx = \left[\frac{1}{1-p} x^{-p+1}(h_{r,a}f)(x)^p\right]_a^b$$
$$+ \int_a^b \frac{x^{-p+1}}{p-1}\left(p(h_{r,a}f(x))^{p-1}f(x) - p(h_{r,a}f(x))^p \frac{r'(x)}{r(x)}\right) dx.$$

Now we notice that the term

$$\left[\frac{x^{1-p}}{1-p}(h_{r,a}f(x))^p\right]_a^b = \frac{b^{1-p}}{1-p}(h_{r,a}f(b))^p$$

is negative, since $p > 1$, $h_{r,a}f \geq 0$ and $b > 0$. Hence,

$$\int_a^b (H_{r,a}f(x))^p dx \leq \int_a^b \frac{x^{-p+1}}{p-1}\left(p(h_{r,a}f(x))^{p-1}f(x) - p(h_{r,a}f(x))^p \frac{r'(x)}{r(x)}\right) dx,$$

or equivalently,

$$\int_a^b \left(\frac{p-1}{p} + \frac{xr'(x)}{r(x)}\right)(H_{r,a}f(x))^p dx \leq \int_a^b (H_{r,af}(x))^{p-1}f(x)dx.$$

Now, using (4.1.3) and Hölder's inequality, we get

$$\frac{1}{\lambda}\int_a^b (H_{r,a}f(x))^p dx \leq \left(\int_a^b (H_{r,a}f(x))^{p-1\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}}$$
$$\leq \left(\int_a^b (H_{r,a}f(x))^p dx\right)^{\frac{p-1}{p}} \left(\int_0^\infty f(x)^p dx\right)^{\frac{1}{p}},$$

that is,

$$\int_a^b (H_{r,a}f(x))^p dx \leq \lambda^p \int_0^\infty f(x)^p dx.$$

If we take $c > a$ then

$$\int_c^b (H_{r,a}f(x))^p dx \leq \int_a^b (H_{r,a}f(x))^p dx \leq \lambda^p \int_0^\infty f(x)^p dx,$$

and, by the Dominated Convergence Theorem, making $a \longrightarrow \infty$ we get

$$\int_c^b (H_r f(x))^p dx \leq \lambda^p \int_0^\infty f(x)^p dx$$

for all $c, b > 0$. Finally, letting $b \longrightarrow \infty$ and $c \longrightarrow 0$,

$$\int_0^\infty (H_r f(x))^p dx \leq \lambda^p \int_0^\infty f(x)^p dx.$$

$\square$

Corollary 4.1.6. *Let $p > 1$ and $f \geq 0$. Let $u(x)$ be an absolutely continuous function defined for $x > 0$. Assume*

$$\frac{p-1}{p} - p\frac{xu'}{u} \geq \frac{1}{\lambda} \tag{4.1.4}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then

$$\int_0^\infty \mathcal{H}f(x)^p u(x) dx \leq \lambda^p \int_0^\infty f(x)^p u(x) dx.$$

Proof. We define $r(x) = \left(\frac{1}{u(x)}\right)^{\frac{1}{p}}$ and then (4.1.4) becomes

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda},$$

since

$$\frac{r'(x)}{r(x)} = -\frac{1}{p}\frac{u'(x)}{u(x)}.$$

Now, we express $f(x) = r(x)g(x)$ for the suitable $g(x) \geq 0$ and we apply Theorem 4.1.5 to g, obtaining

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p u(x)dx \leq \lambda^p \int_0^\infty f(x)^p u(x)dx.$$

$\square$

4.2 Characterization of weighted Hardy inequalities

Some authors like G. Tomaselli, G. Talenti or M. Artola worked in the weighted case, giving some important results. However, B. Muckenhoupt was the first one who gave the complete characterization of this kind of inequalities (cf. [15, Theorem 1]).

Theorem 4.2.1. *Let $1 \leq p \leq \infty$. For a given two weights u and v, we can find a finite constant $C > 0$ such that*

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)dt\right|^p u(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty |f(x)|^p v(x)dx\right)^{\frac{1}{p}} \tag{4.2.1}$$

if, and only if,

$$B = \sup_{r>0} \left(\int_r^\infty \frac{u(x)}{x^p} dx \right)^{\frac{1}{p}} \left(\int_0^r v(x)^{\frac{-1}{p-1}} dx \right)^{\frac{1}{p'}} < \infty.$$

In addition, if C is the sharp constant for (4.2.1), then

$$B \leq C \leq p^{\frac{1}{p}} \, (p')^{\frac{1}{p'}} \, B,$$

if $1 < p < \infty$, and $C = B$ if $p = 1$ or $p = \infty$.

Remark 4.2.2. Equivalently, we can call $U(x) = \frac{1}{x} \, u(x)^{\frac{1}{p}}$, $V(x) = v(x)^{\frac{1}{p}}$ and then (4.2.1) becomes

$$\left(\int_0^\infty \left| U(x) \int_0^x f(t)dt \right|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty |V(x)f(x)|^p \, dx \right)^{\frac{1}{p}} \tag{4.2.2}$$

and B is now

$$B = \sup_{r>0} \left(\int_r^\infty U(x)^p dx \right)^{\frac{1}{p}} \left(\int_0^r V(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

This is the notation used by Muckenhoupt in his paper [15].

Proof. We are going to use the notations given in Remark 4.2.2. We want to prove first that

$$\left(\int_0^\infty \left| U(x) \int_0^x f(t)dt \right|^p dx \right)^{\frac{1}{p}} \leq p^{\frac{1}{p}} \, (p')^{\frac{1}{p'}} \, B \left(\int_0^\infty |V(x)f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $1 < p < \infty$, which would imply that $C \leq p^{\frac{1}{p}} \, (p')^{\frac{1}{p'}} \, B$.

We define $h(x) = \left(\int_0^x |V(t)|^{-p'} dt \right)^{\frac{1}{pp'}}$ and, by Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \left| U(x) \int_0^x f(t)dt \right|^p dx \\
&= \int_0^\infty \left| U(x) \int_0^x f(t) \frac{V(t)h(t)}{V(t)h(t)} dt \right|^p dx \\
&\leq \int_0^\infty |U(x)|^p \left(\int_0^x |f(t)V(t)h(t)|^p dt \right) \left(\int_0^x |V(s)h(s)|^{-p'} ds \right)^{\frac{p}{p'}} dx.
\end{aligned}$$

Now, applying Fubini's theorem we get

$$\begin{aligned}
&\int_0^\infty \int_0^x |U(x)|^p |f(t)V(t)h(t)|^p \left(\int_0^x |V(s)h(s)|^{-p'} ds \right)^{\frac{p}{p'}} dt \; dx \\
&= \int_0^\infty |f(t)V(t)h(t)|^p \left(\int_t^\infty |U(x)|^p \left(\int_0^x |V(s)h(s)|^{-p'} ds \right)^{p-1} dx \right) dt.
\end{aligned}$$

We want to bound

$$\int_0^\infty |f(t)V(t)h(t)|^p \left(\int_t^\infty |U(x)|^p \left(\int_0^x |V(s)h(s)|^{-p'} ds\right)^{p-1} dx\right) dt. \tag{4.2.3}$$

First we notice that, by hypothesis,

$$\left(\int_0^x |V(s)|^{-p'} ds\right)^{\frac{1}{p'}} \leq B \left(\int_x^\infty |U(s)|^p ds\right)^{\frac{-1}{p}},$$

and

$$\left(\int_t^\infty |U(s)|^p ds\right)^{\frac{1}{p}} \leq B \left(\int_0^t |V(s)|^{-p'} ds\right)^{\frac{-1}{p'}} = B|h(t)|^{-p}.$$

Hence

$$\begin{aligned}
\left(\int_0^x |V(s)h(s)|^{-p'} ds\right)^{p-1} &= \left(\int_0^x |V(s)|^{-p'} \left(\int_0^s |V(t)|^{-p'} dt\right)^{\frac{-1}{p}} ds\right)^{p-1} \\
&= \left(\left[p' \left(\int_0^s |V(t)|^{-p'} dt\right)^{\frac{1}{p'}}\right]_{s=0}^{x}\right)^{p-1} \\
&= (p')^{p-1} \left(\int_0^x |V(s)|^{-p'} ds\right)^{\frac{p-1}{p'}} \\
&\leq (Bp')^{p-1} \int_t^\infty |U(x)|^p \left(\int_x^\infty |U(s)|^p ds\right)^{\frac{-1}{p'}} dx \\
&= (Bp')^{p-1} \left[-p \left(\int_x^\infty |U(s)|^p ds\right)^{\frac{1}{p}}\right]_{x=t}^{\infty} \\
&= p(Bp')^{p-1} \left(\int_t^\infty |U(s)|^p ds\right)^{\frac{1}{p}} \\
&\leq p\ B^p\ (p')^{p-1} |h(t)|^{-p}.
\end{aligned}$$

Then, (4.2.3) is bounded by

$$\int_0^\infty |f(t)V(t)h(t)|^p\ p\ B^p\ (p')^{p-1} |h(t)|^{-p} dt = pB^p (p')^{p-1} \int_0^\infty |f(t)V(t)|^p dt,$$

and hence

$$\int_0^\infty \left|U(x) \int_0^x f(t)dt\right|^p dx \leq pB^p (p')^{p-1} \int_0^\infty |f(t)V(t)|^p dt.$$

Now we want to see that $C \leq B$ when $p = 1$ or $p = \infty$. In the case $p = 1$ we have, applying Tonelli's theorem,

$$\int_0^\infty \left|U(x) \int_0^x f(t)dt\right| dx \leq \int_0^\infty \int_0^x |U(x)||f(t)|dt\ dx = \int_0^\infty |f(t)| \int_t^\infty |U(x)|dx\ dt,$$

and since $\frac{1}{|V(t)|}\int_t^\infty |U(x)|dx \leq B$, we conclude that

$$\int_0^\infty \left| U(x)\int_0^x f(t)dt\right| dx \leq B\int_0^\infty |f(t)||V(t)|dt.$$

On the other hand, when $p=\infty$, we have

$$B=\sup_{r>0}\left(\operatorname*{ess\,sup}_{x>r}|U(x)|\right)\left(\int_0^r \frac{1}{|V(x)|}dx\right),$$

and then

$$\begin{aligned}\left| U(x)\int_0^x f(t)dt\right| &\leq |U(x)|\int_0^x |f(t)|\frac{|V(t)|}{|V(t)|}dt\\ &\leq \left(\operatorname*{ess\,sup}_{t>0}|f(t)V(t)|\right)|U(x)|\int_0^x \frac{1}{|V(t)|}dt\\ &\leq B\left(\operatorname*{ess\,sup}_{t>0}|f(t)V(t)|\right).\end{aligned}$$

Finally, we want to see that $B\leq C$. First we notice that for non negative f and for $r>0$ we have

$$\left(\int_r^\infty \left| U(x)\int_0^r f(t)dt\right|^p dx\right)^{\frac{1}{p}} \leq \left(\int_0^\infty \left| U(x)\int_0^x f(t)dt\right|^p dx\right)^{\frac{1}{p}},$$

and taking $V(x)\chi_{(0,r)}(x)$ as $V(x)$, (4.2.1) becomes

$$\left(\int_r^\infty |U(x)|^p dx\right)^{\frac{1}{p}}\left|\int_0^r f(t)dt\right| \leq C\left(\int_0^r |V(x)f(x)|^p dx\right)^{\frac{1}{p}}. \tag{4.2.4}$$

Now we are going to prove that

$$\left(\int_r^\infty |U(x)|^p dx\right)^{\frac{1}{p}}\left(\int_0^r |V(x)|^{-p'}dx\right)^{\frac{1}{p'}} \leq C, \tag{4.2.5}$$

which would imply $B\leq C$.

In the case $p\neq 1$ and $0<\int_0^r |V(x)|^{-p'}dx<\infty$, taking $f(x)=|V(x)|^{-p'}$ we get, from (4.2.4), that

$$\begin{aligned}\left(\int_r^\infty |U(x)|^p dx\right)^{\frac{1}{p}}\left(\int_0^r |V(x)|^{-p'}dx\right) &\leq C\left(\int_0^r |V(x)|^{p-pp'}dx\right)^{\frac{1}{p}}\\ &= C\left(\int_0^r |V(x)|^{-p'}dx\right)^{\frac{1}{p}},\end{aligned}$$

and (4.2.5) holds.

When $p = 1$ and $0 < \operatorname{ess\,sup}_{0<x<r} \frac{1}{|V(x)|} < \infty$, we can take f the characteristic function of the set

$$\left\{ x : \frac{1}{|V(x)|} \geq \frac{-1}{n} + \operatorname*{ess\,sup}_{0<x<r} \frac{1}{|V(x)|} \right\},$$

and, by (4.2.4), we get

$$\begin{aligned}\left(\int_r^\infty |U(x)| dx \right) \left| \int_0^r f(t) dt \right| &\leq C \int_0^r |V(x) f(x)| dx \\ &\leq C \frac{1}{\frac{-1}{n} + \operatorname{ess\,sup}_{0<x<r} \frac{1}{|V(x)|}} \int_0^r f(t) dt.\end{aligned}$$

Therefore,

$$\left(\int_r^\infty |U(x)| dx \right) \left(\frac{-1}{n} + \operatorname*{ess\,sup}_{0<x<r} \frac{1}{|V(x)|} \right) \leq C,$$

and letting $n \longrightarrow \infty$ we get (4.2.5).

Finally, we observe that if $\int_0^r |V(x)|^{-p'} dx = 0$ then (4.2.5) is obvious. If

$$\int_0^r |V(x)|^{-p'} dx = \infty,$$

we can consider the functions

$$f_n(x) = |V(x)|^{-p'} \chi_{A_n}(x),$$

where $A_n := \left\{ x > 0 : \frac{1}{n} \leq |V(x)|^{-p'} \leq n \right\}$. Then, (4.2.4) becomes

$$\left(\int_r^\infty |U(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^r |V(x)|^{-p'} \chi_{A_n}(x) dx \right) \leq C \left(\int_0^r |V(x)|^{-p'} \chi_{A_n}(x) dx \right)^{\frac{1}{p}},$$

that is,

$$\left(\int_r^\infty |U(x)|^p dx \right)^{\frac{1}{p}} \leq C \left(\int_0^r |V(x)|^{-p'} \chi_{A_n}(x) dx \right)^{\frac{-1}{p'}}.$$

Then, by the Monotone Convergence Theorem, if we let $n \longrightarrow \infty$, we get

$$\left(\int_r^\infty |U(x)|^p dx \right)^{\frac{1}{p}} = 0,$$

and so (4.2.5) holds. $\square$

Remarks 4.2.3. The conditions given in Theorem 4.1.2 or Theorem 3.2.1 are equivalent to the condition given in Theorem 4.2.1.

1. If the take, in Theorem 4.2.1, $p = 2$ and $v(x) \equiv 1$, then

$$B = \sup_{r>0} \left(\int_r^\infty \frac{u(x)}{x^2} dx \right)^{\frac{1}{2}} \left(\int_0^r dx \right)^{\frac{1}{2}} = \sup_{r>0} \left(r \int_r^\infty \frac{u(x)}{x^2} dx \right)^{\frac{1}{2}},$$

which is finite if, and only if,

$$\sup_{r>0} \left(r \int_r^\infty \frac{u(x)}{x^2} dx \right)$$

is finite. But this last condition is exactly the one appearing in Theorem 4.1.2.

2. If we take, in Theorem 4.2.1, $u(x) = v(x) = x^\alpha$, for some α real, we have

$$B = \sup_{r>0} \left(\int_r^\infty x^{\alpha-p} dx \right)^{\frac{1}{p}} \left(\int_0^r x^{-\frac{\alpha}{p-1}} dx \right)^{\frac{1}{p'}}.$$

But B is finite if, and only if,

$$\alpha - p < -1 \quad \text{and} \quad \frac{-\alpha}{p-1} > -1,$$

that is, $\alpha < p - 1$, which is exactly the condition in Theorem 3.2.1.

4.3 Weighted Hardy inequalities of (p, q) type

Now we focus in the general case with two different indexes p and q. The following result (cf. [4, Theorem 1]) characterizes the weighted Hardy inequality (p, q) for $q \geq p$ and it is due to J. S. Bradley. It is an extension of Muckenhoupt's diagonal case (cf. Theorem 4.2.1).

Theorem 4.3.1. *Let $1 \leq p \leq q \leq \infty$. Given two weights u and v, we can find a finite constant $C > 0$ such that*

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f(x)^p v(x) dx \right)^{\frac{1}{p}} \tag{4.3.1}$$

for all positive function f if, and only if,

$$B = \sup_{r>0} \left(\int_r^\infty \frac{u(x)}{x^q} dx \right)^{\frac{1}{q}} \left(\int_0^r v(x)^{\frac{-1}{p-1}} dx \right)^{\frac{1}{p'}} < \infty.$$

In addition, if C is the sharp constant for (4.3.1), then

$$B \leq C \leq p^{\frac{1}{q}} \, (p')^{\frac{1}{p'}} \, B,$$

if $1 < p < q < \infty$, and $C = B$ if $p = 1$ or $q = \infty$.

Remark 4.3.2. Equivalently, we can call $U(x) = \frac{1}{x}\, u(x)^{\frac{1}{q}}$, $V(x) = v(x)^{\frac{1}{p}}$ and then (4.3.1) becomes

$$\left(\int_0^\infty \left(U(x)\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty (V(x)f(x))^p\, dx\right)^{\frac{1}{p}} \tag{4.3.2}$$

and B is now

$$B = \sup_{r>0}\left(\int_r^\infty U(x)^q dx\right)^{\frac{1}{q}}\left(\int_0^r V(x)^{-p'}dx\right)^{\frac{1}{p'}} < \infty. \tag{4.3.3}$$

Proof. The proof of (4.3.2) $\Rightarrow$ (4.3.3), that is, $B \leq C$, is analogous to the one in Theorem 4.2.1.

We are going to prove, then, the implication (4.3.3) $\Rightarrow$ (4.3.2), or more precisely, $C \leq p^{\frac{1}{q}}\,(p')^{\frac{1}{p'}}\, B$. First we assume $1 < p < q < \infty$ and we define

$$h(t) = \left(\int_0^t V(s)^{-p'}ds\right)^{\frac{1}{pp'}}.$$

Now, applying Hölder's inequality and Minkowski's integral inequality (cf. Theorem 2.2.3), we get

$$\begin{aligned}
I &= \int_0^\infty \left(U(x)\int_0^x f(t)\frac{V(t)h(t)}{V(t)h(t)}dt\right)^q dx \\
&\leq \int_0^\infty U(x)^q\left(\int_0^\infty (f(t)V(t)h(t))^p\chi_{\{0<t<x\}}(t,x)dt\right)^{\frac{q}{p}}\left(\int_0^x (V(s)h(s))^{-p'}ds\right)^{\frac{q}{p'}} dx \\
&= \int_0^\infty \left(\int_0^\infty U(x)^p(f(t)V(t)h(t))^p\chi_{\{0<t<x\}}(t,x)\left(\int_0^x (V(s)h(s))^{-p'}ds\right)^{\frac{p}{p'}} dt\right)^{\frac{q}{p}} dx \\
&\leq \left(\int_0^\infty (f(t)V(t)h(t))^p\left(\int_t^\infty U(x)^q\left(\int_0^x (V(s)h(s))^{-p'}ds\right)^{\frac{q}{p'}} dx\right)^{\frac{p}{q}} dt\right)^{\frac{q}{p}}.
\end{aligned}$$

To perform the innermost integral, we observe that, since by hypothesis

$$\left(\int_0^x V(u)^{-p'}du\right)^{\frac{1}{p'}} \leq B\left(\int_x^\infty U(s)^q ds\right)^{-\frac{1}{q}},$$

we have that

$$\begin{aligned}
\left(\int_0^x (V(s)h(s))^{-p'}ds\right)^{\frac{q}{p'}} &= \left(\int_0^x V(s)^{-p'}\left(\int_0^s V(u)^{-p'}du\right)^{\frac{-1}{p}} ds\right)^{\frac{q}{p'}} \\
&= \left(\left[p'\left(\int_0^s V(u)^{-p'}du\right)^{\frac{1}{p'}}\right]_{s=0}^{x}\right)^{\frac{q}{p'}} \\
&= \left(p'\left(\int_0^x V(u)^{-p'}du\right)^{\frac{1}{p'}}\right)^{\frac{q}{p'}} \\
&\leq (p'B)^{\frac{q}{p'}}\left(\int_x^\infty U(s)^q ds\right)^{\frac{-1}{p'}}.
\end{aligned}$$

Therefore,

$$I \leq (p'B)^{\frac{q}{p'}}\left(\int_0^\infty (f(t)V(t)h(t))^p\left(\int_t^\infty U(x)^q\left(\int_x^\infty U(s)^q ds\right)^{\frac{-1}{p'}}dx\right)^{\frac{p}{q}}dt\right)^{\frac{q}{p}}.$$

Again, to perform the inner integral we observe that, since by hypothesis

$$\left(\int_t^\infty U(s)^q ds\right)^{\frac{1}{q}} \leq B\left(\int_0^t V(x)^{-p'}dx\right)^{\frac{-1}{p'}} = Bh(t)^{-p},$$

we have that

$$\begin{aligned}
\left(\int_t^\infty U(x)^q\left(\int_x^\infty U(s)^q ds\right)^{\frac{-1}{p'}}dx\right)^{\frac{p}{q}} &= \left(\left[-p\left(\int_x^\infty U(s)^q ds\right)^{\frac{1}{p}}\right]_{x=t}^{\infty}\right)^{\frac{p}{q}} \\
&= p^{\frac{p}{q}}\left(\int_t^\infty U(s)^q ds\right)^{\frac{1}{q}} \\
&\leq p^{\frac{p}{q}}Bh(t)^{-p}.
\end{aligned}$$

Therefore,

$$I \leq (p')^{\frac{q}{p'}}B^q p\left(\int_0^\infty (f(t)V(t))^p dt\right)^{\frac{q}{p}},$$

and hence

$$I^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}}(p)^{\frac{1}{q}}B.$$

Now, for $p=1$, we first observe that, by hypothesis,

$$\left(\int_t^\infty U(x)^q dx\right)^{\frac{1}{q}}\frac{1}{V(s)} \leq B$$

for all $0 < s \leq t$, and hence, by Minkowski's integral inequality (cf. Theorem 2.2.3), we get

$$\begin{aligned}\left(\int_0^\infty \left(U(x)\int_0^x f(t)dt\right)^q dx\right)^{\frac{1}{q}} &\leq \left(\int_0^\infty \left(\int_0^\infty U(x)f(t)\chi_{\{0<t<x\}}(t,x)dt\right)^q dx\right)^{\frac{1}{q}} \\ &\leq \int_0^\infty f(t)\left(\int_t^\infty U(x)^q dx\right)^{\frac{1}{q}} dt \\ &\leq B\int_0^\infty f(t)V(t)dt.\end{aligned}$$

Finally, if $1 \leq p \leq q = \infty$, we apply Hölder's inequality to get

$$\begin{aligned}\int_0^x U(x)f(t)dt = U(x)\int_0^x \frac{1}{V(t)}f(t)V(t)dt \\ &\leq \left(\operatorname*{ess\,sup}_{s\geq x} U(s)\right)\left(\int_0^x \frac{1}{V(t)^{p'}}dt\right)^{\frac{1}{p'}}\left(\int_0^x (f(t)V(t))^p dt\right)^{\frac{1}{p}} \\ &\leq B\left(\int_0^x (f(t)V(t))^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

□

For the case $0 < p < 1$, Q. Lai proved (cf. [12, Theorem 1]) that the weighted Hardy inequality cannot be true, except for trivial situations. His proof is based on a result due to M. M. Day (cf. [8]), which states that the dual space of a Lebesgue space is the zero space when $0 < p < 1$, under some hypothesis on the measure.

Theorem 4.3.3. *(Day) Let μ be a nonatomic measure and let $0 < p < 1$. Then every linear and continuous functional, $T : L^p(\mu) \longrightarrow \mathbb{R}$, must be zero.*

Theorem 4.3.4. *(Lai) Given $0 < q < \infty$, if $0 < p < 1$ then there is no constant $C > 0$ such that*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{4.3.4}$$

for all $f \in L^p(v)$, except for the trivial case $w(x) \equiv 0$ a.e. $x \in (0,\infty)$.

Remark 4.3.5. If we consider $w(x)$ as $\frac{w(x)}{x^q}$ in Theorem 4.3.4, then (4.3.4) becomes

$$\left(\int_0^\infty \left(\int_0^x f(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}. \tag{4.3.5}$$

We prove now Theorem 4.3.4 with the notations of Remark 4.3.5.

Proof. We define

$$c = \inf\{\tau > 0 : w(x) \equiv 0 \quad \text{a.e.} \quad x \in (\tau, \infty)\}.$$

If $c = 0$, then $w(x) \equiv 0$ a.e. $x \in (0,\infty)$ and (4.3.5) trivially holds. We assume $0 < c < \infty$ and hence $w(x) \equiv 0$ a.e. $x \in (c,\infty)$, and

$$\int_b^c w(x)dx > 0 \tag{4.3.6}$$

for all $0 < b < c$.

Now we observe that it has to be $v(x) > 0$ a.e. $x \in (0,c)$. Indeed, if we assume that the set

$$E = \{0 < x < c : v(x) = 0\}$$

has positive measure, then we get a contradiction. It would exist an $x_0 \in (0,c)$ such that $|(0,x_0) \cap E| > 0$ and, if $f(x) = \chi_E(x)$,

$$\int_0^x f(y)dy > 0$$

for all $x \geq x_0$. But we would also have that

$$\int_0^\infty f(y)^p v(y)dy = 0,$$

so if (4.3.5) holds, we would get

$$\int_0^\infty \left(\int_0^x f(y)dy\right)^q w(x)dx \leq 0,$$

and necessarily $w(x) \equiv 0$ a.e. $x \in (x_0,\infty)$, in contradiction with the definition of c.

Given $0 < p < 1$, we define

$$\mathcal{N}'_{p,v}(g) = \sup_{\|f\|_{p,v}\leq 1} \left|\int_0^\infty f(x)g(x)v(x)dx\right|,$$

and we claim that, for any interval (a,b) with $0 \leq a < b < c$,

$$\mathcal{N}'_{p,v}\left(\frac{\chi_{(a,b)}(\cdot)}{v(\cdot)}\right) = \sup_{\|f\|_{p,v}\leq 1} \left|\int_a^b f(x)dx\right| < \infty. \tag{4.3.7}$$

Otherwise, there would exist a sequence $(f_n)_n$ of positive functions such that $\|f_n\|_{p,v} \leq 1$ and

$$\int_a^b f_n(x)dx > n,$$

for all $n \in \mathbb{N}$. But then, if (4.3.5) holds, and since $\|f_n\|_{p,v} = \left(\int_0^\infty f_n(x)^p v(x)dx\right)^{\frac{1}{p}} \leq 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} C &\geq C\left(\int_0^\infty f_n(x)^p v(x)dx\right)^{\frac{1}{p}} \geq \left(\int_0^\infty \left(\int_0^x f_n(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \\ &\geq \left(\int_b^c \left(\int_0^x f_n(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \geq \left(\int_b^c \left(\int_a^b f_n(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \\ &> n\left(\int_b^c w(x)dx\right)^{\frac{1}{q}} \end{aligned}$$

for all $n \in \mathbb{N}$. However, this is only possible if

$$\int_b^c w(x)dx = 0,$$

in contradiction with (4.3.6).

We observe now that the condition (4.3.7) implies that the operator

$$T : L^p(v) \longrightarrow \mathbb{R}$$

defined by

$$f \longmapsto \int_a^b f(x)dx$$

is linear and bounded. Therefore, by Theorem 4.3.3, it must be $T \equiv 0$, that is,

$$\int_a^b f(x)dx = 0$$

for all $f \in L^p(v)$. Obviously, this is false since, for example, we can consider the function $f(x) = \chi_{(a,b)}(x)$, which is in $L^p(v)$ but

$$\int_a^b f(x)dx > 0.$$

So (4.3.5) cannot be satisfied.

Finally, if $c = \infty$, we have that for all $b > 0$ there exists a real number $d > b$ such that $\int_b^d w(x)dx > 0$ and the argument of the case $0 < c < \infty$ can be applied. $\square$

When $q < p$ the situation is slightly more complicated and different arguments are needed. W. Mazya and L. Rozin characterized the case $1 \leq q < p < \infty$ in the eighties, G. Sinnamon characterized (1987) the case $0 < q < 1 < p < \infty$ and the case $0 < q < 1 = p$ is due to G. Sinnamon and V. D. Stepanov (1996).

G. Sinnamon and V. D. Stepanov published a paper [18] where they gave the proof of the $0 < q < 1 = p$ case and a more elementary proof of the case $0 < q < p$, $1 < p < \infty$ (cf. [18, Theorem 2.4]). We give here the proofs presented in this paper. First, in order to deal with the case $0 < q < p$, $1 < p < \infty$ we need some previous results.

Lemma 4.3.6. *Assume α, β and γ are positive functions and γ is increasing and absolutely continuous. If*

$$\int_x^\infty \alpha(t)dt \le \int_x^\infty \beta(t)dt \tag{4.3.8}$$

for all $x > 0$, then

$$\int_0^\infty \gamma(t)\alpha(t)dt \le \int_0^\infty \gamma(t)\beta(t)dt.$$

Proof. It is known that for any positive, increasing and absolutely continuous function γ, we can write

$$\gamma(t) = \gamma(0) + \int_0^t \gamma'(s)ds. \tag{4.3.9}$$

Hence, applying (4.3.9), Tonelli's Theorem and (4.3.8), we get

$$\begin{aligned}\int_0^\infty \gamma(t)\alpha(t)dt &= \gamma(0)\int_0^\infty \alpha(t)dt + \int_0^\infty \int_0^t \gamma'(s)ds\ \alpha(t)dt \\ &= \gamma(0)\int_0^\infty \alpha(t)dt + \int_0^\infty \int_s^\infty \alpha(t)dt\ \gamma'(s)ds \\ &\le \gamma(0)\int_0^\infty \beta(t)dt + \int_0^\infty \int_s^\infty \beta(t)dt\ \gamma'(s)ds \\ &= \int_0^\infty \gamma(t)\beta(t)dt.\end{aligned}$$

□

Proposition 4.3.7. *Let u, b and F be positive functions such that F is increasing and absolutely continuous. We assume also that b satisfies*

$$\int_x^\infty b(t)dt < \infty \quad \forall x > 0 \quad \textit{but} \quad \int_0^\infty b(t)dt = \infty. \tag{4.3.10}$$

If $0 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ then

$$\left(\int_0^\infty F(x)^q u(x)dx\right)^{\frac{1}{q}} \le \left(\frac{r}{p}\right)^{\frac{1}{r}} \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{q}} \left(\int_x^\infty b(t)dt\right)^{-\frac{r}{q}} b(x)dx\right)^{\frac{1}{r}}$$
$$\times \left(\int_0^\infty F(x)^p b(x)dx\right)^{\frac{1}{p}}.$$

Proof. We define $U(x) := \int_x^\infty u(t)dt$ and $B(x) := \int_x^\infty b(t)dt$. We express u as

$u(t) = u(t)^{\frac{q}{r}+\frac{q}{p}}$ and applying Hölder's inequality with $\frac{r}{q}$ and $\frac{p}{q}$, we get

$$
\begin{aligned}
&\left(\int_0^\infty F(x)^q u(x)dx\right)^{\frac{1}{q}}\\
&=\left(\int_0^\infty \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{q}{r}} F(t)^q \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-q}{r}} u(t)dt\right)^{\frac{1}{q}}\\
&\leq \left(\int_0^\infty \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right) u(t)dt\right)^{\frac{1}{r}}\\
&\qquad \times \left(\int_0^\infty F(t)^p \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} u(t)dt\right)^{\frac{1}{p}} = I \times II.
\end{aligned}
$$

Now, applying Tonelli's Theorem we obtain

$$
\begin{aligned}
I &= \left(\int_0^\infty \int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)\ u(t)\ ds\ dt\right)^{\frac{1}{r}}\\
&= \left(\int_0^\infty U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)\int_s^\infty u(t)dtds\right)^{\frac{1}{r}} = \left(\int_0^\infty U(s)^{\frac{r}{q}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{1}{r}}.
\end{aligned}
\tag{4.3.11}
$$

In order to bound II, we want to apply Lemma 4.3.6 with

$$
\alpha(t) = \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} u(t),\ \ \beta(t) = \left(\frac{r}{p}\right)^{\frac{p}{r}} b(t) \text{ and } \gamma(t) = F(t)^q.
$$

By hypothesis, $\gamma(t)$ has to be increasing and it remains to check that $\int_x^\infty \alpha \leq \int_x^\infty \beta$ for all $x > 0$. To see this, we observe that as $\left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}}$ and U are decreasing, we have

$$
\begin{aligned}
\int_x^\infty \alpha(t)dt &= \int_x^\infty \left(\int_0^t U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} u(t)dt\\
&\leq \left(\int_0^x U(s)^{\frac{r}{p}}B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} \int_x^\infty u(t)dt\\
&\leq \left(\int_0^x B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} U(x)^{-1}\int_x^\infty u(t)dt\\
&= \left(\int_0^x B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}},
\end{aligned}
$$

and integration together with (4.3.10) yields

$$
\left(\int_0^x B(s)^{\frac{-r}{q}}b(s)ds\right)^{\frac{-p}{r}} = \left(\frac{p}{r}\left(\int_x^\infty b(t)dt\right)^{\frac{-r}{p}}\right)^{\frac{-p}{r}} = \int_x^\infty \beta(t)dt.
$$

Finally, applying (4.3.11) and Lemma 4.3.6 we conclude

$$\left(\int_0^\infty F(x)^q u(x)dx\right)^{\frac{1}{q}} \leq \left(\int_0^\infty U(s)^{\frac{r}{q}} B(x)^{-\frac{r}{q}} b(x)dx\right)^{\frac{1}{r}} \times \left(\int_0^\infty \left(\frac{r}{p}\right)^{\frac{p}{r}} F(x)^p b(x)dx\right)^{\frac{1}{p}}.$$

□

Proposition 4.3.8. *Suppose that $1 < p < \infty$ and let w be a positive function such that*

$$\int_x^\infty w(t)dt < \infty \quad \forall x > 0 \quad \textit{but} \quad \int_0^\infty w(x)dx = \infty. \tag{4.3.12}$$

Then

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^p \left(\int_0^x w(t)dt\right)^{-p} w(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty f(x)^p w(x)^{1-p}dx\right)^{\frac{1}{p}}, \tag{4.3.13}$$

with $C \leq p'$.

Proof. According to Theorem 4.2.1, the weighted Hardy inequality (4.3.13) holds if, and only if,

$$B = \sup_{r>0} \left(\int_r^\infty \left(\int_0^x w(t)dt\right)^{-p} w(x)dx\right)^{\frac{1}{p}} \left(\int_0^r w(x)dx\right)^{\frac{1}{p'}} < \infty$$

and, moreover, $C \leq p^{\frac{1}{p}} \, (p')^{\frac{1}{p'}} \, B$. But applying (4.3.12) and taking into account that $p > 1$, we have

$$\begin{aligned}
&\left(\int_r^\infty \left(\int_0^x w(t)dt\right)^{-p} w(x)dx\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{p-1}\left(\left(\int_0^r w(t)dt\right)^{1-p} - \left(\int_0^\infty w(t)dt\right)^{1-p}\right)\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{p-1}\right)^{\frac{1}{p}} \left(\int_0^r w(t)dt\right)^{\frac{1}{p}-1} = \left(\frac{1}{p-1}\right)^{\frac{1}{p}} \left(\int_0^r w(t)dt\right)^{\frac{-1}{p'}},
\end{aligned}$$

and hence

$$B = \left(\frac{1}{p-1}\right)^{\frac{1}{p}} < \infty.$$

In addition,

$$C \leq p^{\frac{1}{p}} \, (p')^{\frac{1}{p'}} \left(\frac{1}{p-1}\right)^{\frac{1}{p}} = p'.$$

□

Theorem 4.3.9. *Assume* $0 < q < p$, $1 < p < \infty$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Let* u *and* v *be two weights. Then, the weighted Hardy inequality*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{4.3.14}$$

holds for all positive function f *if, and only if,*

$$D := \left(\int_0^\infty \left(\int_0^x v(t)^{1-p'}dt\right)^{\frac{r}{p'}} \left(\int_x^\infty \frac{u(t)}{t^q}dt\right)^{\frac{r}{p}} \frac{u(x)}{x^q}dx\right)^{\frac{1}{r}} < \infty. \tag{4.3.15}$$

Moreover, if C *is the sharp constant for* (4.3.14)*, then*

$$p'^{\frac{1}{p'}} q^{\frac{1}{p}} \left(1 - \frac{q}{p}\right) D \leq C \leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\ p'^{\frac{1}{p'}} D.$$

Remark 4.3.10. As usual, if we consider $u(x)$ as $\frac{u(x)}{x^q}$ then (4.3.14) becomes

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{4.3.16}$$

and (4.3.15) becomes

$$D := \left(\int_0^\infty \left(\int_0^x v(t)^{1-p'}dt\right)^{\frac{r}{p'}} \left(\int_x^\infty u(t)dt\right)^{\frac{r}{p}} u(x)dx\right)^{\frac{1}{r}} < \infty. \tag{4.3.17}$$

Proof. We use notation introduced in Remark 4.3.10. First we assume that (4.3.16) holds for all $f \geq 0$. We define $w(x) = v(x)^{1-p'}$ and we consider u_0 and w_0 integrable functions such that $0 \leq u_0 \leq u$ and $0 \leq w_0 \leq w$. We consider the function

$$f(x) := \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{pq}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{pq'}} w_0(x),$$

and we observe that

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^x \left(\int_t^\infty u_0(s)ds\right)^{\frac{r}{pq}} \left(\int_0^t w_0(s)ds\right)^{\frac{r}{pq'}} w_0(t)dt \\
&\geq \left(\int_x^\infty u_0(s)ds\right)^{\frac{r}{pq}} \int_0^x \left(\int_0^t w_0(s)ds\right)^{\frac{r}{pq'}} w_0(t)dt \\
&= \left(\int_x^\infty u_0(s)ds\right)^{\frac{r}{pq}} \left[\left(\frac{r}{pq'}+1\right)^{-1} \left(\int_0^t w_0(s)ds\right)^{\frac{r}{pq'}+1}\right]_{t=0}^x \\
&= \frac{p'q}{r}\left(\int_x^\infty u_0(s)ds\right)^{\frac{r}{pq}} \left(\int_0^x w_0(s)ds\right)^{\frac{r}{p'q}}.
\end{aligned}$$

Now, applying this last estimate, (4.3.16) and integrating by parts, we get

$$
\begin{aligned}
&\left(\int_0^\infty \left(\frac{p'q}{r}\right)^q \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{p}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{p'}} u_0(x)dx\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u_0(x)dx\right)^{\frac{1}{q}} \leq \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\
&\leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{q}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{q'}} w_0(x)dx\right)^{\frac{1}{p}} \\
&= C\left(\int_0^\infty \frac{r}{q}\frac{p'}{r} \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{p}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{p'}} u_0(x)dx\right)^{\frac{1}{p}} \\
&= C\left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{p}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{p'}} u_0(x)dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

But since u_0 and w_0 are integrable functions, the right hand side of the inequality is finite and hence

$$
\frac{p'q}{r}\left(\frac{q}{p'}\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_x^\infty u_0(t)dt\right)^{\frac{r}{p}} \left(\int_0^x w_0(t)dt\right)^{\frac{r}{p'}} u_0(x)dx\right)^{\frac{1}{p}} \leq C. \tag{4.3.18}
$$

Finally, approximating u and w from below by an increasing sequence of integrable functions, and applying the Monotone Convergence Theorem, (4.3.18) becomes

$$
\frac{p'q}{r}\left(\frac{q}{p'}\right)^{\frac{1}{p}} D = p'^{\frac{1}{p'}} q^{\frac{1}{p}} \left(1 - \frac{q}{p}\right) D \leq C.
$$

Now assume (4.3.17) holds. For the moment, we will also assume that w satisfies (4.3.12) and we define $W(x) := \int_0^x w(t)dt$. Given a positive function f, we want to apply Proposition 4.3.7 with $b(x) = W(x)^{-p}w(x)$ and $F(x) = \int_0^x f(t)dt$. Notice that b is under the hypothesis of Proposition 4.3.7 since

$$
\int_x^\infty b(t)dt = \int_x^\infty W(t)^{-p}w(t)dt = \frac{1}{p-1}W(x)^{1-p} < \infty
$$

and

$$
\int_0^\infty b(x)dx = \int_0^\infty W(t)^{-p}w(t)dt = \infty.
$$

So applying Proposition 4.3.7, integrating, using Proposition 4.3.8 and integrating by parts, we conclude

$$
\begin{aligned}
&\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}\\
&\leq \left(\frac{r}{p}\right)^{\frac{1}{r}} \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{q}} \left(\int_x^\infty W(t)^{-p}w(t)dt\right)^{-\frac{r}{q}} W(x)^{-p}w(x)dx\right)^{\frac{1}{r}}\\
&\quad \times \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^p W(x)^{-p}w(x)dx\right)^{\frac{1}{p}}\\
&= \left(\frac{r}{p}\right)^{\frac{1}{r}} \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{q}} \left(\frac{1}{p-1}\right)^{-\frac{r}{q}} W(x)^{\frac{r}{q'}}w(x)dx\right)^{\frac{1}{r}}\\
&\quad \times \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^p W(x)^{-p}w(x)dx\right)^{\frac{1}{p}}\\
&\leq \left(\frac{r}{p}\right)^{\frac{1}{r}} \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{r}{q}} \left(\frac{1}{p-1}\right)^{-\frac{r}{q}} W(x)^{\frac{r}{q'}}w(x)dx\right)^{\frac{1}{r}}\\
&\quad \times p' \left(\int_0^\infty f(x)^p w(x)^{1-p}dx\right)^{\frac{1}{p}}\\
&= \left(\frac{r}{p}\right)^{\frac{1}{r}} (p-1)^{\frac{1}{q}} p' \left(\int_0^\infty \frac{p'}{q} \left(\int_x^\infty u(t)dt\right)^{\frac{r}{p}} W(x)^{\frac{r}{p'}}u(x)dx\right)^{\frac{1}{r}}\\
&\quad \times \left(\int_0^\infty f(x)^p w(x)^{1-p}dx\right)^{\frac{1}{p}}\\
&= \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\, p'^{\frac{1}{p'}} D \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

Now we work with a general w (not necessarily satisfying (4.3.12)). We fix u and w and for each $n \in \mathbb{N}$ we define $u_n(x) = u(x)\chi_{(0,n)}(x)$ and $w_n(x) = \min(w(x), n) + \chi_{(n,\infty)}(x)$. Now these functions w_n satisfy (4.3.12) since

$$\int_0^x w_n(t)dt \leq (n+1)x < \infty$$

for all $n \in \mathbb{N}$, $x > 0$ and

$$\int_0^\infty w_n(x)dx = \int_0^\infty \min(w(x), n)dx + \int_n^\infty dx = \infty$$

for all $n \in \mathbb{N}$. So we can apply the previous argument for each pair u_n and w_n to conclude that

$$\begin{aligned} &\left(\int_0^n \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\, p'^{\frac{1}{p'}} \left(\int_0^n \left(\int_0^x \min(w(t),n)dt\right)^{\frac{r}{p'}} \left(\int_x^n u(t)dt\right)^{\frac{r}{p}} u(x)dx\right)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^\infty f(x)^p w_n(x)^{1-p}dx\right)^{\frac{1}{p}}. \end{aligned} \tag{4.3.19}$$

Now, for a given positive function g, we write $f(x) = g(x)\min(w(x),n)^{\frac{1}{p'}}\chi_{(0,n)}(x)$ and (4.3.19) becomes

$$\begin{aligned} &\left(\int_0^n \left(\int_0^x g(t)\min(w(t),n)^{\frac{1}{p'}}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\, p'^{\frac{1}{p'}} \left(\int_0^n \left(\int_0^x \min(w(t),n)dt\right)^{\frac{r}{p'}} \left(\int_x^n u(t)dt\right)^{\frac{r}{p}} u(x)dx\right)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^\infty g(x)^p dx\right)^{\frac{1}{p}} \end{aligned}$$

and letting $n \longrightarrow \infty$ we have, by the Monotone Convergence Theorem, that

$$\begin{aligned} &\left(\int_0^\infty \left(\int_0^x g(t)w(t)^{\frac{1}{p'}}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\, p'^{\frac{1}{p'}} \left(\int_0^\infty \left(\int_0^x w(t)dt\right)^{\frac{r}{p'}} \left(\int_x^\infty u(t)dt\right)^{\frac{r}{p}} u(x)dx\right)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^\infty g(x)^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

In particular, if we take g of the form $g(x) = f(x)w(x)^{-\frac{1}{p'}}$ for a positive function f, we get (4.3.16) with $C \leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}}\, p'^{\frac{1}{p'}}$. □

Now we characterize the weighted Hardy inequality for the case $0 < q < 1 = p$. First we need some previous results.

Definition 4.3.11. Given a positive function v, we define

$$\underline{v}(x) = \operatorname*{ess\,inf}_{0<t<x} v(t) = \sup\{\lambda \in \mathbb{R} : |\{0 < t < x : v(t) < \lambda\}| = 0\}.$$

Remark 4.3.12. $\underline{v}(x)$ is a decreasing function. Indeed, if $x < y$, then

$$\{0 < t < x : v(t) < \lambda\} \subseteq \{0 < t < y : v(t) < \lambda\}$$

and hence

$$|\{0<t<y : v(t)<\lambda\}| = 0 \Rightarrow |\{0<t<x : v(t)<\lambda\}| = 0.$$

Therefore,

$$\{\lambda \in \mathbb{R} : |\{0<t<y : v(t)<\lambda\}| = 0\} \subseteq \{\lambda \in \mathbb{R} : |\{0<t<x : v(t)<\lambda\}| = 0\},$$

and taking supremum in λ we get $\underline{v}(y) \leq \underline{v}(x)$.

The following theorem is a technical result due to G. Sinnamon and V. D. Stepanov (cf. [18, Theorem 3.2]), and states that the weighted Hardy inequality has the same sharp constant when we consider $\underline{v}$ instead of v.

Theorem 4.3.13. *Suppose that* $0<q<\infty$. *Then the best constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\int_0^\infty f(x)v(x)dx, \quad f \geq 0,$$

is unchanged when v *is replaced by* $\underline{v}$.

The next proposition is due to V. D. Stepanov (cf. [20, Proposition 1(b)]). It is a real analysis result which states that the inclusion $L^1(v) \subseteq L^q(u)$ holds when $0<q<1$ under some conditions on the weights u and v.

Proposition 4.3.14. *Assume* $0<q<1$. *Let us consider* C *as the best constant in inequality*

$$\left(\int_0^\infty f(x)^q u(x)dx\right)^{\frac{1}{q}} \leq C\int_0^\infty f(x)v(x)dx, \quad f \geq 0,$$

and consider

$$E_0 = \left(\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{q}{1-q}} \left(\int_x^\infty v(t)dt\right)^{\frac{q}{q-1}} u(x)dx\right)^{\frac{1-q}{q}}.$$

Then $E_0 \approx C$.

Finally, we give the characterization of Hardy's inequality when $0<q<1=p$ (cf. [18, Thoerem 3.3]).

Theorem 4.3.15. *Suppose that* $0<q<1$. *Let us consider* C *as the best constant in inequality*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\int_0^\infty f(x)v(x)dx, \quad f \geq 0, \qquad (4.3.20)$$

and define

$$E = \left(\int_0^\infty \underline{v}(x)^{\frac{q}{q-1}} \left(\int_x^\infty \frac{u(t)}{t^q}dt\right)^{\frac{q}{1-q}} \frac{u(x)}{x^q}dx\right)^{\frac{1-q}{q}}.$$

Then $E \approx C$.

Remark 4.3.16. Again, if we consider $u(x)$ as $\frac{u(x)}{x^q}$ then (4.3.20) becomes

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C \int_0^\infty f(x)v(x)dx, \quad f \ge 0,$$

and E is now

$$E = \left(\int_0^\infty \underline{v}(x)^{\frac{q}{q-1}} \left(\int_x^\infty u(t)dt\right)^{\frac{q}{1-q}} u(x)dx\right)^{\frac{1-q}{q}}.$$

Proof. We use the notation introduced in Remark 4.3.16. By Theorem 4.3.13 C is also the best constant in inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C \int_0^\infty f(x)\underline{v}(x)dx, \tag{4.3.21}$$

with $f \ge 0$. We consider first that $\underline{v}$ is of the form $\underline{v}(x) = \int_x^\infty b(t)dt$ where b satisfies

$$\int_x^\infty b(t)dt < \infty \quad \forall x > 0 \quad \text{but} \quad \int_0^\infty b(t)dt = \infty. \tag{4.3.22}$$

By Tonelli's Theorem, (4.3.21) becomes

$$\begin{aligned}\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} &\le C \int_0^\infty f(x) \int_x^\infty b(t)dt\, dx \\ &= C \int_0^\infty \left(\int_0^t f(x)dx\right) b(t)dt,\end{aligned}$$

and, by Proposition 4.3.14, $E = E_0 \approx C$.

Now we consider the case of a general $\underline{v}$. For each $n \in \mathbb{N}$, we define the function $\underline{v}_n(x) = \underline{v}(x)\chi_{(0,n)}(x)$, which is finite, decreasing (cf. Remark 4.3.12) and tends to 0 when $x \longrightarrow \infty$. Fixed $n \in \mathbb{N}$, we can approximate $\underline{v}_n$ from above by functions of the form $\int_x^\infty b(t)dt$ with b satisfying (4.3.22). Let us consider a decreasing sequence of such functions $(v_m)_m$ converging pointwise to $\underline{v}_n$ at almost every $x > 0$. We define also the function $u_n(x) = u(x)\chi_{(0,n)}(x)$ for each $n \in \mathbb{N}$. Then, by the first part of the proof, the inequality

$$\begin{aligned}\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u_n(x)dx\right)^{\frac{1}{q}} &\lesssim \left(\int_0^\infty v_m(x)^{\frac{q}{q-1}} \left(\int_x^\infty u_n(t)dt\right)^{\frac{q}{1-q}} u_n(x)dx\right)^{\frac{1-q}{q}} \\ &\quad \times \int_0^\infty f(x)v_m(x)dx\end{aligned}$$

holds for all $f \ge 0$. Hence, expressing f as $f(x) = \frac{g(x)}{v_m(x)}$, where $g \ge 0$, we have

$$\begin{aligned}\left(\int_0^\infty \left(\int_0^x \frac{g(x)}{v_m(x)}dt\right)^q u_n(x)dx\right)^{\frac{1}{q}} &\lesssim \left(\int_0^\infty v_m(x)^{\frac{q}{q-1}} \left(\int_x^\infty u_n(t)dt\right)^{\frac{q}{1-q}} u_n(x)dx\right)^{\frac{1-q}{q}} \\ &\quad \times \int_0^\infty g(x)dx\end{aligned}$$

and, as both $(v_m)^{-1}$ and $(v_m)^{\frac{q}{q-1}}$ are increasing, by Monotone Convergence Theorem we have, letting $m \longrightarrow \infty$,

$$\left(\int_0^\infty \left(\int_0^x \frac{g(x)}{\underline{v}_n(x)}dt\right)^q u_n(x)dx\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \underline{v}_n(x)^{\frac{q}{q-1}} \left(\int_x^\infty u_n(t)dt\right)^{\frac{q}{1-q}} u_n(x)dx\right)^{\frac{1-q}{q}} \times \int_0^\infty g(x)dx.$$

Using Monotone Convergence Theorem again, the last expression becomes, letting $n \longrightarrow \infty$,

$$\left(\int_0^\infty \left(\int_0^x \frac{g(x)}{\underline{v}(x)}dt\right)^q u(x)dx\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty \underline{v}(x)^{\frac{q}{q-1}} \left(\int_x^\infty u(t)dt\right)^{\frac{q}{1-q}} u(x)dx\right)^{\frac{1-q}{q}} \times \int_0^\infty g(x)dx.$$

Expressing g as $g(x) = f(x)\underline{v}(x)$ with $f \geq 0$ in the last expression, we get $C \approx E$. $\square$

Chapter 5

Hardy inequalities for monotone functions

In this chapter we are going to study the weighted Hardy inequalities presented in previous sections but, instead of working just with positive functions, we will consider monotone functions.

Some authors as G. H. Hardy already considered the study of Hardy inequalities when we impose the restriction of monotony. In Section 5.1 we will give some of the first results concerning monotone functions. However, the study of the Hardy inequality in the cone of the monotone functions started to generate a real interest when M. A. Ariño and B. Muckenhoupt proved, while they were studying the boundedness of the Hardy Littlewood maximal operator, that the maximal operator is bounded between Lorentz spaces if, and only if, the weighted Hardy inequality restricted to positive and decreasing functions holds. We will see this fact in Section 5.2. Then it is interesting to study the classical Hardy operator in the cone of monotone functions, which will be the main goal in Section 5.3. Finally, in Section 5.4 we give some applications of the study of the Hardy inequalities in the cone of monotone functions. We study when the Lorentz spaces $\Lambda^p(w)$ and the weak-type Lorenz spaces $\Lambda^{p,\infty}(w)$ are Banach, and we relate the weak boundedness of the maximal operator with the weak boundedness of the Hardy operator.

5.1 First results for monotone functions

As has been said in the introduction of the chapter, the Hardy inequalities in the cone of monotone functions were considered before M. A. Ariño and B. Muckenhoupt's approach. In particular, we have a similar result to Theorem 3.2.1 with an estimate from bellow for monotone functions (cf. [9, Theorem 2]) .

Theorem 5.1.1. *If $p \geq 1$, $\alpha < p-1$ and f is a positive decreasing function, then*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \geq \frac{p}{p-1-\alpha}\int_0^\infty f(x)^p x^\alpha dx.$$

Proof. We define $F(x) = \int_0^x f(t)dt$ and, by Lebesgue Differentiation Theorem, we have

$$\frac{d}{dt}(F(t))^p = pf(t)(F(t))^{p-1} \geq pf(t)(f(t)^{p-1}t^{p-1}) = pt^{p-1}f(t)^p$$

for almost every t. Integrating from 0 to x, we get

$$F(x)^p \geq p\int_0^x t^{p-1}f(t)^p dt.$$

Finally, applying Tonelli's Theorem we conclude

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx &= \int_0^\infty F(x)^p x^{\alpha-p} dx \\
&\geq \int_0^\infty \left(p\int_0^x t^{p-1}f(t)^p dt\right) x^{\alpha-p} dx \\
&= p\int_0^\infty \left(\int_t^\infty x^{\alpha-p} dx\right) t^{p-1}f(t)^p dt \\
&= p\int_0^\infty \frac{t^{\alpha+1-p}}{p-\alpha-1} t^{p-1}f(t)^p dt \\
&= \frac{p}{p-\alpha-1}\int_0^\infty f(t)^p t^\alpha dt
\end{aligned}$$

□

Remark 5.1.2. If we apply both Theorem 3.2.1 and Theorem 5.1.1 when $p > 1$ and $\alpha = 0$, we get that, for any positive decreasing function f,

$$p' \int_0^\infty f(x)^p dx \leq \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq (p')^p \int_0^\infty f(x)^p dx.$$

Under some additional conditions, we can get that the integrals

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx$$

and

$$\int_0^\infty f^p(x) x^\alpha dx$$

are comparable (cf. [9, Theorem 2]).

Theorem 5.1.3. *If $0 < p < 1$, $\alpha < p - 1$ and f is a positive monotone function, then*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \approx \int_0^\infty f(x)^p x^\alpha dx.$$

Proof. First we assume f to be decreasing. Then one of the inequalities is Theorem 3.2.1 and the other one is consequence of $\frac{1}{x}\int_0^x f(t)dt \geq f(x)$. Now we assume that f is increasing. First we notice that, for any real number $\alpha \leq 0$, we have

$$\begin{aligned}
\int_0^\infty f(x)x^\alpha dx &= \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} f(x)x^\alpha dx \leq \sum_{k=-\infty}^{\infty} f(2^k)2^{k-1}2^{\alpha(k-1)} \\
&= \frac{1}{2^{\alpha+1}} \sum_{k=-\infty}^{\infty} f(2^k)2^{k(\alpha+1)} = \frac{1}{2^{2\alpha+1}} \sum_{k=-\infty}^{\infty} f(2^k)2^{\alpha(k+1)}2^k \\
&\leq \frac{1}{2^{2\alpha+1}} \sum_{k=-\infty}^{\infty} f(2^k) \int_{2^k}^{2^{k+1}} x^\alpha dx \leq \frac{1}{2^{2\alpha+1}} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} f(x)x^\alpha dx \\
&= \frac{1}{2^{2\alpha+1}} \int_0^\infty f(x)x^\alpha.
\end{aligned}$$

Now we observe that if $f(x)$ is increasing then so is $\frac{1}{x}\int_0^x f(t)dt$. Then, if $0 < p < 1$ and $\alpha < p-1$, we have

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx &\approx \sum_{k=-\infty}^{\infty} \left(\int_0^{2^k} f(t)dt\right)^p 2^{k(\alpha-p+1)} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{k} \int_{2^{m-1}}^{2^m} f(t)dt\right)^p 2^{k(\alpha-p+1)} \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{k} \left(\int_{2^{m-1}}^{2^m} f(t)dt\right)^p 2^{k(\alpha-p+1)} \\
&\leq \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{k} f(2^m)^p 2^{p(m-1)} 2^{k(\alpha-p+1)} \\
&= \sum_{m=-\infty}^{\infty} \left(\sum_{k=m}^{\infty} 2^{k(\alpha-p+1)}\right) f(2^m)^p 2^{p(m-1)} \\
&= \frac{1}{1-2^{\alpha-p+1}} \sum_{m=-\infty}^{\infty} 2^{m(\alpha-p+1)} f(2^m)^p 2^{p(m-1)} \\
&= \frac{1}{2^p} \frac{1}{1-2^{\alpha-p+1}} \sum_{m=-\infty}^{\infty} f(2^m)^p 2^{m(\alpha+1)} \\
&\approx \int_0^\infty f(x)^p x^\alpha dx.
\end{aligned}$$

∎

5.2 Maximal operator and the Hardy inequality for monotone functions

We present in this section M. A. Ariño and B. Muckenhoupt's approach to Hardy inequalities in the cone of monotone functions. We first recall the definition of the maximal operator.

Definition 5.2.1. Given f a locally integrable function in $\mathbb{R}^n$, we define the Hardy-Littlewood maximal operator as

$$Mf(x) := \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where $x \in \mathbb{R}^n$ and the supremum is taken over all cubes in $\mathbb{R}^n$ containing x.

We have the following bound (cf. [3, Theorem III.3.3]) for the nonincreasing rearrangement of the Hardy-Littlewood maximal operator.

Proposition 5.2.2. *If f is an integrable function in $\mathbb{R}^n$, then*

$$t(Mf)^*(t) \leq 4^n \|f\|_{L^1},$$

with $t > 0$.

The following technical lemma (cf. [3, Lemma III.3.7]), known as the Calderón-Zygmund covering Lemma, is needed later on.

Lemma 5.2.3. *Let Ω be an open subset of $\mathbb{R}^n$ with finite measure. Then there is a sequence of dyadic cubes $Q_1, Q_2, ...,$ with pairwise disjoint interiors, that covers Ω and satisfies*

(i) $Q_k \cap \Omega \neq \emptyset$ *for all* $k = 1, 2, ...$

(ii) $|\Omega| \leq \sum_{k=1}^{\infty} |Q_k| \leq 2^n|\Omega|.$

Now we present a theorem (cf. [3, Theorem III.3.8]) stating that the nonincreasing rearrangement of the Hardy Littlewood maximal operator $(Mf)^*$ is equivalent to the Hardy operator applied to the rearrangement of f, that is, f^{**}.

Theorem 5.2.4. *There exist constants c and c' such that*

$$c(Mf)^*(t) \leq f^{**}(t) \leq c'(Mf)^*(t) \tag{5.2.1}$$

for any $t > 0$ and any locally integrable function f in $\mathbb{R}^n$. The constants c and c' only depend on n.

Proof. We prove first the left-hand inequality in (5.2.1). Let us fix $t > 0$ and assume that $f^{**}(t) < \infty$ (otherwise there is nothing to prove). By Theorem 2.3.11, for any $\epsilon > 0$ there are functions $g_t \in L^1$ and $h_t \in L^\infty$ such that $f = g_t + h_t$ and

$$\|g_t\|_{L^1} + t\|h_t\|_{L^\infty} \leq tf^{**}(t) + \epsilon.$$

Now, using Proposition 2.3.5 (ii) and (iii), Proposition 5.2.2 and taking into account that $\|Mf\|_{L^\infty} \leq \ \|f\|_{L^\infty}$, we get

$$\begin{aligned}(Mf)^*(s) &\leq (Mg_t)^*\left(\frac{s}{2}\right) + (Mh_t)^*\left(\frac{s}{2}\right) \leq \frac{c}{s}\|g\|_{L^1} + \|h_t\|_{L^\infty}\\ &= \frac{c}{s}\left(\|g\|_{L^1} + s\|h_t\|_{L^\infty}\right)\end{aligned}$$

for all $s > 0$. Hence, putting $s = t$, we get

$$(Mf)^*(s) \leq \frac{c}{t}\left(\|g\|_{L^1} + t\|h_t\|_{L^\infty}\right) \leq cf^{**}(t) + \frac{c}{t}\epsilon.$$

Letting $\epsilon \longrightarrow 0$, we get the left-hand inequality in (5.2.1).

Let us show now the right-hand inequality in (5.2.1). Again, we assume $(Mf)^*$ is finite since, otherwise, there is nothing to prove. We define the set

$$\Omega = \{x \in \mathbb{R}^n : (Mf)(x) > (Mf)^*(t)\}.$$

This set is open because if $x \in \Omega$, then

$$\sup_{x\in Q} \frac{1}{|Q|}\int_Q f(y)dy > (Mf)^*(t)$$

and we can find a cube $Q_0 \ni x$ such that

$$\frac{1}{|Q_0|}\int_{Q_0} f(y)dy > (Mf)^*(t),$$

that is, $Q_0 \subseteq \Omega$. Then Ω is a measurable set and, applying Proposition 2.3.5 (iv) we deduce $|\Omega| = \lambda_{Mf}((Mf)^*(t)) =\leq t$. Now, applying Lemma 5.2.3 we know that there is a sequence of cubes Q_1, Q_2,..., with pairwise disjoint interiors, that covers Ω, satisfying

$$Q_k \cap \Omega \neq \emptyset, \tag{5.2.2}$$

for all $k = 1, 2...$, and

$$\sum_{k=1}^{\infty} |Q_k| \leq 2^n|\Omega| \leq 2^n t. \tag{5.2.3}$$

We define the set $F = (\cup_{k=1}^{\infty} Q_k)^c$ and the functions

$$g = f\chi_{F^c} = \sum_{k=1}^{\infty} f\chi_{Q_k} \quad \text{and} \quad h = f\chi_F,$$

so that $f = g + h$. Now, by Proposition 2.3.10 and Proposition 2.3.7 we get

$$f^{**}(t) \leq g^{**}(t) + h^{**}(t) = \frac{1}{t}\int_{\mathbb{R}^n} |g(s)|ds + h^*(0) \leq \frac{1}{t}\|g\|_{L^1} + \|h\|_{L^\infty}. \tag{5.2.4}$$

We notice that by (5.2.2) each cube Q_k contains a point in Ω^c and, at these points, the maximal function is bounded by $(Mf)^*(t)$ (because of the definition of Ω). Hence,

$$\frac{1}{|Q_k|}\int_{Q_k} |f(y)|dy \leq (Mf)^*(t)$$

for all $k = 1, 2, ...$ Therefore, using this last estimate and (5.2.3), we get

$$\|g\|_{L^1} = \sum_{k=1}^{\infty}\int_{Q_k} |f(y)|dy \leq \sum_{k=1}^{\infty} |Q_k|(Mf)^*(t) = 2^n t(Mf)^*(t). \tag{5.2.5}$$

On the other hand, F is contained in Ω^c and hence the maximal function is bounded by $(Mf)^*(t)$ in F. Now, as $|f(x)| \leq (Mf)(x)$ for almost every $x \in \mathbb{R}^n$, we deduce

$$\|h\|_{L^\infty} = \|f\chi_F\|_{L^\infty} \leq \|(Mf)\chi_F\|_{L^\infty} \leq (Mf)^*(t). \tag{5.2.6}$$

Finally, using (5.2.5) and (5.2.6) in (5.2.4), we conclude

$$f^{**}(t) \leq (2^n t + 1)(Mf)^*(t),$$

and the right-hand inequality in (5.2.1) holds with $c' = 2^n t + 1$. □

We define the cone of decreasing functions on $L^p(w)$.

Definition 5.2.5. Given a weight w in $\mathbb{R}^+$ and $0 < p < \infty$, we define the cone of decreasing functions as

$$L^p_{\text{dec}}(w) := \left\{ 0 \leq f \downarrow : \left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}} < \infty \right\}.$$

We already know that $(Mf)^* \approx f^{**} = \mathcal{H}f^*$ and the following step is to see that from this fact we can deduce that the boundedness of the Hardy-Littlewood maximal operator $M : \Lambda^p(v) \longrightarrow \Lambda^q(u)$ between weighted Lorentz spaces (cf. Definition 2.3.12) is equivalent to the boundedness of the classical Hardy operator $\mathcal{H} : L^p_{\text{dec}}(v) \longrightarrow L^q(u)$ in the cone of monotone functions.

First we need the following lemma, stating that every positive decreasing function on $\mathbb{R}^+$ is the nonincreasing rearrangement function of a function on $\mathbb{R}^n$.

Lemma 5.2.6. *Let f be a positive decreasing function and consider*

$$g(x) := f(A|x|^n), \ x \in \mathbb{R}^n,$$

where $A = |B_1(0)|$. Then $g^*(t) = f(t)$ for almost every $t > 0$.

Proof. First we consider that f is of the form

$$f(x)=\sum_{j=1}^{m}\alpha_j\chi_{[a_j,a_{j+1})}(t), \tag{5.2.7}$$

with $\alpha_1>...>\alpha_m$ and $0=a_1<a_2<...<a_{m+1}$. Then

$$\lambda_g(t)=|\{x\in\mathbb{R}^n : f(A|x|^n)>t\}|=\begin{cases}0, & \text{if } t\geq\alpha_1,\\ a_1, & \text{if } \alpha_2\leq t<\alpha_1,\\ a_2, & \text{if } \alpha_3\leq t<\alpha_2,\\ \vdots & \\ a_{m+1}, & \text{if } 0\leq t<\alpha_m,\end{cases}$$

that is,

$$\lambda_g(t)=\sum_{j=1}^{m+1}a_j\chi_{[\alpha_{j+1},\alpha_j)}(x),$$

where we define $\alpha_{m+1}=0$. Then

$$g^*(s)=\begin{cases}0 & \text{if } s\geq a_{m+1},\\ \alpha_m & \text{if } a_m\leq s<a_{m+1},\\ \vdots & \\ \alpha_1 & \text{if } 0<s<a_1.\end{cases}=f(s).$$

Now we consider a positive decreasing function f. Then we can find a sequence of functions as the one in (5.2.7), say $(s_m)_m$, such that $s_m(x)\nearrow f(x)$ for almost every $x>0$. Then if we define $g_m(x)=s_m(A|x|^n)$ we have that $g_m(x)\nearrow f(A|x|^n)=g(x)$ for almost every $x\in\mathbb{R}^n$. By Proposition 2.3.5 (vi), we have

$$g_m^*(t)\longrightarrow g^*(t)$$

for all $t>0$. But, as $s_m(t)\longrightarrow f(t)$ for almost every $t>0$, we conclude that $g^*(t)=f(t)$ for almost every $t>0$. □

Finally, we see that the boundedness of $M:\Lambda^p(v)\longrightarrow\Lambda^q(u)$ is equivalent to the boundedness of $\mathcal{H}:L^p_{\text{dec}}(v)\longrightarrow L^q(u)$.

Theorem 5.2.7. *The boundedness of the Hardy-Littlewood maximal operator*

$$M:\Lambda^p(v)\longrightarrow\Lambda^q(u)$$

is equivalent to the boundedness of the classical Hardy operator

$$\mathcal{H}:L^p_{\text{dec}}(v)\longrightarrow L^q(u)$$

restricted to positive functions, that is, the weights u and v for which the inequality

$$\left(\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{p}}\leq C\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}$$

holds for all positive and decreasing functions f.

Proof. First we assume that $M : \Lambda^p(v) \longrightarrow \Lambda^q(u)$ is bounded. Given a positive and decreasing function f, we consider g the function defined in Lemma 5.2.6. Then, by Theorem 5.2.4, we deduce that

$$\|\mathcal{H}f\|_{q,u} = \|\mathcal{H}g^*\|_{q,u} \approx \|(Mg)^*\|_{q,u} \lesssim \|g^*\|_{p,v} = \|f\|_{p,v},$$

and $\mathcal{H} : L^p_{\text{dec}}(v) \longrightarrow L^q(u)$ is bounded.

Conversely, if $\mathcal{H} : L^p_{\text{dec}}(v) \longrightarrow L^q(u)$ is bounded, using Theorem 5.2.4 we get that

$$\|Mf\|_{\Lambda^q(u)} = \|(Mf)^*\|_{q,u} \approx \|\mathcal{H}f^*\|_{q,u} \lesssim \|f^*\|_{p,v} = \|f\|_{\Lambda^p(v)}$$

for all $f \in \Lambda^q(u)$ and, therefore, $M : \Lambda^p(v) \longrightarrow \Lambda^q(u)$ is bounded. $\square$

5.3 Characterization of the weighted Hardy inequality for monotone functions

The problem presented in Section 5.2 makes necessary to study Hardy inequalities in the cone of monotone functions. This problem was solved by M. A. Ariño and B. Muckenhoupt [1] for the diagonal case ($p > 1$) in 1990 and by E. Sawyer [17] in the most general situation ($p, q \geq 1$), in the same year.

First of all we consider the diagonal case, that is, the boundedness of the classical Hardy operator $\mathcal{H} : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$ when $p > 1$.

Remark 5.3.1. One can think that the inequality

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}}, \quad f \geq 0, \ f\downarrow, \tag{5.3.1}$$

holds if, and only if, the Muckenhoupt condition in Theorem 4.2.1,

$$B = \sup_{r>0}\left(\int_r^\infty \frac{w(x)}{x^p}dx\right)^{\frac{1}{p}}\left(\int_0^r w(x)^{\frac{-p'}{p}}dx\right)^{\frac{1}{p'}} < \infty, \tag{5.3.2}$$

is satisfied. It is obvious that if the weight w satisfies (5.3.2) then (5.3.1) holds. However, under the restriction of considering positive decreasing functions, the converse is not necessarily true, i.e., if (5.3.1) is satisfied then the weight w might not satisfy (5.3.2). For example (cf. [1]), we can consider the weight

$$w(x) = \frac{1}{\sqrt{x}}\chi_{(1,2)^c}(x).$$

If $r < 1$, we have

$$\begin{aligned}\left(\int_r^1 x^{-\frac{1}{2}-p}dx + \int_2^\infty x^{-\frac{1}{2}-p}dx\right)\left(\int_0^r x^{\frac{p'}{2p}}dx\right)^{\frac{p}{p'}} &= \left(\frac{r^{\frac{1}{2}-p}}{p-\frac{1}{2}} + C_p\right)\left(\frac{r^{\frac{p'}{2p}}+1}{\frac{p'}{2p}+1}\right)^{\frac{p}{p'}} \\ &= C_p'\,\frac{1}{r} + C_p\, r^{\frac{1}{2}+\frac{p}{p'}} \underset{r\to 0}{\longrightarrow} \infty,\end{aligned}$$

where C_p and C'_p are constants which only depend on p. Hence $B = \infty$ and (5.3.2) does not hold. However, by Theorem 3.2.1 and as f is decreasing, we have that

$$\begin{aligned}
\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p}} &\leq \left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^{-\frac{1}{2}}dx\right)^{\frac{1}{p}} \\
&\leq C\left(\int_0^\infty f(x)^p x^{-\frac{1}{2}}dx\right)^{\frac{1}{p}} \\
&\leq C\left(2\int_0^1 f(x)^p x^{-\frac{1}{2}}dx + \int_2^\infty f(x)^p x^{-\frac{1}{2}}dx\right)^{\frac{1}{p}} \\
&\leq C'\left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}}
\end{aligned}$$

and (5.3.1) holds.

We want to characterize now the weighted Hardy inequality for positive monotone functions in the diagonal case, with the proof given by M. J. Carro and J. Soria (cf. [7]), which is easier than the original one of M. A. Ariño and B. Muckenhoupt.

First we need a previous result which is also due to M. J. Carro and J. Soria (cf. [6, Theorem 2.1]).

Theorem 5.3.2. *Let us consider a measure space (X, μ) and a weight w. Then, if $0 < p < \infty$, the identity*

$$\int_0^\infty f^*(t)^p w(t)dt = p\int_0^\infty y^{p-1}\left(\int_0^{\lambda_f(y)} w(t)dt\right)dy \tag{5.3.3}$$

holds.

Proof. First we assume that f is a simple function, that is, of the form

$$f(x) = \sum_{i=1}^N a_i \chi_{A_i}(x), \tag{5.3.4}$$

where $|a_1| > ... > |a_N| > \ 0$ and the sets A_i are pairwise disjoint and have finite measure. A direct computation as the ones in Example 2.3.2 and Example 2.3.6 give

$$\lambda_f(y) = \sum_{k=1}^N \alpha_k \chi_{\{|a_{k+1}|\leq y<|a_k|\}}(y),$$

and

$$f^*(t) = \sum_{k=1}^N |a_k| \chi_{\{\alpha_{k-1}\leq t<\alpha_k\}}(t),$$

where $\alpha_k := \sum_{i=1}^k \mu(A_i)$, $a_{N+1} := 0$ and $\alpha_0 := 0$. Then

$$\begin{aligned}
\int_0^\infty f^*(t)^p w(t)dt &= \sum_{k=1}^N |a_k|^p \int_{\alpha_{k-1}}^{\alpha_k} w(t)dt = \sum_{k=1}^N |a_k|^p \left(\int_0^{\alpha_k} w(t)dt - \int_0^{\alpha_{k-1}} w(t)dt\right) \\
&= \sum_{k=1}^N (|a_k|^p - |a_{k+1}|^p) \int_0^{\alpha_k} w(t)dt \\
&= p\sum_{k=1}^N \int_{|a_{k+1}|}^{|a_k|} y^{p-1} \left(\int_0^{\alpha_k} w(t)dt\right) dy \\
&= p\sum_{k=1}^N \int_{|a_{k+1}|}^{|a_k|} y^{p-1} \left(\int_0^{\lambda_f(y)} w(t)dt\right) dy \\
&= p\int_0^\infty y^{p-1} \left(\int_0^{\lambda_f(y)} w(t)dt\right) dy.
\end{aligned}$$

Now any measurable function f can be pointwise approximated (almost everywhere) by an increasing sequence of functions as the one in (5.3.4), say $(s_m)_m$. Furthermore, by Proposition 2.3.3 (i) and Proposition 2.3.5 (iii) we know that $\lambda_{s_m} \leq \lambda_f$ and $s_m^* \leq f^*$. Then (5.3.3) follows from the Monotone Convergence Theorem. □

Theorem 5.3.3. *Let $1 < p < \infty$. The inequality*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}} \tag{5.3.5}$$

holds for all positive decreasing functions if, and only if, there exists a constant $C' > 0$ such that

$$\int_r^\infty \frac{w(x)}{x^p}dx \leq C'\frac{1}{r^p}\int_0^r w(x)dx, \tag{5.3.6}$$

for all $r > 0$.

Definition 5.3.4. We denote by B_p the class of weights which satisfy (5.3.6), or we say that the weights for which (5.3.6) is true satisfy a B_p-condition.

Proof. First we assume that (5.3.5) holds for all positive decreasing functions. Fixed $r > 0$, we can choose $f(x) = \chi_{(0,r)}(x)$ and hence

$$\left(\int_0^r w(x)dx + r^p\int_r^\infty \frac{w(x)}{x^p}dx\right)^{\frac{1}{p}} \leq C\left(\int_0^r w(x)dx\right)^{\frac{1}{p}},$$

or equivalently,

$$\int_r^\infty \frac{w(x)}{x^p}dx \leq C'\frac{1}{r^p}\int_0^r w(x)dx,$$

with $C' = C^p - 1$.

Conversely, we assume that (5.3.6) holds for all $r > 0$. Let us observe that

$$\left(\int_0^x f(t)dt\right)^p = p\int_0^x \left(\int_0^t f(s)ds\right)^{p-1} f(t)dt$$
$$= p\int_0^x \left(\frac{1}{t}\int_0^t f(s)ds\right)^{p-1} f(t)t^{p-1}dt,$$

and if we define $g(t) := \left(\frac{1}{t}\int_0^t f(s)ds\right)^{p-1} f(t)$, then

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p}} = p^{\frac{1}{p}}\left(\int_0^\infty \int_0^x g(t)t^{p-1}dt\ \frac{w(x)}{x^p}dx\right)^{\frac{1}{p}}.$$

Now we observe that, if f is decreasing, then the function $\frac{1}{t}\int_0^t f(s)ds$ decreases with t and hence, g is decreasing. Then we know that, by Theorem 5.2.6, $g = h^*$ with $h(x) := g(A|x|^n)$. Applying Theorem 5.3.2, we get that

$$\int_0^x g(t)t^{p-1}dt = \int_0^\infty \int_0^{\lambda_h(y)} t^{p-1}\chi_{(0,x)}(t)dt\ dy = \frac{1}{p}\int_0^\infty \min(\lambda_h(y), x)^p dy.$$

Finally, applying Tonelli's Theorem, (5.3.6), Theorem 5.3.2 and Hölder's Inequality, we conclude that

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx &= \int_0^\infty \int_0^\infty \min(\lambda_h(y), x)^p dy\ \frac{w(x)}{x^p}dx\\
&= \int_0^\infty \left(\int_0^{\lambda_h(y)} w(x)dx + \lambda_h(y)^p \int_{\lambda_h(y)}^\infty \frac{w(x)}{x^p}dx\right) dy\\
&\le (1+C')\int_0^\infty \int_0^{\lambda_h(y)} w(x)dx\ dy\\
&= (1+C')\int_0^\infty g(x)w(x)dx\\
&= (1+C')\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-1} f(x)w(x)dx\\
&\le (1+C')\left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}}\\
&\quad \times \left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p'}},\end{aligned}$$

that is,

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty f(x)^p w(x)dx\right)^{\frac{1}{p}},$$

with $C = 1 + C'$. □

Now it remains to study the diagonal case when $0 < p \leq 1$. The conclusion will be the same, that is, the B_p condition will be also necessary and sufficient. It is a consequence of the following theorem (cf. [7, Theorem 2.4]).

Theorem 5.3.5. *Let $(\mathcal{N}, d\sigma)$ and $(\mathcal{M}, d\mu)$ be two σ-finite measure spaces. Given a measurable function f in $\mathcal{N}$ we can define $Tf(x) = T_k(f^*_\sigma)(x)$ for every $x \in \mathcal{M}$, where $T_k g(x) := \int_0^\infty k(x,t)g(t)dt$ with $k(x,t)$ a positive kernel, and f^*_σ denotes the nonincreasing rearrangement function of f with respect to the measure σ. Let σ_0 denote another σ-finite measure in $\mathcal{N}$ and w_0 a weight in $\mathbb{R}^+$. Then, if $0 < p_0 \leq 1$ and $p_1 \geq p_0$, the operator $T : \Lambda^{p_0}_{\sigma_0}(w_0) \longrightarrow L^{p_1}(d\mu)$ is bounded if, and only if, there exists a constant $C > 0$ such that*

$$\left(\int_{\mathcal{M}} \left(\int_0^{\sigma(A)} k(x,t)dt \right)^{p_1} d\mu(x) \right)^{\frac{1}{p_1}} \leq C \left(\int_0^{\sigma_0(A)} w_0(x)dx \right)^{\frac{1}{p_0}}, \tag{5.3.7}$$

for all measurable sets A in $\mathcal{N}$.

We characterize then the Hardy inequality in the cone of monotone functions for $p < 1$ in the diagonal case (cf. [7, Proposition 2.5]).

Corollary 5.3.6. *Let $0 < p \leq 1$ and w a weight in $\mathbb{R}^+$. Then tha classical Hardy operator $\mathcal{H} : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$ is bounded if, and only if, w satisfies a B_p condition.*

Proof. If $\mathcal{H} : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$ is bounded then

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p w(x)dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty f(x)^p w(x)dx \right)^{\frac{1}{p}}$$

holds for all positive decreasing functions. In particular, given $r > 0$, we can consider the characteristic function $f(t) = \chi_{(0,r)}(t)$ and then

$$\int_0^r w(x)dx + \int_r^\infty \frac{r^p}{x^p} w(x)dx = \int_0^\infty \left(\frac{1}{x} \min(r,x) \right)^p w(x)dx \leq C \int_0^r w(x)dx,$$

which is the B_p condition.

Conversely, we take $p_0 = p_1 = p$, $w_0 = w$, both σ and σ_0 Lebesgue measure, $d\mu = w(x)dx$ and $k(x,t) = \frac{1}{x}\chi_{(0,x)}(t)$ in Theorem 5.3.5. Then B_p condition is (5.3.7) and therefore the operator $T : \Lambda^p(w) \longrightarrow L^p(w)$ is bounded. As $L^p_{\text{dec}}(w)$ is a subspace of $\Lambda^p(w)$, then $T : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$ is also bounded. Finally, observe that for decreasing functions, $T = \mathcal{H}$. □

For the non-diagonal case, the characterization of the weighted Hardy inequality in the cone of monotone functions is based on a duality principle proved by Sawyer in 1990 (cf. [17, Theorem 1]). We already now (cf. Proposition 2.2.1) that

$$\sup_{f \geq 0} \frac{\left| \int_0^\infty f(x)g(x)dx \right|}{\left(\int_0^\infty f(x)^p v(x)dx \right)^{\frac{1}{p}}} = \left(\int_0^\infty |g(x)|^{p'} v(x)^{1-p'} dx \right)^{\frac{1}{p'}}. \tag{5.3.8}$$

Sawyer result's is the analogue of (5.3.8) when we consider the supremum over positive and decreasing functions.

Theorem 5.3.7. *Suppose that $1 < p < \infty$ and let v and g be positive measurable functions on $(0,\infty)$ such that $0 < \int_0^t v(x)dx < \infty$ for all $t > 0$. Then*

$$\sup_{0\le f\downarrow} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds}dt\right)^{p'} v(x)dx\right)^{\frac{1}{p'}}$$
$$\approx \left(\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'-1} \left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx\right)^{\frac{1}{p'}}$$
$$\approx \left(\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} + \frac{\int_0^\infty g(x)dx}{\left(\int_0^\infty v(x)dx\right)^{\frac{1}{p}}}. \tag{5.3.9}$$

Proof. We assume first that g has compact support in $(0,\infty)$ and $\int_0^\infty g(x)dx = 1$. We define

$$\varphi(x) := \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds}dt\right)^{p'-1},$$

with $0 < x < \infty$. Integrating by parts we get that

$$\int_0^\infty \varphi(x)^p v(x)dx = \left[\varphi(x)^p \int_0^x v(t)dt\right]_0^\infty - p\int_0^\infty \varphi(x)^{p-1}\varphi'(x)\left(\int_0^x v(t)dt\right)dx$$
$$= p'\int_0^\infty \varphi(x)g(x)dx.$$

Therefore, we have that

$$\frac{\int_0^\infty \varphi(x)g(x)dx}{\left(\int_0^\infty \varphi(x)^p v(x)dx\right)^{\frac{1}{p}}} = \frac{1}{p'}\left(\int_0^\infty \varphi(x)^p v(x)dx\right)^{\frac{1}{p'}},$$

and since φ is a positive decreasing function, the supremum in (5.3.9) is bounded by

$$\frac{1}{p'}\left(\int_0^\infty \varphi(x)^p v(x)dx\right)^{\frac{1}{p'}} = \frac{1}{p'}\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds}dt\right)^{p'} v(x)dx\right)^{\frac{1}{p'}}. \tag{5.3.10}$$

Conversely, if f is positive and decreasing, then we apply Tonelli's Theorem and

Hölder's inequality to get

$$
\begin{aligned}
\int_0^\infty f(x)g(x)dx &= \int_0^\infty f(x)\frac{g(x)}{\int_0^x v(t)dt}\left(\int_0^x v(t)dt\right)dx \\
&= \int_0^\infty \left(\int_t^\infty \frac{f(x)g(x)}{\int_0^x v(t)dt}dx\right)v(t)dt \\
&\leq \int_0^\infty f(t)\left(\int_t^\infty \frac{g(x)}{\int_0^x v(t)dt}dx\right)v(t)dt \\
&\leq \left(\int_0^\infty f(t)^p v(t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty \left(\int_t^\infty \frac{g(x)}{\int_0^x v(t)dt}dx\right)^{p'} v(t)dt\right)^{\frac{1}{p'}}.
\end{aligned}
\tag{5.3.11}
$$

Using (5.3.10) and (5.3.11) we deduce the first equivalence in (5.3.9).

Now, since $\int_0^\infty g(x)dx = 1$, we can define $(x_j)_{-\infty<j<0}$ as the real numbers satisfying $\int_0^{x_j} g(x)dx = 2^j$, and $x_0 := \infty$. Then

$$
\begin{aligned}
\int_0^\infty \varphi(x)^p v(x)dx &\approx \int_0^\infty \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds}dt\right)^{p'-1} g(x)dx \\
&= \sum_{j=-\infty}^{0} \int_{x_{j-1}}^{x_j} \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds}dt\right)^{p'-1} g(x)dx \\
&\geq \sum_{j=-\infty}^{-1} \left(\int_{x_j}^{x_{j+1}} g(x)dx\right)^{p'-1}\left(\int_0^{x_{j+1}} v(x)dx\right)^{1-p'}\left(\int_{x_{j-1}}^{x_j} g(x)dx\right) \\
&\gtrsim \sum_{j=-\infty}^{-2} \left(\int_0^{x_{j+2}} g(x)dx\right)^{p'-1}\left(\int_0^{x_{j+1}} v(x)dx\right)^{1-p'}\left(\int_{x_{j+1}}^{x_{j+2}} g(x)dx\right) \\
&\gtrsim \sum_{j=-\infty}^{-2} \int_{x_{j+1}}^{x_{j+2}} \left(\int_0^x g(t)dt\right)^{p'-1}\left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx \\
&= \int_0^\infty \left(\int_0^x g(t)dt\right)^{p'-1}\left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx.
\end{aligned}
\tag{5.3.12}
$$

Conversely, we define N as the largest integer such that $2^{N-1} < \int_0^\infty v(x)dx$ (or $N = \infty$ if v is not integrable). Then we define $(x_j)_{-\infty<j<N}$ as the real numbers satisfying $\int_0^{x_j} v(x)dx = 2^j$, and $x_N := \infty$. First of all we observe that, if a_k are

positive numbers then applying Hölder's inequality for series we get that

$$
\begin{aligned}
\sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k\geq j} a_k\right)^{p'} &= \sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k\geq j} 2^{\frac{j-k}{pp'}} 2^{\frac{k-j}{pp'}} a_k\right)^{p'} \\
&\leq \sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k\geq j} 2^{\frac{j-k}{p'}}\right)^{p'-1} \left(\sum_{k\geq j} 2^{\frac{k-j}{p}} a_k^{p'}\right) \\
&\lesssim \sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k\geq j} 2^{\frac{k-j}{p}} a_k^{p'}\right) \\
&= \sum_{j=-\infty}^{N-1} 2^{\frac{k}{p}} a_k^{p'} \sum_{j\leq k} 2^{\frac{j}{p'}} \lesssim \sum_{j=-\infty}^{N-1} 2^k a_k^{p'}.
\end{aligned}
\tag{5.3.13}
$$

Therefore, applying (5.3.13) with $a_k = 2^{-k}\int_{x_k}^{x_{k+1}} g(x)dx$, we have that

$$
\begin{aligned}
\int_0^\infty \varphi(x)^p v(x)dx &= \sum_{j=-\infty}^{N-1} \int_{x_j}^{x_{j+1}} \left(\int_x^\infty \frac{g(t)}{\int_0^t v(s)ds} dt\right)^{p'} v(x)dx \\
&\leq \sum_{j=-\infty}^{N-1} \left(\int_{x_j}^{x_{j+1}} v(x)dx\right) \left(\sum_{k\geq j} \frac{\int_{x_k}^{x_{k+1}} g(x)dx}{\int_0^{x_k} v(x)dx}\right)^{p'} \\
&= \sum_{j=-\infty}^{N-1} 2^j \left(\sum_{k\geq j} 2^{-k} \int_{x_k}^{x_{k+1}} g(x)dx\right)^{p'} \\
&\lesssim \sum_{j=-\infty}^{N-1} 2^j \left(2^{-j} \int_{x_j}^{x_{j+1}} g(x)dx\right)^{p'} \\
&= \sum_{j=-\infty}^{N-1} \left(\int_0^{x_j} v(x)dx\right)^{1-p'} \int_{x_j}^{x_{j+1}} p' \left(\int_{x_j}^{x} g(t)dt\right)^{p'-1} g(x)dx \\
&\lesssim \sum_{j=-\infty}^{N-1} \int_{x_j}^{x_{j+1}} \left(\int_0^x g(t)dt\right)^{p'-1} \left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx \\
&= \int_0^\infty \left(\int_0^x g(t)dt\right)^{p'-1} \left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx.
\end{aligned}
\tag{5.3.14}
$$

Using (5.3.12) and (5.3.14) we deduce the second equivalence in (5.3.9).

Finally, for the third equivalence in (5.3.9), we just integrate by parts, getting

$$\begin{aligned}
&\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'-1} \left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx \\
&= \left[\frac{1}{p'}\left(\int_0^x g(t)dt\right)^{p'} \left(\int_0^x v(t)dt\right)^{1-p'}\right]_0^\infty \\
&\quad + \frac{1}{p}\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'} \left(\int_0^x v(t)dt\right)^{-p'} v(x)dx \\
&= \frac{1}{p}\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx + \frac{1}{p'}\frac{\left(\int_0^\infty g(x)dx\right)^{p'}}{\left(\int_0^\infty v(x)dx\right)^{p'-1}}.
\end{aligned}$$

At the beginning of the proof we have assumed that g has compact support in $(0,\infty)$ and $\int_0^\infty g(x)dx = 1$. If g has not compact support, we can consider the functions $g\chi_{(0,n)}(x)$, $n \in \mathbb{N}$, which have compact support, and apply the Monotone Convergence Theorem in the equivalences we have seen. In addition, if $\int_0^\infty g(x)dx \neq 1$, we can just divide g by its L^1-norm, which does not modify the equivalences. $\square$

As a consequence of Theorem 5.3.7 we can characterize the weighted Hardy inequality in the cone of monotone functions for the non-diagonal case. For simplicity, we will assume that v in Theorem 5.3.7 satisfies $\int_0^\infty v(x)dx = \infty$, in such a way that (5.3.9) becomes

$$\sup_{0\le f\downarrow} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}}.$$

Definition 5.3.8. We define the adjoint of the classical Hardy operator as

$$(\tilde{\mathcal{H}}f)(x) := \int_x^\infty \frac{1}{y} f(y)dy.$$

Theorem 5.3.9. *Let $1 < p, q < \infty$ and u, v be weight functions with v satisfying $\int_0^x v(t)dt > 0$ for all $x > 0$ and $\int_0^\infty v(x)dx = \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}$$

holds for all positive and decreasing functions f if, and only if,

$$\left(\int_0^\infty \left(\int_0^x (\tilde{\mathcal{H}}g)(t)dt\right)^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} \lesssim \left(\int_0^\infty g(x)^{q'} u(x)^{1-q'}dx\right)^{\frac{1}{q'}}. \tag{5.3.15}$$

Proof. First we assume that (5.3.15) holds. If we apply Theorem 5.3.7 with g replaced by $\tilde{\mathcal{H}}g$ then, by hypothesis,

$$\begin{aligned}\sup_{0\le f\downarrow}\frac{\int_0^\infty f(x)(\tilde{\mathcal{H}}g)(x)dx}{\|f\|_{p,v}} &\lesssim \left(\int_0^\infty\left(\int_0^x(\tilde{\mathcal{H}}g)(t)dt\right)^{p'}\frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}}\\ &\lesssim\left(\int_0^\infty g(x)^{q'}u(x)^{1-q'}dx\right)^{\frac{1}{q'}}=\|g\|_{q',u^{1-q'}}.\end{aligned}\tag{5.3.16}$$

Now we observe that, by Tonelli's Theorem,

$$\begin{aligned}\int_0^\infty f(x)(\tilde{\mathcal{H}}g)(x)dx &= \int_0^\infty f(x)\int_x^\infty\frac{1}{y}g(y)dy\,dx=\int_0^\infty g(y)\frac{1}{y}\int_0^y f(x)dx\,dy\\ &=\int_0^\infty(\mathcal{H}f)(x)g(x)dx,\end{aligned}$$

and (5.3.16) becomes

$$\sup_{0\le f\downarrow}\frac{\int_0^\infty(\mathcal{H}f)(x)g(x)dx}{\|f\|_{p,v}}\lesssim\|g\|_{q',u^{1-q'}}.\tag{5.3.17}$$

Finally, by Proposition 2.2.1 and (5.3.17), we have that for all $0\le f\downarrow$,

$$\|\mathcal{H}f\|_{q,u}=\sup_{g\ge0}\frac{\int_0^\infty(\mathcal{H}f)(x)g(x)dx}{\|g\|_{q',u^{1-q'}}}\lesssim\sup_{g\ge0}\frac{\|f\|_{p,v}\|g\|_{q',u^{1-q'}}}{\|g\|_{q',u^{1-q'}}}=\|f\|_{p,v}.$$

For the converse, we proceed in a similar way. First of all we notice that by Theorem 5.3.7,

$$\begin{aligned}\left(\int_0^\infty\left(\int_0^x(\tilde{\mathcal{H}}g)(t)dt\right)^{p'}\frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} &\lesssim\sup_{0\le f\downarrow}\frac{\int_0^\infty f(x)(\tilde{\mathcal{H}}g)(x)dx}{\|f\|_{p,v}}\\ &=\sup_{0\le f\downarrow}\frac{\int_0^\infty(\mathcal{H}f)(x)g(x)dx}{\|f\|_{p,v}}.\end{aligned}\tag{5.3.18}$$

Now, by Proposition 2.2.1 and the hypothesis, we get that

$$\sup_{g\ge0}\frac{\int_0^\infty(\mathcal{H}f)(x)g(x)dx}{\|g\|_{q',u^{1-q'}}}=\|\mathcal{H}f\|_{q,u}\lesssim\|f\|_{p,v},$$

and hence, (5.3.18) becomes

$$\left(\int_0^\infty\left(\int_0^x(\tilde{\mathcal{H}}g)(t)dt\right)^{p'}\frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}}\lesssim\sup_{0\le f\downarrow}\frac{\|f\|_{p,v}\|g\|_{q',u^{1-q'}}}{\|f\|_{p,v}}=\|g\|_{q',u^{1-q'}}.$$

$\square$

Remark 5.3.10. (cf [10, Example 6.7]) We can get an equivalent expression for (5.3.15) which is easier to use. Indeed, we observe that

$$\begin{aligned}\int_0^x (\tilde{\mathcal{H}}g)(t)dt &= \int_0^x \int_t^\infty \frac{g(y)}{y}dy\ dt = \int_0^x \left(\int_t^x \frac{g(y)}{y}dy + \int_x^\infty \frac{g(y)}{y}dy\right) dt \\ &= \int_0^x g(t)dt + x\int_x^\infty \frac{g(t)}{t}dt,\end{aligned}$$

and hence condition (5.3.15) holds if, and only if, both conditions

$$\left(\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} \lesssim \|g\|_{q',u^{1-q'}} \tag{5.3.19}$$

and

$$\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right)^{p'} x^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} \lesssim \|g\|_{q',u^{1-q'}} \tag{5.3.20}$$

are satisfied. If $1 < p \leq q < \infty$, we know by Theorem 4.3.1 that (5.3.19) holds if, and only if,

$$\sup_{r>0}\left(\int_r^\infty \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} \left(\int_0^r u(x)^{(1-q')(1-q)}dx\right)^{\frac{1}{q}} < \infty,$$

and if $1 < q \leq p < \infty$, we know by Theorem 4.3.9 that (5.3.19) holds if, and only if

$$\left(\int_0^\infty \left(\int_x^\infty \frac{v(t)}{\left(\int_0^t v(s)ds\right)^{p'}}dt\right)^{\frac{r}{p'}} \left(\int_0^x u(t)dt\right)^{\frac{r}{p}} u(x)dx\right)^{\frac{1}{r}} < \infty.$$

Inequality (5.3.20) is known as the dual of Hardy's inequality. We have not studied this kind of inequalities in this project but they have a characterization (cf [10, p. 4]). Namely, if $1 < p \leq q < \infty$ then (5.3.20) holds if, and only if,

$$\sup_{r>0}\left(\int_0^r x^{p'} \frac{v(x)}{\left(\int_0^x v(t)dt\right)^{p'}}dx\right)^{\frac{1}{p'}} \left(\int_r^\infty x^{-q}u(x)dx\right)^{\frac{1}{q}} < \infty,$$

and if $1 < q \leq p < \infty$ then (5.3.20) holds if, and only if,

$$\left(\int_0^\infty \left(\int_0^x t^{p'} \frac{v(t)}{\left(\int_0^t v(s)ds\right)^{p'}}dt\right)^{\frac{r}{p'}} \left(\int_x^\infty t^{-q}u(t)dt\right)^{\frac{r}{p}} x^{-q}u(x)dx\right)^{\frac{1}{r}} < \infty.$$

5.4 Applications

The characterization of the weighted Hardy inequality in the cone of the monotone functions has several applications. As we have seen, it characterizes the boundedness of the maximal operator. But, furthermore, we can study the normability of Lorentz spaces and weak-type Lorentz spaces in terms of the Hardy inequalities for positive monotone functions as well as characterizing the weak boundedness of the maximal operator in terms of the weak boundedness of the Hardy operator acting in the cone of monotone functions.

5.4.1 Normability of Lorentz spaces

In this section we are going to see when the classical Lorentz spaces $\Lambda_p(v)$ are Banach spaces, i.e., when they have a norm equivalent to the quasi-norm defining the space. In particular, M. A. Ariño and B. Muckenhoupt's result (cf. Theorem 5.3.3) jointly with Theorem 5.3.7 of E. Sawyer can be used to determine when the classical Lorentz space $\Lambda_p(v)$ is a Banach space for $p > 1$ (cf. [17, Theorem 4]).

We will use the following space introduced by E. Sawyer in [17].

Definition 5.4.1. We define the space

$$\Gamma_p(v) := \left\{ f \text{ measurable on } \mathbb{R}^n : \left(\int_0^\infty f^{**}(x)^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Theorem 5.4.2. *Suppose $1 < p < \infty$ and $v(x)$ is a positive measurable function on $(0, \infty)$. The following conditions are equivalent:*

(i) $\Lambda_p(v)$ *is a Banach space.*

(ii) $\Lambda_p(v) = \Gamma_p(v)$ *and* $\|f\| \approx \left(\int_0^\infty f^*(x)^p v(x) dx\right)^{\frac{1}{p}}$ *with* $\|f\| := \left(\int_0^\infty f^{**}(x)^p v(x) dx\right)^{\frac{1}{p}}$.

(iii) The inequality

$$\left(\int_0^r v(x) dx \right)^{\frac{1}{p}} \left(\int_0^r \left(\frac{1}{x} \int_0^x v(t) dt \right)^{1-p'} dx \right)^{\frac{1}{p'}} \leq Cr$$

holds for all $r > 0$.

(iv) The inequality

$$\left(\int_r^\infty \frac{v(x)}{x^p} dx \right)^{\frac{1}{p}} \leq \frac{B}{r} \left(\int_0^r v(x) dx \right)^{\frac{1}{p}} \tag{5.4.1}$$

holds for all $r > 0$, that is, the weight v satisfies a B_p-condition.

Proof. We are going to prove $(i) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$, $(iv) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$.

We start by assuming that (i) holds, i.e., there exists a norm $\|.\|$ on $\Lambda_p(v)$ and positive constants C_1 and C_2 such that

$$C_1\|f\| \le \left(\int_0^\infty f^*(x)^p v(x)dx\right)^{\frac{1}{p}} \le C_2\|f\|, \tag{5.4.2}$$

for all $f \in \Lambda_p(v)$. Let f be a positive decreasing function on $[0,\infty)$ with

$$\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} < \infty$$

and $g(x) = \chi_{[0,r)}$, for a fixed $r > 0$. Then we define the functions $\tilde{f}(y) = f(A|y|^n)$ and $\tilde{g}(y) = g(A|y|^n)$, $y \in \mathbb{R}^n$, being A the Lebesgue measure of the unit ball in $\mathbb{R}^n$, in such a way that $\tilde{f}^* = f$ and $\tilde{g}^* = g$ (cf. Lemma 5.2.6). First of all we observe that

$$(\tilde{f} * \tilde{g})(y) \ge C\int_0^r f(x)dx$$

for all $|y| \le (A^{-1}r)^{\frac{1}{n}}$, that is,

$$(\tilde{f} * \tilde{g})(y) \ge C\int_0^r f(x)dx\ \chi_{B\left(0,(A^{-1}r)^{\frac{1}{n}}\right)}(y).$$

Now we notice that if we denote

$$h(y) = C\int_0^r f(x)dx\ \chi_{B\left(0,(A^{-1}r)^{\frac{1}{n}}\right)}(y)$$

then

$$\lambda_h(t) = r\chi_{[0,C\int_0^r f(x)dx)}(t)$$

and

$$h^*(s) = C\int_0^r f(x)dx\ \chi_{[0,r)}(s).$$

Hence, by Proposition 2.3.5 (iii),

$$(\tilde{f} * \tilde{g})^*(s) \ge h^*(s) = C\int_0^r f(x)dx\ \chi_{[0,r)}(s). \tag{5.4.3}$$

Therefore, applying (5.4.3) and (5.4.2) we have that

$$\begin{aligned}
C\left(\int_0^r f(x)dx\right)\left(\int_0^r v(x)dx\right)^{\frac{1}{p}} &\le \left(\int_0^\infty (\tilde{f} * \tilde{g})^*(s)^p v(s)ds\right)^{\frac{1}{p}} \le C_2\|\tilde{f} * \tilde{g}\| \\
&= C_2\left\|\int_{\mathbb{R}^n} \tilde{f}(x-y)\tilde{g}(y)dy\right\| \\
&\le C_2\int_{\mathbb{R}^n} \tilde{g}(y)\|\tilde{f}(\cdot - y)\|dy \\
&\le \frac{C_2}{C_1} r\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}},
\end{aligned}$$

where in the last inequality we have used that $f = \tilde{f}^*$ and that $(\tilde{f}(\cdot - y))^* = \tilde{f}^*$, since

$$\{x \in \mathbb{R}^n : |f(x-y)| > t\} = \{z \in \mathbb{R}^n : |f(x)| > t\}.$$

Therefore, we get that

$$\left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \frac{\int_0^r f(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \leq Cr,$$

for all positive and decreasing function f. If we take the supremum over all positive and decreasing functions f we have that, by Theorem 5.3.7,

$$\begin{aligned} Cr &\geq \left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \sup_{0\leq f\downarrow} \frac{\int_0^r f(x)dx}{\left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \\ &\approx \left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_0^x g(t)dt\right)^{p'-1} \left(\int_0^x v(t)dt\right)^{1-p'} g(x)dx\right)^{\frac{1}{p'}} \\ &= \left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \left(\int_0^r \left(\frac{1}{x}\int_0^x v(t)dt\right)^{1-p'} dx\right)^{\frac{1}{p'}}, \end{aligned}$$

which is (iii).

In the second place we assume that (iii) holds, and we want to prove (iv). We define $A_k = \left(2^{-kp}\int_0^{2^k} v(x)dx\right)^{1-p'}$, with $k \in \mathbb{Z}$. Then, using (iii) with $r = 2^m$ we get that

$$\begin{aligned} \sum_{k=-\infty}^{m} A_k - \sum_{k=-\infty}^{m-1} A_k = A_m &= \left(2^{-mp}\int_0^{2^m} v(x)dx\right)^{1-p'} \\ &\geq C\int_0^{2^m} \left(\frac{1}{x^p}\int_0^x v(t)dt\right)^{1-p'} \frac{dx}{x} \\ &= C\sum_{k=-\infty}^{m} \int_{2^{k-1}}^{2^k} \left(\frac{1}{x^p}\int_0^x v(t)dt\right)^{1-p'} \frac{dx}{x} \\ &\geq C\sum_{k=-\infty}^{m} A_k. \end{aligned} \tag{5.4.4}$$

Hence, we have that

$$\sum_{k=-\infty}^{m-1} A_k \leq \beta \sum_{k=-\infty}^{m} A_k, \tag{5.4.5}$$

where $0 < \beta = 1 - C < 1$. Now, iterating (5.4.5) and applying (5.4.4) again we get that

$$A_j \leq \sum_{j=-\infty}^{j} A_k \leq \beta^{m-j} \sum_{k=-\infty}^{m} A_k \leq C\beta^{m-j} A_m, \tag{5.4.6}$$

for all $-\infty < j \leq m < \infty$. Finally, applying (5.4.6) and using that $0 < \beta < 1$, we have that for all $j \in \mathbb{Z}$,

$$\begin{aligned}
\int_{2^j}^{\infty} x^{-p} v(x)dx &\leq C \sum_{m=j+1}^{\infty} 2^{-mp} \int_{2^{m-1}}^{2^m} v(x)dx \leq C \sum_{m=j+1}^{\infty} 2^{-mp} \int_0^{2^m} v(x)dx \\
&= C \sum_{m=j+1}^{\infty} A_m^{1-p} \leq C \sum_{m=j+1}^{\infty} \beta^{(p-1)(m-j)} A_j^{1-p} \leq C A_j^{1-p} \\
&= \frac{C}{(2^j)^p} \int_0^{2^j} v(x)dx,
\end{aligned}$$

which is (5.4.1) with $r = 2^j$. To see (5.4.1) we notice that for $r > 0$, we can find $j \in \mathbb{Z}$ such that $2^j < r < 2^{j+1}$. Then

$$\begin{aligned}
\int_r^{\infty} \frac{1}{x^p} v(x)dx &\leq \int_{2^j}^{\infty} \frac{1}{x^p} v(x)dx \leq \frac{C}{(2^j)^p} \int_0^{2^j} v(x)dx \leq \frac{C}{(2^j)^p} \int_0^r v(x)dx \\
&= \frac{C}{(2^{j+1})^p} 2^p \int_0^r v(x)dx \leq \frac{C}{r^p} \int_0^r v(x)dx.
\end{aligned}$$

Now we suppose that (iv) holds. Then, by Theorem 5.3.3 we know that

$$\left(\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t)dt \right)^p v(x)dx \right)^{\frac{1}{p}} \leq C \left(\int_0^{\infty} f(x)^p v(x)dx \right)^{\frac{1}{p}}$$

holds for all positive decreasing function. In particular, given any $f \in \Lambda_p(v)$, we have that

$$\begin{aligned}
\left(\int_0^{\infty} f^{**}(x)^p v(x)dx \right)^{\frac{1}{p}} &= \left(\int_0^{\infty} \left(\frac{1}{x} \int_0^x f^*(x)dx \right)^p v(x)dx \right)^{\frac{1}{p}} \\
&\leq C \left(\int_0^{\infty} f^*(x)^p v(x)dx \right)^{\frac{1}{p}}.
\end{aligned}$$

In addition, as f^* is decreasing, we have that

$$x f^*(x) \leq \int_0^x f^*(t)dt,$$

that is, $f^*(x) \leq f^{**}(x)$, and therefore,

$$\left(\int_0^{\infty} f^*(x)^p v(x)dx \right)^{\frac{1}{p}} \leq \left(\int_0^{\infty} f^{**}(x)^p v(x)dx \right)^{\frac{1}{p}}.$$

Finally, we assume that (ii) holds. We have to see that $\|f\| := \left(\int_0^{\infty} f^{**}(x)^p v(x)dx \right)^{\frac{1}{p}}$ defines a norm on $\Lambda_p(v)$. The only non-trivial property is the triangular inequality.

We observe that using Proposition 2.3.10 and Hölder's inequality we have that

$$
\begin{aligned}
\|(f+g)^{**}\|_{p,v}^{p} &= \int_0^\infty (f+g)^{**}(x)^p v(x)dx \\
&= \int_0^\infty (f+g)^{**}(x)(f+g)^{**}(x)^{p-1}v(x)dx \\
&\leq \int_0^\infty f^{**}(x)(f+g)^{**}(x)^{p-1}v(x)dx \\
&\quad + \int_0^\infty g^{**}(x)(f+g)^{**}(x)^{p-1}v(x)dx \\
&\leq \|f^{**}\|_{p,v}\left(\int_0^\infty (f+g)^{**}(x)^p v(x)dx\right)^{\frac{1}{p'}} \\
&\quad + \|g^{**}\|_{p,v}\left(\int_0^\infty (f+g)^{**}(x)^p v(x)dx\right)^{\frac{1}{p'}} \\
&= \|(f+g)^{**}\|_{p,v}^{\frac{p}{p'}}\left(\|f^{**}\|_{p,v}+\|g^{**}\|_{p,v}\right),
\end{aligned}
$$

that is, $\|(f+g)^{**}\|_{p,v}^{p\left(1-\frac{1}{p'}\right)} \leq \|f^{**}\|_{p,v}+\|g^{**}\|_{p,v}$. Therefore

$$\|f+g\| = \|(f+g)^{**}\|_{p,v} \leq \|f^{**}\|_{p,v}+\|g^{**}\|_{p,v} = \|f\|+\|g\|.$$

□

Now we focus on the Lorentz space $\Lambda^1(v)$. In this context the B_p condition is not going to characterize the normability of this space. We will need a new class of weights, called $B_{1,\infty}$.

Definition 5.4.3. We say that a weight v is in $B_{1,\infty}$ if there exists a constant $C>0$ such that for all $0<s\leq r<\infty$,

$$\frac{1}{r}\int_0^r v(t)dt \leq C\frac{1}{s}\int_0^s v(t)dt.$$

Remark 5.4.4. If we write $V(x)=\int_0^x v(t)dt$, then $v\in B_{1,\infty}$ is equivalent to say that V is quasi-concave.

In order to study the normability of $\Lambda^1(v)$, we will need the following result, which is due to M. J. Carro and J. Soria (cf. [7, Theorem 3.3]).

Theorem 5.4.5. *Consider the integral operator $T_kf(x):=\int_0^\infty k(x,t)f(t)dt$ where $k(x,t)$ is a positive kernel and assume that if f is decreasing so is T_kf. Then the operator $T_kf: L^{p_0}_{\mathrm{dec}}(w_0)\longrightarrow \Lambda^{p_1,\infty}(w_1)$ with $p_0\geq 1$ is bounded if, and only if,*

$$\sup_{x>0}\sup_{r>0}\int_0^r k(x,t)dt\left(\int_0^r w_0(t)dt\right)^{-\frac{1}{p_0}}\left(\int_0^x w_1(t)dt\right)^{\frac{1}{p_1}}<\infty.$$

The following theorem (cf. [5, Theorem 2.3]), due to M. J. Carro, A. García del Amo and J. Soria, characterizes the weights v for which the space $\Lambda^1(v)$ is normable. This characterization coincide with the boundedness of the maximal operator between $\Lambda^1(v)$ and its weak version (cf. Definition 2.3.13) as well as with the boundedness of the classical Hardy operator between the decreasing functions on $L^1(v)$ and the weak-type space $L^{1,\infty}(v)$.

Theorem 5.4.6. *The following conditions are equivalent.*

(i) $\Lambda^1(v)$ *is a Banach space.*

(ii) $v \in B_{1,\infty}$.

(iii) $M : \Lambda^1(v) \longrightarrow \Lambda^{1,\infty}(v)$ *is bounded.*

(iv) $\mathcal{H} : L^1_{\text{dec}}(v) \longrightarrow L^{1,\infty}(v)$ *is bounded.*

Proof. We prove first that (iii) is equivalent to (iv). So we assume (iii) holds and we consider $f \in L^1_{\text{dec}}(v)$ and a function g such that $g^* = f$ (cf. Lemma 5.2.6). Then, using Theorem 5.2.4 we have that

$$\begin{aligned}\|\mathcal{H}f\|_{L^{1,\infty}(v)} = \|\mathcal{H}g^*\|_{L^{1,\infty}(v)} &= \sup_{t>0}\,(\mathcal{H}g^*)(t)\int_0^t v(s)ds \lesssim \sup_{t>0}\,(Mg)^*(t)\int_0^t v(s)ds \\ &= \|Mg\|_{\Lambda^{1,\infty}(v)} \lesssim \|g\|_{\Lambda^1(v)} = \|f\|_{L^1(v)},\end{aligned}$$

which is (iv). Conversely, we take $f \in \Lambda^1(v)$ and we apply Theorem 5.2.4 again, getting

$$\begin{aligned}\|Mf\|_{\Lambda^{1,\infty}(v)} &= \sup_{t>0}\,(Mf)^*(t)\int_0^t v(s)ds \lesssim \sup_{t>0}\,(\mathcal{H}f^*)(t)\int_0^t v(s)ds = \|\mathcal{H}f^*\|_{L^{1,\infty}(v)} \\ &\lesssim \|f^*\|_{L^1(v)} = \|f\|_{\Lambda^1(v)}.\end{aligned}$$

Now we prove that (ii) and (iv) are equivalent. If we take $k(x,t) = \frac{1}{x}\chi_{(0,x)}(t)$, $w_0 = w_1 = v$ and $p_0 = p_1 = 1$ in Theorem 5.4.5 then we have that the classical Hardy operator $\mathcal{H} : L^1_{\text{dec}}(v) \longrightarrow \Lambda^{1,\infty}(v)$ is bounded if, and only if,

$$\sup_{x>0}\,\sup_{r>0}\,\frac{\min(x,r)}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt < \infty. \tag{5.4.7}$$

On one hand the boundedness of $\mathcal{H} : L^1_{\text{dec}}(v) \longrightarrow \Lambda^{1,\infty}(v)$ is equivalent to the boundedness of $\mathcal{H} : L^1_{\text{dec}}(v) \longrightarrow L^{1,\infty}(v)$, since if f is a decreasing function, then $\|\mathcal{H}f\|_{L^{1,\infty}(v)} = \|\mathcal{H}f\|_{\Lambda^{1,\infty}(v)}$. On the other hand, condition (5.4.7) is equivalent to (ii). Indeed, suppose that (5.4.7) holds. Then

$$\frac{\min(x,r)}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt \le C$$

for some $C > 0$ and for all $x, r > 0$. In particular,

$$\frac{r}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt \leq C$$

for all $0 < r \leq x < \infty$, which is (ii). Conversely, if (ii) holds we fix an $x > 0$. Then, if $r \leq x$,

$$\frac{\min(x,r)}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt = \frac{r}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt \leq C,$$

and if $r > x$,

$$\frac{\min(x,r)}{x}\left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt = \left(\int_0^r v(t)dt\right)^{-1}\int_0^x v(t)dt \leq 1,$$

and (5.4.7) holds.

We prove finally that (i) and (ii) are equivalent. First of all we see that (i) implies (ii). So we assume that there exist a norm $\|.\|$ on $\Lambda^1(v)$ and constants $C_1, C_2 > 0$ such that

$$C_1\|f\| \leq \|f\|_{\Lambda^1(v)} \leq C_2\|f\|,$$

for all $f \in \Lambda^1(v)$. Then, given $N \in \mathbb{N}$ and functions $g_1, ..., g_N$ in $\Lambda^1(v)$, we have that

$$\|g_1 + ... + g_N\|_{\Lambda^1(v)} \leq C_2\left(\|g_1\| + ... + \|g_N\|\right) \leq \frac{C_2}{C_1}\left(\|g_1\|_{\Lambda^1(v)} + ... + \|g_N\|_{\Lambda^1(v)}\right). \tag{5.4.8}$$

We want to see that

$$\frac{V(r)}{r} \leq C\frac{V(s)}{s}$$

for all $0 < s \leq r < \infty$. We fix $s > 0$ and we consider $r = 2^k s$, with $k \in \mathbb{N}$. We define the functions $f(x) := \chi_{(0,2^k s)}(x)$ and $f_j(x) := \chi_{(js,(j+1)s)}(x)$, with $0 \leq j \leq 2^k - 1$. Then, taking into account Example 2.3.6 and Lemma 5.2.6, we can consider functions F and F_j defined in $\mathbb{R}^n$ such that $F^* = f$ and $F_j^* = f_j^* = \chi_{(0,s)}$ and $F = \sum_j F_j$. Now, using (5.4.8) we have that

$$\begin{aligned} V(2^k s) &= \int_0^\infty \chi_{(0,2^k s)}(t)v(t)dt = \|F\|_{\Lambda^1(v)} \leq \frac{C_2}{C_1}\sum_{j=0}^{2^k-1}\|F_j\|_{\Lambda^1(v)} = \frac{C_2}{C_1}\sum_{j=0}^{2^k-1}V(s) \\ &= \frac{2^k C_2}{C_1}V(s), \end{aligned} \tag{5.4.9}$$

which is (ii) when r is of the form $r = 2^k s$, $k \in \mathbb{N}$. For a general $r > s$, $s > 0$ fixed, there exists $k \in \mathbb{N}$ such that $2^{k-1}s \leq r \leq 2^k s$. Then, using (5.4.9), we get that

$$\frac{V(r)}{r} \leq \frac{V(2^k s)}{2^{k-1}s} \leq \frac{C_2}{C_1}\frac{2^k}{2^{k-1}}\frac{V(s)}{s} = \frac{2C_2}{C_1}\frac{V(s)}{s},$$

which is (ii). Finally, if we assume that (ii) holds then, Theorem 1.1 in [11, §II] claims that there exists a decreasing w such that if $W(x) = \int_0^x w(t)dt$, then $V \approx W$. Thus $\Lambda^1(v) = \Lambda^1(w)$. The proof is over because $\|.\|_{\Lambda^1(w)}$ is a norm if, and only if, w is decreasing (cf. [14]). □

As in Theorem 5.4.6 the normability of the Lorentz space $\Lambda^p(w)$ can be also characterized in terms of the maximal operator when $p > 1$. We are going to see that $\Lambda^p(w)$ is normable if, and only if, the maximal operator $M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$ is bounded. We first consider the following result due to M. J. Carro and J. Soria (cf. [7, Theorem 3.3]).

Theorem 5.4.7. *Let $T_k f(x) = \int_0^\infty k(x,t)f(t)dt$ and let us assume that $T_k f$ is a nonincreasing function whenever f is a nonincreasing function. Then, the operator $T_k : L^p_{\text{dec}}(w) \longrightarrow \Lambda^{p,\infty}(w)$ is bounded if, and only if,*

$$\sup_{z>0} \left(\left(\int_0^\infty \left(\int_0^y k(z,t)dt \right)^{p'} \left(\int_0^y w(t)dt \right)^{-p'} w(y)dy \right)^{\frac{1}{p'}} + \int_0^\infty k(z,t)dt \left(\int_0^\infty w(s)ds \right)^{\frac{-1}{p}} \right) \left(\int_0^z w(s)ds \right)^{\frac{1}{p}} < \infty. \tag{5.4.10}$$

Proof. The operator $T_k : L^p_{\text{dec}}(w) \longrightarrow \Lambda^{p,\infty}(w)$ is bounded if, and only if,

$$\sup_{y>0} y \left(\int_0^{\lambda_{T_k f}(y)} w(x)dx \right)^{\frac{1}{p}} \le C \|f\|_{L^p(w)}$$

for all $f \in L^p_{\text{dec}}(w)$. The functions in $L^p_{\text{dec}}(w)$ decrease and hence $T_k f$ decreases for all $f \in L^p_{\text{dec}}(w)$. Therefore for the previous supremum it is enough to take y of the form $T_k(z)$ for $z > 0$ and the operator $T_k : L^p_{\text{dec}}(w) \longrightarrow \Lambda^{p,\infty}(w)$ is bounded if, and only if,

$$\sup_{z>0} T_k f(z) \left(\int_0^z w(x)dx \right)^{\frac{1}{p}} \le C \|f\|_{L^p(w)}$$

for all $f \in L^p_{\text{dec}}(w)$, where we have used that $\lambda_{T_k f}(T_k z) = z$ (since $T_k f$ is decreasing). Equivalently,

$$\sup_{z>0} \frac{T_k f(z)}{\|f\|_{L^p(w)}} \left(\int_0^z w(x)dx \right)^{\frac{1}{p}} \le C$$

for all $f \in L^p_{\text{dec}}(w)$, that is,

$$\sup_{z>0} \sup_{f \in L^p_{\text{dec}}(w)} \frac{T_k f(z)}{\|f\|_{L^p(w)}} \left(\int_0^z w(x)dx \right)^{\frac{1}{p}} \le C.$$

Now notice that by Theorem 5.3.7 (taking $g(t) = k(x,t)$), the previous quantity is comparable to (5.4.10). □

Theorem 5.4.8. *The following are equivalent:*

(i) $\Lambda^p(w)$ *is a Banach space.*

(ii) $\mathcal{H} : L^p_{\text{dec}}(w) \longrightarrow L^{p,\infty}(w)$.

(iii) $M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$.

Proof. To prove $(i) \Leftrightarrow (ii)$ take $k(x,t) = \frac{1}{x}\chi_{(0,x)}(t)$ in Theorem 5.4.7. Then (5.4.10) becomes

$$\left(\int_0^z \left(\frac{1}{y}\int_0^y w(t)dt\right)^{-p'} w(y)dy\right)^{\frac{1}{p'}} \left(\int_0^z w(t)dt\right)^{\frac{1}{p}} \leq Cz,$$

for all $z > 0$, which is equivalent to the B_p condition (cf. Theorem 5.4.2) and, hence, to the normability of $\Lambda^p(w)$. Notice also that $\mathcal{H}f$ is always decreasing since f is decreasing and, then, the image of $\mathcal{H}$ is a subspace of $\Lambda^{p,\infty}(w)$.

The equivalence $\mathcal{H} : L^p_{\text{dec}}(w) \longrightarrow L^{p,\infty}(w) \Leftrightarrow M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$ can be proved in the same way as $\mathcal{H} : L^{p,\infty}_{\text{dec}}(w) \longrightarrow L^{p,\infty}(w) \Leftrightarrow M : \Lambda^{p,\infty}(w) \longrightarrow \Lambda^{p,\infty}(w)$ (see later on the proof $(i) \Leftrightarrow (ii)$ in Theorem 5.4.17). □

5.4.2 Normability of weak Lorentz spaces

In this section we deal with the analogous problem of Section 5.4.1 for the weak-type Lorentz space $\Lambda^{p,\infty}(v)$. We will see that the normability will be equivalent to having the weight in B_p and, in this context, we can study all the cases at the same time considering just $0 < p < \infty$. We follow in this section J. Soria's article [19].

The following proposition (cf. [19, Proposition 2.1]) shows that, in some sense, $\Lambda^{p,\infty}(v)$ does not depend on p.

Proposition 5.4.9. *Let* $0 < p < \infty$ *and* $\alpha > -1$. *For a given weight* v, *define*

$$v_\alpha(t) := (1+\alpha)V(t)^\alpha v(t).$$

Then $\|.\|_{\Lambda^{p,\infty}(v)} = \|.\|_{\Lambda^{p(\alpha+1),\infty}(v_\alpha)}$ *and hence* $\Lambda^{p,\infty}(v) = \Lambda^{p(\alpha+1),\infty}(v_\alpha)$.

Remark 5.4.10. If we take $\alpha = \frac{1}{p} - 1 > -1$ in Proposition 5.4.9, we have that

$$\Lambda^{p,\infty}(v) = \Lambda^{1,\infty}(v_{\frac{1}{p}-1}),$$

in such a way that we have moved the p from the space to the weight.

Proof. Since

$$\left(\int_0^t v_\alpha(r)dr\right)^{\frac{1}{p(\alpha+1)}} = \left(\int_0^t (1+\alpha)V(r)^\alpha v(r)dr\right)^{\frac{1}{p(\alpha+1)}} = \left(\left[v(r)^{\alpha+1}\right]_{r=0}^t\right)^{\frac{1}{p(\alpha+1)}}$$
$$= V(t)^{\frac{1}{p}},$$

we have that

$$\|f\|_{\Lambda^{p(\alpha+1),\infty}(v_\alpha)} = \sup_{t>0} f^*(t) \left(\int_0^t v_\alpha(r)dr \right)^{\frac{1}{p(\alpha+1)}} = \sup_{t>0} f^*(t) V(t)^{\frac{1}{p}} = \|f\|_{\Lambda^{p,\infty}(v)}.$$

□

We will need also the following theorem (cf [19, Theorem 2.5]), which is a characterization of a B_p weight v in terms of its primitive V.

Theorem 5.4.11. *Let $0 < p < \infty$. The following are equivalent:*

(i) $v \in B_p$.

(ii) $\int_0^r \frac{t^{p-1}}{V(t)} dt \le C \frac{r^p}{V(r)}$.

(iii) $\int_r^\infty \frac{V(t)}{t^{p+1}} dt \le C \frac{V(r)}{r^p}$.

Proof. A weight $v \in B_p$ if and only if there exists a constant $C > 0$ such that

$$\int_r^\infty \frac{v(t)}{t^p} dt \le C \frac{V(r)}{r^p}$$

for all $r > 0$. Equivalently,

$$\int_r^s \frac{v(t)}{t^p} dt \le C \frac{V(r)}{r^p} \tag{5.4.11}$$

for all $s > r > 0$. Integrating by parts,

$$\int_r^s \frac{v(t)}{t^p} dt = \frac{V(s)}{s^p} - \frac{V(r)}{r^p} + C \int_r^s \frac{V(t)}{t^{p+1}} dt,$$

and then (5.4.11) is equivalent to

$$\frac{V(s)}{s^p} \le C \frac{V(r)}{r^p} \quad \text{and} \quad \int_r^\infty \frac{V(t)}{t^{p+1}} dt \le C \frac{V(r)}{r^p}.$$

But now observe that the first condition is consequence of the second one since

$$\int_r^\infty \frac{V(t)}{t^{p+1}} dt \ge \int_s^\infty \frac{V(t)}{t^{p+1}} dt \ge V(s) \int_s^\infty \frac{1}{t^{p+1}} dt = \frac{V(s)}{s^p}.$$

Therefore 5.4.11 is just equivalent to

$$\int_r^\infty \frac{V(t)}{t^{p+1}} dt \le C \frac{V(r)}{r^p},$$

which is (iii). Finally, to see equivalence between (ii) and (iii) we use the following result in [16]: if m is a positive function, $0 < r < \infty$, then

$$\int_0^r m(s) \frac{ds}{s} \approx m(r)$$

if, and only if,

$$\int_r^\infty \frac{1}{m(s)} \frac{ds}{s} \approx \frac{1}{m(r)}.$$

The equivalence between (ii) and (iii) follows taking $m(r) = \frac{r^p}{V(r)}$. □

Definition 5.4.12. Given a weight v, we denote $V(t)=\int_0^t v(s)ds$. For $0<p<\infty$, we say that V is a p quasi-concave function if there exists a constant $C>0$ such that for all $0<s\le r<\infty$,

$$\frac{V(r)}{r^p}\le C\frac{V(s)}{s^p}. \tag{5.4.12}$$

The next proposition (cf. [19, Proposition 2.6]) allows us to construct B_p weights.

Proposition 5.4.13. *Let $0<p<\infty$, $\alpha>-1$ and $\epsilon>0$. If V satisfies (5.4.12), then $v_\alpha\in B_{p(\alpha+1)+\epsilon}$.*

Remark 5.4.14. If we take $\alpha=0$ in Proposition 5.4.13, then $v\in B_{p+\epsilon}$ for all $\epsilon>0$, that is,

$$v\in\cap_{q>p}B_q.$$

Proof. We observe that

$$\int_0^r v_\alpha(s)ds=V(r)^{\alpha+1}$$

and, by Theorem 5.4.11 (*iii*), it is enough to prove that

$$\int_r^\infty \frac{V(s)^{\alpha+1}}{s^{p(\alpha+1)+\epsilon+1}}ds\le C\frac{V(r)^{\alpha+1}}{r^{p(\alpha+1)+\epsilon}}.$$

But we notice that, integrating by parts,

$$\int_r^\infty \frac{V(s)^{\alpha+1}}{s^{p(\alpha+1)+\epsilon+1}}ds=C'\frac{V(r)^{\alpha+1}}{r^{p(\alpha+1)+\epsilon}}+C''\int_r^\infty\frac{V(s)^{\alpha}}{s^{p(\alpha+1)+\epsilon}}v(s)ds,$$

so it is enough to see that

$$\int_r^\infty\frac{V(s)^{\alpha}}{s^{p(\alpha+1)+\epsilon}}v(s)ds\le C\frac{V(r)^{\alpha+1}}{r^{p(\alpha+1)+\epsilon}}.$$

Finally, using the p quasi-concavity of V, we conclude that

$$\begin{aligned}
\int_r^\infty\frac{V(t)^{\alpha}}{t^{p(\alpha+1)+\epsilon}}v(t)dt &= \int_r^\infty\left(\frac{V(t)^{\frac{1}{p}}}{t}\right)^{p(\alpha+1)+\epsilon}V(t)^{-1-\frac{\epsilon}{p}}v(t)dt\\
&\le C\left(\frac{V(r)^{\frac{1}{p}}}{r}\right)^{p(\alpha+1)+\epsilon}\int_r^\infty V(t)^{-1-\frac{\epsilon}{p}}v(t)dt\\
&= C\left(\frac{V(r)^{\frac{1}{p}}}{r}\right)^{p(\alpha+1)+\epsilon}\left[-\frac{p}{\epsilon}V(t)^{-\frac{\epsilon}{p}}\right]_{t=r}^\infty\\
&\le C\left(\frac{V(r)^{\frac{1}{p}}}{r}\right)^{p(\alpha+1)+\epsilon}V(r)^{-\frac{\epsilon}{p}}=C\frac{V(r)^{\alpha+1}}{r^{p(\alpha+1)+\epsilon}}.
\end{aligned}$$

□

Before studying the normability of $\Lambda^{p,\infty}(v)$, it is necessary to study the boundedness of the classical Hardy operator $\mathcal{H}$ between weak-type Lebesgue spaces $L^{p,\infty}(v)$ restricted to positive decreasing functions. We need first the next proposition (cf. [19, Proposition 2.7]), which shows that for certain monotone operators, the boundedness on the class $L^{p,\infty}_{\mathrm{dec}}(v)$ of decreasing functions in $L^{p,\infty}(v)$ is determined by the action on a particular function.

Proposition 5.4.15. *Let X be a class of functions in $\mathbb{R}^+$. Consider a functional N defined on functions on $\mathbb{R}^+$ satisfying*

(i) $f \in X$ if, and only if, $N(f) < \infty$,

(ii) there exists $C > 0$ such that for all $f, g \in X$ with $0 \leq f \leq g$, $N(f) \leq CN(g)$,

(iii) there exists $C > 0$ such that for all $\lambda > 0$, $N(\lambda f) \leq C\lambda N(f)$.

Let $0 < p < \infty$ and let T be a positive operator defined on $L^{p,\infty}_{\mathrm{dec}}(v)$ satisfying that

(iv) there exists a constant $C > 0$ so that for all $f, g \in L^{p,\infty}_{\mathrm{dec}}(v)$ with $0 \leq f \leq g$ we have $T(f) \leq CT(g)$.

Then $T : L^{p,\infty}_{\mathrm{dec}}(v) \longrightarrow X$ if, and only if, $N(T(V^{-\frac{1}{p}})) < \infty$.

Proof. First of all we notice that

$$\left\| V^{-\frac{1}{p}} \right\|_{L^{p,\infty}(v)} = \sup_{t>0} V^{-\frac{1}{p}} V(t)^{\frac{1}{p}} = 1 < \infty,$$

an hence $V^{-\frac{1}{p}} \in L^{p,\infty}(v)$. If $T : L^{p,\infty}_{\mathrm{dec}}(v) \longrightarrow X$, then $T(V^{-\frac{1}{p}}) \in X$ or, equivalently, $N(T(V^{-\frac{1}{p}})) < \infty$.

Conversely, we notice that if $f \in L^{p,\infty}_{\mathrm{dec}}(v)$, then

$$f(t) \leq \|f\|_{L^{p,\infty}(v)} V(t)^{-\frac{1}{p}}$$

for all $t > 0$. Then, using the properties of T we get that

$$T(f) \leq C\|f\|_{L^{p,\infty}(v)} T\left(V(t)^{-\frac{1}{p}}\right)$$

and using the properties of N, we conclude that

$$N(Tf) \leq C\|f\|_{L^{p,\infty}(v)} N\left(T\left(V(t)^{-\frac{1}{p}}\right)\right).$$

$\square$

We characterize now (cf. [19, Theorem 2.8]) the weights for which the operator $\mathcal{H} : L^{p,\infty}_{\mathrm{dec}}(v) \longrightarrow L^{p,\infty}(v)$ is bounded and, in addition, we state a $p - \epsilon$ condition for the corresponding weights.

Theorem 5.4.16. *Let $0 < p < \infty$. Then*

(i) $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$ *if, and only if,*

$$\int_0^r \frac{1}{V(t)^{\frac{1}{p}}} dt \leq C \frac{r}{V(r)^{\frac{1}{p}}} \tag{5.4.13}$$

for all $r > 0$.

(ii) If v *satisfies* (5.4.13)*, then there exists* $q > p$ *such that* $\mathcal{H} : L^{q,\infty}_{\text{dec}}(v) \longrightarrow L^{q,\infty}(v)$.

Proof. To prove (i) we take $T = \mathcal{H}$, $X = L^{p,\infty}(v)$ and $N = \|.\|_{L^{p,\infty}(v)}$ in Proposition 5.4.15. Then

$$\left\| \mathcal{H}(V^{-\frac{1}{p}}) \right\|_{L^{p,\infty}(v)} = \sup_{t>0} \frac{1}{t} \left(\int_0^t V(s)^{-\frac{1}{p}} ds \right) V(t)^{\frac{1}{p}} < \infty,$$

which is (5.4.13).

Now we prove (ii). Let $f(t) = V(t)^{-\frac{1}{p}}$ in such a way that $\|f\|_{L^{p,\infty}(v)} = 1$ (cf. proof of Proposition 5.4.15). We define

$$\mathcal{H}^k f(t) := \mathcal{H}(\mathcal{H}^{k-1} f)(t),$$

with $\mathcal{H}^1 f(t) = (\mathcal{H} f)(t)$. We notice that if v satisfies (5.4.13), then by (i) we have that $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$, and therefore $\mathcal{H} f \in L^{p,\infty}_{\text{dec}}(v)$. So $\mathcal{H}^k$ is well-defined. Now we claim that

$$\mathcal{H}^k f(t) = \frac{1}{t} \int_0^t \frac{f(r)}{(k-1)!} \log^{k-1} \left(\frac{t}{r} \right) dr.$$

Indeed, for $k = 2$ we have that, integrating by parts,

$$\mathcal{H}^2 f = \frac{1}{t} \int_0^t \frac{1}{r} \int_0^r f(s) ds \, dr = \frac{1}{t} \int_0^t f(s) ds \log(t) - \frac{1}{t} \quad = \frac{1}{t} \int_0^t \frac{f(r)}{1!} \log^{2-1} \left(\frac{t}{r} \right) dr.$$

In addition, if we assume the formula hols for k then, applying Fubini's Theorem,

$$\begin{aligned} \mathcal{H}^{k+1} f(t) &= \frac{1}{t} \int_0^t \frac{1}{r} \int_0^r \frac{f(s)}{(k-1)!} \log^{k-1} \left(\frac{r}{s} \right) ds \, dr \\ &= \frac{1}{t} \int_0^t f(s) \int_s^t \frac{1}{r} \frac{1}{(k-1)!} \log^{k-1} \left(\frac{r}{s} \right) dr \, ds \\ &= \frac{1}{t} \int_0^t f(s) \frac{1}{k!} \log^k \left(\frac{t}{s} \right) ds. \end{aligned}$$

By (i), we know there exists $A > 0$ such that $\|\mathcal{H}^k f\|_{L^{p,\infty}(v)} \leq A^k \|f\|_{L^{p,\infty}(v)} = A^k$. If we take $0 < \epsilon < \frac{1}{A}$, then

$$\sum_{k=1}^{\infty} \|\epsilon^k \mathcal{H}^k f\|_{L^{p,\infty}(v)} \leq \sum_{k=1}^{\infty} (\epsilon A)^k < \infty,$$

and hence

$$\sum_{k=1}^{\infty} \|\epsilon^k \mathcal{H}^k f\|_{L^{p,\infty}(v)} \leq C$$

for some $C > 0$. Now notice that given $\alpha = \frac{2}{p} - 1$ we have that, by Proposition 5.4.9, $\|g\|_{L^{p,\infty}_{\text{dec}}(v)} = \|g\|_{L^{2,\infty}_{\text{dec}}(v_\alpha)}$ for all function $g \in L^{p,\infty}_{\text{dec}}(v)$. Then, as $L^{2,\infty}_{\text{dec}}(v_\alpha)$ is a Banach space,

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} \epsilon^k \mathcal{H}^k f\right\|_{L^{p,\infty}_{\text{dec}}(v)} &= \left\|\sum_{k=1}^{\infty} \epsilon^k \mathcal{H}^k f\right\|_{L^{2,\infty}_{\text{dec}}(v_\alpha)} \leq C \sum_{k=1}^{\infty} \|\epsilon^k \mathcal{H}^k f\|_{L^{2,\infty}_{\text{dec}}(v_\alpha)} \\ &= C \sum_{k=1}^{\infty} \|\epsilon^k \mathcal{H}^k f\|_{L^{p,\infty}_{\text{dec}}(v)} \leq C.\end{aligned} \tag{5.4.14}$$

Now we observe that

$$\begin{aligned}\left(\sum_{k=1}^{\infty} \epsilon^k \mathcal{H}^k f\right)(t) &= \frac{1}{t} \sum_{k=0}^{\infty} \epsilon^{k+1} \int_0^t \frac{V(r)^{-\frac{1}{p}}}{k!} \log^k\left(\frac{t}{r}\right) dr \\ &= \frac{\epsilon}{t} \int_0^t V(r)^{-\frac{1}{p}} \sum_{k=0}^{\infty} \frac{\left(\epsilon \log\left(\frac{t}{r}\right)\right)^k}{k!} dr \\ &= \frac{\epsilon}{t} \int_0^t V(r)^{-\frac{1}{p}} e^{\epsilon \log \frac{t}{r}} dr \\ &= \frac{\epsilon}{t} \int_0^t V(r)^{-\frac{1}{p}} \left(\frac{t}{r}\right)^{\epsilon} dr\end{aligned}$$

and (5.4.14) gives

$$\left(\frac{\epsilon}{t} \int_0^t V(r)^{-\frac{1}{p}} \left(\frac{t}{r}\right)^{\epsilon} dr\right) V(t)^{\frac{1}{p}} \leq C \tag{5.4.15}$$

for all $t > 0$. Now from (5.4.13) we have that

$$\frac{s}{V(s)^{\frac{1}{p}}} \leq \int_0^t \frac{1}{V(r)^{\frac{1}{p}}} dr \leq C \frac{t}{V(t)^{\frac{1}{p}}}$$

for all $s > t > 0$, which is the p quasi-concave condition. Applying this to (5.4.15) we get that

$$\frac{V(t)^{\frac{\epsilon}{p}}}{t} V(t)^{\frac{1}{p}} \int_0^t V(r)^{-\frac{1}{p}} V(r)^{-\frac{\epsilon}{p}} dr \leq C,$$

which is

$$\int_0^t V(r)^{-\frac{1+\epsilon}{p}} dr \leq C \frac{t}{V(t)^{\frac{1+\epsilon}{p}}}.$$

This implies that $\mathcal{H} : L^{q,\infty}_{\text{dec}}(v) \longrightarrow L^{q,\infty}(v)$ with $q = \frac{p}{1+\epsilon}$. $\square$

Finally, we present the desired result (cf. [19, Theorem 3.1]). The normability of the weak-type Lorentz space $\Lambda^{p,\infty}(v)$ will be equivalent to the boundedness of the Hardy operator $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$, the maximal operator $M : \Lambda^{p,\infty}(v) \longrightarrow \Lambda^{p,\infty}(v)$ or to have the weight v in B_p.

Theorem 5.4.17. *Let $0 < p < \infty$ and let v be a weight in $\mathbb{R}^+$. Then, the following are equivalent.*

(i) $M : \Lambda^{p,\infty}(v) \longrightarrow \Lambda^{p,\infty}(v)$.

(ii) *v satisfies (5.4.13), that is,*

$$\int_0^r \frac{1}{V(t)^{\frac{1}{p}}} dt \leq C \frac{r}{V(r)^{\frac{1}{p}}}$$

for all $r > 0$.

(iii) $v \in B_p$.

(iv) *If* $\|f\|^*_{\Lambda^{p,\infty}(v)} := \sup_{t>0} (\mathcal{H}f^*)(t) V(t)^{\frac{1}{p}}$, *then* $\|.\|_{\Lambda^{p,\infty}(v)} \approx \|.\|^*_{\Lambda^{p,\infty}(v)}$.

(v) *$\Lambda^{p,\infty}(v)$ is a Banach space.*

Proof. We will prove $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$.

We start proving $(i) \Leftrightarrow (ii)$. First of all we are going to see that the boundedness of $M : \Lambda^{p,\infty}(v) \longrightarrow \Lambda^{p,\infty}(v)$ is equivalent to $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$. So we assume (i) holds and we take $f \in L^{p,\infty}_{\text{dec}}(v)$. Then, by Lemma 5.2.6, we can find a function $\tilde{f}$ such that $\tilde{f}^* = f$. Therefore, applying Theorem 5.2.4,

$$\begin{aligned} \|\mathcal{H}f\|_{L^{p,\infty}(v)} &= \|\mathcal{H}\tilde{f}^*\|_{L^{p,\infty}(v)} = \sup_{t>0} (\mathcal{H}\tilde{f}^*)(t) V(t)^{\frac{1}{p}} \lesssim \sup_{t>0} (M\tilde{f})^*(t) V(t)^{\frac{1}{p}} \\ &= \|M\tilde{f}\|_{\Lambda^{p,\infty}(v)} \lesssim \|\tilde{f}\|_{\Lambda^{p,\infty}(v)} = \|f\|_{L^{p,\infty}(v)}. \end{aligned}$$

Conversely, assume $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$ and take $f \in \Lambda^{p,\infty}(v)$. Then, applying again Theorem 5.2.4,

$$\begin{aligned} \|Mf\|_{\Lambda^{p,\infty}(v)} &= \sup_{t>0} (Mf)^*(t) V(t)^{\frac{1}{p}} \lesssim \sup_{t>0} (\mathcal{H}f^*)(t) V(t)^{\frac{1}{p}} = \|\mathcal{H}f^*\|_{L^{p,\infty}(v)} \\ &\lesssim \|f^*\|_{L^{p,\infty}(v)} = \|f\|_{\Lambda^{p,\infty}(v)}. \end{aligned}$$

Finally, by Theorem 5.4.16, $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$ is equivalent to (ii).

We prove now $(ii) \Leftrightarrow (iv)$. Assume first that (iv) holds and take $f \in L^{p,\infty}_{\text{dec}}(v)$. As before, we can find $\tilde{f}$ such that $\tilde{f}^* = f$. Then,

$$\|\mathcal{H}f\|_{L^{p,\infty}(v)} = \sup_{t>0} (\mathcal{H}\tilde{f}^*)(t) V(t)^{\frac{1}{p}} = \|\tilde{f}\|^*_{\Lambda^{p,\infty}(v)} \lesssim \|\tilde{f}\|_{\Lambda^{p,\infty}(v)} = \|f\|_{L^{p,\infty}(v)},$$

which means that $\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$, which is equivalent to (ii) by Theorem 5.4.16. Conversely, assume (ii) holds and, by Theorem 5.4.16, we have that

$\mathcal{H} : L^{p,\infty}_{\text{dec}}(v) \longrightarrow L^{p,\infty}(v)$. On one hand, as $f^* \leq f^{**}$, $\|f\|_{L^{p,\infty}(v)} \leq \|f\|^*_{L^{p,\infty}(v)}$. On the other hand,

$$\|f\|^*_{L^{p,\infty}(v)} = \sup_{t>0}(\mathcal{H}f^*)(t)V(t)^{\frac{1}{p}} = \|\mathcal{H}f^*\|_{L^{p,\infty}(v)} \lesssim \|f^*\|_{L^{p,\infty}(v)} = \|f\|_{\Lambda^{p,\infty}(v)}.$$

Since $\|.\|^*_{\Lambda^{p,\infty}(v)}$ is a norm (recall that $(f+g)^{**} \leq f^{**} + g^{**}$) clearly (iv) implies (v).

We prove now $(ii) \Leftrightarrow (iii)$. Suppose that v satisfies (ii). Then, by Theorem 5.4.16, there exists $q > p$ such that $\mathcal{H} : L^{q,\infty}_{\text{dec}}(v) \longrightarrow L^{q,\infty}(v)$. Then v satisfies (ii) with q instead of p and hence (cf. proof of Theorem 5.4.16) V satisfies the q quasi-concave condition (5.4.12). By Proposition 5.4.13, $v \in B_p$. Now assume (iii) is satisfied. If $0 < p \leq 1$ then V satisfies the p quasi-concave condition and, by Theorem 5.4.11 (ii),

$$\left(\frac{t}{V(t)^{\frac{1}{p}}}\right)^{p-1} \int_0^t \frac{1}{V(r)^{\frac{1}{p}}} dr \leq \int_0^t \left(\frac{r}{V(r)^{\frac{1}{p}}}\right)^{p-1} \frac{1}{V(r)^{\frac{1}{p}}} dr \leq C\frac{t^p}{V(t)},$$

which is (ii). If $p > 1$ we take $1 < q < p$ such that $v \in B_q$ (cf. [2, Theorem 1.1]) and $s = \frac{q-1}{p-1}$. Then, using Hölder's Inequality and Theorem 5.4.11 (ii),

$$\begin{aligned}\int_0^t \frac{1}{V(r)^{\frac{1}{p}}} dr &= \int_0^t \frac{r^s}{V(r)^{\frac{1}{p}}} \frac{1}{r^{\frac{s}{p}}} \frac{1}{s^{\frac{s}{p'}}} \leq C\left(\int_0^t \frac{r^{ps}}{V(r)} \frac{dr}{r^s}\right)^{\frac{1}{p}} t^{\frac{1-s}{p'}} \\ &= C\left(\int_0^t \frac{r^{q-1}}{V(r)} dr\right)^{\frac{1}{p}} t^{1-\frac{q}{p}} \leq C\frac{t}{V(t)^{\frac{1}{p}}}.\end{aligned}$$

Finally, we prove that (v) implies (ii). So we assume $\Lambda^{p,\infty}(v)$ is a Banach space. Using Proposition 5.4.9, we can assume that $p = 1$. Proceeding as in the proof of $(i) \Rightarrow (ii)$ in Theorem 5.4.6, we have that V is quasi-concave. Now we claim that given $A > 0$, we can find $B > 0$ such that for all $t > 0$,

$$V(At) \leq BV(t). \tag{5.4.16}$$

Indeed, using the quasi concavity of V,

$$V(At) \leq V((A+1)t) \leq \frac{C(A+1)t}{t} V(t) = BV(t).$$

Now fix $N \in \mathbb{N}$ even and $\epsilon > 0$. Define $f_j(x) = \frac{1}{V(|\epsilon j - |x|^n|)}$ with $x \in \mathbb{R}^n$, $j = 1, ..., N$. We want to show that there exists $C > 0$ independent of ϵ and j such that

$$\|f_j\|_{\Lambda^{1,\infty}(v)} \leq C. \tag{5.4.17}$$

With this aim we define $m := \inf_{t>0} \frac{1}{V(t)}$. We notice that $m > 0$ if, and only if, v is integrable on $\mathbb{R}^n$. Indeed, if v is not integrable then there exists t_0 such that $V(t) = \infty$ for all $t \geq t_0$, and hence $m = 0$. Conversely, if v is integrable in $\mathbb{R}^n$, then

$$\frac{1}{V(t)} \geq \frac{1}{\|v\|_{L^1(\mathbb{R}^n)}} > 0.$$

When $m=0$, we take $0<s<\infty$ and define

$$\alpha(s) := \inf\left\{r : \frac{1}{V(r)} = s\right\}.$$

Notice that by the continuity of v, $V(\alpha(s)) = \frac{1}{s}$. We have the following estimate for the distribution function of f_j:

$$\begin{aligned}
\lambda_{f_j}(s) &= \left|\left\{x : \frac{1}{V(|\epsilon j - |x|^n|)} > s\right\}\right| = C\int_{\left\{\rho: \frac{1}{V(|\epsilon j - |\rho|^n|)} > s\right\}} \rho^{n-1}d\rho \\
&= C\int_{\{\rho: |\epsilon j - \rho^n| < \alpha(s)\}} \rho^{n-1}d\rho = C\int_{\{\rho:\rho^n > \epsilon j - \alpha(s) \text{ and } \rho^n < \epsilon j + \alpha(s)\}} \rho^{n-1}d\rho \\
&= \begin{cases} C\int_0^{(\epsilon j+\alpha(s))^{\frac{1}{n}}} \rho^{n-1}d\rho, & \text{if } \epsilon j - \alpha(s) \le 0, \\ C\int_{(\epsilon j-\alpha(s))^{\frac{1}{n}}}^{(\epsilon j+\alpha(s))^{\frac{1}{n}}} \rho^{n-1}d\rho, & \text{if } \epsilon j - \alpha(s) > 0, \end{cases} \\
&= \begin{cases} C\frac{1}{n}(\epsilon j + \alpha(s)), & \text{if } \epsilon j - \alpha(s) \le 0, \\ \frac{C}{n}2\alpha(s), & \text{if } \epsilon j - \alpha(s) > 0, \end{cases} \\
&\le C\alpha(s).
\end{aligned}$$

Hence, using (5.4.16),

$$\|f_j\|_{\Lambda^{1,\infty}(v)} = \sup_{t>0} tV(\lambda_{f_j}(t)) \le \sup_{t>0} tV(C\alpha(t)) \le C\sup_{t>0} tV(\alpha(t)) = C.$$

For the case $m>0$ we proceed similarly. The problem would be that for the real numbers satisfying $s < \frac{1}{\|v\|_{L^1}}$ we would have $\lambda_{f_j}(s) = \infty$. But notice that for $s < \frac{1}{\|v\|_{L^1}}$, $sV(\lambda_{f_j}(s)) \le \frac{1}{\|v\|_{L^1}}\|v\|_{L^1} = 1$, and this would not affect that $\|f_j\|_{\Lambda^{1,\infty}(v)}$ is bounded.

Now, given $x \in \mathbb{R}^n$ with $|x|^n < N\epsilon$, we take the $k \in \{0, ..., N-1\}$ satisfying $\epsilon k \le |x|^n < \epsilon(k+1)$. Then

$$\sum_{j=1}^{N} \frac{1}{V(|\epsilon j - |x|^n|)} = \sum_{j=1}^{k} \frac{1}{V(|\epsilon j - |x|^n|)} + \sum_{j=k+1}^{N} \frac{1}{V(|\epsilon j - |x|^n|)} = I + II.$$

In I, since $|x|^n - \epsilon j \le \epsilon(k+1-j)$, then

$$I \ge \sum_{j=1}^{k} \frac{1}{V(\epsilon(k+1-j))} = \sum_{j=1}^{k} \frac{1}{V(\epsilon j)}.$$

Similarly, applying in II that $\epsilon j - |x|^n \le \epsilon(j-k)$, we have that

$$II \ge \sum_{j=k+1}^{N} \frac{1}{V(\epsilon(j-k))} = \sum_{j=1}^{N-k} \frac{1}{V(\epsilon j)}.$$

Then either $k \geq \frac{N}{2}$ or $N-k \geq \frac{N}{2}$, so

$$\sum_{j=1}^{N} \frac{1}{V(|\epsilon j - |x|^n|)} \geq \sum_{j=1}^{k} \frac{1}{V(\epsilon j)} + \sum_{j=1}^{N-k} \frac{1}{V(\epsilon j)} \geq \sum_{j=1}^{\frac{N}{2}} \frac{1}{V(\epsilon j)} \tag{5.4.18}$$

for all $|x|^n < N\epsilon$. Now, using (5.4.18), (5.4.16), the hypothesis and (5.4.17), we have that

$$\left(\sum_{j=1}^{\frac{N}{2}} \frac{1}{V(\epsilon j)}\right) V(\epsilon N) \leq C \left\|\sum_{j=1}^{N} f_j\right\|_{\Lambda^{1,\infty}(v)} \leq CN,$$

for all $|x|^n < N\epsilon$. Then, as $\frac{1}{V}$ decreases,

$$\left(\frac{1}{\epsilon}\int_{\epsilon}^{\frac{\epsilon N}{2}} \frac{dr}{V(r)}\right) V(\epsilon N) \leq \left(\sum_{j=1}^{\frac{N}{2}} \frac{1}{V(\epsilon j)}\right) V(\epsilon N) \leq CN. \tag{5.4.19}$$

Now if we take $s = \frac{\epsilon N}{2}$ in (5.4.19) then

$$N\int_{\frac{2s}{N}}^{s} \frac{dt}{W(t)} \leq CN\frac{2s}{W(2s)} \leq CN\frac{2s}{2W(s)} = CN\frac{s}{W(s)},$$

and taking $N \longrightarrow \infty$,

$$\int_0^s \frac{dt}{W(t)} \leq C\frac{s}{W(s)},$$

which is (ii). □

5.4.3 Maximal operator and normability of Lorentz spaces

We can summarize what has been seen in Sections 5.4.1 and 5.4.2 in the following way. Using Theorem 5.2.7, Theorem 5.3.3, Corollary 5.3.6 and Theorem 5.4.17, we have that, for $p > 0$,

$$\begin{aligned} M : \Lambda^p(u) \longrightarrow \Lambda^p(u) &\Leftrightarrow \mathcal{H} : L^p_{\text{dec}}(u) \longmapsto L^p(u) \\ &\Leftrightarrow u \in B_p \\ &\Leftrightarrow \Lambda^{p,\infty}(u) \text{ is Banach.} \end{aligned} \tag{5.4.20}$$

That means that the strong boundedness of the maximal operator is equivalent to the normability of the weak-type Lorentz space. Similarly, using Theorem 5.4.2, Theorem 5.4.6 and Theorem 5.4.8, we get that

$$\begin{aligned} M : \Lambda^p(u) \longrightarrow \Lambda^{p,\infty}(u) &\Leftrightarrow \mathcal{H} : L^p_{\text{dec}}(u) \longmapsto L^{p,\infty}(u) \\ &\Leftrightarrow u \in B_p \\ &\Leftrightarrow \Lambda^p(u) \text{ is Banach,} \end{aligned} \tag{5.4.21}$$

if $p > 1$, and

$$\begin{aligned} M : \Lambda^1(u) \longrightarrow \Lambda^{1,\infty}(u) &\Leftrightarrow \mathcal{H} : L^1_{\text{dec}}(u) \longmapsto L^{1,\infty}(u) \\ &\Leftrightarrow u \in B_{1,\infty} \\ &\Leftrightarrow \Lambda^1(u) \text{ is Banach.} \end{aligned} \tag{5.4.22}$$

Example 5.4.18. Now, for example, if we take $p = 1$ and $u \equiv 1$ in (5.4.20), we get

$$M : L^1 \longrightarrow L^1 \Leftrightarrow L^{1,\infty} \text{ is Banach.}$$

Since the maximal operator is not bounded in L^1, we can deduce that the weak-type Lebesgue space $L^{1,\infty}$ is not normable. Similarly, if we take $u \equiv 1$ in (5.4.21), we get that

$$M : L^p \longrightarrow L^{p,\infty} \Leftrightarrow L^p \text{ is Banach,}$$

and the weak boundedness of the maximal operator on L^p, $p > 1$, can be deduced. Furthermore, if we take $u \equiv 1$ in (5.4.22), we have that

$$M : L^1 \longrightarrow L^{1,\infty} \Leftrightarrow L^1 \text{ is Banach,}$$

and the weak boundedness of the maximal operator on L^1 can be deduced.

Bibliography

[1] M. A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 727–735.

[2] J. Bastero and F. J. Ruiz, *Elementary reverse Hölder type inequalities with application to operator interpolation theory*, Proc. Amer. Math. Soc. **124** (1996), no. 10, 3183–3192.

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[4] J. S. Bradley, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21** (1978), no. 4, 405–408.

[5] M. J. Carro, A. García del Amo and J. Soria, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 849–857.

[6] M. J. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. **112** (1993), no. 2, 480–494.

[7] M. J. Carro and J. Soria, *Boundedness of some integral operators*, Canad. J. Math. **45** (1993), no. 6, 1155–1166.

[8] M. M. Day, *The spaces L^p with* $0 < p < 1$, Bull. Amer. Math. Soc. **46** (1940), 816–823.

[9] A. Kufner, L. Maligranda and L.-E. Person, *The Hardy Inequality: About its history and some related results*, Pilsen, 2007.

[10] A. Kufner and L.-E. Person, *Weighted inequalities of Hardy type*, World Scientific, 2003.

[11] S. G. Krein, Ju. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Transl. Math. Monographs, vol. 54, Amer. Math. Soc., Providence, RI, 1982.

[12] Q. Lai, *Weighted inequality for some classical integral operators:* $0 < p < 1$, Math. Inequal. Appl **3** (2000), no. 2, 253–258.

[13] N. Levinson, *Generalizations of an inequality of Hardy,* Duke Math. J. **31** (1964), 389–394.

[14] G. G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.

[15] B. Muckenhoupt, *Hardy's inequality with weights,* Studia Math. **44** (1972), 31–38.

[16] Y. Sagher, *Real interpolation with weights,* Indiana Univ. Math. J. **30** (1981), 113–121.

[17] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces,* Studia Math. **96** (1990), no. 2, 145–158.

[18] G. Sinnamon and V. D. Stepanov, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. (2) **54** (1996), no. 1, 89–101.

[19] J. Soria, *Lorentz spaces of weak-type,* Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 193, 93–103.

[20] V. D. Stepanov, *The weighted Hardy's inequality for nonincreasing functions,* Trans. Amer. Math. Soc. **338** (1993), no. 1, 173–186.