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MÀSTER DE MATEMÀTICA AVANÇADA

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A rational family of singular perturbations. The Trichotomy Theorem.

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1 Introduction

The main focus of this paper will be studying the so called Escape Trichotomy for the singular perturbation family of functions. However, to be able to understand it, there will first be necessary to know some concepts and results about complex dynamical systems and more particularly, the asymptotic behaviour of rational maps on the complex sphere.

First, we will introduce the fundamental dynamical systems concepts, such as orbits, fixed and periodic points, attracting and repelling cycles and so on.

Then, we will introduce a simpler way of studying a dynamical systems, conjugacies, by establishing if its behaviour under iterations may be similar to another, better known one.

We will continue by introducing the concept of critical points and why they are a major point of interest in dynamics.

Some famous results about normal families will be considered, to allow us to understand which points in the dynamical plane may behave chaotically (these points will constitute the Julia set). Of course, its complementary, called the Fatou set, will also be a crucial notion in the understanding of the dynamics. From now on, we only focus on rational maps and start by highlighting some helpful properties of the Julia and the Fatou set.

Then, we will see the different types of fixed points and their connection to the Julia and Fatou sets. Then, by considering the maximal connected components of the Fatou set (Fatou components), we will see how the dynamics on rational maps are actually quite restricted.

We can finally speak about the main result. The family of maps that we are going to study is the family of singularly perturbed maps, $F_\lambda(z) = z^n + \frac{\lambda}{z^d}$. It is a rational family obtained by adding a singular perturbation to the universally studied family of $P_c(z) = z^n + c$, $n \geq 2$. This family, specifically for the case $n = 2$, has been a major topic of study in complex dynamics in the 20th century. The Dichotomy Theorem, proved by Julia and Fatou, says that the Julia set (that is, the set of points where the function behaves "chaotically") of a quadratic polynomial is either connected or a Cantor set, depending on whether the orbit of 0 (0, $P_c(0)$, $P_c(P_c(0)), \ldots$) is bounded or not. The set of the parameters c for which the Julia set of P_c is connected is called the Mandelbrot set.

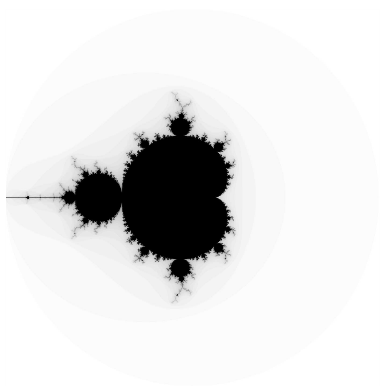

Figure 1: The Mandelbrot set.

Thanks to the work of Douady and Hubbard it is now known, for example, that the Mandelbrot set is connected (even though Mandelbrot himself has initially conjectured that it is not, because of the very small filaments which could not be seen in any picture). However, there even for this basic set, properties such as local connectivity have yet to be proven or disproven.

The family that we are going to study adds a perturbation which coincides with the critical point at 0. It was first proven by McMullen that, for small values of the parameter λ the Julia set is a Cantor set of Jordan curves. So, the topology of the Julia sets will be much more complicated and obviously the same can be said for the parameter plane.

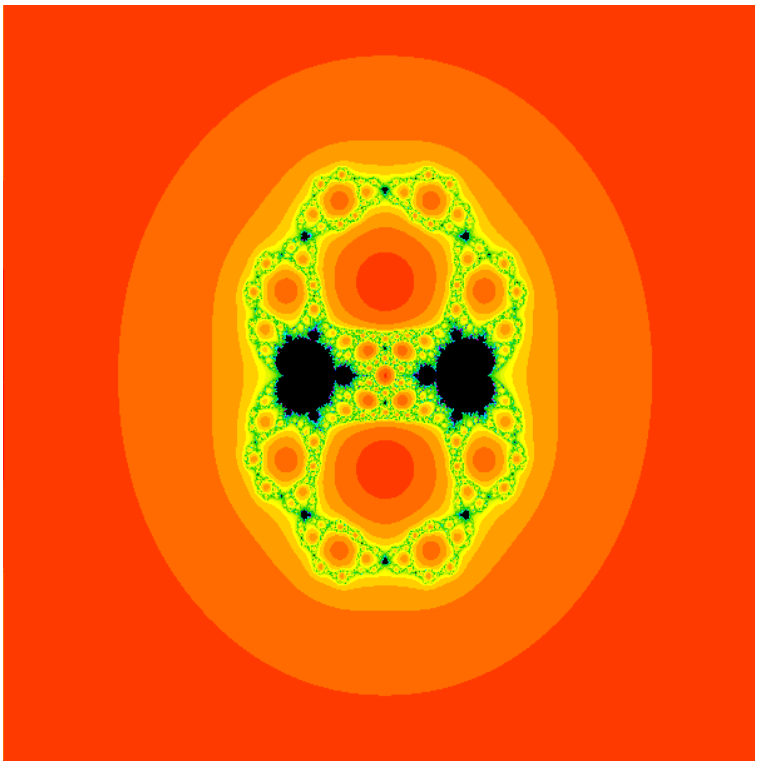

Figure 2: The parameter plane for $z \to z^3 + \frac{1}{z^3}$.

The connection to the quadratic polynomials can instantly be recognized by seeing the 2 so called principal Mandelbrot sets. There are actually infinitely many "copies" of the Mandelbrot set in the parameter plane and this, of course, makes studying certain characteristics not very straightforward.

For $n = d = 3$, we can still remark some symmetries, but what if n and d do not coincide? Then, the parameter plane is vastly modified. In the next figures, we can see the parameters planes for $n = 3$, $d = 4$ and $n = 4$, $d = 3$ and observe the change of topology around the origin or the apparition of a new principal Mandelbrot set.

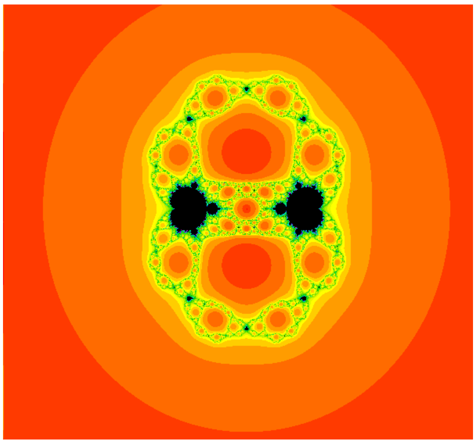

Figure 3: The parameter plane for $z \to z^3 + \frac{1}{z^4}$.

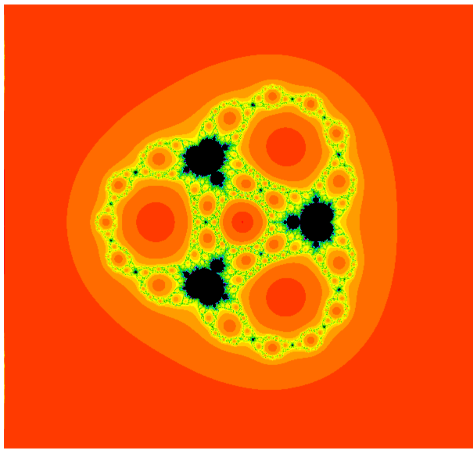

Figure 4: The parameter plane for $z \to z^4 + \frac{1}{z^3}$.

However, what we are going to show next is that assuming that the orbits of the critical points go towards ∞ (F_λ would be a hyperbolic map), there exist only three possible topologies for the Julia set, that is a Cantor set, a Cantor set of Jordan curves or a Sierpinski curve. This result is known as the Escape Trichotomy and it was proven by Devaney, Look and Uminsky.

We will first prove some particular symmetries of the singular perturbation maps and then prove the main result, which establishes the topology of the Julia set depending on the Fatou component in which the critical values are located. We will use different techniques for every one of them, like quasiconformal surgery, studying the properties of curves with help of external rays, Bottcher coordinates and other standard dynamical arguments.

Doing all of these will allow studying a specific transcendental family, which takes values very close to the singular perturbation ones.

2 Preliminaries

2.1 Basics in iteration theory

Let $f : X \to X$ be a continous map, where X is a metric space. Discrete dynamical systems are represented in this context by the iteration of f on a given $x_0 \in X$; the sequences $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$, where $x_0 \in X$.

The goal is having a good understanding of the asymptotic behaviour for those sequences for any initial seed.

Definition 2.1. *Let $x \in X$. Then the orbit $O_f(x)$ of x through f is given by:*

$$O_f(x) = \bigcup_{t\in\mathbb{N}} f^n(x).$$

Definition 2.2. *A point $x \in X$ whose orbit is constant, that is, $f(x) = x$, is called a fixed point.*

Example. *Let $f_c : \mathbb{C} \to \mathbb{C}$, $f_c(z) = z^2 + c$, for some $c \in \mathbb{C}$. The fixed points of f_c are represented by the solutions of the quadratic equation $z^2 + c = z$, that is $z_c = \frac{1\pm\sqrt{1-4c}}{2}$ for $c < \frac{1}{4}$, $z_c = \frac{1}{2}$, for $c = \frac{1}{4}$ and $z_c = \frac{1\pm i\sqrt{4c-1}}{2}$ for $c > \frac{1}{4}$.*

Apart from the fixed points, all the points with finite orbits, the so called periodic points, will play an important role in this work.

Definition 2.3. *A periodic point of period p ($p \in \mathbb{N}, p \geq 2$) is a point $x \in X$ such that $\forall\, i = 1, ..., p-1$, $f^i(x) \neq x$ and $f^p(x) = x$.*

Definition 2.4. *A point $x \in X$ is called pre-periodic if there exists no $p \in \mathbb{N}^*$ for which $f^p(x) = x$, but there exist $m, n \in \mathbb{N}^*$, $m \neq n$ such that $f^m(x) = f^n(x)$.*

Example. *Consider $f : \mathbb{S}^1 \to \mathbb{S}^1$, $f(\theta) = 2\theta$. Then 0 is the unique fixed point, but the points of period $p \geq 2$ are the points θ_k, where $\theta_k = \frac{k}{2^p-1}$ and $gcd(k, 2^p - 1) = 1$.*

The set of preperiodic points is made of all of the the rational points from $[0, 1)$, which are not fixed or periodic.

For instance the periodic points of period 3 are $\{\frac{1}{7}, \frac{2}{7}, ..., \frac{6}{7}\}$, while $\frac{1}{2}$ is a pre-periodic point since its orbit is $O_f(\frac{1}{2}) = \{\frac{1}{2}, \overline{0}\}$

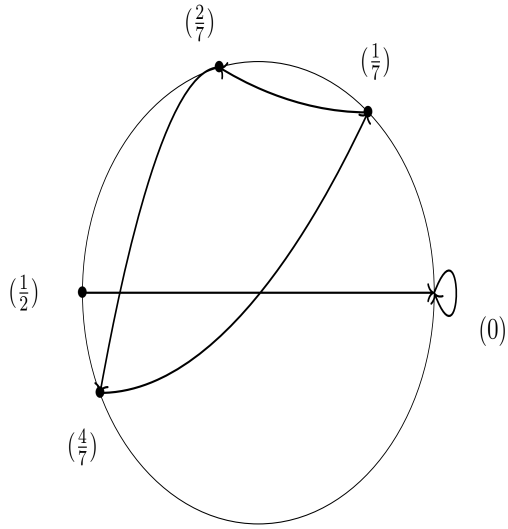

Figure 5: Here we may observe the orbit of the pre-periodic point $\frac{1}{2}$ and the $(\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$ cycle for the map $f : \mathbb{S}^1 \to \mathbb{S}^1$, $f(\theta) = 2\theta$.

Definition 2.5. *Let $f : X \to X$ be a continous map and assume that $x_0 \in X$ is a fixed point of f ; $f(x_0) = x_0$. We say that x_0 is an attracting fixed point of f if and only if:*

$$\exists \delta > 0 \text{ such that if } d(x, x_0) < \delta, \text{ then } f^n(x) \to x_0, \text{ when } n \to \infty.$$

If f is invertible and f^{-1} is continous at $x_0 \in X$, we say that x_0 is a repelling point for f if it is attracting for f^{-1}.

An attracting fixed point x_0 by definition always has a small neighborhood where all points tend to x_0 under iteration. But of course, in general, there would be other points in the phase space with the same property.

Definition 2.6. *The basin of attraction of a fixed point $x_0 \in X$ is the set of points whose orbits under iteration go to x. That is,*

$$A(x_0) = \{x \in X | \, f^n(x) \to x_0 \text{ when } n \to \infty\}.$$

It is an open set (follows easily from the continuity of f and the fact that for a small enough neighborhood of x_0 all points tend to x_0 under iteration). The connected component of $A(x_0)$ containing x_0, $A^*(x_0)$ is called the immediate basin of attraction. If $A(x_0) = X$, then x_0 is called a global attracting point.

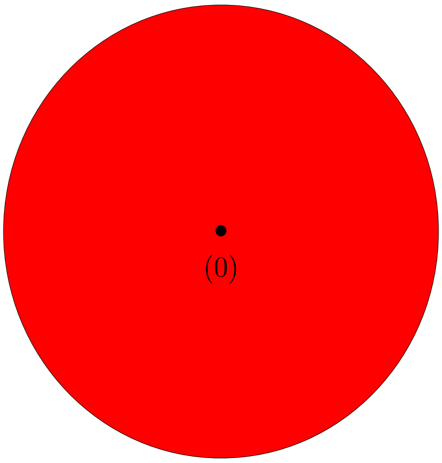

Figure 6: For the map $z \to z^2$, 0 is an attracting fixed point and its basin of attraction is exactly the disk $D(0,1)$.

Definition 2.7. *Let $f : X \to X$ be a continous map. Assume x_0 is a periodic point of period $p \geq 2$. We say that x_0 is an attracting periodic point (of period $p \geq 2$) if it is an attracting fixed point of f^p.*

Remark. *If x_0 is an attracting periodic point of period $p \geq 2$ of f and $O_f(x_0 = \{x_0, ..., x_{p-1}\}$, then every point x_j in the orbit of x_0 is also an attracting periodic point of period $p \geq 2$ of f.*

Indeed , by the (ϵ, δ) definition of continuity, we have that for any ϵ , there exists a δ such that f maps $B(x_0, \delta)$ into $B(f(x_0), \epsilon)$. From ϵ definition of the limit, if x tends to x_0 by f^p, there exists some rank n_{ϵ_1} such that for $n \geq n_{\epsilon_1}$, $f^{np}(x) \in B(x_0, \epsilon_1)$.

Now we just take $\epsilon_1 = \delta$ and we obtain that for any $\epsilon > 0$, there exists n_{ϵ_1} such that $\forall\, n_{\epsilon_1}$, $f^{np+1}(x) \in B(x_1, \epsilon)$.

Analogously, it is shown that all the other points of the cycle are attracting points of period $p \geq 2$.

Remark. *From the previous remark, if x_0 is an attracting periodic point of period $p \geq 2$, then we will say that $O_f(x)$ is an attracting cycle of period $p \geq 2$.*

Definition 2.8. *The basin of attraction of the periodic cycle of the periodic point x_0 of period p is:*

$$A(O_f(x_0)) = \{x \in X| \quad f^{np+l}(x) \to x_l,\, l = \overline{0, p-1}\}$$

x_0 is called an attracting periodic point of period p. Similarly as before , the immediate basin of attraction $A^*(O_f(x_0))$ of the cycle is represented by the reunion of the connected components of the basin which contain the points.

Sometimes, the direct study of the dynamical behaviour of f may be too difficult to be done. But, if there exists a homeomorphism $h : X \to Y$ and $g : Y \to Y$ a map (where Y is a metric space, not necessarily different from X), we may do so by studying the behaviour of g. Given two maps f and g we want to say that they represent the same dynamical system if the orbit induced by the iterates of f and g can be related through a homeomorphism of the whole space.

Definition 2.9. *Let $f : X \to X$, $g : Y \to Y$, where f and g are continous maps and X and Y are metrical spaces. Then (f, X) and (g, Y) are topologically conjugated dynamical systems if there exists $h : X \to Y$ a homeomorphism such that*

$$h \circ f(x) = g \circ h(x), \; \forall x \in X.$$

If h is of class C^r, we say that f and g are C^r-conjugate. If h is linear, we say that f and g are linearly conjugated.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] & Y \end{tikzcd}$$

Figure 7: Of course, for a topological conjugacy, the diagram commutes.

Example. *We are going to start with a conjugacy on the real line between some maps belonging to the logistic family of maps and some of the quadratic maps.*

Let us consider $g_a(x) = ax(1-x)$, where $a \in [0,4]$ and $f_c(z) = z^2 + c$, for $c \in [-2,2]$. Then, there exists h affine (which, if not constant, is a homeomorphism) and $\forall x \in \mathbb{R}$ we have that $h \circ g_a(x) = f_c \circ h(x)$ that is, g_a and f_c are topologically conjugated. We will prove that if $c = \frac{a}{2} - \frac{a^2}{4}$ then $h(x) = \frac{a}{2} - ax$ is the corresponding conjugacy.

First, since $a \in [0,\ 4]$, we have that $c \in [-2,\ 2]$.

Since $h \circ g_a(x) = h(ax(1-x)) = \frac{a}{2} - a * ax(1-x) = \frac{a}{2} - a^2x + a^2x^2$ and

$f_c \circ h(x) = f_c(\frac{a}{2} - ax) = \frac{a^2}{4} - a^2x + a^2x^2 + \frac{a}{2} - \frac{a^2}{4} = \frac{a}{2} - a^2x + a^2x^2$ hold $\forall x \in \mathbb{R}$, we have that h is a conjugacy between g_a and f_c.

Example. *We claim without proof that $f(x) = 2x$ and $g(x) = 3x$ ($f, g : \mathbb{R} \to \mathbb{R}$) are topologically conjugated, but they are not C^r-conjugated for any $r \geq 1$, because otherwise:*

$$h'(f(x)) \cdot f'(x) = g'(h(x) \cdot h'(x)$$

and by evaluating at 0 we get:

$$h'(0)f'(0) = g'(0)h'(0)$$

from where we get $h'(0) = 0$, so h would not be a homeomorphism, contradiction.

Now we will give without proof a lemma which will help us find out if some specific sets are invariant under iteration.

Lemma 2.10. *(The Topological Lemma, see Theorem 3.2.3 from [1])*

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a continous and open map and let Ω be a completely invariant subset of $\hat{\mathbb{C}}$ it in respect to the iteration of f. Then Ω^C, $\partial\Omega$, and $Int(\Omega)$ are also completely invariant in respect to the iteration of f.

2.2 Normal families and Montel's theorem

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (which is in fact the Riemann sphere $\mathbb{S}^2$). Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. It can be proven that f is in fact a rational map i.e

$f(z) = \frac{P(z)}{Q(z)}$, where P, Q are polynomials in the z variable.

Remark. *By the Uniformization Theorem, every Riemann surface is isomorphic to either* $\mathbb{D}, \mathbb{C}$, *or* $\hat{\mathbb{C}}$.

Definition 2.11. *A point $z_0 \in \hat{\mathbb{C}}$ is called regular if there exists a neighborhood U of z_0 such that $f : U \to V$ is a homeomorphism. Otherwise, we say that z_0 is a critical point(observe that when f admits derivative at a point z_0, z_0 is a critical point if $f'(z_0) = 0$).*

Example. *For $f_c : \mathbb{C} \to \mathbb{C}$, $f_c(z) = z^2+c$, for $c \in \mathbb{C}$, 0 is a critical point (since $f_c(z) = f_c(-z)$ and if z lies in a ball centered in 0, then $-z$ lies in the same ball). The point that the critical point 0 maps into, $c = f_c(0)$ is then a critical value of f_c.*

Remark. *In other words, the map f is locally a homeomorphism except at the critical values (that is, the image of the critical points). Such points are singularities of the inverse map and they play a crucial role in the global dynamics. To illustrate this fact we state without proof the Dichotomy Theorem.*

Theorem 2.12. *Let $Q_c = z^2 + c$ be the quadratic family of maps (observe that $z = 0$ is the unique critical point) and $K(Q_c) = \{z \in \mathbb{C} |\, |Q_c^n(z)| < 4, \quad \forall n \geq 0\}$, that is, the set of points whose orbits under any number of iterations remain bounded. Then one of the following holds:*

- *If $0 \in K(Q_c)$, then $K(Q_c)$ is connected.*
- *If $0 \notin K(Q_c)$, then $K(Q_c)$ is totally disconnected (it is actually a Cantor set).*

A main tool that has been extensively used to study the iteration of rational maps is the concept of normal family. The idea behind this is to divide the phase space $\hat{\mathbb{C}}$ depending on the normality of the family of iterations $\{f^n\}_n$.

Definition 2.13. *Let Ω be a domain in $\hat{\mathbb{C}}$ and $\Phi \subset \mathbb{H}(\Omega)$ a collection of holomorphic functions in Ω. We say that Φ is normal if for any sequence $\{f_n\}_{n=1}^{\infty} \subset \Phi$ there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \to f$ uniformly over compact sets of Ω (observe that f needs to be holomorphic in Ω).*

Now, we are able to start studying some more interesting properties of dynamics like whether a map behaves chaotically or not on some specific domain , but first it is necessary to introduce some results about normal families of functions.

Theorem 2.14. *(Ascoli) Let K be a compact set and $\{f_n\}_n \subset C(K)$ be a sequence of uniformly bounded functions, that is:*

$$\forall \epsilon > 0, \exists \delta > 0, \textit{such that}\ \forall n \in \mathbb{N}, \quad |f_n(x)| \leq M < +\infty$$

which are also equicontinous:

$$\forall \epsilon > 0, \exists \delta > 0, \textit{such that } \forall n \in \mathbb{N}, \quad |x-y| < \delta \textit{ implies that } |f_n(x) - f_n(y)| < \epsilon.$$

Then there exists $\{f_{n_k}\}_k$ *such that* $f_{n_k} \to f$ *uniformly on* K*.*

Theorem 2.15. *(Montel) Let* $\Omega \subset \mathbb{C}$ *be a domain, and* $\Phi \subset \mathbb{H}(\Omega)$ *with the following property:*

$$\forall\, K \subset \Omega \, compact, \quad \exists M_K > 0 \textit{ such that } \quad \forall f \in \Phi \sup_{z \in K} |f(z)| \leq M_K.$$

Then Φ *is a normal family.*

Proof. Let $\{f_n\}_n \subset \Phi$ be a sequence of holomorphic functions from Φ. The objective is to show that there exists a sequence uniformly convergent over all compact sets of Ω.
First, we consider a family $\{K_n\}_n \subset K(\Omega))$ of compact sets in Ω such that: $K_1 \subset \mathrm{Int}(K_2) \subset K_2 \subset \mathrm{Int}(K_3) \subset K_3 \subset ...$ and $\bigcup_{n=1}^{\infty} K_n = \Omega$. This family satisfies that $\forall K \in K(\Omega)$ there exists $m \in \mathbb{N}^*$ such that $K \subset K_m$. Consider $\{f_n\}_n \subset \Phi$, and K_j as above.
By hypothesis, $\{f_n\}_n$ is equibounded in K_j.
For equicontinuity we consider δ such that $2\delta < d(K_j, \mathbb{C} - K_{j+1})$, such that

$$\forall z \in K_j, \quad \mathbb{B}(z, 2\delta) \subset K_{j+1}.$$

Now, for $x, y \in K_j$ and $|x-y| < \delta$, we have :

$$|f_n(x) - f_n(y)| \leq |x-y| \sup_{\xi \in I(x,y)} |f_n^{'}(\xi)| \leq \delta \sup_{\xi \in K_{j+1}} |f_n^{'}(\xi)| \leq c\delta M_{k_{j+2}}$$

by using the Cauchy integral formula. Having a bound which does not depend on n, it is now enough to consider a small enough δ to obtain

$$\forall x, y \in K_j \quad |x-y| < \delta \quad \text{ implies } \quad |f_n(x) - f_n(y)|.$$

By the Ascoli theorem we can obtain a partial sequence $\{f_n^{(j)}\}_n$ uniformly convergent on K_j. By repeating this argument we obtain $\{f_n^{(j+1)}\}_n \subset \{f_n^{(j)}\}_n$ for K_{j+1}. By a diagonal argument, $G_k = f_k^{(k)}$ we obtain a subsequence which converges uniformly in every compact set. So Φ is normal. □

Theorem 2.16. *(Fundamental normality test, a stronger version of Montel's theorem) Let* Φ *be a family of holomorphic maps on* Ω*. If there exist 3 points* $a, b, c \in \hat{\mathbb{C}}$*, for which*

$$f(z) \neq \{a, b, c\}, \quad \forall f \in \Phi, \forall z \in \Omega$$

then Φ *is normal in* Ω*.*

Normality is a concept for general families of holomorphic maps on a domain Ω. The main use of Montel's theorem is the particular case where $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map and we consider the family of functions given by the iterates of f; that is

$$f^n = \underbrace{f \circ f \circ ... \circ f}_{n}, \quad n \in \mathbb{N}.$$

Definition 2.17. *We say that a rational map* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is normal at* $z \in \hat{\mathbb{C}}$ *if there exists a neighborhood* U *of* $z \in \hat{\mathbb{C}}$ *such that* $\{f^n_{|U}\}$ *is normal. Of course,* f *is not normal at* $z \in \hat{\mathbb{C}}$ *if it is not normal at any neighborhood* U *of* z.

Now, given $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map, we can divide the phase space $\hat{\mathbb{C}}$ in two sets, Fatou and Julia as follows in the next section.

3 Rational iteration

3.1 The Julia and Fatou sets

Definition 3.1. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The Fatou set for a holomorphic function R is defined as follows:*

$$F(R) = \{z \in \hat{\mathbb{C}} \,|\, \{R^n\}_n \text{ is normal at } z\}$$

Definition 3.2. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The Julia set for a holomorphic function R is defined as follows:*

$$J(R) = \{z \in \hat{\mathbb{C}} \,|\, \{R^n\}_n \text{ is not normal at } z\}$$

Remark. *The Fatou and Julia set are complementary. That is, $\hat{\mathbb{C}} = F(R) \cup J(R)$.*

Example. *Let $R(z) = z^2$.*
Now consider $z_1 \in \hat{\mathbb{C}}$ such that $|z_1| > 1$, so there exists a neighborhood U of z_1 such that $\forall z \in U$, $|z| > 1$. Trivially, under iterations, $R^n(z) \to \infty$, $\forall z \in U$, so for any z_1 outside the closed unit disk, $\{R^n\}_n$ is normal.
Now, consider $z_2 \in \hat{\mathbb{C}}$ such that $|z_2| < 1$ so there exists a neighborhood V of z_2 such that $\forall z \in V$, $|z| < 1$. We have $R^n(z) \to 0$, $\forall z \in V$, so in the interior of the unit disk, $\{R^n\}_n$ is also normal.
Finally, consider z_3 on the unit circle. In any of its neighborhoods, there exist an element a such that $|a| > 1$ and b such that $|b| < 1$. Since under iterations a goes to ∞ and b goes to 0, then f is not normal at z_3.
We have obtained that R is not normal only on the unit circle (the Julia set is the unit circle) and normal everywhere else (the Fatou set is the entire Riemann sphere, except for the unit circle).

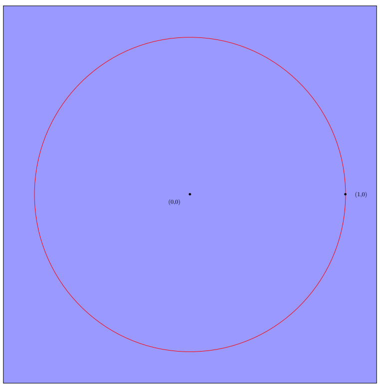

Figure 8: For the map $z \to z^2$, the Julia set is the circle $C(0, 1)$ (in red) and the Fatou set is the rest of the Riemann sphere .

Lemma 3.3. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then $F(R)$ and $J(R)$ are completely invariant.*

Proof. The idea of the proof is to show that an element from the Fatou, respectively Julia, set can only map into the Fatou, respectively Julia, set and since they are complementary, the sets will be completely invariant.

Let $z_1 \in F(R)$ be an arbitrary point in the Fatou set. Then, by definition, the family $\{R^n\}_n$ is normal in a neighborhood U of z_1. So there exists a subsequence $\{R^{n_k}\}_{k\geq 1}$ and g holomorphic such that $R^{n_k} \to g$ uniformly over compact sets. The same can be said by removing the first element of this subsequence. But $\{R^{n_k}\}_{k\geq 2}$ on U is also part of the family of iterations of R in a neighborhood of z_1, so R is normal in a neighborhood of $R(z_1)$ so $R(z_1)$ is in the Fatou set. So any point from the Fatou set maps in the Fatou set.

Analogously, for some arbitrary point z_2 in the Julia set, since the family $\{R^n\}_n$ is not normal in any neighborhood of z_2. This will further imply that there exists no subsequence to satisfy the uniform convergence over compact sets, so the subsequence of iterations around the neighborhood of z_2 will also not have such a subsequence, so the family is not normal around z_2, which finally means that z_2 is in the Julia set. So any point from the Julia set maps in the Julia set.

The proof is concluded by observing that the Julia set and the Fatou set are complementary.

□

Lemma 3.4. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then $F(R)$ is open and $J(R)$ is closed.*

Proof. We will prove that the Fatou set is open and since the Julia set is its complementary, the Julia set will be closed.

Let z_0 be a point in the Fatou set. This implies that the family of iteration $\{R^n\}_n$ is normal in a neighborhood U of z_0. Let us consider $B(z_0, r) \subset\subset U$. Now for z such that $|z - z_0| \leq \frac{r}{3}$, consider the balls $B(z, \frac{r}{3})$. They are fully contained in U so the family of iterations is normal on the balls. This implies that around every z such that $|z - z_0| \leq \frac{r}{3}$ there exists a neighborhood where the family of iterations is normal.

So all the points z with the property $|z - z_0| \leq \frac{r}{3}$ are in the Fatou set, which is equivalent to $B(z_0, \frac{r}{3})$ is in the Fatou set.

So we have obtained that for any point in the Fatou set there exists a neighborhood around that point which is entirely in the Fatou set.

In conclusion, the Fatou set is open and the Julia set, being its complementary, is closed.

□

We will give the following lemma without proof.

Lemma 3.5. *The Julia set is the smallest closed, invariant set of the dynamical plane, which has at least three points (this is called the minimality of the Julia set).*

Lemma 3.6. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then for any $n \in \mathbb{N}^*$, $J(R^n) = J(R)$.*

Proof. Consider $z_0 \in F(R^n)$. We have that the family of iterations $\{(R^n)^k\}_{k\geq 1}$ is normal in a neighborhood around z_0.

Since it is a subsequence of $\{f^p\}_{p\geq 1}$, we have that the family $\{f^p\}_{p\geq 1}$ is normal in a neighborhood of z_0. So $z_0 \in F(R)$, which leads to $F(R^n) \subset F(f)$.

By the complementarity of the Fatou and Julia set we get that $J(R) \subset J(R^n)$ and by the minimality of the Julia set, we also have that $J(R^n) \subset J(R)$. The double inclusion results in $J(R) = J(R^n)$.

□

Lemma 3.7. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then $J(R)$ either has empty interior or it is $\hat{\mathbb{C}}$ (see Theorem 4.2.3 from [1]).*

Proof. Since the Fatou and Julia sets are complementary and the Julia set is closed, we may split the dynamical plane in three sets, $F(R), J(R)$ and $\partial J(R) = \partial F(R)$. By the Topological Lemma 2.10, since $F(R)$ is completely invariant, then so are $\partial F(R)$ and $Int(J(R)$.

Now, if the Fatou set is not empty (that is, the Julia set is not the entire Riemann sphere), $A = F(R) \cup \partial F(R)$ is a closed set and since the interior of the Julia set is completely invariant, so is A. By the minimality of the Julia set we have $J(R) \subset A$ and since $J(R) \cap F(R) = \emptyset$ we have that $J(R) \subset \partial J(R)$, so the Julia set has empty interior.

□

Now we will give without proof some more fundamental results about the Julia set of a rational map.

Lemma 3.8. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Then $J(R)$ has infinitely many points.*

Lemma 3.9. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then $J(R)$ is perfect and uncountable.*

Lemma 3.10. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. Then $J(R)$ is the closure of the repelling periodic points of R.*

3.2 Local theory

Assume $z_0 \in \hat{\mathbb{C}}$ is a fixed point of the rational map R, $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We want to distinguish the local behaviour of points in a neighborhood of z_0, depending on the multiplier $R'(z_0)$ of R at z_0.

Lemma 3.11. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $z_0 \in \hat{\mathbb{C}}$ a fixed point. Then, if $|R'(z_0)| < 1$, z_0 is attracting and if $R'(z_0)| > 1$, z_0 is repelling.*

Proof. First, let us consider that $|R'(z_0)| < 1$. Then there exists $B(z_0, r)$ and $c \in (0, 1)$ such that $|R'(z)| < c < 1$, $\forall z \in B(z_0, r)$. Now let $z \in B(z_0, r)$. Then, by integrating R' on the segment from z_0 to z, we get

$$|R(z) - R(z_0)| \leq |z - z_0| \sup_{a \in B(z_0,r)} |R'(a)|$$

which by considering that z_0 is a fixed point for R leads to:

$$|R(z) - z_0| < c|z - z_0|$$

So by iterating n times we would get

$$|R^n(z) - z_0| < c^n |z - z_0|$$

which tends to 0 when $n \to \infty$, so $\lim_{nto\infty} R^n(z) = z_0$. This holds for any z inside $B(z_0, r)$, so z_0 is an attracting fixed point.

Now assume $R'(z_0)| > 1$. Then there exists $B(z_0, r)$ such that $|R'(z)| > 1$, $\forall z \in B(z_0, r)$ and no critical points or singularities are inside the ball ($z_0 \in \mathbb{C}$ cannot be a singularity since it is a fixed point). So R is continous and there exists a domain U in C so that its restriction on the ball mapped into U is a bijection. So it has an inverse, which is also rational and continous. Now, by the (ϵ, δ) definition of continuity there exists δ such that $R(B(z_0, \delta)) \subset B(z_0, r)$. Now consider $m = \min(\delta, r)$.

If $z \in B(z_0, m)$ then we have $|R'(R^{-1}(z))| > 1$(since its preimage is $x \in B(z_0, r)$, with $|R'(x)| > 1$, which by using the formula for the derivative of the inverse gives us that $|(R^{-1})'(z)| < 1$, if $z \in B(z_0, m)$, where z_0 is a fixed point for the rational map R^{-1}.

So by the first part of this lemma, z_0 is an attracting fixed point for R^{-1}, which implies that by definition, z_0 is a repelling fixed point for R. □

An interesting categorization of fixed points appears depending on whether a holomorphic function may be or not linearizable (there exists a conjugacy to a linear function in a neighborhood of the fixed point). Depending on the multiplier of the fixed point several possibilities arise. By considering a local conjugacy of a fixed point, we can just consider from now on that our fixed point will be 0.

Theorem 3.12. *(Koenigs linearization, see [2])*

Consider $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational function such that 0 *is a fixed point and let $\lambda = R'(0)$ be its multiplier. If $|\lambda| \neq 0, 1$ then there exists some local holomorphic change of coordinates $y = h(z)$ such that $h(0) = 0$ and h conjugates R in a neighborhood of* 0 *to $y \to \lambda y$ in a neighborhood of* 0*, which is going to be unique up to a multiplication by a constant.*

Proof. We start by proving the existence for $|\lambda| < 1$.

First we observe that since $R(0) = 0$, the Taylor series for a point in the neighborhood of (0) will be

$$R(z) = \lambda z + cz^2 + ...$$

So in a neighborhood of the origin we have $R(z) = \lambda z + O(z^2)$ which means that there exists R positive number such that $|R(z) - \lambda z| < c|z^2$, for $|z| < R$.

Now, for any $z \in \mathbb{D}(0, R)$ we consider $z_n = R^n(z)$. We immediately get that $z_n < Rc^n$, which is also inside the disk so we have:

$$|z_{n+1} - z_n| \leq cz_n^2 \leq r^2 c^{2n+1}.$$

If we write $m = \frac{cR^2}{\lambda}$ we have that the sequence $y_n = \frac{z_n}{\lambda^n}$ satisfies

$$|y_{n+1} - y_n| \leq m(\frac{c^2}{|\lambda|^n})^n.$$

So we have that $z \to y_n$ form a sequence of holomorphic functions which converge uniformly to some $h(z) = \lim_{n\to\infty} \frac{R^n(z)}{\lambda^n}$. From here we have that $h \circ R(z) = \lambda h(z)$ in a neighborhood of the origin.

For $|\lambda| > 1$ we have that there exists a neighborhood with no critical points and consider R^{-1} which will have a multiplier $k = \frac{1}{\lambda}$ such that $0 < |k| < 1$ and so there exists some h such that $h \circ R^{-1} \circ h^{-1}(z) = k$. So by taking the inverses it follows that R is conjugated to $y \to \lambda y$.

Now to prove the uniqueness, we assume that h_1 and h_2 are both conjugacies, which immediately gives us that $h_1 \circ h_2^{-1} \circ (\lambda z) = \lambda h_1 \circ h_2^{-1}$. Since the conjugacies have a fixed point at the origin, we will have a Taylor series of the form:

$$h_1 \circ h_2^{-1}(z) = a_1 z + a_2 z^2 + ... \quad .$$

By considering the Taylor expansions in the previous identity we get that $\lambda a_n = a_n \lambda^n$, $n \geq 1$ and since $|\lambda| \neq 0, 1$ we have that a_2, a_3, and so on are identically zero which leads to $h_1(z) = a_1 h_2(z)$.

□

Remark. *By using conjugacies, it can be shown that the Koenigs linearization theorem holds for any fixed point of a rational map, with the multiplier λ such that $|\lambda| = 0, 1$.*

Definition 3.13. *Let $z \in \hat{\mathbb{C}}$ be a fixed point of R such that $|f'(z)| = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Such a point is called a rationally indifferent point. If there exists a linearization of R around it , the point is called a Siegel point. Otherwise, it is called a Cremer point.*

Proposition 3.14. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $z_0 \in \hat{\mathbb{C}}$ a fixed point. If z_0 is attracting or a Siegel point, then $z_0 \in F(f)$. If z_0 is repelling, parabolic or Cremer, then $z_0 \in J(f)$.*

Proof. For attracting fixed points, since by definition there exists a neighborhood such that all points under iteration tend towards the point, we have that the family of the iterations is normal so the point is in the Fatou set.

For Siegel points, since the rational map is conjugated to irrational rotation on a small disk, since the family of iterations of irrational rotation on a disk is normal, we have that the family of iterations of the rational map is also normal, so the point is in the Fatou set.

By the lemma 3.10 the repelling fixed points (considered as periodic points of period 1) belong to the Julia set.

Now, for a Cremer fixed point z_0, assume that there exists some neighborhood around it, where the family of iterations is equibounded (which is equivalent to normality and so, being in the Fatou set).

Consider the family of functions $\{h_n\}_n$, defined by:

$$h_n(z) = \sum_{j=0}^{n-1} \frac{R^j(z)}{\lambda^j}.$$

Observe that $h_n \circ R(z) = \lambda h_n(z) + \frac{1}{n}(-\lambda z + \frac{R^n}{\lambda^{n-1}})$.

Since it is an equibounded family of analytic functions, by the Weierstrass Theorem, there exists a subsequence which is uniformly convergent. By considering the limit h of this sequence, we have that $h(R(z)) = \lambda h(z)$ so the map is linearizable in a neighborhood of z_0. But z_0 is Cremer, which is by definition not linearizable, so the family of iterations is not normal, which implies that the Cremer fixed point must lie in the Julia set.

□

3.3 Fatou components

The main result that will be presented in Section 4 expresses different kinds of topologies, depending on the connected component of the Fatou set in which the critical values are located, so these components will play a major role.

Definition 3.15. *A Fatou component is a connected component of the Fatou set (we already know that the Fatou set is open, from Lemma 3.4).*

Lemma 3.16. *If $U \subset F(f)$ is a Fatou component. Then $f(U) \subset V$, where V is a Fatou component.*

Proof. By Lemma 3.3, F(f) is completely invariant, so $F(U) \subset F(f)$. Since f is continous and open U will be then mapped in an open connected set contained in $F(f)$, so $f(U)$ is contained in an open component of $F(f)$, that is, some Fatou component V.

□

Remark. *If V does not contain any omitted value, then $f(U) = V$.*

From the previous lemma, a Fatou component is either eventually periodic or not. A major result regarding the Fatou components of a rational map is the following theorem by Sullivan.

Theorem 3.17. *(No Wandering Domains theorem) Every Fatou component of a nonlinear rational map is eventually periodic (there are no wandering domains for nonlinear rational maps).*

Theorem 3.18. *Let R be a rational map of degree $d \geq 2$ and let Ω be a Fatou component of period $p \geq 1$ for R. Then exactly one of the following holds:*

- *Ω contains an attracting p-periodic point z_0 and $R^{np}(z) \to z_0$, for $z \in \Omega$, as $n \to \infty$ (Ω is an immediate attractive basin).*

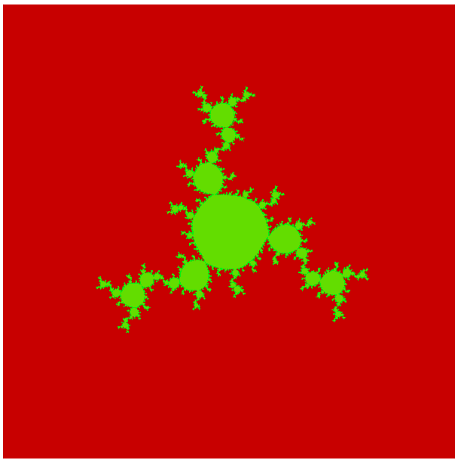

Figure 9: Attracting cycle of period four for $P(z) = z^3 + (-0.2 + 1.083i)$.

- *There exists $z_0 \in \partial\Omega$ such that $R^p(z_0) = z_0$ and such that $R^{np}(z_0) \to z_0$ for $z \in \Omega$, as $n \to \infty$(Ω is an immediate parabolic basin).*

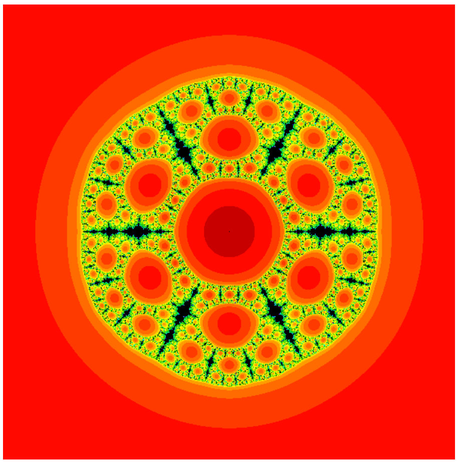

Figure 10: Here we have a rendering of the dynamical plane for $z \to z^3 + \frac{\lambda}{z^3}$ where $\lambda = 0.0486561 - 0.000299745i$. The black region in the middle left is a periodic four parabolic Fatou component.

- *There exists a holomorphic homeomorphism $\phi : \Omega \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, where $\theta \in \mathbb{R}\backslash\mathbb{Q}$. That is, the rational map is conjugated to an irrational rotation on a disk (Ω is called a Siegel disk).*

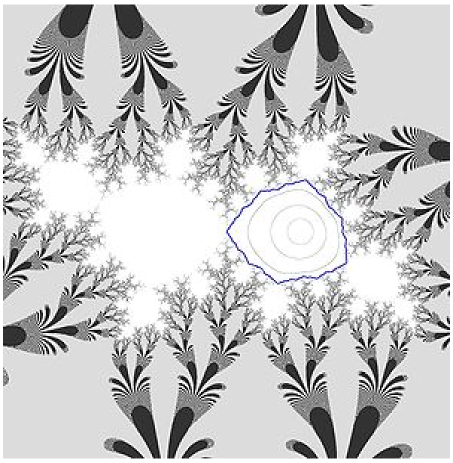

Figure 11: $f(z) = \varphi * a * (\exp(z/a) * (z + 1 - a) + a - 1)$ where φ is the golden ratio and $a = 15 - 15i$.

- *There exists $r > 1$ and a holomorphic homeomorphism $\phi : \Omega \to U$, where $U = \{z \in \mathbb{C} | \, 1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. That is, the rational map is conjugated to an irrational rotation on an annulus (Ω is called a Herman ring).*

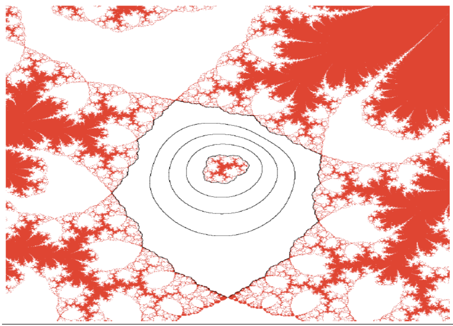

Figure 12: $\lambda * z * \exp((a/2) * (z - 1/z))$ for $\lambda = \exp(2\pi i * 0.6223599)$ and $a = 0.5i$.

3.4 Useful results

Another necessary result,used for computing the number of boundary components of a domain, that will be used several times in proving the main statement is the Riemann-Hurwitz formula.

Theorem 3.19. *(Riemann-Hurwitz formula for rational maps, see [7]) Consider R a proper rational map of degree d from the m-connected domain M to the n-connected domain N,with* *p critical points, counting multiplicity. Then the following holds :*

$$m - 2 = k(n - 2) + p.$$

We finish this chapter with 2 theorems (see chapter 9.8 from [1]) which will be instrumental in proving one part of the main result. But first, we need 2 more lemmas (see chapter 9.2 from [1]).

Lemma 3.20. *Let R be a rational map with degree $d \geq 2$. Consider the family $\{S_n\}_{n\geq 1}$ of single-valued analytic branches of $(R^m)^{-1}$, $m \in \mathbb{N}^*$, in a domain D. Then $\{S_n\}_{n\geq 1}$ is normal in D.*

Proof. Let C_1 and C_2 be two disjoint cycles of R, each of length at least 3. S_n cannot map from outside the cycle A into A, as this would mean that by iterating under R some point inside the cycle at some point would map outside the cycle, impossible. So outside of a finite number of points(A), D is normal. By repeating the same argument we obtain that outside the elements of cycle B, D is normal. Since A and B are disjoint, then D is normal .

□

Lemma 3.21. *Let D and $\{S_n\}_{n\geq 1}$ as in the previous lemma. Then if D intersects $J(R)$, we have that any locally uniform limit of a subsequence of $\{S_n\}_{n\geq 1}$ is constant.*

Proof. First we define the mapping gr such that S_n is a branch of $(R^{gr(n)})^{-1}$ in D. Let us suppose that S_n converge locally uniformly to a non-constant ϕ. So we get that every S_n is univalent in D and implicitly, ϕ is univalent in D. Now consider a point z in both D and $J(R)$ (it exists by hypothesis) and consider a neighborhood of it contained in D. From the uniform convergence, we get that a preimage of D through a branch contains a neighborhood of $\phi(z)$, so this neighborhood must be in J, so we obtain that after a number of iterates the entire Julia set must be contained in D, which is impossible, since otherwise this would actually be true for any subdomain of D which does contain J, but intersects it. So ϕ is constant on D.

□

Theorem 3.22. *Consider R a rational map of degree $d \geq 2$, which has a super attracting fixed point at ∞. If all of the critical points of R are in the immediate basin of attraction of ∞, then $J(R)$ is a Cantor set.*

Proof. Let B be the immediate basin of attraction of ∞. From the No Wandering Domains theorem, we have that every Fatou component eventually maps into a periodic cycle of components, which must attract some critical point. But since all of the critical points of R are in B, we have that all of the components will after some number of iterations be mapped into B and so the orbits of all elements of the Fatou set eventually go to ∞.

Since the Julia set of R is compact and the orbits of the critical points are inside the imediate basin and have ∞ as a limit, there exists a Jordan curve γ which separates the Julia set from the orbits of the critical points.

Consider I the interior and E the exterior of the curve and $S = I \cup \gamma$. Obviously, since $E \cap \gamma$ is a compact set fully contained in the immediate basin of ∞, we may find $n > 0$ such that $R^n(E \cap \gamma) \subset E$ and $R^{-n}(I \cap \gamma) \subset I$.

Now,the critical points of R^n are comprised of the critical points of R(whose orbits are fully in B and, if $n > 1$, prepoles of R, which map to ∞ after 2 iterations of R.So the critical points of R^n are all in S.

From the Lemma 3.6, we have that $J(R) = J(R^n)$ so from now on we will just study $P = R^n$ and consider g its degree.

Now since P is a smooth covering of every component of the preimage of I into I and I is simply connected, the restriction to every of the components of the preimage of I is a homemorphism from the component to I. We consider the branches P_1, P_2, ...P_g of P^{-1} on S and the semigroup generated by them that is

$$P(x_{1,2}, ...x_k) = P_{x_1}...P_{x_k}(S).$$

We get that for any k there exist d^k pairwise disjoint compact sets, whose reunion is the preimage through P^k of S.

Consider P_∞ the limit of this inverse process, that is

$$P_\infty = \bigcap_{k=0}^{\infty} P^{-k}(S).$$

By its definition P_∞ is non-empty compact and perfect. All that remains to be shown is that $P_\infty = J$ and , for it to be a Cantor set, that it is totally disconnected.

Since $J \subset S$, by backwards invariance of the Julia set, we have $J \subset P_\infty$. Now, for $z \in P_\infty$ its orbit is entirely in S so it does not escape to ∞. But since the orbits of all points of the Fatou set go towards ∞, z has to lie in the Julia set, so we have $P_\infty = J$.

Since we are in the conditions of Lemma 3.20, by Lemma 3.21, we have that any locally uniform limit of any sequence of the type

$$P_{x_1},\ P_{x_2}P_{x_1}, ...$$

is constant, so every such sequence converges locally uniformly on S to some point z, so

$$diam[P(x_k,\ ...\ , x_1)] \to 0$$

and J is totally disconnected.

Since the Julia set set is non-empty compact, perfect and totally disconnected, it is a Cantor set.

$\square$

A very common way to characterize chaotic systems is by using symbolic dynamics and this will be the next objective for rational maps.

Definition 3.23. *Let $d \in \mathbb{N}^*$.Consider Π_d the space of infinite sequences with elements in $\{1, ...d\}$. The continous map σ defined by*

$$\sigma(x_1, x_2, x_3...) = (x_2, x_3, ...)$$

is called the shift map on d symbols.

Theorem 3.24. *Consider R as in the hypotheses of the previous theorem. Then it is conjugated to a shift map on d symbols, where d is the degree of the rational map R.*

Proof. To every point from Π_d, $(x_1, x_2, ...x_n)$ we can associate the limit of the sequence

$$P_{x_1}, P_{x_2}, ...$$

and every point of the Julia set can be obtained by this, from the previous theorem. So we have a homeomorphism h from Π_d to J.

Now, since P_{x_i} are defined as branches of R^{-1}, $RP_{x_i} = Id$. All that is left is to compute

$$RP(x_1, ...x_k) = RP_{x_1}..P_{x_k}(S) = P_{x_2}..P_{x_k}(S) = P(x_2, ...x_k)$$

So there exists a homeomorphism such that $R \circ h = h \circ \sigma$, so , by definition, R and h are conjugated. □

The following lemmas (see [2]), that we will give without proof, considered together gives that if the Julia set of a hyperbolic rational map is connected it is also locally connected and this will be critical in proving the local connectivity of the Julia set in the Sierpinski curve case.

Lemma 3.25. *Let U be a simply connected Fatou component of a hyperbolic rational map. Then, the boundary of U, (∂U) is locally connected.*

Lemma 3.26. *Let R be a rational hyperbolic map with a connected Julia set. Then $\forall \epsilon > 0$, there exists a finite number of Fatou components of diameter$> \epsilon$.*

Lemma 3.27. *Let $K \subset \hat{\mathbb{C}}$ be a compact set such that every component of $\hat{\mathbb{C}} \setminus K$ has locally connected boundary and $\forall \epsilon > 0$, there exists a finite number of Fatou components of diameter$> \epsilon$, then K is locally connected.*

Now we are going to introduce a crucial result regarding quasiconformal surgery of Shishikura. This work is not an introduction to quasiconformal maps (see [4]). However, a major result regarding quasiconformal surgery will be used in the final section.

Theorem 3.28. *(First Shishikura principle)*

Consider $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a quasiregular map and $p \geq 1$. Suppose the following exist:
(a) $U = U_1 \cup U_2 \cup ... \cup U_p$, p disjoint open sets of the Riemann sphere such that

$$f(U_i) = U_{i+1}, \quad i = \overline{1, p-1} \text{ and } f(U_p) \subseteq U_1$$

(b) $\psi : U \to \tilde{U}$, a quasiconformal homeomorphism (called the "glueing map") which maps U into $\tilde{U} \subset \hat{\mathbb{C}}$
(c) $G : \tilde{U} \to \tilde{U}$ quasiregular with G^p holomorphic
such that:

$$f_U = \psi^{-1} \circ G \circ \psi \text{ and } \frac{\partial f}{\bar{z}} = 0 \text{ a.e in } f^{-N}(U), \text{ where } N \geq 0.$$

Then f is quasiconformally conjugate to a rational map.

However, to be able to use the Shishikura principle, we will need the following lemma.

Lemma 3.29. *Let $0 < \rho_1, \rho_2$ and two quasisymmetric orientation preserving homeomorphisms $\psi_1 : \mathbb{S}^1 \to \mathbb{S}^1$ and $f_2 : \mathbb{S}^1_{\rho_1} \to \mathbb{S}^1_{\rho_2}$. Then there exists an extension $f : \mathbb{A}_{\rho_1} \to \mathbb{A}_{\rho_2}$, which is quasiconformal in the interior.*

4 Trichotomy

The most basic family of maps studied in holomorphic dynamics is represented by $P_n(z) = z^n + c$, where $n \in \mathbb{N}$, $n \geq 2$. For $n = 2$, for example, we have the famous Dichotomy theorem, which states that the set of parameters for which $J(P_c)$ is connected (that is, the Mandelbrot set) is the same with the set of points for which the orbit of the critical point 0 is bounded.

A more difficult problem appears when a perturbation is added, even more so when the pole of the perturbation coincides with a critical point. So from now on, the family that will be studied is $F_\lambda(z) = z^n + \frac{\lambda}{z^d}$, where $n \geq 2$ and $d \geq 1$, both $n, d \in \mathbb{N}$.

This map has 0 and ∞ as critical points (actually, since $F_\lambda(\infty) = \infty$ and $F'_\lambda(\infty) = \infty$, ∞ is a super attracting fixed point). There are also (n+d) critical points which appear from solving the equation $F'_\lambda(z) = 0$, which is just $nz^{n+d} - d\lambda = 0$.

The immediate basin of attraction of ∞ will have the notation of B. The neighborhood of the origin maps directly into a neighborhood of ∞, which is part of B and so 0 will belong to a Fatou component. For some values of the parameter λ, this Fatou component will not coincide with B, and it will be called the trapdoor, denoted by T, since it is the only way for complex points to escape directly into B (0 is the only point in the complex plane to map directly to ∞).

The main result we are going to use is the escape trichotomy for singularly perturbed rational maps by Devaney-Look-Uminsky :

Theorem 4.1. *Suppose the free critical orbits of F_λ go to ∞(and so the map is hyperbolic). Then exactly one of the following three holds:*
(a) If there exists a critical value in B, then $J(F_\lambda)$ is a Cantor set and the restriction of F_λ on it is a one-sided shift on $(n + d)$ symbols.
(b) If there exists a critical value in T , then $J(F_\lambda)$ is a Cantor set of Jordan curves.
(c) If there exists a critical value in a preimage of T, then $J(F_\lambda)$ is a Sierpinski curve.

The proof of the theorem splits in three parts, according to three possibilities. Before proving the theorem, we will first state and prove some results regarding different kinds of symmetry of the map.

4.1 Preliminaries

Lemma 4.2. *(Dynamical symmetry). Assume ω such that $\omega^{n+d} = 1$. Then $F_\lambda(\omega z) = \omega^n F_\lambda(z)$.*

Proof. The proof of this result follows easily from the following computation:

$$F_\lambda(\omega z) = (\omega z)^n + \frac{\lambda}{(\omega z)^d} = \omega^n(z^n + \frac{\lambda}{z^d \omega^{n+d}}) = \omega^n(z^n + \frac{\lambda}{z^d}) = \omega^n F_\lambda(z).$$

□

Remark. *The critical points which are the solutions of the equation*

$$nz^{n+d} - d\lambda = 0$$

will have their orbits either all go to ∞ or all be attracted to periodic cycles. So the assumption of the theorem, that the free critical orbits go to ∞, is equivalent to the orbit of one of the critical values going to ∞. Since ∞ is a super attractive fixed point, it has an immediate basin of attraction and outside of this Fatou component the only component which may map directly into it is the one containing the origin (they may coincide!). So we have that if the free critical orbits go to ∞, the critical values may only lie in B, T or preimages of T.

Lemma 4.3. *(Symmetry of B and T.) B and T have (n+d)-fold symmetry.*

Proof. Let S be the set of points $z \in B$ for which $\omega z \in B$, which is non empty, as there exist an open neighborhood around ∞ in B. S is also open, since if $x \in \partial S \cap S$, ωx has points in its neighbourhood which belong to B and some who don't, so $x \in \partial B$. But $x \in B$, so we would have a contradiction.
Now, assume $S \neq B$.Let $x \in B \cap \partial S$. So ωx does not belong to B, but there are points in its neighborhood, which do, so x must belong to ∂B. Now we observe that since x is in the immediate basin of attraction of ∞, $F_\lambda^n(x) \to \infty$, but $F_\lambda^n(\omega x)$ does not tend to ∞. However this gives a contradiction, because of the dynamical symmetry lemma.
Analogously, if $z \in T$, then $\omega z \in T$. □

Corollary 4.4. *If one of the critical points lies in B (respectively T), then all of them lie in B (respectively T).*

This corollary combined with the following lemma shows that if a critical point is either in B(or T), then B (or T) surrounds the origin, which as it may be seen further may lead to the appearance of several annuli in the dynamical plane.

Lemma 4.5. *Consider S a Fatou component of F_λ. Now consider $x \in S$ such that there exists $j \in \mathbb{N}^*$, for which $\omega^j \neq 1$ (where $\omega^{n+d} = 1, \omega \neq 1$), so that $\omega^j x \in S$. Then $\forall i \in \mathbb{N}$, $\omega^i x \in S$, so S has $(n+d)$-fold symmetry and surrounds the origin.*

Proof. Consider $i \in \mathbb{N}$ such that $\omega^i x \notin S$. Let C_1 be a continous curve from x to $\omega^j x$, which is entirely in S. Now define $C_2 = \omega^j C_1$. By symmetry, it also lies in a Fatou component. Since $\omega^j x$ lies in the Fatou component S, C_2 must also belong to S. Analogously we can keep constructing $C_n \in S, \quad n \geq 3$.
Now consider the smallest $l \in \mathbb{N}^*$ such that $\omega^{jl} = 1$. Then the reunion of C_i, $i = \overline{1, l}$ is a closed curve C in S, which surrounds the origin. By repeating the previous construction such that $D_n = \omega^i C_n$, we obtain $D = \omega^i C$, which is a closed curve surrounding the origin which lies in the Fatou component $\omega^i S$, which is different from S because $\omega^i x \in \omega^i S$, but $\omega^i x \notin S$. Finally, because both C and D surround the origin and D is a rotation of C around the origin($\omega^i C = D$), we have that they must intersect, so they are in the same Fatou component, contradiction. □

Now we can start the proof Theorem 4.1. We will split in 3 parts, depending on the location of the critical values: in B, T, or preimages of T (they have to lie in one of these since the orbits of the critical points go to ∞ and, if it exists T is the only Fatou component, except for B, to map directly into B).

4.2 Cantor set

The first part that will be proved of the trichotomy will be the Cantor set one. This will be accomplished by showing that all the critical points lie in this in case in B, which allows the use of theorems 3.22 and 3.24 for rational maps to reach the desired result.

Proposition 4.6. *Assume that there exists a critical value of F_λ in B. Then all of the critical points of F_λ are in B.*

Proof. From Lemma 4.3 we get that if there exists a critical point in B, then all of them are in B. Suppose there is no critical point in B.
We have that ∞ is attracting, so we have an analytic homeomorphism Φ_λ, defined in a neighborhood of ∞, where it conjugates F_λ to $z \to z^n$ on the unit disk. We have no critical points in B, by their topological definition it means it can be pulled back by F_λ^{-1}, until the neighborhood, let's call it N, contains a critical value, and by symmetry all of them.
Now consider Green's function associated to Φ_λ and one of its level sets γ, which bounds an open connected set surrounding N,let's call it M. Now consider the preimage of M under F_λ which contains 0. We have that:

- M is simply connected.
- there are $(n + d)$ critical points different from 0 in the preimage of M, since all of the critical values are in M, which have multiplicity 1.
- the preimage contains 0(the preimage of ∞) and it has multiplicity (d-1)
- since in the neighborhood of the origin, F_λ can be conjugated to $z \to z^d$, in the preimage F_λ must have degree d.

So now, we just use the Riemann-Hurwitz formula and get that the preimage of N has $n+d+1$ distinct boundary components, which are all mapped into γ, which is a simple closed curve. Since the critical values are in N, they cannot be on γ, so there are no critical points on the boundary of the preimage of M.
So γ has at least (n+d+1) preimages, but the degree of F_λ is $(n + d)$ which gives us a contradiction. So there exists a critical point in B, and by symmetry, all of them are in B. □

Corollary 4.7. *Since a singularly perturbed map is a rational map of degree $(n + d)$, and by the previous proposition all of its critical points lie in B, from the theorems 3.22 and 3.24 we have that the Julia set of F_λ is indeed a Cantor set in this case and F_λ is conjugated to a shift map on $(n + d)$ symbols.*

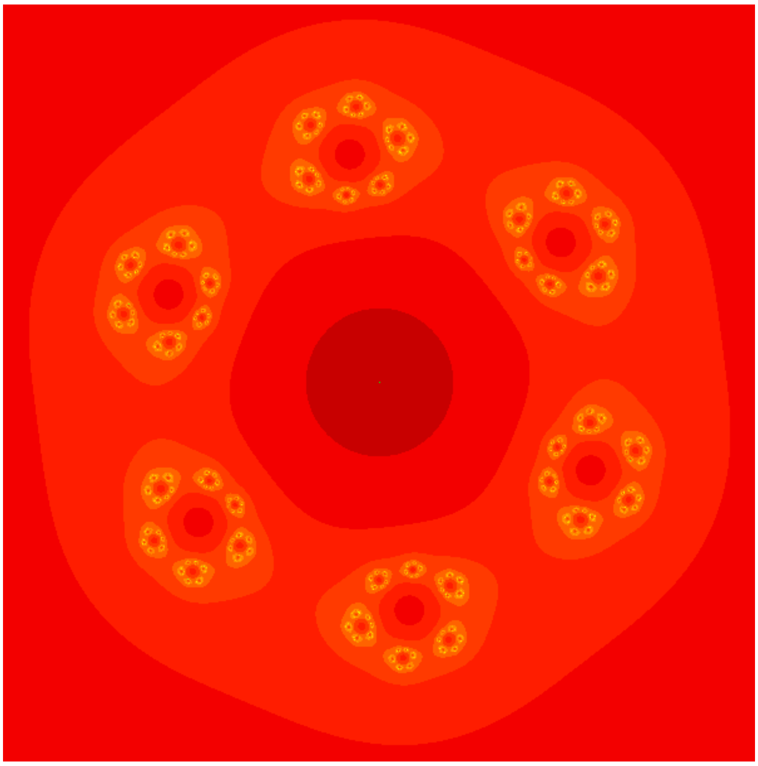

Figure 13: The Julia set for $n = 3$, $d = 3$, $\lambda = 0.4 + 0.4i$ is a Cantor set.

4.3 Cantor set of Jordan curves

The second case of the Trichotomy was first observed by McMullen, for very small values of the parameter λ. Now it is known that the Julia set is a Cantor set of circles when a critical value of F_λ lies in T (so B and T are disjoint) and each connected component in the parameter plane for which this happens is called a McMullen domain.

Proposition 4.8. *Suppose a critical value of F_λ is in T. Then the preimage of T is a connected set containing all of the critical points.*

Proof. Assume there is no component of the preimage of T with at least 2 critical points, so the preimage of T has $(n+d)$ components, each mapping 2-to-1 into T. So we have that each component has 2 prepoles, so we have 2(n+d) prepoles, contradiction.
So there exists a component with 2 critical points. By the Lemma 4.5, all of them lie in the same Fatou component, which is connected and surrounds the origin. □

Proposition 4.9. *If there is a critical value of F_λ in T, then the preimage of $\overline{T}$ is an annulus that divides the region between $\overline{B}$ and $\overline{T}$ into two open subannuli, each mapped into $\hat{\mathbb{C}} - (\overline{B} \cup \overline{T})$.*

Proof. Since $\infty \in B$ and the $(n + d)$ solutions of $z^{n+d} + \lambda = 0$ are in the preimage of T, the only critical point in T is 0, so T is simply connected. Also, since the degree of F_λ on the

preimage of T is $(n+d)$ and it has exactly $(n+d)$ critical points, from the Riemann-Hurwitz formula we get that it has 2 boundary components, so it is an annulus. From the previous proposition, we know that it also surrounds the origin.
Finally, since the preimage of T maps into T, but B and T map into B, we have that the boundary of the preimage maps into ∂T, respectively ∂B and ∂T map into ∂B and considering that the preimage of T surrounds the origin, the proof is finished.

□

Proposition 4.10. *The boundaries of B, T and of the preimages of T are simple closed curves surrounding the origin.*

Proof. The idea is to show that B is a simply connected domain whose boundary is a Jordan curve. As a consequence, the boundaries of T and its preimages are as stated.
Let $A = \mathbb{C} - (\overline{B} \cup \overline{T}$. The preimage of $\overline{T}$ is in A, by the previous proposition. So if we remove the preimage of $\overline{T}$ from A, we have two open components. Let us denote them by A_i and A_e, first one sharing the boundary with T and being d-to-1 on A and the other bordering B and being n-to-1 on A.
Consider γ_1 a Jordan curve in A_i surrounding the origin and γ_2 its preimage in A_e. We have that F_λ maps n-to-1 from γ_2 to γ_1, so it is n-to-1 in the exterior of γ_2.
Let $U = \text{Ext}(\gamma_2)$ and $V = \text{Ext}(\gamma_1)$. Let $\rho < 1$. Let $\phi : \hat{\mathbb{C}} \setminus \overline{V} \to \mathbb{D}_\rho$.

Now consider the homeomorphism $\psi_1 : \gamma_1 \to \mathbb{S}^1_{p^n}$, the restriction to γ_1 of the continous extension of ϕ to γ_1. We can also define $\psi_2 : \gamma_2 \to \mathbb{S}^1_p$ such that $\psi_1 F_\lambda(z) = (\psi_2(z))^n$.

$$\begin{array}{ccc} \partial U & \xrightarrow{F_\lambda} & \partial V \\ {\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ \mathbb{S}^1_\rho & \xrightarrow{z^n} & \mathbb{S}^1_{\rho^n} \end{array}$$

Since ∂U and ∂V are Jordan curves, we may extend the maps ψ_1 and ψ_2 to a homeomorphism ψ of $A_0 = V \setminus U$:

$$\psi : A_0 \to A_p = S^1_p \setminus S^1_{p^n}.$$

such that $\psi|_{\partial V} = \psi_1$ and $\psi|\partial U = \psi_2$ (by the Lemma 3.29). In fact, ψ is quasiconformal in $\mathring{A}_0$. Now, finally we can define:

$$G(z) = \begin{cases} F_\lambda(z), \text{ if } z \in U \\ \phi^{-1}(\psi(z))^n, \text{ if } z \in V \setminus U \\ \phi^{-1}(\phi(z))^n, \text{if } z \in \hat{\mathbb{C}} \setminus V. \end{cases}$$

G is a quasiregular (quasiconformal with some critical points) map whose critical points are 0 and ∞. By the first Shishikura principle and the Bottcher coordinates G is quasiconformally conjugate to $z \to z^n$. So its Julia set will be conjugate to a quasicircle. Since ∂B is invariant under F_λ and in V, it is the Julia set of the quasiconformal map. So ∂B is a quasicircle and so a Jordan curve, which also means that all of its preimages are Jordan curves.

□

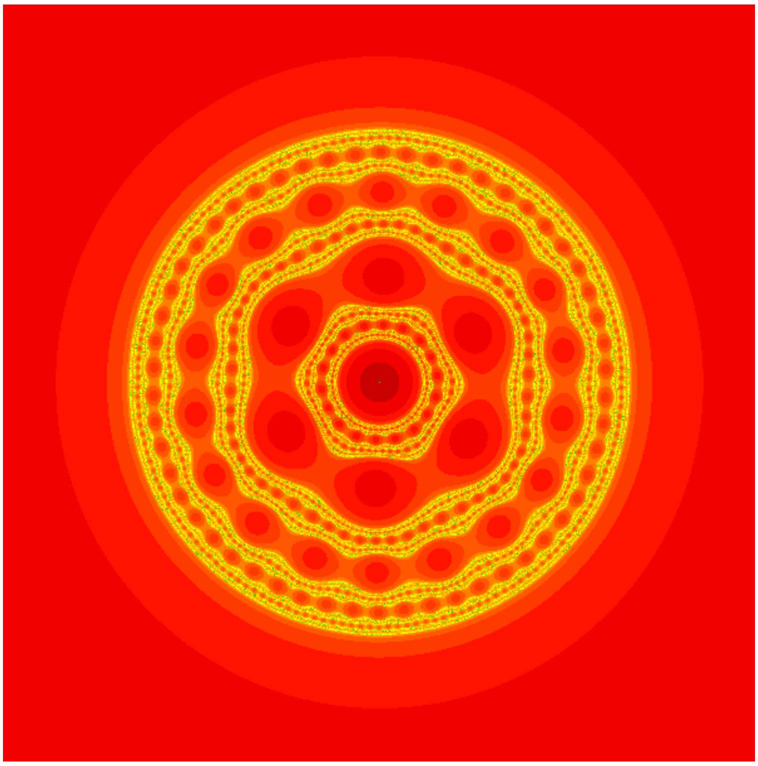

Figure 14: The Julia set for $n = 3$, $d = 3$, $\lambda = 0.00489 - 0.0012i$ is a Cantor set of Jordan curves.

Now, consider that when $\frac{1}{n} + \frac{1}{d} \geq 1$, by a major result of McMullen, there will be no critical values in the trapdoor (see [6]).

Otherwise, the same as before, for every Fatou component situated in an annulus, its preimage will consist of two disjoint annuli. Of course, the annuli corresponding to different numbers of iterations cannot coincide, or there would be some points which would, everyone of them, map to several other points, impossible, by the definition of a map. Now considering the boundaries of these Fatou components, which will be Jordan curves, we obtain that the Julia set is a Cantor set of Jordan curves.

4.4 Sierpinski curve

In the final part of the Trichotomy the critical values lie in a preimage of T, which leade to the Julia set being a Sierpinski curve (a planar set which is homeomorphic to the Sierpinski carpet), that is, it is compact, connected, locally connected, nowhere dense and any two complementary domains are bounded by disjoint Jordan curves.

Proposition 4.11. *Assume there is no critical value in either B or T. Then $\mathbb{C} - \overline{B}$ has a single open, connected component.*

Proof. Suppose there exist more open, connected components and let C_0 be the one contain-

ing the origin. Since T is the Fatou component surrounding the origin, it is entirely in C_0. We now show that there exists a prepole in C_0. Assume not, then by the symmetry lemma,then either everyone of them is in different Fatou component,or all of them are in the same Fatou component .
In the first case, every point from the boundary of T (which is part of ∂B has a preimage from the boundaries of every of the Fatou components and d preimages from its boundary. So a point has $(n + 2d)$ preimages, contradiction.
Now, assuming all of the prepoles are in the same Fatou component, it must surround the origin and hence it separates B from C_0. But since $\partial C_0 \subset \partial B$, so we have a contradiction.

So there is a prepole in C_0 and by symmetry, all of them, so F_λ has degree $(n+d)$ in C_0. So C_0 cannot have preimages outside C_0.
Suppose there exists another component of $\mathbb{C} - \overline{B}$. Consider c in its boundary such that $c \notin \partial C_0$. By Montel's theorem, a neighborhood (which can be chosen such that it does not intersect with C_0 will map forward in W_0, contradiction.

□

Proposition 4.12. *The Julia set of F_λ is a Sierpinski curve.*

Proof. This trichotomy makes the assumption that all of the critical orbits go to ∞, so the Julia set is $\mathbb{C} - \bigcup F_\lambda^{-n}(B)$. So, the Julia set is $\mathbb{C}$ without countably many open, simply connected sets. So it is compact and connected. Because it is not $\mathbb{C}$, it is also nowhere dense. From the hyperbolic lemmas we have that since the orbits of the critical points go to ∞, the Julia set is locally connected. So for the Julia set to be a Sierpinski curve.

We also have that since B is simply connected, ∂B is locally connected.

All that is left to be proven is that he boundary of B and all of its preimages are disjoint simple closed curves.

Because the boundary of B is locally connected, all external rays land at a point in ∂B. To show that it is a simple closed curve it is enough to show that no two external rays land at the same point, as by definition we will be able to build an injective map from a circle to it.
From the previous proposition C_0 is connected and simply connected. Assume there exists $p \in \partial B$, such that $\gamma(t_1)$ and $\gamma(t_2)$ land at p. We have that C_0 lies entirely in one of the two open sets. Consider the other open set, $\gamma(t_1, t_2)$, which is made of all the external rays between t_1 and t_2. Now we use that there have to exist $a, b \in \mathbb{N}$ such that

$$\gamma\Big(\frac{a}{b}, \frac{a+1}{b}\Big) \subset \gamma(t_1, t_2).$$

Otherwise, we have that all the external rays of argument between t_1 and t_2 land at p and we would obtain a contradiction since the set of angles $\theta \in \mathbb{R} - \mathbb{Z}$ that land at a point has measure 0.
Consider such a, b exist and we have that because of the conjugacy with z^n after enough iterations, $\gamma(\frac{a}{b}, \frac{a+1}{b})$ is mapped over all B. So there will be an external ray landing at a point on ∂C_0. So it will have in its neighborhood points in C_0, but its preimage before the iterations did not have was not on ∂C_0 so we have a contradiction, since we'd have a point

outside C_0 mapping, after a finite number of iterations in C_0.
So ∂B and all its preimages are Jordan curves. Now we prove that not two of the curves intersect.
First , for B and T, if there exists a point which belongs to both ∂B and ∂T, there exists an external ray in B and a preimage of an external ray , in T, landing at the point, so it is a critical point, contradiction, because the critical points go to ∞, they are not part of ∂B. For the other cases, if the preimages which intersect need the same number of iterations to reach B, are iterated until they reach it, we once again get a critical point on ∂B, or , if they ened a different number, map them until the one who needs more iterations gets to T and we have arrived to the first case. $\square$

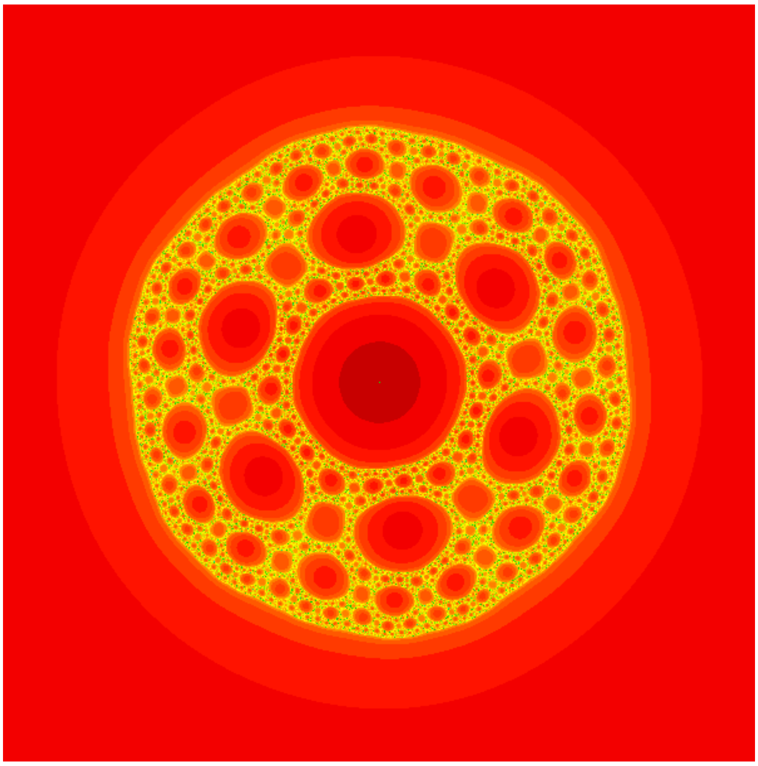

Figure 15: The Julia set for $n = 3$, $d = 3$, $\lambda = 0.02525 + 0.03348i$ is a Sierpinski curve.

References

[1] Beardon, Alan F. Iteration of Rational Functions: Complex Analytic Dynamical Systems. New York: Springer, 2000. Print.

[2] Milnor, John W. "Dynamics in One Complex Variable" (Annals of Mathematics Studies; No. 160). N.p.: Princeton UP, 2006. Print.

[3] Carleson, Lennart, and Theodore W. Gamelin. "Complex Dynamics". New York NY: Springer New York, 1993. Print.

[4] Branner, Bodil, and Nuria Fagella. "Quasiconformal Surgery in Holomorphic Dynamics". Cambridge: Cambridge U.P., 2014. Print.

[5] Devaney, Robert L., Daniel M. Look, and David Uminsky. "The Escape Trichotomy for Singularly Perturbed Maps". Indiana University Mathematics Journal 54.6 (2005): 1621-1634. Print.

[6] McMullen, Curt. "Automorphisms of Rational Maps". Berkeley, CA: Mathematical Sciences Research Institute, 1986. Print.

[7] Steinmetz, Norbert. "Rational Iteration Complex Analytic Dynamical Systems". Berlin: W. De Gruyter, 1993. Print.