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Hankel operators on vector-valued Bergman spaces

Roc Oliver Vendrell

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Advisor: **Jordi Pau Plana**

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Roc Oliver Vendrell
Departament de Matemàtica i Informàtica
Universitat de Barcelona
roc.oliverv@gmail.com

Certifico que la present memòria
ha estat realitzada per en
Roc Oliver Vendrell,
en el Departament de Matemàtica i Informàtica,
sota la meva direcció,
i amb Joaquim Ortega-Cerdà com a tutor.

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Als pares, als avis i a l'avi Jordi

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Roc

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Introduction

For $0 < p < \infty$ and $\alpha > -1$ the Bergman spaces A^p_α consist of those holomorphic functions f on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}v_\alpha(z) \right)^{1/p} < \infty,$$

where $\mathrm{d}v_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, \mathrm{d}v(z)$ and $\mathrm{d}v$ is the normalized area measure on $\mathbb{D}$. The theory of Bergman spaces has been a central subject of study in complex analysis during many years with many known applications. The book [15] by S. Bergman contains the first systematic treatment of the Hilbert space of square integrable analytic functions with respect to Lebesgue area measure on a domain. When attention was later directed to the spaces A^p over the unit disk, it was natural to call them Bergman spaces. At the end of the 20th century it came a period of intense research on Bergman spaces and, nowadays, there are many excellent text books of the topic of Bergman spaces [37, 42, 82, 83].

Let $\mathbb{B}_n$ be the unit ball of the n-dimensional complex plane $\mathbb{C}^n$ $(n \geq 1)$. As the reader may have already noticed, in this dissertation we want to study a further generalization of the theory: we focus on vector-valued Bergman spaces on $\mathbb{B}_n$ with the same weight, in contrast to the scalar-valued Bergman spaces mentioned earlier. Usually, we write X for a general complex Banach space. A vector-valued function $f \colon \mathbb{B}_n \to X$ is a function that takes values in some Banach space X. For $0 < p < \infty$ and $\alpha > -1$, the vector-valued Bergman spaces $A^p_\alpha(X)$ are defined as the space of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{A^p_\alpha(X)} = \left(\int_{\mathbb{B}_n} \|f(z)\|_X^p \, \mathrm{d}v_\alpha(z) \right)^{1/p} < \infty.$$

Now $\mathrm{d}v_\alpha(z) = c_\alpha(1-|z|^2)^\alpha \, \mathrm{d}v(z)$, where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}_n$ and c_α is a normalizing constant so that $v_\alpha(\mathbb{B}_n) = 1$. If $X = \mathbb{C}$ we obtain the previous scalar-valued Bergman spaces, so clearly, this generalizes the standard Bergman spaces defined above.

Although the vector-valued Bergman spaces have similar properties to the scalar-valued Bergman spaces, this generalization makes the theory much more complicated, sometimes even the most simple thing in the classical case presents new difficulties and then new ideas are required. The theory of vector-valued functions is by now classical, but the incorporation of Bergman spaces in the equation is much more recent. There are many authors that are working nowadays on vector-valued Bergman spaces, see [7, 19–21, 23, 27–32, 49, 51, 63, 77] for example.

The main goal of this work is to study vector-valued Bergman spaces and to obtain the weak factorization of these spaces. In order to do that we need to study small Hankel operators with operator-valued holomorphic symbols. We also study the big Hankel operator acting on vector-valued Bergman spaces.

This dissertation is structured into seven chapters.

In **Chapter 1** we collect all the previous results and notations needed to follow the rest of the manuscript. More concretely, some of the topics covered in this chapter are the Bochner integral, the integral for vector-valued functions appearing first in [22] by Bochner; the Bergman metric, results of the metric used in $\mathbb{B}_n$; harmonic and subharmonic function; basic notions of differentiation, where the differential operators $R^{\alpha,t}$ are presented which is important in the next chapters and in the final section we recall some topics on Banach spaces, as the Rademacher type and cotype of a Banach space and some other related results.

Having all that in mind, in **Chapter 2**, the vector-valued Bergman spaces are presented. First of all, we show the standard properties of this kind of theory, such as that the point evaluations $z \mapsto f(z)$ are bounded linear operators in Theorem 2.1.1. As in the scalar case, we still have an integral representation of each function $f \in A^1_\alpha(X)$ as

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, dv_\alpha(w), \qquad z \in \mathbb{B}_n.$$

The vector-valued polynomials $\mathcal{P}(X)$ are still dense in the vector-valued Bergman spaces $A^p_\alpha(X)$, for $1 \leq p < \infty$. In Section 2.2, we basically prove the boundedness of the Bergman projection, defined as

$$P_\beta f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} \, dv_\beta(w), \qquad z \in \mathbb{B}_n, f \in L^1_\alpha(X), \beta > -1,$$

on the vector-valued Bergman spaces. That is, the projection $P_\beta \colon L^p_\alpha(X) \to A^p_\alpha(X)$ is bounded for $1 \leq p < \infty$ if $p(\beta + 1) > \alpha + 1$. All these properties and results are used later in subsequent chapters. We want to mention that most of the results of this chapter are certainly known to experts, and if new, the proof of the result follows easily with the same argument as the scalar case with apropriate modifications. One example of this is the atomic decomposition of vector-valued Bergman spaces on $\mathbb{B}_n$ obtained in Theorem 2.3.5. Some especial case appears in [27]. Atomic decomposition for scalar-valued Bergman spaces was obtained by Coifman and Rochberg [24] and after that was a central topic in complex analysis.

It is known that Bloch spaces are intimately related to Bergman spaces. Recall that the scalar-valued Bloch space $\mathcal{B}$ is defined as the set of holomorphic functions

f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\,|f'(z)|<\infty.$$

The generalization to the case of vector-valued functions on $\mathbb{B}_n$ is done in a similar way: changing the derivative by the Laplacian and the absolute value by the norm on the Banach space X.

One of the main reasons of the relationship between Bloch and Bergman spaces is that, roughly speaking, the Bloch space can be thought of as the limit case of the Bergman spaces A^p_α as $p\to\infty$, see [82, Theorem 2.16] for more understanding on that. In fact, the Bloch space $\mathcal{B}$ can be naturally identified with the dual space of A^1_α. The integral representation for the Bloch space and the duality between A^1_α and the Bloch space can be found in many different papers/books, including [37, 68, 69, 82].

Another important reason is that Bloch type spaces appear naturally when we work with Hankel operators. A classical theorem says that the small Hankel operator is bounded on the Bergman spaces if and only if the symbol belongs to some Bloch space [6, 8]. As we deal with Hankel operators, Bloch type spaces will be very important for us. Moreover, the Bloch space is also interesting in its own right. In fact, the Bloch space has been studied much earlier that the Bergman spaces. In particular, the Bloch space of the unit disk plays an important role in classical geometric function theory. See [3, 4, 8, 9, 83] for more information on them. Later on, serious research on the Bloch space of the unit ball began with Timoney's papers [72, 73] and by now it is very developed, see, for example, [82].

The vector-valued Bloch type spaces play a similar role and therefore we dedicate one full chapter to these spaces. **Chapter 3** is devoted to present and characterize the vector-valued Bloch type spaces. In Section 3.1 we characterize the vector-valued Bloch spaces $\mathcal{B}(X)$ in terms of some specific function and we show its integral representation in Theorem 3.1.3. More concretely, we show that a holomorphic X-valued function f is in $\mathcal{B}(X)$ if and only if $f=P_\alpha g$ for some $g\in L^\infty(X)$, which is again equivalent to that the function $(1-|z|^2)^t R^{\alpha,t}f(z)$ is in $L^\infty(X)$, for some $t>0$. The scalar case of this result was proved in [72, 82]. In this section we also prove that the dual of $A^1_\alpha(X)$ can be identified with $\mathcal{B}(X^*)$ in Theorem 3.1.6. This duality is also proved in [7] by J. L. Arregui and O. Blasco, but our proof here is more similar to the classical one in Zhu's book [82]. In Section 3.2 we prove the same characterization and integral representation of the more generalized Bloch type space $\mathcal{B}_\gamma(X)$, see [82, Chapter 7] for more details on the scalar case of these spaces. More concretely, in Theorem 3.2.1 we can find the generalization of Theorem 3.1.3 for these spaces and as a consequence we obtain another characterization better suited for our purpose in Corollary 3.2.3 saying that a holomorphic X-valued function f is in $\mathcal{B}_\gamma(\mathbb{B}_n,X)$ if and only if the function $(1-|z|^2)^{\gamma+t}R^{\alpha,t+1}f(z)$ is in $L^\infty(\mathbb{B}_n,X)$, for some $t\geq 0$ and some conditions on α and γ. Note that all this results are important for the next chapters.

A very important result of this section is also Theorem 3.2.6 that we state here:

Theorem 3.2.6. *For any $z \in \mathbb{B}_n$ we have*

$$(1-|z|^2)\,\|Rf(z)\|_X \leq (1-|z|^2)\,\|\nabla f(z)\|_{X^n} \leq \left\|\widetilde{\nabla} f(z)\right\|_{X^n}, \qquad f \in \mathcal{H}(X).$$

The same result for the scalar case is straightforward to prove (see [82, Lemma 2.14]). However on the vector-valued case this is non trivial anymore. This theorem allows us, among other things, to fully characterize the general vector-valued Bloch spaces in Theorem 3.2.8 and connect it to the characterization of Hankel operators. Thus, it is important to note again that all this material is also used later on especially for the characterization of the Hankel operators on the vector-valued Bergman spaces.

Since we mention Hankel operators, in **Chapter 4** we prove the characterization of the boundedness of the small Hankel operator with analytic operator-valued symbols between vector-valued Bergman spaces (of different type). We explain what this means in the following. First of all, we consider $1 < p, q < \infty$ and X and Y Banach spaces, the small Hankel operator $h_T \colon A^p_\alpha(X) \to A^q_\alpha(Y)$ with a holomorphic operator-valued symbol $T \colon \mathbb{B}_n \to \mathcal{L}(\overline{X}, Y)$ is defined as

$$h_T f(z) = \int_{\mathbb{B}_n} \frac{T(w)\overline{f(w)}}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, \mathrm{d}v_\alpha(w),$$

where $\overline{X}$ is the complex conjugate of the Banach space X (do not confuse with the dual), see Section 1.1 for more details on $\overline{X}$. The scalar case of this small Hankel operator is defined as $h_\varphi f = \overline{P_\alpha}(\varphi f)$, for φ a holomorphic scalar function. Classically, in the study of small Hankel operators, it turns out that it is more convenient to study $h_{\overline{\varphi}}$ instead of h_φ, maybe the reason is the use of $\overline{P_\alpha}$. Then, note that, roughly speaking, $\overline{h_{\overline{\varphi}}}$ is similar to h_T defined above, so clearly this definition generalizes the scalar case (see Section 4.4 for more details). Hankel operators have been extensively studied and developed by many authors during many years. Small Hankel operators on Bergman spaces are closer in spirit to Hankel operators on Hardy spaces, which is also a well-known important theory. Initially, some authors tried to prove the boundedness of the small Hankel operator $h_{\overline{\varphi}}$ when $p = q = 2$. Then many others authors tried to generalize it for other exponents p and q. It is known that for values of $p \leq q$ the methods and characterizations are very close and similar, but when $p > q$ the picture changes completely and usually the characterizations are known to be different and more difficult to prove.

In this chapter we completely characterize the boundedness of the small Hankel operator h_T in terms of its operator-valued symbol T for all the different values of p and q. We begin in Section 4.1 with some preliminaries and basic properties needed later on in this chapter. Next, in Sections 4.2 and 4.3 the actual characterizations of h_T in terms of its symbol T are made. The characterization when $p = q$ is in Theorem 4.2.1 which we state here:

Theorem 4.2.1. *For $1 < p < \infty$, the small Hankel operator $h_T \colon A^p_\alpha(X) \to A^p_\alpha(Y)$ is bounded if and only if $T \in \mathcal{B}(\mathcal{L}(\overline{X}, Y))$ (with equivalent norms).*

Here you can see the relation between Bloch and Bergman spaces, and this is a result very similar to the scalar case. You can find the scalar case of this result when $p = 2$ in [83]. In Theorem 4.2.2 this result is generalized and improved in order to include the cases of $p < q$. Although both theorems and cases can be proved together, we have decided to divide into two different theorems because the first one is easier to prove and to follow, then the other is a generalization of that. Also the scalar case of the first theorem is much more classical, so it is much easier to compare. For the remaining cases of $p > q$ it is proved in Theorem 4.3.3, which is the following:

Theorem 4.3.3. *For $1 < q < p < \infty$ and Y with finite cotype, the small Hankel operator $h_T\colon A^p_\alpha(X) \to A^q_\alpha(Y)$ is bounded if and only if $T \in A^t_\alpha(\mathcal{L}(\overline{X}, Y))$ (with equivalent norms) where*

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}.$$

As we said the characterization here changes completely and the symbol T is in terms of a different vector-valued Bergman space, which is a similar result of the scalar case. In fact, for the proof, we use similar ideas to the scalar case [60] that in turn use the Rademancher functions, this is why the condition of the cotype appears (see Section 1.7 for more details on Rademancher functions and cotype of a Banach space). Finally, in Section 4.4 we expose some examples and applications of these results. For example, in Theorem 4.4.5 the generalization of the small Hankel operator of A. Aleman and O. Constantin [1] is made.

Another very important consequence of the boundedness of the small Hankel operator between vector-valued Bergman spaces is shown in **Chapter 5**. We establish the weak factorization of the vector-valued Bergman spaces. Factorization of analytic functions is a very big topic and many people worked on it during many years and it is known to have many applications. It is a classic result of functional analysis that the Hardy space H^1 can be factored into two functions in the Hardy space H^2. This factorization is sometimes called "strong". After that many authors tried to generalize it to different contexts. Horowitz [43] proved the strong factorization of weighted Bergman spaces on the unit disk and Gowda in [40] showed that this strong factorization is no longer possible to obtain for Bergman spaces in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ of dimension $n \geq 2$. Then, Coifman, Rochberg and Weiss [25] enlighten us that it is still possible to obtain some factorization, which was called weak factorization. The proof of Coifman, Rochberg and Weiss was done for the Hardy space H^1. The fact that the scalar Bergman spaces A^1_α admit a weak factorization is a direct consequence of the atomic decomposition mentioned earlier. More recently, in [60], Pau and Zhao proved the weak factorization of the scalar-valued Bergman spaces A^p_α with $p > 1$ as a consequence of the boundedness of the small Hankel operator, which where we take the ideas. Basically, this weak factorization for the scalar-valued Bergman space A^q_α, for $1 < q < \infty$, consists of the following: every $f \in A^q_\alpha$ can be written as $f = \sum_k g_k h_k$, where the sequences $\{g_k\}_k \subset A^{p_1}_\alpha$ and

$\{h_k\}_k \subset A_\alpha^{p_2}$ and $p_1, p_2 > 1$ satisfying that $1/q = 1/p_1 + 1/p_2$, and

$$\sum_k \|g_k\|_{A_\alpha^{p_1}} \|h_k\|_{A_\alpha^{p_2}} \leq C \|f\|_{A_\alpha^q}.$$

It is natural to ask if this weak factorization can be extended to the setting of vector-valued Bergman spaces. One of the problems for the vector-valued case is that in general it makes no sense to "multiply" an X-valued function f with an Y-valued function g. This is why we consider a generalized product $x \boxdot y$ defined for $x \in X$ and $y \in Y$. Then, the product type space (sometimes denoted weakly factored space in the scalar case) $X \widehat{\boxdot} Y$ is the completion of finite sums

$$\sum_k x_k \boxdot y_k, \qquad \{x_k\}_k \subset X, \{y_k\}_k \subset Y,$$

with the following norm

$$\|u\|_{X \widehat{\boxdot} Y} := \inf \left\{ \sum_k \|x_k\|_X \|y_k\|_Y : u = \sum_k x_k \boxdot y_k \right\}.$$

It is remarkable to say that it already exists a lot of different types and examples of products of this kind, as for example, the product tensor product space $X \widehat{\otimes} Y$, even the same weakly factored space $X \odot Y$ denoted in [60], the point evaluations or any well defined product of two Banach spaces, see Section 5.1 for more details. In fact, in Section 5.1 we can find the main result of this chapter, which is the weak factorization of vector-valued Bergman spaces in Theorem 5.1.4 using this kind of product. The result is the following:

Theorem 5.1.4. *For $1 < q < \infty$ and Y^* with finite cotype, we have*

$$A_\alpha^q(X \widehat{\boxdot} Y) = A_\alpha^{p_1}(X) \widehat{\boxdot} A_\alpha^{p_2}(Y),$$

(with equivalent norms) for any $p_1, p_2 > 1$ satisfying $1/q = 1/p_1 + 1/p_2$.

Note that the case of $q = 1$ is also proved in Theorem 5.1.5 which basically says that

$$A_\alpha^1(X \widehat{\boxdot} Y) = A_\alpha^p(X) \widehat{\boxdot} A_\alpha^{p'}(Y),$$

where $p > 1$ and p' is the conjugate exponent of p. Note that in this particular case we do not need the condition of cotype of one of the Banach spaces. The reason behind that is because we use Theorem 4.2.1 instead of Theorem 4.3.3, which is the used to prove Theorem 5.1.4. Meanwhile, we also show the characterization of the boundedness of some Hankel form which is known to be very related to the weak factorization. In Section 5.2 we show different applications of this weak factorization, like a generalization of a theorem of O. Constantin in [28] that is Theorem 5.2.2, where she works on the particular Banach spaces, the Schatten classes $\mathcal{S}^t$.

The other different and well-known Hankel operator is the big Hankel operator. Big Hankel operators are closely related to Toeplitz operators and they are also well-known and well established between scalar-valued function spaces, see [83] for a general reference. The scalar setting of this big Hankel operator is defined as $H_\varphi f = (I - P_\alpha)(\varphi f)$ and it is known that $H_\varphi = 0$ when φ is analytic. This means that it makes sense to study big Hankel operators with anti-analytic symbols, which means studying $H_{\overline{\varphi}}$ such that φ is analytic. Our general approach is the following: considering $1 < p, q < \infty$, X and Y Banach spaces. Let $T \colon \mathbb{B}_n \to \mathcal{L}(X, Y)$ be a holomorphic operator-valued symbol, then the big Hankel operator $H_T \colon A^p_\alpha(X) \to L^q_\alpha(Y)$ is defined as

$$H_T f(z) = \int_{\mathbb{B}_n} \frac{(T(z) - T(w))\overline{f(w)}}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w).$$

Note the similarity with the scalar setting: the big Hankel operator $\overline{H_{\overline{\varphi}}}$ is, roughly speaking, very similar to H_T, so H_T is a generalization of the scalar setting. The first major progress made in the study of big Hankel operators on scalar-valued Bergman spaces is Axler's paper [8], in which it is shown the boundedness and compactness of big Hankel operators with anti-analytic symbols on the scalar-valued Bergman spaces on $\mathbb{D}$ in the case of $p = q = 2$. Later on, Axler's result was generalized in [5,6] to weighted scalar-valued Bergman spaces on $\mathbb{B}_n$. In the case of general symbols, Zhu [80] is the first to exhibit the connection between size estimates of a Hankel operator and the mean oscillation of the symbol in the Bergman metric. This idea is then generalized and developed systematically in [13,14] and [11] in the context of a bounded symmetric domain. For the other cases, Pau, Zhao and Zhu [61] solved the problem for $p \leq q$, which they also solved for different weights among other things.

Therefore, in **Chapter 6** we fully characterize the boundedness of the big Hankel operator on vector-valued Bergman spaces in terms of its operator-valued holomorphic symbol for all cases of $p > 1$ and $q > 1$, and so we solve and generalize the previous problem. In Section 6.1 there are some preliminary results included, and in Section 6.2, the important results of this chapter are shown. Theorem 6.2.1 and Corollary 6.2.2 deal with the cases of $p \leq q$. The following result is only the case $p = q$ but we think it is enough to include here for the sake of clarity:

Theorem 6.2.2. *For $1 < p < \infty$, the Hankel operator $H_T \colon A^p_\alpha(X) \to A^q_\alpha(Y)$ is bounded if and only if $T \in \mathcal{B}_\gamma(\mathcal{L}(\overline{X}, Y))$ (with equivalent norms).*

For the remaining case, when $q < p$, we have Theorem 6.2.3 which is the following:

Theorem 6.2.3. *For $1 < q < p < \infty$, the Hankel operator $H_T \colon A^p_\alpha(X) \to A^q_\alpha(Y)$ is bounded if and only if $T \in A^t_\alpha(\mathcal{L}(\overline{X}, Y))$ where $1/t = 1/q - 1/p$ (with equivalent norms).*

Note that both cases are mirrors to the scalar case, as predicted.

Finally, in **Chapter 7** we discuss some open problems we have not been able to solve, as well as some other interesting problems in the same line as this work in order to look on the future.

We also want to mention that during the process of this dissertation, building on his master thesis, the author had the opportunity to work with D. Pascuas and publish the article [56] on Toeplitz operators on generalized Fock spaces. However, we have decided to not include it in this manuscript.

Chapter 1

Preliminaries

In this chapter we collect most of the information that we need for the rest of the manuscript. The results of this chapter are well known, but we include it for completeness. More concretely, here we present basic concepts of the unit ball of the complex plane, the Bergman metric, holomorphic vector-valued functions, the Bochner integral, some needed inequalities and finally we discuss briefly the notions of type and cotype of Banach spaces.

1.1 Basic Concepts and Notations

Let $\mathbb{C}$ be the set of complex numbers. Throughout this manuscript we fix a positive integer $n \in \mathbb{N}, n \geq 1$ and let

$$\mathbb{C}^n = \mathbb{C} \times \overset{n)}{\cdots} \times \mathbb{C}$$

denote the complex Euclidean space of dimension n. The usual operations on $\mathbb{C}^n$, addition, scalar multiplication, and conjugation, are defined componentwise. For

$$z = (z_1, \cdots, z_n), \qquad w = (w_1, \cdots, w_n),$$

in $\mathbb{C}^n$, we define

$$\langle z, w \rangle = \langle z, w \rangle_{\mathbb{C}^n} = z_1 \overline{w_1} + \cdots + z_n \overline{w_n},$$

where $\overline{w_k}$ is the complex conjugate of w_k. We also write the norm of $z \in \mathbb{C}^n$ as

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The space $\mathbb{C}^n$ becomes an n-dimensional Hilbert space when endowed with the inner product above $\langle \cdot, \cdot \rangle$. The standard basis for $\mathbb{C}^n$ consist of $\{e_k\}_{k=1}^n$ where

$$e_k = (0, \cdots, 0, \overset{k)}{1}, 0, \cdots, 0), \qquad k \in \{1, \ldots, n\}.$$

We denote the open unit ball in $\mathbb{C}^n$ by

$$\mathbb{B}_n := \{z \in \mathbb{C}^n \colon \ |z| < 1\}.$$

The open unit disk of $\mathbb{C}$, $\mathbb{B}_1$, is usually denoted by $\mathbb{D}$. The boundary of $\mathbb{B}_n$, $\partial\mathbb{B}_n$, will be denoted by $\mathbb{S}_n$ and is called the unit sphere in $\mathbb{C}^n$. Thus

$$\mathbb{S}_n := \partial\mathbb{B}_n = \{z \in \mathbb{C}^n\colon\ |z| = 1\}.$$

Occasionally, we will also need the closed unit ball

$$\overline{\mathbb{B}}_n = \mathbb{B}_n \cup \mathbb{S}_n = \{z \in \mathbb{C}^n\colon\ |z| \leq 1\}.$$

Finally, we usually use the notation $B(z, r)$ for the Euclidean ball of radius r and center $z \in \mathbb{C}^n$. The Euclidean ball $B(0, r)$ sometimes is denoted by $r\mathbb{B}_n$.

For a complex Banach space X, we denote by $\|\cdot\|_X : X \to \mathbb{C}$ the norm of X. Let X^* represent the topological dual space (or sometimes called continuous dual) of X, i.e., the elements $x^* \in X^*$ are functionals $x^*\colon X \to \mathbb{C}$ linear and bounded. Let (Ω, Σ, μ) be a measure space ($\Omega \subset \mathbb{C}^n$). In a general sense, we define a vector-valued function f (sometimes called X-valued function) as a function that takes values to the Banach space X, that is, $f\colon \Omega \to X$.

This manuscript works basically with vector-valued holomorphic functions in a complex Banach space X. There are many equivalent definitions of holomorphic functions in several complex variables. Basically, a function f is holomorphic in several complex variables if and only if it is holomorphic in each variable separately. Thus a function $f\colon \Omega \to X$ is said to be holomorphic in Ω if for every point $z \in \Omega$ and for every $k \in \{1, 2, \cdots, n\}$ the limit

$$\frac{\partial f}{\partial z_k}(z) := \lim_{\lambda \to 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}$$

exists (and is finite), where $\lambda \in \mathbb{C}$ and is denoted the partial derivative of f respect to z_k. For simplicity, sometimes we use the following notation $\partial_k f := \partial f/\partial z_k$. We denote $\mathcal{H}(\Omega, X)$ to be the set of vector-valued holomorphic functions $f\colon \Omega \to X$. If $X = \mathbb{C}$ we write $\mathcal{H}(\Omega) := \mathcal{H}(\Omega, \mathbb{C})$, the classical set of scalar-valued holomorphic functions. It is well-known that a vector-valued function $f\colon \Omega \to X$ is holomorphic if and only if it is weakly holomorphic, that is, $x^* \circ f \in \mathcal{H}(\Omega)$, for any $x^* \in X^*$.

We denote by $X^n := X \times \cdots \times X$ the Banach space product of n copies of X with norm

$$\|x\|_{X^n} := \left(\sum_{i=1}^{n} \|x_i\|_X^2\right)^{\frac{1}{2}}, \qquad x = (x_1, \ldots, x_n) \in X^n.$$

Following the same notation, we will write $\mathbb{B}_n(X)$ for the ball of radii 1 of X^n, that is,

$$\mathbb{B}_n(X) := \{x \in X^n\colon\ \|x\|_{X^n} < 1\},$$

and denote by $\mathbb{B}(X) := \mathbb{B}_1(X)$. It is clear that $\mathbb{B}_n = \mathbb{B}_n(\mathbb{C})$.

Let $x \in X$, $x^* \in X^*$ and $\lambda \in \mathbb{C}$ a scalar value. We define

$$(\lambda x^*)(x) := \overline{\lambda} \cdot x^*(x).$$

We also use throughout this manuscript the following notation

$$\langle x, x^* \rangle_X := x^*(x)$$

that represent the "inner product" in the Banach space X. Notice that for any $\lambda \in \mathbb{C}$

$$\langle \lambda x, x^* \rangle_X = x^*(\lambda x) = \lambda \cdot x^*(x) = (\overline{\lambda} x^*)(x) = \langle x, \overline{\lambda} x^* \rangle_X$$

so we have the regular rule of inner product. Note that when $X = H$ is a Hilbert space (a Banach space endowed with an inner product) we have that $\langle \cdot, \cdot \rangle_H$ is the usual inner product of H. We also know, by duality (a consequence of Hahn-Banach), that for every element $x \in X$,

$$\|x\|_X = \sup_{\|x^*\|_{X^*}=1} |\langle x, x^* \rangle_X| .$$

We need also the notion of the "conjugate" of a Banach space X. For an element $x \in X$, we denote $\overline{x}$ the functional on X^* such that $\overline{x}(x^*) = \overline{\langle x, x^* \rangle_X}$, for any $x^* \in X^*$. We call $\overline{x}$ the complex conjugate to x. Then, the set of all these complex conjugates is $\overline{X}$, that is,

$$\overline{X} := \{\overline{x} \colon x \in X\} .$$

We can see that $\overline{X}$ is a Banach space with norm $\|\overline{x}\|_{\overline{X}} := \sup_{\|x^*\|_{X^*}=1} |\overline{x}(x^*)|$. Moreover, it follows that $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, for every $x \in X$, so that X and $\overline{X}$ are isometrically anti-isomorphic. We were not able to find so much information of that in the literature but one can find some in [55, Section 1].

For any two Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. In some cases we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$, that is, $\mathcal{L}(X) := \mathcal{L}(X, X)$. Recall that when Y is a Banach space, the space of linear bounded operators $\mathcal{L}(X, Y)$ is also a Banach space endowed with the following norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y , \qquad T \in \mathcal{L}(X, Y).$$

Note that $X^* = \mathcal{L}(X, \mathbb{C})$, and since $\mathbb{C}$ is always Banach space (with the absolute value as a norm) we have that X^* is always a Banach space, even if X is only a normed vector space. We refer to [82,83] for more information and general reference about this and more.

We let $\mathrm{d}v$ denote the normalized volume measure on $\mathbb{B}_n$, so that $v(\mathbb{B}_n) = 1$. The surface measure on $\mathbb{S}_n$ will be denoted by $\mathrm{d}\sigma$ and we normalize σ so that $\sigma(\mathbb{S}_n) = 1$. The measures v and σ are related by

$$\int_{\mathbb{B}_n} f(z)\,\mathrm{d}v(z) = 2n \int_0^1 r^{2n-1} \int_{\mathbb{S}_n} f(r\zeta)\,\mathrm{d}\sigma(\zeta)\,\mathrm{d}r. \tag{1.1.1}$$

Sometimes this is called integration in polar coordinates, see also [82, Lemma 1.8]. If $\alpha \in \mathbb{R}$, then the measure $(1 - |z|^2)^\alpha\,\mathrm{d}v(z)$ is finite if and only if $\alpha > -1$. Thus, we consider a real parameter $\alpha > -1$ and define a weighted volume measure $\mathrm{d}v_\alpha$ on $\mathbb{B}_n$ by

$$\mathrm{d}v_\alpha(z) := c_\alpha(1 - |z|^2)^\alpha \,\mathrm{d}v(z), \tag{1.1.2}$$

where c_α is a normalizing constant taken so that $v_\alpha(\mathbb{B}_n) = 1$.

In order to motivate the following section when $\Omega = \mathbb{B}_n$ (mostly our case) and for a scalar function $f \in \mathcal{H}(\mathbb{B}_n)$ we recall the well-known mean value property for holomorphic functions, that is

$$f(0) = \int_{\mathbb{S}_n} f(r\zeta)\,\mathrm{d}\sigma(\zeta) \tag{1.1.3}$$

for any $0 \le r < 1$. This particular equality is very important in our theory, so it will be very interesting to get it for vector-valued functions. In fact, it is not difficult to assume it, the only difference here is the integral of a vector-valued function. Even for a general vector-valued function $f \in \mathcal{H}(\Omega, X)$, the integral "$\int_\Omega f$" has no meaning since the functions that we consider here take values in a Banach space X and the Lebesgue integral only makes sense for scalar functions. Thus, we need to understand what an integral of a vector-valued function $f \in \mathcal{H}(\Omega, X)$ means. Perhaps you already noticed that this concern also is valid for (1.1.1).

It is also very interesting to know if we have similar properties that the Lebesgue integral have, like linearity, measurability and, for example, when the norm of the integral and the integral of the norm is comparable and in which sense.

Therefore, in order to get the notion of an integral on the vector-valued setting and other properties one needs to extend the definition of Lebesgue integral to functions tanking values in a Banach space X, namely the Bochner integral.

1.2 The Bochner Integral

In this section we deal with the integrability of vector-valued functions, so we introduce the Bochner integral and some properties in a general way. Let (Ω, Σ, μ) be a measure space and X be a Banach space. The Bochner integral is defined in much the same way as the Lebesgue integral. We refer to [33, 34] for an account of all of this. First, a simple function is any finite sum of the form

$$s_n(z) = \sum_{k=1}^{n} x_k \chi_{E_k}(z), \quad z \in \Omega$$

where $\{E_k\}_k$ is a sequence of pairwise disjoint members of the σ-algebra Σ and $\{x_k\}_k \subset X$ is a sequence of elements of X. The χ_E denotes the characteristic function of $E \in \Sigma$. We say that the simple function s_n is *Bochner integrable with respect to μ* whenever

$$\sum_{k=1}^{n} \|x_k\|_X \,\mu(E_k) < \infty,$$

that is, if $\mu(E_k)$ is finite whenever $x_k \neq 0$. In this case, we define the *Bochner integral* of s_n over $E \in \Sigma$ to be

$$\int_E s_n(z)\,\mathrm{d}\mu(z) = \int_E \left[\sum_{k=1}^n x_k \chi_{E_k}(z)\right] \mathrm{d}\mu(z) := \sum_{k=1}^n x_k \mu(E \cap E_k)$$

exactly as it is for the ordinary Lebesgue integral.

A vector-valued function $f\colon \Omega \to X$ is *μ-measurable* whenever there exists a sequence $\{s_n\}_n$ of X-valued, Σ-simple functions defined on Ω such that

$$s_n(z) \longrightarrow f(z), \qquad \mu\text{-almost every } z \in \Omega. \tag{1.2.1}$$

A vector-valued function $f\colon \Omega \to X$ is *Bochner integrable with respect to* μ when f is μ-measurable and whenever there exists a sequence of Bochner integrable simple functions $\{s_n\}_n$ of the above form such that

$$\lim_{n\to\infty} \int_\Omega \|f(z) - s_n(z)\|_X \,\mathrm{d}\mu(z) = 0 \tag{1.2.2}$$

where the integral is an ordinary Lebesgue integral. In this case, the *Bochner integral* of f over $E \in \Sigma$ is defined by

$$\int_E f(z)\,\mathrm{d}\mu(z) := \lim_{n\to\infty} \int_E s_n(z)\,\mathrm{d}\mu(z).$$

It is common to call $L^1_\mu(\Omega, X)$ the set of Bochner integrable functions respect to μ. It can be shown, see [34] for a proof, that a μ-measurable function $f\colon \Omega \to X$ is Bochner integrable with respect to μ if and only if

$$\int_\Omega \|f(z)\|_X \,\mathrm{d}\mu(z) < \infty.$$

From now on, a μ-Bochner integrable function it refers to a function that is Bochner integrable with respect to μ.

Lemma 1.2.1. *Let X and Y be Banach spaces, and $T\colon X \to Y$ be a bounded linear operator. If f is a μ-Bochner integrable X-valued function, then Tf is μ-Bochner integrable and*

$$\int_E Tf(z)\,\mathrm{d}\mu(z) = T\left(\int_E f(z)\,\mathrm{d}\mu(z)\right) \tag{1.2.3}$$

for every $E \in \Sigma$.

Proof. Let us see first that Tf is μ-Bochner integrable. Let $\{s_n\}_n$ be a sequence of μ-Bochner integrable simple functions such that (1.2.1) and (1.2.2) holds. Now take $\{Ts_n\}_n$ as a "candidate" sequence for Tf to be μ-Bochner integrable. Indeed, $\{Ts_n\}_n$ is a sequence of μ-Bochner integrable simple functions on Y because $\{s_n\}_n$ is on X and T is a bounded linear operator. In addition, for the same reason, we have that $\|Tf(z) - Ts_n(z)\|_Y \leq \|T\|_{X\to Y} \|f(z) - s_n(z)\|_X$, for every $z \in \Omega$. From that we can easily deduce with (1.2.1) that Tf is μ-measurable on Y. Moreover, by (1.2.2), we have that

$$\int_\Omega \|Tf(z) - Ts_n(z)\|_Y \, \mathrm{d}\mu(z) \leq \|T\|_{X\to Y} \int_\Omega \|f(z) - s_n(z)\|_X \, \mathrm{d}\mu(z) \longrightarrow 0,$$

as $n \to \infty$. Now, let us prove the equality (1.2.3). Using the fact that T is a bounded linear operator and the μ-Bochner integrability of f

$$\begin{aligned} T\left(\int_\Omega f(z)\,\mathrm{d}\mu(z)\right) &= \lim_{n\to\infty} T\left(\int_\Omega s_n(z)\,\mathrm{d}\mu(z)\right) \\ &= \lim_{n\to\infty} T\left(\int_\Omega \left[\sum_{k=1}^n x_k \chi_{E_k}(z)\right] \mathrm{d}\mu(z)\right) \\ &= \lim_{n\to\infty} T\left(\sum_{k=1}^n x_k \mu(E_k)\right) = \lim_{n\to\infty} \sum_{k=1}^n T(x_k)\mu(E_k) \end{aligned}$$

and, on the other hand, the μ-Bochner integrability of Tf shows that

$$\begin{aligned} \int_\Omega Tf(z)\,\mathrm{d}\mu(z) &= \lim_{n\to\infty} \int_\Omega Ts_n(z)\,\mathrm{d}\mu(z) = \lim_{n\to\infty} \int_\Omega \left[\sum_{k=1}^n T(x_k)\chi_{E_k}(z)\right] \mathrm{d}\mu(z) \\ &= \lim_{n\to\infty} \sum_{k=1}^n T(x_k)\mu(E_k). \end{aligned}$$

□

Proposition 1.2.2. *Let X be a Banach space and let $f\colon \Omega \to X$ be a μ-Bochner integrable vector-valued function. Then the inequality*

$$\left\| \int_E f(z)\,\mathrm{d}\mu(z) \right\|_X \leq \int_E \|f(z)\|_X \, \mathrm{d}\mu(z)$$

holds for all $E \in \Sigma$.

Proof. By duality we have

$$\left\| \int_E f(z)\,\mathrm{d}\mu(z) \right\|_X = \sup\left\{ \left| T\left(\int_E f(z)\,\mathrm{d}\mu(z)\right)\right| : T \in X^*, \|T\|_{X^*} \leq 1 \right\}.$$

Now, using Lemma 1.2.1 and the fact that T is bounded, we get that

$$\left| T\left(\int_E f(z)\,\mathrm{d}\mu\right)\right| \leq \int_E |Tf(z)| \, \mathrm{d}\mu(z) \leq \|T\|_{X^*} \int_E \|f(z)\|_X \, \mathrm{d}\mu(z).$$

Then, it directly follows that

$$\left\| \int_E f(z)\,\mathrm{d}\mu(z) \right\|_X \leq \int_E \|f(z)\|_X \, \mathrm{d}\mu(z).$$

□

In this manuscript we usually work with $\Omega = \mathbb{B}_n$ and $\mu = v_\alpha$ defined in (1.1.2). Since we already have defined the Bochner integral, we know that any vector-valued holomorphic function fulfills the mean value property in (1.1.3) and the integration in polar coordinates in (1.1.1), taking into account that the integral is the Bochner integral.

Let $0 < p < \infty$. We already defined $L^1_\mu(\mathbb{B}_n, X)$ and, in general, we can define the so-called *vector-valued Lebesgue spaces* or *Bochner-Lebesgue spaces* $L^p_\mu(\mathbb{B}_n, X)$ as the μ-measurable functions $f\colon \mathbb{B}_n \to X$ such that

$$\int_{\mathbb{B}_n} \|f(z)\|_X^p \, \mathrm{d}\mu(z) < \infty.$$

For $p = \infty$ we consider $L^\infty_\mu(\mathbb{B}_n, X)$ which consist of μ-measurable functions $f\colon \mathbb{B}_n \to X$ such that $\|f\|_X$ is μ-essentially bounded in $\mathbb{B}_n$.

When μ is the counting measure, the Bochner-Lebesgue space is denoted by $\ell^p(X)$, which consist of sequences $\{x_k\}_k$ of X such that

$$\|\{x_k\}_k\|_{\ell^p(X)} = \left(\sum_{k=1}^\infty \|x_k\|_X^p\right)^{1/p} < \infty, \qquad 0 < p < \infty.$$

When $p = \infty$, $\|\{x_k\}_k\|_{\ell^\infty(X)} := \sup_k \|x_k\|_X < \infty$.

Another important example is when $\mu = v_\alpha$, for $\alpha > -1$, where v_α is defined in (1.1.2). So we consider the *weighted vector-valued Lebesgue spaces* $L^p_\alpha(\mathbb{B}_n, X) := L^p_{v_\alpha}(\mathbb{B}_n, X)$ which consist of v_α-measurable functions $f\colon \mathbb{B}_n \to X$ such that

$$\|f\|_{p,\alpha,X} = \|f\|_{L^p_\alpha(\mathbb{B}_n,X)} := \left(\int_{\mathbb{B}_n} \|f(z)\|_X^p \, \mathrm{d}v_\alpha(z)\right)^{1/p} < \infty, \qquad 0 < p < \infty.$$

When $X = \mathbb{C}$ we obtain the classical scalar-valued Lebesgue spaces $L^p_\alpha(\mathbb{B}_n)$ (or simply L^p_α) and the norm is denoted by $\|\cdot\|_{p,\alpha}$. From now on, basically we consider $\Omega = \mathbb{B}_n$ and we will write $x \lesssim y$ when there is a positive constant $C > 0$ such that $x \le Cy$. If both $x \lesssim y$ and $y \lesssim x$, then we write $x \simeq y$.

1.3 The Bergman Metric

It is important to define the Bergman metric in $\mathbb{B}_n$. Before that we need to introduce the automorphisms of $\mathbb{B}_n$, which are also very important in what follows. The characterization of the automorphisms in $\mathbb{B}_n$ is well-known and we refer to [82, Section 1.2] for more details, and also for the rest of the section.

A mapping $F\colon \mathbb{B}_n \to \mathbb{B}_n$ is said to be bi-holomorphic if F is bijective, holomorphic and F^{-1} is also holomorphic. The automorphism group of $\mathbb{B}_n$, denoted by $\mathrm{Aut}(\mathbb{B}_n)$, consists of all bi-holomorphic mappings of $\mathbb{B}_n$. It is clear that $\mathrm{Aut}(\mathbb{B}_n)$ is a group with composition being the groups operation. Traditionally, bi-holomorphic mapping are also called automorphisms. Basically, all the automorphisms consist of unitary transformations of $\mathbb{C}^n$ and involutions. Indeed, the characterization of the unitary transformations of $\mathbb{C}^n$, in [82, Lemma 1.1], is the following: φ of $\mathrm{Aut}(\mathbb{B}_n)$ is a unitary transformation of $\mathbb{C}^n$ if and only if $\varphi(0) = 0$.

On the other hand, the other class of automorphisms consist of symmetries of $\mathbb{B}_n$, called also involutive automorphisms or involutions. Thus, for any $a \in \mathbb{B}_n \setminus \{0\}$ we define

$$\varphi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n, \tag{1.3.1}$$

where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [a]$. It is clear that

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a, \qquad z \in \mathbb{C}^n,$$

and

$$Q_a(z) = z - \frac{\langle z, a\rangle}{|a|^2} a, \qquad z \in \mathbb{C}^n.$$

When $a = 0$, we simply define $\varphi_0(z) = -z$. It is obvious that each φ_a is a holomorphic mapping from $\mathbb{B}_n$ into $\mathbb{C}^n$. Also in [82, Theorem 1.4] we have a complete characterization of automorphisms in $\mathbb{B}_n$, that is, every $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ is of the form

$$\varphi = U\varphi_a = \varphi_b V,$$

where U and V are unitary transformations of $\mathbb{C}^n$, and φ_a and φ_b are involutions. For each $a \in \mathbb{B}_n$ the mapping φ_a satisfies

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}, \qquad z \in \mathbb{B}_n,$$

and

$$\varphi_a \circ \varphi_a(z) = z, \qquad z \in \mathbb{B}_n.$$

In particular, each φ_a is an automorphism of $\mathbb{B}_n$ that interchanges the points 0 and a.

The Bergman distance between two points $z, w \in \mathbb{B}_n$, is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|},$$

where φ_z is the involutive automorphism of $\mathbb{B}_n$ that interchanges 0 and z. A simple calculation shows that $|\varphi_z(w)| = \tanh \beta(z, w)$, for every $z, w \in \mathbb{B}_n$. It is well-known that the Bergman metric is invariant under automorphisms, that is,

$$\beta(\varphi(z), \varphi(w)) = \beta(z, w),$$

for all $z, w \in \mathbb{B}_n$ and $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

For $z \in \mathbb{B}_n$ and $r > 0$ we let $D(z, r)$ denote the Bergman metric ball centered at z and radius r, that is,

$$D(z, r) := \{w \in \mathbb{B}_n \colon \beta(z, w) < r\}.$$

It is also well known that, for any $r > 0$, we have that

$$v(D(z, r)) \simeq (1 - |z|^2)^{n+1}$$

and

$$v_\alpha(D(z, r)) \simeq (1 - |z|^2)^{n+1+\alpha}.$$

1.4 Some Interesting Inequalities

The following result is basic but at the same time very important. We will see that it is used intensively in many situations.

Theorem 1.4.1 ([82, Theorem 1.12]). *For $\alpha > -1$ and $\beta \in \mathbb{R}$ let*

$$I_{\alpha,\beta}(z) := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha \, dv(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+\beta}}, \qquad z \in \mathbb{B}_n.$$

(i) If $\beta = 0$, there exists $C > 0$ such that

$$I_{\alpha,\beta}(z) \le C \log \frac{1}{1 - |z|^2}.$$

(ii) If $\beta > 0$, there exists $C > 0$ such that

$$I_{\alpha,\beta}(z) \le \frac{C}{(1 - |z|^2)^\beta}.$$

(iii) If $\beta < 0$, there exists $C > 0$ such that

$$I_{\alpha,\beta}(z) \le C.$$

The best constants C in the previous inequalities has been recently obtained in [50]. We also have a similar integral estimate with an extra unbounded factor $\beta(z, w)$ which is easy to prove.

Lemma 1.4.2. *Let $\alpha > -1$, $t > 0$ and set*

$$J_{\alpha,t}(z) := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha \beta(z, w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+t}} \, dv(w),$$

then $J_{\alpha,t}(z) \lesssim (1 - |z|^2)^{-t}$, for every $z \in \mathbb{B}_n$.

Proof. Since $\beta(0,z)$ has logarithmic behavior we know that

$$\beta(z,w)=\beta(\varphi_z(z),\varphi_z(w))=\beta(0,\varphi_z(w))\lesssim(1-|\varphi_z(w)|^2)^\gamma=\frac{(1-|z|^2)^\gamma(1-|w|^2)^\gamma}{|1-\langle z,w\rangle|^{2\gamma}}$$

for small $\gamma<0$ that will be determined later. Then, by Theorem 1.4.1 we have that

$$\begin{aligned}J_{\alpha,t}(z)&\lesssim(1-|z|^2)^\gamma\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha+\gamma}}{|1-\langle z,w\rangle|^{n+1+\alpha+t+2\gamma}}\,\mathrm{d}v(w)\\&\lesssim(1-|z|^2)^\gamma(1-|z|^2)^{-(t+\gamma)}=(1-|z|^2)^{-t}\end{aligned}$$

provided that $\alpha+\gamma>-1$ and $\gamma+t>0$. This constraints are fulfilled if we choose γ such that

$$\max\{-t,-(1+\alpha)\}<\gamma<0$$

and the result is proved. □

Another interesting result is the following change of variables formula.

Proposition 1.4.3. *Let $\alpha>-1$ and $f\in L^1_\alpha(\mathbb{B}_n,X)$. Then*

$$\int_{\mathbb{B}_n}f\circ\varphi(z)\,\mathrm{d}v_\alpha(z)=\int_{\mathbb{B}_n}f(z)\frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}}\,\mathrm{d}v_\alpha(z)$$

where $\varphi\in\mathrm{Aut}(\mathbb{B}_n)$ and $a=\varphi(0)$.

Proof. This is proved in the same way as in the scalar case, see [82, Proposition 1.13]. □

Lemma 1.4.4 ([82, Lemma 2.20]). *For each $r>0$ there exists a positive constant C_r such that*

$$C_r^{-1}\le\frac{(1-|a|^2)}{(1-|z|^2)}\le C_r$$

and

$$C_r^{-1}\le\frac{(1-|a|^2)}{|1-\langle a,z\rangle|}\le C_r,$$

for all a and z in $\mathbb{B}_n$ with $\beta(a,z)<r$. Moreover, if r is bounded above, then we may choose C_r to be independent of r.

A sequence $\{a_k\}_k$ of points in $\mathbb{B}_n$ is called a *separated sequence* (in the Bergman metric) if there exists a positive constant $\delta>0$ such that $\beta(a_i,a_j)>\delta$ for any $i\neq j$. We also need the following well known discrete version of Theorem 1.4.1.

Lemma 1.4.5. *Let $\{z_k\}_k$ be a separated sequence in $\mathbb{B}_n$, and let $n<t<s$. Then there exists a positive constant C such that*

$$\sum_{k=1}^\infty\frac{(1-|z_k|^2)^t}{|1-\langle z,z_k\rangle|^s}\le C(1-|z|^2)^{t-s},\qquad z\in\mathbb{B}_n.$$

Lemma 1.4.5 can be deduced from Theorem 1.4.1 after noticing that, if a sequence $\{z_k\}_k$ is separated, then there is a constant $r>0$ such that the Bergman metric balls $D(z_k,r)$ are pairwise disjoints.

1.5 Harmonic and Subharmonic Functions

A *harmonic function* is a continuous function that fulfills the mean value property. It is well known that any holomorphic function in Ω is harmonic in Ω. In particular, if $f \in \mathcal{H}(\mathbb{B}_n, X)$ then (1.1.3) holds.

It is known that we also have the area version of the mean value property

$$f(0) = \frac{1}{v_\alpha(R\mathbb{B}_n)} \int_{R\mathbb{B}_n} f(w)\,\mathrm{d}v_\alpha(w). \tag{1.5.1}$$

for any $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $0 < R < 1$.

A function $f\colon \mathbb{B}_n \to [-\infty, \infty)$ is said to be *upper semi-continuous* if

$$\limsup_{z \to z_0} f(z) \leq f(z_0)$$

for every $z_0 \in \mathbb{B}_n$. An upper semi-continuous function $f \in \mathbb{B}_n \to [-\infty, \infty)$ is said to be *subharmonic* if

$$f(z) \leq \int_{\mathbb{S}_n} f(z + r\zeta)\,\mathrm{d}\sigma(\zeta)$$

for all $z \in \mathbb{B}_n$ and $0 \leq r < 1 - |z|$. Sometimes this property is called the sub-mean value property. It is interesting to notice that if f is subharmonic, we also have

$$f(0) \leq \frac{1}{v_\alpha(R\mathbb{B}_n)} \int_{R\mathbb{B}_n} f(w)\,\mathrm{d}v_\alpha(w), \tag{1.5.2}$$

for any $0 < R < 1$, which is the area version of the sub-mean value property. In our case is much more convenient to work with the area versions of these properties.

The next result is well known [64, Lemma 6.4.1] but, for completeness, we give the proof.

Lemma 1.5.1. *If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then the function $z \mapsto \log \|f(z)\|_X$ is subharmonic.*

Proof. Clearly $\log \|f\|_X$ is upper semi-continuous since it is continuous. Now fix $z \in \mathbb{B}_n$. By Hahn-Banach, there exists $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$ such that

$$\|f(z)\|_X = |\langle f(z), x^* \rangle_X| \,.$$

Recall that $\langle f(z), x^* \rangle_X = x^*(f(z))$ is also an holomorphic function in $\mathbb{C}$. Then, by the scalar-valued case, we have

$$\begin{aligned} \log \|f(z)\|_X &= \log |\langle f(z), x^* \rangle_X| \\ &\leq \int_{\mathbb{S}_n} \log |\langle f(z + r\zeta), x^* \rangle_X| \,\mathrm{d}\sigma(\zeta) \\ &\leq \int_{\mathbb{S}_n} \log \|f(z + r\zeta)\|_X \,\mathrm{d}\sigma(\zeta) \end{aligned}$$

for any $0 \leq r < 1 - |z|$. This proves that $\log \|f\|_X$ is subharmonic. $\square$

From this fact one can deduce the following particular and well known sub-mean value property. (This can be proved also using the scalar-valued case of the sub-mean value property, see [7] for example).

Theorem 1.5.2. *If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then the function $z \mapsto \|f(z)\|_X^p$ is subharmonic, for every $0 < p < \infty$. In particular, if $0 < p < \infty$, $\alpha > -1$ and $r > 0$ then there exists $C > 0$ such that*

$$\|f(z)\|_X^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w),$$

for any $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$.

Proof. Define $u(z) := \log \|f(z)\|_X$, for $z \in \mathbb{B}_n$, and $\varphi(t) := e^{pt}$, for $t \in \mathbb{R}$. Then, clearly,

$$\|f(z)\|_X^p = e^{p \log \|f(z)\|_X} = \varphi \circ u(z),$$

for every $z \in \mathbb{B}_n$. Since u is subharmonic, by previous Lemma 1.5.1, and φ is an increasing convex function, we have using Jensen's inequality that $\|f\|_X^p$ is subharmonic as well. So we will have, by (1.5.2), that

$$\|f(0)\|_X^p \leq \frac{1}{v_\alpha(R\mathbb{B}_n)} \int_{R\mathbb{B}_n} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w),$$

for any $0 < R < 1$. If we take $R := \tanh r < 1$ we have that $D(0,r)$ is a Euclidean ball centered at the origin with Euclidean radius R. Then

$$\|f(0)\|_X^p \leq \frac{1}{v_\alpha(D(0,r))} \int_{D(0,r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w).$$

Pick $z \in \mathbb{B}_n$. If we replace f by $f \circ \varphi_z$ and we change variables according to Proposition 1.4.3 we obtain that

$$\|f(z)\|_X^p \leq \frac{1}{v_\alpha(D(0,r))} \int_{D(z,r)} \|f(w)\|_X^p \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)}} \, \mathrm{d}v_\alpha(w).$$

By applying the equivalences in Lemma 1.4.4 we obtain that there exists $C = C(R, p, \alpha) > 0$ such that

$$\|f(z)\|_X^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w).$$

Note that $v_\alpha(D(z,r)) \simeq (1-|z|^2)^{n+1+\alpha}$. □

1.6 Several Notions of Differentiation

An important concept of differentiation on the unit ball is that of the radial derivative, which is based on the usual partial derivatives of a holomorphic function. Thus for a holomorphic function $f \colon \mathbb{B}_n \to X$ we write

$$Rf(z) := \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z).$$

If the homogeneous expansion of the function f is given by

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

where f_k are homogeneous holomorphic polynomials of degree k with coefficients on X, then

$$Rf(z) = \sum_{k=0}^{\infty} k f_k(z) = \sum_{k=1}^{\infty} k f_k(z).$$

More generally, for any two real parameters α and t with the property that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, we define an invertible operator

$$R^{\alpha,t} \colon \mathcal{H}(\mathbb{B}_n, X) \to \mathcal{H}(\mathbb{B}_n, X)$$

as follows. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of f, then

$$R^{\alpha,t} f(z) := \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\,\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\,\Gamma(n+1+k+\alpha)} f_k(z),$$

where Γ is the classical Gamma function. The inverse of $R^{\alpha,t}$, denoted by $R_{\alpha,t}$ is given by

$$R_{\alpha,t} f(z) := \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+t)\,\Gamma(n+1+k+\alpha)}{\Gamma(n+1+\alpha)\,\Gamma(n+1+k+\alpha+t)} f_k(z).$$

The following result gives an alternative description of these operators.

Proposition 1.6.1. *Let $\alpha > -1$ and $t \geq 0$. Then the operator $R^{\alpha,t}$ is the unique continuous linear operator on $\mathcal{H}(\mathbb{B}_n, X)$ satisfying*

$$R^{\alpha,t}\left(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right) = \frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}$$

for any $w \in \mathbb{B}_n$ and $x \in X$. Similarly, the operator $R_{\alpha,t}$ is the unique continuous linear operator on $\mathcal{H}(\mathbb{B}_n, X)$ satisfying

$$R_{\alpha,t}\left(\frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}\right) = \frac{x}{(1-\langle z,w\rangle)^{n+1+\alpha}}$$

for any $w \in \mathbb{B}_n$ and $x \in X$.

Proof. Just follow the proof given in [82, Proposition 1.14]. □

We also want to recall the notion of (invariant) gradient. Let X be a Banach space and $f \in \mathcal{H}(\mathbb{B}_n, X)$. Then

$$\widetilde{\nabla} f(z) := \nabla(f \circ \varphi_z)(0)$$

and we call $\|\widetilde{\nabla} f(z)\|_{X^n}$ the invariant gradient of f at $z \in \mathbb{B}_n$, and $\|\nabla f(z)\|_{X^n}$ is the holomorphic gradient of f at $z \in \mathbb{B}_n$, where ∇ is the "complex" gradient

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right).$$

1.7 Type and Cotype of a Banach Space

In this final section we define some advanced notations and topics on Banach spaces. As usual, we consider X any complex Banach space.

Let $r_k(t) := \mathrm{sgn}(\sin(2^{\mathrm{k}}\pi \mathrm{t}))$ be the *Rademacher functions*. There are many well known applications of these functions. Related with these functions we have the well-known Khintchine's inequality that can be found in many places, e.g. [78, p.12] or [84]. Let $0 < p < \infty$. There exist positive constants $A_p, B_p > 0$ such that

$$A_p \left(\sum_{k=1}^{N} |x_k|^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{k=1}^{N} r_k(t) x_k \right|^p \mathrm{d}t \right)^{1/p} \leq B_p \left(\sum_{k=1}^{N} |x_k|^2 \right)^{1/2}, \qquad (1.7.1)$$

for any $N \geq 1$ and $x_1, \ldots, x_N \in \mathbb{C}$. A direct consequence is that for any $0 < p, q < \infty$ we have that

$$\left(\int_0^1 \left| \sum_{k=1}^{N} r_k(t) x_k \right|^p \mathrm{d}t \right)^{1/p} \simeq \left(\int_0^1 \left| \sum_{k=1}^{N} r_k(t) x_k \right|^q \mathrm{d}t \right)^{1/q},$$

with the constants only depending on p and q. A result due to Kahane generalizes Khinchine's inequality (or the consequence) to arbitrary Banach spaces and provides estimates between L^p norms of Rademacher means (see [78, p.95] or [46] for the original result).

Theorem 1.7.1 (Khintchine-Kahane Inequality). *Let $0 < p, q < \infty$. Then for any $N \geq 1$ there exists $C_{p,q} > 0$ such that*

$$\left(\int_0^1 \left\| \sum_{k=1}^{N} r_k(t) x_k \right\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} \leq C_{p,q} \left(\int_0^1 \left\| \sum_{k=1}^{N} r_k(t) x_k \right\|_X^q \mathrm{d}t \right)^{\frac{1}{q}}$$

for every Banach space X and $\{x_k\}_{k=1}^N \subset X$.

In general, a Banach space X does not fulfill Khintchine's inequality, but one can introduce a property that allows us to apply similar ideas. A Banach space X has *Rademacher type* s (or simply type s), for $1 \leq s \leq 2$, if there is a constant $C > 0$ such that

$$\left(\int_0^1 \left\| \sum_{k=1}^{N} r_k(t) x_k \right\|_X^2 \mathrm{d}t \right)^{\frac{1}{2}} \leq C \left(\sum_{k=1}^{N} \|x_k\|_X^s \right)^{\frac{1}{s}} \qquad (1.7.2)$$

no matter how we select $N \geq 1$ and a finitely many vectors $\{x_k\}_{k=1}^N \subset X$. We say that X has *Rademacher cotype* s (or simply cotype s), for $2 \leq s < \infty$, if there exists $C > 0$ such that

$$\left(\sum_{k=1}^{N} \|x_k\|_X^s \right)^{\frac{1}{s}} \leq C \left(\int_0^1 \left\| \sum_{k=1}^{N} r_k(t) x_k \right\|_X^2 \mathrm{d}t \right)^{\frac{1}{2}} \qquad (1.7.3)$$

no matter how we select $N \geq 1$ and a finitely many vectors $\{x_k\}_{k=1}^N \subset X$. The reason to put bounds in the values of type and cotype is to avoid trivial concepts since scalars does not satisfy neither (1.7.2) for $s > 2$ nor (1.7.3) for $s < 2$ (see Khintchine's inequality (1.7.1)). These concepts first appear in [53] and have become a basic fact for many situations. Take a look at [78] and [52] for historical background and applications.

Some examples of spaces with known type and cotype are the Lebesgue spaces L^p and the Schatten ideals $\mathcal{S}^p$. The spaces $L^p(\Omega, X)$, $1 \leq p \leq \infty$ are of type $\min(2, p)$ and of cotype $\max(2, p)$, see [78, p.98]. The spaces of p-Schatten class $\mathcal{S}^p$, for $1 \leq p < \infty$, are known to have a finite cotype [74]. In particular, $\mathcal{S}^p$ for $p \geq 2$ is of type 2 and for $p \leq 2$ is of cotype 2. Also we know that if a Banach space X has type s then X^* has cotype r, where $1/s + 1/r = 1$, see [78, p.97]. For example, since $\mathcal{S}^p$ for $p \geq 2$ is of type 2, $(\mathcal{S}^p)^* = \mathcal{S}^{p'}$ is of cotype 2, where $1/p + 1/p' = 1$. There are other known examples of spaces with non trivial type and cotype like the Musielak-Orlicz spaces [47].

Chapter 2

Vector-valued Bergman Spaces

The theory of Bergman spaces has been a central subject of study in complex analysis during many years. It is well known that Bergman spaces are very related to Hardy spaces. In fact, the theory of Hardy spaces was appeared first in the literature and it is a well established theory with many applications. With the emergence of functional analysis, the Bergman spaces become also popular. Stefen Bergman, the forefather of the theory of Bergman spaces, in [15] had developed an elegant theory of Hilbert spaces of analytic functions. His study deals mainly with the case $p = 2$ and is concerned with more general domains in the plane or in higher dimensional complex spaces, even so later on these spaces will be called Bergman spaces. In the next years many authors studied this theory trying to mimic the theory of Hardy spaces. However, it soon became apparent that Bergman spaces are in many aspects much more complicated than their Hardy spaces cousins. The theory of Bergman spaces is well developed and first established for the unit disk $\mathbb{D}$ and for scalar functions, see [36, 37, 42, 83]. Later on, for the unit ball of $\mathbb{C}^n$ you can see [65, 82]. We recommend [82] for more advanced topics on Bergman spaces and more.

In this chapter we present the next step of the theory of Bergman spaces, the theory of vector-valued Bergman spaces in the unit ball of $\mathbb{C}^n$. In Section 2.1 we show the basic properties of vector-valued Bergman spaces which are the similar ones of the scalar case. In Section 2.2 we prove basically the boundedness of the Bergman projection, again the results here present similarities with the scalar case, and which it is well known that it has many applications. Finally, in the last Section 2.3 we develop the new atomic decomposition for vector-valued Bergman spaces, used also in the next chapters. Atomic decomposition for Bergman spaces was initially due to Coifman and Rochberg [24] and after that was a central topic in complex analysis. Our proof here is a modified version from the originial [24] and [82].

If we do not say the contrary, in this chapter we consider X any complex Banach space and $\alpha > -1$ a scalar value.

2.1 Definition and Basic Properties

We begin defining one of the main ingredients in this manuscript. The *weighted vector-valued Bergman spaces* $A^p_\alpha(\mathbb{B}_n, X)$ are defined to be

$$A^p_\alpha(\mathbb{B}_n, X) := \mathcal{H}(\mathbb{B}_n, X) \cap L^p_\alpha(\mathbb{B}_n, X), \qquad 0 < p < \infty.$$

It is easy to see that $A^p_\alpha(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$, for all $1 < p < \infty$.

Recall the sub-mean value property of functions $f \in \mathcal{H}(\mathbb{B}_n, X)$ in Theorem 1.5.2: for any $0 < p < \infty$ and $r > 0$ we have

$$\|f(z)\|_X^p \lesssim \frac{1}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_X^p \, dv_\alpha(w),$$

for every $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$. An immediate consequence is that any vector-valued Bergman function f satisfies the pointwise estimate

$$\|f(z)\|_X \le \frac{C}{(1-|z|^2)^{(n+1+\alpha)/p}} \|f\|_{p,\alpha,X} \,,$$

for some positive constant C independent of the point $z \in \mathbb{B}_n$ and the function f. As in the scalar case, one can get the constant C to be one.

Theorem 2.1.1. *Let $0 < p < \infty$ and $\alpha > -1$. Then*

$$\|f(z)\|_X \le \frac{\|f\|_{p,\alpha,X}}{(1-|z|^2)^{(n+1+\alpha)/p}},$$

for any $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$.

Proof. Let $f \in A^p_\alpha(\mathbb{B}_n, X) \subset \mathcal{H}(\mathbb{B}_n, X)$ and $0 < p < \infty$. By Theorem 1.5.2, $\|f\|_X^p$ is a subharmonic function and using the area version of it in (1.5.2) with $R = 1$, we have that

$$\|f(0)\|_X^p \le \int_{\mathbb{B}_n} \|f(w)\|_X^p \, dv_\alpha(w).$$

This proves the desired result when $z = 0$.

In general, fix $z \in \mathbb{B}_n$ and consider the function

$$F(w) := (f \circ \varphi_z)(w) \frac{(1-|z|^2)^{(n+1+\alpha)/p}}{(1-\langle w, z\rangle)^{2(n+1+\alpha)/p}}, \qquad w \in \mathbb{B}_n.$$

Changing variables according to Proposition 1.4.3, we can see that

$$\|F\|_{p,\alpha,X} = \|f\|_{p,\alpha,X} \,.$$

The desired result then follows from $\|F(0)\|_X \le \|F\|_{p,\alpha,X}$. $\square$

One important consequence is that, for any $z \in \mathbb{B}_n$, the point evaluations

$$\begin{array}{rccc} T_z : & A^p_\alpha(\mathbb{B}_n, X) & \longrightarrow & X \\ & f & \mapsto & f(z) \end{array}$$

are bounded linear operators for all $1 \leq p < \infty$. This allows one to prove that the vector-valued Bergman spaces $A^p_\alpha(\mathbb{B}_n, X)$ are closed subspaces of $L^p_\alpha(\mathbb{B}_n, X)$ and, therefore, they are Banach spaces. In particular, $A^2_\alpha(\mathbb{B}_n, X)$ is a Hilbert space only if X is a Hilbert space as well. We use the following notation

$$\langle f, g \rangle_{\alpha,X} := \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_X \, \mathrm{d}v_\alpha(z). \tag{2.1.1}$$

to represent the "inner product" for $f \in A^2_\alpha(\mathbb{B}_n, X)$ and $g \in A^2_\alpha(\mathbb{B}_n, X^*)$. When we refer to the classical scalar-valued Bergman spaces we will denote $\langle \cdot, \cdot \rangle_\alpha = \langle \cdot, \cdot \rangle_{\alpha,\mathbb{C}}$ for the inner product of L^2_α. Recall that in the scalar Bergman spaces, A^2_α is also a reproducing kernel Hilbert space but in the vector-valued scheme this is not clear, but we still have an integral representation of each vector-valued Bergman function that we still call it, the reproducing formula.

Proposition 2.1.2. *Let $\alpha > -1$ and $f \in A^1_\alpha(\mathbb{B}_n, X)$. Then*

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w),$$

for any $z \in \mathbb{B}_n$.

Proof. Since $f \in A^1_\alpha(\mathbb{B}_n, X) \subset \mathcal{H}(\mathbb{B}_n, X)$, we have, by (1.5.1) with $R = 1$, that

$$f(0) = \int_{\mathbb{B}_n} f(w) \, \mathrm{d}v_\alpha(w).$$

Now fix $z \in \mathbb{B}_n$. If we replace f by $f \circ \varphi_z$ and we change variables according to Proposition 1.4.3 we obtain that

$$f(z) = \int_{\mathbb{B}_n} f \circ \varphi_z(w) \, \mathrm{d}v_\alpha(w) = \int_{\mathbb{B}_n} f(w) \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} \, \mathrm{d}v_\alpha(w).$$

Replacing again $f(w)$ by $f(w)(1 - \langle w, z \rangle)^{n+1+\alpha}$ we get that

$$f(z)(1 - |z|^2)^{n+1+\alpha} = (1 - |z|^2)^{n+1+\alpha} \int_{\mathbb{B}_n} f(w) \frac{(1 - \langle w, z \rangle)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} \, \mathrm{d}v_\alpha(w).$$

Then, simplifying we arrive at the desired reproducing formula. □

We have some important consequences which are the following (recall the definitions of $R^{\alpha,t}$ and $R_{\alpha,t}$ given in Section 1.6).

Proposition 2.1.3. *Let $\alpha > -1$, $f \in A^1_\alpha(\mathbb{B}_n, X)$ and $t > 0$. Then*

$$R^{\alpha,t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha+t}} \, \mathrm{d}v_\alpha(w)$$

and

$$R_{\alpha,t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}} \, \mathrm{d}v_{\alpha+t}(w),$$

for any $z \in \mathbb{B}_n$.

Proof. Taking the reproducing formula in Proposition 2.1.2 and applying $R^{\alpha,t}$ under the integral sign we get the first identity using Proposition 1.6.1. The case $R_{\alpha,t}$ is the same proof but applying the reproducing formula in Proposition 2.1.2 with $\alpha + t$ instead of α. □

The following density property is well known and extremely useful in many situations.

Lemma 2.1.4. *Let $\alpha > -1$. The vector-valued holomorphic polynomials, $\mathcal{P}(\mathbb{B}_n, X)$, are dense in $A^p_\alpha(\mathbb{B}_n, X)$, for any $1 \le p < \infty$.*

Proof. Given a function $f \in A^p_\alpha(\mathbb{B}_n, X)$, let $f_\rho(z) = f(\rho z)$ be its dilations, where $0 < \rho < 1$. Each function f_ρ is analytic in a larger disk, so it can be approximated uniformly in $\mathbb{B}_n$ by polynomials, the partial sums of its Taylor series. Thus it will be enough to prove that f can be approximated in $A^p_\alpha(\mathbb{B}_n, X)$ norm by its dilations, i.e., $\|f - f_\rho\|_{p,\alpha,X} \to 0$ as $\rho \to 1$. First, we recall that the integral means

$$M_p(r, f) := \left(\int_{\mathbb{S}_n} \|f(r\zeta)\|^p_X \, \mathrm{d}\sigma(\zeta) \right)^{1/p}, \qquad 0 \le r < 1,$$

are increasing with r, see [82, Corollary 4.21], and observe that $M_p(r, f_\rho) = M_p(r\rho, f)$. Therefore,

$$M^p_p(r, f - f_\rho) \le 2^p (M^p_p(r, f) + M^p_p(r, f_\rho)) \le 2^{p+1} M^p_p(r, f).$$

But the hypothesis that $f \in A^p_\alpha(\mathbb{B}_n, X)$ is equivalent to saying that $M^p_p(r, f)$ is integrable over the interval $[0, 1)$ with respect the measure $2nc_\alpha r^{2n-1}(1 - r^2)^\alpha \mathrm{d}r$, see (1.1.1), and it is clear that $f_\rho(z) \to f(z)$ uniformly on compact subsets of $\mathbb{B}_n$ as $\rho \to 1$, which implies that $M^p_p(r, f - f_\rho) \to 0$ for each $r \in [0, 1)$. Thus by (1.1.1) and the Lebesgue dominated convergence theorem, we may conclude that

$$\|f - f_\rho\|^p_{p,\alpha,X} = 2nc_\alpha \int_0^1 M^p_p(r, f - f_\rho) r^{2n-1} (1 - r^2)^\alpha \, \mathrm{d}r \longrightarrow 0,$$

as $\rho \to 1$, which completes the proof. □

Lemma 2.1.5. *Let $\alpha > -1$ and suppose $t > 0$. Then*

$$\int_{\mathbb{B}_n} f(z) \cdot \overline{g(z)} \, \mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} R^{\alpha,t} f(z) \cdot \overline{g(z)} \, \mathrm{d}v_{\alpha+t}(z),$$

for every $f \in A^1_\alpha(\mathbb{B}_n, X)$ and $g \in A^1_\alpha$.

Proof. Let $f \in \mathcal{P}(\mathbb{B}_n, X)$ and $g \in \mathcal{P}$ be polynomials. Then, by Proposition 2.1.3, Fubini's theorem and the integral representation in Proposition 2.1.2 we have that

$$\begin{aligned}\int_{\mathbb{B}_n} R^{\alpha,t} f(z) \cdot \overline{g(z)}\, \mathrm{d}v_{\alpha+t}(z) &= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{f(w)\, \mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}} \right) \cdot \overline{g(z)}\, \mathrm{d}v_{\alpha+t}(z) \\ &= \int_{\mathbb{B}_n} f(w) \cdot \int_{\mathbb{B}_n} \frac{\overline{g(z)}\, \mathrm{d}v_{\alpha+t}(z)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}\, \mathrm{d}v_\alpha(w) \\ &= \int_{\mathbb{B}_n} f(w) \cdot \overline{g(w)}\, \mathrm{d}v_\alpha(w).\end{aligned}$$

Since f and g are polynomials we can apply Theorem 1.4.1 to see that the assumption of Fubini's theorem is fulfilled. Therefore, by density and Lemma 2.1.4, we obtain the general result. □

Let $w \in \mathbb{B}_n$. Recall the scalar-valued Bergman reproducing kernels are defined as

$$K_w(z) := \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \qquad z \in \mathbb{B}_n.$$

It is not difficult to see, with the help of Theorem 1.4.1 and Lemma 1.4.4, that

$$\|K_w\|_{p,\alpha} \simeq (1-|w|^2)^{-\frac{(p-1)}{p}(n+1+\alpha)}, \qquad 0 < p < \infty.$$

So, for $0 < p < \infty$, we recall also the scalar-valued p-normalized reproducing kernels

$$k_{p,w}(z) := \frac{K_w(z)}{\|K_w\|_{p,\alpha}}, \qquad z \in \mathbb{B}_n.$$

Since we are working on vector-valued setting we will define the vector-valued reproducing kernels on $x \in X \setminus \{0\}$ by $K_w^x(z) = xK_w(z)$ and the vector-valued p-normalized reproducing kernels on $x \in X \setminus \{0\}$ as $k_{p,w}^x(z) = x/\|x\|_X\, k_{p,w}(z)$, for every $z \in \mathbb{B}_n$.

2.2 Projections and Duality

For $\alpha > -1$, the Bergman projection operator P_α, is the integral type operator defined by

$$P_\alpha f(z) := \int_{\mathbb{B}_n} \frac{f(w)\, \mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \tag{2.2.1}$$

for any $f \in L^1_\alpha(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$. It is easy to see that $P_\alpha f \in \mathcal{H}(\mathbb{B}_n, X)$, for every $f \in L^1_\alpha(\mathbb{B}_n, X)$. Indeed, Lemma 1.2.1 implies that $\langle P_\alpha f, x^*\rangle_X = P_\alpha(\langle f, x^*\rangle_X)$, for any $f \in L^1_\alpha(\mathbb{B}_n, X)$ and $x^* \in X^*$. Since P_α acting on scalar-valued functions is holomorphic, we have that P_α is weakly holomorphic and then vector-valued holomorphic. Moreover, it is easy to see by Fubini's theorem and Theorem 1.4.1 that $P_\gamma \colon L^1_\alpha(\mathbb{B}_n, X) \to A^1_\alpha(\mathbb{B}_n, X)$ is a bounded linear operator, for all $\gamma > \alpha$. We have a more general result on the boundedness of integral type operators acting on $L^p_\alpha(\mathbb{B}_n, X)$.

Theorem 2.2.1. *Let $1 \le p < \infty$, $\alpha, \gamma > -1$ and β a real parameter. If*

$$-p\beta < \alpha + 1 < p(\gamma + 1) \tag{2.2.2}$$

then the operator defined by

$$Tf(z) = (1-|z|^2)^{\beta} \int_{\mathbb{B}_n} \frac{f(w)\, \mathrm{d}v_\gamma(w)}{(1-\langle z, w\rangle)^{n+1+\gamma+\beta}}, \qquad z \in \mathbb{B}_n$$

is bounded on $L^p_\alpha(\mathbb{B}_n, X)$.

Proof. Let $f \in L^p_\alpha(\mathbb{B}_n, X)$. If $p = 1$ it directly follows by Tonelli's theorem and Theorem 1.4.1. Indeed,

$$\begin{aligned}
\int_{\mathbb{B}_n} \|Tf(z)\|_X \, \mathrm{d}v_\alpha(z) &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|f(w)\|_X \frac{(1-|z|^2)^{\alpha+\beta}(1-|w|^2)^{\gamma}}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \, \mathrm{d}v(w)\, \mathrm{d}v(z) \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X (1-|w|^2)^{\gamma} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha+\beta}\, \mathrm{d}v(z)}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \right) \mathrm{d}v(w) \\
&\lesssim \int_{\mathbb{B}_n} \|f(w)\|_X (1-|w|^2)^{\gamma}(1-|w|^2)^{\alpha-\gamma} \, \mathrm{d}v(w) \lesssim \|f\|_{1,\alpha,X},
\end{aligned}$$

taking into account that $\gamma - \alpha > 0$ and $\alpha + \beta > -1$ which is true by the hypothesis (2.2.2).

Now suppose $1 < p < \infty$ and let $\varepsilon > 0$ that will be specified later. By Proposition 1.2.2, we have that

$$\begin{aligned}
\|Tf(z)\|_X^p &\lesssim (1-|z|^2)^{p\beta} \left(\int_{\mathbb{B}_n} \|f(w)\|_X \frac{(1-|w|^2)^{\gamma}}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \, \mathrm{d}v(w) \right)^p \\
&= (1-|z|^2)^{p\beta} \left(\int_{\mathbb{B}_n} \|f(w)\|_X \frac{(1-|w|^2)^{\frac{\alpha+\varepsilon}{p}+\gamma-\frac{\alpha+\varepsilon}{p}}}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \, \mathrm{d}v(w) \right)^p.
\end{aligned}$$

The last integral over p, using Hölder's inequality, is less or equal than

$$\left(\int_{\mathbb{B}_n} \|f(w)\|_X^p \frac{(1-|w|^2)^{\alpha+\varepsilon}\, \mathrm{d}v(w)}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \right) \left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{(\gamma p-\alpha-\varepsilon)/(p-1)}\, \mathrm{d}v(w)}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \right)^{p-1}.$$

Then, if

$$\frac{\gamma p - \alpha - \varepsilon}{p-1} > -1 \tag{2.2.3}$$

and

$$\gamma + \beta - \frac{\gamma p - \alpha - \varepsilon}{p-1} > 0, \tag{2.2.4}$$

by Theorem 1.4.1, we have that

$$\left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{(\gamma p-\alpha-\varepsilon)/(p-1)}\, \mathrm{d}v(w)}{|1-\langle z, w\rangle|^{n+1+\gamma+\beta}} \right)^{p-1} \lesssim (1-|z|^2)^{\gamma-\beta(p-1)-\alpha-\varepsilon}.$$

So, we get

$$\|Tf(z)\|_X^p \lesssim (1-|z|^2)^{\gamma+\beta-\alpha-\varepsilon}\left(\int_{\mathbb{B}_n}\|f(w)\|_X^p \frac{(1-|w|^2)^{\alpha+\varepsilon}\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\gamma+\beta}}\right).$$

Finally, applying Tonelli's theorem and Theorem 1.4.1 again, if

$$\gamma+\beta-\varepsilon > -1, \tag{2.2.5}$$

we obtain that

$$\begin{aligned}\int_{\mathbb{B}_n}\|Tf(z)\|_X^p\,\mathrm{d}v_\alpha(z) &\lesssim \int_{\mathbb{B}_n}\|f(w)\|_X^p\,(1-|w|^2)^{\varepsilon}\left(\int_{\mathbb{B}_n}\frac{\mathrm{d}v_{\gamma+\beta-\varepsilon}(z)}{|1-\langle z,w\rangle|^{n+1+\gamma+\beta}}\right)\mathrm{d}v_\alpha(w)\\ &\lesssim \int_{\mathbb{B}_n}\|f(w)\|_X^p\,(1-|w|^2)^{\varepsilon}(1-|w|^2)^{-\varepsilon}\,\mathrm{d}v_\alpha(w) \lesssim \|f\|_{p,\alpha,X}^p\,.\end{aligned}$$

It only remains to show that (2.2.3), (2.2.4) and (2.2.5) are fulfilled. For this we need to take $\varepsilon > 0$ such that

$$\max\{0,\gamma-\alpha-\beta(p-1)\} < \varepsilon < \min\{\gamma+\beta+1, p(\gamma+1)-(\alpha+1)\}$$

which is clearly possible by the hypothesis (2.2.2) and we are done. □

As an immediate consequence, we have the following result on the boundedness of the Bergman projection on $L^p_\alpha(\mathbb{B}_n, X)$.

Theorem 2.2.2. *Let $1 \le p < \infty$ and $\alpha,\gamma > -1$. If $p(\gamma+1) > \alpha+1$, then the Bergman projection $P_\gamma\colon L^p_\alpha(\mathbb{B}_n, X) \to A^p_\alpha(\mathbb{B}_n, X)$ defined in (2.2.1) is a bounded linear operator.*

Proof. It is a direct consequence of Theorem 2.2.1 where $\beta = 0$. □

Proposition 2.2.3. *Let $\alpha > -1$. For any $1 < p < \infty$, if p' is the conjugate exponent of p, then*

$$\langle P_\alpha f, g\rangle_{\alpha,X} = \langle f, P_\alpha g\rangle_{\alpha,X},$$

for every $f \in L^p_\alpha(\mathbb{B}_n, X)$ and $g \in L^{p'}_\alpha(\mathbb{B}_n, X^*)$.

Proof. Let $f \in L^p_\alpha(\mathbb{B}_n, X)$ and $g \in L^{p'}_\alpha(\mathbb{B}_n, X^*)$. Then, by Lemma 1.2.1 and Fubini's theorem, we have that

$$\begin{aligned}\langle P_\alpha f, g\rangle_{\alpha,X} &= \int_{\mathbb{B}_n}\langle P_\alpha f(z), g(z)\rangle_X\,\mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} g(z)(P_\alpha f(z))\,\mathrm{d}v_\alpha(z)\\ &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{g(z)(f(w))}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,\mathrm{d}v_\alpha(w)\,\mathrm{d}v_\alpha(z)\\ &= \int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\frac{g(z)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,\mathrm{d}v_\alpha(z)\right)(f(w))\,\mathrm{d}v_\alpha(w)\\ &= \int_{\mathbb{B}_n}(P_\alpha g(w))\,(f(w))\,\mathrm{d}v_\alpha(w) = \int_{\mathbb{B}_n}\langle f(w), P_\alpha g(w)\rangle_X\,\mathrm{d}v_\alpha(w)\\ &= \langle f, P_\alpha g\rangle_{\alpha,X}.\end{aligned}$$

Note that the application of Fubini's theorem is correct because of Hölder's inequality and the boundedness of the scalar positive Bergman projection P_α^+. □

Observe that Theorem 2.2.2 and Proposition 2.1.2 show that P_α is a bounded projection of $L^p_\alpha(\mathbb{B}_n, X)$ onto $A^p_\alpha(\mathbb{B}_n, X)$ for any $1 < p < \infty$ (that is, $P_\alpha \colon L^p_\alpha(\mathbb{B}_n, X) \to A^p_\alpha(\mathbb{B}_n, X)$ is a bounded linear operator such that $P_\alpha \circ P_\alpha = P_\alpha$ and $P_\alpha L^p_\alpha(\mathbb{B}_n, X) = A^p_\alpha(\mathbb{B}_n, X)$). As a consequence of that fact we will obtain results on duality of vector-valued Bergman spaces.

It is well-known that, for any $1 < p < \infty$, the dual space of $L^p_\alpha(\mathbb{B}_n, X)$ can be identified with $L^{p'}_\alpha(\mathbb{B}_n, X^*)$, where p' is the conjugate exponent of p, with some restriction on the Banach space X, that is, when X^* has the Radon-Nikodym property respect to the measure v_α, see [35, p. 98]. This condition is satisfied if, for example, X^* is separable [35, p. 79]. Fortunately, for vector-valued Bergman spaces, we have the same duality structure but without any restriction on the Banach space X. The case of unit disk is proved in [7, Theorem 3.9] and the same proof works for the unit ball $\mathbb{B}_n$.

Theorem 2.2.4. *Let $1 < p < \infty$ and $\alpha > -1$. If p' be the conjugate exponent of p, then $(A^p_\alpha(\mathbb{B}_n, X))^*$ can be identified with $A^{p'}_\alpha(\mathbb{B}_n, X^*)$ with equivalent norms under the integral pairing defined by (2.1.1).*

2.3 Atomic Decomposition

In this section we show that every function in the Bergman space $A^p_\alpha(\mathbb{B}_n, X)$ can be decomposed into a series of very nice functions (called atoms). First of all, we show some well-known preliminary results which are classical.

Lemma 2.3.1 ([82, Lemma 2.27]). *For any $R > 0$ and any real b there exists a constant $C > 0$ such that*

$$\left| \frac{(1 - \langle z, u\rangle)^b}{(1 - \langle z, v\rangle)^b} - 1 \right| \le C\beta(u, v)$$

for all $z, u, v \in \mathbb{B}_n$ with $\beta(u, v) \le R$.

Recall that for $r > 0$ and $z \in \mathbb{B}_n$ the set

$$D(z, r) = \{w \in \mathbb{B}_n \colon \beta(z, w) < r\}$$

is a Bergman metric ball at z.

Theorem 2.3.2 ([82, Theorem 2.23]). *There exists a positive integer N such that for any $0 < r \le 1$ we can find a sequence $\{a_k\}_k$ in $\mathbb{B}_n$ with the following properties:*

(i) $\mathbb{B}_n = \cup_k D(a_k, r)$.

(ii) The sets $D(a_k, r/4)$ are mutually disjoint.

(iii) Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, R)$, for any $R > r/4$.

In the remainder of this section we fix a sequence $\{a_k\}_k$ chosen according to Theorem 2.3.2. We are going to call r the separation constant for the sequence $\{a_k\}_k$, and we are going to call $\{a_k\}_k$ an r-lattice in the Bergman metric.

Lemma 2.3.3 ([82, Lemma 2.28]). *For each $k \geq 1$ there exists a Borel set D_k satisfying the following conditions:*

(a) $D(a_k, r/4) \subset D_k \subset D(a_k, r)$ *for every* k.

(b) $D_k \cap D_j = \emptyset$ *for* $k \neq j$.

(c) $\mathbb{B}_n = \bigcup_k D_k$.

We need to further partition the sets $\{D_k\}$ in Lemma 2.3.3. We let η denote a positive radius that is much smaller than the separation constant r, in the sense that the quotient η/r is small. We fix a finite sequence $\{z_1, \ldots, z_J\}$ in $D(0,r)$, depending on η, such that $\{D(z_j, \eta)\}_j$ cover $D(0,r)$ and that $\{D(z_j, \eta/4)\}_j$ are disjoint. We then enlarge each set $D(z_j, \eta/4) \cap D(0,r)$ to a Borel set E_j in such a way that $E_j \subset D(z_j, \eta)$ and that

$$D(0,r) = \bigcup_{j=1}^{J} E_j.$$

is a disjoint union (see proof of [82, Lemma 2.28]).

For $k \geq 1$ and $1 \leq j \leq J$ we define $a_{kj} := \varphi_{a_k}(z_j)$ and

$$D_{kj} := D_k \bigcap \varphi_{a_k}(E_j).$$

It is clear that $a_{kj} \in D(a_k, r)$ for all $k \geq 1$ and $1 \leq j \leq J$. Since

$$D_k = \bigcup_{j=1}^{J} D_{kj}$$

is a disjoint union for every k, we obtain a disjoint decomposition

$$\mathbb{B}_n = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{J} D_{kj} \tag{2.3.1}$$

of $\mathbb{B}_n$.

We also fix a real parameter $b > n$ and let $\beta = b - (n+1)$ (or, equivalently, we fix $\beta > -1$ and let $b = \beta + n + 1$). We define an operator S on $\mathcal{H}(\mathbb{B}_n, X)$ as follows

$$Sf(z) := \sum_{k=1}^{\infty} \sum_{j=1}^{J} \frac{v_\beta(D_{kj}) \widehat{f}_\beta(a_{kj})}{(1 - \langle z, a_{kj} \rangle)^b}$$

where

$$\widehat{f}_\beta(a_{kj}) = \frac{1}{v_\beta(D_{kj})} \int_{D_{kj}} f(w) \, \mathrm{d}v_\beta(w)$$

is the averaging function of f in the disk D_{kj}.

The following lemma is the key for the atomic decomposition for vector-valued Bergman spaces.

Lemma 2.3.4. *For any $p > 0$ and $\alpha > -1$ there exists a constant $C > 0$, independent of the separation constants r and η, such that*

$$\|f(z) - Sf(z)\|_X \leq C\eta \sum_{k=1}^{\infty} \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{|1-\langle z, a_k\rangle|^b} \left(\int_{D(a_k,2r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w) \right)^{\frac{1}{p}}$$

for every $r \geq 1$, $z \in \mathbb{B}_n$, and $f \in \mathcal{H}(\mathbb{B}_n, X)$.

Proof. Without loss of generality we may assume that $f \in A^1_\beta(\mathbb{B}_n, X)$. Then, by Proposition 2.1.2,

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)\,\mathrm{d}v_\beta(w)}{(1-\langle z, w\rangle)^b}, \qquad z \in \mathbb{B}_n.$$

Using the partition $\{D_{kj}\}_{k,j}$ of $\mathbb{B}_n$ in (2.3.1), we write

$$f(z) - Sf(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \int_{D_{kj}} f(w) \left(\frac{1}{(1-\langle z, w\rangle)^b} - \frac{1}{(1-\langle z, a_{kj}\rangle)^b} \right) \mathrm{d}v_\beta(w).$$

Therefore, by Proposition 1.2.2, we have that

$$\|f(z) - Sf(z)\|_X \leq \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{I_{kj}}{|1-\langle z, a_{kj}\rangle|^b}$$

where

$$I_{kj} := \int_{D_{kj}} \|f(w)\|_X \left| \frac{(1-\langle z, a_{kj}\rangle)^b}{(1-\langle z, w\rangle)^b} - 1 \right| \mathrm{d}v_\beta(w).$$

Now, Lemma 2.3.1 and the fact that for each $w \in D_{kj}$ the quantity $1-|w|^2$ is comparable to $1-|a_k|^2$, see Lemma 1.4.4, shows that

$$I_{kj} \lesssim \eta(1-|a_k|^2)^\beta \int_{D_{kj}} \|f(w)\|_X \, \mathrm{d}v(w).$$

For each $w \in D_{kj}$ we also have $D(w, r) \subset D(a_k, 2r)$. Then, by Theorem 1.5.2, we obtain that

$$\begin{aligned} \|f(w)\|_X &\lesssim \frac{1}{(1-|w|^2)^{(n+1+\alpha)/p}} \left(\int_{D(w,r)} \|f(u)\|_X^p \, \mathrm{d}v_\alpha(u) \right)^{\frac{1}{p}} \\ &\lesssim \frac{1}{(1-|a_k|^2)^{(n+1+\alpha)/p}} \left(\int_{D(a_k,2r)} \|f(u)\|_X^p \, \mathrm{d}v_\alpha(u) \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that

$$I_{kj} \lesssim \eta(1-|a_k|)^{\beta-(n+1+\alpha)/p} v(D_{kj}) \left(\int_{D(a_k,2r)} \|f(u)\|_X^p \, \mathrm{d}v_\alpha(u) \right)^{\frac{1}{p}}.$$

Since

$$\sum_{j=1}^{J} v(D_{kj}) = v(D_k) \leq v(D(a_k, r)) \lesssim (1 - |a_k|^2)^{n+1}$$

we have that

$$\begin{aligned}\sum_{j=1}^{J} I_{kj} &\lesssim \eta (1 - |a_k|)^{\beta - (n+1+\alpha)/p + (n+1)} \left(\int_{D(a_k,2r)} \|f(u)\|_X^p \, dv_\alpha(u) \right)^{\frac{1}{p}} \\ &= \eta (1 - |a_k|)^{b - (n+1+\alpha)/p} \left(\int_{D(a_k,2r)} \|f(u)\|_X^p \, dv_\alpha(u) \right)^{\frac{1}{p}}.\end{aligned}$$

Finally, by Lemma 2.3.1, we know that $|1 - \langle z, a_{kj} \rangle|$ is comparable to $|1 - \langle z, a_k \rangle|$ for all $z \in \mathbb{B}_n$, then

$$\begin{aligned}\|f(z) - Sf(z)\|_X &\lesssim \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{J} I_{kj}}{|1 - \langle z, a_k \rangle|^b} \\ &\lesssim \eta \sum_{k=1}^{\infty} \frac{(1 - |a_k|)^{b - (n+1+\alpha)/p}}{|1 - \langle z, a_k \rangle|^b} \left(\int_{D(a_k,2r)} \|f(u)\|_X^p \, dv_\alpha(u) \right)^{\frac{1}{p}}.\end{aligned}$$

Hence, the proof of the lemma is complete. □

We are in the situation to prove the atomic decomposition of the vector-valued Bergman functions.

Theorem 2.3.5. *Suppose $p > 0$, $\alpha > -1$ and let b be a parameter such that*

$$b > n \max\left(1, \frac{1}{p}\right) + \frac{\alpha + 1}{p}.$$

Then we have that

(i) For any separated sequence $\{a_k\}_k \subset \mathbb{B}_n$ and $\{\lambda_k\}_k \in \ell^p(X)$ the function

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1 - |a_k|^2)^{b - (n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}$$

belongs to $A^p_\alpha(\mathbb{B}_n, X)$ and $\|f\|_{p,\alpha,X} \lesssim \|\{\lambda_k\}_k\|_{\ell^p(X)}$.

(ii) If $f \in A^p_\alpha(\mathbb{B}_n, X)$ then exists a r-lattice $\{a_k\}_k \subset \mathbb{B}_n$ and a sequence $\{\lambda_k\}_k \in \ell^p(X)$ such that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1 - |a_k|^2)^{b - (n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}$$

holds and $\|\{\lambda_k\}_k\|_{\ell^p(X)} \lesssim \|f\|_{p,\alpha,X}$.

Proof. (i) It is easy to see that the series converges uniformly on compact subsets of $\mathbb{B}_n$ and, therefore, f defines an analytic function on $\mathbb{B}_n$. We set $t := b-(n+1+\alpha)/p$.

If $0 < p \leq 1$, it is easy to see, using Theorem 1.4.1, that

$$\begin{aligned}\|f\|_{p,\alpha,X}^p &= \int_{\mathbb{B}_n} \left\| \sum_{k=1}^{\infty} \lambda_k \frac{(1-|a_k|^2)^t}{(1-\langle z, a_k\rangle)^b} \right\|_X^p \,\mathrm{d}v_\alpha(z) \\ &\lesssim \sum_{k=1}^{\infty} \|\lambda_k\|_X^p \,(1-|a_k|^2)^{tp} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha}{|1-\langle z, a_k\rangle|^{bp}} \,\mathrm{d}v(z) \\ &\lesssim \sum_{k=1}^{\infty} \|\lambda_k\|_X^p \,\frac{(1-|a_k|^2)^{tp}}{(1-|a_k|^2)^{bp-(n+1+\alpha)}} = \|\{\lambda_k\}_k\|_{\ell^p(X)}^p .\end{aligned}$$

Now, let $p > 1$ and let $\varepsilon > 0$ be a parameter that will be specified later. Then, by Hölder's inequality, we have that

$$\begin{aligned}\|f\|_{p,\alpha,X}^p &= \int_{\mathbb{B}_n} \left\| \sum_{k=1}^{\infty} \lambda_k \frac{(1-|a_k|^2)^t}{(1-\langle z, a_k\rangle)^b} \right\|_X^p \,\mathrm{d}v_\alpha(z) \\ &\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|\lambda_k\|_X^p \,\frac{(1-|a_k|^2)^{t-\varepsilon}}{|1-\langle z, a_k\rangle|^b} \right) \left(\sum_{k=1}^{\infty} \frac{(1-|a_k|^2)^{t+\varepsilon/(p-1)}}{|1-\langle z, a_k\rangle|^b} \right)^{p-1} \mathrm{d}v_\alpha(z).\end{aligned}$$

Suppose that $n < t + \varepsilon/(p-1) < b$. By Lemma 1.4.5, we obtain that

$$\left(\sum_{k=1}^{\infty} \frac{(1-|a_k|^2)^{t+\varepsilon/(p-1)}}{|1-\langle z, a_k\rangle|^b} \right)^{p-1} \lesssim (1-|z|^2)^{t(p-1)+\varepsilon-b(p-1)} = (1-|z|^2)^{b-(n+1+\alpha)-t+\varepsilon}.$$

Then

$$\|f\|_{p,\alpha,X}^p \lesssim \sum_{k=1}^{\infty} \|\lambda_k\|_X^p \,(1-|a_k|^2)^{t-\varepsilon} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{b-(n+1)-t+\varepsilon}}{|1-\langle z, a_k\rangle|^b} \,\mathrm{d}v(z).$$

If $b-(n+1)-t+\varepsilon > -1$ and $t-\varepsilon > 0$, then we can apply Theorem 1.4.1 obtaining that the last integral is

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{b-(n+1)-t+\varepsilon}}{|1-\langle z, a_k\rangle|^b} \,\mathrm{d}v(z) \lesssim (1-|a_k|^2)^{\varepsilon-t}$$

and therefore

$$\|f\|_{p,\alpha,X}^p \lesssim \sum_{k=1}^{\infty} \|\lambda_k\|_X^p .$$

It is straightforward to see that the conditions on ε can be fulfilled for some $\varepsilon > 0$. That is, we choose $\varepsilon > 0$ such that

$$\max\left(t-pt', n - \frac{n+1+\alpha}{p}\right) < \varepsilon < \min\left(n+1+\alpha - \frac{n+1+\alpha}{p}, t\right),$$

where $t' := b - n - (1+\alpha)/p$.

(ii) Fix a r-lattice $\{a_k\}_k \subset \mathbb{B}_n$ and let $\{D_{kj}\}_{kj}$ be the finer partition. By Lemma 2.3.4 and the first part of this proof (i) with

$$\lambda_k = \left(\int_{D(a_k,2r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w) \right)^{\frac{1}{p}},$$

we have that there exists $C > 0$ (not depending on r or η) such that

$$\int_{\mathbb{B}_n} \|f(z) - Sf(z)\|_X^p \, \mathrm{d}v_\alpha(z) \le C\eta \sum_{k=1}^{\infty} \int_{D(a_k,2r)} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w).$$

Since each $w \in \mathbb{B}_n$ can be at most N of the sets $D(a_k, 2r)$, see Theorem 2.3.2, we have that

$$\int_{\mathbb{B}_n} \|f(z) - Sf(z)\|_X^p \, \mathrm{d}v_\alpha(z) \le CN\eta \int_{\mathbb{B}_n} \|f(w)\|_X^p \, \mathrm{d}v_\alpha(w).$$

If we take η small enough so that $CN\eta < 1$ we obtain that $I - S$ is a bounded operator on $A^p_\alpha(\mathbb{B}_n, X)$ with norm less than 1, where I is the identity operator on $A^p_\alpha(\mathbb{B}_n, X)$. In that case, it follows from standard functional analysis that the operator S is invertible on $A^p_\alpha(\mathbb{B}_n, X)$. Therefore, every $f \in A^p_\alpha(\mathbb{B}_n, X)$ admits a representation

$$f(z) = \sum_{k,j} \lambda_{kj} \frac{(1 - |a_{kj}|^2)^t}{(1 - \langle z, a_{kj} \rangle)^b},$$

where

$$\lambda_{kj} = \frac{v_\beta(D_{kj}) \widehat{g}_\beta(a_{kj})}{(1 - |a_{kj}|^2)^t},$$

$g = S^{-1} f$ and $\beta = b - (n+1)$. We know that

$$v_\beta(D_{kj}) \le v_\beta(D_k) \simeq (1 - |a_k|^2)^{n+1+\beta} = (1 - |a_k|^2)^b.$$

Since $1 - |a_{kj}|^2$ is comparable to $1 - |a_k|^2$ by Lemma 1.4.4, we have that

$$\sum_{k,j} \|\lambda_{k,j}\|_X^p \lesssim \sum_{k,j} (1 - |a_k|^2)^{n+1+\alpha} \|\widehat{g}_\beta(a_{kj})\|_X^p .$$

With a similar ideas than the proof of Lemma 2.3.4 we have that

$$\begin{aligned}
\|\widehat{g}_\beta(a_{kj})\|_X^p &= \left\| \frac{1}{v_\beta(D_{kj})} \int_{D_{kj}} g(w) \, \mathrm{d}v_\beta(w) \right\|_X^p \\
&\le \left(\frac{1}{v_\beta(D_{kj})} \int_{D_{kj}} \|g(w)\|_X \, \mathrm{d}v_\beta(w) \right)^p \\
&\lesssim \left(\frac{(1 - |a_k|^2)^{-(n+1+\alpha)/p}}{v_\beta(D_{kj})} \int_{D_{kj}} \left(\int_{D(a_k,2r)} \|g(u)\|_X^p \, \mathrm{d}v_\alpha(u) \right)^{1/p} \mathrm{d}v_\beta(w) \right)^p \\
&= (1 - |a_k|^2)^{-(n+1+\alpha)} \int_{D(a_k,2r)} \|g(u)\|_X^p \, \mathrm{d}v_\alpha(u).
\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k,j} \|\lambda_{k,j}\|_X^p &\lesssim J \sum_{k=1}^{\infty} \int_{D(a_k,2r)} \|g(u)\|_X^p \,\mathrm{d}v_\alpha(u) \leq JN \int_{\mathbb{B}_n} \|g(u)\|_X^p \,\mathrm{d}v_\alpha(u) \\ &\leq JN \left\|S^{-1}\right\|^p \int_{\mathbb{B}_n} \|f(u)\|_X^p \,\mathrm{d}v_\alpha(u).\end{aligned}$$

Hence the proof of the theorem is complete. $\square$

Chapter 3

Vector-valued Bloch Type Spaces

It is well known that Bloch type spaces are very related with Bergman spaces. If we do not say the contrary, in this chapter, we fix X any complex Banach space. Usually, we also denote $\alpha > -1$ a real parameter. Roughly speaking, the Bloch space can be thought of as the limit case of the Bergman spaces $A^p_\alpha(\mathbb{B}_n, X)$ as $p \to \infty$. In particular, we will see that the Bloch space can be naturally identified with the dual space of $A^1_\alpha(\mathbb{B}_n, X)$. Then, it is clear that the Bloch space is intimately related to the Bergman spaces and to the Bergman metric as well; it consists exactly of those holomorphic functions that are Lipschitz from $\mathbb{B}_n$ with the Bergman metric to X with the X-norm, see Corollary 3.2.9. Moreover, the Bloch space is also interesting in its own right. In fact, the Bloch space has been studied much earlier that the Bergman spaces. In particular, the Bloch space of the unit disk plays an important role in classical geometric function theory. See [3, 4, 8, 9, 83] for more information on them.

Later on, serious research on the Bloch space of the unit ball began with Timoney's papers [72, 73]. In particular, the scalar case of Theorem 3.1.3 was proved in [72, 82]. We recommend [82] for more information on Bloch spaces on the unit ball of $\mathbb{C}^n$ and more.

The integral respresentation for the Bloch space and the duality between A^1_α and the Bloch space can be found in many different papers/books, including [37, 68, 69, 82]. The case of vector-valued holomorphic functions it is done in [7]. Our proof here is more similar to the classical one.

Therefore, in this chapter we are going to go further in these investigations and we study the Bloch type spaces on the unit ball acting on vector-valued holomorphic functions. In particular, we are going to introduce and characterize the vector-valued Bloch type spaces (the more basic and the more general one) and we prove various characterizations of the Bloch type spaces and some important estimate results needed in the next chapters.

3.1 Vector-valued Bloch space

In this section we introduce the vector-valued Bloch space and some properties of them. It is well-known that Bloch type spaces play an important role in characterizing Hankel operators on Bergman spaces. The *vector-valued Bloch space* $\mathcal{B}(\mathbb{B}_n, X)$ is defined as the space of vector-valued holomorphic functions $f \in \mathcal{H}(\mathbb{B}_n, X)$ such that

$$\|f\|_{\mathcal{B}(\mathbb{B}_n,X)} := \sup_{z\in\mathbb{B}_n} (1-|z|^2)\,\|\nabla f(z)\|_{X^n} < \infty.$$

Under the norm

$$\|f\| = \|f(0)\|_X + \|f\|_{\mathcal{B}(\mathbb{B}_n,X)}$$

it becomes a Banach space. Sometimes is very useful to have other characterizations different from the definition. Before providing these descriptions of the vector-valued Bloch space, we need a preliminary result.

Lemma 3.1.1. *Suppose $\alpha > -1, \beta > 0$ and $t \geq 0$. If*

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^\beta}, \qquad z \in \mathbb{B}_n,$$

for some $g \in A^1_\alpha(\mathbb{B}_n, X)$, then

$$\|\nabla f(z)\|_{X^n} \leq \beta \int_{\mathbb{B}_n} \frac{\|R^{\alpha,t}g(w)\|_X\,\mathrm{d}v_{\alpha+t}(w)}{|1-\langle z,w\rangle|^{\beta+1}},$$

for every $z \in \mathbb{B}_n$.

Proof. Fix $z \in \mathbb{B}_n$. By Lemma 2.1.5 we have that

$$f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,t}g(w)\,\mathrm{d}v_{\alpha+t}(w)}{(1-\langle z,w\rangle)^\beta}.$$

Now, differentiating under the integral sign we obtain that

$$\frac{\partial}{\partial z_k} f(z) = \beta \int_{\mathbb{B}_n} \frac{\overline{w_k} R^{\alpha,t}g(w)\,\mathrm{d}v_{\alpha+t}(w)}{(1-\langle z,w\rangle)^{\beta+1}},$$

and this implies that

$$\nabla f(z) = \beta \int_{\mathbb{B}_n} \frac{\overline{w} R^{\alpha,t}g(w)\,\mathrm{d}v_{\alpha+t}(w)}{(1-\langle z,w\rangle)^{\beta+1}}.$$

Note that this is a Bochner integral in the Banach space X^n. Then, applying Proposition 1.2.2 with X^n shows the result. □

For $f \in \mathcal{H}(\mathbb{B}_n, X)$ we define the functions

$$f_{\alpha,t}(z) := (1-|z|^2)^t R^{\alpha,t} f(z), \qquad z \in \mathbb{B}_n. \tag{3.1.1}$$

We will see that these functions play an important role on the vector-valued Bloch and Bergman spaces. Next result is an example.

Theorem 3.1.2. *Suppose $\alpha > -1$, $t > 0$ and $0 < p < \infty$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then $f \in A^p_\alpha(\mathbb{B}_n, X)$ if and only if the function $f_{\alpha,t} \in L^p_\alpha(\mathbb{B}_n, X)$ (defined in (3.1.1)). Moreover, $\|f\|_{p,\alpha,X} \simeq \|f_{\alpha,t}\|_{p,\alpha,X}$.*

Proof. Let $\beta := \alpha + N$, where N is a sufficiently large positive integer. Since $R^{\alpha,t}$ and $R^{\beta,t}$ are comparable, see [82, p. 54], we have that the function $f_{\alpha,t} \in L^p_\alpha(\mathbb{B}_n, X)$ if and only if $f_{\beta,t} \in L^p_\alpha(\mathbb{B}_n, X)$. Then, we only need to prove that $f \in A^p_\alpha(\mathbb{B}_n, X)$ if and only if $f_{\beta,t} \in L^p_\alpha(\mathbb{B}_n, X)$.

Suppose first that $f_{\beta,t} \in L^p_\alpha(\mathbb{B}_n, X)$. Let $z \in \mathbb{B}_n$. By [82, Corollary 2.3], Fubini's theorem and the reproducing formula in Proposition 2.1.2 we have that

$$\begin{aligned} P_\beta f_{\beta,t}(z) &= \int_{\mathbb{B}_n} \frac{f_{\beta,t}(w)\, dv_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}} \\ &= \frac{c_\beta}{c_{\beta+t}} \int_{\mathbb{B}_n} \frac{R^{\beta,t} f(w)\, dv_{\beta+t}(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}} \\ &= \frac{c_\beta}{c_{\beta+t}} \lim_{r\to 1} \int_{\mathbb{B}_n} \frac{f_r(w)\, dv_\beta(w)}{(1 - \langle z, w\rangle)^{n+1+\beta}} = \frac{c_\beta}{c_{\beta+t}} f(z). \end{aligned} \tag{3.1.2}$$

Then, if $1 \le p < \infty$ and $N > (\alpha + 1)(1/p - 1)$, by Theorem 2.2.2,

$$\|f\|_{p,\alpha,X} = \frac{c_{\beta+t}}{c_\beta} \|P_\beta f_{\beta,t}\|_{p,\alpha,X} \le \frac{c_{\beta+t}}{c_\beta} \|P_\beta\| \, \|f_{\beta,t}\|_{p,\alpha,X},$$

showing that $f \in A^p_\alpha(\mathbb{B}_n, X)$. When $0 < p < 1$, by (3.1.2) and passing the norm inside we have

$$\|f(z)\|_X \lesssim \int_{\mathbb{B}_n} \|g_z(w)\|_X \,(1 - |w|^2)^{\beta+t}\, dv(w), \qquad z \in \mathbb{B}_n,$$

where

$$g_z(w) := \frac{R^{\beta,t} f(w)}{(1 - \langle w, z\rangle)^{n+1+\beta}}, \qquad w \in \mathbb{B}_n.$$

We further assume that N is large enough so that

$$\beta + t = \frac{n + 1 + \alpha'}{p} - (n + 1)$$

for some $\alpha' > -1$. Observe that

$$(n + 1 + \beta)p = n + 1 + \alpha' - pt = n + 1 + \alpha + (\alpha' - pt - \alpha),$$

and that we may assume that N is so large that

$$\alpha' - pt - \alpha > 0.$$

Then, it is clear, by hypothesis, that $g_z \in A^p_{\alpha'}(\mathbb{B}_n, X)$, for every $z \in \mathbb{B}_n$. Therefore, using Theorem 2.1.1, we have

$$\begin{aligned} \|f(z)\|^p_X &\lesssim \left(\int_{\mathbb{B}_n} \|g_z(w)\|^p_X \, \|g_z(w)\|^{1-p}_X \,(1 - |w|^2)^{\beta+t}\, dv(w) \right)^p \\ &\lesssim \|g_z\|^{(1-p)p}_{p,\alpha',X} \left(\int_{\mathbb{B}_n} \|g_z(w)\|^p_X \frac{(1 - |w|^2)^{\beta+t}}{(1 - |w|^2)^{(n+1+\alpha')(1/p-1)}}\, dv(w) \right)^p \\ &\simeq \|g_z\|^{p-p^2}_{p,\alpha',X} \, \|g_z\|^{p^2}_{p,\alpha',X} = \|g_z\|^p_{p,\alpha',X}. \end{aligned}$$

But, Fubini's theorem and Theorem 1.4.1 imply that

$$\int_{\mathbb{B}_n} \|g_z\|_{p,\alpha',X}^p \, dv_\alpha(z) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\left\|R^{\beta,t} f(w)\right\|_X^p}{|1-\langle w,z\rangle|^{(n+1+\beta)p}} \, dv_{\alpha'}(w) \, dv_\alpha(z) \lesssim \|f_{\beta,t}\|_{p,\alpha,X},$$

which in turn imply that $\|f\|_{p,\alpha,X} \lesssim \|f_{\beta,t}\|_{p,\alpha,X}$.

Next assume that $f \in A_\alpha^p(\mathbb{B}_n, X)$. Since $\beta > \alpha$ it implies that $f \in A_\beta^p(\mathbb{B}_n, X)$ and then, by Proposition 2.1.3, we have

$$R^{\beta,t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\beta(w)}{(1-\langle z,w\rangle)^{n+1+\beta+t}},$$

and so

$$(1-|z|^2)^t R^{\beta,t} f(z) = (1-|z|^2)^t \int_{\mathbb{B}_n} \frac{f(w)\, dv_\beta(w)}{(1-\langle z,w\rangle)^{n+1+\beta+t}}, \qquad z \in \mathbb{B}_n.$$

If $1 \le p < \infty$ and $N > (\alpha+1)(1/p-1)$ it follows from Theorem 2.2.1. If $0 < p < 1$ we use the same strategy as before. We write

$$\beta = \frac{n+1+\alpha'}{p} - (n+1).$$

Here we assume that N is large enough so that $\alpha' > \alpha$. Then, using Theorem 2.1.1 again, we have

$$\begin{aligned}
\left\|R^{\beta,t} f(z)\right\|_X^p &\le \left(\int_{\mathbb{B}_n} \frac{\|f(w)\|_X \, dv_\beta(w)}{|1-\langle z,w\rangle|^{n+1+\beta+t}}\right)^p = \left(\int_{\mathbb{B}_n} \|h_z(w)\|_X \, dv_\beta(w)\right)^p \\
&= \left(\int_{\mathbb{B}_n} \|h_z(w)\|_X^p \, \|h_z(w)\|_X^{1-p} \, dv_\beta(w)\right)^p \\
&\lesssim \|h_z\|_{p,\alpha',X}^p,
\end{aligned}$$

where

$$h_z(w) := \frac{f(w)}{(1-\langle w,z\rangle)^{n+1+\beta+t}}, \qquad w \in \mathbb{B}_n.$$

Therefore, by Tonelli's theorem and Theorem 1.4.1 again,

$$\begin{aligned}
\|f_{\beta,t}\|_{p,\alpha,X}^p &\lesssim \int_{\mathbb{B}_n} (1-|z|^2)^{pt} \int_{\mathbb{B}_n} \frac{\|f(w)\|_X^p \, dv_{\alpha'}(w)}{|1-\langle z,w\rangle|^{(n+1+\beta+t)p}} \, dv_\alpha(z) \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X^p \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{pt}\, dv_\alpha(z)}{|1-\langle z,w\rangle|^{n+1+\alpha'+pt}}\right) dv_{\alpha'}(w) \\
&\lesssim \|f\|_{p,\alpha,X}^p.
\end{aligned}$$

Thus, the proof of the theorem is completed. □

The following theorem gives us several conditions that are equivalent to but more easily verifiable than the definition.

Theorem 3.1.3. *Suppose $t > 0$ and $\alpha > -1$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$, then the following conditions are equivalent:*

(i) The function f is in $\mathcal{B}(\mathbb{B}_n, X)$.

(ii) We have $f = P_\alpha g$ for some $g \in L^\infty(\mathbb{B}_n, X)$.

(iii) The function $f_{\alpha,t}(z) = (1-|z|^2)^t R^{\alpha,t} f(z)$ is in $L^\infty(\mathbb{B}_n, X)$.

Moreover,

$$\|f\|_{\mathcal{B}(\mathbb{B}_n,X)} \simeq \|g\|_{\infty,X} \simeq \|f_{\alpha,t}\|_{\infty,X}\,.$$

Proof.
(i) $\Rightarrow$ (ii): By Cauchy-Schwarz inequality we have that

$$\|Rf(z)\|_X^2 \le \left(\sum_{i=1}^n |z_i|\,\|\partial_i f(z)\|_X\right)^2 \le |z|^2\,\|\nabla f(z)\|_{X^n}^2\,. \tag{3.1.3}$$

In particular $\|Rf(z)\|_X \le \|\nabla f(z)\|_{X^n}$, for every $z \in \mathbb{B}_n$, then condition (i) implies that $(1-|z|^2)Rf(z)$ is in $L^\infty(\mathbb{B}_n, X)$. Consider the function

$$g(z) := \frac{c_{\alpha+1}}{c_\alpha}(1-|z|^2)\int_{\mathbb{B}_n}\frac{f(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+2+\alpha}}, \qquad z\in\mathbb{B}_n.$$

It is easy to see (see proof of [82, Theorem 3.4]) that

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha}\left[(1-|z|^2)f(z) + \frac{(1-|z|^2)Rf(z)}{n+1+\alpha}\right].$$

So to show that $g \in L^\infty(\mathbb{B}_n, X)$ we only need to prove that $(1-|z|^2)\,\|f(z)\|_X$ is bounded on $\mathbb{B}_n$. But the identity

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t}\,\mathrm{d}t \tag{3.1.4}$$

shows that f grows at most as fast as $-\log(1-|z|^2)$. Indeed, using (3.1.3), the hypothesis and the fact that $1-|z| \le 1-|z|^2$, for $z\in\mathbb{B}_n$, we have that

$$\begin{aligned}\|f(z)-f(0)\| &\le \int_0^1 \frac{|z|\,\mathrm{d}t}{(1-t^2\,|z|^2)} \le \int_0^1 \frac{|z|\,\mathrm{d}t}{(1-t\,|z|)}\\ &= -\log(1-t\,|z|)\Big|_0^1 = -\log(1-|z|) \le -\log(1-|z|^2).\end{aligned}$$

Then it follows that g is in $L^\infty(\mathbb{B}_n, X)$. Moreover, for every $z\in\mathbb{B}_n$, by Fubini's theorem (using [57, Lemma 2.5]) and the reproducing property in Proposition 2.1.2, we get that

$$\begin{aligned} P_\alpha g(z) &= \int_{\mathbb{B}_n} \frac{g(w)\, dv_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} = \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{f(u)\, dv_\alpha(u)}{(1-\langle w,u\rangle)^{n+2+\alpha}} \right) \frac{dv_{\alpha+1}(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} \\ &= \int_{\mathbb{B}_n} f(u) \overline{\int_{\mathbb{B}_n} \frac{(1-\langle w,z\rangle)^{-(n+1+\alpha)}}{(1-\langle u,w\rangle)^{n+2+\alpha}}\, dv_{\alpha+1}(w)}\, dv_\alpha(u) \\ &= \int_{\mathbb{B}_n} f(u)\overline{(1-\langle u,z\rangle)^{-(n+1+\alpha)}}\, dv_\alpha(u) = f(z) \end{aligned}$$

and then $f = P_\alpha g$.

(ii) $\Rightarrow$ (iii): If there exists a function $g \in L^\infty(\mathbb{B}_n, X)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, dv_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}},$$

by Proposition 1.6.1, we have that

$$R^{\alpha,t} f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, dv_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}.$$

Then, by Proposition 1.2.2 and Theorem 1.4.1, we obtain that

$$\left\| R^{\alpha,t} f(z) \right\|_X \lesssim \|g\|_{\infty,X} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha\, dv(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+t}} \lesssim \|g\|_{\infty,X} \frac{1}{(1-|z|^2)^t},$$

which also imply that $\|f_{\alpha,t}\|_{\infty,X} \lesssim \|g\|_{\infty,X}$.

(iii) $\Rightarrow$ (i): We can assume that $f \in A^1_\alpha(\mathbb{B}_n, X)$. Indeed, it is a direct consequence of Theorem 3.1.2. If $f_{\alpha,t} \in L^\infty(\mathbb{B}_n, X) \subset L^p(\mathbb{B}_n, X)$, for any $p \geq 1$, then $f \in A^p_\alpha(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$. Therefore, applying Proposition 2.1.2, we have that

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \qquad z \in \mathbb{B}_n.$$

Then, an application of Lemma 3.1.1 and Theorem 1.4.1 shows that

$$\|\nabla f(z)\| \lesssim \|f_{\alpha,t}\|_{\infty,X} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha\, dv(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+1}} \lesssim \frac{\|f_{\alpha,t}\|_{\infty,X}}{(1-|z|^2)},$$

for every $z \in \mathbb{B}_n$. This completes the proof of the theorem. □

Now we continue with some properties of vector-valued Bloch spaces.

Lemma 3.1.4. *Let $\alpha > -1$. Then $\mathcal{B}(\mathbb{B}_n, X) \subset A^p_\alpha(\mathbb{B}_n, X)$, for any $1 \leq p < \infty$.*

Proof. This is immediate from Theorems 3.1.2 and 3.1.3. □

Next result gives a pointwise estimate for vector-valued Bloch functions.

Proposition 3.1.5. *If $f \in \mathcal{B}(\mathbb{B}_n, X)$ then there exists $C > 0$ such that*

$$\|f(z)\|_X \leq C \|f\|_{\mathcal{B}(\mathbb{B}_n,X)} \log \frac{1}{1-|z|^2},$$

for any $z \in \mathbb{B}_n$.

Proof. Let $t > 0$ and $f \in \mathcal{B}(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$. If we apply Proposition 2.1.2 and Lemma 2.1.5 we get that

$$f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,t} f(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}} \, dv_{\alpha+t}(w).$$

Therefore, by Proposition 1.2.2 and Theorems 3.1.3 and 1.4.1, we have that

$$\begin{aligned} \|f(z)\|_X &\lesssim \|f\|_{\mathcal{B}(\mathbb{B}_n,X)} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{(1-\langle z, w\rangle)^{n+1+\alpha}} \, dv(w) \\ &\lesssim \|f\|_{\mathcal{B}(\mathbb{B}_n,X)} \log \frac{1}{1-|z|^2}. \end{aligned}$$

□

To finish this section we include the duality of vector-valued Bloch spaces used later on. It is well-known that in the scalar-valued case $(A^1_\alpha)^*$ can be identified with $\mathcal{B}$. In this case we have similar result.

Theorem 3.1.6. *Let X be a Banach space and $\alpha > -1$. The space $(A^1_\alpha(\mathbb{B}_n, X))^*$ can be identified with $\mathcal{B}(\mathbb{B}_n, X^*)$ under the integral pairing (2.1.1).*

Proof. If $g \in \mathcal{B}(\mathbb{B}_n, X^*)$, then by Theorem 3.1.3 we have that there exist $h \in L^\infty(\mathbb{B}_n, X^*)$ such that

$$g(z) = P_\alpha h(z) = \int_{\mathbb{B}_n} \frac{h(w) \, dv_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}}$$

and $\|h\|_{\infty,X^*} \lesssim \|g\|_{\mathcal{B}(\mathbb{B}_n,X^*)}$. Let $f \in A^1_\alpha(\mathbb{B}_n, X)$, by Fubini's theorem and the integral representation in Proposition 2.1.2 we have

$$\begin{aligned} \langle f, g\rangle_{\alpha,X} &= \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_X \, dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\langle f(z), h(w)\rangle_X \, dv_\alpha(w)}{(1-\langle w, z\rangle)^{n+1+\alpha}} \, dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle f(w), h(w)\rangle_X \, dv_\alpha(w). \end{aligned}$$

Then,

$$\begin{aligned} \left|\langle f, g\rangle_{\alpha,X}\right| &\leq \int_{\mathbb{B}_n} \|f(w)\|_X \, \|h(w)\|_{X^*} \, dv_\alpha(w) \\ &\leq \|h\|_{\infty,X^*} \|f\|_{1,\alpha,X} \lesssim \|g\|_{\mathcal{B}(\mathbb{B}_n,X^*)} \|f\|_{1,\alpha,X}, \end{aligned}$$

which implies that g induces a bounded linear functional on $A^1_\alpha(\mathbb{B}_n, X)$.

Conversely, let Λ be a bounded linear functional on $A^1_\alpha(\mathbb{B}_n, X)$. Since $A^2_\alpha(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$, we have that

$$|\Lambda(f)| \leq \|\Lambda\| \, \|f\|_{1,\alpha,X} \leq \|\Lambda\| \, \|f\|_{2,\alpha,X},$$

which implies that Λ also defines a bounded linear functional on $A^2_\alpha(\mathbb{B}_n, X)$ as well. Moreover, by the duality of the vector-valued Bergman spaces in Theorem 2.2.4, we have that $(A^2_\alpha(\mathbb{B}_n, X))^* = A^2_\alpha(\mathbb{B}_n, X^*)$ under the same integral pairing (2.1.1). Then, there exists an unique $g \in A^2_\alpha(\mathbb{B}_n, X^*)$ such that

$$\Lambda(f) = \langle f, g\rangle_{\alpha,X} = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_X \, \mathrm{d}v_\alpha(z), \tag{3.1.5}$$

for any $f \in A^2_\alpha(\mathbb{B}_n, X)$. Fix $t > 0$, $x \in X$ and $z \in \mathbb{B}_n$. Define

$$f^x_z(w) := \frac{x}{(1 - \langle w, z\rangle)^{n+1+\alpha+t}}, \qquad w \in \mathbb{B}_n.$$

Notice that $f^x_z \in A^2_\alpha(\mathbb{B}_n, X)$ and since $g \in A^2_\alpha(\mathbb{B}_n, X^*)$, Theorem 3.1.2 imply that $g_{\alpha,t} \in L^2_\alpha(\mathbb{B}_n, X^*)$ which means that $R^{\alpha,t}g \in A^2_{\alpha+t}(\mathbb{B}_n, X^*)$. Then, by Lemma 2.1.5 and the integral representation in Proposition 2.1.2 we have that

$$\begin{aligned}
\Lambda(f^x_z) &= \int_{\mathbb{B}_n} \langle f^x_z(w), g(w)\rangle_X \, \mathrm{d}v_\alpha(w) = \left\langle x, \int_{\mathbb{B}_n} \frac{g(w)\, \mathrm{d}v_\alpha(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha+t}} \right\rangle_X \\
&= \left\langle x, \int_{\mathbb{B}_n} \frac{R^{\alpha,t}g(w)\, \mathrm{d}v_{\alpha+t}(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha+t}} \right\rangle_X = \left\langle x, R^{\alpha,t}g(z)\right\rangle_X.
\end{aligned}$$

Therefore, using Theorem 1.4.1,

$$\begin{aligned}
\left\|R^{\alpha,t}g(z)\right\|_{X^*} &= \sup_{\|x\|_X \leq 1} \left|\left\langle x, R^{\alpha,t}g(z)\right\rangle_X\right| = \sup_{\|x\|_X \leq 1} |\Lambda(f^x_z)| \\
&\leq \sup_{\|x\|_X \leq 1} \|\Lambda\| \, \|f^x_z\|_{1,\alpha,X} \leq \|\Lambda\| \int_{\mathbb{B}_n} \frac{\mathrm{d}v_\alpha(z)}{|1 - \langle z, w\rangle|^{n+1+\alpha+t}} \\
&\lesssim \|\Lambda\| \, (1 - |z|^2)^{-t}.
\end{aligned}$$

Theorem 3.1.3 gives us that $g \in \mathcal{B}(\mathbb{B}_n, X^*)$. It remains to prove that (3.1.5) remains true for functions on $A^1_\alpha(\mathbb{B}_n, X)$. First, it is easy to see that $A^2_\alpha(\mathbb{B}_n, X)$ is dense in $A^1_\alpha(\mathbb{B}_n, X)$. Indeed, since $\mathcal{P}(\mathbb{B}_n, X) \subset A^2_\alpha(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$ and $\mathcal{P}(\mathbb{B}_n, X)$ are dense in $A^1_\alpha(\mathbb{B}_n, X)$ it implies that $A^2_\alpha(\mathbb{B}_n, X)$ is dense in $A^1_\alpha(\mathbb{B}_n, X)$. Now, let $f \in A^1_\alpha(\mathbb{B}_n, X)$. Then, by the fact that Λ is bounded and the Lebesgue dominated convergence theorem we have

$$\begin{aligned}
\Lambda(f) &= \lim_{n\to\infty} \Lambda(f_n) = \lim_{n\to\infty} \int_{\mathbb{B}_n} \langle f_n(z), g(z)\rangle_X \, \mathrm{d}v_\alpha(z) \\
&= \int_{\mathbb{B}_n} \lim_{n\to\infty} \langle f_n(z), g(z)\rangle_X \, \mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n} \langle f(z), g(z)\rangle_X \, \mathrm{d}v_\alpha(z).
\end{aligned}$$

Note that we can apply Lebesgue dominated convergence theorem since $A^2_\alpha(\mathbb{B}_n, X)$ is dense in $A^1_\alpha(\mathbb{B}_n, X)$, that is, there exists $\{f_n\}_n \subset A^2_\alpha(\mathbb{B}_n, X)$ such that $f_n \to f$ uniformly in X as $n \to \infty$, i.e., for every $\varepsilon > 0$ we have

$$\|f_n(z) - f(z)\|_X < \varepsilon/2, \qquad z \in \mathbb{B}_n,$$

and $\mathcal{P}(\mathbb{B}_n, X)$ is dense in $A^1_\alpha(\mathbb{B}_n, X)$, that is, there exists $\{P_n\}_n \subset \mathcal{P}(\mathbb{B}_n, X)$ such that

$$\|f(z) - P_n(z)\|_X \leq \varepsilon/2, \qquad z \in \mathbb{B}_n.$$

Therefore, by the triangle inequality,

$$\begin{aligned}|\langle f_n(z), g(z)\rangle_X| &\leq |\langle f_n(z) - f(z), g(z)\rangle_X| + |\langle f(z) - P_n(z), g(z)\rangle_X| \\ &\qquad + |\langle P_n(z), g(z)\rangle_X| \\ &\leq (\varepsilon + C)\,\|g(z)\|_{X^*},\end{aligned}$$

which is a integrable function since $g \in A^2_\alpha(\mathbb{B}_n, X^*)$. This completes the proof of the theorem. □

You can see a completely different proof in [7, Corollary 3.17]. See also [19] for more information.

3.2 Vector-valued γ-Bloch space

In this section we are going to define and characterize the more general type of vector-valued Bloch spaces, the γ-Bloch spaces $\mathcal{B}_\gamma(\mathbb{B}_n, X)$ for $\gamma > 0$, defined as the space of vector-valued holomorphic functions $f \in \mathcal{H}(\mathbb{B}_n, X)$ such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \left\| \frac{\partial f}{\partial z_k}(z) \right\|_X < \infty,$$

for any $1 \leq k \leq n$. It is easy to see that $f \in \mathcal{H}(\mathbb{B}_n, X)$ belongs to $\mathcal{B}_\gamma(\mathbb{B}_n, X)$ if and only if

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, X)} := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \, \|\nabla f(z)\|_{X^n} < \infty.$$

Taking $\|f\| = \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, X)} + \|f(0)\|_X$ we have a norm. It is clear that $\mathcal{B}_1(\mathbb{B}_n, X) = \mathcal{B}(\mathbb{B}_n, X)$. We also have some equivalent characterizations of these spaces more suitable for our purpose.

Theorem 3.2.1. *Let X be any Banach space. Suppose $\gamma > 0$, $\beta > -1$ such that $\gamma + \beta > 0$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then the following conditions are equivalent:*

(i) $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$.

(ii) There exists a function $g \in L^\infty(\mathbb{B}_n, X)$ such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, \mathrm{d}v_\beta(w)}{(1 - \langle z, w\rangle)^{n+\beta+\gamma}}, \qquad z \in \mathbb{B}_n.$$

(iii) The function $(1-|z|^2)^\gamma R^{\beta+\gamma-1,1}f(z)$ *is in* $L^\infty(\mathbb{B}_n, X)$.

Moreover,

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} \simeq \|g\|_{\infty,X} \simeq \sup_{z\in\mathbb{B}_n} (1-|z|^2)^\gamma \left\|R^{\beta+\gamma-1,1}f(z)\right\|_X .$$

Proof.

(i) $\Rightarrow$ (ii): We use similar ideas of [82, Theorem 7.1]. If (i) holds, then the function $(1-|z|^2)^\gamma Rf(z)$ is in $L^\infty(\mathbb{B}_n, X)$, since $\|Rf(z)\|_X \le \|\nabla f(z)\|_{X^n}$, for every $z \in \mathbb{B}_n$. Therefore, the function defined by

$$g(z) := \frac{c_{\beta+\gamma}}{c_\beta}(1-|z|^2)^\gamma \left(f(z) + \frac{Rf(z)}{n+\beta+\gamma}\right)$$

is in $L^\infty(\mathbb{B}_n, X)$ as well (see proof of Theorem 3.1.3). Now let $z \in \mathbb{B}_n$. If we consider the holomorphic function

$$h(z) = \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_\beta(w)}{(1-\langle z,w\rangle)^{n+\beta+\gamma}} = \int_{\mathbb{B}_n} \left(f(w) + \frac{Rf(w)}{n+\beta+\gamma}\right) \frac{\mathrm{d}v_{\beta+\gamma}(w)}{(1-\langle z,w\rangle)^{n+\beta+\gamma}},$$

and we apply the differential operator $R^{\beta+\gamma-1,1}$ under the integral sign, then using Proposition 1.6.1 and the reproducing formula in Proposition 2.1.2, we obtain

$$R^{\beta+\gamma-1,1}h = f + \frac{Rf}{n+\beta+\gamma}.$$

Note that the application of the reproducing formula in Proposition 2.1.2 is correct because the function $(1-|z|^2)^\gamma Rf(z)$ is in $L^\infty(\mathbb{B}_n, X)$, so clearly $Rf \in A^1_{\beta+\gamma}(\mathbb{B}_n, X)$, and using (3.1.4), (3.1.3) and the hypothesis we have that

$$\begin{aligned}
\int_{\mathbb{B}_n} \|f(z)-f(0)\|_X \,\mathrm{d}v_{\beta+\gamma}(z) &\lesssim \int_{\mathbb{B}_n} \left(\int_0^1 \frac{\|Rf(tz)\|_X}{t}\,\mathrm{d}t\right)(1-|z|^2)^{\beta+\gamma}\,\mathrm{d}v(z) \\
&\lesssim \int_{\mathbb{B}_n} \left(\int_0^1 |z|\,\|\nabla f(tz)\|_X (1-|tz|^2)^\gamma\,\mathrm{d}t\right)(1-|z|^2)^\beta\,\mathrm{d}v(z) \\
&\lesssim \int_{\mathbb{B}_n} dv_\beta(z) < \infty,
\end{aligned}$$

which imply that $f \in A^1_{\beta+\gamma}(\mathbb{B}_n, X)$ as well. Now recall that if

$$f = \sum_{k=0}^\infty f_k,$$

is the homogeneous expansion of the function f, where f_k are homogeneous holomorphic polynomials of degree k with coefficients on X, then

$$Rf = \sum_{k=0}^\infty k f_k.$$

Moreover, by definition

$$R_{\beta+\gamma-1,1}f = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\beta+\gamma)\Gamma(n+k+\beta+\gamma)}{\Gamma(n+\beta+\gamma)\Gamma(n+1+k+\beta+\gamma)} f_k = \sum_{k=0}^{\infty} \frac{n+\beta+\gamma}{n+k+\beta+\gamma} f_k,$$

and

$$R_{\beta+\gamma-1,1}Rf = \sum_{k=0}^{\infty} \frac{n+\beta+\gamma}{n+k+\beta+\gamma} k f_k.$$

Therefore, since $h = R_{\beta+\gamma-1,1}R^{\beta+\gamma-1,1}h$, a calculation using the definitions shows that

$$\begin{aligned} h &= R_{\beta+\gamma-1,1}\left(f + \frac{Rf}{n+\beta+\gamma}\right) \\ &= \sum_{k=0}^{\infty} \frac{n+\beta+\gamma}{n+k+\beta+\gamma} f_k + \sum_{k=0}^{\infty} \frac{k}{n+k+\beta+\gamma} f_k \\ &= \sum_{k=0}^{\infty} f_k = f. \end{aligned}$$

(ii) $\Rightarrow$ (iii): Let $z \in \mathbb{B}_n$. We have that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, dv_\beta(w)}{(1-\langle z, w\rangle)^{n+\beta+\gamma}},$$

for some $g \in L^\infty(\mathbb{B}_n, X)$. If we apply $R^{\beta+\gamma-1,1}$ with Proposition 1.6.1 we obtain that

$$R^{\beta+\gamma-1,1}f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, dv_\beta(w)}{(1-\langle z, w\rangle)^{n+1+\beta+\gamma}}.$$

By Proposition 1.2.2 and Theorem 1.4.1 we have that

$$\left\|R^{\beta+\gamma-1,1}f(z)\right\|_X \lesssim \|g\|_{\infty,X} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\beta\, dv(w)}{(1-\langle z, w\rangle)^{n+1+\beta+\gamma}} \lesssim \|g\|_{\infty,X} \frac{1}{(1-|z|^2)^\gamma}.$$

(iii) $\Rightarrow$ (i): First, notice that $f \in A^1_{\beta+\gamma-1}(\mathbb{B}_n, X)$. Indeed, by Theorem 3.1.2,

$$\begin{aligned} \|f\|_{1,\beta+\gamma-1,X} &\simeq \int_{\mathbb{B}_n} (1-|z|^2) \left\|R^{\beta+\gamma-1,1}f(z)\right\|_X \, dv_{\beta+\gamma-1}(z) \\ &\lesssim \int_{\mathbb{B}_n} (1-|z|^2)^\beta\, dv(z) < \infty, \end{aligned}$$

and so, by Proposition 2.1.2, we have that

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_{\beta+\gamma-1}(w)}{(1-\langle z, w\rangle)^{n+\beta+\gamma}}, \qquad z \in \mathbb{B}_n.$$

A direct application of Lemma 3.1.1 gives that

$$\|\nabla f(z)\|_{X^n} \leq (n+\beta+\gamma) \int_{\mathbb{B}_n} \frac{\left\|R^{\beta+\gamma-1,1} f(w)\right\| \, \mathrm{d}v_{\beta+\gamma}(w)}{|1-\langle z,w\rangle|^{n+1+\beta+\gamma}}.$$

Finally, using our assumption and Theorem 1.4.1, we see that $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$. The proof is complete. □

Next lemma gives some kind of shift of the radial derivatives characterizations.

Lemma 3.2.2. *Let $\gamma > 0$ and $\alpha > -1$ such that $\alpha - \gamma > -2$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then the following conditions are equivalent:*

(a) The function $(1-|z|^2)^\gamma R^{\alpha,1} f(z)$ is in $L^\infty(\mathbb{B}_n, X)$;

(b) The function $(1-|z|^2)^{\gamma+t} R^{\alpha,t+1} f(z)$ is in $L^\infty(\mathbb{B}_n, X)$ for some $t \geq 0$.

Moreover,

$$C_1 := \sup_{z\in\mathbb{B}_n} (1-|z|^2)^\gamma \left\|R^{\alpha,1} f(z)\right\|_X \simeq \sup_{z\in\mathbb{B}_n} (1-|z|^2)^{\gamma+t} \left\|R^{\alpha,t+1} f(z)\right\|_X =: C_2.$$

Proof. First of all, we will prove that if $(1-|z|^2)^{\gamma+t} R^{\alpha,t+1} f(z)$ is in $L^\infty(\mathbb{B}_n, X)$, for some $t \geq 0$, then $f \in A^1_\alpha(\mathbb{B}_n, X)$. Indeed, by Theorem 3.1.2 and the hypothesis, we have that

$$\begin{aligned} \|f\|_{1,\alpha,X} &\simeq \int_{\mathbb{B}_n} (1-|z|^2)^{t+1} \left\|R^{\alpha,t+1} f(z)\right\|_X \mathrm{d}v_\alpha(z) \\ &\leq c_\alpha C_2 \int_{\mathbb{B}_n} (1-|z|^2)^{\alpha-\gamma+1} \, \mathrm{d}v(z) < \infty. \end{aligned}$$

(a) ⇒ (b): Let $t > 0$. By Proposition 2.1.3 and Lemma 2.1.5 we have that

$$R^{\alpha,t+1} f(z) = \int_{\mathbb{B}_n} \frac{f(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t+1}} = \int_{\mathbb{B}_n} \frac{R^{\alpha,1} f(w)\,\mathrm{d}v_{\alpha+1}(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t+1}}.$$

Now, by Proposition 1.2.2 and Theorem 1.4.1, we obtain that

$$\begin{aligned} \left\|R^{\alpha,t+1} f(z)\right\|_X &\lesssim C_1 \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\alpha-\gamma+1}\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+t+1}} \\ &\lesssim \frac{C_1}{(1-|z|^2)^{\gamma+t}} \end{aligned}$$

and it implies that $C_2 \lesssim C_1 < \infty$.

(b) ⇒ (a): Using the same as before we can get that

$$R^{\alpha,1} f(z) = \int_{\mathbb{B}_n} \frac{f(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+1}} = \int_{\mathbb{B}_n} \frac{R^{\alpha,t+1} f(w)\,\mathrm{d}v_{\alpha+t+1}(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+1}}.$$

Again, passing the norm inside the integral and applying Theorem 1.4.1, we obtain that

$$\begin{aligned}\left\|R^{\alpha,1}f(z)\right\|_X &\lesssim C_2 \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\alpha-\gamma+1}\,\mathrm{d}v(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+1}} \\ &\lesssim \frac{C_2}{(1-|z|^2)^{\gamma}}\end{aligned}$$

and it implies that $C_1 \lesssim C_2 < \infty$. □

As a corollary we get the following result which is actually the most used in what follows.

Corollary 3.2.3. *Let $\alpha > -1$ and $\gamma > 0$ such that $\alpha-\gamma > -2$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$ then $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$ if and only if the function $(1-|z|^2)^{\gamma+t}R^{\alpha,t+1}f(z)$ is in $L^\infty(\mathbb{B}_n, X)$, for some $t \geq 0$. Moreover,*

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} \simeq \sup_{z\in\mathbb{B}_n}(1-|z|^2)^{\gamma+t}\left\|R^{\alpha,t+1}f(z)\right\|_X.$$

Proof. Notice that with the same notations of above theorems, if we set $\alpha := \beta+\gamma-1$, condition $\beta > -1$ is equivalent to say that $\alpha-\gamma > -2$ and condition $\gamma+\beta > 0$ is equivalent to say that $\alpha > -1$. This means that, in fact, under these conditions, statement (iii) of Theorem 3.2.1 is equivalent to statement (a) of Lemma 3.2.2. So, Theorem 3.2.1 and Lemma 3.2.2 gives the result. □

As a consequence we also have some basic properties of these spaces.

Corollary 3.2.4. *Let $\gamma > 0$ and $1 \leq p < \infty$. Then $\mathcal{B}_\gamma(\mathbb{B}_n, X) \subset A^p_\alpha(\mathbb{B}_n, X)$ for all $\alpha > -1$ with $\alpha > p(\gamma-1)-1$.*

Proof. Let $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$, $1 < p < \infty$ and $\alpha > -1$. By Theorems 3.1.2 and 3.2.1, we have that

$$\begin{aligned}\|f\|_{p,\alpha,X} &\simeq \left(\int_{\mathbb{B}_n}(1-|z|^2)^{\gamma p+(1-\gamma)p}\left\|R^{\alpha,1}f(z)\right\|_X^p\,\mathrm{d}v_\alpha(z)\right)^{1/p} \\ &\lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}\left(\int_{\mathbb{B}_n}(1-|z|^2)^{\alpha+(1-\gamma)p}\,\mathrm{d}v(z)\right)^{1/p}.\end{aligned}$$

The condition $\alpha > p(\gamma-1)-1$ ensures that the last integral is finite. Since $A^p_\alpha(\mathbb{B}_n, X) \subset A^1_\alpha(\mathbb{B}_n, X)$, for all $1 < p < \infty$, we are done. □

Then we also have the following pointwise estimate for vector-valued γ-Bloch functions.

Proposition 3.2.5. *Suppose $\gamma > 1$. If $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$ then there exists $C > 0$ such that*

$$\|f(z)\|_X \leq \frac{C\,\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}}{(1-|z|^2)^{\gamma-1}},$$

for any $z \in \mathbb{B}_n$.

Proof. Let $t > 0$ and $\alpha > -1$ big enough such that $\alpha - \gamma > -2$. This imply that $\gamma < \alpha + 2$, so any $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$ will belong to $A^1_\alpha(\mathbb{B}_n, X)$ by Corollary 3.2.4. Thus, using Proposition 2.1.2 and Lemma 2.1.5, we have

$$f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,t+1} f(w)\, dv_{\alpha+t+1}(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}.$$

By Proposition 1.2.2, Corollary 3.2.3 and Theorem 1.4.1 we obtain that

$$\|f(z)\|_X \lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\alpha-\gamma+1}\, dv(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \lesssim \frac{\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}}{(1 - |z|^2)^{\gamma-1}}.$$

Note that the application of Theorem 1.4.1 is correct because $\gamma > 1$ and $\alpha - \gamma > -2$. ☐

Note that the particular case of $z = 0$ we have that $\|f(0)\|_X \lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}$, for any $\gamma > 0$. We have already seen the pointwise estimate of vector-valued γ-Bloch functions for the particular case of $\gamma = 1$ in Proposition 3.1.5. Then, it is coherent to ask what happen for the remaining cases of $0 < \gamma < 1$. In this case, when $0 < \gamma < 1$, we need to introduce the vector-valued Lipschitz Spaces.

For $0 < \gamma < 1$ we define $\Lambda_\gamma(\mathbb{B}_n, X)$ to be the space of vector-valued holomorphic function $f \colon \mathbb{B}_n \to X$ such that

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n,X)} = \sup\left\{ \frac{\|f(z) - f(w)\|_X}{|z - w|^\gamma} : z, w \in \mathbb{B}_n, z \neq w \right\} < \infty.$$

As in the scalar case, it follows that $\mathcal{B}_\gamma(\mathbb{B}_n, X) = \Lambda_{1-\gamma}(\mathbb{B}_n, X)$ with equivalent norms, when $0 < \gamma < 1$, see [82, Chapter 7] for more information.

As a consequence we have the following pointwise estimate: there exists $C > 0$ such that

$$\|f(z) - f(w)\|_X \leq C \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} |z - w|^{1-\gamma},$$

for every $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$ when $0 < \gamma < 1$ and $z, w \in \mathbb{B}_n$. In that case, we have $\mathcal{B}_\gamma(\mathbb{B}_n, X) \subset H^\infty(\mathbb{B}_n, X)$, the space of all vector-valued bounded analytic functions on $\mathbb{B}_n$.

We need another characterization of the γ-Bloch spaces in terms of the invariant gradient. Before that we need two new results about the vector-valued invariant gradient.

Recall that the invariant gradient of an holomorphic function f is

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0), \qquad z \in \mathbb{B}_n.$$

In the scalar case, it is easy to see [82, Lemma 2.14] that

$$(1 - |z|^2)\, |Rf(z)| \leq (1 - |z|^2)\, |\nabla f(z)| \leq \left|\widetilde{\nabla} f(z)\right|,$$

where Rf denotes the radial derivative of f. In the Banach space case, the first inequality is still obvious using just Cauchy-Schwarz, see (3.1.3). The second inequality is not trivial anymore, but we still have the following result.

Theorem 3.2.6. *Let X be a Banach space and $f \in \mathcal{H}(\mathbb{B}_n, X)$. For any $z \in \mathbb{B}_n$ we have*

$$(1-|z|^2)\,\|Rf(z)\|_X \le (1-|z|^2)\,\|\nabla f(z)\|_{X^n} \le \left\|\widetilde{\nabla} f(z)\right\|_{X^n}.$$

Proof. The first inequality is just (3.1.3). The second inequality is more involved. First of all, we have that

$$\left\|\widetilde{\nabla} f(z)\right\|_{X^n}^2 = \sum_{j=1}^n \|\partial_j (f\circ\varphi_z)(0)\|_X^2 \ge \sum_{j=1}^n \left|\langle \partial_j (f\circ\varphi_z)(0), w_j\rangle_X\right|^2,$$

for any $w_j \in X^*$ with $\|w_j\|_{X^*} = 1$ that will be determined later. On the other hand, with the notation $s_z := \sqrt{1-|z|^2}$, it is elementary to see that for any $j \in \{1,\dots,n\}$, we have

$$\partial_j \varphi_z(0) = -\left(s_z e_j - (s_z - s_z^2)\frac{\overline{z_j} z}{|z|^2}\right),$$

see the proof of [82, Lemma 1.6]. Then, for any $j \in \{1,\dots,n\}$,

$$\begin{aligned}
\partial_j (f\circ\varphi_z)(0) &= \sum_{i=1}^n \partial_i f(z)\,(\partial_j\varphi_z(0))_i \\
&= -\sum_{i=1}^n \partial_i f(z)\left(s_z\delta_{ij} - (s_z - s_z^2)\frac{\overline{z_j} z_i}{|z|^2}\right) = \sum_{i=1}^n \alpha_{ij}\partial_i f(z),
\end{aligned}$$

where

$$\alpha_{ij} := \left(s_z\delta_{ij} - (s_z - s_z^2)\frac{\overline{z_j} z_i}{|z|^2}\right).$$

Therefore, we have that

$$\begin{aligned}
\left|\langle \partial_j (f\circ\varphi_z)(0), w_j\rangle_X\right|^2 &= \langle \partial_j (f\circ\varphi_z)(0), w_j\rangle\,\overline{\langle \partial_j (f\circ\varphi_z)(0), w_j\rangle} \\
&= \sum_{i=1}^n\sum_{k=1}^n \alpha_{ij}\overline{\alpha_{kj}}\,\langle \partial_i f(z), w_j\rangle\,\overline{\langle \partial_k f(z), w_j\rangle}.
\end{aligned}$$

An elementary computation gives

$$\begin{aligned}
\alpha_{ij}\overline{\alpha_{kj}} &= \alpha_{ij}\alpha_{jk} \\
&= s_z^2\delta_{ij}\delta_{kj} - \delta_{ij}s_z(s_z - s_z^2)\frac{z_j\overline{z_k}}{|z|^2} - \delta_{kj}s_z(s_z - s_z^2)\frac{z_i\overline{z_j}}{|z|^2} + (s_z - s_z^2)^2\frac{z_i\overline{z_k}\,|z_j|^2}{|z|^2}.
\end{aligned}$$

Then, we get that $\sum_{j=1}^n \left|\langle \partial_j (f\circ\varphi_z)(0), w_j\rangle_X\right|^2$ is equal to

$$\begin{aligned}
s_z^2\sum_{j=1}^n |\langle \partial_j f(z), w_j\rangle|^2 &- \frac{s_z(s_z - s_z^2)}{|z|^2}\sum_{j=1}^n\sum_{k=1}^n z_j\overline{z_k}\,\langle \partial_j f(z), w_j\rangle\,\overline{\langle \partial_k f(z), w_j\rangle} \\
&- \frac{s_z(s_z - s_z^2)}{|z|^2}\sum_{j=1}^n\sum_{i=1}^n z_i\overline{z_j}\,\langle \partial_i f(z), w_j\rangle\,\overline{\langle \partial_j f(z), w_j\rangle} \\
&+ \frac{(s_z - s_z^2)^2}{|z|^2}\sum_{j=1}^n\frac{|z_j|^2}{|z|^2}\sum_{i=1}^n\sum_{k=1}^n z_i\overline{z_k}\,\langle \partial_i f(z), w_j\rangle\,\overline{\langle \partial_k f(z), w_j\rangle}.
\end{aligned}$$

Set $\lambda_i = \lambda_i(z) := z_i \partial_i f(z)$ and recall that $\sum_{i=1}^n \lambda_i(z) = Rf(z)$, so we can rewrite the previous formula as

$$\begin{aligned} s_z^2 \sum_{j=1}^n |\langle \partial_j f(z), w_j \rangle|^2 &- \frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^n \langle \lambda_j, w_j \rangle \overline{\langle Rf(z), w_j \rangle} \\ &- \frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^n \langle Rf(z), w_j \rangle \overline{\langle \lambda_j, w_j \rangle} \\ &+ \frac{(s_z - s_z^2)^2}{|z|^2} \sum_{j=1}^n \frac{|z_j|^2}{|z|^2} \langle Rf(z), w_j \rangle \overline{\langle Rf(z), w_j \rangle}. \end{aligned}$$

Grouping a little more we obtain that it is equal to

$$s_z^2 \sum_{j=1}^n |\langle \partial_j f(z), w_j \rangle|^2 - \frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^n \beta_j + \frac{(s_z - s_z^2)^2}{|z|^2} \sum_{j=1}^n \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 ,$$

where $\beta_j = \beta_j(z) := \langle \lambda_j, w_j \rangle \overline{\langle Rf(z), w_j \rangle} + \langle Rf(z), w_j \rangle \overline{\langle \lambda_j, w_j \rangle}$. We want to point out that

$$\begin{aligned} \beta_j &= \langle \lambda_j, w_j \rangle \overline{\langle Rf(z), w_j \rangle} + \langle Rf(z), w_j \rangle \overline{\langle \lambda_j, w_j \rangle} = 2\mathrm{Re}(\langle \lambda_j, w_j \rangle \overline{\langle Rf(z), w_j \rangle}) \\ &\leq 2 |\langle \lambda_j, w_j \rangle| |\langle Rf(z), w_j \rangle| = 2 \frac{|z|}{|z_j|} |\langle \lambda_j, w_j \rangle| \frac{|z_j|}{|z|} |\langle Rf(z), w_j \rangle| \\ &\leq \frac{|z|^2}{|z_j|^2} |\langle \lambda_j, w_j \rangle|^2 + \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 \\ &= |z|^2 |\langle \partial_j f(z), w_j \rangle|^2 + \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 . \end{aligned}$$

Therefore, as $s_z - s_z^2 \geq 0$,

$$-\frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^n \beta_j \geq (s_z^3 - s_z^2) \sum_{j=1}^n |\langle \partial_j f(z), w_j \rangle|^2 - \frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^n \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 .$$

Finally, grouping all terms, we get

$$\begin{aligned} \sum_{j=1}^n |\langle \partial_j (f \circ \varphi_z)(0) \rangle|^2 &\geq s_z^3 \sum_{j=1}^n |\langle \partial_j f(z), w_j \rangle|^2 \\ &\quad + \left[\frac{(s_z - s_z^2)^2}{|z|^2} - \frac{s_z(s_z - s_z^2)}{|z|^2} \right] \sum_{j=1}^n \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 \\ &= s_z^3 \sum_{j=1}^n |\langle \partial_j f(z), w_j \rangle|^2 + \frac{s_z^4 - s_z^3}{|z|^2} \sum_{j=1}^n \frac{|z_j|^2}{|z|^2} |\langle Rf(z), w_j \rangle|^2 . \end{aligned}$$

Now, since $s_z^4 - s_z^3 < 0$, the fact that $|\langle Rf(z), w_j\rangle| \leq \|Rf(z)\|_X$ and (3.1.3) we have that

$$\begin{aligned}\sum_{j=1}^{n} |\langle \partial_j (f\circ\varphi_z)(0)\rangle|^2 &\geq s_z^3 \sum_{j=1}^{n} |\langle \partial_j f(z), w_j\rangle|^2 + \frac{s_z^4 - s_z^3}{|z|^2} \sum_{j=1}^{n} \frac{|z_j|^2}{|z|^2} \|Rf(z)\|_X^2 \\ &\geq s_z^3 \sum_{j=1}^{n} |\langle \partial_j f(z), w_j\rangle|^2 + (s_z^4 - s_z^3) \|\nabla f(z)\|_{X^n}^2 .\end{aligned}$$

Since $\{w_j\}_j \subset X^*$ are arbitrary, we can take $w_j \in X^*$ such that

$$\langle \partial_j f(z), w_j\rangle = \|\partial_j f(z)\|_X = \sup_{w\in X^*} \left|\langle \partial_j f(z), w\rangle_X\right|,$$

then

$$\begin{aligned}\sum_{j=1}^{n} |\langle \partial_j (f\circ\varphi_z)(0)\rangle|^2 &\geq s_z^3 \|\nabla f(z)\|_{X^n}^2 + (s_z^4 - s_z^3) \|\nabla f(z)\|_{X^n}^2 \\ &= (1-|z|^2)^2 \|\nabla f(z)\|_{X^n}^2 ,\end{aligned}$$

and the proof of the theorem is finished. □

A similar result of Lemma 3.1.1 for the invariant gradient $\widetilde{\nabla}$ is the following.

Lemma 3.2.7. *Suppose $\alpha > -1, \beta > 0$ and $t \geq 0$. If*

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z, w\rangle)^\beta}, \qquad z \in \mathbb{B}_n,$$

for some $g \in A_\alpha^1(\mathbb{B}_n, X)$, then

$$\left\|\widetilde{\nabla} f(z)\right\|_{X^n} \leq \sqrt{2}\beta(1-|z|^2)^{1/2} \int_{\mathbb{B}_n} \frac{\|R^{\alpha,t} g(w)\|_X\,\mathrm{d}v_{\alpha+t}(w)}{|1-\langle z, w\rangle|^{\beta+\frac{1}{2}}}$$

for every $z \in \mathbb{B}_n$.

Proof. Fix $z \in \mathbb{B}_n$. By Lemma 2.1.5 we have that

$$f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,t} g(w)\,\mathrm{d}v_{\alpha+t}(w)}{(1-\langle z, w\rangle)^\beta}.$$

Fix $a \in \mathbb{B}_n$ and make the change of variable $w \mapsto \varphi_a(w)$ in the identity

$$f\circ\varphi_a(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,t} g(w)\,\mathrm{d}v_{\alpha+t}(w)}{(1-\langle \varphi_a(z), w\rangle)^\beta},$$

to obtain

$$f\circ\varphi_a(z) = c_{\alpha+t} \int_{\mathbb{B}_n} \frac{R^{\alpha,t} g\circ\varphi_a(w)(1-|\varphi_a(w)|^2)^{\alpha+t}}{(1-\langle \varphi_a(z), \varphi_a(w)\rangle)^\beta} J_a(w)\,\mathrm{d}v(w) \tag{3.2.1}$$

where

$$J_a(w) = \frac{(1-|a|^2)^{n+1}}{|1-\langle w,a\rangle|^{2(n+1)}}.$$

Recall that, see [82, Lemma 1.3],

$$1-\langle \varphi_a(z),\varphi_a(w)\rangle = \frac{(1-|a|^2)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)}.$$

Note that the differential in z at 0 of $[(1-\langle z,w\rangle)/(1-\langle z,a\rangle)]^\beta$ is $\beta(\overline{w}-\overline{a})$ so differentiating in z at 0 under the integral sign of (3.2.1) then produces

$$\widetilde{\nabla} f(a) = c_{\alpha+t}\beta \int_{\mathbb{B}_n} \frac{(\overline{w}-\overline{a})(1-\langle a,w\rangle)^\beta}{(1-|a|^2)^\beta} R^{\alpha,t} g\circ\varphi_a(w) J_a(w)(1-|\varphi_a(w)|^2)^{\alpha+t}\, dv(w).$$

Make the change of variables $w\mapsto \varphi_a(w)$ again, we get

$$\widetilde{\nabla} f(a) = \beta \int_{\mathbb{B}_n} \frac{(\overline{\varphi_a(w)}-\overline{a})R^{\alpha,t}g(w)}{(1-\langle a,w\rangle)^\beta}\, dv_{\alpha+t}(w).$$

Then, applying Proposition 1.2.2 with X^n, we obtain

$$\left\|\widetilde{\nabla} f(a)\right\|_{X^n} = \beta \int_{\mathbb{B}_n} \frac{|\varphi_a(w)-a|\, \|R^{\alpha,t}g(w)\|_X}{|1-\langle a,w\rangle|^\beta}\, dv_{\alpha+t}(w). \tag{3.2.2}$$

The rest is the same as the last part of the proof of [82, Lemma 3.3]. Let $s_a := \sqrt{1-|a|^2}$. By the definition of φ_a in (1.3.1), we have that

$$\begin{aligned}
|\varphi_a(w)-a| &= \frac{\left|a\langle w,a\rangle (s_a-1+|a|^2)/|a|^2 - ws_a\right|}{|1-\langle w,a\rangle|}\\
&= \frac{s_a\left|a\langle w,a\rangle(1-\frac{1-|a|^2}{s_a})/|a|^2 - w\right|}{|1-\langle w,a\rangle|}\\
&= \frac{s_a\left|a\langle w,a\rangle(1-s_a)/|a|^2 - w\right|}{|1-\langle w,a\rangle|}.
\end{aligned}$$

But, by elementary computations, for $w,a\in\mathbb{B}_n$, we get

$$\begin{aligned}
\left|a\langle w,a\rangle(1-s_a)/|a|^2 - w\right|^2 &= |\langle w,a\rangle|^2\frac{(1-s_a)^2}{|a|^2} - 2|\langle w,a\rangle|^2\frac{1-s_a}{|a|^2} + |w|^2\\
&= |\langle w,a\rangle|^2\frac{s_a^2-1}{|a|^2} + |w|^2\\
&= |w|^2 - |\langle w,a\rangle|^2 \le 1-|\langle w,a\rangle|^2\\
&\le 2\,|1-\langle w,a\rangle|.
\end{aligned}$$

This implies that

$$|\varphi_a(w)-a| \le \sqrt{2}\frac{s_a}{|1-\langle w,a\rangle|^{1/2}}.$$

Putting in (3.2.2) completes the proof of the lemma. □

Finally, the last characterization is the following.

Theorem 3.2.8. *Let X be a Banach space. If $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $\gamma > \frac{1}{2}$, then $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$ if and only if the function*

$$(1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} f(z) \right\|_{X^n}$$

is in $L^\infty(\mathbb{B}_n, X)$. Moreoever, $\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} \simeq \sup_{z\in\mathbb{B}_n}(1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} f(z) \right\|_{X^n}$.

Proof. Recall from Theorem 3.2.6 that

$$(1-|z|^2) \|\nabla f(z)\|_{X^n} \leq \left\| \widetilde{\nabla} f(z) \right\|_{X^n}, \qquad z \in \mathbb{B}_n.$$

So the boundedness of $(1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} f(z) \right\|_{X^n}$ implies that of $(1-|z|^2)^{\gamma} \|\nabla f(z)\|_{X^n}$. Further we have that

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)} = \sup_{z\in\mathbb{B}_n}(1-|z|^2)^{\gamma} \|\nabla f(z)\|_{X^n} \leq \sup_{z\in\mathbb{B}_n}(1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} f(z) \right\|_{X^n}.$$

The first implication is done.

On the other hand, if $f \in \mathcal{B}_\gamma(\mathbb{B}_n, X)$, then by Theorem 3.2.1,

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)\, \mathrm{d}v(w)}{(1-\langle z, w\rangle)^{n+\gamma}}, \qquad z \in \mathbb{B}_n,$$

where g is a function in $L^\infty(\mathbb{B}_n, X)$ and $\|g\|_{\infty,X} \lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}$. An application of Lemma 3.2.7 and Theorem 1.4.1 gives

$$\begin{aligned}
\left\| \widetilde{\nabla} f(z) \right\|_{X^n} &\leq \sqrt{2}(n+\gamma)(1-|z|^2)^{\frac{1}{2}} \|g\|_{\infty,X} \int_{\mathbb{B}_n} \frac{\mathrm{d}v(w)}{|1-\langle z, w\rangle|^{n+\gamma+\frac{1}{2}}} \\
&\lesssim \sqrt{2}(n+\gamma)(1-|z|^2)^{\frac{1}{2}} \|g\|_{\infty,X} \frac{1}{(1-|z|^2)^{\gamma-\frac{1}{2}}},
\end{aligned}$$

which imply that

$$(1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} f(z) \right\|_{X^n} \lesssim \sqrt{2}(n+\gamma) \|g\|_{\infty,X} \lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n,X)}.$$

Then, applying supremum over $z \in \mathbb{B}_n$ we complete the proof of the theorem. $\square$

To finish this chapter we have the following consequence of the vector-valued Bloch space $\mathcal{B}(\mathbb{B}_n, X)$, you can follow the same proof of [7, Corollary 5.3], but we include another simpler proof for completeness.

Corollary 3.2.9. *If $f \in \mathcal{B}(\mathbb{B}_n, X)$ then there exists $C > 0$ such that*

$$\|f(z) - f(w)\|_X \leq C\beta(z, w) \|f\|_{\mathcal{B}(\mathbb{B}_n,X)},$$

for any $z, w \in \mathbb{B}_n$.

Proof. Let $w \in \mathbb{B}_n$. It follows from Proposition 3.1.5 and the fact that $\log \frac{1}{1-|z|^2} \lesssim \beta(z,0)$ that

$$\|f(z) - f(0)\|_X \lesssim \beta(z,0)\, \|f\|_{\mathcal{B}(\mathbb{B}_n,X)}\,.$$

Replacing f by $f \circ \varphi_w$ and using Theorem 3.2.8 we have that

$$\|f \circ \varphi_w(z) - f(w)\|_X \lesssim \beta(z,0)\, \|f \circ \varphi_w\|_{\mathcal{B}(\mathbb{B}_n,X)} \lesssim \beta(z,0) \sup_{\zeta \in \mathbb{B}_n} \left\| \widetilde{\nabla}(f \circ \varphi_w)(\zeta) \right\|_{X^n}.$$

Now, we know that the invariant gradient and β are invariant under automorphisms. So, using this fact, a change of variable $z \mapsto \varphi_w(z)$ and Theorem 3.2.8 again, we get that

$$\begin{aligned} \|f(z) - f(w)\|_X &\lesssim \beta(\varphi_w(z), \varphi_w(w)) \sup_{\zeta \in \mathbb{B}_n} \left\| \widetilde{\nabla}(f \circ \varphi_w)(\zeta) \right\|_{X^n} \\ &= \beta(z,w) \sup_{\zeta \in \mathbb{B}_n} \left\| \widetilde{\nabla} f(\zeta) \right\|_{X^n} \lesssim \beta(z,w)\, \|f\|_{\mathcal{B}(\mathbb{B}_n,X)} \end{aligned}$$

and we are done. $\square$

Chapter 4

Small Hankel Operators

The history of Hankel operators begin with the Hardy spaces and the Hankel matrices. Immediately they began very popular and they have known very good applications [62]. Since then many other generalizations are made.

It is well known also that Hankel operators are closely related to Toeplitz operators, another well known type of operator. In fact, many problems of Toeplitz operators can be reformulated in terms of Hankel operators, and vice versa. There are only one type of Hankel operators for Hardy spaces. In the Bergman setting there are two very different Hankel operators, the small (or little) Hankel operator and the big Hankel operator. The main characteristic of the small Hankel operator is that it only depends on the conjugate analytic part of the symbol (for scalar case).

In this chapter we are going to introduce the small Hankel operator with operator-valued symbol and we will characterize the boundedness of these Hankel operators between one vector-valued Bergman space to another different vector-valued Bergman space. Before that we also make some properties of these operators and in the final section we are going to make some conclusions. Note that throughout this chapter, we refer to X or Y any complex Banach space and we fix $\alpha > -1$.

4.1 Basic Properties

In this section we are going to introduce the basic concepts of the small (or little) Hankel operator. In the scalar case the small Hankel operator $h_\varphi \colon A^2_\alpha \to \overline{A^2_\alpha}$ with symbol $\varphi \in L^\infty(\mathbb{C})$ is defined by

$$h_\varphi f(z) := \overline{P_\alpha}(\varphi f)(z), \qquad f \in A^2_\alpha.$$

Since

$$\overline{P_\alpha} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w), \qquad f \in L^2_\alpha$$

is an integral operator we have that h_φ is also an integral operator with the form

$$h_\varphi f(z) = \int_{\mathbb{B}_n} \frac{f(w)\varphi(w)}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w), \qquad f \in A^2_\alpha.$$

The characterization of this operator is widely studied and established, see [83] for example, and it is well-known that has many applications, mostly in physics. We are interested not only in vector-valued functions but in a more generalization of the symbol.

From now on, we fix $T\colon \mathbb{B}_n \to \mathcal{L}(\overline{X}, Y)$ a holomorphic function, i.e., $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, and then the most general small Hankel operator h_T with symbol T is defined as

$$h_T f(z) := \int_{\mathbb{B}_n} \frac{T(w)\overline{f(w)}}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w), \qquad f \in \mathcal{P}(\mathbb{B}_n, X). \tag{4.1.1}$$

It is clear that if T satisfies

$$\int_{\mathbb{B}_n} \frac{\|T(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1-\langle w, z\rangle|^{n+1+\alpha}}\, dv_\alpha(w) < \infty, \qquad \text{for every } z \in \mathbb{C}, \tag{4.1.2}$$

then the small Hankel operator h_T is well-defined for polynomials, which is a dense subspace of $A^p_\alpha(\mathbb{B}_n, X)$ for $1 \leq p < \infty$, by Lemma 2.1.4, meaning h_T is densely defined on $A^p_\alpha(\mathbb{B}_n, X)$ for any $1 \leq p < \infty$. Notice that since

$$|1-\langle z, w\rangle| \geq |1-|\langle z, w\rangle|| \geq (1-|z|), \qquad w \in \mathbb{B}_n, z \in \mathbb{C}$$

if $T \in A^r_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, for some $1 \leq r < \infty$, the condition (4.1.2) also holds. The following technical lemma is useful for what follows.

Lemma 4.1.1. *Let $f \in A^1_\alpha(\mathbb{B}_n, X)$ and $g \in A^1_\alpha(\mathbb{B}_n, Y^*)$. If $T \in A^r_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, for some $1 < r < \infty$, then*

$$\langle h_T f, g\rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \left\langle T(z)\overline{f(z)}, g(z)\right\rangle_Y dv_\alpha(z).$$

Proof. Let $f \in \mathcal{P}(\mathbb{B}_n, X)$ and $g \in \mathcal{P}(\mathbb{B}_n, Y^*)$. By Fubini's theorem and Proposition 2.1.2 we have that

$$\begin{aligned}
\langle h_T f, g\rangle_{\alpha,Y} &= \int_{\mathbb{B}_n} \langle h_T f(w), g(w)\rangle_Y \, dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} g(w) \left(\int_{\mathbb{B}_n} \frac{T(z)\overline{f(z)}}{(1-\langle w, z\rangle)^{n+1+\alpha}}\, dv_\alpha(z) \right) dv_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{g(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w) \right) (T(z)\overline{f(z)})\, dv_\alpha(z) \\
&= \int_{\mathbb{B}_n} \left\langle T(z)\overline{f(z)}, g(z)\right\rangle_Y dv_\alpha(z).
\end{aligned}$$

It is clear that if $T \in A^r_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, for some $1 < r < \infty$, the hypothesis of Fubini's theorem is fulfilled. Indeed, let $1 < r < \infty$ and r' be the conjugate exponent of r. Since f and g are polynomials and using Tonelli's theorem, Theorem 1.4.1 and Hölder's inequality we have that

$$\begin{aligned}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left|g(w)\left(T(z)\overline{f(z)}\right)\right|}{|1-\langle w,z\rangle|^{n+1+\alpha}}\,\mathrm{d}v_\alpha(z)\,\mathrm{d}v_\alpha(w) &\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\|T(z)\|_{\mathcal{L}(\overline{X},Y)}\,\mathrm{d}v_\alpha(z)}{|1-\langle w,z\rangle|^{n+1+\alpha}}\,\mathrm{d}v_\alpha(w)\\ &\lesssim \int_{\mathbb{B}_n}\|T(z)\|_{\mathcal{L}(\overline{X},Y)}\log\frac{1}{1-|z|^2}\,\mathrm{d}v_\alpha(z)\\ &\leq \|T\|_{r,\alpha,\mathcal{L}(\overline{X},Y)}\left(\int_{\mathbb{B}_n}\log^{r'}\frac{1}{1-|z|^2}\,\mathrm{d}v_\alpha(z)\right)^{1/r'}.\end{aligned}$$

Since the last integral is finite we are done. Therefore, the general result follows by the density of the polynomials and Lemma 2.1.4. □

The aim of this chapter is to characterize the general small Hankel operator h_T in terms of properties of the symbol T. The techniques used here are similar to [1]. In order to do that, we need to show the following generalization of Lemma 2.1.5.

Lemma 4.1.2. *Let X be a Banach space. Suppose $t>0$, then*

$$\begin{aligned}\int_{\mathbb{B}_n}\langle f(z),g(z)\rangle_X\,\mathrm{d}v_\alpha(z) &= \int_{\mathbb{B}_n}\left\langle R^{\alpha,t}f(z),g(z)\right\rangle_X\,\mathrm{d}v_{\alpha+t}(z)\\ &= \int_{\mathbb{B}_n}\left\langle f(z),R^{\alpha,t}g(z)\right\rangle_X\,\mathrm{d}v_{\alpha+t}(z)\end{aligned}$$

for any $f\in A^1_\alpha(\mathbb{B}_n,X)$ and $g\in A^1_\alpha(\mathbb{B}_n,X^*)$.

Proof. Let $f\in\mathcal{P}(\mathbb{B}_n,X)$ and $g\in\mathcal{P}(\mathbb{B}_n,X^*)$. By Proposition 2.1.3 we have that the second integral is equal to

$$\int_{\mathbb{B}_n}(g(z))\left(\int_{\mathbb{B}_n}\frac{f(w)\,\mathrm{d}v_\alpha(w)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}\right)\mathrm{d}v_{\alpha+t}(z).$$

Then Lemma 1.2.1, Fubini's theorem and Proposition 2.1.2 show that it is equal to

$$\begin{aligned}\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\frac{g(z)\,\mathrm{d}v_{\alpha+t}(z)}{(1-\langle w,z\rangle)^{n+1+\alpha+t}}\right)(f(w))\,\mathrm{d}v_\alpha(w) &= \int_{\mathbb{B}_n}(g(w))\,(f(w))\,\mathrm{d}v_\alpha(w)\\ &= \int_{\mathbb{B}_n}\langle f(w),g(w)\rangle_X\,\mathrm{d}v_\alpha(w).\end{aligned}$$

Here the hypothesis of Fubini's theorem is fulfilled due to Theorem 1.4.1 and the fact that f and g are polynomials. The general result follows by the density of the polynomials and Lemma 2.1.4. Note that the second identity is proved in the same way. □

Corollary 4.1.3. *Let X,Y be two Banach spaces. Suppose $t>0$ and $1<p<\infty$. If $T\in A^{p'}_\alpha(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))$, where p' is the conjugate exponent of p, then the equality*

$$\int_{\mathbb{B}_n}\left\langle T(z)\overline{f(z)},g(z)\right\rangle_Y\mathrm{d}v_\alpha(z) = \int_{\mathbb{B}_n}\left\langle R^{\alpha,t}T(z)\overline{f(z)},g(z)\right\rangle_Y\mathrm{d}v_{\alpha+t}(z)$$

holds whenever $f\in A^p_\alpha(\mathbb{B}_n,X)$ and $g\in A^1_\alpha(\mathbb{B}_n,Y^*)$.

Proof. It is a direct consequence of Lemma 4.1.2 and observing that $R^{\alpha,t}(T(z)\overline{f(z)}) = (R^{\alpha,t}T(z))(\overline{f(z)})$ by definition and the fact that $T\overline{f} \in A^1_\alpha(\mathbb{B}_n, Y)$ when $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $T \in A^{p'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ by Hölder's inequality. □

4.2 Boundedness. Case $p \leq q$

Here we present one of the most important results of this chapter, the boundedness of the small Hankel operators h_T with symbol $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)$. In this section we are going to characterize bounded small Hankel operators h_T between $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ for $1 < p \leq q < \infty$. In order to do that we begin with the case $p = q$, which is the most simple case.

Recall the small Hankel operator h_T is defined as

$$h_T f(z) := \int_{\mathbb{B}_n} \frac{T(w)\overline{f(w)}}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w),$$

for any $f \in \mathcal{P}(\mathbb{B}_n, X)$.

Theorem 4.2.1. *Let $1 < p < \infty$. The small Hankel operator $h_T\colon A^p_\alpha(\mathbb{B}_n, X) \to A^p_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if $T \in \mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Moreover, $\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^p_\alpha(\mathbb{B}_n,Y)} \simeq \|T\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}$.*

Proof. Let $1 < p < \infty$ and let p' be the conjugate exponent of p. Suppose first that h_T is a bounded linear operator on $A^p_\alpha(\mathbb{B}_n, X)$ to $A^p_\alpha(\mathbb{B}_n, Y)$ with norm $\|h_T\| := \|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^p_\alpha(\mathbb{B}_n,Y)}$. Let $x \in X$ and $t > 0$. Fix $z \in \mathbb{B}_n$ and put

$$f(w) := \frac{x}{(1 - \langle w, z\rangle)^t}, \qquad w \in \mathbb{B}_n.$$

It is clear that $f \in A^p_\alpha(\mathbb{B}_n, X)$ if $t > 0$ is large enough. Indeed, it is easy to see that $f \in \mathcal{H}(\mathbb{B}_n, X)$. Moreover, by Theorem 1.4.1, we have that

$$\begin{aligned}
\|f\|_{p,\alpha,X} &= \left(\int_{\mathbb{B}_n} \|f(w)\|^p_X\, dv_\alpha(w)\right)^{1/p}\\
&\lesssim \|x\|_X \left(\int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha\, dv(w)}{|1 - \langle z, w\rangle|^{tp}}\right)^{1/p}\\
&\lesssim \frac{\|x\|_X}{(1 - |z|^2)^{t-(n+1+\alpha)/p}},
\end{aligned}$$

if $t > (n + 1 + \alpha)/p$. Therefore, with this particular f and Proposition 2.1.3, we have that

$$\begin{aligned}
h_T f(z) &= \int_{\mathbb{B}_n} \frac{T(w)\overline{f(w)}}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w)\\
&= \int_{\mathbb{B}_n} \frac{T(w)(\overline{x})}{(1 - \langle z, w\rangle)^{n+1+\alpha+t}}\, dv_\alpha(w) = R^{\alpha,t}T(z)(\overline{x}).
\end{aligned}$$

Then, by Theorem 2.1.1 and the hypothesis, it implies that

$$\left\|R^{\alpha,t}T(z)(\overline{x})\right\|_Y = \|h_T f(z)\|_Y \leq \frac{\|h_T f\|_{p,\alpha,Y}}{(1-|z|^2)^{(n+1+\alpha)/p}} \lesssim \frac{\|h_T\| \, \|x\|_X}{(1-|z|^2)^t}.$$

Since $x \in X$ is arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$, we get that

$$\left\|R^{\alpha,t}T(z)\right\|_{\mathcal{L}(\overline{X},Y)} \lesssim \frac{\|h_T\|}{(1-|z|^2)^t}.$$

Finally, by Theorem 3.1.3, this implies that T is in the Bloch space $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ with $\|T\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \lesssim \|h_T\|$.

Conversely, suppose that $T \in \mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $g \in A^{p'}_\alpha(\mathbb{B}_n, Y^*)$ and let $t > 0$. By Lemma 3.1.4 and Lemma 4.1.1, we have that

$$\langle h_T f, g\rangle_{\alpha,Y} = \int_{\mathbb{B}_n} \left\langle T(z)\overline{f(z)}, g(z)\right\rangle_Y \, \mathrm{d}v_\alpha(z).$$

Then, by Lemma 3.1.4 again, Corollary 4.1.3, Theorem 3.1.3 and Hölder's inequality, we obtain that

$$\begin{aligned}
\left|\langle h_T f, g\rangle_{\alpha,Y}\right| &= \left|\int_{\mathbb{B}_n} \left\langle T(z)\overline{f(z)}, g(z)\right\rangle_Y \, \mathrm{d}v_\alpha(z)\right| \\
&= \left|\int_{\mathbb{B}_n} \left\langle R^{\alpha,t}T(z)\overline{f(z)}, g(z)\right\rangle_Y \, \mathrm{d}v_{\alpha+t}(z)\right| \\
&\lesssim \int_{\mathbb{B}_n} \left\|R^{\alpha,t}T(z)\right\|_{\mathcal{L}(\overline{X},Y)} \left\|\overline{f(z)}\right\|_{\overline{X}} \|g(z)\|_{Y^*} (1-|z|^2)^t \, \mathrm{d}v_\alpha(z) \\
&\lesssim \|T\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \int_{\mathbb{B}_n} \|f(z)\|_X \, \|g(z)\|_{Y^*} \, \mathrm{d}v_\alpha(z) \\
&\leq \|T\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \|f\|_{p,\alpha,X} \, \|g\|_{p',\alpha,Y^*}.
\end{aligned}$$

Therefore, by duality, we obtain

$$\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^p_\alpha(\mathbb{B}_n,Y)} = \sup_{\substack{\|f\|_{p,\alpha,X}=1 \\ \|g\|_{p',\alpha,Y^*}=1}} \left|\langle h_T f, g\rangle_{\alpha,Y}\right| \lesssim \|T\|_{\mathcal{B}(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.$$

Hence the proof of the theorem is complete. □

The next theorem is the characterization of bounded small Hankel operators h_T between $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ for $1 < p < q < \infty$.

Theorem 4.2.2. *Let $1 < p < q < \infty$. The small Hankel operator $h_T \colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ where*

$$\gamma := 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right).$$

Moreover, $\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \simeq \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}$.

Proof. Let $1 < p < q < \infty$ and, p' and q' be the conjugate exponents of p and q respectively. Suppose first that h_T is a bounded linear operator on $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ with norm $\|h_T\| := \|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)}$. Let $x \in X$ and $t > 0$. Fix $z \in \mathbb{B}_n$ and put

$$f(w) := \frac{x}{(1-\langle w, z\rangle)^t}, \qquad w \in \mathbb{B}_n.$$

Let us see that $f \in A^p_\alpha(\mathbb{B}_n, X)$ for $t > 0$ big enough. Indeed, it is clear that $f \in \mathcal{H}(\mathbb{B}_n, X)$. Moreover, by Theorem 1.4.1, we have that

$$\begin{aligned}\|f\|_{p,\alpha,X} &= \left(\int_{\mathbb{B}_n} \|f(w)\|_X^p \, dv_\alpha(w)\right)^{1/p}\\ &\lesssim \|x\|_X \left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{|1-\langle z, w\rangle|^{tp}} \, dv(w)\right)^{1/p}\\ &\lesssim \frac{\|x\|_X}{(1-|z|^2)^{t-(n+1+\alpha)/p}}\end{aligned}$$

only if $t > (n+1+\alpha)/p$ holds. Then, by Proposition 2.1.3 it is easy to see that

$$h_T f(z) = R^{\alpha,t}T(z)(\overline{x}).$$

Therefore, by Theorem 2.1.1, we have

$$\begin{aligned}\left\|R^{\alpha,t}T(z)(\overline{x})\right\|_Y = \|h_T f(x)\|_Y &\le \frac{\|h_T f\|_{q,\alpha,Y}}{(1-|z|^2)^{(n+1+\alpha)/q}}\\ &\le \frac{\|h_T\|\,\|f\|_{p,\alpha,X}}{(1-|z|^2)^{(n+1+\alpha)/q}} \lesssim \frac{\|h_T\|\,\|x\|_X}{(1-|z|^2)^{t+(n+1+\alpha)(1/q-1/p)}}\end{aligned}$$

Since $x \in X$ is arbitrary and $\|x\|_X = \|\overline{x}\|_{\overline{X}}$ we obtain that

$$\left\|R^{\alpha,t}T(z)\right\|_{\mathcal{L}(\overline{X},Y)} \lesssim \frac{\|h_T\|}{(1-|z|^2)^{\gamma+t-1}}$$

and it implies that

$$\sup_{z\in\mathbb{B}_n} (1-|z|^2)^{\gamma+t-1} \left\|R^{\alpha,t}T(z)\right\|_{\mathcal{L}(\overline{X},Y)} \lesssim \|h_T\|$$

By Corollary 3.2.3 this means that $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and $\|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \lesssim \|h_T\|$.

Conversely, suppose that $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $g \in A^{q'}_\alpha(\mathbb{B}_n, Y^*)$ and let $t > 0$. Since $0 < \gamma < 1$, Corollary 3.2.4 implies that $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)) \subset A^{p'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, so by Lemma 4.1.1 and Corollaries 4.1.3 and

3.2.3, we have that

$$
\begin{aligned}
|\langle h_T f, g\rangle|_{\alpha,Y} &= \left| \int_{\mathbb{B}_n} \left\langle T(z)\overline{f(z)}, g(z) \right\rangle_Y \, \mathrm{d}v_\alpha(z) \right| \\
&= \left| \int_{\mathbb{B}_n} \left\langle R^{\alpha,t+1} T(z)\overline{f(z)}, g(z) \right\rangle_Y \, \mathrm{d}v_{\alpha+t+1}(z) \right| \\
&\lesssim \int_{\mathbb{B}_n} \left\| R^{\alpha,t+1} T(z) \right\|_{\mathcal{L}(\overline{X},Y)} \left\| \overline{f(z)} \right\|_{\overline{X}} \|g(z)\|_{Y^*} (1-|z|^2)^{\alpha+t+1} \, \mathrm{d}v(z) \\
&\lesssim \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \int_{\mathbb{B}_n} \|f(z)\|_X \|g(z)\|_{Y^*} (1-|z|^2)^{\alpha-\gamma+1} \, \mathrm{d}v(z).
\end{aligned}
$$

By Hölder's inequality the last integral is less or equal than

$$
\left(\int_{\mathbb{B}_n} \|f(z)\|_X^q (1-|z|^2)^{\alpha-q(\gamma-1)} \, \mathrm{d}v(z) \right)^{1/q} \left(\int_{\mathbb{B}_n} \|g(z)\|_{Y^*}^{q'} (1-|z|^2)^{\alpha} \, \mathrm{d}v(z) \right)^{1/q'}.
$$

Now, since $p < q$ and using Theorem 2.1.1, we have that

$$
\begin{aligned}
\|f(z)\|_X^q = \|f(z)\|_X^p \|f(z)\|_X^{q-p} &\leq \frac{\|f(z)\|_X^p \|f\|_{p,\alpha,X}^{q-p}}{(1-|z|^2)^{(q-p)(n+1+\alpha)/p}} \\
&= \frac{\|f(z)\|_X^p \|f\|_{p,\alpha,X}^{q-p}}{(1-|z|^2)^{-q(\gamma-1)}}.
\end{aligned}
$$

Then

$$
\left(\int_{\mathbb{B}_n} \|f(z)\|_X^q (1-|z|^2)^{\alpha-q(\gamma-1)} \, \mathrm{d}v(z) \right)^{1/q}
$$

is less or equal than

$$
\|f\|_{p,\alpha,X}^{1-p/q} \left(\int_{\mathbb{B}_n} \|f(z)\|_X^p \frac{(1-|z|^2)^{\alpha-q(\gamma-1)}}{(1-|z|^2)^{-q(\gamma-1)}} \, \mathrm{d}v(z) \right)^{1/q} \lesssim \|f\|_{p,\alpha,X}.
$$

Summing up, we obtain that

$$
\left| \langle h_T f, g\rangle_{\alpha,Y} \right| \lesssim \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))} \|f\|_{p,\alpha,X} \|g\|_{q',\alpha,Y^*}.
$$

Therefore, by duality, we finally obtain that

$$
\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} = \sup_{\substack{\|f\|_{p,\alpha,X}=1 \\ \|g\|_{q',\alpha,Y^*}=1}} \left| \langle h_T f, g\rangle_{\alpha,Y} \right| \lesssim \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}.
$$

Hence the proof of the theorem is complete. □

4.3 Boundedness. Case $p > q$

In this section we are going to characterize boundedness of small Hankel operator h_T between $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ for $1 < q < p < \infty$. Before that we need some preliminary results. The following is a characterization lemma of vector-valued Bergman spaces.

Lemma 4.3.1. *Let $t \geq 0$ be a fixed number and $1 < p < \infty$. Suppose that $f \in \mathcal{H}(\mathbb{B}_n, X)$. If there is a constant $C > 0$ such that for all $0 < r \leq 1$ and any r-lattice $\{a_k\}_k \subset \mathbb{B}_n$ the inequality*

$$\sum_{k=1}^{\infty} (1 - |a_k|^2)^{tp+(n+1+\alpha)} \left\| R^{\alpha,t} f(a_k) \right\|_X^p \leq C^p \tag{4.3.1}$$

holds, then $f \in A^p_\alpha(\mathbb{B}_n, X)$ and $\|f\|_{p,\alpha,X} \lesssim C$.

Proof. Suppose that (4.3.1) holds and let p' the conjugate exponent of p. Let also $g \in A^{p'}_\alpha(\mathbb{B}_n, X^*)$. Since

$$t \geq 0 > -\frac{1+\alpha}{p} = \frac{1+\alpha}{p'} - (1+\alpha)$$

we have that

$$t + (n+1+\alpha) > n + \frac{1+\alpha}{p'}.$$

Then, by Theorem 2.3.5 (ii) (with $b = t + (n+1+\alpha)$), we have that there exist a r-lattice $\{a_k\}_k \subset \mathbb{B}_n$ and a sequence $\{\lambda_k\}_k \in \ell^{p'}(X^*)$ such that

$$g(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1 - |a_k|^2)^{t+(n+1+\alpha)-(n+1+\alpha)/p'}}{(1 - \langle z, a_k \rangle)^{n+1+\alpha+t}} = \sum_{k=1}^{\infty} \lambda_k \frac{(1 - |a_k|^2)^{t+(n+1+\alpha)/p}}{(1 - \langle z, a_k \rangle)^{n+1+\alpha+t}}$$

and $\|\{\lambda_k\}_k\|_{\ell^{p'}(X^*)} \lesssim \|g\|_{p',\alpha,X^*}$. If $f \in \mathcal{H}(\mathbb{B}_n, X)$, by Fubini's theorem and Proposition 2.1.3, then

$$\begin{aligned}
\langle f, g \rangle_{\alpha,X} &= \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_X \, dv_\alpha(z) \\
&= \int_{\mathbb{B}_n} \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{t+(n+1+\alpha)/p}}{(1 - \langle a_k, z \rangle)^{n+1+\alpha+t}} \langle f(z), \lambda_k \rangle_X \, dv_\alpha(z) \\
&= \sum_{k=1}^{\infty} (1 - |a_k|^2)^{t+(n+1+\alpha)/p} \lambda_k \left(\int_{\mathbb{B}_n} \frac{f(z)}{(1 - \langle a_k, z \rangle)^{n+1+\alpha+t}} \, dv_\alpha(z) \right) \\
&= \sum_{k=1}^{\infty} (1 - |a_k|^2)^{t+(n+1+\alpha)/p} \lambda_k \left(R^{\alpha,t} f(a_k) \right) \\
&= \sum_{k=1}^{\infty} (1 - |a_k|^2)^{t+(n+1+\alpha)/p} \left\langle R^{\alpha,t} f(a_k), \lambda_k \right\rangle_X .
\end{aligned}$$

Then, by Hölder's inequality and the hypothesis (4.3.1), we have that

$$
\begin{aligned}
|\langle f, g\rangle_{\alpha,X}| &\leq \sum_{k=1}^{\infty} \|\lambda_k\|_{X^*} (1-|a_k|^2)^{t+(n+1+\alpha)/p} \left\|R^{\alpha,t} f(a_k)\right\|_X \\
&\leq \|\{\lambda_k\}_k\|_{\ell^{p'}(X^*)} \left(\sum_{k=1}^{\infty} (1-|a_k|^2)^{tp+(n+1+\alpha)} \left\|R^{\alpha,t} f(a_k)\right\|_X^p\right)^{\frac{1}{p}} \\
&\leq \|\{\lambda_k\}_k\|_{\ell^{p'}(X^*)} C.
\end{aligned}
$$

Therefore, by duality in Theorem 2.2.4,

$$
\|f\|_{p,\alpha,X} = \sup_{g \in A^{p'}_\alpha(\mathbb{B}_n, X^*)} \frac{|\langle f, g\rangle_{\alpha,X}|}{\|g\|_{p',\alpha,X^*}} \leq C \sup_{g \in A^{p'}_\alpha(\mathbb{B}_n, X^*)} \frac{\|\{\lambda_k\}_k\|_{\ell^{p'}(X^*)}}{\|g\|_{p',\alpha,X^*}} \lesssim C.
$$

This completes the proof of the lemma. □

Lemma 4.3.2. *Let $1 < q < p < \infty$ and $\mu = \{\mu_k\}_k \subset \mathcal{L}(X,Y)$ be a sequence. If there exists $C > 0$ such that*

$$
\|\{\mu_k(\lambda_k)\}_k\|_{\ell^q(Y)} = \left(\sum_{k=1}^{\infty} \|\mu_k(\lambda_k)\|_Y^q\right)^{\frac{1}{q}} \leq C \|\{\lambda_k\}_k\|_{\ell^p(X)} \tag{4.3.2}
$$

holds for any sequence $\{\lambda_k\}_k \in \ell^p(X)$, then $\mu \in \ell^t(\mathcal{L}(X,Y))$ where

$$
\frac{1}{t} = \frac{1}{q} - \frac{1}{p}
$$

and $\|\mu\|_{\ell^t(\mathcal{L}(X,Y))} \lesssim C$.

Proof. For any $k \in \mathbb{N}$ we have that for all $\varepsilon > 0$ there is $\lambda_k \in X$ with $\|\lambda_k\|_X = 1$ such that

$$
(1-\varepsilon) \|\mu_k\|_{\mathcal{L}(X,Y)} \leq \|\mu_k(\lambda_k)\|_Y .
$$

Fix $\varepsilon = 1/2$ and a positive large integer $N > 1$. We define the sequence $\lambda^N = \{\lambda_k^N\}_k \subset X$ by

$$
\lambda_k^N := \lambda_k \|\mu_k\|_{\mathcal{L}(X,Y)}^{\frac{q}{p-q}}
$$

if $k \leq N$ and 0 otherwise. Since this sequence has a finite number of elements it is clear that it belongs to $\ell^p(X)$. Thus, applying (4.3.2) to this sequence, taking into account that

$$
\|\lambda^N\|_{\ell^p(X)}^q = \left(\sum_{k=1}^{\infty} \|\lambda_k^N\|_X^p\right)^{q/p} = \left(\sum_{k=1}^{N} \|\lambda_k\|_X^p \|\mu_k\|_{\mathcal{L}(X,Y)}^t\right)^{q/p} = \left(\sum_{k=1}^{N} \|\mu_k\|_{\mathcal{L}(X,Y)}^t\right)^{q/p}
$$

and

$$
\sum_{k=1}^{\infty} \|\mu_k(\lambda_k^N)\|_Y^q = \sum_{k=1}^{N} \|\mu_k\|_{\mathcal{L}(X,Y)}^{\frac{q^2}{p-q}} \|\mu_k(\lambda_k)\|_Y^q \geq \frac{1}{2^q} \sum_{k=1}^{N} \|\mu_k\|_{\mathcal{L}(X,Y)}^{\frac{q^2}{p-q}+q} = \frac{1}{2^q} \sum_{k=1}^{N} \|\mu_k\|_{\mathcal{L}(X,Y)}^t ,
$$

we obtain that

$$\frac{1}{2^q}\sum_{k=1}^{N}\|\mu_k\|_{\mathcal{L}(X,Y)}^t \leq \sum_{k=1}^{\infty}\left\|\mu_k(\lambda_k^N)\right\|_Y^q \leq C^q\left\|\lambda_k^N\right\|_{\ell^p(X)}^q = C^q\left(\sum_{k=1}^{N}\|\mu_k\|_{\mathcal{L}(X,Y)}^t\right)^{q/p}$$

which is equivalent to

$$\left(\sum_{k=1}^{N}\|\mu_k\|_{\mathcal{L}(X,Y)}^t\right)^{\frac{p-q}{p}} \leq (2C)^q.$$

The result follows from this and tending $N \to \infty$. □

Now we are in the situation to prove the remaining case of $q < p$.

Theorem 4.3.3. *Let $1 < q < p < \infty$. If Y has finite cotype, then the small Hankel operator $h_T\colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ where*

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}.$$

Moreover, $\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \simeq \|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)}$.

Proof. Suppose first that $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$ and let q' be the conjugate exponent of q. By duality in Theorem 2.2.4, Lemma 4.1.1 and Hölder's inequality two times, (first with (q, q') then with $(p/q, p/(p-q))$), we have that

$$\begin{aligned}\|h_T f\|_{q,\alpha,Y} &= \sup_{\|g\|_{q',\alpha,Y^*}=1}\left|\langle h_T f, g\rangle_{\alpha,Y}\right| \leq \sup_{\|g\|_{q',\alpha,Y^*}=1}\int_{\mathbb{B}_n}\left|\left\langle T(z)\overline{f(z)}, g(z)\right\rangle_Y\right| \,\mathrm{d}v_\alpha(z)\\ &\leq \sup_{\|g\|_{q',\alpha,Y^*}=1}\|g\|_{q',\alpha,Y^*}\left(\int_{\mathbb{B}_n}\left\|T(z)\overline{f(z)}\right\|_Y^q \,\mathrm{d}v_\alpha(z)\right)^{\frac{1}{q}}\\ &\leq \|f\|_{p,\alpha,X}\,\|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)}\,.\end{aligned}$$

Then, h_T is bounded on $A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ and $\|h_T\|_{A^p_\alpha(\mathbb{B}_n,X)\to A^q_\alpha(\mathbb{B}_n,Y)} \leq \|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)}$.

Conversely, suppose that $h_T\colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ is bounded and let $r_k(t) := \mathrm{sgn}(\sin(2^{\mathrm{k}}\pi \mathrm{t}))$ be the Rademacher functions. Let also b be large enough so that

$$b > n + \frac{1+\alpha}{p} \geq 0.$$

Fix $r > 0$ and consider $\{a_k\}_k \subset \mathbb{B}_n$ an r-lattice and $\{D_k\}_k$ the associated sets in Lemma 2.3.3. By Theorem 2.3.5 (i), we have that for every sequence $\{\lambda_k\}_k \in \ell^p(X)$ the function

$$g_t(z) := \sum_{k=1}^{\infty}\lambda_k r_k(t)\frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{(1-\langle z, a_k\rangle)^b}, \qquad z \in \mathbb{B}_n,$$

belongs to $A^p_\alpha(\mathbb{B}_n, X)$ with $\|g_t\|_{p,\alpha,X} \lesssim \|\{\lambda_k r_k(t)\}_k\|_{\ell^p(X)} \lesssim \|\{\lambda_k\}_k\|_{\ell^p(X)}$, for almost every $t \in (0,1)$. Denote by

$$g_k(z) := \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{(1-\langle z, a_k\rangle)^b}, \qquad z \in \mathbb{B}_n.$$

Since $h_T \colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ is bounded, we have that

$$\begin{aligned}\|h_T g_t\|^q_{q,\alpha,Y} &= \int_{\mathbb{B}_n} \left\| \sum_{k=1}^{\infty} r_k(t) h_T(\lambda_k g_k)(z) \right\|^q_Y \mathrm{d}v_\alpha(z) \\ &\leq \|h_T\|^q \, \|g_t\|^q_{p,\alpha,X} \lesssim \|h_T\|^q \, \|\{\lambda_k\}_k\|^q_{\ell^p(X)},\end{aligned}$$

for almost every $t \in (0,1)$. Integrating both sides respect to t from 0 to 1 and using Tonelli's theorem we obtain that

$$\int_{\mathbb{B}_n} \int_0^1 \left\| \sum_{k=1}^{\infty} r_k(t) h_T(\lambda_k g_k)(z) \right\|^q_Y \mathrm{d}t \, \mathrm{d}v_\alpha(z) \lesssim \|h_T\|^q \, \|\{\lambda_k\}_k\|^q_{\ell^p(X)}.$$

Let $2 \leq s < \infty$ be the cotype of Y (see Section 1.7 for more information). By the definition of cotype (1.7.3) and Kahane inequality in Theorem 1.7.1 we get

$$\begin{aligned}\left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|^s_Y \right)^{q/s} &\lesssim \left(\int_0^1 \left\| \sum_{k=1}^{\infty} r_k(t) h_T(\lambda_k g_k)(z) \right\|^s_Y \mathrm{d}t \right)^{q/s} \\ &\lesssim \int_0^1 \left\| \sum_{k=1}^{\infty} r_k(t) h_T(\lambda_k g_k)(z) \right\|^q_Y \mathrm{d}t,\end{aligned}$$

which implies that

$$\int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|^s_Y \right)^{q/s} \mathrm{d}v_\alpha(z) \lesssim \|h_T\|^q \, \|\{\lambda_k\}_k\|^q_{\ell^p(X)}. \tag{4.3.3}$$

Now, we claim that

$$\sum_{k=1}^{\infty} \int_{D_k} \|h_T(\lambda_k g_k)(z)\|^q_Y \, \mathrm{d}v_\alpha(z) \lesssim \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|^s_Y \right)^{q/s} \mathrm{d}v_\alpha(z). \tag{4.3.4}$$

Indeed, if $s/q \leq 1$, it directly follows from Tonelli's theorem

$$\begin{aligned}\sum_{k=1}^{\infty} \int_{D_k} \|h_T(\lambda_k g_k)(z)\|^q_Y \, \mathrm{d}v_\alpha(z) &\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|^q_Y \right)^{\frac{s}{q}\frac{q}{s}} \mathrm{d}v_\alpha(z) \\ &\leq \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|^s_Y \right)^{q/s} \mathrm{d}v_\alpha(z).\end{aligned}$$

If $s/q > 1$, notice first that by Tonelli's theorem

$$\sum_{k=1}^{\infty} \int_{D_k} \|h_T(\lambda_k g_k)(z)\|_Y^q \, dv_\alpha(z) = \int_{\mathbb{B}_n} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|_Y^q \, \chi_{D_k}(z) \right)^{\frac{s}{q}\frac{q}{s}} dv_\alpha(z)$$

then we can apply Hölder's inequality, with $(s/q, s/(s-q))$, to obtain that

$$\begin{aligned}\left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|_Y^q \, \chi_{D_k}(z) \right)^{\frac{s}{q}\frac{q}{s}} &\leq \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|_Y^s \right)^{\frac{q}{s}} \left(\sum_{k=1}^{\infty} \chi_{D_k}(z) \right)^{1-\frac{q}{s}} \\ &\leq N^{1-q/s} \left(\sum_{k=1}^{\infty} \|h_T(\lambda_k g_k)(z)\|_Y^s \right)^{q/s}.\end{aligned}$$

Recall that each $z \in \mathbb{B}_n$ belongs to at most $N \in \mathbb{N}$ of the sets D_k, see Theorem 2.3.2 and Lemma 2.3.3. Then, (4.3.4) holds. On the other hand, by Theorem 1.5.2, we have that

$$\|h_T(\lambda_k g_k)(a_k)\|_Y^q \lesssim \frac{1}{(1-|a_k|^2)^{n+1+\alpha}} \int_{D_k} \|h_T(\lambda_k g_k)(z)\|_Y^q \, dv_\alpha(z).$$

Note that $h_T f = P_\alpha(T\overline{f})$ so $h_T f \in \mathcal{H}(\mathbb{B}_n, Y)$ for any $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Then, combining it with (4.3.4) and (4.3.3) we obtain that

$$\sum_{k=1}^{\infty} (1-|a_k|^2)^{n+1+\alpha} \|h_T(\lambda_k g_k)(a_k)\|_Y^q \lesssim \|h_T\|^q \, \|\{\lambda_k\}_k\|_{\ell^p(X)}^q .$$

If we calculate, using Proposition 2.1.3, we get that

$$\begin{aligned} h_T(\lambda_k g_k)(a_k) &= \int_{\mathbb{B}_n} \frac{T(z)(\overline{\lambda_k g_k(z)})}{(1-\langle a_k, z\rangle)^{n+1+\alpha}} \, dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p} T(z)(\overline{\lambda_k})}{(1-\langle a_k, z\rangle)^{n+1+\alpha+b}} \, dv_\alpha(z) \\ &= (1-|a_k|^2)^{b-(n+1+\alpha)/p} \left(\int_{\mathbb{B}_n} \frac{T(z) \, dv_\alpha(z)}{(1-\langle a_k, z\rangle)^{n+1+\alpha+b}} \right) (\overline{\lambda_k}) \\ &= (1-|a_k|^2)^{b-(n+1+\alpha)/p} R^{\alpha,b} T(a_k)(\overline{\lambda_k}), \end{aligned}$$

then we have that

$$\sum_{k=1}^{\infty} (1-|a_k|^2)^{(b+(n+1+\alpha)/t)q} \left\| R^{\alpha,b} T(a_k)(\overline{\lambda_k}) \right\|_Y^q \lesssim \|h_T\|^q \, \|\{\lambda_k\}_k\|_{\ell^p(X)}^q$$

which is equivalent to

$$\sum_{k=1}^{\infty} \left\| (1-|a_k|^2)^{b+(n+1+\alpha)/t} R^{\alpha,b} T(a_k)(\overline{\lambda_k}) \right\|_Y^q \lesssim \|h_T\|^q \, \left\| \{\overline{\lambda_k}\}_k \right\|_{\ell^p(\overline{X})}^q .$$

Since the sequence $\{\overline{\lambda_k}\}_k \in \ell^p(\overline{X})$ is arbitrary, Lemma 4.3.2 implies that the sequence $\{(1-|a_k|^2)^{b+(n+1+\alpha)/t} R^{\alpha,b}T(a_k)\}_k \in \ell^t(\mathcal{L}(\overline{X},Y))$ and

$$\sum_{k=1}^{\infty}(1-|a_k|^2)^{bt+(n+1+\alpha)} \left\|R^{\alpha,b}T(a_k)\right\|^t_{\mathcal{L}(\overline{X},Y)} \lesssim \|h_T\|^t .$$

Finally, by Lemma 4.3.1, we have that $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$ and $\|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)} \lesssim \|h_T\|$. This completes the proof of the theorem. □

4.4 Conclusions and Examples

In this section we summarize all the important results we have obtained in the previous sections and then we apply it to different contexts and examples. The general theorem is the following.

Theorem 4.4.1. *Let $1 < p < \infty$ and $1 < q < \infty$. Suppose $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$. Let the small Hankel operator $h_T \colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, Y)$ be the linear operator defined by (4.1.1).*

(i) If $1 < p \le q < \infty$, then h_T is bounded if and only if $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$ where

$$\gamma = 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right)$$

and, moreover, $\|h_T\| \simeq \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n,\mathcal{L}(\overline{X},Y))}$.

(ii) If $1 < q < p < \infty$ and Y has finite cotype, then h_T is bounded if and only if $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$ where

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}$$

and, moreover, $\|h_T\| \simeq \|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)}$.

Proof. The first case follows from Theorems 4.2.1 and 4.2.2. The second case is Theorem 4.3.3. □

Notice that since X is isometrically equivalent to $\overline{X}$ we also have the same result if we replace $\overline{X}$ by X. Now we present some important examples of small Hankel operators. For example, if $Y = \mathbb{C}$ we get the small Hankel operator $h_{\overline{\varphi}}$ with symbol $\varphi \in \mathcal{H}(\mathbb{B}_n, X^*)$. The small Hankel operator $h_{\overline{\varphi}} \colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha$ is the linear operator defined as

$$h_{\overline{\varphi}}f(z) = \int_{\mathbb{B}_n} \frac{\langle f(w), \varphi(w)\rangle_X}{(1-\langle w,z\rangle)^{n+1+\alpha}}\, dv_\alpha(w), \qquad z \in \mathbb{B}_n, f \in \mathcal{P}(\mathbb{B}_n, X).$$

Theorem 4.4.2. *Let $1 < p < \infty$ and $1 < q < \infty$. Suppose $\varphi \in \mathcal{H}(\mathbb{B}_n, X^*)$.*

(i) If $1 < p \leq q < \infty$*, then* $h_{\overline{\varphi}}\colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha$ *is bounded if and only if* $\varphi \in \mathcal{B}_\gamma(\mathbb{B}_n, X^*)$ *where*

$$\gamma = 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right)$$

and, moreover, $\|h_{\overline{\varphi}}\| \simeq \|\varphi\|_{\mathcal{B}_\gamma(\mathbb{B}_n, X^*)}$.

(ii) If $1 < q < p < \infty$*, then* $h_{\overline{\varphi}}\colon A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha$ *is bounded if and only if* $\varphi \in A^t_\alpha(\mathbb{B}_n, X^*)$ *where*

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}$$

and, moreover, $\|h_{\overline{\varphi}}\| \simeq \|\varphi\|_{t,\alpha,X^*}$.

Notice that when $\varphi \in \mathcal{H}(\mathbb{B}_n, X^*)$ and by definition we have that

$$\overline{h_{\overline{\varphi}} f(z)} = \int_{\mathbb{B}_n} \frac{\overline{\langle f(w), \varphi(w)\rangle_X}}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w) = \int_{\mathbb{B}_n} \frac{\overline{f(w)}\varphi(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w),$$

for any $z \in \mathbb{B}_n$ and $f \in \mathcal{P}(\mathbb{B}_n, X)$. Then, we define $T_\varphi \in \mathcal{H}(\mathbb{B}_n, \overline{X}^*)$ by

$$T_\varphi(z)(\overline{x}) := \overline{x}(\varphi(z)), \tag{4.4.1}$$

for every $z \in \mathbb{B}_n$ and $x \in X$. This way we get that

$$\overline{h_{\overline{\varphi}} f(z)} = \int_{\mathbb{B}_n} \frac{\overline{f(w)}\varphi(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w) = \int_{\mathbb{B}_n} \frac{T_\varphi(w)\overline{f(w)}}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dv_\alpha(w) = h_{T_\varphi} f(z),$$

for all $z \in \mathbb{B}_n$ and $f \in \mathcal{P}(\mathbb{B}_n, X)$. It is clear that the boundedness of $\overline{h_{\overline{\varphi}}}$ is equivalent to the boundedness of h_{T_φ} (with equivalent norms), then the proof of Theorem 4.4.2 is a direct consequence of the general Theorem 4.4.1 and the following proposition.

Proposition 4.4.3. *Let* $\varphi \in \mathcal{H}(\mathbb{B}_n, X^*)$ *and* $T_\varphi \in \mathcal{H}(\mathbb{B}_n, \overline{X}^*)$ *defined in* (4.4.1)*. Then:*

(1) $\|\varphi(z)\|_{X^*} = \|T_\varphi(z)\|_{\overline{X}^*}$*, for every* $z \in \mathbb{B}_n$*;*

(2) For $\alpha > -1$ *and* $p \geq 1$*, one has* $\varphi \in A^p_\alpha(\mathbb{B}_n, X^*)$ *if and only if* $T_\varphi \in A^p_\alpha(\mathbb{B}_n, \overline{X}^*)$*. Moreover,* $\|\varphi\|_{p,\alpha,X^*} = \|T_\varphi\|_{p,\alpha,\overline{X}^*}$*;*

(3) For $\gamma > 0$*, one has* $\varphi \in \mathcal{B}_\gamma(\mathbb{B}_n, X^*)$ *if and only if* $T_\varphi \in \mathcal{B}_\gamma(\mathbb{B}_n, \overline{X}^*)$*. Morever,* $\|\varphi\|_{\mathcal{B}_\gamma(\mathbb{B}_n, X^*)} = \|T_\varphi\|_{\mathcal{B}_\gamma(\mathbb{B}_n, \overline{X}^*)}$.

Proof. Fix $z \in \mathbb{B}_n$. Then, by definition, we have

$$\begin{aligned}\|T_\varphi(z)\|_{\overline{X}^*} &= \sup_{\|\overline{x}\|_{\overline{X}}=1} \left|\langle \overline{x}, T_\varphi(z)\rangle_{\overline{X}}\right| = \sup_{\|\overline{x}\|_{\overline{X}}=1} |\overline{x}(\varphi(z))| \\ &= \sup_{\|x\|_X=1} |\langle x, \varphi(z)\rangle_X| = \|\varphi(z)\|_{X^*}.\end{aligned}$$

This proves (1). The statement (2) follows directly by (1). Let us prove the statement (3). First of all, by Corollary 3.2.4 and (2) we can suppose that $\varphi \in A^1_\alpha(\mathbb{B}_n, X^*)$ and $T_\varphi \in A^1_\alpha(\mathbb{B}_n, \overline{X}^*)$, for some $\alpha > -1$ big enough. Let $t > 0$, by (1), we have

$$\left\| R^{\alpha,t}\varphi(z) \right\|_{X^*} = \| T_{R^{\alpha,t}\varphi}(z) \|_{\overline{X}^*}, \qquad z \in \mathbb{B}_n.$$

Take $x \in X$ and $z \in \mathbb{B}_n$. Then, by definition and Proposition 2.1.3, we get

$$\begin{aligned} \langle \overline{x}, T_{R^{\alpha,t}\varphi}(z) \rangle_{\overline{X}} &= \overline{x}(R^{\alpha,t}\varphi(z)) \\ &= \int_{\mathbb{B}_n} \frac{\overline{x}(\varphi(w))}{(1-\langle z, w\rangle)^{n+1+\alpha+t}} \, dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{T_\varphi(w)(\overline{x})}{(1-\langle z, w\rangle)^{n+1+\alpha+t}} \, dv_\alpha(w) \\ &= R^{\alpha,t} T_\varphi(z)(\overline{x}). \end{aligned}$$

Since $x \in X$ and $z \in \mathbb{B}_n$ are arbitrary we obtain

$$\| T_{R^{\alpha,t}\varphi}(z) \|_{\overline{X}^*} = \left\| R^{\alpha,t} T_\varphi(z) \right\|_{\overline{X}^*}, \qquad z \in \mathbb{B}_n.$$

Therefore, by Theorem 3.2.1 this implies (3) and we are done. □

Another example, if $X = Y = H$ is a complex Hilbert space we get the small Hankel operator Γ_T, with symbol $T : \mathbb{D} \to \mathcal{L}(H)$. You can see some results on them from Aleman and Constantin [1], the following result is from them.

Theorem 4.4.4 ([1, Theorem 3.1])**.** *The small Hankel operator Γ_T extends to a bounded linear operator on $A^2_\alpha(\mathbb{D}, H)$ if and only if*

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \| T'(z) \| < \infty.$$

Clearly the last condition is the same as the Bloch condition $T \in \mathcal{B}(\mathbb{D}, H)$ and using Lemma 4.1.1 we have that their Hankel operator is the same as we defined us in a more general way. Then, we can generalize it in the following way.

Theorem 4.4.5. *Let $1 < p < \infty$ and $1 < q < \infty$. Suppose $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}))$. Let the small Hankel operator $\Gamma_T : A^p_\alpha(\mathbb{B}_n, X) \to A^q_\alpha(\mathbb{B}_n, \overline{X})$ be the linear operator.*

(i) If $1 < p \le q < \infty$, then Γ_T is bounded if and only if $T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(X))$ where

$$\gamma = 1 + (n + 1 + \alpha) \left(\frac{1}{q} - \frac{1}{p} \right)$$

and, moreover, $\|\Gamma_T\| \simeq \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(X))}$.

(ii) If $1 < q < p < \infty$, then Γ_T is bounded if and only if $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(X))$ where

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}$$

and, moreover, $\|\Gamma_T\| \simeq \|T\|_{t,\alpha,\mathcal{L}(X)}$.

Chapter 5

Factorization Results

In this chapter we present some results in factorization, an application of the boundedness of the small Hankel operator. One important application of the boundedness of the little Hankel operator is the weak factorization of the vector-valued Bergman spaces. The ideas here comes from [60] and its bibliography therein. It is well-known that strong factorization for Bergman spaces in the unit ball of $\mathbb{C}^n$ of dimension $n \geq 2$ are no longer possible to obtain [40] (in [43] Horowitz proved the strong factorization of weighted Bergman spaces on the unit disk). Clearly this fact extend for vector-valued Bergman spaces, but it is still possible to obtain some weak factorization for functions in these spaces.

5.1 Weak Factorizations and Hankel Forms

Given two Banach spaces X and Y, let $x \boxdot y$ be a product defined for $x \in X$ and $y \in Y$. The product type space $X \widehat{\boxdot} Y$ is the completion of finite sums

$$\sum_k x_k \boxdot y_k, \qquad \{x_k\}_k \subset X, \{y_k\}_k \subset Y,$$

with the following norm

$$\|u\|_{X \widehat{\boxdot} Y} := \inf \left\{ \sum_k \|x_k\|_X \|y_k\|_Y : u = \sum_k x_k \boxdot y_k \right\}.$$

We have plenty different types of products of this kind. For example, we can consider the algebraic tensor product $x \otimes y$. Then we get the projection tensor product space $X \widehat{\otimes} Y$. The tensor product $x \otimes y$ can be viewed as an operator from Y^* to X defined by

$$(x \otimes y)(w) = w(y) \cdot x, \qquad w \in Y^*.$$

We refer to [35, Chapter VIII] and [67] for more on tensor products of Banach spaces.

As a second example, let X and Y be two Banach spaces of functions where the pointwise multiplication $f \cdot g$; $f \in X$, $g \in Y$, is well defined, or X and Y be spaces of operators with the usual product of operators. Then we obtain the weakly factored space $X \odot Y$ defined as

$$X \odot Y := \left\{ f = \sum_k g_k \cdot h_k \colon \{g_k\}_k \subset X, \{h_k\}_k \subset Y \right\},$$

with the following norm

$$\|f\|_{X \odot Y} = \inf \left\{ \sum_k \|g_k\|_X \, \|h_k\|_Y \, : \, f = \sum_k g_k \cdot h_k \right\}.$$

For instance we can take $X = A^p_\alpha$ and $Y = A^q_\alpha$, the scalar Bergman spaces or we can take even more complicated spaces like the vector-valued Bergman spaces; $X = A^p_\alpha(\mathbb{D}, A^r_\alpha(\mathbb{D}^n))$ and $Y = A^q_\alpha(\mathbb{D}, A^s_\alpha(\mathbb{D}^n))$. We also can take the Schatten class ideals $X = \mathcal{S}^p$ and $Y = \mathcal{S}^q$.

Another example is $X = \mathcal{L}(H)$ and $Y = H$, where H is a Hilbert space, with the natural product

$$T \boxdot x = T(x).$$

In this section we establish weak factorizations for vector-valued Bergman spaces $A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$, with $1 \leq q < \infty$ and $\alpha > -1$, into two other Bergman spaces, where X, Y are any two Banach spaces.

It is well-known that to obtain weak factorization results is equivalent to give a characterization of the boundedness of certain Hankel forms. Given $\alpha > -1$ and $\varphi \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$, we define the associated Hankel type bilinear form B_φ for polynomials $f \in \mathcal{P}(\mathbb{B}_n, X)$ and $g \in \mathcal{P}(\mathbb{B}_n, Y)$ by

$$B_\varphi(f, g) = \langle f \boxdot g, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y}.$$

Note that for any $z \in \mathbb{B}_n$, we define $(f \boxdot g)(z) := f(z) \boxdot g(z)$. Since the polynomials are dense in the vector-valued Bergman spaces (see Lemma 2.1.4) the Hankel form B_φ is densely defined on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ for any $p_1, p_2 \geq 1$. Recall that the Hankel form B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ if and only if there exists $C > 0$ such that

$$|B_\varphi(f, g)| \leq C \, \|f\|_{p_1, \alpha, X} \, \|g\|_{p_2, \alpha, Y},$$

for any $f \in A^{p_1}_\alpha(\mathbb{B}_n, X)$ and $g \in A^{p_2}_\alpha(\mathbb{B}_n, Y)$, moreover, $\|B_\varphi\|$ is the infimum of $C > 0$ of such above inequalities. The principal ideas of this result comes from [60] and the fact that $(X \widehat{\otimes} Y)^* = \mathcal{L}(X, Y^*)$ ([35, Chapter VIII. Section 2]) in the case of the projection tensor product.

Proposition 5.1.1. *Let $\alpha > -1$ and $1 \leq q < \infty$, $1 < p_1, p_2 < \infty$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}. \tag{5.1.1}$$

The following statements are equivalent:

(1) *$A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y) = A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)$, with equivalent norms.*

(2) *For any $\varphi \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$, B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ if and only if $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*$. Moreover, $\|B_\varphi\| \simeq \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*}$.*

Proof. First of all, we notice that by the duality of the vector-valued Bergman spaces in Theorems 2.2.4 and 3.1.6 we identify $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*$ with Λ_φ where $\Lambda_\varphi(f) = \langle f, \varphi\rangle_{\alpha, X \widehat{\boxdot} Y}$, for any $f \in A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$.

(1) $\Rightarrow$ (2): Suppose first that B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ and we want to prove that $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*$. By duality we get

$$\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} = \sup_{\|f\|_{q,\alpha,X \widehat{\boxdot} Y}=1} \left|\langle f, \varphi\rangle_{\alpha, X \widehat{\boxdot} Y}\right|.$$

But by hypothesis (1), any $f \in A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$ belongs to $A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)$ with

$$\|f\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)} \lesssim \|f\|_{q,\alpha,X \widehat{\boxdot} Y}.$$

Thus, for any $\varepsilon > 0$, we can find sequences $\{g_k\}_k \subset A^{p_1}_\alpha(\mathbb{B}_n, X)$ and $\{h_k\}_k \subset A^{p_2}_\alpha(\mathbb{B}_n, Y)$ such that $f = \sum_{k \geq 1} g_k \boxdot h_k$ and we have that

$$\sum_{k\geq 1} \|g_k\|_{p_1,\alpha,X} \|h_k\|_{p_2,\alpha,Y} \leq \|f\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)} + \varepsilon \lesssim \|f\|_{q,\alpha,X \widehat{\boxdot} Y} + \varepsilon.$$

Therefore, by the boundedness of B_φ,

$$\begin{aligned}
\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} &= \sup_{\|f\|_{q,\alpha,X \widehat{\boxdot} Y}=1} \left|\sum_{k\geq 1} \langle g_k \boxdot h_k, \varphi\rangle_{\alpha, X \widehat{\boxdot} Y}\right| \\
&= \sup_{\|f\|_{q,\alpha,X \widehat{\boxdot} Y}=1} \left|\sum_{k\geq 1} B_\varphi(g_k, h_k)\right| \\
&\leq \|B_\varphi\| \sup_{\|f\|_{q,\alpha,X \widehat{\boxdot} Y}=1} \sum_{k\geq 1} \|g_k\|_{p_1,\alpha,X} \|h_k\|_{p_2,\alpha,Y} \\
&\lesssim \|B_\varphi\| (1 + \varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we are done.

Conversely suppose $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*$. Take any $g \in A^{p_1}_\alpha(\mathbb{B}_n, X)$ and $h \in A^{p_2}_\alpha(\mathbb{B}_n, Y)$. Then, by hypothesis, $g \boxdot h \in A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y) \subset A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$ with $\|g \boxdot h\|_{q,\alpha,X \widehat{\boxdot} Y} \lesssim \|g \boxdot h\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)}$. Therefore

$$\begin{aligned}
|B_\varphi(g, h)| = \left|\langle g \boxdot h, \varphi\rangle_{\alpha, X \widehat{\boxdot} Y}\right| &\leq \|g \boxdot h\|_{q,\alpha,X \widehat{\boxdot} Y} \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \\
&\lesssim \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \|g \boxdot h\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)} \\
&\leq \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \|g\|_{p_1,\alpha,X} \|h\|_{p_2,\alpha,Y}
\end{aligned}$$

which imply that B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ with

$$\|B_\varphi\| \lesssim \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y)\right)^*}.$$

(2) $\Rightarrow$ (1): For the inclusion $A^{p_1}_\alpha(\mathbb{B}_n, X) \,\widehat{\boxdot}\, A^{p_2}_\alpha(\mathbb{B}_n, Y) \subset A^q_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y)$ with the estimate $\|f\|_{q,\alpha,X \widehat{\boxdot} Y} \lesssim \|f\|_{A^{p_1}_\alpha(\mathbb{B}_n,X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n,Y)}$ we use similar ideas as before. Take $f \in A^{p_1}_\alpha(\mathbb{B}_n, X) \,\widehat{\boxdot}\, A^{p_2}_\alpha(\mathbb{B}_n, Y)$. Then, for all $\varepsilon > 0$, there exist sequences $\{g_k\}_k \subset A^{p_1}_\alpha(\mathbb{B}_n, X)$ and $\{h_k\}_k \subset A^{p_2}_\alpha(\mathbb{B}_n, Y)$ such that $f = \sum_{k \geq 1} g_k \boxdot h_k$ and we have that

$$\sum_{k \geq 1} \|g_k\|_{p_1,\alpha,X} \|h_k\|_{p_2,\alpha,Y} \leq \|f\|_{A^{p_1}_\alpha(\mathbb{B}_n,X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n,Y)} + \varepsilon.$$

Therefore, by duality and the hypothesis, we have that

$$\begin{aligned}
\|f\|_{q,\alpha,X \widehat{\boxdot} Y} &= \sup_{\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*}=1} \left| \langle f, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y} \right| \leq \sup_{\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*}=1} \sum_k |B_\varphi(g_k, h_k)| \\
&\leq \sup_{\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*}=1} \|B_\varphi\| \sum_k \|g_k\|_{p_1,\alpha,X} \|h_k\|_{p_2,\alpha,Y} \\
&\lesssim \|f\|_{A^{p_1}_\alpha(\mathbb{B}_n,X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n,Y)} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we are done.

On the other hand, by the atomic decomposition in Theorem 2.3.5, we have the inclusion $A^q_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y) \subset A^{p_1}_\alpha(\mathbb{B}_n, X) \,\widehat{\boxdot}\, A^{p_2}_\alpha(\mathbb{B}_n, Y)$. Indeed, any function $f \in A^q_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y)$ can be written as

$$f(z) = \sum_{k=1}^{\infty} \lambda_k \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/q}}{(1 - \langle z, a_k \rangle)^b},$$

where $\{\lambda_k\}_k \subset X \,\widehat{\boxdot}\, Y$ and some b large enough (see notation in Theorem 2.3.5). Then, for any $k \geq 1$, there exist sequences $\{x^k_j\}_j \subset X$ and $\{y^k_j\}_j \subset Y$ such that $\lambda_k = \sum_j x^k_j \boxdot y^k_j$ and, therefore

$$f(z) = \sum_{k=1}^{\infty} \sum_j x^k_j \boxdot y^k_j \frac{(1 - |a_k|^2)^{b-(n+1+\alpha)/q}}{(1 - \langle z, a_k \rangle)^b} = \sum_{k,j} g^k_j(z) \boxdot h^k_j(z),$$

where

$$g^k_j(z) := x^k_j \frac{(1 - |a_k|^2)^{bq/p_1-(n+1+\alpha)/p_1}}{(1 - \langle z, a_k \rangle)^{bq/p_1}}$$

and

$$h^k_j(z) := y^k_j \frac{(1 - |a_k|^2)^{bq/p_2-(n+1+\alpha)/p_2}}{(1 - \langle z, a_k \rangle)^{bq/p_2}}.$$

Moreover, it is easy to see, using Theorem 1.4.1, that

$$\|g^k_j\|^{p_1}_{p_1,\alpha,X} = \int_{\mathbb{B}_n} \|x^k_j\|^{p_1}_X \frac{(1 - |a_k|^2)^{bq-(n+1+\alpha)}}{|1 - \langle z, a_k \rangle|^{bq}} \, dv_\alpha(z) \lesssim \|x^k_j\|^{p_1}_X,$$

which imply that $\|g_j^k\|_{p_1,\alpha,X} \lesssim \|x_j^k\|_X < \infty$ and $g_j^k \in A_\alpha^{p_1}(\mathbb{B}_n, X)$. Similary, we can see that $\|h_j^k\|_{p_2,\alpha,Y} \lesssim \|y_j^k\|_Y < \infty$ and $h_j^k \in A_\alpha^{p_2}(\mathbb{B}_n, Y)$. This would give us that $f \in A_\alpha^{p_1}(\mathbb{B}_n, X) \widehat{\boxdot} A_\alpha^{p_2}(\mathbb{B}_n, Y)$ and we finally obtain the inclusion

$$A_\alpha^q(\mathbb{B}_n, X \widehat{\boxdot} Y) \subset A_\alpha^{p_1}(\mathbb{B}_n, X) \widehat{\boxdot} A_\alpha^{p_2}(\mathbb{B}_n, Y).$$

Now, in order to have the corresponding estimate for the norms, we will show that, for any bounded linear functional F on the space $A_\alpha^{p_1}(\mathbb{B}_n, X) \widehat{\boxdot} A_\alpha^{p_2}(\mathbb{B}_n, Y)$, there is a unique function $\varphi_F \in \left(A_\alpha^q(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*$ with $\|\varphi_F\|_{\left(A_\alpha^q(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \lesssim \|F\|$ such that $F(f) = \langle f, \varphi_F \rangle_{\alpha, X \widehat{\boxdot} Y}$, for any $f \in A_\alpha^q(\mathbb{B}_n, X \widehat{\boxdot} Y)$. This would give

$$\begin{aligned} \|f\|_{A_\alpha^{p_1}(\mathbb{B}_n,X) \widehat{\boxdot} A_\alpha^{p_2}(\mathbb{B}_n,Y)} &= \sup_{\|F\|=1} |F(f)| \\ &\le \sup_{\|F\|=1} \|\varphi_F\|_{\left(A_\alpha^q(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \|f\|_{q,\alpha,X \widehat{\boxdot} Y} \lesssim \|f\|_{q,\alpha,X \widehat{\boxdot} Y} . \end{aligned}$$

Thus, suppose $F \in (A_\alpha^{p_1}(\mathbb{B}_n, X) \widehat{\boxdot} A_\alpha^{p_2}(\mathbb{B}_n, Y))^*$ with norm $\|F\|$. Fix $g \in A_\alpha^{p_1}(\mathbb{B}_n, X)$. Define

$$\begin{array}{rccc} \Lambda_g \colon & A_\alpha^{p_2}(\mathbb{B}_n, Y) & \longrightarrow & \mathbb{C} \\ & h & \longmapsto & F(g \boxdot h). \end{array}$$

Then, we have that

$$|\Lambda_g(h)| = |F(g \boxdot h)| \le \|F\| \, \|g\|_{p_1,\alpha,X} \, \|h\|_{p_2,\alpha,Y} , \qquad h \in A_\alpha^{p_2}(\mathbb{B}_n, Y),$$

which imply that Λ_g is lineal and bounded with norm $\|\Lambda_g\| \le \|F\| \, \|g\|_{p_1,\alpha,X}$. Thus, by duality of vector-valued Bergman spaces, see Theorem 2.2.4, there exists an unique $\varphi_g \in A_\alpha^{p_2'}(\mathbb{B}_n, Y^*)$ such that $\Lambda_g(h) = \langle h, \varphi_g \rangle_{\alpha,Y}$, for any $h \in A_\alpha^{p_2}(\mathbb{B}_n, Y)$, where p_2' is the conjugate exponent of p_2. Then, define $\varphi = \varphi_F \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ as

$$\varphi(z)\,(x \boxdot y) := \varphi_{g_x}(z)\,(y) = \langle y, \varphi_{g_x}(z) \rangle_Y , \qquad z \in \mathbb{B}_n, \tag{5.1.2}$$

for any $x \in X$ and $y \in Y$, where $g_x(z) := x$, for every $z \in \mathbb{B}_n$. Therefore, if we take any vector-valued polynomial $f \in \mathcal{P}(\mathbb{B}_n, X \widehat{\boxdot} Y)$ we have that there exist $\{g_k\}_k \subset \mathcal{P}(\mathbb{B}_n, X)$ and $\{h_k\}_k \subset \mathcal{P}(\mathbb{B}_n, Y)$ such that $f = \sum_k g_k \boxdot h_k$ and, moreover, using the relation (5.1.2), we obtain

$$\begin{aligned} \langle f, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y} &= \sum_k \int_{\mathbb{B}_n} \langle (g_k \boxdot h_k)(z), \varphi(z) \rangle_{X \widehat{\boxdot} Y} \, dv_\alpha(z) \\ &= \sum_k \int_{\mathbb{B}_n} \langle h_k(z), \varphi_{g_k}(z) \rangle_Y \, dv_\alpha(z) = \sum_k \langle h_k, \varphi_{g_k} \rangle_{\alpha,Y} \\ &= \sum_k \Lambda_{g_k}(h_k) = \sum_k F(g_k \boxdot h_k) = F(f). \end{aligned}$$

Now, for polynomials $g \in \mathcal{P}(\mathbb{B}_n, X)$ and $h \in \mathcal{P}(\mathbb{B}_n, Y)$ we have

$$\begin{aligned} |B_\varphi(g,h)| &= \left| \langle g \boxdot h, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y} \right| = |F(g \boxdot h)| \\ &\leq \|F\| \, \|g \boxdot h\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)} \\ &\leq \|F\| \, \|g\|_{p_1, \alpha, X} \, \|h\|_{p_2, \alpha, Y} \,, \end{aligned}$$

which shows that B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ with $\|B_\varphi\| \leq \|F\|$, using the density of polynomials in Lemma 2.1.4. By hypothesis, we know that $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y) \right)^*$ with $\|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*} \lesssim \|B_\varphi\| \leq \|F\|$. Hence $\Lambda(f) = \langle f, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y}$ defines a bounded linear form on $A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$ and coincides with F on polynomials. By density again (see Lemma 2.1.4) we have that $F(f) = \Lambda(f)$, for every $f \in A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)$, and the proof is complete. □

It is remarkable to note that if $1/q = 1/p_1 + 1/p_2$ the inclusion

$$A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y) \subset A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y),$$

with the estimate $\|f\|_{q, \alpha, X \widehat{\boxdot} Y} \lesssim \|f\|_{A^{p_1}_\alpha(\mathbb{B}_n, X) \widehat{\boxdot} A^{p_2}_\alpha(\mathbb{B}_n, Y)}$, is a direct consequence of Minkowski and Hölder inequalities, then it is true whether the other equivalence is true or not.

Now, we can see the relation of the small Hankel operator and the Hankel forms. We fix $\varphi \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ and we define the holomorphic operator $T_\varphi \colon \mathbb{B}_n \to \mathcal{L}(\overline{X}, Y^*)$ by

$$T_\varphi(z)(\overline{x}) = M^*_x(\varphi(z)), \qquad z \in \mathbb{B}_n, x \in X, \tag{5.1.3}$$

where $M_x \colon Y \to X \widehat{\boxdot} Y$ is the multiplication operator $M_x(y) = x \boxdot y$, for $y \in Y$. Thus $M^*_x \colon (X \widehat{\boxdot} Y)^* \to Y^*$ and then we have

$$\begin{aligned} \langle y, T_\varphi(z)(\overline{x}) \rangle_Y &= \langle y, M^*_x(\varphi(z)) \rangle_Y \\ &= \langle M_x(y), \varphi(z) \rangle_{X \widehat{\boxdot} Y} = \langle x \boxdot y, \varphi(z) \rangle_{X \widehat{\boxdot} Y} \,, \end{aligned} \tag{5.1.4}$$

for any $z \in \mathbb{B}_n, x \in X$ and $y \in Y$. Then, if $T_\varphi \in A^r_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$, for some $1 < r < \infty$, by Lemma 4.1.1 and (5.1.4), we have

$$\begin{aligned} \left\langle g, h_{T_\varphi} f \right\rangle_{\alpha, Y} &= \int_{\mathbb{B}_n} \left\langle g(z), T_\varphi(z) \overline{f(z)} \right\rangle_Y \, \mathrm{d}v_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle f(z) \boxdot g(z), \varphi(z) \rangle_{X \widehat{\boxdot} Y} \, \mathrm{d}v_\alpha(z) \\ &= \langle f \boxdot g, \varphi \rangle_{\alpha, X \widehat{\boxdot} Y} = B_\varphi(f, g), \end{aligned} \tag{5.1.5}$$

for any $f \in \mathcal{P}(\mathbb{B}_n, X)$ and $g \in \mathcal{P}(\mathbb{B}_n, Y)$. Then, it is clear the relation between the operators B_φ and h_{T_φ}. The Hankel form B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ if and only if $h_{T_\varphi} \colon A^{p_1}_\alpha(\mathbb{B}_n, X) \to A^{p_2'}_\alpha(\mathbb{B}_n, Y^*)$ is bounded (with equivalent norms), where p_2' is the conjugate exponent of p_2. Moreover, the symbols of these operators T_φ and φ have also a relation, they have similar growth in the vector-valued setting.

Proposition 5.1.2. *Let* $\varphi \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ *and* $T_\varphi \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$ *defined in* (5.1.3)*. Then:*

(1) $\|\varphi(z)\|_{(X \widehat{\boxdot} Y)^*} \simeq \|T_\varphi(z)\|_{\mathcal{L}(\overline{X}, Y^*)}$*, for every* $z \in \mathbb{B}_n$*;*

(2) For $\alpha > -1$ *and* $p \geq 1$*, one has* $\varphi \in A^p_\alpha(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ *if and only if* $T_\varphi \in A^p_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$*;*

(3) $\varphi \in \mathcal{B}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ *if and only if* $T_\varphi \in \mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$*.*

Proof. Fix $z \in \mathbb{B}_n$. Then, by (5.1.4), we have

$$\begin{aligned} \|T_\varphi(z)\|_{\mathcal{L}(\overline{X}, Y^*)} &= \sup_{\substack{\|\overline{x}\|_{\overline{X}}=1, \\ \|y\|_Y=1}} \left|\langle y, T_\varphi(z)(\overline{x})\rangle_Y\right| \\ &= \sup_{\substack{\|x\|_X=1, \\ \|y\|_Y=1}} \left|\langle x \boxdot y, \varphi(z)\rangle_{X \widehat{\boxdot} Y}\right| \leq \|\varphi(z)\|_{(X \widehat{\boxdot} Y)^*}. \end{aligned}$$

On the other hand, by duality, we have that

$$\|\varphi(z)\|_{(X \widehat{\boxdot} Y)^*} = \sup_{\|\lambda\|_{X \widehat{\boxdot} Y}=1} \left|\langle \lambda, \varphi(z)\rangle_{X \widehat{\boxdot} Y}\right|.$$

We also know that if $\lambda \in X \widehat{\boxdot} Y$, for every $\varepsilon > 0$, there exist sequences $\{x_k\}_k \subset X$ and $\{y_k\}_k \subset Y$ such that $\lambda = \sum_k x_k \boxdot y_k$ and, moreover,

$$\sum_k \|x_k\|_X \|y_k\|_Y \leq \|\lambda\|_{X \widehat{\boxdot} Y} + \varepsilon.$$

Therefore, by (5.1.4), we get

$$\begin{aligned} \|\varphi(z)\|_{(X \widehat{\boxdot} Y)^*} &\leq \sup_{\|\lambda\|_{X \widehat{\boxdot} Y}=1} \sum_k \left|\langle x_k \boxdot y_k, \varphi(z)\rangle_{X \widehat{\boxdot} Y}\right| \\ &= \sup_{\|\lambda\|_{X \widehat{\boxdot} Y}=1} \sum_k \left|\langle y_k, T_\varphi(z)(\overline{x_k})\rangle_Y\right| \\ &= \sup_{\|\lambda\|_{X \widehat{\boxdot} Y}=1} \sum_k \|x_k\|_X \|y_k\|_Y \left|\left\langle \frac{y_k}{\|y_k\|_Y}, T_\varphi(z)\left(\frac{\overline{x_k}}{\|\overline{x_k}\|_{\overline{X}}}\right)\right\rangle_Y\right| \\ &\leq (1+\varepsilon) \|T_\varphi(z)\|_{\mathcal{L}(\overline{X}, Y^*)}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary this proves (1). The statement (2) is a direct consequence of (1). Indeed, by (1), we have

$$\begin{aligned} \|\varphi\|^p_{p,\alpha,(X \widehat{\boxdot} Y)^*} &= \int_{\mathbb{B}_n} \|\varphi(z)\|^p_{(X \widehat{\boxdot} Y)^*} \, dv_\alpha(z) \\ &\simeq \int_{\mathbb{B}_n} \|T_\varphi(z)\|^p_{\mathcal{L}(\overline{X}, Y^*)} \, dv_\alpha(z) = \|T_\varphi\|^p_{p,\alpha,\mathcal{L}(\overline{X}, Y^*)}, \end{aligned}$$

which implies (2). The statement (3) is more involved. First of all, by Lemma 3.1.4 and (2) we can suppose that $\varphi \in A^1_\alpha(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ and $T_\varphi \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$, for some $\alpha > -1$. Now, let $t > 0$, by (1), we have

$$\left\| R^{\alpha,t}\varphi(z) \right\|_{(X \boxdot Y)^*} \simeq \left\| T_{R^{\alpha,t}\varphi}(z) \right\|_{\mathcal{L}(\overline{X},Y^*)}, \qquad z \in \mathbb{B}_n.$$

Take $x \in X$, $y \in Y$ and $z \in \mathbb{B}_n$. Then, by (5.1.4) and Proposition 2.1.3, we get

$$\begin{aligned}
\langle y, T_{R^{\alpha,t}\varphi}(z)(\overline{x}) \rangle_Y &= \left\langle M_x(y), R^{\alpha,t}\varphi(z) \right\rangle_{X \widehat{\boxdot} Y} \\
&= \left\langle M_x(y), \int_{\mathbb{B}_n} \frac{\varphi(w)\, \mathrm{d}v_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right\rangle_{X \widehat{\boxdot} Y} \\
&= \int_{\mathbb{B}_n} \frac{\langle M_x(y), \varphi(w) \rangle_{X \widehat{\boxdot} Y}}{(1 - \langle w, z \rangle)^{n+1+\alpha+t}}\, \mathrm{d}v_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{\langle y, T_\varphi(w)(\overline{x}) \rangle_Y}{(1 - \langle w, z \rangle)^{n+1+\alpha+t}}\, \mathrm{d}v_\alpha(w) \\
&= \left\langle y, \left(\int_{\mathbb{B}_n} \frac{T_\varphi(w)\, \mathrm{d}v_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right)(\overline{x}) \right\rangle_Y \\
&= \left\langle y, R^{\alpha,t} T_\varphi(z)(\overline{x}) \right\rangle_Y .
\end{aligned}$$

Since $x \in X$ and $y \in Y$ are arbitrary we obtain

$$\left\| T_{R^{\alpha,t}\varphi}(z) \right\|_{\mathcal{L}(\overline{X},Y^*)} = \left\| R^{\alpha,t} T_\varphi(z) \right\|_{\mathcal{L}(\overline{X},Y^*)}, \qquad z \in \mathbb{B}_n.$$

Therefore, by Theorem 3.1.3 this implies (3) and we are done. □

Then, as a conclusion, we can deduce the following result of the characterization of the Hankel form B_φ.

Theorem 5.1.3. *Let $\varphi \in \mathcal{H}(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$. Suppose $1 \le q < \infty$ and $1 < p_1, p_2 < \infty$ satisfying (5.1.1). We further suppose that Y^* has finite cotype if $q > 1$. Then B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ if and only if $\varphi \in \left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y) \right)^*$. Moreover, $\|B_\varphi\| \simeq \|\varphi\|_{\left(A^q_\alpha(\mathbb{B}_n, X \widehat{\boxdot} Y)\right)^*}$.*

Proof. By (5.1.5), the Hankel form B_φ is bounded on $A^{p_1}_\alpha(\mathbb{B}_n, X) \times A^{p_2}_\alpha(\mathbb{B}_n, Y)$ if and only if $h_{T_\varphi} \colon A^{p_1}_\alpha(\mathbb{B}_n, X) \to A^{p_2'}_\alpha(\mathbb{B}_n, Y^*)$ is bounded (with equivalent norms), where p_2' is the conjugate exponent of p_2.

If $q > 1$, condition (5.1.1) imply that $1 < p_2' < p_1 < \infty$ and $1/q' = 1/p_2' - 1/p_1$, where q' is the conjugate exponent of q. Then, using Theorem 4.3.3 and the hypothesis, we have that $h_{T_\varphi} \colon A^{p_1}_\alpha(\mathbb{B}_n, X) \to A^{p_2'}_\alpha(\mathbb{B}_n, Y^*)$ is bounded if and only if $T_\varphi \in A^{q'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$ with equivalent norms. By Proposition 5.1.2 (2), $T_\varphi \in A^{q'}_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$ if and only if $\varphi \in A^{q'}_\alpha(\mathbb{B}_n, (X \widehat{\boxdot} Y)^*)$ (with equivalent norms) and the duality of vector-valued Bergman spaces in Theorem 2.2.4 gives the result.

Similarly, if $q = 1$, condition (5.1.1) imply that $p_2' = p_1$. Then, using Theorem 4.2.1, we obtain that $h_{T_\varphi} \colon A^{p_1}_\alpha(\mathbb{B}_n, X) \to A^{p_2'}_\alpha(\mathbb{B}_n, Y^*)$ is bounded if and only if $T_\varphi \in \mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$ with equivalent norms. Finally, by Proposition 5.1.2 (3) and the duality in Theorem 3.1.6 finish the proof. □

Finally, we can prove one of the main results of this chapter, the weak factorization of vector-valued Bergman spaces.

Theorem 5.1.4. *Suppose X, Y two Banach spaces where Y^* has finite cotype. Let $1 < q < \infty$ and $\alpha > -1$. Then*

$$A^q_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y) = A^{p_1}_\alpha(\mathbb{B}_n, X) \,\widehat{\boxdot}\, A^{p_2}_\alpha(\mathbb{B}_n, Y),$$

(with equivalent norms) for any $p_1, p_2 > 1$ satisfying (5.1.1).

Proof. Follows directly by Proposition 5.1.1 and Theorem 5.1.3. □

We have also the case $q = 1$ that does not have any restriction on the Banach spaces.

Theorem 5.1.5. *Suppose X, Y two Banach spaces and let $\alpha > -1$. Then*

$$A^1_\alpha(\mathbb{B}_n, X \,\widehat{\boxdot}\, Y) = A^p_\alpha(\mathbb{B}_n, X) \,\widehat{\boxdot}\, A^{p'}_\alpha(\mathbb{B}_n, Y),$$

for any $1 < p < \infty$, where p' is the conjugate exponent of p.

Proof. Follows directly by Proposition 5.1.1 and Theorem 5.1.3. □

5.2 Applications

In this section we are going to introduce different applications of the weak factorization of the vector-valued Bergman spaces. First of all, let $\mathcal{S}^t = \mathcal{S}^t(H)$ be the t-Schatten class of operators acting on some Hilbert space H. The first application is a generalization of a Sarason's theorem with operator-valued Bergman spaces. Sarason in [66, Theorem 4] showed that $H^1(\mathbb{D}, \mathcal{S}^1) = H^2(\mathbb{D}, \mathcal{S}^2)H^2(\mathbb{D}, \mathcal{S}^2)$, where $H^p(\mathbb{D}, X)$ are the vector-valued Hardy spaces. Constantin in [28, Theorem 2.1] proved a result of Sarason type on Bergman spaces; a weak factorization of operator-valued Bergman spaces. Namely, the theorem says the following.

Theorem 5.2.1 ([28, Theorem 2.1])**.** *Let $t \geq 1, t_1, t_2, p > 1$ such that*

$$\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}, \tag{5.2.1}$$

then we have that

$$A^1_\alpha(\mathbb{D}, \mathcal{S}^t) = A^p_\alpha(\mathbb{D}, \mathcal{S}^{t_1}) \odot A^{p'}_\alpha(\mathbb{D}, \mathcal{S}^{t_2}),$$

(with equivalent norms) where p' is the conjugate exponent of p.

Taking into account that for $t \geq 1$ and $t_1, t_2 > 1$ satisfying (5.2.1) we have that

$$\mathcal{S}^t = \mathcal{S}^{t_1} \cdot \mathcal{S}^{t_2},$$

we can generalize it, by using Theorem 5.1.5, in order to get the same result for $\mathbb{B}_n$. Moreover, if we use Theorem 5.1.4 instead, we get a more generalization of this result.

Theorem 5.2.2. *Let $\alpha > -1$ and $q, p_1, p_2 > 1$ satisfying (5.1.1). Let also $t \geq 1$ and $t_1, t_2 > 1$ satisfying (5.2.1). Then, we have that*

$$A^q_\alpha(\mathbb{B}_n, \mathcal{S}^t) = A^{p_1}_\alpha(\mathbb{B}_n, \mathcal{S}^{t_1}) \odot A^{p_2}_\alpha(\mathbb{B}_n, \mathcal{S}^{t_2}),$$

with equivalent norms.

Proof. It is a direct application of Theorem 5.1.4 and we only need to prove that $(\mathcal{S}^{t_1})^*$ or $(\mathcal{S}^{t_2})^*$ has finite cotype. This is immedate since $(\mathcal{S}^{t_1})^* = \mathcal{S}^{t_1'}$ and $(\mathcal{S}^{t_2})^* = \mathcal{S}^{t_2'}$, where t_1' and t_2' are the conjugate exponents of t_1 and t_2 respectively. But we know that $\mathcal{S}^t$ have finite cotype, for any $t > 1$. □

The other application appears also in [28, Theorem 4.1] which is a weak factorization of the scalar-valued Bergman spaces on the polydisc. The key for the proof is that we can identify $A^p_\alpha(\mathbb{D}^n)$ with the vector-valued Bergman space $A^p_\alpha(\mathbb{D}, A^p_\alpha(\mathbb{D}^{n-1}))$. Constantin proved the following theorem.

Theorem 5.2.3 ([28, Theorem 4.1])**.** *Let $\alpha > -1$. We have*

$$A^1_\alpha(\mathbb{D}^n) = A^2_\alpha(\mathbb{D}^n) \odot A^2_\alpha(\mathbb{D}^n),$$

with equivalent norms.

Then, we can generalize it using Theorem 5.1.5 in the following way.

Theorem 5.2.4. *Let $\alpha > -1$ and $p > 1$. We have*

$$A^1_\alpha(\mathbb{D}^n) = A^p_\alpha(\mathbb{D}^n) \odot A^{p'}_\alpha(\mathbb{D}^n),$$

with equivalent norms, where p' is the conjugate exponent of p.

Proof. We use Theorem 5.1.5 and the fact that $A^p_\alpha(\mathbb{D}^n) = A^p_\alpha(\mathbb{D}, A^p_\alpha(\mathbb{D}^{n-1}))$. □

We can even generalize further using Theorem 5.1.4.

Theorem 5.2.5. *Let $\alpha > -1$ and $q, p_1, p_2 > 1$ satisfying condition (5.1.1). Then, we have*

$$A^q_\alpha(\mathbb{D}^n) = A^{p_1}_\alpha(\mathbb{D}^n) \odot A^{p_2}_\alpha(\mathbb{D}^n),$$

with equivalent norms.

Proof. Notice that $A^p_\alpha(\mathbb{D}^n) = A^p_\alpha(\mathbb{D}, A^p_\alpha(\mathbb{D}^{n-1}))$. We shall prove the theorem by induction over n. For $n = 1$ it is a direct application of Theorem 5.1.4 with $X = Y = \mathbb{C}$. Suppose the result holds for $n-1$, that is,

$$A^q_\alpha(\mathbb{D}^{n-1}) = A^{p_1}_\alpha(\mathbb{D}^{n-1}) \odot A^{p_2}_\alpha(\mathbb{D}^{n-1}), \tag{5.2.2}$$

and we want to prove it for n. Since $A^q_\alpha(\mathbb{D}^n) = A^q_\alpha(\mathbb{D}, A^q_\alpha(\mathbb{D}^{n-1}))$, we use the hypothesis of induction (5.2.2) to apply Theorem 5.1.4 and we deduce that

$$\begin{aligned} A^q_\alpha(\mathbb{D}^n) = A^q_\alpha(\mathbb{D}, A^q_\alpha(\mathbb{D}^{n-1})) &= A^{p_1}_\alpha(\mathbb{D}, A^{p_1}_\alpha(\mathbb{D}^{n-1})) \odot A^{p_2}_\alpha(\mathbb{D}, A^{p_2}_\alpha(\mathbb{D}^{n-1})) \\ &= A^{p_1}_\alpha(\mathbb{D}^n) \odot A^{p_2}_\alpha(\mathbb{D}^n), \end{aligned}$$

and we are done. □

As a final example, taking $Y = \mathbb{C}$, we obtain the inclusion $A^p_\alpha(\mathbb{B}_n, X) \subset A^{p_1}_\alpha(\mathbb{B}_n, X) \odot A^{p_2}_\alpha$, a result that we isolate next.

Theorem 5.2.6. *Let X be a Banach space, $\alpha > -1$, and $1 \leq p < \infty$. Then any function $f \in A^p_\alpha(\mathbb{B}_n, X)$ admits a decomposition $f = \sum_k f_k g_k$ with $\{f_k\}_k \subset A^{p_1}_\alpha(\mathbb{B}_n, X)$ and $\{g_k\}_k \subset A^{p_2}_\alpha$ for any $p_1, p_2 \geq 1$ satisfying $1/p = 1/p_1 + 1/p_2$. Moreover, one has the estimate*

$$\sum_k \|f_k\|_{p_1,\alpha,X} \|g_k\|_{p_2,\alpha} \leq C \|f\|_{p,\alpha,X},$$

for some constant $C > 0$.

Chapter 6

Big Hankel Operators

Recall that the (big) Hankel operator with symbol φ between scalar-valued Bergman space A^2_α is defined as follows. Let φ be a bounded function on $\mathbb{C}$ then

$$H_\varphi f := (I - P_\alpha)(\varphi f), \qquad f \in A^2_\alpha.$$

Thanks to the integral representation of each $f \in A^2_\alpha$ and the integral form of P_α we can write it as

$$H_\varphi f(z) = \int_{\mathbb{B}_n} \frac{(\varphi(z) - \varphi(w))f(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w).$$

It is well-known that $H_\varphi = 0$ when φ is analytic then we are going to study Hankel operators with anti-analytic symbols. In this chapter we always consider X and Y be two complex Banach spaces. In order to generalize this definition to vector-valued function $f \in \mathcal{H}(\mathbb{B}_n, X)$ we first take an operator-valued symbol $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and then define the Hankel operator as

$$H_T f(z) = \int_{\mathbb{B}_n} \frac{(T(z) - T(w))\overline{f(w)}}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w), \qquad f \in A^2_\alpha(\mathbb{B}_n, X). \tag{6.0.1}$$

It is clear that H_T is well-defined in the set of polynomials $\mathcal{P}(\mathbb{B}_n, X)$ if $T \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Indeed, for fixed $z \in \mathbb{B}_n$ and $f \in \mathcal{P}(\mathbb{B}_n, X)$, by Proposition 2.1.2, there exists a constant $C > 0$ such that

$$\begin{aligned} \|H_T f(z)\|_Y &\le \left\| T(z)\overline{f(z)} \right\|_Y + \int_{\mathbb{B}_n} \frac{\left\| T(w)\overline{f(w)} \right\|_Y}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w) \\ &\le \left\| T(z)\overline{f(z)} \right\|_Y + C \int_{\mathbb{B}_n} \|T(w)\|_{\mathcal{L}(\overline{X}, Y)} \, \mathrm{d}v_\alpha(w), \end{aligned}$$

so if $T \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ then H_T is densely defined on the set of polynomials, see also Lemma 2.1.4.

Similarly, it is easy to see that this definition matches with the scalar-valued, that is, $H_T f = (I - \overline{P_\alpha})(T\overline{f})$. Indeed, for $f \in A^1_\alpha(\mathbb{B}_n, X)$, using Lemma 1.2.1 and the integral representation Proposition 2.1.2, we have

$$H_T f(z) = T(z)\overline{f(z)} - \int_{\mathbb{B}_n} \frac{T(w)\overline{f(w)}}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, \mathrm{d}v_\alpha(w) = (I - \overline{P_\alpha})(T(z)\overline{f(z)}), \quad z \in \mathbb{B}_n.$$

In this chapter we are going to study the boundedness of these generalization of (big) Hankel operators H_T between vector-valued Bergman spaces in terms of its symbol T. Before that we need some preliminary results that are needed later.

6.1 Preliminary Results

We begin with a well-known result for the scalar case. Let $w \in \mathbb{B}_n$. Recall that the vector-valued p-normalized reproducing kernels on $x \in X \setminus \{0\}$, for $0 < p < \infty$, is defined as $k^x_{p,w} = x/\|x\|_X \, k_{p,w}$, where

$$k_{p,w}(z) = \frac{K_w(z)}{\|K_w\|_{p,\alpha}}, \qquad z \in \mathbb{B}_n.$$

Remember also that $K^x_w = xK_w$, where K_w is the scalar-valued Bergman reproducing kernel (see Section 2.1).

Lemma 6.1.1. *Let $1 < p < \infty$, $x \in X$ and $w \in \mathbb{B}_n$. If $T \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ then*

$$H_T k^x_{p,w}(z) = (T(z) - T(w))(\overline{k^x_{p,w}(z)}),$$

for every $z \in \mathbb{B}_n$.

Proof. Since $K^x_z \in A^1_\alpha(\mathbb{B}_n, X)$ and $T\overline{K^x_z} \in A^1_\alpha(\mathbb{B}_n, X)$ for any $x \in X$ and $z \in \mathbb{B}_n$ (because $T \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and K^x_z is bounded), using the integral representation Proposition 2.1.2 we have that

$$\begin{aligned} H_T K^x_\zeta(z) &= \int_{\mathbb{B}_n} \frac{(T(z) - T(w))(\overline{K^x_\zeta(w)})}{(1 - \langle w, z\rangle)^{n+1+\alpha}} \, dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{(T(z) - T(w))(K^{\overline{x}}_z(w))}{(1 - \langle \zeta, w\rangle)^{n+1+\alpha}} \, dv_\alpha(w) \\ &= (T(z) - T(\zeta))(K^{\overline{x}}_z(\zeta)) = (T(z) - T(\zeta))(\overline{K^x_\zeta(z)}). \end{aligned}$$

Multiplying both sides by $\|K^x_\zeta\|^{-1}_{p,\alpha}$ we get the result. $\square$

For values $1 < p, q < \infty$ and an operator-valued $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, we define the integrals

$$\begin{aligned} I_{p,q}T(z) &:= \int_{\mathbb{B}_n} \|T(w) - T(z)\|^q_{\mathcal{L}(\overline{X},Y)} \frac{(1 - |z|^2)^{(n+1+\alpha)q(1-\frac{1}{p})}}{|1 - \langle z, w\rangle|^{(n+1+\alpha)q}} \, dv_\alpha(z) \\ &\simeq \int_{\mathbb{B}_n} \|T(w) - T(z)\|^q_{\mathcal{L}(\overline{X},Y)} |k_{p,z}(w)|^q \, dv_\alpha(w). \end{aligned}$$

Lemma 6.1.2. *Let $1 < p, q < \infty$ and $\alpha > -1$. If $T \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ then*

$$(1 - |z|^2)^{(n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right)} \left\|\widetilde{\nabla} T(z)\right\|_{(\mathcal{L}(\overline{X},Y))^n} \lesssim I_{p,q}T(z)^{\frac{1}{q}}.$$

Proof. Take $0 < \delta < 1$. Then the dilatations, $T_\delta(z) = T(\delta z)$ for every $z \in \mathbb{B}_n$, $T_\delta \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and so, by Proposition 2.1.2, we have

$$T(\delta z) = \int_{\mathbb{B}_n} \frac{T(\delta w)\, \mathrm{d}v_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}}, \qquad z \in \mathbb{B}_n.$$

Changing variables $w \mapsto w/\delta$ and replacing δz by z we obtain

$$T(z) = \delta^2 \int_{\delta\mathbb{B}_n} \frac{T(w)\, \mathrm{d}v_\alpha(w)}{(\delta^2-\langle z, w\rangle)^{n+1+\alpha}}.$$

Since $\delta < 1$ is arbitrary we take $\delta = \tanh r < 1$ so that $\delta\mathbb{B}_n = D(0,r)$. Moreover, let $1 \le k \le n$, differentiating under the integral sign we get that

$$\partial_k T(z) = \frac{\partial}{\partial z_k} T(z) = \delta^2 (n+1+\alpha) \int_{D(0,r)} \frac{T(w)\overline{w_k}\, \mathrm{d}v_\alpha(w)}{(\delta^2-\langle z, w\rangle)^{n+1+\alpha+1}},$$

and substituting in $z = 0$ we have that

$$\partial_k T(0) = C_r \int_{D(0,r)} T(w)\overline{w_k}\, \mathrm{d}v_\alpha(w).$$

where $C_r = \frac{(n+1+\alpha)}{\delta^{2(n+1+\alpha)}}$. With no confusion, we take a fixed $z \in \mathbb{B}_n$ and we replace T by $T \circ \varphi_z - T(z)$ then we get

$$\widetilde{\nabla} T(z) = C_r \int_{D(0,r)} (T \circ \varphi_z(w) - T(z))\overline{w}\, \mathrm{d}v_\alpha(w).$$

Changing variables according to Proposition 1.4.3 we obtain that

$$\widetilde{\nabla} T(z) = C_r \int_{D(z,r)} (T(w) - T(z)) \frac{\overline{\varphi_z(w)}(1-|z|^2)^{n+1+\alpha}}{(1-\langle z, w\rangle)^{2(n+1+\alpha)}}\, \mathrm{d}v_\alpha(w).$$

Now, if q' is the conjugate exponent of q then by Proposition 1.2.2, Hölder's inequality and the integral estimate in Theorem 1.4.1 we have that

$$\begin{aligned}
\left\|\widetilde{\nabla} T(z)\right\|_{(\mathcal{L}(\overline{X},Y))^n} &\lesssim \int_{D(z,r)} \|T(w) - T(z)\|_{\mathcal{L}(\overline{X},Y)} \frac{(1-|z|^2)^{n+1+\alpha}\, \mathrm{d}v_\alpha(w)}{|1-\langle z, w\rangle|^{2(n+1+\alpha)}} \\
&\le (I_{p,q}T(z))^{\frac{1}{q}} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{(n+1+\alpha)q'/p}}{|1-\langle z, w\rangle|^{(n+1+\alpha)q'}}\, \mathrm{d}v_\alpha(w) \right)^{\frac{1}{q'}} \\
&\lesssim (I_{p,q}T(z))^{\frac{1}{q}} (1-|z|^2)^{(n+1+\alpha)\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}$$

This proves this lemma. ∎

Using previous results we obtain the following.

Lemma 6.1.3. *Let $1 < p, q < \infty$. If H_T is well-defined (so that $T \in A^1_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ and bounded from $A^p_\alpha(\mathbb{B}_n, X)$ to $L^q_\alpha(\mathbb{B}_n, Y)$ with norm $\|H_T\|$ then*

$$(1-|z|^2)^{(n+1+\alpha)\left(\frac{1}{q}-\frac{1}{p}\right)} \left\|\widetilde{\nabla} T(z)\right\|_{(\mathcal{L}(\overline{X},Y))^n} \lesssim \|H_T\|,$$

for any $z \in \mathbb{B}_n$.

Proof. There exists $\overline{x} \in \overline{\mathbb{B}(\overline{X})}$ such that

$$\begin{aligned} I_{p,q}T(z) &\lesssim \int_{\mathbb{B}_n} \left\|(T(w)-T(z))\overline{k^x_{p,z}(w)}\right\|^q_Y \, \mathrm{d}v_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\|H_T k^x_{p,z}(w)\right\|^q_Y \, \mathrm{d}v_\alpha(w) \\ &= \left\|H_T k^x_{p,w}\right\|^q_{q,\alpha,Y} \le \|H_T\|^q \left\|k^x_{p,w}\right\|^q_{p,\alpha,X} \lesssim \|H_T\|^q . \end{aligned}$$

using Lemma 6.1.1 in the first equality. The rest follows from Lemma 6.1.2. □

This previous lemma tell us that if H_T is bounded, automatically the function $(1-|\cdot|^2)^{(n+1+\alpha)\left(\frac{1}{q}-\frac{1}{p}\right)} \left\|\widetilde{\nabla} T\right\|_{(\mathcal{L}(\overline{X},Y))^n}$ is in L^∞. We end this section with a series of results needed later. The following is taken from [60, Lemma 9].

Lemma 6.1.4. *Let X be a Banach space. Then, for $1 < p < \infty$, $\alpha > -1$ and $n+1+\alpha < b$, we have that*

$$\int_{\mathbb{B}_n} \frac{\|f(z)-f(w)\|^p_X}{|1-\langle w,z\rangle|^b} \, \mathrm{d}v_\alpha(z) \lesssim \int_{\mathbb{B}_n} \left\|\widetilde{\nabla} f(z)\right\|^p_{X^n} \frac{\mathrm{d}v_\alpha(z)}{|1-\langle w,z\rangle|^b},$$

for every $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $w \in \mathbb{B}_n$.

Proof. The proof works exactly as the proof of [60, Lemma 9] but we need to use Theorem 3.2.6 instead of the scalar case. □

For a holomorphic vector-valued function $f \in \mathcal{H}(\mathbb{B}_n, X)$ and a complex direction u (a unit vector in $\mathbb{C}^n$) we define the complex directional derivative

$$\frac{\partial f}{\partial u}(z) := \lim_{\lambda \to 0} \frac{f(z+\lambda u)-f(z)}{\lambda},$$

where $\lambda \in \mathbb{C}$ and $z \in \mathbb{B}_n$. A simple application of the chain rule gives

$$\frac{\partial f}{\partial u}(z) = \sum_{k=1}^n u_k \frac{\partial f}{\partial z_k}(z),$$

where $u = (u_1, \dots, u_n)$. More generally, for $n > 1$, $f \in \mathcal{H}(\mathbb{B}_n, X)$, and $z \in \mathbb{B}_n \setminus \{0\}$ we will write

$$\|\nabla_T f(z)\|_X := \sup_{u \in [z]^T, |u|=1} \left\|\frac{\partial f}{\partial u}(z)\right\|_X,$$

and call it the complex tangential gradient of f at z.

For $f \in \mathcal{H}(\mathbb{B}_n, X)$ and $z \in \mathbb{B}_n$, we also define the following value

$$Q_f(z) := \sup_{\|x^*\|_{X^*}=1} \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\sum_{k=1}^n w_k \langle \partial_k f(z), x^*\rangle_X\right|}{\langle B(z)w, w\rangle^{1/2}},$$

where $w = (w_1, \dots, w_n) \in \mathbb{C}^n \setminus \{0\}$ and $B(z)$ is the Bergman matrix of $\mathbb{B}_n$ well defined in [82, Section 1.5].

Theorem 6.1.5. *Let $n > 1$ and $z \in \mathbb{B}_n \setminus \{0\}$, then*

$$(1-|z|^2)^{1/2} \|\nabla_T f(z)\|_X \le \left\|\widetilde{\nabla} f(z)\right\|_{X^n},$$

for every $f \in \mathcal{H}(\mathbb{B}_n, X)$.

Proof. We will prove, in fact, that we have that

$$(1-|z|^2)^{1/2} \|\nabla_T f(z)\|_X \le Q_f(z) \le \left\|\widetilde{\nabla} f(z)\right\|_{X^n}.$$

Notice that, using [82, Proposition 1.18], $(1-|z|^2)^{-1}$ is an eigenvalue of $B(z)$ with eigenspace $[z]^T$. Therefore,

$$\begin{aligned}
Q_f(z) &\ge \sup_{\|x^*\|=1} \sup_{w\in[z]^T} \frac{\left|\sum_{k=1}^n w_k \langle \partial_k f(z), x^*\rangle_X\right|}{\langle B(z)w, w\rangle^{1/2}} \\
&= (1-|z|^2)^{1/2} \sup_{w\in[z]^T} \sup_{\|x^*\|=1} \left| \left\langle \sum_{k=1}^n \frac{w_k}{|w|} \partial_k f(z), x^* \right\rangle_X \right| \\
&= (1-|z|^2)^{1/2} \sup_{w\in[z]^T} \left\| \sum_{k=1}^n \frac{w_k}{|w|} \partial_k f(z) \right\|_X = (1-|z|^2)^{1/2} \|\nabla_T f(z)\|_X .
\end{aligned}$$

On the other side, replacing w by $B^{-1/2}(z)w$ in the definition of $Q_f(z)$ (so that $w_k = \sum_{j=1}^n B_{k,j}^{-1/2}(z) w_j$) and using Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
Q_f(z) &= \sup_{\|x^*\|_{X^*}=1} \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\sum_{k=1}^n \sum_{j=1}^n B_{k,j}^{-1/2}(z) w_j \langle \partial_k f(z), x^*\rangle_X\right|}{|w|} \\
&\le \sup_{\|x^*\|_{X^*}=1} \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{\sum_{j=1}^n \left|\sum_{k=1}^n B_{k,j}^{-1/2}(z) \langle \partial_k f(z), x^*\rangle_X\right| |w_j|}{|w|} \\
&\le \sup_{\|x^*\|_{X^*}=1} \left(\sum_{j=1}^n \left| \sum_{k=1}^n B_{k,j}^{-1/2}(z) \langle \partial_k f(z), x^*\rangle_X \right|^2 \right)^{1/2} \\
&= \sup_{\|x^*\|_{X^*}=1} \left(\sum_{j=1}^n \sum_{k=1}^n \sum_{i=1}^n B_{k,j}^{-1/2}(z) \overline{B_{i,j}^{-1/2}(z)} \langle \partial_k f(z), x^*\rangle_X \overline{\langle \partial_i f(z), x^*\rangle_X} \right)^{1/2} .
\end{aligned}$$

Recall that $\overline{B_{i,j}^{-1/2}(z)} = B_{j,i}^{-1/2}(z)$ and using [82, Proposition 1.18(b)] we have that

$$\sum_{j=1}^{n} B_{k,j}^{-1/2}(z) B_{j,i}^{-1/2}(z) = B_{k,i}^{-1}(z) = (1-|z|^2)(\delta_{ki} - z_k \overline{z_i}).$$

Therefore,

$$\begin{aligned} Q_f(z) &\le \sup_{\|x^*\|_{X^*}=1} (1-|z|^2)^{1/2} \left(\sum_{k=1}^{n} \sum_{i=1}^{n} (\delta_{ki} - z_k\overline{z_i}) \langle \partial_k f(z), x^* \rangle_X \overline{\langle \partial_i f(z), x^* \rangle_X} \right)^{1/2} \\ &= (1-|z|^2)^{1/2} \sup_{\|x^*\|_{X^*}=1} \left(\sum_{k=1}^{n} |\langle \partial_k f(z), x^* \rangle_X|^2 - |\langle Rf(z), x^* \rangle_X|^2 \right)^{1/2} . \end{aligned} \tag{6.1.1}$$

On the other hand, we have that

$$\left\| \widetilde{\nabla} f(z) \right\|_{X^n}^2 = \sum_{j=1}^{n} \|\partial_j (f \circ \varphi_z)(0)\|_X^2 \ge \sum_{j=1}^{n} \left| \langle \partial_j (f\circ\varphi_z)(0), x^* \rangle_X \right|^2,$$

for any $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$. Following the same proof and notations of Theorem 3.2.6 we obtain that $\sum_{j=1}^{n} \left| \langle \partial_j (f\circ\varphi_z)(0), x^* \rangle_X \right|^2$ is equal to

$$s_z^2 \sum_{j=1}^{n} \left| \langle \partial_j f(z), x^* \rangle_X \right| - \frac{s_z(s_z - s_z^2)}{|z|^2} \sum_{j=1}^{n} \beta_j + \frac{(s_z - s_z^2)^2}{|z|^2} \sum_{j=1}^{n} \frac{|z_j|^2}{|z|^2} |\langle Rf(z), x^* \rangle_X|^2,$$

where, in this case, $\beta_j = \beta_j(z) = \langle \lambda_j, x^* \rangle_X \overline{\langle Rf(z), x^* \rangle_X} + \langle Rf(z), x^* \rangle_X \overline{\langle \lambda_j, x^* \rangle_X}$. Clearly (recall that $\lambda_j = z_j \partial_j f(z)$, so that $\sum_{j=1}^{n} \lambda_j = Rf(z)$), we have that

$$\sum_{j=1}^{n} \beta_j = 2 |\langle Rf(z), x^* \rangle_X|^2$$

and, since $s_z = (1-|z|^2)^{1/2}$,

$$-2\frac{s_z(s_z - s_z^2)}{|z|^2} + \frac{(s_z - s_z^2)^2}{|z|^2} = \frac{-s_z^2(1-s_z^2)}{|z|^2} = -s_z^2.$$

Therefore,

$$\sum_{j=1}^{n} \left| \langle \partial_j (f\circ\varphi_z)(0), x^* \rangle_X \right|^2 = s_z^2 \sum_{j=1}^{n} \left| \langle \partial_j f(z), x^* \rangle_X \right| - s_z^2 |\langle Rf(z), x^* \rangle_X|^2,$$

which imply that

$$\begin{aligned} \left\| \widetilde{\nabla} f(z) \right\|_{X^n} &\ge \left(\sum_{j=1}^{n} \left| \langle \partial_j (f\circ\varphi_z)(0), x^* \rangle_X \right|^2 \right)^{1/2} \\ &= s_z \left(\sum_{j=1}^{n} \left| \langle \partial_j f(z), x^* \rangle_X \right| - |\langle Rf(z), x^* \rangle_X|^2 \right)^{1/2} \\ &= (1-|z|^2)^{1/2} \left(\sum_{j=1}^{n} \left| \langle \partial_j f(z), x^* \rangle_X \right| - |\langle Rf(z), x^* \rangle_X|^2 \right)^{1/2} . \end{aligned}$$

Since $x^* \in X^*$ is arbitrary, where $\|x^*\|_{X^*} = 1$, we can apply supremums and we get the desired result using (6.1.1) and this theorem is completed. $\square$

The next one is [61, Theorem 2.8].

Theorem 6.1.6. *Let $n > 1$ and $f \in \mathcal{H}(\mathbb{B}_n, X)$. If $\|\nabla_T f(z)\|_X \to 0$ as $|z| \to 1^-$, then f is constant.*

Proof. The proof is exactly as in the proof of [61, Theorem 2.8]. $\square$

Corollary 6.1.7. *Let $n > 1$ and $f \in \mathcal{H}(\mathbb{B}_n, X)$. If $(1-|z|^2)^{-1/2} \left\| \widetilde{\nabla} f(z) \right\|_{X^n} \to 0$ as $|z| \to 1^-$, then f is constant.*

Proof. It directly follows by Theorems 6.1.5 and 6.1.6. $\square$

In the next section we will see more characterization of bounded Hankel operators.

6.2 Bounded Hankel Operators

In this section we characterize the boundedness of the Hankel operator H_T in terms of its symbol T.

Theorem 6.2.1. *Let $1 < p \le q < \infty$ and X, Y two Banach spaces. The Hankel operator $H_T \colon A^p_\alpha(\mathbb{B}_n, X) \to L^q_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if*

$$A_T := \sup_{z \in \mathbb{B}_n} (1-|z|^2)^{\gamma-1} \left\| \widetilde{\nabla} T(z) \right\|_{(\mathcal{L}(\overline{X},Y))^n} < \infty,$$

where

$$\gamma = 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right).$$

Moreover, $\|H_T\| \simeq A_T$.

Proof. Suppose first that $A_T < +\infty$. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$, $z \in \mathbb{B}_n$ and $\varepsilon > 0$ that we will specify later. If q' is the conjugate exponent of q then, by Hölder's inequality, we have

$$\|H_T f(z)\|_Y \le \int_{\mathbb{B}_n} \frac{\|f(w)\|_X \, \|T(z) - T(w)\|_{\mathcal{L}(\overline{X},Y)}}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \, dv_\alpha(w) \le I_1(z)^{\frac{1}{q}} I_2(z)^{\frac{1}{q'}},$$

where

$$I_1(z) := \int_{\mathbb{B}_n} \frac{\|f(w)\|_X^q \, (1-|w|^2)^{\varepsilon q}}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \, dv_\alpha(w)$$

and

$$I_2(z) := \int_{\mathbb{B}_n} \frac{\|T(z) - T(w)\|_{\mathcal{L}(\overline{X},Y)}^{q'} \, (1-|w|^2)^{-\varepsilon q'}}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \, dv_\alpha(w).$$

In one hand, by Lemma 6.1.4, the hypothesis and the integral estimate Theorem 1.4.1, we have that

$$\begin{aligned}
I_2(z)^{\frac{1}{q'}} &\lesssim \left(\int_{\mathbb{B}_n} \left\|\widetilde{\nabla} T(w)\right\|^{q'}_{(\mathcal{L}(\overline{X},Y))^n} \frac{dv_{\alpha-\varepsilon q'}(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\right)^{\frac{1}{q'}} \\
&\leq A_T \left(\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{-q'(\gamma-1)}\, dv_{\alpha-\varepsilon q'}(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\right)^{\frac{1}{q'}} \\
&\lesssim A_T (1-|z|^2)^{-\varepsilon-(\gamma-1)}.
\end{aligned}$$

On the other hand, when $p<q$ using the norm estimate of Theorem 2.1.1, we obtain

$$\begin{aligned}
I_1(z) &= \int_{\mathbb{B}_n} \frac{\|f(w)\|^p_X \, \|f(w)\|^{q-p}_X \, (1-|w|^2)^{\varepsilon q}}{|1-\langle z,w\rangle|^{n+1+\alpha}} \, dv_\alpha(w) \\
&\leq \|f\|^{q-p}_{p,\alpha,X} \int_{\mathbb{B}_n} \frac{\|f(w)\|^p_X \, (1-|w|^2)^{\varepsilon q+q(\gamma-1)}}{|1-\langle z,w\rangle|^{n+1+\alpha}} \, dv_\alpha(w).
\end{aligned}$$

If $q=p$ then $\gamma=1$, and $I_1(z)$ equals the last expression. Therefore, by Tonelli's theorem and Theorem 1.4.1 again, we have

$$\begin{aligned}
\|H_T f\|^q_{q,\alpha,Y} &= \int_{\mathbb{B}_n} \|H_T f(z)\|^q_Y \, dv_\alpha(z) \leq \int_{\mathbb{B}_n} I_1(z) I_2(z)^{\frac{q}{q'}} \, dv_\alpha(z) \\
&\lesssim A_T^q \, \|f\|^{q-p}_{p,\alpha,X} \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\|f(w)\|^p_X \, dv_{\alpha+q(\varepsilon+\gamma-1)}(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} \, dv_{\alpha-q(\varepsilon+\gamma-1)}(z) \\
&= A_T^q \, \|f\|^{q-p}_{p,\alpha,X} \int_{\mathbb{B}_n} \|f(w)\|^p_X \int_{\mathbb{B}_n} \frac{dv_{\alpha-q(\varepsilon+\gamma-1)}}{|1-\langle w,z\rangle|^{n+1+\alpha}} \, dv_{\alpha+q(\varepsilon+\gamma-1)}(w) \\
&\lesssim A_T^q \, \|f\|^q_{p,\alpha,X}.
\end{aligned}$$

If we take $\varepsilon>0$ fulfilling all the requirements needed in the applications of Theorem 1.4.1, that is,

$$1-\gamma < \varepsilon < 1-\gamma + \min\left(\frac{\alpha+1}{q}, \frac{\alpha+1}{q'}\right)$$

then we have proved that H_T is bounded on $A^p_\alpha(\mathbb{B}_n,X) \to A^q_\alpha(\mathbb{B}_n,Y)$ with norm $\|H_T\| \lesssim A_T < \infty$.

The other implication is a direct consequence of Lemma 6.1.3. Suppose that $H_T\colon A^p_\alpha(\mathbb{B}_n,X) \to L^q_\alpha(\mathbb{B}_n,Y)$ is bounded with norm $\|H_T\|$. By Lemma 6.1.3, we have that

$$(1-|z|^2)^{\gamma-1} \left\|\widetilde{\nabla} T(z)\right\|_{(\mathcal{L}(\overline{X},Y))^n} \lesssim \|H_T\|.$$

which implies that $A_T \lesssim \|H_T\| < \infty$ and the proof of the theorem is complete. □

A direct consequence is the following result.

Corollary 6.2.2. *Let $1 < p \le q < \infty$ and X, Y two Banach spaces. Let*

$$\gamma := 1 + (n+1+\alpha)\left(\frac{1}{q} - \frac{1}{p}\right).$$

The Hankel operator $H_T\colon A^p_\alpha(\mathbb{B}_n, X) \to L^q_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if

(a) *$T \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$, if $\gamma > 1/2$. Moreover, $\|H_T\| \simeq \|T\|_{\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}$.*

(b) *T is constant, if $\gamma < 1/2$.*

(c) *T fulfills the following condition*

$$A_T := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{-1/2} \left\| \widetilde{\nabla} T(z) \right\|_{(\mathcal{L}(\overline{X}, Y))^n} < \infty,$$

if $\gamma = 1/2$. Moreover, $\|H_T\| \simeq A_T$.

Proof. It follows directly by Theorems 6.2.1 and 3.2.8 and Corollary 6.1.7. □

The following theorems is for the remaining cases $1 < q < p < \infty$.

Theorem 6.2.3. *Let $1 < q < p < \infty$. The Hankel operator $H_T\colon A^p_\alpha(\mathbb{B}_n, X) \to L^q_\alpha(\mathbb{B}_n, Y)$ is a bounded linear operator if and only if $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ where*

$$\frac{1}{t} = \frac{1}{q} - \frac{1}{p}.$$

Moreover, $\|H_T\| \simeq \|T\|_{t,\alpha,\mathcal{L}(\overline{X}, Y)}$.

Proof. Let $f \in A^p_\alpha(\mathbb{B}_n, X)$ and suppose that $T \in A^t_\alpha(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$. Then

$$H_T f = T\overline{f} - \overline{P_\alpha}(T\overline{f}).$$

So, the boundedness of the projection P_α (which imply the boundedness of $\overline{P_\alpha}$), see Theorem 2.2.2, shows that

$$\begin{aligned}\|H_T f\|_{q,\alpha,Y} &\le \left\|T\overline{f}\right\|_{q,\alpha,Y} + \left\|\overline{P_\alpha}(T\overline{f})\right\|_{q,\alpha,Y} \\ &\le (1 + \|\overline{P_\alpha}\|) \left\|T\overline{f}\right\|_{q,\alpha,Y}.\end{aligned}$$

On the other hand, by Hölder's inequality with $(p/q, p/(p-q))$, we have that

$$\begin{aligned}\left\|T\overline{f}\right\|^q_{q,\alpha,Y} &= \int_{\mathbb{B}_n} \left\|T(z)\overline{f(z)}\right\|^q_Y \, \mathrm{d}v_\alpha(z) \\ &\le \left(\int_{\mathbb{B}_n} \|f(z)\|^p_X \, \mathrm{d}v_\alpha(z)\right)^{\frac{q}{p}} \left(\int_{\mathbb{B}_n} \|T(z)\|^t_{\mathcal{L}(\overline{X}, Y)}\right)^{\frac{p-q}{p}} \\ &= \|f\|^q_{p,\alpha,X} \|T\|^q_{t,\alpha,\mathcal{L}(\overline{X}, Y)}.\end{aligned}$$

It implies then that $H_T\colon A^p_\alpha(\mathbb{B}_n,X)\to L^q_\alpha(\mathbb{B}_n,Y)$ is bounded and $\|H_T\|\lesssim\|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)}$.

Conversely, we suppose that $H_T\colon A^p_\alpha(\mathbb{B}_n,X)\to L^q_\alpha(\mathbb{B}_n,Y)$ is bounded with norm $\|H_T\|$. Following the same argument as Theorem 4.3.3 we get that

$$\sum_{k=1}^\infty\int_{D(a_k,r)}\left\|H_T(k^{\lambda_k}_{p,a_k})(z)\right\|^q_Y\,\mathrm{d}v_\alpha(z)\lesssim\|H_T\|^q\,\|\{\lambda_k\}_k\|^q_{\ell^p(X)},$$

where $\{\lambda_k\}_k\subset\ell^p(X)$. By Lemma 6.1.1 we have

$$H_T(k^{\lambda_k}_{p,a_k})(z)=(T(z)-T(a_k))(\overline{k^{\lambda_k}_{p,a_k}})(z),\qquad z\in\mathbb{B}_n,$$

so, we obtain that

$$\sum_{k=1}^\infty\int_{D(a_k,r)}\left\|(T(z)-T(a_k))(\overline{k^{\lambda_k}_{p,a_k}})(z)\right\|^q_Y\,\mathrm{d}v_\alpha(z)\lesssim\|H_T\|^q\,\|\{\lambda_k\}_k\|^q_{\ell^p(X)}.$$

Arguing as in the proof of Lemma 6.1.2 we have that

$$\left\|\widetilde{\nabla}T(w)(\overline{x})\right\|_{Y^n}\lesssim I^x_{p,q}T(w)^{1/q}(1-|w|^2)^{(n+1+\alpha)(1/p-1/q)},$$

for any $x\in X$ and $w\in\mathbb{B}_n$, where

$$I^x_{p,q}T(w)=\int_{D(w,r)}\left\|(T(z)-T(w)(\overline{k^x_{p,w}})(z)\right\|^q_Y\,\mathrm{d}v_\alpha(z).$$

Therefore,

$$\begin{aligned}\sum_{k=1}^\infty(1-|a_k|^2)^{(n+1+\alpha)(1-q/p)}\left\|\widetilde{\nabla}T(a_k)(\overline{\lambda_k})\right\|^q_{Y^n}&\lesssim\sum_{k=1}^\infty I^{\lambda_k}_{p,q}T(a_k)\\&\lesssim\|H_T\|^q\,\|\{\lambda_k\}_k\|^q_{\ell^p(X)}.\end{aligned}$$

Writing it as

$$\sum_{k=1}^\infty\left\|(1-|a_k|^2)^{(n+1+\alpha)(1/q-1/p)}\widetilde{\nabla}T(a_k)(\overline{\lambda_k})\right\|^q_{Y^n}\lesssim\|H_T\|^q\,\|\{\lambda_k\}_k\|^q_{\ell^p(X)}$$

and since the sequence $\{\overline{\lambda_k}\}_k\subset\ell^p(\overline{X})$ is arbitrary, Lemma 4.3.2 implies that

$$\{(1-|a_k|^2)^{(n+1+\alpha)(1/q-1/p)}\widetilde{\nabla}T(a_k)\}_k\subset\ell^t((\mathcal{L}(\overline{X},Y))^n)$$

and

$$\sum_{k=1}^\infty(1-|a_k|^2)^{n+1+\alpha}\left\|\widetilde{\nabla}T(a_k)\right\|^t_{(\mathcal{L}(\overline{X},Y))^n}\lesssim\|H_T\|^t.$$

Theorem 3.2.6 says that

$$(1-|a_k|^2)\,\|RT(a_k)\|_{\mathcal{L}(\overline{X},Y)}\leq\left\|\widetilde{\nabla}T(a_k)\right\|$$

so, we obtain

$$\sum_{k=1}^{\infty}(1-|a_k|^2)^{n+1+\alpha+t}\,\|RT(a_k)\|^t_{\mathcal{L}(\overline{X},Y)} \lesssim \|H_T\|^t\,.$$

Applying Lemma 4.3.1 (with $t=0$ in the lemma, do not confuse with the t of this theorem) we have that $RT \in A^t_{\alpha+t}(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$ and $\|RT\|_{t,\alpha+t,\mathcal{L}(\overline{X},Y)} \lesssim \|H_T\|$. Therefore, by [82, Theorem 2.16] we get $T \in A^t_{\alpha}(\mathbb{B}_n, \mathcal{L}(\overline{X},Y))$ and $\|T\|_{t,\alpha,\mathcal{L}(\overline{X},Y)} \lesssim \|H_T\|$. Note that the proof of [82, Theorem 2.16] for vector-valued case is the same as the scalar-valued case and with this the theorem is complete. □

Chapter 7

Open Questions and Future Research

In our opinion, we think that we have done a satisfactory work in order to get a better understanding of the function properties of vector-valued Bergman spaces and the action of Hankel operators acting on them. We hope that this work is going to attract many other researchers to this area, and expect that the study of vector-valued Bergman spaces is going to experience a period of intensive research in the next years. However, we have not been able to attach all the problems we had in mind, and in this last chapter we discuss some open problems we left, as well as some other problems we think it can be interesting to look on the future.

7.1 Compact Hankel operators

After our study on bounded small and big Hankel operators acting on vector-valued Bergman space, the next step is to look for a characterization of the compactness. According to the results for the scalar case that can be found in Zhu's book [82, Chapter 8], and the result obtained by O. Constantin [28] in the case of dimension $n = 1$, $X = H$ being a Hilbert space, and $q = p = 2$, one may expect that, for an holomorphic operator-valued function $T\colon \mathbb{B}_n \to \mathcal{L}(X, Y)$, the small Hankel operator $h_T\colon A^p_\alpha(\mathbb{B}_n, X) \to A^p_\alpha(\mathbb{B}_n, Y)$ is going to be compact if and only if T belongs to the little Bloch space $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(X, Y))$. Here, $\mathcal{K}(X, Y)$ denotes the space of all compact operators from X to Y, and for a Banach space Z, the little Bloch space $\mathcal{B}_0(\mathbb{B}_n, Z)$ is the subspace of $\mathcal{B}(\mathbb{B}_n, Z)$ consisting of those Z-valued Bloch functions f with

$$\lim_{|z|\to 1^-} (1 - |z|^2) \|\nabla f(z)\|_{Z^n} = 0.$$

Similar results are expected to hold for the compactness of h_T when $q \neq p$. It is also interesting to complete the results obtained in Chapter 6 by describing the compactness of the big Hankel operator H_T from $A^q_\alpha(\mathbb{B}_n, X)$ to $L^q_\alpha(\mathbb{B}_n, Y)$ for Banach spaces X and Y. Results on compactness for the big Hankel operator on the scalar case can be found, for example, in Zhu's book [82, Chapter 8], [81] and [61].

7.2 Schatten class Hankel operators

It is now a classical result of Arazy, Fisher and Peetre (see [6]) that the big Hankel operator with conjugate analytic symbol $H_{\bar{f}}$ acting on the scalar Bergman space A^2_α, is in the Schatten ideal $\mathcal{S}_p$ if and only if f belongs to the analytic Besov space B_p, $1 < p < \infty$ (the case of dimension $n > 1$ can be found in [5]). In our case, it turns out that, when H is a Hilbert space, the vector-valued Bergman space $A^2_\alpha(\mathbb{B}_n, H)$ is also a Hilbert space, and it makes sense to study when the big Hankel operator H_T with an holomorphic operator-valued symbol T, acting on $A^2_\alpha(\mathbb{B}_n, H)$, belongs to the Schatten class $\mathcal{S}_p$. In view of the results for the scalar case, it seems clear that vector-valued analytic Besov spaces are going to enter in action, so that apart from dominating the necessary techniques on Schatten class operators, a further study of analytic vector-valued Besov spaces is necessary. One can also look on the problem of describing the membership in $\mathcal{S}_p$ of the small Hankel operator h_T acting on $A^2_\alpha(\mathbb{B}_n, H)$.

7.3 Weak Factorization of Hardy Spaces in the Bessel Setting

It is worth mention that the theory of Hardy spaces has been studied and developed extensively in harmonic analysis and more precisely the theory of Hardy spaces on the Euclidean setting has been shown to have many applications, see [26, 38, 39, 70] for an instance of general references.

The real-variable Hardy space theory on n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$, plays an important role in harmonic analysis and has been systematically developed [26, 39]. There are many equivalent definitions of the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$. It is well-known that when $p > 1$, the actual definition of $H^p(\mathbb{R}^n)$ makes it equivalent to $L^p(\mathbb{R}^n)$, but when $p \in (0, 1]$, these spaces are much better suited to ask questions about harmonic analysis than are the $L^p(\mathbb{R}^n)$ spaces, see [41, 70] for an account of all of this.

It is well known that the Hardy spaces of 1-dimensional Euclidean space $H^p(\mathbb{R})$ admits a strong factorization. As we already said, in [40], Gowda discovered that this strong factorization is no longer to obtain for superior dimensions $n \geq 2$. In the case of the real-variable Hardy space $H^1(\mathbb{R}^n)$, Coifman, Rochberg and Weiss [25] provided a factorization that works in studying function theory and operator theory of $H^1(\mathbb{R}^n)$ which was called the weak factorization. This weak factorization for $H^1(\mathbb{R}^n)$ consist of the following: every $f \in H^1(\mathbb{R}^n)$ can be written as

$$f = \sum_{j=1}^{\infty} \sum_{i=1}^{n} (g^i_j R_i h^i_j + h^i_j R_i g^i_j),$$

where $\{g^i_j\}_{i,j}, \{h^i_j\}_{i,j} \in H^2(\mathbb{R}^n)$ and R_i are the Riesz transforms on $\mathbb{R}^n$ and

$$\|f\|_{H^1(\mathbb{R}^n)} \simeq \inf \left\{ \sum_{j=1}^{\infty} \sum_{i=1}^{n} \left\|g^i_j\right\|_{L^2(\mathbb{R}^n)} \left\|g^i_j\right\|_{L^2(\mathbb{R}^n)} \right\},$$

with the infimum taken over all possible representations of f as above. Later, Uchiyama [75] found an algorithmic way to generalize this weak factorization for Hardy spaces $H^p(\mathbb{R}^n)$ with values of $p \in (0, 1]$, but close to 1. On the other side, it is also well-known, as pointed in [25], that this weak factorization is closely related with the boundedness of some commutator on L^p spaces. Since then, many authors generalized the boundedness of this commutator between different L^p spaces [45, 48, 58, 59].

The theory of the classical Hardy space $H^p(\mathbb{R}^n)$ is intimately connected to the Laplacian Δ. Changing the differential operator introduces new challenge and directions to explore. In 1965, Muckenhoupt and Stein in [54] introduced the notion of conjugacy associated with this Bessel operator Δ_λ, $\lambda > 0$, which is defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x}\frac{d}{dx} f(x), \qquad x > 0.$$

They developed a theory in the setting of Δ_λ which parallels the classical one associated to Δ. Results on $L^p(\mathbb{R}_+, dm_\lambda)$-boundedness of conjugate functions and fractional integrals associated with Δ_λ were obtained, where $p \in (1, \infty)$, $\mathbb{R}_+ := (0, \infty)$ and $dm_\lambda(x) := x^{2\lambda} dx$. Since then, many problems based on the Bessel context were studied; see, for example, [16, 18, 71, 76, 79]. In particular, the properties and L^p boundedness ($1 < p < \infty$) of Riesz transforms

$$R_{\Delta_\lambda} f := \partial_x (\Delta_\lambda)^{-1/2} f$$

related to Δ_λ have been studied in [54, 76]. The related Hardy space

$$H^1(\mathbb{R}_+, dm_\lambda) := \left\{ f \in L^1(\mathbb{R}_+, dm_\lambda) \colon R_{\Delta_\lambda} f \in L^1(\mathbb{R}_+, dm_\lambda) \right\}$$

with norm $\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)} := \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)} + \|R_{\Delta_\lambda} f\|_{L^1(\mathbb{R}_+, dm_\lambda)}$ has been studied by Betancor et al. in [17] where they established the characterizations of the atomic Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ associated with Δ_λ in terms of the Riesz transform and the radial maximal function associated with the Hankel convolution of a class $Z^{[\lambda]}$ of functions, which includes the Poisson semigroup and the heat semigroup as special cases. Duong, Li, Wick and Yang [71] used Uchiyama's algorithm to prove the weak factorization on the Bessel setting of Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ in terms of the Riesz transform R_{Δ_λ}.

Therefore, the open question is building up a weak factorization for Hardy spaces $H^p(\mathbb{R}_+, dm_\lambda)$ for $p \in (0, 1)$ in terms of a bilinear form related to R_{Δ_λ}, like [71] did, using the same method of Uchiyama. As a second question will be also to prove that if this weak factorization implies the characterization of the corresponding commutator. In order to do that we first propose the question for values of p near 1, see Yang and Yang [79] for a characterization of the atomic Hardy spaces $H^p(\mathbb{R}_+, dm_\lambda)$ for values of $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right]$. More concretely, the problem will be the following.

Problem 7.3.1. *Let $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right]$ and $q, r > 0$ such that*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \tag{7.3.1}$$

Prove that for any $f \in H^p(\mathbb{R}_+, dm_\lambda)$, there exists numbers $\{\alpha_j^k\}_{j,k}$, functions $\{g_j^k\}_{j,k} \subset L^q(\mathbb{R}_+, dm_\lambda)$ and $\{h_j^k\}_{j,k} \subset L^r(\mathbb{R}_+, dm_\lambda)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \, \Pi(g_j^k, h_j^k) \tag{7.3.2}$$

in $H^p(\mathbb{R}_+, dm_\lambda)$ where Π is defined as

$$\Pi(g, h) := g R_{\Delta_\lambda} h - h \widetilde{R_{\Delta_\lambda}} g,$$

where $\widetilde{R_{\Delta_\lambda}}$ is the adjoint operator of R_{Δ_λ}. Moreover, there should exists a positive constant C independent of f such that

$$C^{-1} \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} \leq \inf \left\{ \left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|^p \, \|g_j^k\|_q^p \, \|h_j^k\|_r^p \right)^{\frac{1}{p}} : \right.$$
$$\left. f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \, \Pi(g_j^k, h_j^k) \right\} \leq C \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)}.$$

Note that the case $p = 1$ is exactly as Duong, Li, Wick and Yang [71] did, so the contribution here will be the cases of $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right)$.

We are studying this problem in collaboration with Brett D. Wick in Washington University in St. Louis. We made a huge advance on it, but there are still some technical details that by now we are not able to do. We hope to be able to arrange these technical problem soon.

7.4 The Bergman Projection on Weighted Vector-valued Bergman Spaces

A classical result of Békollé and Bonami [12] describes the weights ω such that the Bergman projection is bounded on $L^p(\mathbb{D}, \omega \, \mathrm{d}A)$, for $1 < p < \infty$. These weights have been characterized by the following Békollé–Bonami condition

$$\sup_S \left(\frac{1}{A(S)} \int_S \omega(z) \, \mathrm{d}A(z) \right) \left(\frac{1}{A(S)} \int_S \omega(z)^{-p'/p} \, \mathrm{d}A(z) \right)^{p/p'} < \infty,$$

where the supremum is over all the Carleson sectors $S = S(I)$, and p' is the conjugate exponent of p. This result also has been generalized to higher dimension, on the unit ball of $\mathbb{C}^n$ [10]. The Békollé–Bonami theorem is the analog result of the Bergman spaces of the well known characterization of the weights ω such that the Hilbert transform is bounded in $L^p(\mathbb{R}, \omega \, \mathrm{d}x)$, weights that are well described for the Muckenhoupt condition A_p (a classical result of the Harmonic Analysis [44]).

It is interesting then to study the analog of the Békollé–Bonami theorem in the case of vector-valued Bergman spaces. More concretely, the problem to develop is the following: for $1 < p < \infty$, describe the positive operator-valued weights W such that the Bergman projection is bounded on $L^p(\mathbb{D}, W\,\mathrm{d}A)$. Let H be a Hilbert space. Given a positive operator-valued weight $W\colon \mathbb{D} \to \mathcal{L}(H)$, the corresponding space L^p of vector-valued measurable functions in $\mathbb{D}$ is denoted by $L^p(\mathbb{D}, W\,\mathrm{d}A)$ and has the following norm

$$\|f\|_{p,W}^p := \int_{\mathbb{D}} \left\|W^{1/p}(z) f(z)\right\|_H^p \,\mathrm{d}A(z).$$

All the proofs known about the Békollé–Bonami theorem does not work for the context of vector-valued Bergman spaces, and so it is necessary to develop new techniques to tackle this problem. It is remarkable that a characterization for the case $p = 2$ has been obtained recently by Aleman and Constantin in [2], but their method again can not be extended for the other cases of $p \neq 2$. Therefore, it is still an open problem when $p \neq 2$.

We just begin to study this problem. First of all, we find another proof of the scalar case much more simple and flexible than the original one which it has more possibilities that the method and techniques will work for the vector-valued case with appropriate modifications. However, it is unclear that this proof will generalize to the vector-valued setting.

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