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# Conformal Cartographic Representations

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# Abstract

Maps are a useful tool to display information. They are used on a daily basis to locate places, orient ourselves or present different features, such as weather forecasts, population distributions, etc. However, every map is a representation of the Earth that actually distorts reality. Depending on the purpose of the map, the interest may rely on preserving different features. For instance, it might be useful to design a map for navigation in which the directions represented on the map at a point concide with the ones the map reader observes at that point. Such map projections are called conformal.

This dissertation aims to study different conformal representations of the Earth. The shape of the Earth is modelled by a regular surface. As both the Earth and the flat piece of paper onto which it is to be mapped are two-dimensional surfaces, the map projection may be described by the relation between their coordinate systems. For some mathematical models of the surface of the Earth it is possible to define a parametrization that verifies the conditions $E = G$ and $F = 0$, where $E$, $F$ and $G$ denote the coefficients of the first fundamental form. In this cases, the mapping problem is shown to reduce to the study of conformal functions from the complex plane onto itself. In particular, the Schwarz-Christoffel formula for the mapping of the upper half-plane on a polygon is applied to cartography.

# Contents

# 1 Introduction

Evidence of maps used to describe our surroundings or to orient ourselves dates back as far as our knowledge of human civilization. Nowadays we encounter maps on a daily basis. From street maps to weather maps, they are a powerful tool to display information. Whatever its purpose, every map is a representation of something that is larger or more complex. This dissertation is concerned with maps of the Earth.

Before making maps of the world we are called upon to determine its size and shape. These studies comprise the subject of geodesy. Section 3.1 is meant as a basic introduction to the physics of the Earth. The geoid, from the greek "Earth-shaped", is defined as the surface of gravitational equipotential, the shape of the Earth abstracted from topographic features. Althought it is the common definition of our world's shape, its complexity makes the mathematical or computational treatment unfeasible. Thus, simpler mathematical models are required. Among these we may highlight the sphere and the spheroid. The spherical model of the Earth is used whenever the region to be mapped is large or the whole Earth and the size of the map is small. On the other hand, the spheroid is a more accurate model and therefore used for smaller regions and larger maps. Furthermore, this section presents the geographic coordinate system, a parametrization of the surface defined by the latitude and the longitude.

Regardless of the model used, there exist a large amount of different maps, from the simple and sublime to the complex and confusing. For instance, the same region over the surface of the Earth might be represented with different appearance, as the reader can appreciate in Figure 6, on page 33. This results from the fact that the Earth is a curved object, while the map is not. Therefore, it cannot be flattened unless compressed, streched or torn apart. In mathematics, this results is known as Egregium Theorem. It states that a local isometry between two surfaces exists if and only if both surfaces have the same Gaussian curvature at corresponding points. Section 3.2 is concerned with this result.

The Egregium theorem is of great importance in cartography, since it implies that no ideal map of the Earth can be constructed. It is impossible to represent, even a little portion of the surface of the Earth, with full fidelity. This leads to the introduction of the concept of distortion, whose undertanding is a crutial issue in the proper selection of a map projection depending on its purpose. Section 3.3 aims to familiarize the reader with one the most broadly used measures for distortion in cartography: the scale factor. It quantifies the distortion, regarding that if the projection was an isometry, the scale factor would be equal 1 and the greater its value, the greater the distortion. We shall see that it depends on the point and on the direction.

Section 3.4 briefly covers the use of auxiliar surfaces that conceptually help the developement of certain map projections: the cone, the cylinder and the plane. It also contains a brief discussion about the different aspects of a map projection: normal, transverse and oblique, and both the tangential and secant cases. This terminology describes the vocabulary commonly used in cartography and the definitions provide with a classification of the map projections.

One of the geometric features that a map projection may preserve are angles. Such projections are called conformal. The roots of conformal mapping lie in the early nineteenth century. Gauss considered in the 1820s conformal maps between surfaces. Since then, a great effort has been made on the development of the mathematics for conformal mapping. The aim of this dissertation is to apply conformal mapping to cartography, i.e., to study conformal representations of the Earth.

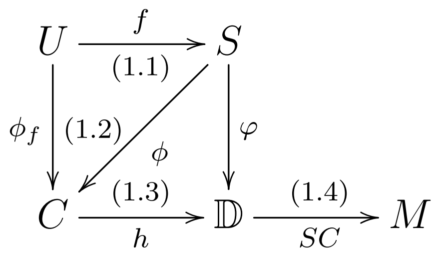
Section 2.2 and 2.3 are intended to prove that any conformal map between a regular surface $S$ and the plane is the composition of the so-called isometric map $\phi : S \to \mathbb{R}^2$ and a conformal function $h : \mathbb{C} \to \mathbb{C}$.

Section 2.2 is concerned with the study of the isometric map. It maps each point of the regular surface onto a set of rectangular coordinates, denoted by $(p, q)$, such that the metric over the surface under this rectangular coordinates is $\rho(p, q)\mathbb{1}$, where $\rho$ is a real-valued non-zero function and $\mathbb{1}$ denotes the size 2 identity matrix. Such rectangular coordinates are called isometric coordinates in cartography and the $pq$-plane, the isometric plane. This notation and terminology will be used throughout this dissertation. Example 2.18 computes the isometric coordinates for the sphere and Proposition 3.5 for the spheroid.

Section 2.3 deals with conformal maps from the complex plane onto intself. In particular we show that a function $h : \mathbb{C} \to \mathbb{C}$ is conformal if and only if it is holomorphic and its derivative does not vanish. This section concludes with the statement of Riemann's mapping theorem. This theorem was stated in Riemann's celebrated doctoral dissertation of 1851 and ensures the existence of a conformal map between any two simply connected domains of the complex plane, provided that neither is the entire plane. Thus, if for a given regular surface we are able to compute its isometric coordinates, we can construct a conformal map projection onto any simply connected domain of the complex plane. However, Riemann's theorem does not give any clue about how this conformal map is or how to construct it.

The Schwarz-Christoffel (SC) formula was discovered independently by Christoffel in 1867 and Schwarz in 1869. It provides with a conformal map from the upper half-plane $\mathbb{H}$ onto the interior of a simple polygon $P$. The main idea relies on the fact that if the derivative of a function $f : \mathbb{H} \to P$ is of the form $f'(z) = (z - z_0)^{\alpha-1}$, $z_0 \in \mathbb{R}$, then $f(z)$ maps the real segment $(-\infty, z_0)$ onto a straight line that forms an angle of $(\alpha - 1)\pi$ with the real axis and the segment $(z_0, \infty)$ onto a line parallel to the real axis. Generalizing, it is possible to find a set of prevertices $z_i \in \partial\mathbb{H}$, $i = 1, \ldots, n$ such that the real axis is mapped to the boundary of the polygon and $\mathbb{H}$ onto its interior. Notice that in the vertices of the polygon conformality is not preserved. Three of the $n$ prevertices are ours to choose, while the others are determined by the position of the vertices of the polygon. Thus, for a polygon with more than three vertices the so-called parameter problem needs to be solved before applying the SC formula. These matters are adressed in Section 2.4.

Moreover, the SC formula can be adapted for the use of the unit disk $\mathbb{D}$ as canonical domain. Thus, whenever a conformal map from $\mathbb{D}$ onto a polygon is found, its composition with any map projection within the unit disk $\varphi$ defines a conformal map from the surface of the Earth onto the polygon. The following diagram represents the idea. Here, $S$ denotes the surface, $M$ the resulting map, $U$ denotes the geographic coordinates, $C$ the isometric plane, $\phi$ the isometric map and $h$ is a conformal map.




The isometric map is the well-known Mercator projection. Published by Gerardus Mercator in about 1569 for a spherical model of the Earth, it was the result of the seek of a map projection showing curves of constant bearing, also called loxodromes, as straight lines. At a time when transoceanic journeys were begining and the technology to determine a rhumb was based on the compass, this map fulfilled a basic need for navigation. Section 4.1 explains its normal aspect, the one derived by Mercator. In the following Section 4.2, spherical trigonometry is used to establish a relation between the coordinates of a rotated sphere from the transverse to the normal aspect, and find the formulas for the transverse aspect.

For historical reasons, most of the projections are expressed in geographic coordinates and defined for a spherical model. In Section 3.1.3, we shall see that there exists a conformal map from the spheroid onto the sphere such that, if a map projection is defined for the sphere and expressed in isometric coordinates, by replacing the isometric coordinates of the sphere by those of the spheroid, a conformal map from the spheroid is obtained. However, the properties and classification might not be preserved. This also called double-conformal projection was developed by Gauss.

For each of the most used auxiliar surfaces, the cylinder, the cone and the plane (azimuthal projection), there exists a unique conformal projection, called Mercator, Lambert conformal conic and stereographic projection respectively. In all of them, the double-conformal map is used to determine the map projection formulas in the ellipsoidal case. However, Johann Heinrich Louis Krueger suggested in 1923 a direct transformation from the spheroid for the transverse Mercator projection using series. This projection, explained in Section 4.2, is also known under the name of Gauss-Krueger projection and is the base for the UTM (Universal Transverse Mercator) coordinate system, broadly used nowadays.

A usual technique for map projection developement in cartography is the generalization of the formulas. We shall see that the Lambert conformal conic is a general case of Mercator and stereographic projection. Johann Heinrich Lambert contributions to cartography have been declared as a breackthrough. His ideas, based on the use of the isometric coordinates, were further depeloped by Joseph Louis Lagrange, which culminated in the Lagrange projection, a projection of the whole world within a disk. This projection became a fundamental step for the mathematical development of projections, since the whole world conformally mapped to the unit disk is a convenient basis for further transformations.

A map projection, when used to further developements, is usually referred as parent projection. It is represented by $\varphi$ in the diagram. Within this dissertation we shall limit ourselves to the use of the stereographic, in its equatorial and polar aspects, and Lagrange projections as parent projection. However, this restriction is not necessary. Section 5.1 briefly recalls, under some small modifications (rotations and reflexions), the formulas for these three map projections.

I would like to remark that there are many different formulas and paths to describe a map projection. The parent projection itself allows many definitions, based on the orientation and the flexibility provided by the complex trigonometric functions. In most of the cases I have tried to justify the prevertices selection and the reason why some rotation or reflexions are performed, mostly by means of a verbal description of the position of representative parallels and meridians such as the equator or the poles. Moreover, the SC formula admits scaling and translation. I have choosen these parameters in an attempt to provide with the most simple derivations. However, most of the bibliography is rather concerned with the most straightforward formulas or suitable equations for computation purposes. Thus, the reader might find different formulas for the same (other than scaling, translation, etc.) map projection when consulting different books.

Some of the maps presented in this dissertation were made with the matplotlib basemap toolkit, a library for plotting 2D data on maps in Python. The grid transformations on Section 5 were created using the Schwarz-Christoffel Toolbox for Matlab.

The SC formula is computed for three polygons: the triangle, the square and the rectangle. For the triangle and the rectangle, we use as canonical domain the upper half-plane and compose the obtained SC map with an appropriate Moebius transformation from $\mathbb{D}$ onto $\mathbb{H}$. On the other hand, the conformal map onto the square is computed directly from the unit disk.

The computation of the SC formula for an equilateral triangle leads to the incomplete beta function. By means of this function, two map projections within an equilater triangle are discussed. The first, developed by O.S. Adams, projects one hemisphere of the sphere and the second, described by L.P. Lee, represents the whole world. These can be found in Section 5.2.

In Section 5.3, we discuss three famous map projections of one hemisphere within the square: Pierce quincucial, Adams and Guyou projections. The three of them are different aspects of the same projection, although none of the authors recognized it. Moreover, all of them were developed without using the SC formula, by means of elliptic coordinates and elliptic functions, which we shall not discuss in this dissertation. After that, two projections of the whole world within a square described by Adams are studied.

In Section 5.4, a brief explanation of the parameter problem for the rectangle (that can also be applied to the square, as a particular case of the rectangle) is followed by the evaluation of the SC formula, which leads to the incomplete elliptic integral of first kind. This function is used to describe a map of the whole world within a rectangle.

The rectangle $R$ can also be used further to be mapped onto an ellipse. Section 5.5 deals with a map projection of the whole world within an ellipse. This is based on the function $\sin w$, $w \in R$. With a proper normalization of the coordinates $w$, $\sin w$ provides with a conformal map onto an ellipse with foci at $-1$ and 1. The projection was described by Adams and presents a slit at the two segments over the real axis that connect the focis with the extreme of the ellipse. Over this segments, conformality fails.

Littrow's projection may be found by adapting the SC formula to the exterior of a polygon, which for this projection is simply a segment that may be placed over the real axis. This map projection is discussed in Section 5.6. The SC formula is applied in fact twice. The first projection uses as parent projection the stereographic projection centered at the north pole, which is represented in the upper half plane of the map. The second, uses the stereographic projection centered at the south pole and represented on the lower half plane. However, both projections lead to the same formula so the projection may be defined by a single equation. Littrow's projection is also retroazimuthal, i.e., it preserves directions from any point over the map to the central point.

The last two projections that this dissertation deals with are Eisenlohr and August projections. For their developement we shall not use the SC formula. Eisenlohr's projection is the result of the seek of a minimal distorted map projection. He argued that a map presents minimal geodetic distortion when the scale factor has constant value over the boundary of the map. Moreover, since the map projection is conformal, it is harmonic. This leads to a Laplace equation that may be solved by means of a Green function. This matters are discussed in Section 5.7.

However, the computations for Eisenlohr's map projection are complicated. August projection's aspect is similar to Eisenlohr's and was found as an alternative to it. It maps the whole world onto the interior of an epycicloid, a curve defined by a point on a circle rolling without sliding around another fixed circle. The two-cusped epicycloid with radi $\frac{1}{2}$ may be described by the equation $f(z) = \frac{1}{2}(3z - z^3)$, $z = e^{i\theta}$, $\theta \in [0, 2\pi)$. Since the function $f$ is analytic in the interior of the unit disk, it maps conformally its interior onto the interior of the simply connected domain bounded by the epicycloid. Section 5.8 is concerned with this map projection.

Many other conformal map projections have been developed, such as the projections onto regular polygons or the polyhedral maps, among others. Despite this, I hope this dissertation gives a general idea about how conformal mapping is applied to cartography and the important paper of mathematics in the developement of maps of the Earth.

# 2 Preliminaries

This is an introductory chapter intended to review the basic mathematical concepts that are about to be applied to cartography in the following sections. If the reader is already familiar with basic concepts on riemannian surfaces and conformal mapping, she/he may prefer to skip this section and go directly to Section 3, Elements of map projections.

## 2.1 Surfaces

The surface of the Earth, that shall be denoted by $S$, is a Riemannian surface embeded in the Euclidean space $\mathbb{R}^3$. Such a surface is locally 2-dimensional. An injection $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ whose range is an open subset of $\mathbb{R}^3$ is called a *chart* when $f$ is differentiable and the Jacobian matrix has rank 2. This requirements are known as the regularity condition. Such a chart defines on its domain $U$ a set of coordinate $(u, v) \in U$ so that a point of the surface is defined by $f(u, v) = (f^1(u, v), f^2(u, v), f^3(u, v))$,

$$U \xrightarrow{f} S.$$

There may be many different coordinate possibilities for the same surface.

**Example 2.1.** Let $R = \{(x, y, z) \in \mathbb{R}^3 \ s.t. \ z = 0, |x| < 1, |y| < 2\}$ be a rectangle in the xy-plane and let $f : (-1, 1) \times (-2, 2) \to \mathbb{R}^3$ be defined by $f(u, v) = (u, v, 0)$. Then $f$ is a coordinate chart of the rectangle.

It is generally not possible to construct a suitable chart for a set $S$ whose image is the whole of $S$. The best that we can do is to define a collection of charts for $S$ whose images together cover $S$. Such a collection $\mathscr{A}$ is called an *atlas* if, when $f_1$ and $f_2$ are any two charts of $\mathscr{A}$ whose domains intersect, the change of coordinates $f_2 \circ f_1^{-1}$ is a diffeomorphism, this is, an injection such that both the function and its inverse are $C^\infty$ functions.

**Example 2.2.** Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$ and let $\mathbb{D} = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ denote the open unit disk in $\mathbb{R}^2$. The image of the chart $f : \mathbb{D} \to \mathbb{S}^2$

$$f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

covers only the upper hemisphere of $\mathbb{S}^2$. The mapping $f$ is differentiable and the Jacobian is of rank 2, and so the parametrization is regular. The charts $f_i : \mathbb{D} \to \mathbb{S}^2$, $i = 2, \ldots, 6$

$$\begin{aligned}
&f_2(u, v) = (u, v, -\sqrt{1 - u^2 - v^2}),\\
&f_3(u, v) = (u, \sqrt{1 - u^2 - v^2}, v), \ \ f_4(u, v) = (u, -\sqrt{1 - u^2 - v^2}, v)\\
&f_5(u, v) = (\sqrt{1 - u^2 - v^2}, u, v), \ \ f_6(u, v) = (-\sqrt{1 - u^2 - v^2}, u, v)
\end{aligned}$$

are all regular parametrizations and cover respectively the lower hemisphere, the east-west and the back-front of the sphere. The set $\{f_i, i = 1, \ldots, 6\}$ is an atlas of $\mathbb{S}^2$ and defines a coordinates system over the sphere.

The sphere is a model of the surface of the Earth broadly used in cartography.

The curves

$$u \mapsto f(u, v_0),$$
$$v \mapsto f(u_0, v)$$

are called the *coordinate curves.*

A large set of regular surfaces can be found by means of the implicit function theorem:

**Theorem 2.3.** *If $\phi : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in \phi(U)$ is a regular value, that is rank$(J_\phi(x)) = 1$ for all $x \in U$ such that $\phi(x) = a$, then $S := \phi^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.*

**Example 2.4.** The ellipsoid

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

where $a, b, c \neq 0$, is a regular surface provided that $\Sigma = \phi^{-1}(0)$ where

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

is differentiable and 0 is a regular value, since $\phi(x, y, z) = 0$ for

$$(x, y, z) \in \{(\pm a, 0, 0), (0, \pm b, 0), (0, 0, \pm c)\}$$

and the jacobian matrix

$$J_\phi(x, y, z) = \begin{pmatrix} \frac{2x}{a^2} & \frac{2y}{b^2} & \frac{2z}{c^2} \end{pmatrix}$$

has rank 1 for these points.

The ellipsoid is used in cartography to model $S$, the surface of the Earth.

### 2.1.1 Tangent plane and normal vector to the surface: The Gaussian curvature.

We shall denote the partial derivatives of a chart $f$ by $f_u(p) = \frac{\partial f}{\partial u}(p)$, and $f_v(p) = \frac{\partial f}{\partial v}(p)$. Observe that $f_u$ and $f_v$ are the tangent vectors of the coordinate curves.

The *tangent plane* of a surface in $p \in S$, $p \in f(U)$ for some chart $f$ of the surface $S$, is defined as

$$\mathrm{Im}(d_p f) = \{w \in \mathbb{R}^3 \mid \exists x = (u,v) \in U,\ w = d_p f(u,v)\}.$$

It is a vector subspace of dimension 2 of $\mathbb{R}^3$ that does not depend on the choice of the coordinate chart $f$. A base of the tangent plane is formed by $f_u$ and $f_v$. We denote de tangent plane by $T_p f$. Observe that $T_p f$ contains all the tangent vectors of a regular curve $\gamma$ over $S$, $p \in \gamma$. Moreover, if $v \in T_p f$, then there exists a curve over $S$ such that $\gamma(t_0) = p$ and $\gamma'(t_0) = v$.

A map $f : U \to S$ satisfies the regularity condition at $(u,v) \in U$, if and only if $f_u \times f_v \neq 0$. Therefore we can define the normal direction $f_u \times f_v$.

**Definition 2.5.** *The Gauss map $N : S \to \mathbb{S}^2$ is defined as*

$$N(p) = \frac{f_u(p) \times f_v(p)}{\|f_u(p) \times f_v(p)\|}, \quad p \in S$$

Notice that the tangent plane at $p$ is the vector subspace of $\mathbb{R}^3$ of vectors normal to $N(p)$. It can be proved that under a change of coordinates, $|N|$ is preserved, so that the only variation is the sign of $N$. This is consistent with the fact that $N$ is the normal vector to the tangent plane which is invariant under change of coordinates and $\pm N$ defines the same plane.

**Definition 2.6.** *Let $f$ be a chart of the surface $S$ containing the point $p \in S$. The Gaussian curvature is defined as*

$$K(p) = \frac{\det(N_x, N_y, N)}{\det(f_x, f_y, N)}. \tag{2.1}$$

We shall see in the next chapter that the Gaussian curvature is an intrinsic property of the surface and does not depend on the way the surface is embedded within the euclidean space, although the definition of the Gaussian curvature of a surface certainly depends on the way in which the surface is located in space.

**Example 2.7.** [Surface of revolution]

Let $\gamma : I \to \mathbb{R}^3$ be a parametric regular curve such that $\gamma(I)$ is contained on a plane, and assume without loss of generality, that the plane is the xz-plane, $x > 0, y = 0$, i.e., $\gamma(t) = (a(t), 0, b(t))$, $a, b : I \to \mathbb{R}$. A surface of revolution is a surface created by rotating the curve $\gamma$ around the z-axis and its parametrization is $f : I \times \mathbb{R} \to \mathbb{R}^3$

$$f(u,v) = (a(u)\cos(v), a(u)\sin(v), b(u)).$$

It is a regular surface, since by a straightforward computation,

$$\|f_u \times f_v\| = |a(u)|\sqrt{a'(u)^2 + b'(u)^2} \neq 0 \quad \forall u \in I,$$

where $a(u) > 0$ for all $u$ since we imposed $x > 0$.

The Gauss map is

$$N = \frac{(-b' \cos v, -b' \sin v, a')}{\sqrt{(b')^2 + (a')^2}}.$$

Another computation shows that the Gaussian curvature is

$$K = \frac{b'(b''a' - b'a'')}{a(a'^2 + b'^2)^2}.$$

Particular examples of surfaces of revolution that are important in cartography are the sphere and the ellipsoid of revolution.

**Sphere:** The sphere of radi $R$, without the meridian $\pi$ and the poles, is a surface of revolution generated by the curve

$$\gamma(u) = (R \cos u, 0, R \sin u),\ u \in (\frac{-\pi}{2}, \frac{\pi}{2}).$$

This curve leads to a parametrization

$$f(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin v), \quad u \in (\frac{-\pi}{2}, \frac{\pi}{2}), \quad v \in (-\pi, \pi).$$

Its Gaussian curvature is

$$K = \frac{1}{R^2} \tag{2.2}$$

**Ellipsoid of revolution:** An ellipsoid of revolution or spheroid is a type of ellipsoid in which two of the three semiaxes are equal. It can be found by rotating an ellipse with semiaxes $a$ y $b$:

$$\gamma(t) = (a \cos t, 0, b \sin t),\ t \in (\frac{-\pi}{2}, \frac{\pi}{2}).$$

The resulting parametrization of the spheroid is then

$$f(u, v) = (a \cos u \cos v, a \cos u \sin v, b \sin u), \quad u \in (\frac{-\pi}{2}, \frac{\pi}{2}), \quad v \in (-\pi, \pi),$$

and the Gaussian curvature is

$$K = \frac{b^2}{\left(a^2 \sin^2 v + b^2 \cos^2 b\right)^2}. \tag{2.3}$$

The coordinate system for the sphere and the spheroid derived in Example 2.7 is called in geodesy *geocentric coordinate system*.

### 2.1.2 The metric

The Riemannian metric or first fundamental form allows the developement of geometric properties of the surface, a non euclidean space. It defines an inner product within the surface as the induced inner product by the dot product on $\mathbb{R}^3$ in which the surface is embedded.

**Definition 2.8.** *For a point $p \in S$, $S$ a regular surface, the* first fundamental form *is the bilinear symmetric form*

$$g_p : T_pS \times T_pS \to \mathbb{R}$$

*defined by $g_p(w_1, w_2) = \langle d_pf(w_1), d_pf(w_2)\rangle$, where $f$ is a coordinate chart in a neighbourhood of $p$. The coefficients of the associate matrix with respect to the standard basis $\mathbb{R}^2$, $\{e_1 = (1,0), e_2 = (0,1)\}$ are called the* coefficients of the first fundamental form *and denoted by $E$, $F$ and $G$. Hence*

$$g(w_1, w_2) = w_1^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_2, \quad w_1, w_2 \in T_pS,$$

*where*

$$E = g(e_1, e_1) = \langle f_x, f_x\rangle\,, \quad F = g(e_1, e_2) = \langle f_x, f_y\rangle\,, \quad G = g(e_2, e_2) = \langle f_y, f_y\rangle$$

*Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^3$.*

**Example 2.9.** The coefficients of the first fundamental form of the sphere and the spheroid under the parametrization proposed in Example 2.7 are

$$E_{\mathbb{S}^2} = \langle f_u, f_u\rangle = R^2, \quad F_{\mathbb{S}^2} = \langle f_u, f_v\rangle = 0, \quad G_{\mathbb{S}^2} = \langle f_v, f_v\rangle = R^2 \cos^2 u$$

$$E_\Sigma = a^2 \sin^2 u + b^2 \cos^2 u, \quad F_\Sigma = 0, \quad G_\Sigma = a^2 \cos^2 u$$

The first fundamental form allows the measurement of lenghts, angles and areas over the surface analougsly as in the euclidean space.

**Definition 2.10.** *Let $\gamma : I \to \mathbb{R}^3$ be a regular curve over a regular surface $S$ with chart $f : U \to S$ at a neighbourhood of $p \in f(U)$, $p \in \gamma(I)$ , $\gamma = f \circ \tilde{\gamma}$, $\tilde{\gamma} : I \to U$. Let $a, b \in I$ within the neighbourhood of $p$. Then the lenght of $\gamma$ between $a$ and $b$ is*

$$L_{(a,b)}(\gamma) = \int_a^b ||\dot{\gamma}|| = \int_a^b \sqrt{g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})} \tag{2.4}$$

**Example 2.11.** Recalling the geocentric parametrization of the sphere discussed in Example 2.7 we may use equation (2.4) to compute the longitude of the coordinate curve $v \mapsto (u_0, v)$, also called parallels in geodesy, and that we shall denote by $p_{u_0}$, $p_{u_0}(t) = f \circ \tilde{p}_{u_0}(t) = f(u_0, t)$. The tangent to the curve is $\tilde{p}'_{u_0}(t) = (0,1)^T$, and using the coefficients found in Example 2.9 we obtain

$$L^{\mathbb{S}^2}_{(-\pi,\pi)}(p_{u_0}) = \int_{-\pi}^{\pi} \sqrt{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} R^2 & 0 \\ 0 & R^2\cos^2 u_0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} dt = 2\pi R \cos u_0 \tag{2.5}$$

where $u_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This is the lenght of a circumference of radi $R\cos u_0$, and regarding that $u$ is the angle between a point over the sphere and the xy-plane, it is also the radii of the circumference defined by a parallel of latitude $u$. Thus, by applying simple trigonometry we obtain the same result.

**Definition 2.12.** *Let $\gamma$ and $\delta$ be regular curves defined over the surface $S$ in a neighbourhood of $p \in S$ that allows a coordinate chart $f$ and let $\gamma(t_0) = \delta(t_1) = p$ for some $t_0, t_1 \in \mathbb{R}$. The angle $\alpha$ between the two curves is defined by*

$$\cos\alpha = \frac{\langle \dot{\gamma}(t_0), \dot{\delta}(t_1)\rangle}{\|\dot{\gamma}(t_0)\|\|\dot{\delta}(t_1)\|} = \frac{g(\dot{\tilde{\gamma}}(t_0), \dot{\tilde{\delta}}(t_1))}{g(\dot{\tilde{\gamma}}(t_0), \dot{\tilde{\gamma}}(t_0))g(\dot{\tilde{\delta}}(t_1), \dot{\tilde{\delta}}(t_1))} \tag{2.6}$$

**Example 2.13.** We may compute the angle between the coordinate curves of the geocentric coordinates of the sphere defined in Example 2.7, $\tilde{p}_{u_0} = (u_0, t)^T$, $\tilde{m}_{v_0} = (t, v_0)^T$, using (2.6). The tangent vectors to the coordiante curves are

$$\tilde{p}_{u_0} = (0,1)^T, \quad \tilde{m}_{v_0} = (1,0)^T$$

and therefore

$$\langle p'_{u_0}, m'_{v_0}\rangle = (0,1)\begin{pmatrix} R^2 & 0 \\ 0 & R^2\cos^2 u_0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \quad \forall (u,v) \in U.$$

Hence,

$$\cos\alpha = 0,$$

so the geocentric system over the sphere is orthogonal.

**Definition 2.14.** *The area of a region $R \subset S$ in a neighbourhood of $p \in R$ with coordinate chart $f : U \to R$ is*

$$A(R) = \int_U det(f_u, f_v, N)dudv = \int_U \|f_u \times f_v\|dudv.$$

The area can be computed in terms of the coefficients of the first fundamental form

$$A(f) = \int_U \sqrt{EG - F^2}dudv,$$

regarding that

$$\det(f_u, f_v, N)^2 = \det((f_u, f_v, N)^T(f_u, f_v, N)) = \begin{vmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{vmatrix} = EG - F^2.$$

## 2.2 Conformal mapping between surfaces

So far, basic geometric properties of a surface $S$ embedded in $\mathbb{R}^3$ have been studied. This section aims to study conformality between surfaces.

### 2.2.1 Differentiability and conformality between surfaces

Any map from one surface to another has a representation in coordinate terms. Let $f$ and $f'$ be charts from the surfaces $S$ and $S'$ and with domains $U$ and $V$ respectively, and let $\phi$ be a map between the surfaces $\phi : S \to S'$. Then, $\phi_f = f'^{-1} \circ \phi \circ f : U \to V$ is well defined and usually called the *representation of $\phi$ in terms of the charts $f$* and $f'$.

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & S \\ {\scriptstyle \phi_f}\downarrow & & \downarrow{\scriptstyle \phi} \\ V & \xrightarrow[\ f'\ ]{} & S' \end{array}$$

In particular, the representation in coordinate terms of a map between surfaces allows the definition of differentiability. A function $\phi : S \to S'$ is said to be *differentiable in $p \in S$* if $\phi_f$ is differentiable in $f^{-1}(p)$. A function $\phi : S \to S'$ is said to be *differentiable* if it is differentiable at every point of its domain.

As a change of coordinates is a diffeomorphism, the differentiability of $\phi$ is independent of the coordinate system chosen.

A diffeomorphism between surfaces $\phi : S \to S'$ is an injection such that $\phi$ and its inverse $\phi^{-1}$ are both differentiable. Two surfaces are said to be *diffeomorphic* if there exists a global diffeomorphism of $S$ onto $S'$.

We are now ready to study the conformal maps between surfaces.

**Definition 2.15.** *Let $S$ and $S'$ be regular surfaces and $\phi : S \to S'$ a diffeomorphism. Then, $\phi$ is said to be conformal if $\phi$ preserves angles locally, i.e., if for all $\gamma : I \to S$ and $\delta : I' \to S$ regular curves over $S$ so that they intersect at $p \in S$, $p = \gamma(t_0) = \delta(t_1)$ with angle $\alpha$, then their images by $\phi$, $\gamma' = \phi(\gamma)$ and $\delta' = \phi(\delta)$ intersect at $\phi(p) = \phi(\gamma(t_0)) = \phi(\delta(t_1))$ with angle $\alpha$.*

**Proposition 2.16.** *Let $\phi : S \to S'$ be a diffeomorphism. Then $\phi$ is conformal if and only if there exists a nonzero function $\rho : S \to \mathbb{R}$ such that for all $p \in S$ and for all $w_1, w_2 \in T_p(S)$,*

$$g(w_1, w_2) = \rho^2(p) g'(d\phi_p(w_1), d\phi_p(w_2))$$

*Proof.* $\Leftarrow$) Let $\phi : S \to S'$ be a diffeomorfism and assume that there exists a nonzero function $\rho : S \to \mathbb{R}$ satisfying the stated equation. Let $w_1$ and $w_2$ be the tangent vectors of two regular curves on $S$ that intersect forming an angle $\alpha$ and let $\alpha'$ be the angle between the images of the curves by $\phi$ in $S'$. Then,

$$\begin{aligned}
\cos\alpha &= \frac{g(w_1, w_2)}{\sqrt{g(w_1, w_1)g(w_2, w_2)}} \\
&= \frac{\rho^2(p)g'(d\phi_p(w_1), d\phi_p(w_2))}{\sqrt{\rho^2(p)g'(d\phi_p(w_1), d\phi_p(w_1))\rho^2 g'(d\phi_p(w_2), d\phi_p(w_2))}} \\
&= \frac{g'(df_p(w_1), df_p(w_2))}{\sqrt{g'(d\phi_p(w_1), d\phi_p(w_1))g'(d\phi_p(w_2), d\phi_p(w_2))}} = \cos\alpha'
\end{aligned}$$

Thus, $\phi$ is conformal.

$\Rightarrow$) Let $\phi : S \to S'$ be a conformal mapping. Because $\phi$ is a diffeomorphism, a chart $f : U \subset \mathbb{R}^2 \to S$ determines a chart on $S'$, namely, $f' = \phi \circ f : U \to S'$. Thus, we can apply the same coordinates on each surface. Furthermore, the local expression for a curve on $S$, $\gamma(t) = f \circ \tilde{\gamma} : I \to S$, $\gamma(t) = f(u(t), v(t))$, carries over to $S'$ via the mapping $\phi$ as $\phi \circ \gamma(t) = f'(u(t), v(t))$. Thus

$$d_p\phi\left(f_u \frac{du}{dt} + f_v \frac{dv}{dt}\right) = f'_u \frac{du}{dt} + f'_v \frac{dv}{dt}$$

Let $w_1,\ w_2 \in T_pS$ be orthogonal unit vectors. Because $\phi$ is conformal, $g'(d_p\phi(w_1), d_p\phi(w_2)) = 0$. Let $\|d_p\phi(w_1)\| = c_1$ and $\|d_p\phi(w_2)\| = c_2$. By the linearity of the inner product we find

$$\begin{aligned}
\frac{1}{\sqrt{2}} &= \frac{g(w_1, w_1 + w_2)}{\sqrt{g(w_1, w_1)g(w_1 + w_2, w_1 + w_2)}} \\
&= \frac{g(d_pf(w_1), d_pf(w_1) + d_pf(w_2))}{\sqrt{g(d_pf(w_1), d_pf(w_1))g(d_pf(w_1) + d_pf(w_2), d_pf(w_1) + d_pf(w_2))}} \\
&= \frac{c_1^2}{c_1\sqrt{c_1^2 + c_2^2}}
\end{aligned}$$

Thereofore $c_1 = c_2$. Define $\rho : S \to \mathbb{R}$ by $\rho(p) = c_1$. At $p \in S$ write $f_u = aw_1 + bw_2$. Then $d_p\phi(f_u) = ad_p\phi(w_1) + bd_\phi(w_2)$ and the coefficients of the first fundamental Gaussian form are

$$\begin{aligned}
\overline{E} &= g(d_p\phi(f_u), d_p\phi(f_u)) = a^2 d_p\phi(w_1)d_p\phi(w_1) + 2ab d_p\phi(w_1)d_p\phi(w_2) + b^2 d_p\phi(w_2)d_p\phi(w_2) \\
&= \rho(p)(a^2 + b^2) = \rho^2(p)E
\end{aligned}$$

Analogously $\overline{F} = \rho^2(p)F$ and $\overline{G} = \rho^2(p)G$. Varying the orthonormal basis smoothly in a small neighborhood around $p$, we obtain this relationship between component functions for all points near $p$. □

### 2.2.2 The isometric plane

We are interested in map projections and therefore we want the surface $S'$ to be contained in a plane $\Pi$. The coefficients of the first fundamental form in $\Pi$ are given by $E = G = 1$ and $F = 0$ for some orthogonal coordinates over the plane, say $p, q$. By Proposition 2.16, a map $\phi$ from the surface of the Earth $S$ to the plane is conformal if we can define a non zero function $\rho : S \to \Pi$ satisfying $g_S = \rho^2 g_\Pi$. Because $g_\Pi$ is the unit matrix, we require $E = G = \rho^2$ and $F = 0$.

Suppose a chart on the surface $S$ with orthogonal coordinates $(u, v)$, this is, the corresponding coefficient $F$ is 0. If $E$, $G$ can be written as

$$E = \frac{\rho^2}{U}, \quad G = \frac{\rho^2}{V},$$

where $U$ is a function of $u$ alone and $V$ a function of $v$ alone, both non vanishing in its domain, then we can define $\phi(u, v) = (p(u), q(v))$ such that

$$\left(\frac{dp}{du}\right)^2 = \frac{1}{U}, \quad \left(\frac{dq}{dv}\right)^2 = \frac{1}{V}$$

so that the metric in terms of $(p, q)$ over the surface is

$$\begin{aligned} E(p,q) &= \left(\frac{dp}{du}\right)^2 E(u,v) = \rho^2 \\ F(p,q) &= 0 \\ G(p,q) &= \left(\frac{dq}{dv}\right)^2 G(u,v) = \rho^2, \end{aligned}$$

and therefore the map $\phi$ is a conformal representation from the surface to the plane.

**Definition 2.17.** *The coordinates $(p, q)$ over a surface that verify $E = G$ and $F = 0$ are called* isometric coordinates. *Respectively, the pq-plane is called* isometric plane.

**Example 2.18.** Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-\pi, \pi) \to \mathbb{R}^3$ be the parametrization of the sphere considered in Example 2.7

$$f(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u).$$

The first fundamental form under this parametrization was computed in Example 2.9:

$$E = R^2, \quad F = 0, \quad G = R^2 \cos^2 u$$

Define

$$\rho(u, v) = R \cos u. \tag{2.7}$$

Notice that we can write

$$E = \rho^2 \frac{1}{\cos^2 u}, \quad G = \rho^2$$

Let $\phi : S^2 \to \mathbb{R}^2$ be defined by $(u, v) \mapsto (p(u), q(v))$, where

$$\left(\frac{dp}{du}\right)^2 = \frac{1}{\cos^2 u}, \quad q = v \tag{2.8}$$

Then, in the $pq$-plane

$$E(p,q) = \left(\frac{dp}{du}\right)^2 E = \rho^2 \quad F(p,q) = \left(\frac{dp}{du}\right) F = 0, \quad G(p,q) = G = \rho^2$$

Therefore, the map $\phi(u,v) = (p(u), q(v))$ is a conformal map between the sphere and the plane. We can find $p(u)$ by integrating (2.8):

$$p = \int \sec u \; du = \ln|\tan\left(\frac{u}{2} + \frac{\pi}{4}\right)|.$$

The coordinates $(p,q)$ computed above are the isometric coordinates over the sphere.

The conformal map $\phi$ is known in cartography as the Mercator projection, a map projection that will be studied in Section 4.

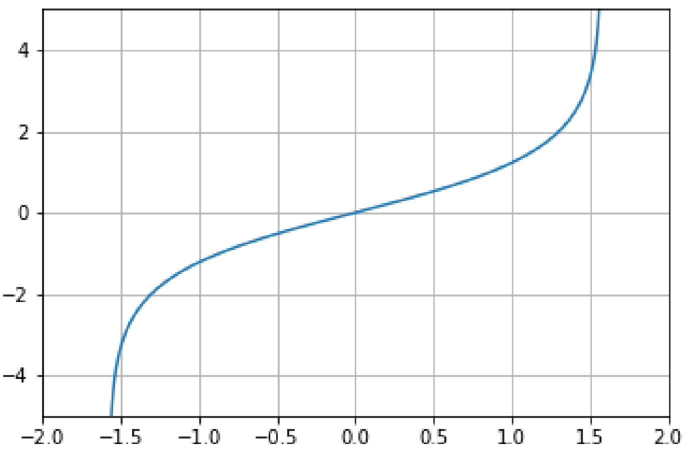
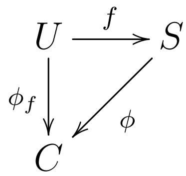



Figure 1: The isometric latitude $p$ as a function of $u$ in radians.

We shall call $\phi$ as the isometric map to the isometric plane denoted by $C$.




## 2.3 Holomorphy and conformality: Riemann's Theorem

Now that we have mapped the surface onto the plane conformally, we shall use conformal functions from the plane onto the plane (notice that the composition of conformal maps is also conformal) to create different maps. In this section we shall study the conditions for a function $h : \mathbb{R}^2 \to \mathbb{R}^2$ to be conformal.

In order to do so, we shall use complex notation.

$$x + iy = h(p + iq).$$

### 2.3.1 Holomorphy and conformality

**Definition 2.19.** *Let $h$ be a function defined over an open domain $S \subset \mathbb{C}$ and let $z_0 \in S$. It is said that $h$ has a derivative $h'(z_0)$ in $z_0$ or is holomorphic in $z_0$ if it exists the limit*

$$\lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = h'(z_0).$$

The values of the complex function $f$ can be written by $x(z) + iy(z)$, where $x$ and $y$ are real valued functions of a complex variable $z = p + iq$. By the canonical isomorphism $\mathbb{C} \simeq \mathbb{R}^2$ given by the relation $z = p + iq \mapsto (p, q)$ the function can be identified as a real function in two variables $h = x(p, q) + iy(p, q)$.

Let $x, y \in \mathscr{C}^1(S)$. Then, $h = x + iy$ is holomorphic in $z_0 = p_0 + iq_0$ if and only if

$$\frac{\partial x}{\partial p}(p_0, q_0) = \frac{\partial y}{\partial q}(p_0, q_0), \quad \frac{\partial x}{\partial q}(p_0, q_0) = -\frac{\partial y}{\partial p}(p_0, q_0)$$

The former equations are known under the name of *Cauchy-Riemman equations.*

Equivalently, $h$ is holomorphic in $z_0$ if and only if

$$\frac{\partial h}{\partial \overline{z}}(z_0) = 0.$$

We shall now derive the condition for conformality in terms of $f$.

**Proposition 2.20.** *A function $h : S \subset \mathbb{C} \to \mathbb{C}$ is conformal in $z_0$ if and only if $h$ is holomorphic in $z_0$ and $h'(z_0) \neq 0$. Moreover, the differential $dh(z_0)$ represents a rotation of angle arg$(h'(z_0))$ and a dilation of magnitude $|h'(z_0)|$.*

*Proof.* Let $\gamma_1(t)$ and $\gamma_2(t)$ be two regular curves such that $\gamma_1(0) = \gamma_2(0) = z_0$. The angle at which both curves intersect in $z_0$ is $\alpha = |\arg(\dot{\gamma}_1(0)) - \arg(\dot{\gamma}_2(0))| = \arg(\dot{\gamma}_1(0)\overline{\dot{\gamma}_2(0)})$. Let $\tilde{\gamma}_i = h(\gamma_i)$ for $i = 1, 2$. Then, the tangent vector to $\tilde{\gamma}_i$ at $h(z_0)$ is $\dot{\tilde{\gamma}}_i(0) = (h \circ \tilde{\gamma}_i)'(0) = dh(z_0)(\gamma_i'(0))$ for $i = 1, 2$. Therefore, the angle is

$$\arg(dh(z_0)(\gamma_1'(0))\overline{dh(z_0)(\gamma_2'(0))})$$

Assume $h$ to be holomorphic at $z_0$ and that $h'(z_0) \neq 0$. Then,

$$dh(z_0)(\gamma_i'(0)) = h'(z_0)\gamma_i'(0)$$

and thus,

$$\arg(h'(z_0)\gamma_1'(0)\overline{h'(z_0)\gamma_2'(0)}) = \arg(|h'(z_0)|^2\gamma_1'(0)\overline{\gamma_2'(0)}) = \arg(\gamma_1'(0)\overline{\gamma_2'(0)}).$$

This is, $h$ is conformal at $z_0$. Note that because $h$ is holomorphic, $|dh(z_0)(\gamma'(0))| = |h'(z_0)||\gamma'(0)|$, and therefore $h$ applies a dilatation independent of the direction and of magnitude $|h'(z_0)|$.

Assume now that $h$ is conformal at $z_0$. Then,

$$\frac{dh(z_0)(w)}{w} = \frac{\partial h}{\partial z}(z_0) + \frac{\partial h}{\partial \overline{z}}(z_0)\frac{\overline{w}}{w}, \quad w \in \mathbb{C} \setminus 0$$

has constant argument. Let

$$h'(z_0) = \frac{\partial h}{\partial z}(z_0) = Re^{i\phi} \quad R > 0$$

Taking $w = e^{i\theta}$ with $\theta \in [0, 2\pi]$, we obtain that

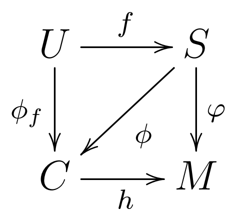
$$\frac{dh(z_0)(e^{i\theta})}{e^{i\theta}} = Re^{i\phi} + \frac{\partial h}{\partial \overline{z}}(z_0)e^{-2i\theta}$$

has constant argument. Thus, dividing by $e^{i\phi}$ we obtain that

$$Re^{-i\phi}\frac{dh(z_0)(e^{i\theta})}{e^{i\theta}} = R + \frac{\partial h}{\partial \overline{z}}(z_0)e^{-i(2\theta+\phi)}$$

has constant argument while varying $\theta$. This is only possible if $\frac{\partial h}{\partial \overline{z}}(z_0) = 0$, and therefore $h$ is holomorphic at $z_0$ and $h'(z_0) \neq 0$. □

Therefore, any conformal map $\varphi$ from a regular surface onto the plane is a composition of the isometric map $\phi$ followed by a conformal function $h$ from the complex plane onto itself. We represent this by the diagram




### 2.3.2 Riemann's Theorem

The Riemann mapping theorem was first stated in Riemann's celebrated doctoral dissertation of 1851: any simply connected region in the complex plane can be conformally mapped onto any other, provided that neither is the entire plane.

**Theorem 2.21** (Riemann's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $a \in \Omega$. Then there exist a unique conformal map $\varphi : \Omega \to \mathbb{D}$ such that $\varphi(a) = 0$ and $\varphi'(a) > 0$.*

Riemman's Theorem ensures that any two simply connected domains are conformaly equivalent. Despite the importance of this result, it is an existence theorem and it gives no clue about how this conformal map looks like or how to construct it.

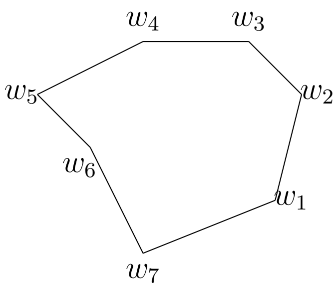
## 2.4 The Schwarz Christoffel Formula

The Schwarz Christoffel (SC) formula was discovered independently be Elwin Bruno Christoffel between 1868 and 1870 and by Hermann Amandus Schwarz in the late 1860s. The formula provides a method to find explicitely the conformal map between the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$, or the unit disk $\mathbb{D}$, and some simply connected domains such as polygons that Riemann's Theorem claims to exist.

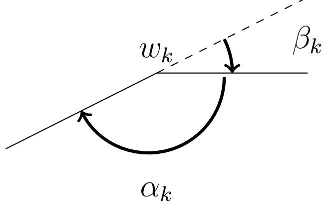
### 2.4.1 Polygons

**Definition 2.22.** *A* linear curve *is a closed curve formed by a concatenation of finitely many line segments $\gamma_k$. The point at which two segments meet is a* vertex $w_k$. *A simple linear curve, i.e., a linear curve that does not intersect itself, is called a* polygon $\Gamma$.

The polygon $P$ separates the complex plane in two regions. We shall denote by $P$ the interior of the polygon. The numbering of the vertices is taken in counterclockwise order. We allow the vertices to be the point at infinity. Their treatment may not be easy but is feasible.




**Definition 2.23.** *The* interior angle $\alpha_k\pi$ *at the vertex* $w_k$ *is the angle created by sweeping counterclockwise from the outgoing side to the incoming side. Similarly, the* outer angle $\beta_k\pi$ *is defined as the angle created by extending the incoming side and sweeping from there towards the outgoing side.*




Finite vertices have inner angles in the range $(0, 2\pi)$ given by $\frac{w_{k-1}-w_k}{w_{k+1}-w_k}$, where $w_{n+1} = w_1$. On the other hand, vertices at the infinity verify $\alpha_k \in [-2, 0]$. The value $\alpha = 2$ defines a slit and the outgoing and incoming sides are collinear. The map $f : \mathbb{H} \to \mathbb{D}$ is still one-to-one in the interior of the polygon although the function can no longer be extended one-to-one to the boundary of the polygon.

Notice that

$$\alpha_k + \beta_k = 1, \ \forall k = 1, \ldots, n.$$

If the polygon has finite vertices, and because it is a closed linear curve, it does a total turn of $2\pi$, this is

$$\sum_{k=1}^{n} \beta_k = 2.$$

In terms of the interior angles

$$\sum_{k=1}^{n} \alpha_k = n - 2.$$

Note that linear curves that are not polygons do not need to abide by this rule.

The interior of the polygon is a simply connected domain. By Riemman's Theorem, a conformal map from a simply connected domain to the polygon exists. We shall refer to the mapped domain as canonical domain. The SC formula was first developed for the canonical domain $\mathbb{H}$. We shall therefore introduce the formula using $\mathbb{H}$ as the canonical domain. Nevertheless, we will be interested in studying the conformal functions from $\mathbb{D}$, as we will project the sphere in the unit disk. We shall use different Moebius transformation from $\mathbb{D}$ onto $\mathbb{H}$ such as

$$\mu(z) = \frac{z + i}{iz + 1}$$

which identifies

$$\mu(1) = 1, \quad \mu(i) = \infty, \quad \mu(-1) = -1, \quad \mu(-i) = 0.$$

Then, is we find a comformal map $f$ from $\mathbb{H}$ onto a polygon, the map $f \circ \mu$ will map conformally the unit disk onto that same polygon.

$$\mathbb{D} \xrightarrow{\ \mu\ } \mathbb{H} \xrightarrow{\ SC\ } P$$

### 2.4.2 The Schwarz Christoffel Idea

The underlying idea of the SC transformation is to consider the derivative of conformal transformation $f : \mathbb{H} \to P$ with the form $f'(z) = (z - z_0)^{\alpha - 1}$ near $z_0$. Then $f'(z)$ can be expressed as

$$f'(z) = |(z - z_0)|^{\alpha - 1} \exp\left(i(\alpha - 1)\arg(z - z_0)\right)$$

The geometrical significance of this formula is that

$$\arg(f'(z)) = (\alpha - 1)\arg(z - z_0).$$

Whenever a function $f : \mathbb{C} \to \mathbb{C}$ is multi-valued, the introduction of brach cuts is needed in order to allow for analyticity. The standard branch cut used in complex analysis is $\{x \mid x < 0\}$ which restricts all complex arguments between $-\pi$ and $\pi$. However, because the branch $\{yi|y < 0\}$ is not in the domain of $\mathbb{H}$ we shall use this branch, so that the arguments of the complex numbers shall be restricted between $\frac{-\pi}{2}$ and $\frac{3\pi}{2}$.

Let us consider $z_0 \in \mathbb{R}$. The evaluation of the former expression in the interval $z \in (-\infty, z_0)$ is

$$\arg(f'(z)) = (\alpha - 1)\pi$$

and therefore, the conformal function $f(z)$ maps $z \in (-\infty, z_0)$ to a linear segment in the complex plane that forms an angle of $(\alpha - 1)$ with the real axis. Analogously, for $z \in (z_0, \infty)$

$$\arg(f'(z)) = 0$$

and $f(z)$ maps $z \in (z_0, \infty)$ to a linear segment that is parallel to the real axis.

Repeating this for all the vertices, let us consider $z_1, \dots, z_n \in \mathbb{R}$ and the derivative

$$f'(z) = \prod_{k=0}^{n} (z - z_k)^{\alpha - 1}.$$

For $z \in (z_k, z_{k+1})$

$$\arg(f'(z)) = \sum_{i=k}^{n} (\alpha_i - 1)\pi.$$

The function $f'(z)$ has piecewise constant argument for values with specific jumps. Therefore, $f(z)$ maps the real axis onto a polygon with $n$ vertices and interior angles $\alpha_k$.

Furthermore, by multiplying the proposed derivative by a complex number $C \in \mathbb{C}$ we allow the argument of $f'(z)$ to be non zero for $z \in (z_n, \infty)$.

$$\arg(f'(z)) = \arg(C) + \sum_{i=k}^{n} (\alpha_i - 1)\pi$$

Notice that $C$ can be interpreted as a rotation and dilation of the polygon provided by $f'(z) = \prod_{k=0}^{n} (z - z_k)^{\alpha - 1}$, since

$$\arg(Cf'(z)) = \arg(\mathrm{C}) + \arg(f'(z)).$$

### 2.4.3 The Schwarz Christoffel Formula

**Theorem 2.24.** *Let $P$ be the interior of a polygon $\Gamma$ having vertices $w_1, \dots, w_n$ and interior angles $\alpha_1, \dots, \alpha_n$ in counterclockwise order. Let $f$ be any conformal map from the upper half-plane $\mathbb{H}$ to $P$ with $f(\infty) = w_n$. Then*

$$f(z) = A + C \int_0^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta$$

*for some complex constants $A$ and $C$, and $z_1 < z_1 < \cdots < z_{n-1}$ are real numbers satisfying $w_k = f(z_k)$ for $k = 1, \ldots, n-1$.*

*The function $f$ is called* Schwarz-Christoffel transformation.

Before proving the theorem some considerations may be useful.

The change of the lower integration limit affects only the value of $A$, which allows a translation of the polygon throughout the complex plane. The value of $A$ is related with the image of zero, also known as the conformal center.

The function $f$, although defined in $\mathbb{H}$, can be continuosly extended to the real axis.

In the SC formula we assumed that $z_n = \infty$. This can be done without loss of generality because, as infinity lies on the real axis, its image by $f$ is in the boundary of the polygon. Then, in case infinity is not a prevertex, we could introduce its image as a new vertex $f(z_{n+1}) = w_{n+1}$ with the interior angle $\alpha_{n+1} = 1$.

The map fails to be conformal at the vertices, as $f'(z_k) = 0$.

*Proof.* Let $z_1, \ldots, z_n$ be finite prevertices for the SC formula. By the Schwarz reflection principle, the mapping function $f$ can be analytically continued into the lower half-plane. The image continues into the reflection of $P$ about one of the sides of $\Gamma$. By reflecting again about a side of the new polygon, we can return analytically to $\mathbb{H}$. The same can be done for any even number of reflections of $P$, each time creating a new branch of $f$. The image of each branch must be a translated and rotated copy of $P$. Now, if $A$ and $C$ are any complex constants, then

$$\frac{(A + Cf(z))''}{(A + Cf(z))'} = \frac{f''(z)}{f'(z)}.$$

Therefore, the function $\frac{f''}{f'}$ can be defined by continuation as a single-valued analytic function everywhere in the closure of $\mathbb{H}$, except at the prevertices of $\Gamma$. Similarly, considering odd numbers of reflections, we see that $\frac{f''}{f'}$ is single-valued and analytic in the lower half-plane as well. At a prevertex $z_k$,

$$f'(z) = (z - z_k)^{\alpha_k - 1}\phi(z)$$

for a function $\phi(z)$ analytic in a neighborhood of $z_k$ and $\phi(z_k) \neq 0$. Therefore, $\frac{f''}{f'}$ has a simple pole at $z_k$ with residue $\alpha_k - 1$, and

$$\frac{f''(z)}{f'(z)} - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - z_k}$$

is an entire function. Because all the prevertices are finite, $f$ is analytic at $z = \infty$, and a Laurent expansion there implies that $\frac{f''}{f'} \to 0$ as $z \to 0$. By Liouville's theorem, it follows that the former expresion is identically zero. Expressing $\frac{f''}{f'}$ as $(\log f')'$ and integrating twice results in the SC formula. □

It has been shown that the SC formula maps $\mathbb{H}$ to some polygon with interior angles $\alpha_k\pi$. However, the length of each side of the polygon is determined by the choice of the prevertices to compute the formula. Different prevertices provide different values of the vertices. Usually we are interested in mapping the upper half-plane to a concrete polygon, this is, we want to fix the vertices $w_k$ so that their positions in the complex plane are some desired ones. This requires the knowledge of those prevertices that allow this to happen. Determining the correct values of the prevertices for a given polygon $\Gamma$, $\alpha_k$, $w_k$ is known as the *parameter problem*. In the majority of the cases there is no analytic solution for the prevertices, which depend nonlinearly on the side lengths of $\Gamma$.

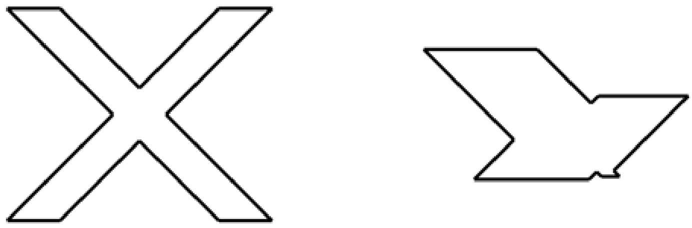


Figure 2: The parameter problem: on the left, the desired polygon $\Gamma$. On the right, the image by the function, $f(\mathbb{R} \cup \infty)$, for arbitrary prevertices.

The correct selection of the prevertices provides a correct side lenght ratio of the polygon. Then, the complex constants $A$ and $C$ translate and rotate, dilate respectively the polygon generating any similar polygon to the one computed by the integral.

On the other hand, the map $f(z)$ has three degrees of freedom that allow the arbitrary choice of three prevertices. This can be seen by considering the Moebius transformation that maps conformally any three points on the upper half plane $z_1, z_2, z_3$ to any other three points $w_1, w_2, w_3$. Therefore, for $n \leq 3$, no parameter problem needs to be solved.

In addition, the evaluation of the integral may require numerical computation.

#### 2.4.4 Adaptation of the Formula

The SC formula can be adapted to maps from different canonical domains, to exterior maps, to maps with branch points, to doubly connected regions, to regions bounded by circular arcs, and even to piecewise analytic boundaries. Particular interest for this thesis has the adaptation of the formula to the canonical domain $\mathbb{D}$.

**Theorem 2.25.** *Let $P$ be the interior of a polygon $\Gamma$ having prevertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ in counterclockwise order. Let $F$ be any conformal map from the unit disk to $P$. Then,*

$$F(z) = A + C \int_0^z \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1} \, d\zeta$$

*for some complex constants $A$ and $C$, where $w_k = f(z_k)$ for $k = 1, \dots, n$.*

The appearence of the formula is similar to that for $\mathbb{H}$. However, the prevertices are defined on the boundary of $\mathbb{D}$ and the product runs over all the prevertices, since none can be infinity. In general, the adaptation of the formula to other simply connected domains leads to different formulas.

*Proof.* Let $\mu(z) = i\dfrac{z+1}{1-z}$, which maps $\mathbb{D}$ to $\mathbb{H}$ and let $f(z)$ be the SC formula using as canonical domain $\mathbb{H}$. Observe that $\mu'(z) = \dfrac{2i}{(z-1)^2}$ Let $F = f \circ g$, then

$$\begin{aligned} F'(z) = \mu'(z) f'(\mu(z)) &= \mu'(z) \prod_{k=1}^{n} (\mu(z) - \mu(z_k))^{\alpha_k - 1} \\ &= \frac{2i}{(z-1)^2} \prod_{k=1}^{n} \left( i\frac{z+i}{1-z} - i\frac{z_k+1}{1-z_k+1} \right)^{\alpha_k - 1} \\ &= \frac{2}{(z-i)^2} \prod_{k=1}^{n} \left( \frac{2i(z-z_k)}{(1-z)(1-z_k)} \right)^{\alpha_k - 1} \end{aligned}$$

Using that $\sum_{k=1}^{n} (\alpha_k - 1) = -2$ we see that

$$F'(z) = -\frac{i}{2} \prod_{k=1}^{n} (1-z_k)^{1-\alpha_k} \prod_{k=1}^{n} (z-z_k)^{\alpha_k - 1} = \overline{C} \prod_{k=1}^{n} (z-z_k)^{\alpha_k - 1}$$

□

# 3 Elements of map projections

Before we can make maps of the world, we are called upon to determine the size and the shape of the Earth. These studies comprise the subject of geodesy. In this section we shall first discuss different models for the surface of the earth: the sphere, the spheroid and the geoid. Then, we apply the theory developed in Section 2.1 to define the geographic coordinates of the sphere and the spheroid and use Section 2.2 to derive the isometric coordinates for the spheroid and a conformal map between the sphere and the spheroid. After that, we shall see by means of the Egregium Theorem that no ideal map can be constructed and introduce the notion of distortion. Then, different measures of the distortion are studied. In the last section we shall introduce some basic elements in map projections and provide a classification.

## 3.1 The surface of the Earth

Back at least to the sixth century B.C., the early Greeks's notion of the Earth shape ranged from the flat disc to the sphere. The first serious attemp to measure the size of this sphere was the classic experiment carried out by Eratosthenes in the third century. Claudius Ptolemy's work *Geography* also describes the shape of the Earth as a sphere along with the Earth's dimensions.

The gravitation laws ensure that an evenly distributed mass forms a sphere, as it is the object with minimum energy. But the Earth rotates along an axis binding the north and south pole and therefore the centrifugal force appears, flattening the earth towards the poles. Isaac Newton proved in his famous *Principia*, back in the late seventeenth century, that a rotating self-gravitating fluid body in equilibrium takes the form of an oblate ellipsoid of revolution, also called spheroid. The difference is small, the equatorial diameter is about 12761.47 km, and the pole diameter is 12718.68 km, about 40 km shorter. The amount of polar flattening may be expressed by

$$f = \frac{a-b}{a},$$

where $a$ and $b$ are the lenghts of the major and minor semi-axes of the ellipsoid of revolution. The value of the flattening is always expressed as a fraction. For the Earth this value is close to $\frac{1}{298}$. As it is a very small value, usually the reciprocal flattening $\frac{1}{f}$ is used instead.

However, the intern mass of the Earth is unevenly distributed. Moreover, it is time-dependent. Mass shifts around inside the planet alter the gravitational field. Thus, the surface of the Earth is not a perfect oblate ellipsoid. A new surface that encounters the combination of the Earth's mass attraction, the gravitational force, and the centrifugal force of the Earth's rotation is defined: the geoid.

The *geoid* is defined in geodesy as the exact shape that the Earth's ocean would adopt in the absence of land and perturbations. Esentially, the figure of the Earth abstracted from its topographical features. It is the surface of gravitational equipotential: an idealized (in the absence of currents, air preassure variation, tides etc.) equilibrium surface of sea water called the mean sea level surface. It is a surface to which the force of gravity is everywhere perpendicular.

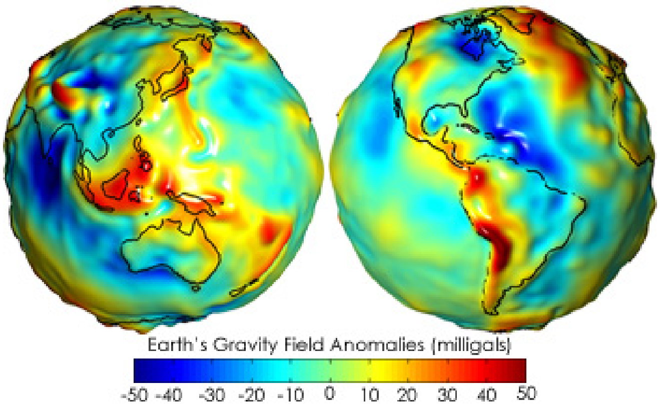



Figure 3: Three-dimensional visualization of geoid ondulations, using units of gravity.

The primarly interest of the geoid is that modern technology allows precise measurements of its shape. Unfortunally, the amount of irregularities of the geoid is too big for a feasible mathematical or computational treatment. The most widely used solution is to find a spheroid to approximate the surface of the geoid, this is, define the parameters of an oblate ellipsoid that ressemble the shape of the geoid. Such ellipsoid is called *reference ellipsoid*.

Different attemps to find the "best fitting" ellipsoid for measurements have been developed along centuries. Table 1 gives some examples of some reference ellipsoids used in different times and places. As a particular spheroid can fit with minimal errors in a certain region of the geoid but not in another, different so-called *local ellipsoids* were used for different regions. Each country or continent used their own parameters for the spheroid that best fits the specific region for measurements. This approach has been abandoned, leading to a global reference ellipsoid that has a shape and size such that it is a best-fit model of the geoid in a least-squares sense, together with a selected offset.

By specifying different offsets for the same reference ellipsoid, the precision in a given region is adjusted. An ellipsoid of reference with an specified offset is known under the name of *datum*. While a spheroid approximates the shape of the earth, a datum defines the position of the spheroid relative to the center of the earth. A datum provides a frame of reference for measuring locations on the surface of the earth. Many different datums have been developed. The WGS84, the North America datum NAD83 or the European datum ED50 are some examples.

| Reference ellipsoid name | Equatorial radius | Polar radius | Inverse flattening |
|---|---|---|---|
| Airy (1830) | 6,377,563.396 | 6,356,256.909 | 299.3249646 |
| Clarke (1866) | 6,378,206.4 | 6,356,583.8 | 294.9786982 |
| Haydorf (1910) | 6,378,388 | 6,356,911.946 | 297 |
| GRS-67 (1967) | 6,378,160 | 6,356,774.516 | 298.247167427 |
| GRS-80 (1979) | 6,378,137 | 6,356,752.3141 | 298.257222101 |
| WGS-84 (1984) | 6,378,137 | 6,356,752.3142 | 298.257223563 |

Table 1: Reference ellipsoids

The spherical Earth is still used in geodesy as a rough and first order aproximation of the surface of the Earth. Its use simplifies the computations, and the errors assumed are negligible when we map a large region or the whole Earth using a very small *global scale*.

The global scale can be understood as a reduction of the measures of the surface before applying the map projection. It is a constant factor that resizes the surface and it is usually denoted by 1:1000 meaning that the value of the parameters describing the surface of the earth are multiplied by a factor of 1/1000. In further analysis we shall denote the resized parameters that describe the surface by its non-resized denotation. This should not led to confusion.

The radius of the sphere has a value of

$$R = 6,371 \text{ km},$$

encountered by averanging the polar and equatorial radius, $6,358$ km and $6,378$ km respectively.

Conversely, for very small areas and when the topographic features are more important than the Earth's curvature, the surface can be approximated by a plane. However, for precise maps of small regions, the spheriod is the model to be used.

Another quantity that describes the shape of the spheroid is the eccentricity.

$$\epsilon^2 = \frac{a^2 - b^2}{a^2}$$

Henceforth, we shall reserve the letters $a$, $b$, $f$, $\epsilon$ and $R$ to denote the parameters discussed in this section.

### 3.1.1 Geographic coordinate system

A primary use of the model choosen is to serve as a basis for a coordinate system.

**Definition 3.1.** *The moment at which the sun reaches its zenith is called local solar noon. This event occurs simultaneously at all points along a semi-circular arc, called a* Meridian. *Where two meridians come together at the poles, they form an angle that is the basis for determining the* longitude. *The origin of the longitude is an arbitrary choice and is called the Prime meridian.*

Although it is an arbitrary choice, we are generally accustomed to the use as the Prime Meridian the meridian passing through the former site of the Royal Observatory at Greenwich. The use of the Greenwich Meridian was agred internationally in 1884, and this remains.

Longitudes are usually denoted by $\theta$ and, by definition, taking always the smallest angle, they range between $-\pi$ and $\pi$.

$$\theta \in (-\pi, \pi).$$

**Definition 3.2.** *The* latitude *of any point over the Earth is equal to the difference between the angle made by the sun at noon of the same day at the equator, the curve over the spheroid or sphere that devides the surface into two symmetric parts, the hemispheres.* Paralells *are curves over the surface with constant latitude.*

We shall denote the latitude by $\lambda$. Notice that the maximum variation of the angle that defines a latitude is equal $\frac{\pi}{2}$. Latitude, therefore, ranges between $-\frac{\pi}{2}$ to 0 for points in the south hemisphere, and from 0 to $\frac{\pi}{2}$ for points in the north hemisphere. Thus,

$$\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

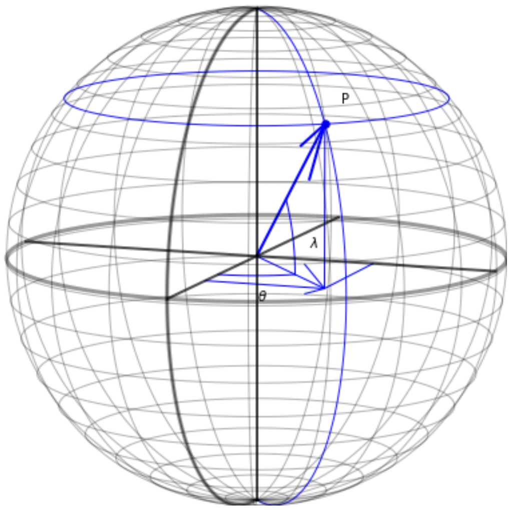



Figure 4: The point $P$ over the sphere has longitude $\theta$ and latitude $\lambda$.

Observe that latitude and longitude together give a complete system for locating points on the Earth. They are called *geodetic or geographic coordinates.* Morover, parallels and meridians are the coordinate curves for this parametrization. A parallel is a curve over the surface with constant latitude.

$$p_{\lambda_0}(t) = f(\lambda_0, t) \tag{3.1}$$

The meridians are curves of constant longitude.

$$m_{\theta_0}(t) = f(t, \theta_0) \tag{3.2}$$

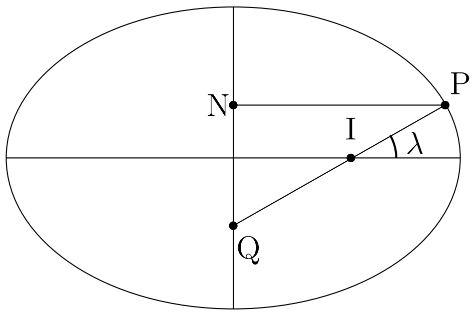



Figure 5: Cross section of the ellipsoid of revolution. The distance from $Q$ to $P$ is equal $N(\lambda)$ and the distance $Q$, $I$ is equal $\epsilon^2 N(\lambda)$.

**Proposition 3.3.** *The longitude and latitude defined above provide a parametrization of the sphere* $\mathbb{S}^2$ *and the spheroid* $\Sigma$ *(both without accounting the poles and the International Date Line, the meridian placed at* $180°$ *from the Prime meridian). These parametrizations are:*

$$\begin{aligned} f_{\mathbb{S}^2}(\lambda, \theta) &= R(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda) \\ f_{\Sigma}(\lambda, \theta) &= (N(\lambda)\cos\lambda\cos\theta, N(\lambda)\cos\lambda\sin\theta, N(\lambda)(1-\epsilon^2)\sin\lambda) \\ \lambda &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \theta \in (-\pi, \pi) \end{aligned}$$

*where*

$$N(\lambda) = \frac{a}{\sqrt{1 - \epsilon^2 \sin^2\lambda}}.$$

*Proof.* Notice that at noon, the sun incides over the surface in the direction of the normal to the surface.

In the case of the spherical surface, the normal direction $N$ is the radial direction $\forall p \in \mathbb{S}^2$. Therefore the latitude is the angle measured at the centre of the Earth between the plane of the equator and the radius drawn to the point. Figure 4 shows the geographic coordinate system over the sphere. Notice that this parametrization is the geocentric coordinate system defined in Example 2.7.

In the spheroid, the geodetic latitude is the angle between the major axis of the spheroid and the normal to the tangent plane at any point on the surface of the spheroid, measured at the point of intersection of the normal with the equatorial plane, as shown in Figure 5.

We introduce $u(\lambda)$ to be the distance $PN$ of a point $P$ from the central axis and we also set $v(\lambda)$ for the lenght $QP$ of the normal at $P$ to its intersection with the symmetry axis. Therefore

$$u(\lambda) = v(\lambda)\cos\lambda = \sqrt{x^2 + y^2}.$$

The equation of any meridian ellipse is

$$\frac{u^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Differentiating this equation with respect to $u$ gives

$$\frac{dz}{du} = -\frac{ub^2}{za^2}.$$

Since the normal and tangent are perpendicular, the product of their gradients is $-1$ and therefore the gradient of the normal is

$$\tan\lambda = -\left(\frac{dz}{du}\right)^{-1} = \frac{za^2}{ub^2} = \frac{z}{u(1-\epsilon^2)}.$$

Cancelling $z$ gives

$$u^2[1 + (1-\epsilon^2)\tan^2\lambda] = a^2.$$

Thus,

$$PN = u(\lambda) = \frac{a\cos\lambda}{(1-\epsilon^2\sin^2\lambda)^{\frac{1}{2}}}$$

$$z(\lambda) = \frac{a(1-\epsilon^2)\sin\lambda}{(a-\epsilon^2\sin^2\lambda)^{\frac{1}{2}}} = PI\sin\lambda.$$

Since $v = QP = PN\sec\lambda = u\sec\lambda$ we have

$$QP = \frac{a}{(1-\epsilon^2\sin^2\lambda)^{\frac{1}{2}}}.$$

□

**Remark 3.4.** Under this parametrization, the coefficients of the first fundamental form of the spheroid are

$$E = M^2, \quad F = 0, \quad G = N^2\cos^2\lambda$$
$$M = \frac{a(1-\epsilon^2)}{\left(1-\epsilon^2\sin^2\lambda\right)^{\frac{3}{2}}}.$$

### 3.1.2 Isometric coordinates of the spheroid

The importance of the isometric coordinates of a surface has been shown in Section 2.2. In Example 2.18 we derived the isometric coordinates of the sphere. In this section we shall use the same argument to derive the isometric coordinates of the spheroid.

**Proposition 3.5.** *The isometric coordinates of the spheroid are*

$$p = \ln\left[\tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right)\left(\frac{1-\epsilon\sin\lambda}{1+\epsilon\sin\lambda}\right)^{\frac{\epsilon}{2}}\right], \quad q = \theta.$$

*Proof.* Note that $(\lambda, \theta)$ are orthogonal coordinates, since under this parametrization $F = 0$, and that we can write the other two coefficients of the first fundamental form as

$$E = N^2 \cos^2 \lambda \left( \frac{M^2}{N^2 \cos^2 \lambda} \right), \quad G = N^2 \cos^2 \lambda.$$

Defining

$$\rho = N \cos \lambda, \tag{3.3}$$

we may write

$$E = \rho^2 \left( \frac{M^2}{N^2} \sec^2 \lambda \right), \quad G = \rho^2.$$

Using Proposition 2.16, the map $\phi(\lambda, \theta) = (p(\lambda), q(\theta))$ defined by

$$\left( \frac{dp}{d\lambda} \right)^2 = \frac{M^2}{N^2} \sec^2 \lambda, \quad q(\theta) = \theta$$

is a conformal map from the spheroid to the plane. By integration we deduce that

$$p = \int \frac{M}{N} \sec \lambda \, d\lambda = \ln | \tan \left( \frac{\lambda}{2} + \frac{\pi}{4} \right) \left( \frac{1 - \epsilon \sin \lambda}{1 + \epsilon \sin \lambda} \right)^{\epsilon/2} |$$

□

### 3.1.3 A conformal map between the sphere and the spheroid

The spheroid can be conformally mapped upon the sphere by means of the function $\varphi = \phi_{\mathbb{S}^2}^{-1} \circ \phi_\Sigma$, where $\phi$ denotes the isometric map from the surface onto the plane. The image by $\varphi$ of a point $(\lambda, \theta) \in \Sigma$ is a point over the sphere whose coordinates are denoted by $(\overline{\lambda}, \overline{\theta}) \in \mathbb{S}^2$,

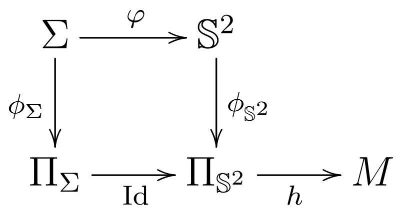
$$(\overline{\lambda}, \overline{\theta}) = \varphi(\lambda, \theta) = \phi_{\mathbb{S}^2}^{-1}(p_\Sigma(\lambda), \theta)$$

and therefore $\overline{\theta} = \theta$ and

$$\ln | \tan \left( \frac{\pi}{4} + \frac{\overline{\lambda}}{2} \right) | = \ln | \tan \left( \frac{\pi}{4} + \frac{\lambda}{2} \right) \left( \frac{1 - \epsilon \sin \lambda}{1 + \epsilon \sin \lambda} \right)^{\epsilon/2} |$$

The coordinate $\overline{\lambda}$ is called *conformal latitude.*

Notice that if we have a conformal map projection $g = h \circ \phi_{\mathbb{S}^2}$ defined over the sphere, then we have defined too a conformal map projection from the spheroid onto the plane, $g_\Sigma = h \circ \phi_{S^2} \circ \varphi$ and it is only needed to substitude in $h$ the isometric latitude of the sphere by the isometric latitude of the spheroid.



## 3.2 Egregium Theorem

In Section 2.2 we have studied the diffeomorphic relation between surfaces and showed that they preserve the analytic structure of the surface. However, two diffeomorphic surfaces can present different geometric properties. When two surfaces have the same geometric properties they are called isometric surfaces.

As an example, a sphere and an ellipsoid are diffeomorphic by the diffeomorphism $\Phi(x, y, z) = (ax, by, cz)$. But they do not have the same parallel length. In Example 2.11 we computed the lenght of a parallel over the sphere: $L_{\mathbb{S}^2}(p_{\lambda_0}) = 2\pi R \cos\lambda_0$. By applying (2.4) to the spheroid we obtain $L_\Sigma(p_{\lambda_0}) = 2\pi N(\lambda_0)\cos\lambda_0$ where $N$ is a function of $\lambda$. Therefore, in general, they do not have the same lenght.

We shown in Section 2.1.2 that the geometric properties of a given surface are described by the first fundamental form. Therefore, in order to have an isometry between surfaces we need the first fundamental forms to be equivalent. As the first fundamental form depends on the point, we talk about local isometry whenever the first fundamental form is equal at corresponding points.

**Definition 3.6.** *Let $S$ and $S'$ be two surfaces with metrics $g$ and $g'$ and let $p \in S$. A map $\phi : S \to S'$ is a local isometry in $p$ if $\phi$ is a local diffeomorphism such that $\forall w_1, w_2 \in T_pS$*

$$g(w_1, w_2) = g'(d_p\phi(w_1), d_p(w_2)).$$

Guass' Theorema Egregium states in which cases a surface can be bend into another without streching it. This is, when as isometry or local isometry can be constructed between surfaces. As an example, we might think of a cone, which can be developed onto a plane, thus, bended into a plane without crumpling, even when they are very different-looking surfaces.

**Theorem 3.7** (Theorema Egregium)**.** *The Gaussian curvature of a surface is invariant under local isometries.*

Egregium is translated from the latin as remakable, extraordinary. This theorem deserves this adjective because it states that the Gaussian curvature can be determined entirely by measuring angles, distances and their rates on the surface itself, although the definition of the Gaussian curvature of a surface depends on the particular way in which the surface is embedded in the ambient 3-dimensional Euclidean space. Thus the Gaussian curvature is an intrinsic invariant of a surface. Another interpretation is that the curvature depends only on the first fundamental form.

This result is of enormous significance for cartography, since it implies that no planar map of the Earth can be perfect, even for a portion of the Earth's surface.

**Corollary 3.8.** *There is no local isometry between the sphere $\mathbb{S}^2$ and the plane $\Pi$, or between the spheroid $\Sigma$ and the plane.*

*Proof.* We have already computed the curvature for the sphere in (2.2) and for the spheroid in (2.3). It is easy to see that the curvature of the plane is equal to zero:

$$K_{S^2} = \frac{1}{R^2}, \quad K_\Pi = 0, \quad K_\Sigma = \frac{b^2}{(a^2 \sin^2 v + b^2 \cos^2 v)^2}$$

Therefore, no local isometry between them can be defined. □

An ideal map projection is such that all relevant geometric features are preserved. Lenghts, angles and areas are carried by an ideal map projection to identical lengths, angles and areas. This result shows that, independently of the model used of the Earth, it is impossible to depict rounded objects on flat surfaces with complete fidelity. Thus, every map presents some kind of distortion and this fact cannot be avoided. In the next chapter we shall study some distortion properties.

## 3.3 Study of the distortion

The fact that every map projection shows some kind of distortion leads to the question "Which map projection is best?" No particular map projection is best for everything. The secret lies in choosing an appropriate projection that will allow the map to retain the most important properties for each particular use. A map can be seen as a visual tool that is to be used for some particular study: data plotting, distance from one city to another, etc. Therefore, depending on the purpose of the map, the interest will rely on the conservation of different properties.

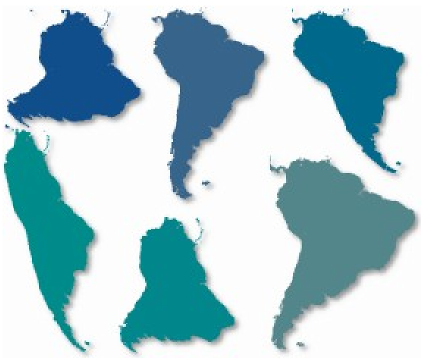


Figure 6: The projection of Africa for different map projections.

Imagine that we are interested in a map to study some features depending on the size of the countries. Then our "best map" could be a map that shows all areas on the surface of the Earth correctly; an equal area map. On the other hand, if we are interested in a map for navigation, our primarly interest might be to present rhumb lines or lines of constant bearing as straight lines. This allows the measuring of the correct angle that will bring you to your destination.

Some of the geometric features that can be measured on the Earth's surface are

- Area

- Shape
- Direction
- Distance

So far we have defined what we mean by area, shape (angles) and distances over surfaces and their projection over the map. The direction of a given curve is usually refered as the *azimuth*, the clockwise angle from the tangent to the meridian to the tangent to the projected curve. On the other hand, the *grid azimuth* is defined as the clockwise angle from the ordinate axis to the tangent to the projected curve. Notice that if the meridians are projected to vertical straight lines these two angles concide. Another important element in map projections is the *meridian convergence*, usually denoted by $\gamma$. The meridian convergence is the angle between the ordinate axis of the xy-plane and the tangent to the projected meridian.

Map projections can be constructed to preserve at least one of these properties, though only in a limited way. An important part of the cartographic process is understanding distortion and choosing the best combination of projection, mapped area and coordinate origin minimizing it for each job.

### 3.3.1 Scale factor

In geography it is of great importance to understand and control the distortion produced by a particular map projection. This fact leads to the developement of different measures of the distortion. One of the most broadly used is the scale factor. We shall see that it measures at each point and each direction the scale that is applied. If the projection was an isometry, the scale applied would be 1 at each point and for all the direction. For this reason, the scale factor can be understood as a measure of how far is a given map projection from an isometry.

Let $S$ be a surface and $p \in S$ with a chart $f : U \to S$. Let $w \in T_pS$. Then $w$ is the tangent vector at $p$ for some regular curve over the surface, $\gamma = f \circ \tilde{\gamma}$, $\gamma(t_0) = p$. We can write $\gamma(t) = f((u(t), v(t))$, and

$$w = \gamma'(t_0) = \left( f_u \frac{du}{dt}, f_v \frac{dv}{dt} \right),$$

that in the basis $(f_u, f_v)$ of the tangent plane in $p$ is

$$w = \tilde{\gamma}'(t_0) = \left( \frac{du}{dt}, \frac{dv}{dt} \right).$$

The lenght of the curve $\gamma$ between $t_0$ and $t$, $\gamma(t)$ in a neighbourhood of $p$, is

$$\begin{aligned} s(t) &= \int_{t_0}^{t} \sqrt{g(\dot{\gamma}(t'), \dot{\gamma}(t'))} dt' \\ &= \int_{t_0}^{t} \sqrt{E \left( \frac{du}{dt'} \right)^2 + 2F \left( \frac{du}{dt'} \right) \left( \frac{dv}{dt'} \right) + G \left( \frac{dv}{dt'} \right)^2} dt', \end{aligned}$$

where $E$, $F$ and $G$ are the coefficients of the first fundamental form.

This expression is usually written

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

where $ds$ is called element of lenght of arc or *line element.* It is invariant under change of chart.

In the euclidean space $\mathbb{R}^2$ the arc lenght element is simply

$$ds^2 = dx^2 + dy^2.$$

The expression can be written in terms of the surface coordinates by the map projection $(x, y) = \phi_f(u, v)$, where

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = J_{\phi_f} \begin{pmatrix} du \\ dv \end{pmatrix}$$

By defining

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 \\ f &= \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial y}{\partial u}\right) \\ g &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 \end{aligned}$$

the element of arc lenght in the plane can be written as

$$ds^2 = edu^2 + 2fdudv + gdv^2$$

**Definition 3.9.** *The scale factor of the map projection $\phi$ at a point $p = f(u, v) \in S$ and direction $w = (du, dv) \in T_pS$ is defined as*

$$\sigma^2 = \frac{ds^2}{dS^2} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}.$$

Notice that the scale factor depends on the point $p$ and on the direction. We shall use the scale factor as a measure of the distortion of the map projection. If a map between surfaces had no distortion, this is, if $\phi$ was an isometry, the scale factor would be 1 for all $p \in S$, independent of the direction.

In geodesy, the adjective true or *standard* usually refers to a projected point or curve that has a scale factor equal to 1. As an example, we shall see that the center of the projection, when it is also the point of tangency, is a point of true scale.

**Example 3.10.** The scale factors along the meridians and parallels are denoted by $k$ and $h$ respectively. Derivating (3.1) and (3.2) we obtain

$$\tilde{m}'_{\theta_0}(t) = (1, 0), \quad \tilde{p}'_{\lambda_0}(t) = (0, 1)$$

and therefore the corresponding scale factors are

$$k = \frac{\sqrt{e}}{\sqrt{E}}, \quad h = \frac{\sqrt{g}}{\sqrt{G}}. \tag{3.4}$$

As studied in Section 2.2 any conformal map projection from a surface $S$ to $\mathbb{R}^2$ can be expressed as the composition of the change of variables to the isometric plane $C$ and an holomorphic function $h$. If the isometric coordinates $(p, q)$ are used, the scale factor is reduced to

$$\sigma^2 = \frac{edp^2 + 2fdpdq + gdq^2}{\rho^2(dp^2 + dq^2)}$$

Moreover, the projection under the isometric coordinates is an holomorphic function and therefore it verifies the Cauchy-Riemann equations, which implies that

$$e = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 = \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 = g,$$
$$f = 0$$

and therefore,

$$\sigma^2 = \frac{e}{\rho^2}.$$

In particular this shows that a map projection is conformal if the scale factor does not depend on the direction.

Notice that

$$h'(p+iq)h'(p-iq) = \left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 = \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2$$

and therefore we may write the scale factor as

$$\sigma = \frac{\sqrt{h'(p+iq)h'(p-iq)}}{\rho}.$$

**Example 3.11.** Recalling the map $\rho$ from equations (2.7) and (3.3) for the sphere and spheroid respectively

$$\rho_{\mathbb{S}^2} = R\cos\lambda, \quad \rho_\Sigma = N\cos\lambda$$

we may write the scale factor of a map projection $h : \mathbb{C} \to \mathbb{C}$ expressed in isometric coordinates as

$$\sigma_{\mathbb{S}^2} = \frac{\sqrt{h'(p+iq)h'(p-iq)}}{R\cos\lambda}, \quad \sigma_\Sigma = \frac{\sqrt{h'(p+iq)h'(p-iq)}}{N(\lambda)\cos\lambda}.$$

### 3.3.2 Distortion visualization

Many different techniques for visualyzing the distortion of a map projection have been developed. We shall study only three of them: the graticule, the isoscale lines and the Tissot indicatrix.

The resulting network of parallels and meridians, which comprise the system of geographical coordinates, is known as the *graticule*, but with reference to the

Earth's surface and to the representation of it on a plane surface by means of a map projection. It is a net of coordinate curves.

The plot of the graticule allows the visualization of the transformed parallels and meridians and is used as a method to visualize the distortion. A verbal description of the graticule is also often used in cartography. We have seen that in order to be a conformal map projection, the scale factor must not depend on the direction at each point. In particular, because the coordinate curves are orthogonal, the graticule at the plane must show parallels and meridians intersecting at angles equal $\frac{\pi}{2}$. Figure 7 shows an example of graticule for a non-conformal projection. The parallels and meridians do not intersect at right angles at all points.

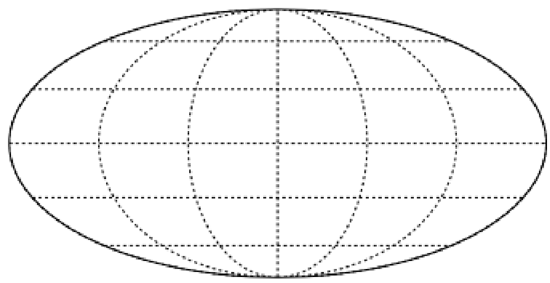


Figure 7: Graticule of the Mollweide projection. The central meridian and the parallels are straight lines while other meridians are curved.

The use of *isolines* for the display of map projection distortion is effective for depicting the magnitude and the distribution of any distortion measurement. Isolines can be used to map any variable that is assumed continuous. In the previous section we defined the scale factor as a measure of the distortion. The evaluation and plot of the *isocols*, this is, lines on the map with constant scale factor, provides absolute values of distortion.

Similar to the isolines, color methods provide a continuous display of distortion across a map. Distortion is symbolized by assigning a color frequency to each distortion value.

The last method we shall see is the Tissot's indicatrix.

### 3.3.3 Tissot's Indicatrix

The main sources of information used in this section are [13] and [18].

In 1881 Nicolas Auguste Tissot published his famous theory of deformation of map projections. Tissot proved that an infinitesimally small circle on the spherical or ellipsoidal Earth will be transformed on the projection plane into an infinitesimally small ellipse. This ellipse is called the Indicatrix and describes the local characteristics of a map projection at and near the point in question.

Tissot stated his theorem as follows:

Whatever the system of projection there are, at every point on one of the surfaces and, if angles are not preserved, there are only two of them, such that the directions

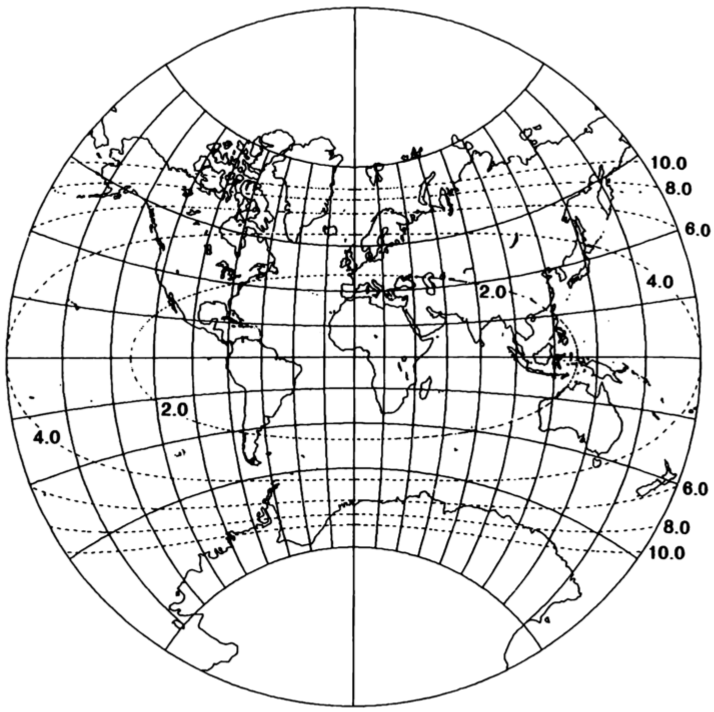



Figure 8: Lagrange conformal projection of the world in a circle, with isocols for p. 20° graticule.

which correspond to them on the one surface also intersect one another at right angles.[15]

The two orthogonal directions are called the *principal directions.* The projection of the infinitesimal small circle on the surface is then an infinitely small ellipse whose semi-axes lie along the two principal directions.

Fix a point $p \in S$ and consider the scale factor at $p$, that depends, in general, on the direction. Consider an infinitesimally small circle of unit radius on the surface. In the (planar) local coordinate system formed by the two orthogonal vectors that are respectively to the parallel and to the meridian on the surface, the equation of that circle may be written

$$ds_\lambda^2 + ds_\theta^2 = 1.$$

This may also be written as a quadratic form

$$\begin{pmatrix} ds_\lambda & ds_\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ds_\lambda \\ ds_\theta \end{pmatrix} = 1$$

The increments from the ellipsoidal distance elements $(ds_\lambda, ds_\theta)$ can be written in terms of the increments in the ellipsoidal curvilinear coordinates $(d\lambda, d\theta)$

$$\begin{pmatrix} ds_\lambda \\ ds_\theta \end{pmatrix} = \begin{pmatrix} N\cos\lambda & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} d\lambda \\ d\theta \end{pmatrix} = K \begin{pmatrix} d\lambda \\ d\theta \end{pmatrix}$$

and the former can be expressed in terms of the desired increments on the map

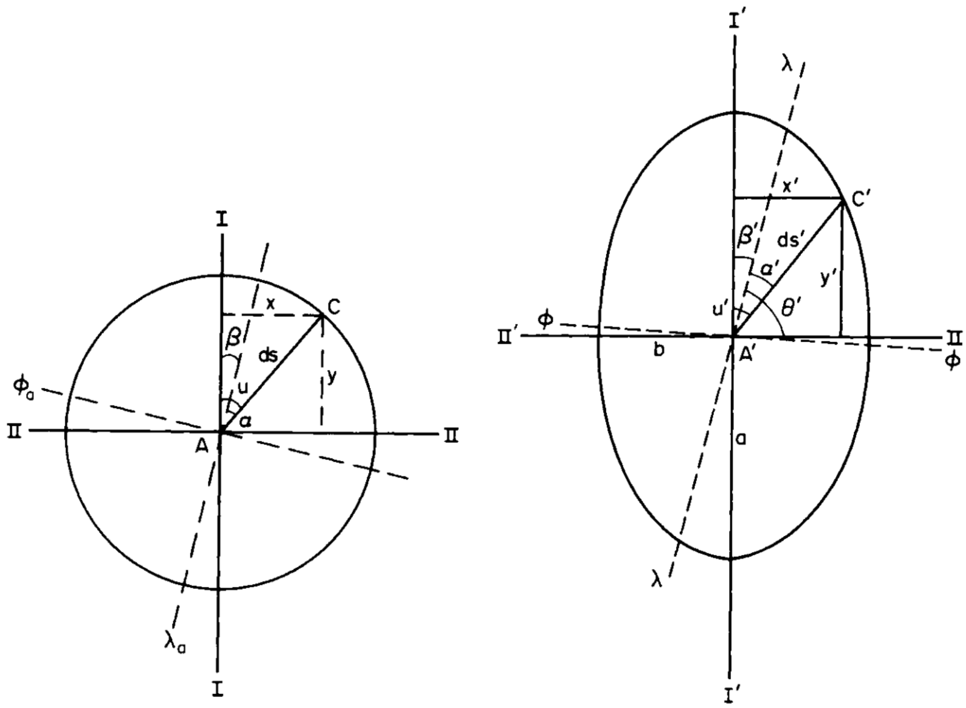



Figure 9: Unit circle upon the surface of the Earth and the indicatrix. I and II are the principal directions.

projection plane $(dx, dy)$ in terms of the Jacobian of the transformation.

$$\begin{pmatrix} ds_\lambda \\ ds_\theta \end{pmatrix} = KJ^{-1} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Note that $C = KJ^{-1}$ is non-singular. Under this change of coordinates, the initial circular quadratic form transforms into

$$\begin{pmatrix} dx, & dy \end{pmatrix} C^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} C \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dx, & dy \end{pmatrix} C^T C \begin{pmatrix} dx \\ dy \end{pmatrix} = 1$$

where $C^T C = (J^{-1})^T K^T K J^{-1}$ is a symmetric positive definite matrix. This is another quadratic, whose geometry depends on the properties of the matrix $C^T C$. It can be shown that this matrix has two positive eigenvalues under the prerequisite that $J$ is a non-singulr matrix. Therefore, the former equation describes an ellipse on the projection surface.

Then, the principal directions are defined by the eigenvectors of the matrix $C^T C$, say $s_1$, $s_2$ and the semi-axes of the Indicatrix are related by

$$a = \frac{1}{|s_1|}, \quad b = \frac{1}{|s_2|}.$$

Now that we know that the image is an ellipse, in order to find the values of $a$ and $b$ we may consider

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = JK^{-1} \begin{pmatrix} ds_\lambda \\ ds_\theta \end{pmatrix}$$

so that the semi-axes are the eigenvalues of $AA^T$ where $A = (a_{ij}) = JK^{-1}$. Notice that $A = C^{-1}$. There are formulas used in algebra for the computation of the eigenvalues of a matrix $AA^T$:

$$ab = \det(AA^T) = kh\sin\theta',$$

where $k^2 = a_{11}^2 + a_{21}^2$ and $h^2 = a_{12}^2 + a_{22}^2$ and $\theta'$ is the angle between the coordinte curves. Thus,

$$a' = \sqrt{k^2 + h^2 + 2ab}$$
$$b' = \sqrt{k^2 + h^2 - 2ab}$$
$$a = \frac{a' + b'}{2}, \quad b = \frac{a' - b'}{2}.$$

In addition to the principal directions, there exist another pair of diameters intersecting in right angles in the infinitessimal circle that differ maximally far from a right angle on the map. This deviation is the maximum angular deformation. If the map is conformal, this equals zero at every point on the map.

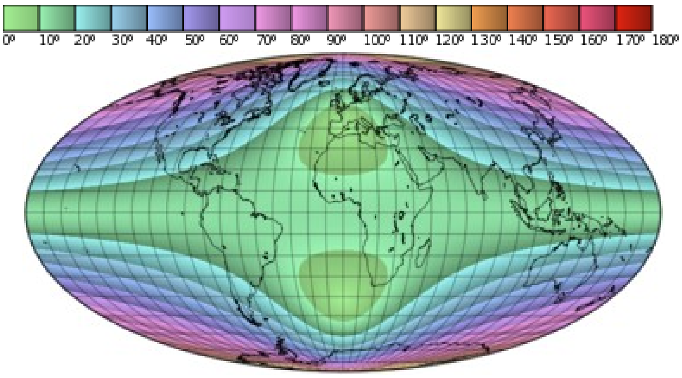



Figure 10: Color map of the maximum angular deformation in the Mollweide projection.

Tissot developed formulas relating the maximum angular distortion $w$ with the parameters of the Indicatrix $a$, $b$.

$$\sin\frac{w}{2} = \frac{a-b}{a+b}.$$

**Example 3.12.** [The Indicatrix in a conformal map projection.]

Recall that in order to define a conformal map projection the map from the surface of the Earth (sphere or spheroid) onto the plane is written as a composition of the function that maps the chart to the isometric plane and an holomorphic function, $f = h \circ \phi$ where $h$ holomorphism and $\phi(u, v) = (p, q)$, $(p, q)$ the isometric coordinates.

The inverse of the Jacobian matrix can be written as

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} \frac{\partial y}{\partial \theta} & -\frac{\partial x}{\partial \theta} \\ -\frac{\partial y}{\partial \lambda} & \frac{\partial x}{\partial \lambda} \end{pmatrix}$$

and therefore $C^T C$ can be computed as

$$C^T C = (J^{-1})^T K^T K J^{-1} =$$
$$\frac{1}{\det^2 J}\begin{pmatrix} G\left(\frac{\partial y}{\partial \theta}\right)^2 + E\left(\frac{\partial y}{\partial \lambda}\right)^2 & -\left(G\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} + E\frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \lambda}\right) \\ -\left(G\frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \theta} + E\frac{\partial y}{\partial \lambda}\frac{\partial x}{\partial \lambda}\right) & G\left(\frac{\partial x}{\partial \theta}\right)^2 + E\left(\frac{\partial x}{\partial \lambda}\right)^2 \end{pmatrix}$$

The characteristic polynomial is

$$p(x) = \frac{1}{\det^2 J}\left(x^2 - x(Ge+Eg) + EG\left(\left(\frac{\partial y}{\partial \theta}\right)\left(\frac{\partial x}{\partial \lambda}\right) - \left(\frac{\partial y}{\partial \lambda}\right)\left(\frac{\partial x}{\partial \theta}\right)\right)^2\right) \tag{3.5}$$

Note that because $h$ is an holomorphism, $h$ over $(p, q)$ verifies the Cauchy-Riemann equations, and therefore

$$\left(\frac{\partial x}{\partial \lambda}\right) = \left(\frac{\partial y}{\partial \theta}\right)\left(\frac{dp}{d\lambda}\right), \quad \left(\frac{\partial y}{\partial \lambda}\right) = -\left(\frac{\partial x}{\partial \theta}\right)\left(\frac{dp}{d\lambda}\right)$$

Thus, we can rewrite equation (3.5)

$$p(x) = \frac{1}{\det^2 J}\left[x^2 - x(Ge+Eg) + EGg^2\left(\frac{dp}{d\lambda}\right)^2\right]$$

Moreover,

$$e(\lambda, \theta) = \left(\frac{dp}{d\lambda}\right)^2 g(\lambda, \theta)$$

and the discriminant of the equation $p(x) = 0$ is

$$\Delta = g^2\left(G\left(\frac{dp}{d\lambda}\right)^2 - E\right)^2.$$

Recalling the definition of the derivative of the isometric latitude we obtain

$$\left(\frac{dp}{d\lambda}\right)^2 = \frac{E}{\rho^2} = \frac{E}{G}$$

and therefore

$$\Delta = 0.$$

This implies that the semi-axes have the same lenght, i.e., the Indicatrix is a circle. Moreover, the double eigenvalue is

$$x = \frac{Ge+Eg}{2\det^2 J} = \frac{G}{g} = \frac{E}{e}$$

and therefore the radius of the indicatrix is

$$r = \frac{1}{|x|} = \frac{g}{G} = \frac{e}{E} = h^2 = k^2$$

The lenght of the differential of arc does not depend on the direction.

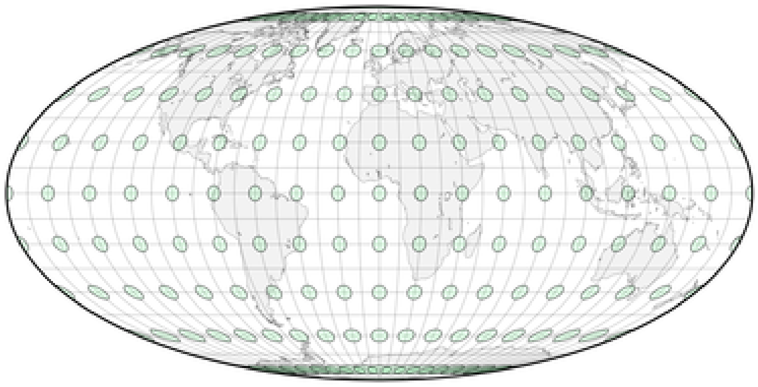


Figure 11: Mollweide Tissot indicatrices.

The indicatrix method for the visualization has been critized because it only describes the infinitesimal areas near the center of the ellipses of distortion, which is not the same as the area of the ellipses indicated visually.

## 3.4 Map Projection Classification

A map projection is usually created with the help of an intermediate surfaces with null Gaussian curvature, so that there exists an isometry between the map's support and the plane. The most spreadly used are the conic and cilindrical surfaces.

1. Azimuthal projection: the map's support is directly a plane.
2. Cylindrical projection: the cylinder is used as an intermediate surface.
3. Conic projection: the auxiliar surface is a cone.

**Example 3.13.** A cylinder with height $h$ and radius $r$ allows a parametrization $f(u, v) = (r \cos u, r \cos u, v)$, $u \in (0, 2\pi)$, $v \in (-h, h)$. To use the cylinder as auxiliar surface would mean to find $\phi(\lambda, \theta) = (u, v)$ and afterwards to cut the cilynder at $u_0$ and develop it onto a plane, $x = r(u - (u_0 + \pi))$, $y = z$.

For example, the Sanson-Flamsteed projection defines $v = R\lambda$ and $u = \cos(\lambda)\theta$. The resulting projection is

$$x = R\theta \cos \lambda, \quad y = R\lambda.$$

The map's support touches the surface of the Earth in one (the surface is tangent) or more (the surface is secant) regions. The points or lines of tangency present no distortion, since the map projection is the identity for these points.

Another key feature is the orientation of the surface with respect to the map's support surface.

1. The polar aspect aligns the north-south axis with the projection system's.

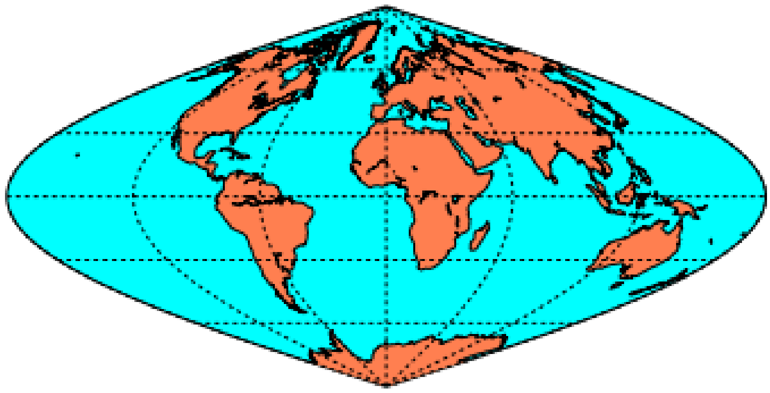


Figure 12: Sanson-Flamsteed projection.

2. The equatorial aspect map projection is centered on the Equator, which is set across one of the map's major axes.

3. The oblique aspect map has neither the polar axis nor the equatorial plane aligned with the projection system.

Also, the normal aspect refers to the conventional, direct or regular aspect. It demands the simplest calculations and produces the most straightforward graticule. The normal aspect for an azimuthal projection is the polar aspect, as for the conic projections, while the equatorial aspect is the normal aspect for cylindrical projections. On the other hand, the transverse aspect is created by rotating the normal axis by $\frac{\pi}{2}$. Therefore, for azimuthal and conic projections is the equatorial aspect while for the cylindre is the polar aspect.

The projection from the surface of the Earth to the auxiliar surface is very often originated by light rays originated at a point in the space called focus. Each beam intersect the surface of the Earth at a unique point $P$ and the auxiliar surface at $P'$. Then $P'$ is the projection of the point $P$. This kind of projections are called perspective projections. On the other hand, many map projections have a purely mathematic description. Some of them are the Azimuthal non-perspective projections, the pseudocylindrical and pseudoconical projections. Example 3.13 is a pseudocylindrical projection.

Another classification of map projections is due to the geometrical properties that they preserve.

1. Equidistant: map projections for which a non-trivial set of well defined standard lines is defined.

   **Example 3.14.** The azimuthal equidistant projection preserve distances along all segments that undergo the center of the projection. The azimuthal equidis-

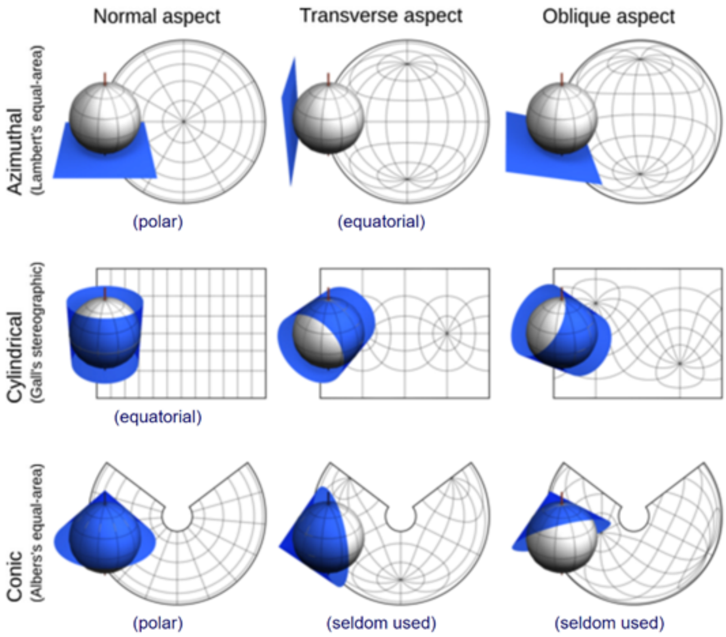



Figure 13: Normal, transverse and oblique aspects for the azimuthal, cylindrical and conic projections, all of them secant to the sphere.

tant polar projection is defined by the following equations:

$$x = \left(\frac{\pi}{2} + \lambda\right) R\cos\theta$$
$$y = \left(\frac{\pi}{2} + \lambda\right) R\sin\theta.$$

In this case, parallels define a set of true lines: all points of constant latitude are at the same distance of the central point.

2. Equal-Area: map projections that preserve areas.

   It can be proved that a map projection is equal area if and only if

$$EG - F^2 = eg - f^2.$$

Both Mollweide projection and Sanson-Flansteed projections are equal area.

**Example 3.15.** The Gall-Peters projections is an equal-area rectangular map projection

$$x = R\theta$$
$$y = 2R\sin\lambda.$$

It is a cylindrical secant projection in equatorial aspect with standard parallels of latitude $\lambda_0 = 45°$ and $\lambda_1 = -45°$.

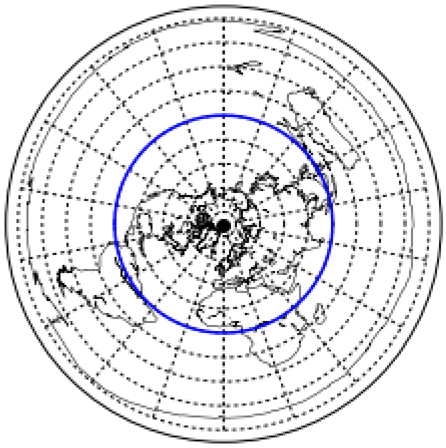


Figure 14: Azimuthal equidistant projection polar aspect. All the points in blue are are the same distant from the central point.

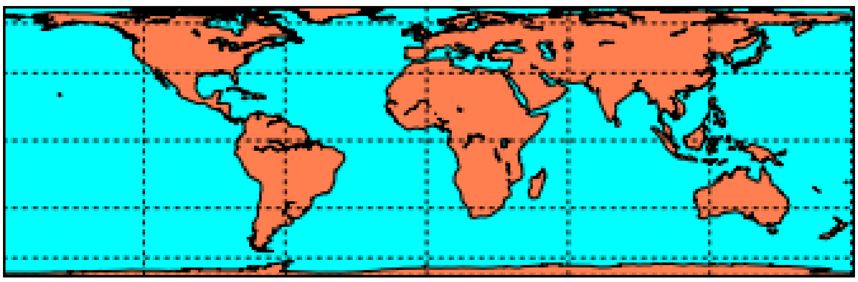


Figure 15: Gall-Peters equal area projection.

3. Conformal: map projections that preserves angles locally.

The aim of this dissertation is to study this kind of map projections. In the following sections we shall study some of the best-known conformal map projections, how to derive them and their properties. In Section 4 the classic conformal projections shall be discussed. After that, in Section 5 the SC formula is applied in order to derivate some conformal maps.

# 4 Classic conformal map projections

For each of the three major projection groups, azimuthal, cylindrical and conic projections there is a single conformal design: the stereographic projection, the Mercator projection and the Lambert conic projection. In this chapter we shall derive this three projections and show that Lambert's conic is a general case of the other two.

## 4.1 The normal Mercator Projection

The Mercator conformal projection was published by Gerardus Mercator, whose original name was Gerhard Kremer, in about 1550 in "Nova et Aucta Orbis Terrae Descriptio ad Usum Navigatium Emendate Accommodata" (A new and enlarged description of the Earth with corrections for use in navigation). Mercator arrived at his map projection formulas empirically in attempting to map the rhumb lines on the globe into straight lines in the map. Edward Wright gave the mathematical formulation about 40 years later.

A rhumb line or loxodrome is a curve over the surface with constant bearing. A compass measures the bearing of the direction in relation to the true north, the pole north. A route over a loxodrome would show constant rhumb in the compass. This was of great interest in navigation because in order to plan a trip between two points using a map that shows loxodromes as straight lines it would be suficient to draw a straight line betweeen both points on the map, compute the bearing and follow the same rhumb. This route is, in general, longer than the shortest path between the points and equal only if it lies over the Equator.

We shall derive first the equations assuming a spherical surface. Afterwards the evaluation of the formula for the spheroid will be discussed.

In order to construct Mercator's rectangular map, notice that if we project the meridians as vertical equispaced straight lines $x = c\theta$, $c \in \mathbb{R}$, then a straight line intersects all the meridians forming the same angle. Moreover, the parallels are wanted to be straight horizontal lines whose spacing $y = g(\lambda)$ we want to determine so that the antiimage of this straight lines are indeed the loxodromes. In order to preserve all of these angles, we clearly want to construct a conformal map. We have already seen that, in particular, the scale factor along meridians $k$, and along paralels $h$, have to be equal. Moreover, the scale factor along a parallel of latitude $\lambda$ is

$$h = \sqrt{\frac{g}{G}} = \frac{\sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2}}{R\cos\lambda} = \frac{c\sec\lambda}{R}$$

For the condition of conformality we need $k = R^{-1}\sec\lambda$. If we depict this parallel as a horizontal line segment at height $y = g(\lambda)$, then, the scale factor along the meridian at any point of the parallel will be $h = R^{-1}g'(\lambda)$. So the function $g(\lambda)$ must satisfy $g'(\lambda) = c\sec\lambda$. We also want to have $g(0) = 0$ so that the equator is mapped to a segment of the x-axis. Together, these conditions imply that

$$g(\lambda) = c\int_0^\lambda \sec(t)dt = c\ln|\sec\lambda + \tan\lambda| = c\ln\tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right).$$

This is the isometric latitude found in Example 2.18 multiplied by a factor $c$.

Moreover, we assume the equator to be a line of true scale. This implies that distances along the image of the equator are to be preserved. In particular, the lenght of the Equator has to be equal in the sphere, computed in (2.5), $L_{(0,2\pi)}(p_0) = 2\pi R$ and in the map $L$, $L = 2\pi c$. Thus, $c = R$ and the equations found are then

$$\begin{cases} x &= R\theta \\ y &= R\ln|\sec\lambda + \tan\lambda|. \end{cases} \tag{4.1}$$

These equations are the definition of the isometric coordinates of the sphere multiplied by a factor $R$. They can be also expressed as

$$x + iy = R(q + ip),$$

where $(p, q)$ are the isometric coordinates.

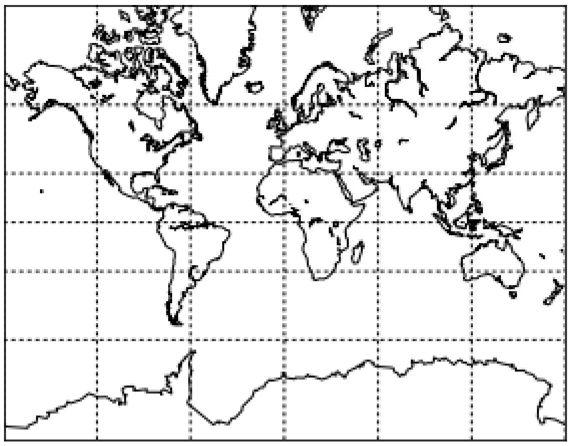


Figure 16: The Mercator projection.

By replacing $q = \theta - \theta_0$, the map can be centered at another arbitrary meridian $\theta_0$, instead of the Greenwhich meridian.

The inverse formulas for the sphere are as follows

$$\lambda = \frac{\pi}{2} - 2\arctan(e^{\frac{y}{R}}) = \arctan[\sinh\left(\frac{y}{R}\right)]$$
$$\theta = \frac{x}{R} + \theta_0.$$

The scale factor is

$$\sigma = \sec\lambda.$$

Notice that it does not depend on the direction. Along the equator there is no distortion, $\sigma = 1$, and the scale factor increases fast towards the poles, which are at infinite distance fro the equator. Therefore, the poles cannot be shown on the map. Greenland appears larger than South America, although it is only $\frac{1}{8}$ the size of South America.

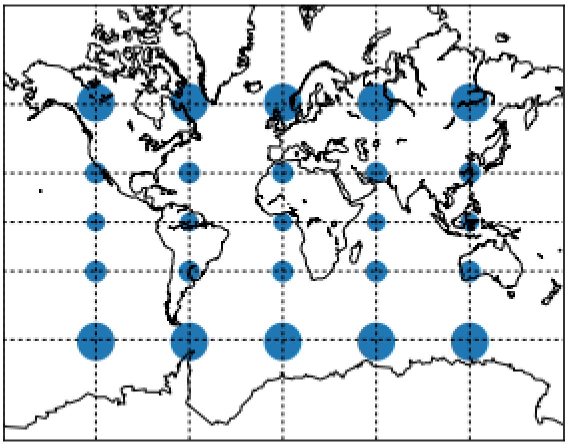


Figure 17: Tissot's indicatrices over the Mercator's map projection.

The normal aspect of Mercator's projection for the spheroid can be found by substituting the isometric latitude of the sphere by the isometric latitude of the spheroid and the lenght of the equator over the surface by $L_{(-\pi,\pi)}(p_0) = 2\pi N(0)\cos(0) = 2\pi a$, where $a$ is the semi-major axis of the ellipsoid. Thus, the map projection equations from the spheroid are,

$$\begin{cases} x &= a\theta \\ y &= a \ln \tan\left(\frac{\pi}{4} + \frac{\lambda}{2}\right)\left(\frac{1-\epsilon\sin\lambda}{1+\epsilon\sin\lambda}\right)^{\epsilon/2} \end{cases}.$$

w

The inverse formulas for the spheroid are as follows

$$\lambda = \frac{\pi}{2} - 2\arctan\left(e^{\frac{y}{a}}\left(\frac{1-\epsilon\sin\lambda}{1+\epsilon\sin\lambda}\right)^{\frac{\epsilon}{2}}\right)$$
$$\theta = \frac{x}{R} + \theta_0.$$

A computation shows that

$$\sigma = \frac{a}{N\cos\lambda}.$$

We want to show that the image of the loxodromes are straight lines under the formulas found for the map projection.

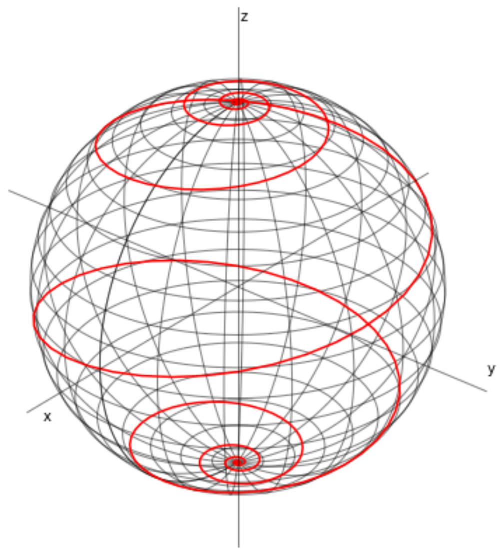



Figure 18: The loxodrome with bearing 80° over the sphere.

At each point $p \in S$ we can define the unitary vectors in the tangent plane that track the parallel and the meridian in $p$:

$$\widehat{\lambda} = \frac{f_\theta}{\|f_\theta\|} = \frac{f_\theta}{\sqrt{G}}$$
$$\widehat{\theta} = \frac{f_\lambda}{\|f_\theta\|} = \frac{f_\lambda}{\sqrt{E}},$$

where $E$ and $G$ are the first and third coefficients of the first fundamental form. The curve that forms an angle $\beta$ with the meridian has unit tangent vector $\widehat{\beta}$ equal

$$\widehat{\beta} = \sin\beta\widehat{\lambda} + \cos\beta\widehat{\theta}.$$

On the other hand, a curve $\gamma(t) = f(\lambda(t), \theta(t))$ over the surface has a tangent vector equal

$$\gamma'(t) = \left(f_\lambda\frac{d\lambda}{dt}, f_\theta\frac{d\theta}{dt}\right) = \widehat{\theta}\sqrt{E}\frac{d\lambda}{dt} + \widehat{\lambda}\sqrt{G}\frac{d\theta}{dt}$$

Because $\widehat{\beta} = \gamma'$ we obtain

$$\frac{d\lambda}{dt} = \frac{\cos\beta}{\sqrt{E}}, \quad \frac{d\theta}{dt} = \frac{\sin\beta}{\sqrt{G}}$$

In order to integrate we can write

$$\frac{d\theta}{d\lambda} = \tan\beta\frac{\sqrt{E}}{\sqrt{G}}$$

In Example 2.18 and Proposition 3.5 we have shown that in the geographic coordinate system for both the sphere and the spheroid, we could write $E = \frac{\rho^2}{U}$ and

$G = \frac{\rho^2}{V}$ where $U$ and $V$ were respectively functions of $\lambda$ and $\theta$ alone. Therefore we can rewrite the expresion above by

$$\frac{1}{\sqrt{V}} d\theta = \tan \beta \frac{1}{\sqrt{U}} d\lambda$$

Moreover, the integral of the right hand side is the isometric latitude and the integral of the left hand side, the isometric longitude. Therefore we have derived the equations for the loxodrome

$$q(\theta) = p(\lambda) \tan \beta.$$

Now it is obvious to see that by the transformation $f(q + ip) = a(q + ip)$ the loxodrome transforms to a straight line with an angle with the vertical equal to $\beta$,

$$f(\gamma(t)) = f(p \tan \beta + ip) = a(p \tan \beta + ip)$$

and therefore $y = \tan \beta x$.

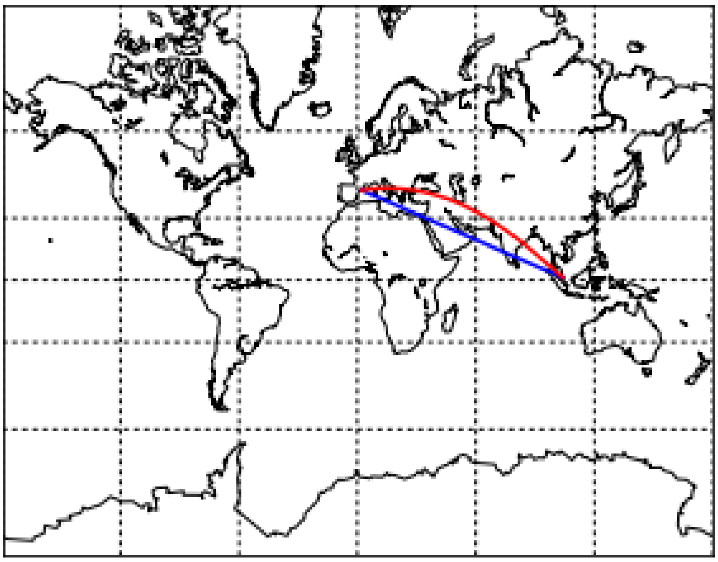


Figure 19: Mercator's map projection and the plot of the loxodrome (blue) and the great circle (red) from Barcelona to Singapore.

#### 4.1.1 The Mercator projection with two standard parallels

In the former section we derived the equations for the Mercator projection assuming the Equator to be a standard parallel. However, it is also possible to choose another parallel as standard, another parallel with true scale. If $\lambda_1$ is the parallel made standard, then the opposite parallel of latitude $-\lambda_1$ is also standard. This is the secant Mercator projection, where the surface is projected to a cylinder which cuts the sphere at two parallels with latitude $\pm\lambda_1$.

In this case, the lenght of the rectangular map $L$ is the lenght of the parallel $\lambda_1$, $L_{(-\pi,\pi)}(p_{\lambda_1}) = 2\pi K \cos\lambda_1$, where $K = R$ for the sphere and $K = N(\lambda_1)$ if we are considering the ellipsoidal case. The equations for the sphere are

$$\begin{aligned} x &= R\cos\lambda_1\theta \\ y &= R\cos\lambda_1 \ln|\tan\left(\frac{\pi}{4}+\frac{\lambda}{2}\right)|, \end{aligned}$$

and for the ellipsoid of revolution

$$\begin{aligned} x &= N(\lambda_1)\cos\lambda_1\theta \\ y &= N(\lambda_1)\cos\lambda_1 \ln|\tan\left(\frac{\pi}{4}+\frac{\lambda}{2}\right)\left(\frac{1-\epsilon\sin\lambda}{1+\epsilon\sin\lambda}\right)^{\epsilon/2}|. \end{aligned}$$

The scale factors are respectively

$$\begin{aligned} \sigma_{\mathbb{S}^2} &= \sec\lambda\cos\lambda_1 \\ \sigma_{\Sigma} &= \frac{N(\lambda_1)\cos\lambda_1}{N(\lambda)\cos\lambda}. \end{aligned}$$

The map will look exactly the same, but the scale will be slightly different, equal to one at the parallels $\lambda_1$ and $-\lambda_1$, increasing towards the poles and decreasing towards the equator from the standard parallels.

## 4.2 The Transverse Mercator Projection

The Transverse Mercator projection, which has become of great importance in modern cartography and is widely used, was invented in its spherical form by Johann Heinrich Lambert in 1772. In 1822 Gauss gave an analytical derivation of the projection for the ellipsoidal case, showing that it was a special case of a conformal mapping of one surface onto another. Then, in 1912, Krueger completed the development of the Transverse Mercator projection by developing the formulas further in order that they would be suitable for numerical calculations.

The requirements for the transverse Mercator projection are

1. The scale shall be true along the central meridian $\theta_0$.
2. The origin of ordinate $y$ is at the equator
3. The origin of the abscissa $x$ is at the central meridian

In the spherical case of a cylindrical projection we can relate the transverse graticule with the normal aspect graticule by means of a rotation. Let the position of a point $p$ be defined by the geographic coordinates $(\lambda, \theta)$ and by the corresponding coordinates $\lambda'$ and $\theta'$ on the rotated system, see Figure 21.

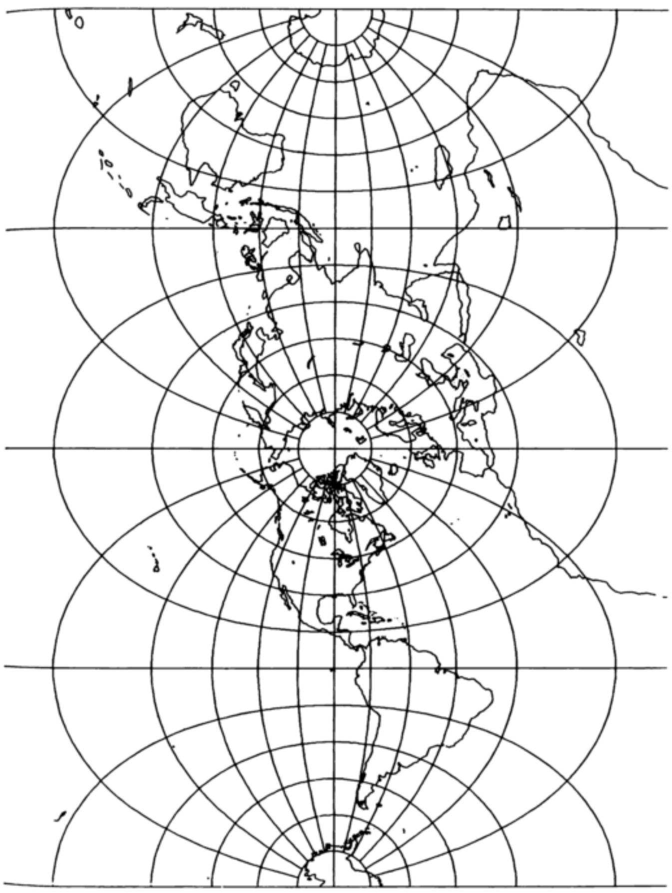


Figure 20: Transverse Mercator Projection with central meridian at 90° and −90°. 15° graticule.

Then, using spherical geometry and from the spherical triangle $NEP$ and $PEM$,

$$\cos\lambda'\cos\theta' = \cos\lambda\cos\theta, \quad \cos\lambda'\sin\theta' = \sin\lambda.$$

Hence,

$$\tan\theta' = \tan\lambda\sec\theta, \quad \sin\lambda' = \cos\lambda\sin\theta.$$

Using the coordinates $\lambda'$ and $\theta'$ we can apply the normal Mercator map projection with latitude $\lambda'$ and longitude $-\theta'$.

$$\begin{cases} x' &= -R\theta' \\ y' &= R\ln|\sec\lambda' + \tan\lambda'|. \end{cases}$$

Notice that

$$\sec\lambda' + \tan\lambda' = \tan\left(\frac{\pi}{4} + \frac{\lambda'}{2}\right) = \sqrt{\frac{1+\sin\lambda'}{1-\sin\lambda'}}, \tag{4.2}$$

and by substituting the values of $\lambda'$ and $\theta'$ and restoring $x = y'$ and $y = -x'$ in order to satisfy the second and third requirements, we finally obtain

$$\begin{cases} x &= \dfrac{R}{2}\ln\left(\dfrac{1+\cos\lambda\sin\theta}{1-\cos\lambda\sin\theta}\right) \\ y &= R\arctan\left(\tan\lambda\sec\theta\right). \end{cases} \tag{4.3}$$

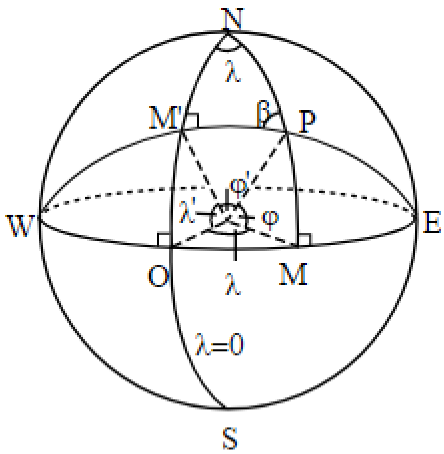



Figure 21: Transverse Mercator spherical geometry.

The equation for $x$ might also be written as

$$x = R\,\mathrm{arctanh}(\cos\lambda\sin\theta).$$

It can be seen after some computations that Equation (4.3) can be expressed as

$$y + ix = 2R\mathrm{arccot}(e^{p+iq}). \tag{4.4}$$

It is clear from Figure 20 that paths of constant compass bearing are no longer straight lines.

The inverse transformation gives

$$\begin{cases} \lambda(x,y) & = \arcsin\left(\mathrm{sech}\frac{x}{R}\sin\frac{y}{R}\right) \\ \theta(x,y) & = \arctan\left(\sinh\frac{x}{R}\sec\frac{y}{R}\right). \end{cases}$$

The scale factor is computed by evaluating the partial derivatives of $x$ and $y$ in $\theta$ and results in:

$$\sigma = \frac{1}{\sqrt{1-\cos^2\lambda\sin^2\theta}}.$$

Notice that for the central meridian, $\theta_0 = 0$, $\sigma = 1$. Hence, it is a standard line and the requirement 1 in page **??** is fullfiled. The scale factor might also be written in terms of the cartesian coordinates:

$$\sigma(x,y) = \cosh\left(\frac{x}{R}\right).$$

This showns that the scale factor depends only on the distance from the central meridian.

The ellipsoidal case is more tricky. Gauss proposed a double conformal projection, using the conformal map between the spheroid and the sphere explained in Section 3.1.3. Then, the map (4.4), where now $p$ denotes the isometric latitude of the spheroid, maps conformally the spheroid onto the plane verifying the three requirements.

On the other hand, Krueger proposed a direct map from the ellipsoid onto the map. Since we want the map projection, say $f$, to remain conformal, $f$ can be written as

$$y + ix = f(p + iq)$$

where $p$ is the isometric latitude and $q = \theta - \theta_0$, $\theta_0$ the longitude of the central meridian.

The third requirement implies that when $\theta = \theta_0$, or analogously $q = 0$, we must have $x = 0$ and therefore, $y = f(p)$.

Moreover, the first requirement is only satisfied if for $q = 0$,

$$f(p(\lambda)) = L_\lambda = \int_0^\lambda \sqrt{E(\lambda')} d\lambda'$$

In the spheroidal case, where $\sqrt{E} = M$, this integral cannot be computed exactly. Moreover, so far we have only obtained the values of $f$ at the $y$-axis. For a point $z = p + iq$ and assuming $q$ to be small, we can expand $f$ in a Taylor series around the point $z_0 = p$, in which the value of the function is known:

$$y + ix = f(z) = f(z_0 + \Delta z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(iq)^n \tag{4.5}$$

Since $i^{4k} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$ and $i^{4k+4} = 1$, $k \in \mathbb{N}$, let us separate the real and the imaginary parts of (4.5):

$$\begin{aligned} x &= f'(p)q - \frac{1}{3!}f'''(p)q^3 + \frac{1}{5!}f^{(v)}(p)q^5 - \frac{1}{7!}f^{(vii)}(p)q^7 + \dots \\ y &= f(q) - \frac{1}{2!}f''(p)q^2 + \frac{1}{4!}f^{(iv)}(p)q^4 - \frac{1}{6!}f^{(vi)}(p)q^6 + \dots \end{aligned}$$

The remaining problem is the evaluation of the derivatives of $f(p) = L_\lambda$. In order to do so, recall that

$$\frac{d\lambda}{dp} = \frac{\rho}{M}, \quad \rho = N\cos\lambda, \quad \frac{d\rho}{d\lambda} = -M\sin\lambda$$

and therefore

$$\begin{aligned} f'(p) &= \frac{dL_\lambda}{dp} = \frac{dL_\lambda}{d\lambda}\frac{d\lambda}{dp} = \rho \\ f''(p) &= \frac{d}{d\lambda}\left(\frac{dL_\lambda}{dp}\right)\frac{d\lambda}{dp} = -\sin\lambda\rho \\ f'''(p) &= \frac{d}{d\lambda}\left(\frac{d^2L_\lambda}{d^2p}\right)\frac{d\lambda}{dp} = -N\cos^3\lambda(1 - \tan^2\lambda + e^2\cos^2\lambda). \end{aligned}$$

In the same way, the higher derivatives can be computed. Substituting on the equation yields,

$$
\begin{aligned}
\frac{x}{N} &= \theta\cos\lambda + \frac{\theta^3\cos^3\lambda}{6}(1 - t^2 + \nu^2) \\
&+ \frac{\theta^5\cos^5\lambda}{120}(5 - 18t^2 + t^4 + 14\nu^2 - 58t^2\nu^2 + 13\nu^4 + 4\nu^6 - 64\nu^4t^2 - 24\nu^6t^2) \\
&+ \frac{\theta^7\cos^7\lambda}{5040}(61 - 479t^2 + 179t^4 - t^6) + \ldots \\
\frac{y}{N} &= \frac{L(m_0)}{N} + \frac{\theta^2}{2}\sin\lambda\cos\lambda + \frac{\theta^4}{24}\sin\lambda\cos^3\lambda(5 - t^2 + 9\nu^2 + 4\nu^4) \\
&+ \frac{\theta^6}{720}\sin\lambda\cos^5\lambda(61 - 58t^2 + t^4 + 270\nu^2 - 330t^2\nu^2 + 445\nu^4 + 324\nu^6 \\
&- 680\nu^4t^2 + 88\nu^8 - 600\nu^6t^2 - 192\nu^8t^2) + \ldots
\end{aligned}
$$

where $L(m_0) = \int_0^\lambda M d\lambda$, $t = \tan\lambda$, $\nu^2 = e'^2\cos^2\lambda$ and $e'^2 = \dfrac{a^2 - b^2}{b^2}$. The above mapping equations yield $x$ and $y$ values accurate to 0.001 meters for $\theta = 3°$. They define only a map projection within a neighbourhood of the central meridian. However, by repeating the process for consecutive meridians placed at certain angular distance from each other, we might obtain a map of the whole world.

### 4.2.1 Universal Transverse Mercator (UTM)

The UTM projection is the family of Transverse Mercator projection with 6° meridian zones. Each zone has 3° on the left side of each central meridian and 3° on the right side. The total number of zones is 60.

This map projection was adopted by the US Army Map Service in 1947 for their use in worldwide mapping.

A factor $k = 0.9996$ is applied. Hence, the projection formulas are

$$x_{UTM} = kx, \quad y_{UTM} = ky$$

where $x$ and $y$ are the rectangular coordinates of the transverse Mercator projection defined in (4.3). This scale factor reduces the overall scale factor,

$$\sigma_{UTM} = k\sigma,$$

where $\sigma$ is the scale factor of the transverse mercator projection. This situation is analogous to that in which the cylinder is secant to the surface instead of tangent.

## 4.3 Stereographic Projection

The stereographic projection is probably the most used azimuthal projection. His polar aspect was already known by Hipparchus (about 150 B.C).

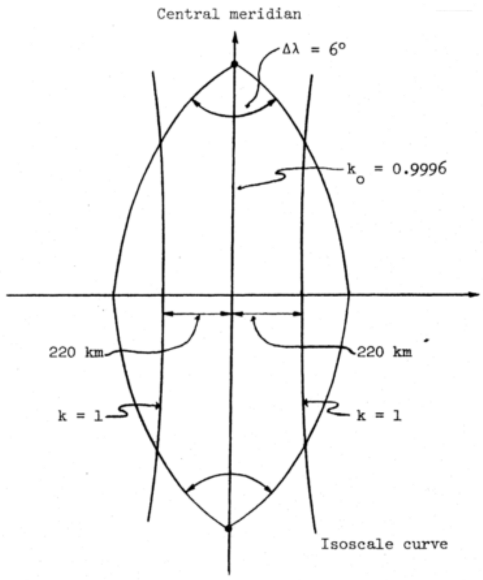



Figure 22: Scale Factor on the UTM.

It is an azimuthal perspective projection that maps the sphere from a focal point placed over the surface $F = (\lambda_F, \theta_F)$ to a plane tangent to the surface at the antipodal point $C$. Let $P \in S$ be a point over the surface. Then the line $FP$ intersects the plane of projection at a unique point $P'$. Then, $P'$ is the projection of the point $P$.

We shall derive the formula for the north polar aspect. Fix the perspective point at $F = (0, 0, -1)$, the south pole, and the point of tangency at $N = (0, 0, 1)$, the north pole. A cross section is shown in Figure 23. Notice that in order to find $P'$, we only need to compute its distance $r = NP'$ from the north pole as a function of the geographic latitude $\lambda$, since the distance is constant for all $\theta$ and it does not depend on the geographic longitude.

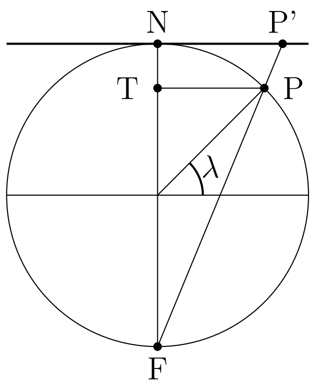



Figure 23: Cross section of the ellipsoid of revolution. The distance from $Q$ to $P$ is equal $N(\lambda)$ and the distance $Q$, $I$ is equal $e^2 N(\lambda)$.

The triangles $\Delta FPT$ and $\Delta FP'N$ are similar and therefore, since $OT = R \sin \lambda$,

$PT = R\cos\lambda$ and the radius of the sphere is $R$,

$$\frac{r}{2R} = \frac{\cos\lambda}{\sin\lambda + 1} = \frac{1}{\tan\lambda + \sec\lambda}$$

and therefore

$$r = \frac{2R}{\sec\lambda + \tan\lambda}.$$

We want to write down the projection in cartesian coordinates $x, y$. In order to do this we choose the Greenwich meridian $\theta = 0$ to project on the negative vertical axis $y < 0$. Moreover, we measure the longitude $\theta$ from it in the counterclockwise direction. Hence, the stereographic projection is given by

$$\begin{cases} x &= r\sin\theta = \dfrac{2R\sin\theta}{\tan\lambda + \sec\lambda} \\ y &= -r\cos\theta = -\dfrac{2R\cos\theta}{\tan\lambda + \sec\lambda}. \end{cases} \tag{4.6}$$

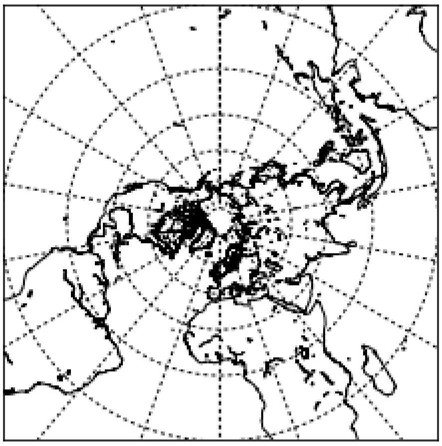


Figure 24: Polar stereographic projection, north pole.

The former equations can be written in terms of the isometric coordinates of the sphere. Notice that $r = 2Re^{-p}$ and using Euler's identity

$$y + ix = -2Re^{-(p+iq)} = f(p + iq)$$

In terms of $\zeta = p + iq$, $f(\zeta) = -2Re^{-\zeta}$. The projection is conformal, since

$$\frac{\partial f}{\partial \overline{\zeta}} = -2R(e^{\zeta} - e^{\zeta}) = 0$$

for all $\zeta$.

The scale factor can be easily computed by derivating Equation (4.6),

$$\sigma = \frac{\sqrt{\left(\dfrac{\partial x}{\partial \theta}\right)^2 + \left(\dfrac{\partial y}{\partial \theta}\right)^2}}{\rho} = \frac{r}{\rho} = \frac{2e^{-p}}{\cos\lambda} = \frac{2}{1 + \sin\lambda}.$$

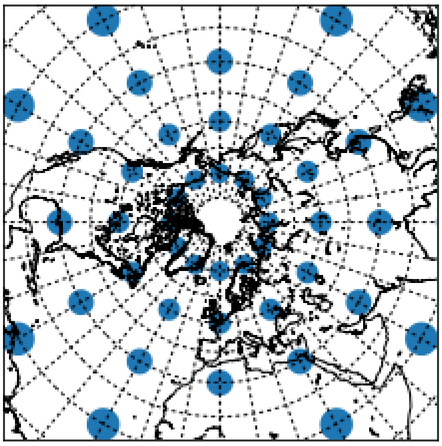


Figure 25: Tissot indicatrices in the polar stereographic projection.

Other aspects of the stereographic projection in its spherical form may be found by rotating the sphere from the point of tangency $(\lambda_0, \theta_0)$ to the north pole and then applying the equations of the polar stereographic projection. Assuming that the point $p$ is placed in the north hemisphere, $\lambda_0 \geq 0$, such a rotation is given by the matrix

$$T = \begin{pmatrix} \cos\theta_0 \sin\lambda_0 & \sin\theta_0 \sin\lambda_0 & -\cos\lambda_0 \\ -\sin\theta_0 & \cos\theta_0 & 0 \\ \cos\theta_0 \cos\lambda_0 & \sin\theta_0 \cos\lambda_0 & \sin\lambda_0 \end{pmatrix}.$$

Let $(x', y', z')$ be the cartesian coordinates of a rotated point $p$ with geographic coordinates $(\lambda, \theta)$. Hence, $(x', y', z') = Tf(\lambda, \theta)$ and the new coordinates of $p$, say $(\lambda', \theta')$ of the rotated sphere are

$$\lambda' = \arcsin z', \quad \theta' = \arctan\left(\frac{y'}{x'}\right).$$

This is, in particular, true for any azimuthal projection from the sphere onto the plane.

In order to find the equatorial aspect of the stereographic projection, we have to rotate the point $(0, \theta_0)$ to the north pole. We can assume without loss of generality that $\theta_0 = 0$. Substituting in $T$ we find

$$T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now the rotated coordinates of a point $(\lambda, \theta)$ verify

$$\sin\lambda' = \cos\lambda\cos\theta, \quad \cos\theta' = \frac{-\sin\lambda}{\sqrt{1 - \cos^2\lambda\cos^2\theta}}, \quad \sin\theta' = \frac{\cos\lambda\sin\theta}{\sqrt{1 - \cos^2\lambda\cos^2\theta}}.$$

Using the same argument as in Section 4.2, Equation (4.2) holds, and therefore,

$$e^{-p'} = \left(\frac{1+\cos\lambda\cos\theta}{1-\cos\lambda\cos\theta}\right)^{-\frac{1}{2}}$$

Applying the stereographic projection described by equation (4.6) to the relative coordinates $\lambda'$ and $\theta'$,

$$\begin{aligned} \frac{x}{2R} &= \frac{\cos\lambda\sin\theta}{1+\cos\lambda\cos\theta} = \frac{\sin\theta}{\cos\theta+\sec\lambda} \\ \frac{y}{2R} &= \frac{\sin\lambda}{1+\cos\lambda\cos\theta} = \frac{\tan\lambda}{\cos\theta+\sec\lambda}. \end{aligned}$$

We want to express the equations in terms of the isometric latitude $p$. The hyperbolic trigonometric functions will help us. We have:

$$\begin{aligned} \cosh p &= \frac{e^p+e^{-p}}{2} = \frac{1}{2}\left(\frac{1+\sin\lambda}{\cos\lambda}+\frac{\cos\lambda}{1+\sin\lambda}\right) = \frac{1}{\cos\lambda} \\ \sinh p &= \sqrt{\cosh^2 p - 1} = \sqrt{\left(\frac{1}{\cos\lambda}\right)^2 - 1} = \tan\lambda. \end{aligned}$$

Finally,

$$\begin{cases} x &= 2R\dfrac{\sin\theta}{\cos\theta+\cosh p} \\ y &= 2R\dfrac{\sinh p}{\cos\theta+\cosh p}. \end{cases}$$

In particular this can be written as follow

$$x+iy = 2R\tan\left(\frac{\theta+ip}{2}\right). \tag{4.7}$$

The scale factor is

$$\sigma = \frac{\sqrt{\left(\dfrac{\partial x}{\partial p}\right)^2+\left(\dfrac{\partial y}{\partial p}\right)^2}}{\rho} = \frac{2}{\cos\lambda(\cos\theta+\cosh p)} = \frac{2}{1+\cos\lambda\cos\theta}.$$

In the ellipsoidal form, only the polar aspect is truly azimutal, but it is not perspective, in order to retain conformality. The oblique and equatorial aspects are neither azimuthal nor perspective. The formulas result from replacing the isometric latitude of the sphere in the spherical equations with the isometric latitude of the spheroid, as explained in Section 3.1.3.

## 4.4 Lambert Conformal Conic Projection

First developed by Lambert in his "Beitraege zum Gebrauche der Mathematik", Berlin, 1772, the projection is used worldwide. The principal source of information used in this section is [5].

The requirements of this projection are:

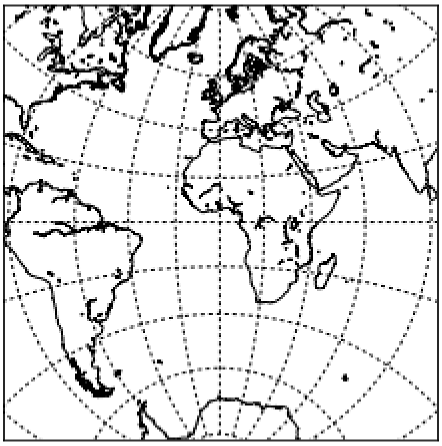


Figure 26: Equatorial aspect of the stereographic projection. Point of tangency $(0,0)$.

1. Parallels are to be part of concentric circles.

2. Meridians are to be radii of concentric circles.

We want to find the conformal function $f(q+ip) = x+iy$ satisfying the requirements above. For convenience, we shall use the polar coordinates $(\rho, \varphi)$ in order to derive the map formulas.

The first requirement implies that $\rho$ is a function of $\lambda$ alone, or analogously, $\rho = \rho(p)$. On the other hand, the second requirement implies $\varphi = \varphi(\theta)$. Finally, assume that the cone has an opening angle $2\pi t$, $t \in (0,1)$. Notice that all the meridians are equidistantly spaced between the angle $-\pi t$ and $\pi t$. Therefore we obtain

$$\varphi(\theta) = t\theta.$$

The cartesian coordinates assuming that the image of the vertex of the cone is place at $(0, \rho_0)$, $\rho_0 = \rho(p_0)$ for an arbitrary $p_0$ are

$$\begin{cases} x(q,p) &= \rho(p)\sin(tq) \\ y(q,p) &= \rho_0 - \rho(p)\cos(tq). \end{cases}$$

The partial derivatives are

$$\frac{\partial x}{\partial q} = \rho t\cos(tq), \quad \frac{\partial x}{\partial p} = \frac{d\rho}{dp}\sin(tq)$$
$$\frac{\partial y}{\partial q} = \rho t\sin(tq), \quad \frac{\partial y}{\partial p} = -\frac{d\rho}{dp}\cos(tq).$$

The Cauchy Riemann equations imply that

$$\frac{d\rho}{dp} = -\rho t,$$

and by integrating we obtain

$$\rho(p) = \rho_0 e^{-t(p-p_0)}. \tag{4.8}$$

The formulas can be written as follows

$$\begin{cases} x(q,p) &= \rho_0 e^{-t(p-p_0)} \sin(tq) \\ y(q,p) &= \rho_0 - \rho_0 e^{-t(p-p_0)} \cos(tq). \end{cases} \tag{4.9}$$

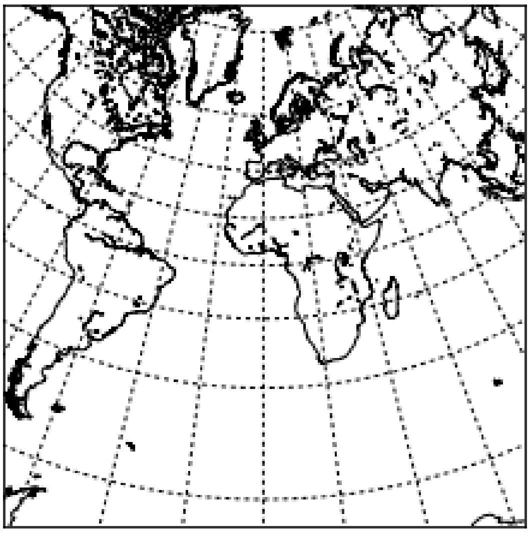


Figure 27: Lambert Conic Projection with one standard parallel $\lambda_0 = 20°$.

Up to this point, both $t$ and $\rho_0$ are parameters whose value can be arbitrary chosen. They can be related with a geometric feature for the projection.

We shall fix $t$ so that the parallel of latitude $\lambda_0$ is a standard parallel. This implies

$$L_{\lambda_0} := L_{(-\pi,\pi)}(p_{\lambda_0}) = 2\pi\rho_0 t,$$

which implies

$$t = \frac{L_{\lambda_0}}{2\pi\rho_0}.$$

Notice that because of the constraint $t \in (0,1)$, $2\pi\rho_0 > L_{\lambda_0}$.

We can choose $\rho_0$ so that $\lambda_0$ is the unique standard parallel, this is, the cone as an auxiliar mapping surface is tangent to the surface at $\lambda_0$, or in order to obtain another standard parallel, namely $\lambda_1$, so that the cone is secant to the surface, intersecting with it at $\lambda_0$ and $\lambda_1$.

### 4.4.1 Tangent cone

A cone that is tangent at latitude $\lambda_0$ has an opening angle equal to the value of the latitude for both the sphere and the spheroid. Hence,

$$\rho_0 \sin\lambda_0 = N \cos\lambda,$$

where $N = R$ is the cone is tangent to a sphere. Therefore,

$$\rho = N \cot \lambda_0 = \frac{L_{\lambda_0}}{2\pi \sin \lambda_0}$$

The parameter $t$ can be written as

$$t = \sin \lambda_0.$$

Substituting

$$x = N \cot \lambda_0 e^{-\sin \lambda_0 (p - p_0)} \sin(\sin \lambda_0 q)$$
$$y = N \cot \lambda_0 (1 - e^{-\sin \lambda_0 (p - p_0)} \cos(\sin \lambda_0 q)),$$

or in terms of the geogrephic latitude

$$x = \frac{N}{\tan \lambda_0} \left( \frac{\tan \lambda_0 + \sec \lambda_0}{\tan \lambda + \sec \lambda} \right)^{\sin \lambda_0} \sin(\sin \lambda_0 \theta)$$
$$y = \frac{N}{\tan \lambda_0} - \frac{N}{\tan \lambda_0} \left( \frac{\tan \lambda_0 + \sec \lambda_0}{\tan \lambda + \sec \lambda} \right)^{\sin \lambda_0} \cos(\sin \lambda_0 \theta)$$

Both the Mercator and the stereographic projections are limit cases of the Lambert conical projection. The formula of the stereographic projection is recovered when the latitude of the standard parallel $\lambda_0 \to \frac{\pi}{2}$. The parameter $t$ tends to 1 and the auxiliary cone transforms into a plane. Analogously, for $\lambda_0 \to 0$, the standard parallel tends to the equator and the cone to a cylinder, $t \to 0$. Therefore, this limit case defines the Mercator Projection.

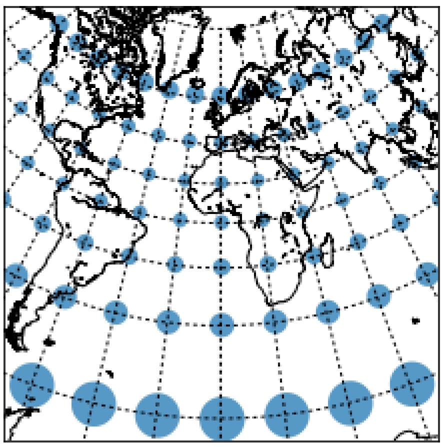


Figure 28: Lambert conic with standard parallel at latitude 20°

### 4.4.2 Secant cone

As mentioned before, it is possible to choose $\rho_0$ such that another parallel $\lambda_1 \neq \lambda_0$ is standard. If $\lambda_1$ is standard, then

$$L_{\lambda_1} = \frac{t}{\rho(\lambda_1)},$$

and as $t = \dfrac{L_{\lambda_0}}{\rho_0}$ we obtain

$$\rho_0 = \frac{L_{\lambda_0}}{L_{\lambda_1}}\rho(\lambda_1) = \frac{N_0 \cos\lambda_0}{N_1 \cos\lambda_1}\rho(\lambda_1).$$

By substituting $\rho_0$ in (4.8)

$$\begin{aligned}\rho(\lambda_1) = \rho_0 \left(\frac{\tan\lambda_0 + \sec\lambda_0}{\tan\lambda_1 + \sec\lambda_1}\right)^{\dfrac{N_0\cos\lambda_0}{\rho_0}} \\ = \quad \rho(\lambda_1)\frac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\left(\frac{\tan\lambda_0 + \sec\lambda_0}{\tan\lambda_1 + \sec\lambda_1}\right)^{\dfrac{N_0\cos\lambda_0}{\rho_0}},\end{aligned}$$

which implies

$$1 = \frac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\left(\frac{\tan\lambda_0 + \sec\lambda_0}{\tan\lambda_1 + \sec\lambda_1}\right)^{\dfrac{N_0\cos\lambda_0}{\rho_0}}.$$

Applying in both sides the logarithm

$$0 = \ln\left(\frac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\right) + \frac{N_0\cos\lambda_0}{\rho_0}\ln\left(\frac{\tan\lambda_0 + \sec\lambda_0}{\tan\lambda_1 + \sec\lambda_1}\right)$$

Isolating $\rho_0$ from the former equation

$$\begin{aligned}\rho_0 = \rho(\lambda_0) = -N_0\cos\lambda_0 \frac{\ln\left(\dfrac{\tan\lambda_0 + \sec\lambda_0}{\tan\lambda_1 + \sec\lambda_1}\right)}{\ln\left(\dfrac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\right)} \\ = \quad -N_0\cos\lambda_0 \frac{p_0 - p_1}{\ln\left(\dfrac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\right)},\end{aligned}$$

and therefore

$$t = \frac{\ln\left(\dfrac{N_0\cos\lambda_0}{N_1\cos\lambda_1}\right)}{p_0 - p_1}.$$

Substituting the found value of $\rho_0$ and $t$ in equation (4.9), the formulas for the conic Lambert's projection with two standard parallels are finally obtained.

## 4.5 Lagrange Projection

This projection is named after Joseph Louis Lagrange, who generalized Lambert's concept developing its ellipsoidal case and thoroughly studied its properties in 1779. Lambert conceptually introduce a set of conformal map projections based on three simple steps:

1. On the sphere, compress or expand every meridian by multiplying its longitude by a constant factor.
2. Still on the sphere, move the parallels along to restore conformality.
3. Apply an azimuthal stereographic projection in the equatorial aspect.

Figure 29: Lagrange projection for $k = 0.5$.

Lambert introduces the idea of using the isometric coordinates to project the surface. Step one multiplies the longitude by a factor $k$

$$\theta \mapsto \tilde{\theta} = k\theta = kq.$$

The second step computes the isometric latitude

$$\lambda \mapsto \tilde{\lambda} = kp.$$

Recalling equation (4.7), the third step is

$$x + iy = \tan \frac{k(q + ip)}{2}.$$

By separating the real and imaginary part

$$\begin{cases} x &= \dfrac{\sin(kq)}{\cos(kq) + \cosh(kp)} \\ y &= \dfrac{\sinh(kp)}{\cos(kq) + \cosh(kp)} \end{cases}$$

The scale factor is

$$\sigma = \frac{k}{\rho(\cos kq + \cosh kp)},$$

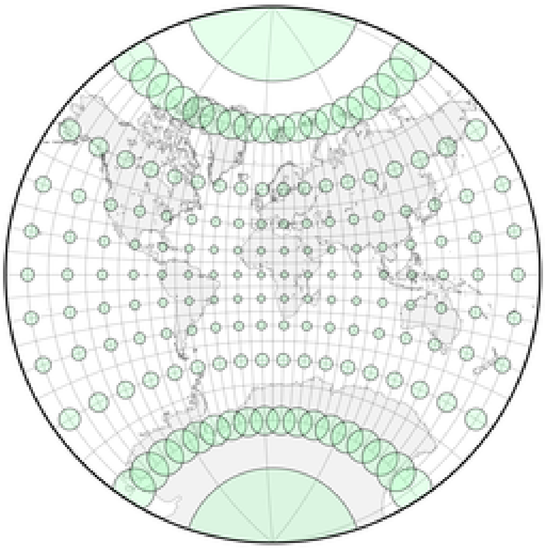


Figure 30: Tissot indicatrices in the polar stereographic projection.

where $N = R$ in the spherical case. The scale factor is large near the poles, which are nonconformal, since the derivative of the map projection function equal zero for $\lambda = \pm\pi$. The isocols are ovals extended along parallels as shown in Figure 8.

For $k = \frac{1}{2}$, the whole world is conformally mapped to a unit disk. This is the case usually refered as Lagrange's projection. It was of great importance in cartography, as it is a convenient basis for further transformations.

# 5 Conformal map based on SC transformation

The SC formula allows us to explicitely find a conformal map between the unit disk and some simply connected domains such as polygons. In Section 4 we studied different projections of the surface of the Earth within the unit disk: Lagrange's projection and the stereographic projection. In this section we shall first review this map projections and afterwards, apply the SC formula in order to find some well-known map projections. Finally, two projections that are not constructed by means of the SC formula are discussed: Eisenlohr and August map projections.

## 5.1 Parent projection

This section briefly recalls the map projections found in Section 4 that conformally map the Earth onto the unit disk and that are about to be used in further sections. The formulas might be slightly different. Although, the only changes describe rotations and reflexions in order to obtain a more convenient orientation of the projection.

### 5.1.1 One hemisphere

Let $p$ be the isometric latitude and $q = \theta - \theta_0$, and let $\zeta = p + iq$.

The stereographic projection allows us to project half the sphere into the unit disk. Recall the equations:

1. Polar Aspect. North pole:

$$x + iy = \exp(-\overline{\zeta}), \quad \lambda \in \left(0, \frac{\pi}{2}\right), \; q \in (-\pi, \pi). \tag{5.1}$$

   The central meridian $q = 0$ has its image in the positive $x$ axis ($x > 0$, $y = 0$) and the meridians grow counterclockwise.

2. Polar Aspect. South pole:

$$x + iy = \exp\zeta, \quad \lambda \in (-\frac{\pi}{2}, 0), \; q \in (-\pi, \pi). \tag{5.2}$$

   where the central meridian $q = 0$ has its image in the positive x axis ($x > 0$, $y = 0$) and the meridians grow counterclockwise.

3. Equatorial Aspect: with central meridian $\theta_0$ and poles in the imaginary axis, and the equator represented on the real axis:

$$x + iy = i\tanh\left(\frac{\overline{\zeta}}{2}\right), \quad \lambda \in (-\frac{\pi}{2}, \frac{\pi}{2}), \; q \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{5.3}$$

Notice that the stereographic projection presents no singularities.

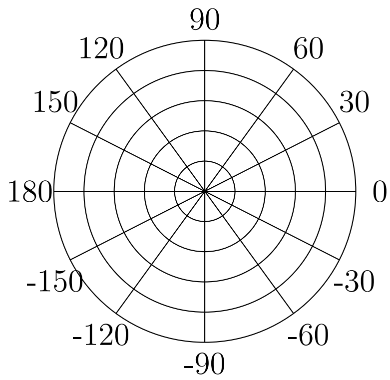



Figure 31: Polar stereographic projection graticule for meridians of longitude indicated in degrees.

### 5.1.2 Whole World

It has been shown that the Lagrange projection maps the whole world within a disc. Let $\zeta = p + iq$. Recall the equations derived in Section 4.5:

$$x + iy = i \tanh\left(\frac{\bar{\zeta}}{4}\right). \tag{5.4}$$

The Equator is mapped onto the real axis and the north and south pole are represented respectively at $z = i$ and $z = -i$. This projection has two singular points, $\lambda = \frac{\pi}{2}$ and $\lambda = \frac{-\pi}{2}$.

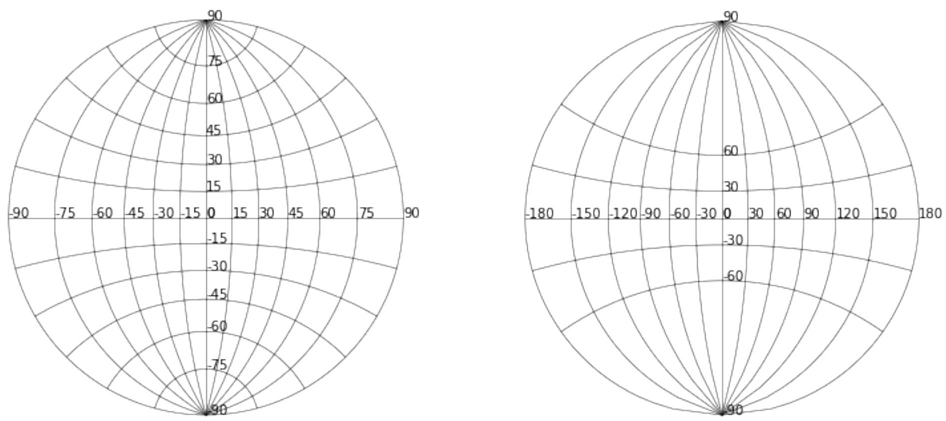



Figure 32: On the left, equatorial stereographic 15° graticule described by Equation (5.3). On the right, Lagrange graticule 30° graticule described by Equation (5.4).

## 5.2 Triangle

Triangles are polygons with $n = 3$ vertices, so no parameter problem needs to be solved. Thus, the prevertices can be choosen arbitrarely. Assume we want to map the upper half plane onto a triangle with finite vertices and inner angles $\pi\alpha_1$, $\pi\alpha_2$ and $\pi\alpha_3$, $\sum_{i=1}^{3} \alpha_i = 1$. Let the prevertices be $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. Then, the SC formula provides a map $F$ from the $\mathbb{H}$ onto the triangle of the form

$$F(z) = A + C \int_0^z \zeta^{\alpha_1 - 1}(\zeta - 1)^{\alpha_2 - 1} d\zeta, \quad A, C \in \mathbb{C}.$$

The integral above is a particular case of the incomplete beta function

$$B(z; a, b) = \int_0^z \zeta^{a-1}(\zeta - 1)^{b-1} d\zeta.$$

Thus,

$$F(z) = A + CB(z; \frac{1}{3}, \frac{1}{3}). \tag{5.5}$$

### 5.2.1 Adams one hemisphere in an equilateral triangle

The value of $A$ and $C$ in Equation 5.5 can be determined by choosing the vertices of the equilateral triangle. Let

$$w_1 = 0, \quad w_2 = 1, \quad w_3 = \frac{1}{2} + \sqrt{\frac{3}{4}} i.$$

This selection of vertices respects the condition of equiangles. Now, $F(0) = 0$ and therefore $A = 0$. Morever, $F(1) = 1$. Thus,

$$1 = CB\left(\frac{1}{3}, \frac{1}{3}\right),$$

where $B(a, b)$ is the complete beta function

$$B(a, b) = \int_0^1 \zeta^{a-1}(\zeta - 1)^{b-1} d\zeta = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a, b > 0.$$

Thus,

$$F(z) = \frac{B\left(z; \frac{1}{3}, \frac{1}{3}\right)}{B\left(\frac{1}{3}, \frac{1}{3}\right)} = I_z\left(\frac{1}{3}, \frac{1}{3}\right) \tag{5.6}$$

where $I_z(a, b)$ is the regularized beta function

$$I_z(a, b) = \frac{B(z; a, b)}{B(a, b)}.$$

Since we want to project one hemisphere we chose as parent projection the stereographic north polar projection, see (5.1). The boundary of the triangle corresponds to the eEquator and the meridians $-150°$, $-30°$ and $90°$ are straight lines that meet at the conformal center, the north pole. Figure 31 shows that these meridians in the stereographic projection intersect the boundary at the points

$$z_1 = e^{i\frac{7\pi}{6}}, \quad z_2 = e^{-i\frac{\pi}{6}}, \quad z_3 = i.$$

To find the projection, we use a Moebius transformation $\mu$ from $\mathbb{D}$ onto $\mathbb{H}$ such that

$$\mu(-\frac{\sqrt{3}}{2} - \frac{i}{2}) = 0, \quad \mu(\frac{\sqrt{3}}{2} - \frac{1}{2}) = 1, \quad \mu(i) = \infty,$$

which is

$$\mu(z) = \frac{(1 - i\sqrt{3}) + (\sqrt{3} - i)z}{2\sqrt{3}(z - i)}.$$

and therefore,

$$G(z) = F(\mu(z))$$

maps conformally the unit disk to the triangle with the equator as boundary and meridians $-150°$, $-30°$ and $90°$ respectively joining the center of the triangle with the vertices $w_1$, $w_2$ and $w_3$.

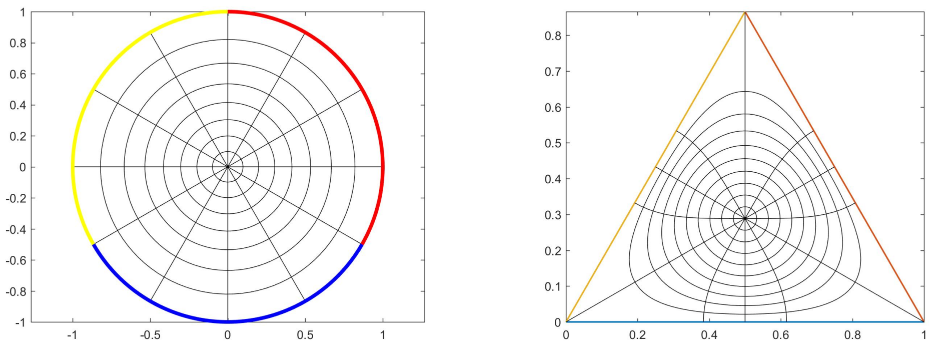


Figure 33: Grid transformation of Adam's equilateral triangle conformal map

The stereographic projection in polar aspect is used as parent projection. Recalling equations (5.1)

$$z = e^{-\bar{\zeta}}, \quad \zeta = p + iq$$

we obtain

$$F(\zeta) = I\left(\frac{(1 - i\sqrt{3}) + (\sqrt{3} - i)e^{-\bar{\zeta}}}{2\sqrt{3}(e^{\overline{-\zeta}} - i)}; \frac{1}{3}, \frac{1}{3}\right) \tag{5.7}$$

### 5.2.2 Lee whole world in an equilateral triangle

By choosing the same prevertices as in the former section, the argument holds and the function that maps $\mathbb{H}$ onto an equilateral triangle is

$$F(z) = I_z\left(\frac{1}{3}, \frac{1}{3}\right)$$

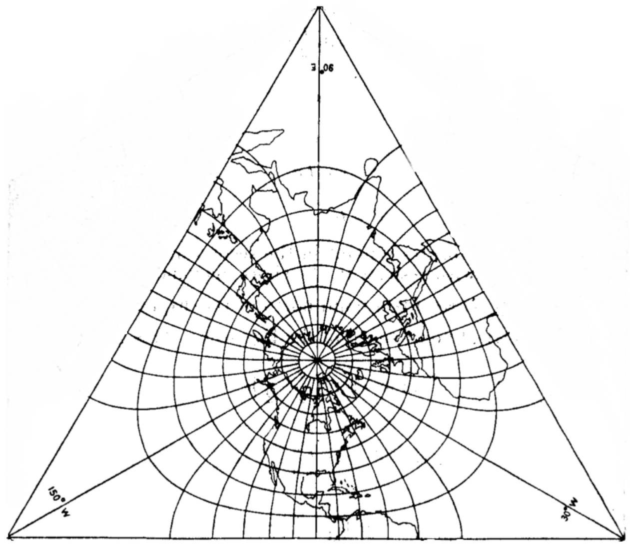


Figure 34: Adams comformal map of one hemisphere in an equilateral triangle.

Since we want to plot the whole world, we select as parent projection Lagrange projection, see (5.4). This projection represent the north pole in $i$ and the south pole in $-i$. The equator is represented in the x-axis. Moreover, Lee's projection within an equilateral triangle represents one pole at one of the vertices of the triangle and the other at the middle of the opposite side to the vertex.

The Moebius transformation

$$\mu(z) = \left(\frac{1-i}{2}\right) \frac{z+1}{z-i}$$

maps $\mathbb{D}$ onto $\mathbb{H}$ satisfying

$$\mu(-1) = 0, \quad \mu(1) = 1, \quad \mu(i) = \infty.$$

Thus, the map

$$G(z) = F(\mu(z)) = I\left(\frac{1-i}{2}\frac{z+1}{z-i}; \frac{1}{3}, \frac{1}{3}\right)$$

maps the unit disk onto an equilateral triangle with the north pole at $w_3$ and the south pole represented at the middle point of the side defined by the vertices $w_1$, $w_2$.

Recalling (5.4) and by substituting we find

$$G(z) = I\left(\left(\frac{1-i}{2}\right)\left(\frac{\tanh\frac{\bar{\zeta}}{4}+1}{\tanh\frac{\bar{\zeta}}{4}-i}\right); \frac{1}{3}, \frac{1}{3}\right).$$

Figure 36 shows the graticule of Lee's projection of the whole world within an equilateral triangle and Figure 37 the resulting map.

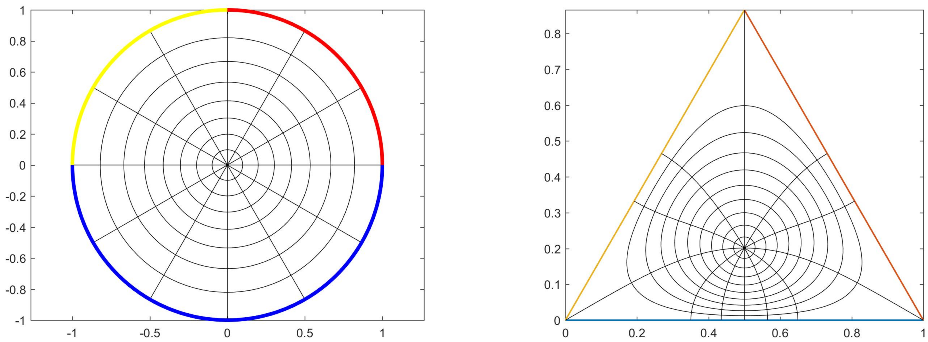


Figure 35: Grid transformation of the conformal map between the disk and the equilateral triangle proposed by Lee.

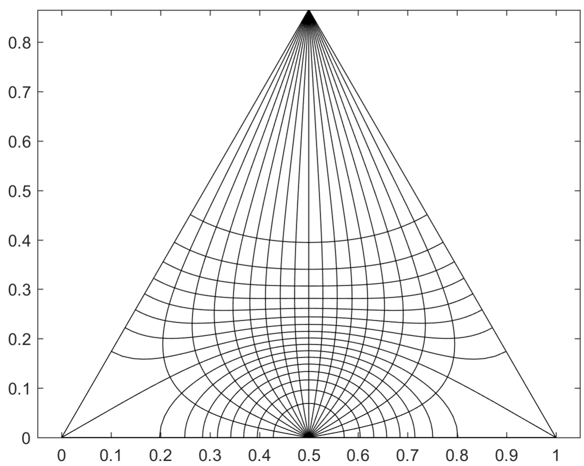


Figure 36: Resulting grid of Lee's map projection of the whole world in an equilateral triangle.

## 5.3 Square

For polygons with $n = 4$ vertices, the parameter problem needs to be solved first. We shall discuss the problem for $n = 4$ in the next section. For now, let's assume that the prevertices $z_1 = i$, $z_2 = -1$, $z_3 = -i$,$z_4 = 1$, $z_1, z_2, z_3, z_4 \in \mathbb{D}$, are a suitable option for the use of the SC formula to the square.

Mapping from $\mathbb{D}$ to the square using prevertices

$$z_1 = i, \quad z_2 = -1, \quad z_3 = -i, \quad z_4 = 1$$

and inner angles $\pi\alpha_j = \frac{\pi}{2}$ for all $j = 1, 2, 3, 4$, the SC formula is

$$f(z) = A + C \int_0^z \prod_{k=1}^{4} (\zeta - z_k)^{\alpha_k - 1} d\zeta = A + C \int_0^z \frac{d\zeta}{\sqrt{1 - \zeta^4}}.$$

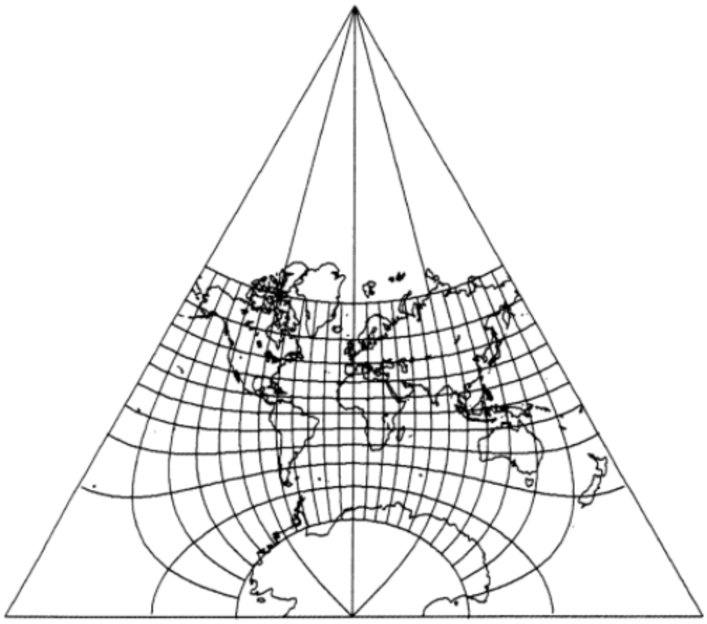


Figure 37: Lee comformal map: the whole world in an equilateral triangle.

For simplicity we shall assume $A = 0$ and $C = 1$. Then, $f(z)$ can be writen as

$$f(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1+\zeta^2)}}$$

The integral above is the incomplete elliptic integral of first kind with *modulus* $k = i$. An elliptic integral of first kind with modulus $k$ is usually denoted by $F(\varphi|k)$ and its general expression is

$$F(\varphi|k) = \int_0^{\sin\varphi} \frac{d\zeta}{\sqrt{(1-k^2\zeta^2)(1-\zeta^2)}}.$$

Thus,

$$f(z) = F(\arcsin z|i) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1+\zeta^2)}}.$$

Another usual notation is $F(\varphi, m)$, where $m = k^2$ is called the *parameter* of the elliptic function. In our case, $m = -1$.

The image of the prevertices by the map $f$ are

$$f(1) = K(-1), \quad f(i) = K(-1)i, \quad f(-1) = -K(-1), \quad f(-i) = -K(-1)i$$

where $K(m)$ is the *complete elliptic integral of first kind*,

$$K(m) = \int_0^1 \frac{d\zeta}{\sqrt{(1-m\zeta^2)(1-\zeta^2)}}.$$

In the case $m = -1$,

$$K(-1) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}}$$

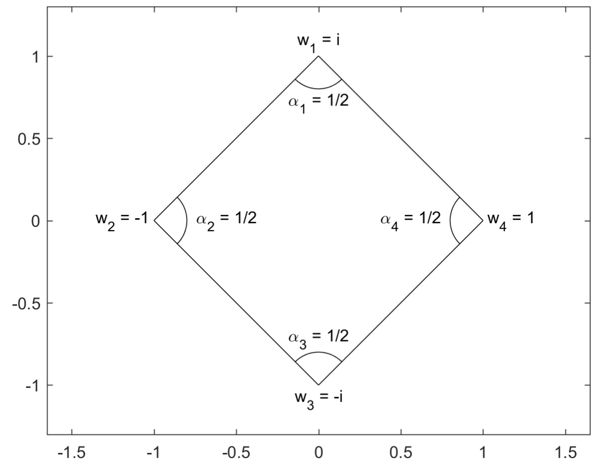



Figure 38: The square as a polygon.

Notice that by applying a factor $C = K(-1)^{-1}$ to the map $f(z)$, we obtain $g(z) = \frac{1}{K(-1)}F(\arcsin z, -1)$, that maps the unit disk onto the rectangle with vertices at 1, $i$, $-1$ and $-i$ as shown in Figure 38.

Let $w = f(z)$; then we may write

$$w = F(\arcsin z, -1).$$

The use of elliptic functions allows us to write the inverse transformation. Since $F^{-1}(w) := \mathrm{am}(w, -1) = \arcsin z$, where am denotes the Jacobi amplitude function, then, using the Jacobi sinus sn we obtain

$$\sin(\mathrm{am}(w, -1)) := \mathrm{sn}(w, -1) = z.$$

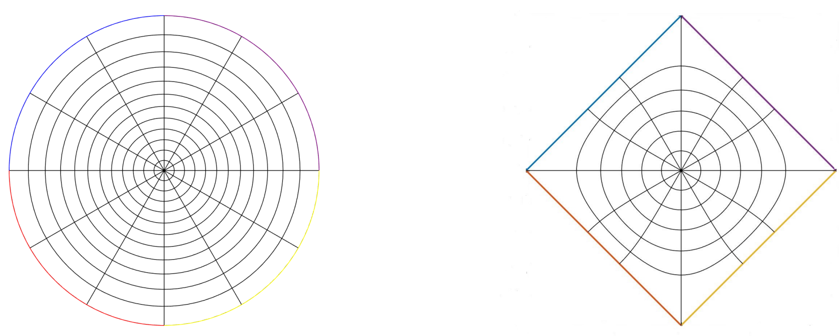


Figure 39: Grid transformation by the conformal map between the disk and the square.

Using the addition formulas of the Jacobi elliptic functions, a negative parameter sinus may be written in terms of the Jacobi elliptic function $\mathrm{sd} = \frac{\mathrm{sn}}{\mathrm{dn}}$ with a positive parameter. The result is

$$\mathrm{sn}(w,-1)=\frac{1}{\sqrt{2}}\mathrm{sd}(\sqrt{2}w,\frac{1}{2}).$$

Thus,

$$\mathrm{sd}(\sqrt{2}w|\frac{1}{\sqrt{2}})=\sqrt{2}z \tag{5.8}$$

allows the computation of the new coordinates $w$ in the square in terms of $z\in\mathbb{D}$.

The origin of coordinates is at the center of the square and the axes are the diagonals of the square. The length of the diagonal of the square is

$$d=\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}},$$

and the side of the square has length $s=\sqrt{2}K(-1)$.

All the projections that are to be studied in the following sections, this is, Pierce's, Adam's and Guyou's projections, are computed by means of (5.8) using different aspects of the stereographic projection. However, none of them used Equation (5.8) directly to derive their projections and they did not recognize that all three are merely different aspects of the same projection. Despite interest due to their mathematical development, the conformal map projections onto squares of Pierce, Guyou and Adams have seldom been used.

### 5.3.1 Pierce quincuncial projection

Pierce published his "quincuncial projection" in 1877. This projection is formed by transforming the stereographic projection in polar aspect by means of the Jacobi cosine elliptic function $cn=\cos am$

$$\cos am(x+iy)(angle\ of\ mod.=\frac{\pi}{4})=\tan\frac{\lambda'}{2}(\cos q+i\sin q),$$

where $\lambda'$ denotes the complementary latitude, $\lambda'=\frac{\pi}{2}-\lambda$, and therefore

$$\tan\frac{\lambda'}{2}=\tan\left(\frac{\pi}{4}-\frac{\lambda}{2}\right)=e^p.$$

This conformal map is equivalent to the conformal map derived in (5.8).

Pierce projected with the polar stereographic projection twice. The north hemisphere and the south hemisphere are mapped onto two different squares by means of (5.8). Both squares represent the respective pole at the center of the square.

Recalling the equations for the north and south polar stereographic, see (5.1) and (5.2) we obtain

$$\mathrm{sd}(\sqrt{2}w, \frac{1}{\sqrt{2}}) = \sqrt{2}e^{-\bar{\zeta}}$$
$$\mathrm{sd}(\sqrt{2}w, \frac{1}{\sqrt{2}}) = \sqrt{2}e^{\zeta}.$$

The values at the boundary of the square are achieved when $p = 0$, which implies $z = e^{iq}$ for both hemispheres for both projections, the north and the south pole. Therefore, for corresponding sides, the two projections present the same values at the boundary. Because of this, Pierce arranged the south hemisphere in four pieces, corresponding respectively to the four simmetric parts of the square defined by the meridians $\theta \in (0, \frac{\pi}{2})$, $\theta \in (\frac{\pi}{2}, \pi)$, $\theta \in (-\pi, -\frac{\pi}{2})$ and $\theta \in (-\frac{\pi}{2}, 0)$ respectively. This arrangement can be described as a reflection along the lines $y = x$, $w = x + iy \mapsto y + ix$, followed by a translation, $w = x + iy \mapsto y + ix + T$, where $T = 1+i$ for the first of the four pieces, $T = -1+i$ for the second, $T = -1-i$ and $T = 1 - i$, for the third and fourth pieces. By doing so, the whole world is mapped within a larger square, as shown in Figure 40.

The projection is arranged in five pieces and for this reason named quincuncial projection.

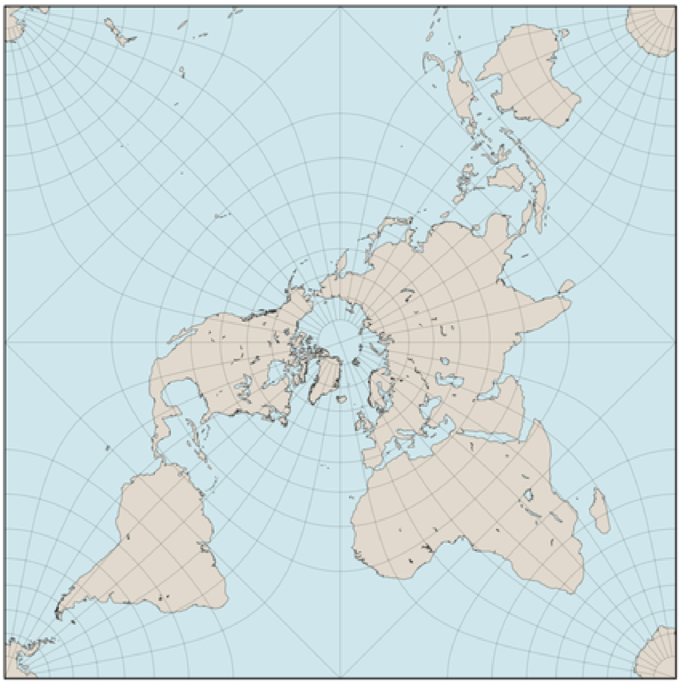


Figure 40: Pierce quincuncial projection within a square.

### 5.3.2 Adams

Adams projection of an hemisphere within a square (1925) uses the stereographic projection in equatorial aspect and applies the SC transformation (5.8). The equatorial stereographic map projection described by the formulas (5.3) represents the equator in the real axis and the north and south pole at $i$ and $-i$ respectively. Therefore, Equation (5.8) maps the poles onto opposite vertices of the square and the equator remains in the real axis. Figure 41 shows the grid transformation of the equatorial stereographic and the obtained grid on the square.

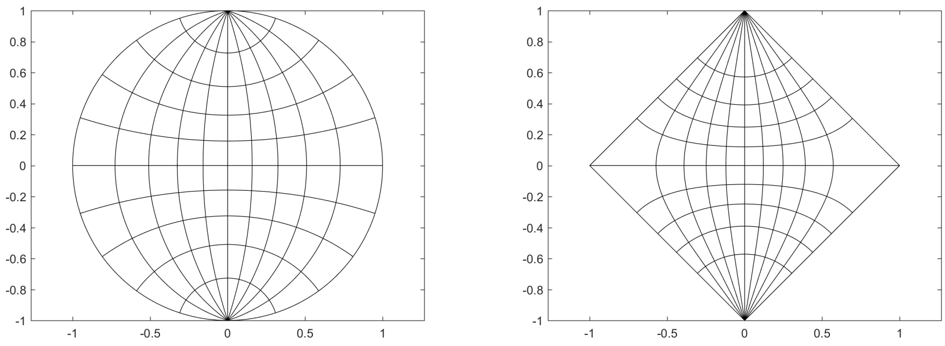



Figure 41: Adams projection grid transformation.

Recalling the equation of the stereographic projection in equatorial aspect we may write Adam's projection as follows

$$\operatorname{sd}(\sqrt{2}w, \frac{1}{\sqrt{2}}) = \sqrt{2}i \tanh \frac{\overline{\zeta}}{2}, \quad \zeta = p + iq.$$

If the central meridian is choosen to be the Greenwich meridian $q_0 = 0$ we obtain the map shown in Figure 42.

Notice that Adam's projection in a square may be described as the transverse aspect of Pierce's quincucial projection.

### 5.3.3 Guyou

In 1887 Guyou presented his projection of an hemisphere within a square. The projection is obtained using the equatorial stereographic projection turned $-\frac{\pi}{4}$ so that the poles are no longer prevertices of the SC transformation, but parallels with latitude $\frac{\pi}{4}$ and $\frac{-\pi}{4}$ respectively and the bounding meridian. After applying the SC formula for the rotated stereographic projection, the square is usually rotated again a total angle of $\frac{\pi}{4}$ in order to show the poles aligned in the imaginary axis. The rotations can be easily computed by multiplying the coordinates by $\frac{1-i}{2}$ and $\frac{1+i}{2}$ respectively.

Guyou produced twice this projection, one centered at the meridian $\frac{-\pi}{2}$ and the other at $\frac{\pi}{2}$, and placed them side by side obtaining thus a map of the whole world in a rectangle $2 \times 1$.

It has been shown that the Lagrange projection maps the whole world within a disk. This projection has two singular points in its boundary that shall map into two singularities in the boundary of the square. If the singular points are mapped

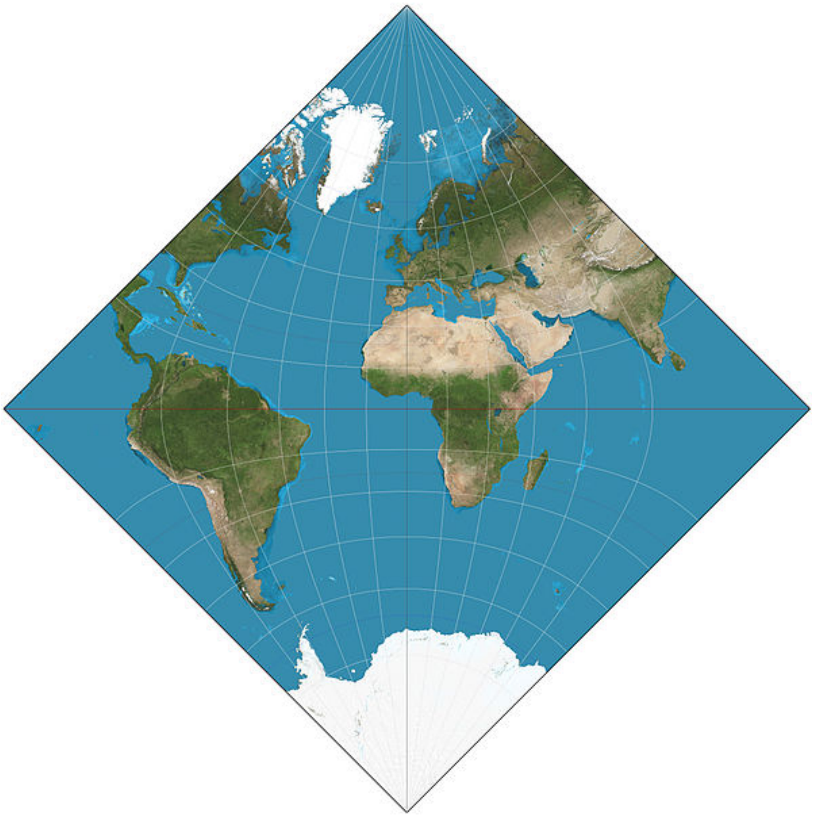


Figure 42: Adams hemisphere in a square. 15° graticule.

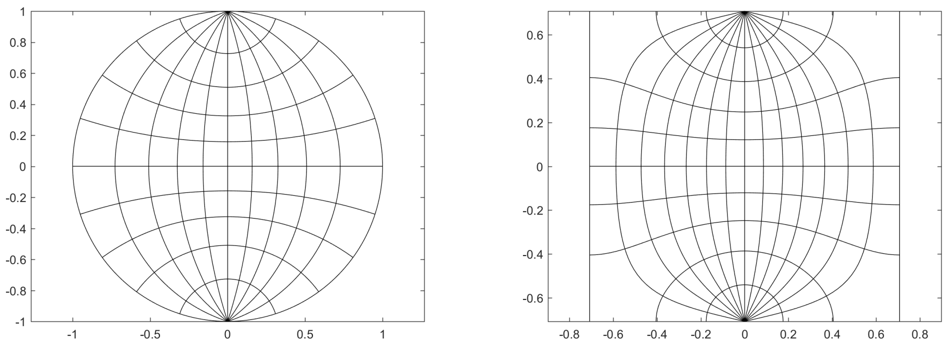



Figure 43: Grid transformation in Guyou's map projection.

to the vertex of the square, the resulting map has four non conformal points at each vertex. If not, new singularity points have to be taken into account, changing the characteristics of the map.

Although many different projections of this kind have been developed, the essence of this projections lies in the application of the SC formula (5.8) to different aspects of the Lagrange projection with different rotations and singularities. Two of the best well-known whole world projections within a square have been selected and developed below.

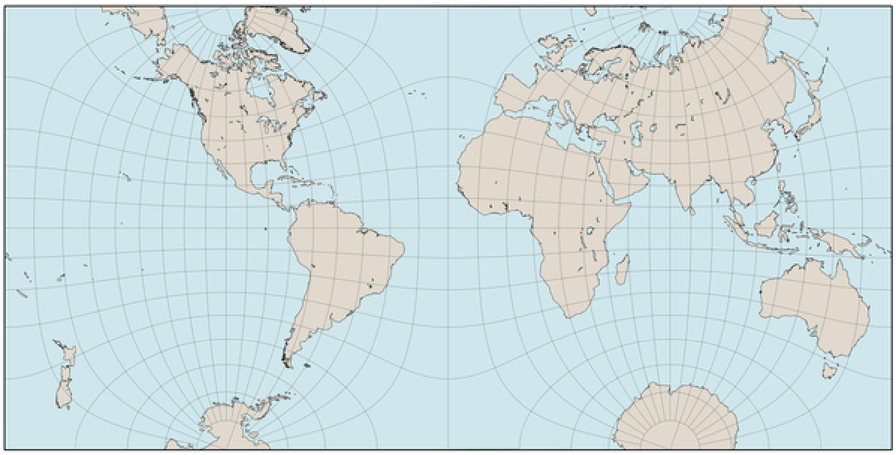


Figure 44: Guyou hemisphere in a square.

### 5.3.4 Adams 1929

In this projection, the poles are placed at oposite vertices of the square. Therefore, the parent projection is the Lagrange projection described by Equation (5.4) and the resulting projection is

$$\mathrm{sd}(\sqrt{2}w|\frac{1}{\sqrt{2}}) = \sqrt{2}i\tanh\frac{\overline{\zeta}}{4}$$

Figure 45 shows the resulting map projection.

### 5.3.5 Adams 1936

As in the Guyou projection, here, the Lagrange projection is first rotated through $\frac{\pi}{4}$. Then the SC formula is applied and the resulting square is again rotated $\frac{-\pi}{4}$. Therefore, the poles are placed at the midpoints of opposite sides of the square.

## 5.4 Rectangle

The rectangle is a special case of a quadrilateral $n = 4$, for which, $\alpha_k = \frac{1}{4},\ k = 1, 2, 3, 4$. Opposite sides have the same lenght. Let $2K$ denote the lenght of one pair of sides and $K'$ the lenght of the other pair.

The first problem encountered in order to find the SC transformation is the parameter problem. Only three of the four prevertices are ours to choose. However, because of the symmetry of the rectangle, the prevertices can be chosen to be

$$z_1 = e^{i\theta}, \quad z_2 = -e^{-i\theta}, \quad z_3 = -e^{i\theta}, \quad z_4 = e^{-i\theta} \quad \in \mathbb{D}$$
$$t_1 = -1, \quad t_2 = 1, \quad t_3 = \frac{1}{k}, \quad t_4 = -\frac{1}{k}, \quad \in \mathbb{H},$$

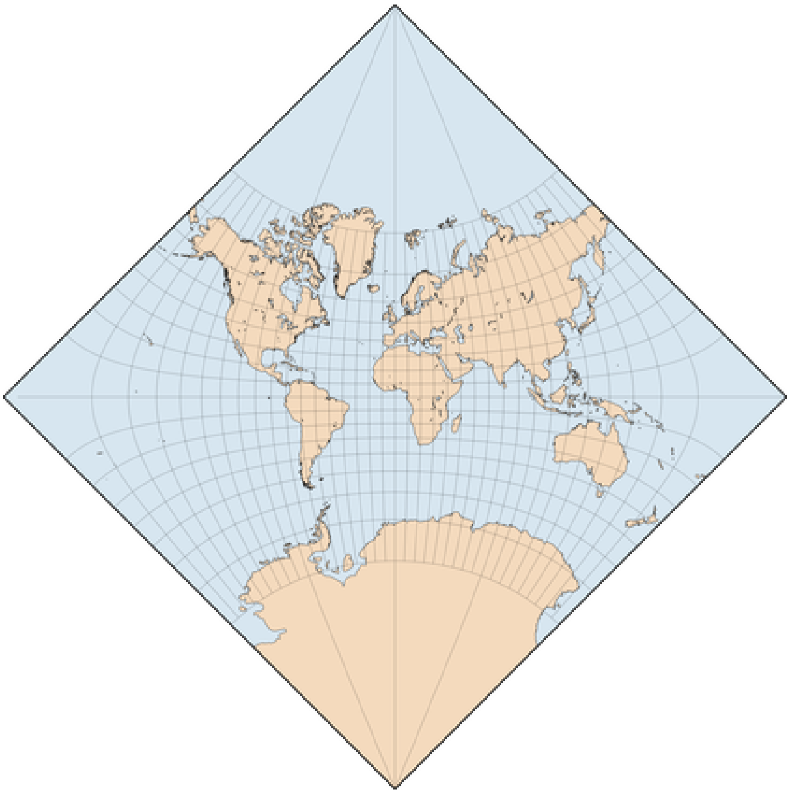


Figure 45: Adams projection of the whole world within a square, 1929.

Figure 46: Adams projection of the whole world within a square, 1936.

where $\theta \in (0, \frac{\pi}{2})$, $k \in (0, 1)$ express respectively the degree of freedom in the choice of the prevertices. These values are related by the equation

$$k = \frac{1 - \sin\theta}{1 + \sin\theta}.$$

The Moebius transformation

$$\mu(z) = -i\frac{z-i}{\sqrt{k}(z+i)} \tag{5.9}$$

maps conformally $\mathbb{D}$ onto $\mathbb{H}$ such that $\mu(z_i) = t_i$.

The degree of freedom is related to the aspect ratio, known under the name of *conformal modulus*. We want to relate the aspect ratio with the parameters $k$ and $\theta$ so that for a given aspect ratio we can find apropiate prevertices in order to solve the parameter problem. Then, the complex constants $A$ and $C$ will allow the construction of any similar rectangle.

Considering the SC formula usign as canonical domain $\mathbb{H}$ and prevertices $-\frac{1}{k}$, $-1$, $1$ and $\frac{1}{k}$ we obtain

$$f(t) = A + C\int_0^t \frac{d\zeta}{\sqrt{(\zeta^2 - \frac{1}{k^2})(\zeta^2-1)}} = A + Ck\int_0^t \frac{d\zeta}{\sqrt{(1-k^2\zeta^2)(1-\zeta^2)}}.$$

Assuming, without loss of generality, $A = 0$ and $C = \frac{1}{k}$, $f(t)$ is the incomplete elliptic integral of first kind with modulus $k$. Using the same notation as in Section 5.3,

$$f(t) = F(\arcsin t|k). \tag{5.10}$$

The inverse transformation might be written in terms of elliptic functions. Let $w = f(z)$, then

$$\mathrm{sn}(w|k) = t. \tag{5.11}$$

Equation (5.10) maps conformally $\mathbb{H}$ onto a rectangle so that

$$t_1 \mapsto w_1 = -K, \quad t_2 \mapsto w_2 = K, \quad t_3 \mapsto w_3 = K + iK', \quad t_4 \mapsto w_4 = -K + iK'$$

where $K(k)$ is the complete elliptic integral of first kind

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and

$$K' = \int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = K(k')$$

where $k' = \sqrt{1-k^2}$ is the *complementary modulus*. Notice that the aspect ratio is equal to $\frac{K(k')}{2K(k)}$.

Using the Moebius transformation (5.9) we obtain a map from $\mathbb{D}$ onto the rectangle of the same aspect ratio

$$w = F(\arcsin\left(-i\frac{z-i}{\sqrt{k}(z+i)}\right)|k). \tag{5.12}$$

The inverse transformation is

$$\mathrm{sn}(w,k) = -i\frac{z-i}{\sqrt{k}(z+i)}. \tag{5.13}$$

### 5.4.1 Projection within a rectangle with a meridian as boundary (Adams 1925)

Adams conformally projected the sphere within a rectangle with an aspect ratio equal to $\frac{1}{2}$, this is, whose width is twice the height. By imposing this, we obtain $K(k) = K(k')$, that implies that

$$k = k' = \frac{1}{2}.$$

Since we want to map the whole sphere, we use as parent projection Lagrange projection. Adams located the poles at the midpoints of the upper and lower sides of the rectangle and in the boundary represented the meridians $\theta = \pm\pi$. Thus, the Lagrange projection needs to represent the north pole at $i$, the south pole at $-i$ and the meridians growing in clockwise order. The equation of such parent projection is

$$z = i \tanh \frac{\overline{\zeta}}{4}$$

and therefore the conformal map from the sphere to the rectangle may be written as

$$w = F(\arcsin\left(-i\frac{\tanh\frac{\overline{\zeta}}{4} - 1}{\sqrt{k}(\tanh\frac{\overline{\zeta}}{4} + 1)}\right)|k)$$

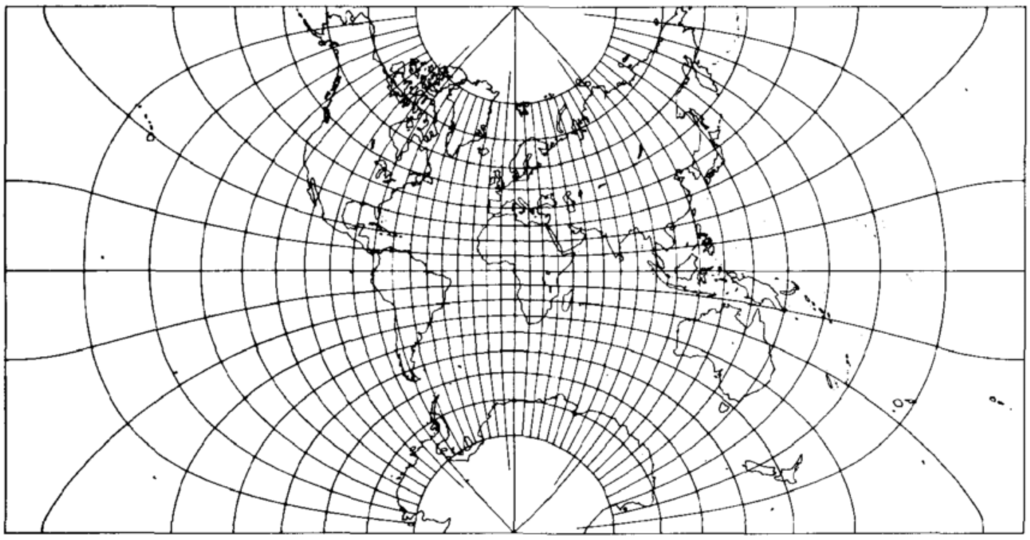


Figure 47: Adams whole world whitin a rectangle, $k = \sin 45°$.

## 5.5 Ellipse

This section mainly follows the articles [11] and [14].

The function $f(w) = \sin w$ maps conformally the rectangle within an ellipse, since

$$\sin w = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

and therefore, for a fixed $y$,

$$\left(\frac{\mathrm{Re}(f(w))}{\cosh y}\right)^2 + \left(\frac{\mathrm{Im}(f(w))}{\sinh y}\right)^2 = 1$$

The semi-major axis of each ellipse is $\cosh y$ and the semi-minor axis $\sinh y$. Moreover, the foci are located at $\pm\sqrt{\cosh^2 y + \sinh^2 y} = \pm 1$.

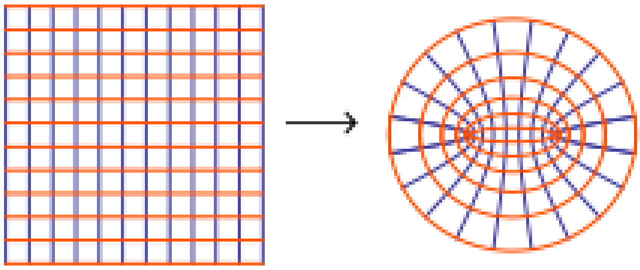


Figure 48: The function $\sin z$.

In the former section we found that the elliptic integral of first kind

$$F(\arcsin t|k)$$

maps conformally the upper-half-plane onto a rectangle with vertices at $-K(k)$, $K(k)$, $K(k) + iK(k')$ and $-K(k) + iK(k')$.

Since

$$F(\frac{z}{k|z|}|k) = \overline{F(z|k)} + iK(k')$$

holds, the function

$$F(\frac{z}{\sqrt{k}}|k)$$

maps the upper half of the unit disk to the rectangle with vertices at $-K(k)$, $K(k)$, $K(k) + i\frac{K(k')}{2}$ and $-K(k) + i\frac{K(k')}{2}$. We can apply the normalization $\frac{\pi}{2K(k)}$, so that

$$f(z) = \frac{\pi}{2K(k)} F(\frac{z}{\sqrt{k}}, k)$$

maps the upper half of $\mathbb{D}$ onto a rectangle with vertices at $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2} + i\frac{\pi}{4}\frac{K(k')}{K(k)}$ and $-\frac{\pi}{2} + i\frac{\pi}{4}\frac{K(k')}{K(k)}$.

The function

$$f(z) = \sin\left(\frac{\pi}{2K(k)} F(\frac{z}{\sqrt{k}}, k)\right) \tag{5.14}$$

maps therefore the upper half of the unit disk onto the upper half of an ellipse with semi-major axis $\cosh\frac{\pi}{4}\frac{K(k')}{K(k)}$ and semi-minor axis $\sinh\frac{\pi}{4}\frac{K(k')}{K(k)}$.

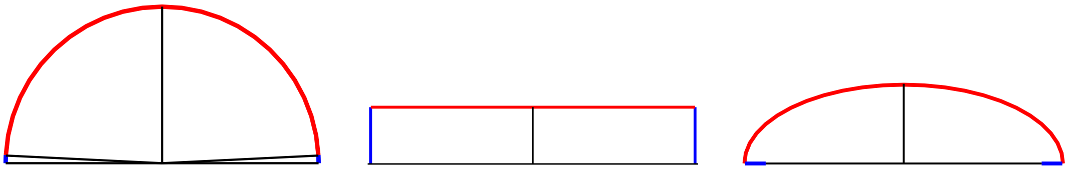


Figure 49: Boundary identifications between the upper half of the unit disk, the rectangle and the ellipse, for $k = \sin(65°)$

By the Schwarz reflection principle, which states that an analytic function with canonical domain $\mathbb{H}$ can be analitically extended by means of $f(\bar{t}) = \overline{f(t)}$ if $f(t) \in \mathbb{R}$ for $t \in \mathbb{R}$, and since $F(t|k)$ maps the interval $[-1, 1]$ onto $[-K(k), K(k)]$, the function $F(t|k)$ can be continued analytically to the slit domain $\mathbb{C} \setminus ((-\infty, -1] \cup ([1, \infty))$. Thus, the map (5.14) present two slits, both at the segment over the real axis that joins the focus to the boundary of the ellipse, drawed in blue in Figure 49, where the map is non-conformal.

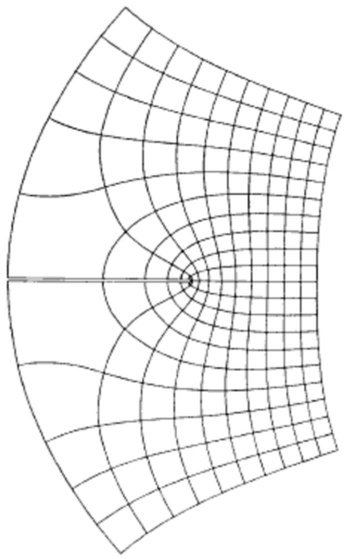


Figure 50: A 1° graticule in the region near the extremity of the major axis in the Adams projection of the world within an ellipse.

Adams suggested a map within an ellipse for which the semi-major axis was about twice the lenght of the semi-minor axis. Thus,

$$\cosh \frac{\pi}{4}\frac{K(k')}{K(k)} = 2\sinh \frac{\pi}{4}\frac{K(k')}{K(k)}$$

which implies

$$\frac{\pi}{4}\frac{K(k')}{K(k)} = \frac{\ln 3}{2},$$

and therefore the intermediant rectangle has an aspect ratio of $\frac{\log(3)}{\pi}$. Adams used a value $k = \sin(65°)$, obtaining an approximation of the desired aspect ratio.

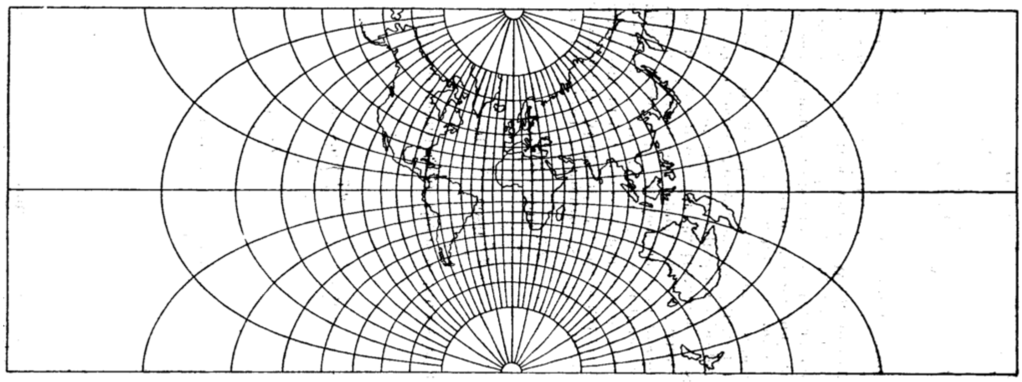


Figure 51: Projection within a rectangle $k = \sin(65°)$ for the projection of the whole world within an ellipse.

The parent projection is Lagrange projection presenting the north pole at $i$ and the south pole at $-i$. Thus,

$$f(\zeta) = \sin\left(\frac{\pi}{2K(\sin(65°))} F\left(i \frac{\tanh \frac{\overline{\zeta}}{4}}{\sqrt{\sin(65°)}} | \sin(65°)\right)\right)$$

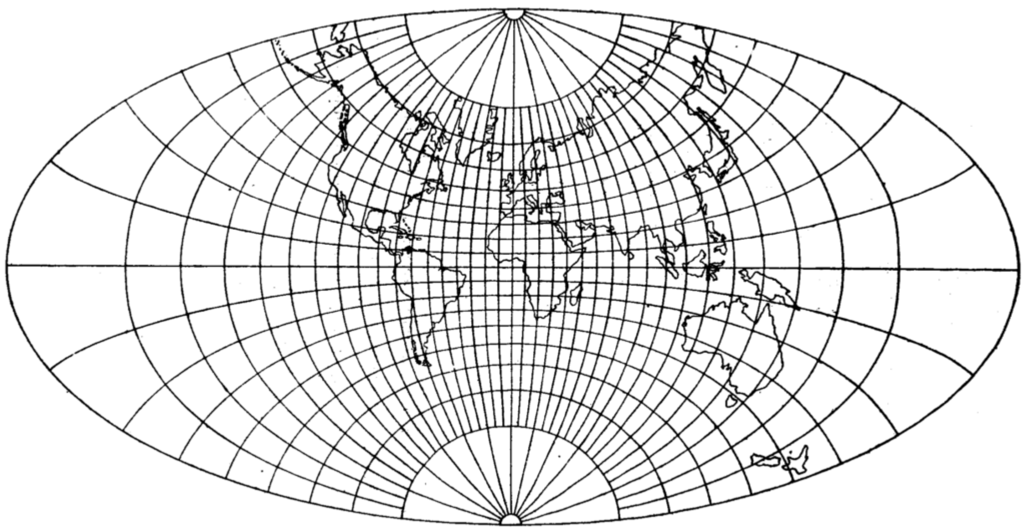


Figure 52: Conformal map of the whole world within an ellipse.

## 5.6 Littrow

Joseph Johann von Littrow developed his map projection in 1833. It is the only conformal projection that is also retroazimuthal. A retroazimuthal projection shows the directions or azimuths correctly from any point of the map to the center of the map.

The equations for the sphere are as follows

$$\begin{cases} x &= R\dfrac{\sin(\theta - \theta_0)}{\cos\lambda} \\ y &= R\cos(\theta - \theta_0)\tan\lambda, \end{cases} \tag{5.15}$$

where $\theta_0$ is the central meridian. We shall assume $\theta_0 = 0$ and derive the equations of this projections using the SC formula for an exterior map.

The SC formula adapted to the exterior of the polygon using as canonical domain $\mathbb{D}$ is

$$f(z) = A + C\int_0^z \zeta^{-2}\prod_{k=1}^{n}(z_k - \zeta)^{1-\alpha_k}\, d\zeta.$$

The Littrow map uses the polygon with two vertices $n = 2$ and angles $\alpha_1 = 0$ and $\alpha_2 = 0$. Note that no parameter problem needs to be solved. Therefore arbitrary prevertices can be used in the SC formula. Let $z_1 = -i$ and $z_2 = i$. The evaluation of the corresponding SC integral is

$$\int_0^z \zeta^{-2}(1 - i\zeta)(1 + i\zeta)d\zeta = \int_0^z \frac{1+\zeta^2}{\zeta^2}d\zeta = z - z^{-1}.$$

By imposing $w_1 = -1$ and $w_2 = 1$ and integrating along the boundary of the unit disk we get:

$$2 = C\int_{-i}^{i}\frac{d\zeta}{1+\zeta^2} = C(2i - \frac{2}{i}) = 4Ci.$$

Thus,

$$C = -\frac{1}{2}i$$

and the function is

$$w = f(z) = -\frac{i}{2}(z - z^{-1})$$

The map $f$ identifies $i \mapsto 1$ and $-i \mapsto -1$. Moreover, the center of the unit disk in mapped to infinity.

Let $z$ be the stereographic polar projection from the north pole, i.e.,

$$z = e^{-\bar{\zeta}}, \quad \zeta = p + iq, \quad \lambda \in (0, \pi/2), \quad \theta \in (-\pi/2, \pi/2).$$

Notice that we have allowed $\theta$ to vary between $-\pi/2$ and $\pi/2$. This means that we have taken just half the stereographic projection, $\mathrm{Re}(z) > 0$, which is mapped onto $\mathrm{Im}(w) > 0$.

Thus,

$$w = f(\zeta) = -\frac{i}{2}(e^{-\bar{\zeta}} - e^{\bar{\zeta}})$$

and since $e^{-p} = -e^{p}$, we may rewrite the expression above as

$$w = f(\zeta) = \frac{i}{2}(e^{\zeta} - e^{-\zeta}).$$

In order to complete the map we shall consider the stereographic projection south pole for $\mathrm{Re}(z) > 0$ and plot it in $\mathrm{Im}(w) < 0$. Thus, $C = \frac{1}{2}i$ and the function is

$$w = f(z) = \frac{i}{2}(z - z^{-1}),$$

where now $z$ is the stereographic polar projection from the south pole:

$$z = e^{\zeta}, \quad \zeta = p + iq, \quad \lambda \in (-\pi/2, 0), \quad \theta \in (-\pi/2, \pi/2)$$

Finally, we can write our map as

$$w = \frac{i}{2}\left(e^{\zeta} - e^{-\zeta}\right), \quad \zeta = p + iq, \quad \lambda \in (-\pi/2, \pi/2), \quad \theta \in (-\pi/2, \pi/2). \tag{5.16}$$

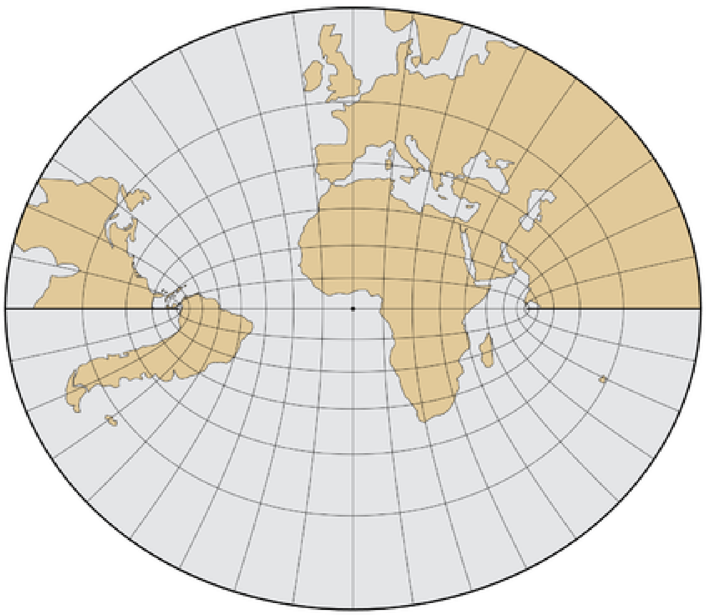


Figure 53: Littrow retroazimuthal projection.

By separating the imaginary and real part of $w = x + iy$ assuming a sphere, we obtain

$$\begin{aligned} -2iw &= e^{p+iq} - e^{-p-iq} \\ &= (\tan\lambda + \sec\lambda)(\cos q + i\sin q) - (\sec\lambda - \tan\lambda)(\cos q - i\sin q) \\ &= 2\tan\lambda\cos q - 2i\sec\lambda\sin q, \end{aligned}$$

and therefore

$$w = \sec\lambda\sin q + i\tan\lambda\cos q.$$

The former equation is analogous to the expression (5.15).

## 5.7 Eisenlohr

Mercator Projection does not represent the poles, the stereographic projection does not map the focal point over the sphere and Lambert conic projection is nonconformal at the vertex of the cone. Also Lagrange projection has two singular points at the poles. Friedrich Eisenlohr published in 1870 a projection that represents the whole world without any singular point. Regardless, Eisenlohr developed his projection as a result of the search of conformal map bounded by two meridians and presenting minimal distortion. The source of information for this section are [8] and [3].

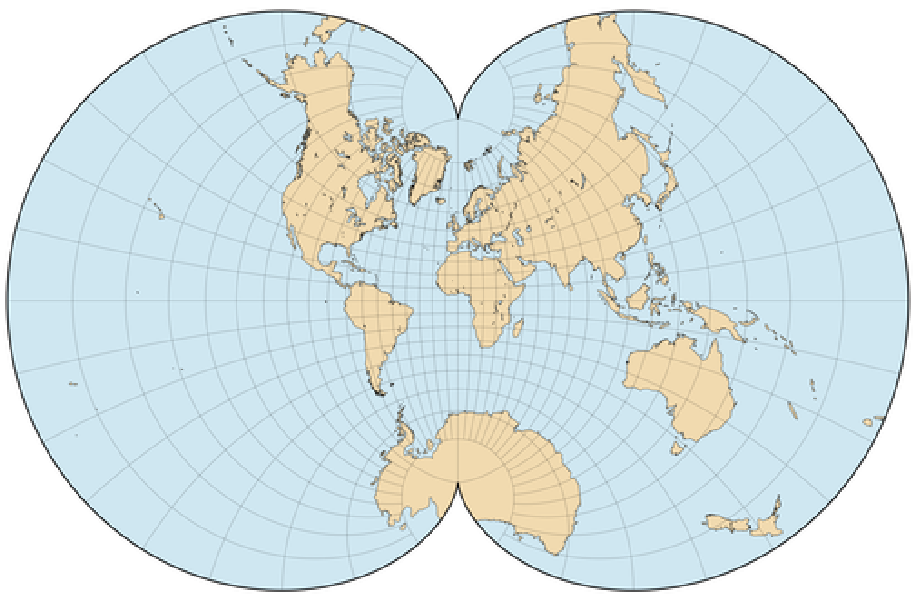


Figure 54: Eisenlohr projection.

Recalling the isometric coordinates $p$ and $q$ and a conformal map projection $z = x + iy = f(q + ip)$, so that the distortion is $\sigma = \rho^{-1}(\lambda)f'(q + ip)$. Taking the logarithm

$$\log\sigma + \log\rho^{-1} = \mathrm{Re}(\log f').$$

In the case of the sphere, $\rho = \cosh p$ and by assuming $z = \log f'$ we may writte

$$z = \log \frac{\sigma}{\cosh p} \tag{5.17}$$

Moreover, the distortion can be related with the Gaussian curvature $K$ of the surface at each point,

$$K = \rho^2 \Delta \log \rho.$$

Thus, $g = \log \sigma$ satisfies

$$\Delta g = \rho^{-2} K > 0$$

and is a subharmonic function. Therefore, it satisfies the strong maximum principle, which states that subharmonic functions cannot assume an interior maximum unless they are constant. Thus, the minimal geodetic distortion is attained when $g$ has the same value at the boundary. Therefore,

$$z = -\log \sinh p \tag{5.18}$$

at the boundary.

Also is $z$ harmonic, so is $\Delta z = 0$ for all the interior points.

Finding $z$ such that $z$ is harmonic and $z$ attains a certain value at the boundary is called the Dirichlet problem. By fixing as boundary the equator, $p = 0$, the solution of the problem leads to the stereographic projection; the map projection with minimal geodetic distortion with the equator as boundary curve.

Eisenlohr solved this problem for the case in which the boundary curves are two meridians. Taking $p$ and $q$ as rectangular coordinates, (the Mercator projection), then the problem is simplified to finding the value of $z$ at each point contained between two parallel lines, where its value at the boundary is equal to 0 and at all interior points $z$ is a continuous functions of $p$ and $q$. This problem is known as Green's potential problem.

The Green function of this problem is the real part of

$$G(p,q) = \log \left( \frac{e^{(P-p+i(q-Q))\frac{1}{4}} - e^{-(P-p+i(q-Q))\frac{1}{4}}}{e^{(P-p+i(q+Q-2\pi))\frac{1}{4}} - e^{-(P-p+i(q-Q-2\pi)\frac{1}{4}}} \right)$$

where $P$ and $Q$ are the coordinates of the point with respect to the middle of the strip. Then,

$$z' = -\frac{1}{2\pi} \int z_0 \frac{\partial G}{\partial n}$$

where $n$ is the normal to the boundary at each point and $z_0$ the value of $z$ at the boundary, (5.18). Explicitly:

$$z' = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log(\cosh(p))\, dp}{\cosh\left(\dfrac{P - p + iQ}{2}\right)}.$$

By integrating we obtain

$$x + iy = \int z' d\zeta = \frac{2}{i}(v + ui) + \sqrt{2}(\sinh(v - ui)),$$

where

$$\tan u = \frac{\sin \frac{\lambda}{2}}{\cos \frac{\lambda}{2} + \cos \frac{q}{2}\sqrt{2 \cos \lambda}}$$

$$v = \frac{1}{2} \log \left( \frac{\cos \frac{\lambda}{2} + \sqrt{\cos \lambda} \cos \left( \frac{\pi - 2q}{4} \right)}{\cos \frac{\lambda}{2} + \sqrt{\cos \lambda} \cos \left( \frac{\pi - 2q}{4} \right)} \right)$$

## 5.8 August

Freidrich August published his projection in 1874 as an alternative to Eisenlohr's design. In this projection, a world map is bounded by an epicycloid, a curve defined by a point on a circle rolling without sliding around another fixed circle. The two-cusped epicycloid with radi $\frac{1}{2}$ may be described by the following equations

$$\begin{cases} x(\varphi) & = \frac{1}{2}(3 \cos \varphi - \cos(3\varphi)) \\ y(\varphi) & = \frac{1}{2}(3 \sin \varphi - \sin(2\varphi)) \end{cases}, \quad \varphi \in [0, 2\pi] \tag{5.19}$$

The map

$$f(z) = \frac{1}{2}(3z - z^3) \tag{5.20}$$

maps the boundary of the unit disk, $e^{i\varphi}$, $\varphi \in [0, 2\pi)$ onto

$$f(e^{i\varphi}) = \frac{1}{2}(3e^{i\varphi} + e^{3i\varphi}) = \frac{1}{2}((3 \cos \varphi - \cos(3\varphi) + i(3 \sin \varphi - \sin(3\varphi)))$$

i.e., the epycicloid.

Moreover, the map (5.20) is analytic in the interior of the unit disk and therefore, it maps conformally the interior of the disk onto the interior of the simply connected domain bounded by the epicycloid.

As we want to map the whole world, we select as parent projection Lagrange's projection. The map projection formula

$$z = \tanh \frac{\overline{\zeta}}{4}$$

maps the north pole to $z = 1$ and the south pole to $z = -1$. Moreover, notice that Equation (5.19) maps the intersections of the two cusps at $(1, 0)$ and $(-1, 0)$. Thus, the map

$$f(z) = \frac{1}{2}(3 \tanh(\overline{\zeta}) - \tanh(\overline{\zeta})^3)$$

Figure 55: August projection.

represents the unit disk onto the interior of an epycicloid where the north and south pole are represented at the intersection of the cusps.

However, we would like the to be at $(0, 1)$ and $(0, -1)$. In order to do so we shall just rotate $f(z)$ a total angle of $\frac{\pi}{2}$. Such rotation can be performed by multiplying by $i$. The resulting map projection is August projection and is described by equation

$$w = \frac{i}{2}(3\tanh(\overline{\zeta}) - \tanh(\overline{\zeta})^3)$$

where $\zeta = p + iq$.

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