Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

The Seifert-Van Kampen theorem via covering spaces

Autor: Roberto Lara Martín

Director:	**Dr. Javier José Gutiérrez Marín**
Realitzat a:	**Departament de Matemàtiques i Informàtica**
Barcelona,	**29 de juny de 2017**

Contents

Introduction

The Seifert-Van Kampen theorem describes a way of computing the fundamental group of a space X from the fundamental groups of two open subspaces that cover X, and the fundamental group of their intersection. The classical proof of this result is done by analyzing the loops in the space X and deforming them into loops in the subspaces. For all the details of such proof see [1, Chapter I].

The aim of this work is to provide an alternative proof of this theorem using covering spaces, sets with actions of groups and category theory. On this version of the theorem we are going to ask more conditions on the topological space than in the 'classical' proof. Nevertheless, the spaces which do not follow those requirements are a bit 'pathological'.

First, we are going to introduce category theory, and all the concepts which will be needed to follow the proof. Then we are going to talk about group actions, focusing on its categorical implications. After we recall the basics of homotopy theory, we are going to see covering spaces and how do they relate with the fundamental group. Finally, we are going to prove the theorem of Seifert-van Kampen using covering spaces.

Acknowledgments

I would like to thank Professor Gutierrez for his advice and time given along the making of this paper.

I would like also to thank Professors Casacuberta and Vicente Navarro, who were my professors of the topology courses taken in the University of Barcelona, and whose passion and knowledge introduced me to topology.

And last but not least, I would like to thank my parents for supporting me everyday with their love and patience.

Chapter 1

Category theory

In short, category theory studies mathematical structures and its relations in an abstract way. Here we are going to see some basic concepts, but we recommend to read [2] to learn about its logical foundations and [3] for its first steps.

1.1 Basic terminology

Definition 1.1. *A category $\mathfrak{C}$ consists of the following:*

1) *a class $Ob(\mathfrak{C})$, whose elements will be called "objects of the category";*

2) *for every pair A, B of objects, a set $\mathfrak{C}(A, B)$, whose elements will be called "morphisms" or "arrows" from A to B;*

3) *for every triple A, B, C of objects, a composition law*

$$\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \longrightarrow \mathfrak{C}(A, C);$$

   the composite of the pair (f, g) will be written $g \circ f$ or just gf;

4) *for every object A, a morphism $1_A \in \mathfrak{C}(A, A)$ called the identity on A.*

These data are subject to the following axioms:

1) *Associativity axiom: given morphisms $f \in \mathfrak{C}(A, B)$, $g \in \mathfrak{C}(B, C)$, $h \in \mathfrak{C}(C, D)$ the following equality holds:*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2) *Identity axiom: given morphisms $f \in \mathfrak{C}(A, B), g \in \mathfrak{C}(B, C)$ the following equalities hold:*

$$1_B \circ f = f, \quad g \circ 1_B = g.$$

Examples 1.2. Here is a list of some obvious examples of categories and the corresponding notation, when it is classical:

1. Sets and functions: **Set**.
2. Topological spaces and continuous maps: **Top**.
3. Groups and groups homomorphisms: **Gr**.
4. Abelian groups and groups homomorphisms: **Ab**.
5. Commutative rings with unit and ring homomorphisms: **Rng**.
6. If R is a commutative ring, R-modules and R-lineal maps: $\mathbf{Mod}_R$.

Definition 1.3. *A functor F from a category $\mathfrak{A}$ to a category $\mathfrak{B}$ consists of the following:*

1) a function

$$Ob(\mathfrak{A}) \longrightarrow Ob(\mathfrak{B})$$

between the classes of objects of $\mathfrak{A}$ and $\mathfrak{B}$; the image of $A \in Ob(\mathfrak{A})$ is written $F(A)$ or just FA;

2) for every pair of objects A, A' of $\mathfrak{A}$, a map

$$\mathfrak{A}(A, A') \longrightarrow \mathfrak{B}(FA, FA');$$

the image of $f \in \mathfrak{A}(A, A')$ is written $F(f)$ or just Ff.

These data are subject to the following axioms:

1) for every pair of morphisms $f \in \mathfrak{A}(A, A'), g \in \mathfrak{A}(A', A'')$

$$F(g \circ f) = F(g) \circ F(f);$$

2) for every object $A \in \mathfrak{A}$

$$F(1_A) = 1_{FA}.$$

Remark 1.4. Given two functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{C}$, a pointwise composition immediately produces a new functor $G \circ F : \mathfrak{A} \to \mathfrak{C}$. This composition law is obviously associative.

Examples 1.5. Some examples of functors include:

1. For any category $\mathfrak{C}$, the identity functor $1_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ that maps each object and morphism of $\mathfrak{C}$ to itself.

2. The "forgetful functor" $U : \mathbf{Ab} \to \mathbf{Set}$ that maps a group $(G, +)$ to the underlying set G and a group homomorphism f to the corresponding map f.

3. Let R be a commutative ring. Tensoring with R produces a functor from the category $\mathbf{Ab}$ of abelian groups to $\mathbf{Mod}_R$:

$$- \otimes R : \mathbf{Ab} \longrightarrow \mathbf{Mod}_R.$$

   An abelian group A is mapped to the group $A \otimes_{\mathbb{Z}} R$ provided with the scalar multiplication induced by the formula

$$(a \otimes r)r' = a \otimes (rr')$$

   A group homomorphism $f : A \to B$ is mapped to the R-linear mapping $f \otimes \mathrm{id}_R$.

4. We obtain a functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ from the category of sets to itself by sending a set A to its power set $\mathcal{P}(A)$ and a map $f : A \to B$ to the "direct image map" from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

Definition 1.6. *A morphism $f : A \to B$ in a category $\mathfrak{C}$ is called an isomorphism when there exists a morphism $g : B \to A$ of $\mathfrak{C}$ which satisfies the relations*

$$f \circ g = 1_B, \quad g \circ f = 1_A.$$

Proposition 1.7. *Every functor preserves isomorphisms.*

Proof. Let $F : \mathfrak{A} \to \mathfrak{B}$ be a functor and $f : A \to B$ an isomorphism in $\mathfrak{A}$. Let g be a morphism such that $f \circ g = 1_B$, $g \circ f = 1_A$. Then

$$1_{FA} = F(1_A) = F(g \circ f) = F(g) \circ F(f),$$
$$1_{FB} = F(1_B) = F(f \circ g) = F(f) \circ F(g).$$

□

Definition 1.8. *Consider two functors $F, G : \mathfrak{A} \rightrightarrows \mathfrak{B}$ from a category $\mathfrak{A}$ to a category $\mathfrak{B}$. A natural transformation $\alpha : F \Rightarrow G$ from F to G is a class of morphisms $(\alpha_A : FA \to GA)_{A \in \mathfrak{A}}$ of $\mathfrak{B}$ indexed by the objects of $\mathfrak{A}$ and such that for every morphism $f : A \to A'$ in $\mathfrak{A}$, the following diagram commutes*

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow[\alpha_{A'}]{} & GA', \end{array}$$

that is $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$.
A natural isomorphism is a natural transformation $\alpha : F \Rightarrow G$ *in which every arrow* α_A *is an isomorphism. In this case, the natural isomorphism may be depicted as* $\alpha : F \cong G$.

Definition 1.9. *An equivalence of categories consists of functors* $F : \mathfrak{A} \leftrightarrows \mathfrak{B} : G$ *together with natural isomorphisms* $\alpha : 1_{\mathfrak{A}} \cong GF$, $\beta : FG \cong 1_{\mathfrak{B}}$. *Two categories* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are equivalent, written* $\mathfrak{A} \cong \mathfrak{B}$, *if there exists and equivalence between them.*

Proposition 1.10. *Consider two functors* $F, G : \mathfrak{A} \rightrightarrows \mathfrak{B}$ *such that* $\alpha : F \cong G$. *If* F *is an equivalence of categories, then so is* G.

Proof. Using the notations of 1.8 we have a commutative diagram

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow[\alpha_{A'}]{} & GA', \end{array}$$

where the α_A are isomorphisms. Denote H the functor such that $\beta : 1_{\mathfrak{A}} \cong HF$ and $\gamma : FH \cong 1_{\mathfrak{B}}$. Composing the above diagram with H and considering the one given by β gives the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\beta_A} & HFA & \xrightarrow{H\alpha_A} & HGA \\ {\scriptstyle 1_{\mathfrak{A}}f = f}\downarrow & & {\scriptstyle HFf}\downarrow & & \downarrow{\scriptstyle HGf} \\ A' & \xrightarrow[\beta_{A'}]{} & HFA' & \xrightarrow[H\alpha_{A'}]{} & HGA', \end{array}$$

with β_A, $\beta_{A'}$, $H\alpha_A$ and $H\alpha_{A'}$ isomorphisms. So, by the outer square of the diagram we see $\beta' : 1_{\mathfrak{A}} \cong HG$. We show in a similar way that $\gamma' : GH \cong 1_{\mathfrak{B}}$. $\square$

Definition 1.11. *Consider a functor* $F : \mathfrak{A} \to \mathfrak{B}$ *and for every pair of objects* $A, A' \in Ob(\mathfrak{A})$, *the map*

$$\mathfrak{A}(A, A') \longrightarrow \mathfrak{B}(FA, FA'), \quad f \mapsto Ff.$$

1) *The functor* F *is faithful when the above-mentioned maps are injective for all* A, A'.

2) *The functor* F *is full when the above-mentioned maps are surjective for all* A, A'.

3) *The functor* F *is essentially surjective (on objects) when each object* $B \in Ob(\mathfrak{B})$ *is isomorphic to an object of the form* FA, $A \in Ob(\mathfrak{A})$.

Theorem 1.12. *Assuming the axiom of choice, given a functor $F : \mathfrak{A} \to \mathfrak{B}$ the following conditions are equivalent:*

1) *F is full, faithful and essentially surjective.*

2) *F is an equivalence of categories, i.e., there exist a functor $G : \mathfrak{B} \to \mathfrak{A}$ and two natural isomorphisms $\alpha : 1_{\mathfrak{A}} \cong GF$, $\beta : FG \cong 1_{\mathfrak{B}}$.*

Proof. See [3, Theorem 1.5.9]. □

Proposition 1.13. *Consider two functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{C}$:*

1) *If F and G are equivalences of cateogries, then so is GF.*

2) *If G and GF are equivalences of categories, then so is F.*

3) *If F and GF are equivalences of categories, then so is G.*

Proof.

1) Consider for every pair of objects $A, A' \in Ob(\mathfrak{A})$ the map

$$\begin{array}{ccccc} \mathfrak{A}(A, A') & \longrightarrow & \mathfrak{B}(FA, FA') & \longrightarrow & \mathfrak{C}(GFA, GFA') \\ f & \longmapsto & Ff & \longmapsto & GFf \end{array}$$

It is injective and surjective because it is the composition of two injective and surjective maps, so GF is full and faithful. Now for each object $C \in Ob(\mathfrak{C})$, we have C is isomorphic to an object of the form GB with $B \in Ob(\mathfrak{B})$, and B is isomorphic of an object of the form FA with $A \in Ob(\mathfrak{A})$, so by Proposition 1.7 C is isomorphic to GFA, i.e. GF is essentially surjective.

2) For every pair of objects $A, A' \in Ob(\mathfrak{A})$ consider the maps

$$\mathfrak{A}(A, A') \longrightarrow \mathfrak{B}(FA, FA'), \quad f \mapsto Ff$$

$$\mathfrak{A}(A, A') \longrightarrow \mathfrak{C}(GFA, GFA'), \quad f \mapsto GFf$$

Fix A and A' and consider $f_1, f_2 \in \mathfrak{A}(A, A')$ such that $Ff_1 = Ff_2$. Then $GFf_1 = GFf_2$. Since GF is an equivalence of categories we have $f_1 = f_2$, so the first map is injective. Now let $g \in \mathfrak{B}(FA, FA')$. We have $Gg \in \mathfrak{C}(GFA, GFA')$. Again since GF is an equivalence of categories we know there exists $f \in \mathfrak{A}(A, A')$ such that $Gg = GFf$, and since G is an equivalence of categories we have $g = Ff$. Thus the first map is surjective. Finally, if $B \in Ob(\mathfrak{B})$, then there exists $A \in Ob(\mathfrak{A})$ such that $GB \in Ob(\mathfrak{C})$ is isomorphic to GFA. Denote as $\phi : GB \to GFA$ such isomorphism. Since G is an equivalence of categories, $G^{-1}\phi : B \to FA$ is an isomorphism, and so F is essentially surjective.

3) The fully faithfulness of G is shown in the same way as in 2). For essential surjectivity, consider $C \in Ob(\mathfrak{C})$. We know there exists $A \in Ob(\mathfrak{A})$ such that $C \cong GFA$, and $B \in Ob(\mathfrak{B})$ such that $B \cong FA$. Thus $GB \cong GFA \cong C$.

□

Corollary 1.14. *Consider the following commutative diagram of functors:*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha} & \mathfrak{B} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathfrak{C} & \xrightarrow[\mu]{} & \mathfrak{D} \end{array}$$

with α, γ and μ equivalences of categories. Then β is an equivalence of categories.

Proof. By Proposition 1.13.1 the functor $\mu\gamma$ is an equivalence of categories. If $\beta\alpha = \mu\gamma$, then by 1.13.3 the functor β is an equivalence of categories. □

1.2 Coproducts

Definition 1.15. *Let I be a set and $(C_i)_{i\in I}$ a family of objects in a given category $\mathfrak{C}$. A coproduct of that family is a pair $(P, (s_i)_{i\in I})$ where*

1) P is an object of $\mathfrak{C}$,

2) for every $i \in I$, $s_i : C_i \to P$ is a morphism of $\mathfrak{C}$,

and this pair is such that for every other pair $(Q, (t_i)_{i\in I})$ where

1) Q is an object of $\mathfrak{C}$,

2) for every $i \in I$, $t_i : C_i \to Q$ is a morphism of $\mathfrak{C}$,

there exists a unique morphism $u : P \to Q$ such that for every index i, $t_i = u \circ s_i$.

$$\begin{array}{ccc} P & & \\ {\scriptstyle s_i}\uparrow & \overset{u}{\searrow} & \\ C_i & \xrightarrow[t_i]{} & Q. \end{array}$$

The following proposition guarantees that the coproduct is well defined:

Proposition 1.16. *When the coproduct of a family of objects exists in a category, it is unique up to an isomorphism.*

Proof. See [2, Proposition 2.2.2]. □

Example 1.17. In the category of groups **Gr** the coproduct of a family of groups $(G_i)_{i\in I}$ exists and it is the free product $\prod_{i\in I} *G_i$.

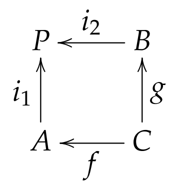
Proof. See [5, Theorem 4.2] . □

1.3 Pushouts

Definition 1.18. *Consider two morphisms $f : C \to A$, $g : C \to B$ in a category $\mathfrak{C}$. A pushout of (f, g) is a triple (P, i_1, i_2) where*

1) P is an object of $\mathfrak{C}$,

2) $i_1 : A \to P$, $i_2 : B \to P$ are morphisms of $\mathfrak{C}$ such that $i_1 \circ f = i_2 \circ g$,

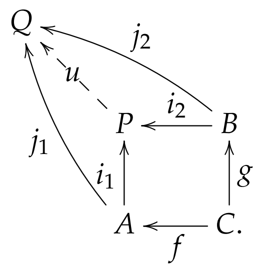


and for every other triple (Q, j_1, j_2) where

1) Q is an object of $\mathfrak{C}$,

2) $j_1 : A \to Q$, $j_2 : B \to Q$ are morphisms of $\mathfrak{C}$ such that $j_1 \circ f = j_2 \circ g$,

there exists a unique morphism $u : P \to Q$ such that $j_1 = u \circ i_1$ and $j_2 = u \circ i_2$



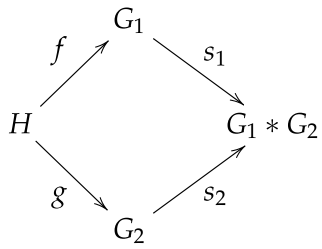
As with the coproduct we have:

Proposition 1.19. *When the pushout exists in a category, it is unique up to an isomorphism.*

Proof. See [2, Metatheorem 1.10.2, Proposition 2.5.2 and page 52] □

Example 1.20. In the category of groups **Gr**, given two morphisms $f : H \to G_1$, $g : H \to G_2$ the pushout of (f, g) exists and it is the free product with amalgamation $G_1 *_H G_2$.

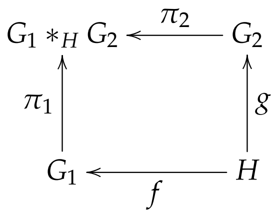
Proof. Consider the coproduct of G_1 and G_2, $(G_1 * G_2, s_1, s_2)$. Let $\phi = s_1 \circ f, \psi = s_2 \circ g$



Recall that the free product with amalgamation is the quotient group

$$G_1 *_H G_2 = {G_1 * G_2}/{N} ,$$

where $N = \{\phi(h)\psi(h)^{-1};\, h \in H\}$. Consider the projection maps $\pi : G_1 * G_2 \to G_1 *_H G_2$, $\pi_1 = \pi \circ s_1 : G_1 \to G_1 *_H G_2$, $\pi_2 = \pi \circ s_2 : G_2 \to G_1 *_H G_2$. For all $h \in H$ we have, by definition, $\pi_1 \circ f(h) = \pi \circ s_1 \circ f(h) = \pi \circ \phi(h) = \pi \circ \psi(h) = \pi \circ s_2 \circ g(h) = \pi_2 \circ g(h)$, i.e. the following diagram

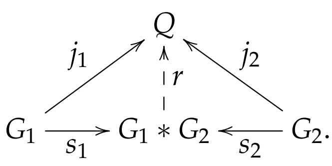


commutes. Now let (Q, j_1, j_2) be another triple such that Q is a group and

$$j_1 \circ f = j_2 \circ g. \tag{1.1}$$

Given that $G_1 * G_2$ is the coproduct of G_1 and G_2 we know that there exists a unique morphism $r : G_1 * G_2 \to Q$ such that

$$j_1 = r \circ s_1, \quad j_2 = r \circ s_2 \tag{1.2}$$

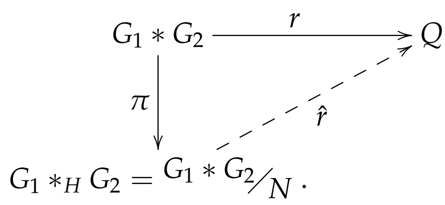


Using (1.1) and (1.2) we get that, for every $h \in H$,

$$r \circ s_1 \circ f(h) = r \circ s_2 \circ g(h) \Longrightarrow r \circ \phi(h) = r \circ \psi(h)$$

$$\Longrightarrow r(\phi(h)\psi(h)^{-1}) = r(1) = 1$$

so $N \subset Ker(r)$ and thus by the fundamental homomorphism theorem there exists a unique $\hat{r} : G_1 *_H G_2 \to Q$ such that $r = \hat{r} \circ \pi$



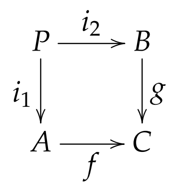
We have now $j_1 = r \circ s_1 = \hat{r} \circ \pi \circ s_1 = \hat{r} \circ \pi_1$ and $j_2 = r \circ s_2 = \hat{r} \circ \pi \circ s_2 = \hat{r} \circ \pi_2$. So $u = \hat{r}$ is the unique morphism we were looking for. □

1.4 Pullbacks

Definition 1.21. *Consider two morphisms $f : A \to C$, $g : B \to C$ in a category $\mathfrak{C}$. A pushout of (f, g) is a triple (P, i_1, i_2) where*

1) P is an object of $\mathfrak{C}$,

2) $i_1 : P \to A$, $i_2 : P \to B$ are morphisms of $\mathfrak{C}$ such that $f \circ i_1 = g \circ i_2$,

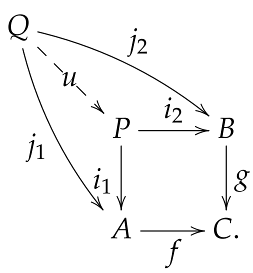


and for every other triple (Q, j_1, j_2) where

1) Q is an object of $\mathfrak{C}$,

2) $j_1 : Q \to A$, $j_2 : Q \to B$ are morphisms of $\mathfrak{C}$ such that $f \circ j_1 = g \circ j_2$,

there exists a unique morphism $u : Q \to P$ such that $j_1 = i_1 \circ u$ and $j_2 = i_2 \circ u$



Remark 1.22. Abusing notation, we denote the pullback as $A \times_C B$.

Again we have:

Proposition 1.23. *When the pullback exists in a category, it is unique up to an isomorphism.*

Proof. See [2, Proposition 2.5.2] □

Examples 1.24.

1. In the category **Set** the pullback given by $f : A \to C$, $g : B \to C$ is

$$A \times_C B = \{(a,b); a \in A, b \in B, f(a) = g(b)\}, \quad i_1(a,b) = a, \quad i_2(a,b) = b.$$

   Obviously $A \times_C B$ is an object of **Set** and $f \circ i_1 = g \circ i_2$. If (Q, j_1, j_2) is another triple as above then we can define for all $q \in Q$ the morphism $u : Q \to A \times_C B$, $u(q) = (j_1(q), j_2(q))$, which is trivially well defined and $j_1 = i_1 \circ u$, $j_2 = i_2 \circ u$. If we have another morphism $u' : Q \to A \times_C B$, $u'(q) = (u'_1(q), u'_2(q))$ verifying the condition, then since $j_1(q) = i_1 \circ u'(q) = u'_1(q)$ and $j_2(q) = i_2 \circ u'(q) = u'_2(q)$ for every $q \in Q$ we get $u = u'$.

2. We can define the pullback in **Top** in a similar way as in **Set**.

1.5 Strict comma category

Definition 1.25. *Consider two functors $F : \mathfrak{A} \to \mathfrak{C}$ and $G : \mathfrak{B} \to \mathfrak{C}$. The "strict comma category" $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}$ is defined in the following way:*

1) *The objects of $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}$ are the triples (A, B, f) where $A \in Ob(\mathfrak{A})$, $B \in Ob(\mathfrak{B})$ and $f : FA \to GB$ is an isomorphism in $\mathfrak{C}$.*

2) *A morphism of $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}$ from (A, B, f) to (A', B', f') is a pair (a, b), where $a : A \to A'$ is a morphism of $\mathfrak{A}$, $b : B \to B'$ is a morphism of $\mathfrak{B}$, and $f' \circ F(a) = G(b) \circ f$, i.e., the following diagram commutes*

$$\begin{array}{ccc} FA & \xrightarrow{f} & GB \\ {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Gb} \\ FA' & \xrightarrow[f']{} & GB'. \end{array}$$

3) The composition law in $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}$ is that induced by the composition laws of $\mathfrak{A}$ and $\mathfrak{B}$, thus

$$(a', b') \circ (a, b) = (a' \circ a, b' \circ b).$$

Theorem 1.26. *Consider two strict comma categories $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}$ and $\mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}'$ defined by the pair of functors (F, G) and (F', G') respectively. Suppose that there exists $\Phi_1 : \mathfrak{A} \to \mathfrak{A}'$, $\Phi_2 : \mathfrak{B} \to \mathfrak{B}'$ and $\varphi : \mathfrak{C} \to \mathfrak{C}'$ three equivalences of categories such that the following diagram commutes*

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{\Phi_1} & \mathfrak{A}' & & \\ & \searrow^{F} & & \searrow^{F'} & \\ & & \mathfrak{C} & \xrightarrow{\varphi} & \mathfrak{C}' \\ & \nearrow^{G} & & \nearrow_{G'} & \\ \mathfrak{B} & \xrightarrow[\Phi_2]{} & \mathfrak{B}' & & \end{array} .$$

Then $\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B} \cong \mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}'$.

Proof. Consider the functor $\Phi : \mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B} \to \mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}'$, defined by $\Phi(A, B, f) = (\Phi_1(A), \Phi_2(B), \varphi(f))$. We are going to check that this functor is well defined. Obviously $\Phi_1(A) \in Ob(\mathfrak{A}')$ and $\Phi_2(B) \in Ob(\mathfrak{B}')$. By hypotesis

$$\varphi(FA) = F'\Phi_1(A) \text{ and } \varphi(GB) = G'\Phi_2(B) \tag{1.3}$$

so $\varphi(f) : F'\Phi_1(A) \to G'\Phi_2(B)$ is well defined and it is an isomorphism of $\mathfrak{C}'$ by Proposition 1.7, therefore $(\Phi_1(A), \Phi_2(B), \varphi(f)) \in Ob(\mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}')$.
Now for any objects $(A_1, B_1, f_1), (A_2, B_2, f_2) \in Ob(\mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B})$ consider the map

$$\begin{array}{ccc} \mathfrak{A} \times_{\mathfrak{C}} \mathfrak{B}((A_1, B_1, f_1), (A_2, B_2, f_2)) & \xrightarrow{\Phi} & \mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}'(\Phi(A_1, B_1, f_1), \Phi(A_2, B_2, f_2)) \\ (a, b) & \mapsto & \Phi((a, b)) = (\Phi_1(a), \Phi_2(b)) \end{array}$$

Recall that $f_2 \circ F(a) = G(b) \circ f_1$, so by composing with φ we get $\varphi(f_2) \circ \varphi(Fa) = \varphi(f_2 \circ F(a)) = \varphi(G(b) \circ f_1) = \varphi(G(b)) \circ \varphi(f_1)$, i.e. the following diagram

$$\begin{array}{ccc} \varphi(FA_1) & \xrightarrow{\varphi(f_1)} & \varphi(GB_1) \\ {\scriptstyle \varphi(Fa)}\downarrow & & \downarrow{\scriptstyle \varphi(Gb)} \\ \varphi(FA_2) & \xrightarrow[\varphi(f_2)]{} & \varphi(GB_2) \end{array} \tag{1.4}$$

commutes. Again by hypotesis we have $F'\Phi_1(a) = \varphi(Fa)$ and $G'\Phi_2(b) = \varphi(Gb)$, so together with (1.3) and (1.4) we get that the diagram

$$\begin{array}{ccc} F'\Phi_1(A_1) & \xrightarrow{\varphi(f_1)} & G'\Phi_2(B_1) \\ {\scriptstyle F'\Phi_1(a)}\downarrow & & \downarrow{\scriptstyle G'\Phi_2(b)} \\ F'\Phi_1(A_2) & \xrightarrow[\varphi(f_2)]{} & G'\Phi_2(B_2) \end{array}$$

is well defined and commutes, thus $(\Phi_1(a), \Phi_2(b)) \in \mathfrak{A}' \times_{\mathfrak{C}'} \mathfrak{B}'(\Phi(A_1, B_1, f_1), \Phi(A_2, B_2, f_2))$. Checking the rest of the axioms is trivial as Φ_1 and Φ_2 are both functors.

Since Φ_1 and Φ_2 are equivalences of categories we can clearly see that Φ is full and faithfull. Essential surjectivity is obvious since Φ_1, Φ_2 and φ are equivalences of categories. □

1.6 Initial objects

Definition 1.27. *An object **0** of a category is initial when every object C is provided with exactly one arrow from **0** to C.*

Examples 1.28.

1. In the category **Set**, the empty set is the initial object. The same holds in the category **Top**.
2. In the categories **Gr** and **Ab**, (0) is the initial object.
3. In the category **Rng**, $\mathbb{Z}$ is the initial object.

Proposition 1.29. *An initial object **0** is unique up to isomorphism.*

Proof. Let $\mathbf{0}$ and $\mathbf{0'}$ be two initial objects in a category. Then there exists exactly one arrow $\varphi : \mathbf{0} \to \mathbf{0'}$ and another $\psi : \mathbf{0'} \to \mathbf{0}$. Consider the compositions $\psi \circ \varphi : \mathbf{0} \to \mathbf{0}$, $\varphi \circ \psi : \mathbf{0'} \to \mathbf{0'}$. Since $\mathbf{0}$ is an initial object, the identity morphism $1_{\mathbf{0}} : \mathbf{0} \to \mathbf{0}$ is the only one that maps from $\mathbf{0}$ to $\mathbf{0}$, so $1_{\mathbf{0}} = \psi \circ \varphi$. We show in a similar way that $1_{\mathbf{0'}} = \varphi \circ \psi$. □

Chapter 2

Groups actions

2.1 Groups acting on sets

Definition 2.1. *Let S be a set and G a group. A left group action of G on the set S is a map from $G \times S$ into S, the image of (g, s) being denoted by $g \cdot s$, such that*

1) $e \cdot s = s$ for e the identity of G and for all $s \in S$,

2) $(g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$ for all $g_1, g_2 \in G$ and for all $s \in S$.

In this situation, we also say that G acts on S or that S is a G-set. We can define a right group action of G on the set S in a similar way:

Definition 2.2. *Let S be a set and G a group. A right group action of G on the set S is a map from $S \times G$ into S, the image of (s, g) being denoted by $s \cdot g$, such that*

1) $s \cdot e = s$ for e the identity of G and for all $s \in S$,

2) $s \cdot (g_1 g_2) = (s \cdot g_1) \cdot g_2$ for all $g_1, g_2 \in G$ and for all $s \in S$.

Example 2.3. Let G be an arbitrary group and let S be an arbitrary set. Let G act on S by letting $g \cdot s = s$ for all $g \in G$ and $s \in S$. This is known as the trivial action of G on S.

Definition 2.4. *A group G acts transitively on a set S if $S = G \cdot s$ for some $s \in S$.*

2.2 The category of G-sets

Definition 2.5. *Let G be a group and S_1, S_2 two G-sets. A morphism of G-sets from S_1 to S_2 is a map $f : S_1 \to S_2$ such that $f(g \cdot s) = g \cdot f(s)$, for any $g \in G$ and $s \in S_1$.*

Proposition 2.6. *Let G be a group and S_1, S_2 and S_3 three G-sets. Let $f : S_1 \to S_2$, $f' : S_2 \to S_3$ be two morphisms of G-sets. Then the composition map $f' \circ f$ is a morphism of G-sets.*

Proof. Let $s \in S_1$ and $g \in G$, then, since f and f' are morphisms of G-sets,

$$(f' \circ f)(g \cdot s) = f'(f(g \cdot s)) = f'(g \cdot f(s)) = g \cdot f'(f(s)) = g \cdot (f' \circ f)(s).$$

□

Given a group G, the last proposition allows us to define what it is known as the category of G-sets, denoted by G**-Sets**, whose objects are G-sets and whose morphisms are morphisms of G-sets.

Let G_1 and G_2 be two groups and $f : G_2 \to G_1$ be a group homomorphism. For every $S \in G_1$**-Sets** we denote $f^*(S)$ the G_2-set we get by considering for every $s \in S$ and $g \in G_2$ the (left) action

$$g \cdot s = f(g) \cdot s.$$

We therefore obtain a functor $f^* : G_1$**-Sets** $\to G_2$**-Sets**.

Proposition 2.7. *The functor f^* is an equivalence of categories if and only if f is an isomorphism of groups.*

Proof. If f is an isomorphism, then we can consider $(f^{-1})^*$, which is the inverse of f^*. We are going to see the converse (recall by Theorem 1.12 that, since f^* is an equivalence of categories, it is full, faithful and essentially surjective):

1) *f^* essentially surjective $\Rightarrow f$ injective.* If $f(g) = e$, then the element g acts trivially over each G_2-set of the form $f^*(S)$, thus on every G_2-set because f^* is essentially surjective. In particular, it acts trivially over G_2 by left translation, so $g = e$.

2) *f^* is full and faithfull $\Rightarrow f$ surjective.* For any two objects $S, S' \in G_1$**-Sets**, every morphism of G_2-sets from S to S' is a morphism of G_1-sets. Take $S = \{s\}$ with the trivial action of G_1 and $S' = G_1 / f(G_2)$ with a G_1 action given by $g_1 \cdot [s'] = [g_1 \cdot s']$ for $g_1 \in G_1$, $[s'] \in S'$. The map $\varphi : S \to S'$ such that $\varphi(S)$ is the class of e the identity element is a morphism of G_2-sets: if $g_2 \in G_2$,

$$\varphi(g_2 \cdot s) = \varphi(f(g_2) \cdot s) = \varphi(s) = [e] = [e] \cdot [e] = [f(g_2)] \cdot \varphi(s).$$

Then φ is a morphism of G_1-sets and e is left fixed by G_1. Since G_1 acts transitively over $G_1 / f(G_2)$ we get

$$G_1 / f(G_2) = \{[e]\} \Longrightarrow f(G_2) = G_1.$$

□

Chapter 3

Homotopy theory

In this chapter we will recall the basics of homotopy theory. From now on we will consider X to be a topological space and $I = [0, 1]$.

3.1 Homotopy of spaces

Definition 3.1. *A path in X is a continuous map $f : I \to X$.*

Definition 3.2. *A homotopy of paths in X is a family $f_t : I \to X$, $0 \leq t \leq 1$, such that*

1) the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t,

2) the associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

3.2 The fundamental group

Definition 3.3. *When two paths f_0 and f_1 are connected in the above way by a homotopy f_t, they are said to be homotopic. The notation for this is $f_0 \simeq f_1$.*

Proposition 3.4. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

Proof. See [1, Proposition 1.2]. □

Definition 3.5. *The equivalence class of a path f under the equivalence relation of homotopy will be denoted $[f]$ and called the homotopy class of f.*

Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, there is a composition or product path $f * g$ that traverses first f and then g, defined by the formula

$$f * g = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

This product operation respects homotopy classes since $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies f_t and g_t, and if $f_0(1) = g_0(0)$ so that $f_0 * g_0$ is defined, then $f_t * g_t$ is defined and provides a homotopy $f_0 * g_0 \simeq f_1 * g_1$.

Definition 3.6. *Suppose we restrict attention to paths $f : I \to X$ with the same starting and ending point $f(0) = f(1) = x_0 \in X$. Such paths are called loops, and the common starting and ending point x_0 is referred to as the basepoint. The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$ and it is called the fundamental group of X at the basepoint x_0.*

Proposition 3.7. *$\pi_1(X, x_0)$ is a group with respect to the product $[f]\,[g] = [f * g]$.*

Proof. See [1, Proposition 1.3]. □

Definition 3.8. *We say that X is path-connnected if any two points $x_0, x_1 \in X$ may be joined by a path f.*

Proposition 3.9. *If X is path-connected, then for any two points $x_0, x_1 \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. See [1, Proposition 1.5]. □

Thus if X is path-connected, the group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of the basepoint x_0. It is usual then to denote the fundamental group of a path-connected space X without taking any point $\pi_1(X)$. We can then state the notion of simply connectedness:

Definition 3.10. *We say that X is simply connected if it is path-connected and its fundamental group is trivial.*

Chapter 4

Covering spaces

4.1 Definition and basic properties

Definition 4.1. *A covering space of X is a topological space Y together with a continuous map $p : Y \to X$ such that for every $x \in X$ there exists a neighbourhood U of x in X, a discrete space F (i.e. a space with the discrete topology) and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ such that the diagram*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow^{p} & \downarrow \pi_1 \\ & & U \end{array}$$

commutes (where π_1 denotes the first projection).

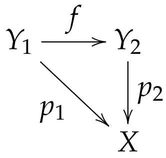
Remark 4.2. In the literature the terms 'covering' and 'cover' are also used for what we call a covering space. Abusing notation, when we refer to a covering space (Y, p) we will just write Y.

Example 4.3. The exponential function $p : \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$ is a covering with discrete space $F = \mathbb{Z}$. For each $t \in \mathbb{R}$ and $p(t) = z$ we have a homeomorphism

$$p^{-1}(S^1 \backslash z) = \sqcup_{n \in Z}(t+n, t+n+1) \cong (t, t+1) \times \mathbb{Z}.$$

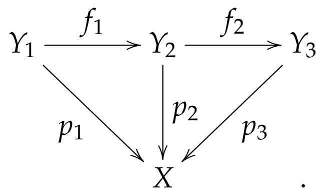
Remark 4.4. Some authors define a covering space as a topological space Y together with a continuous map $p : Y \to X$ subject to the following condition: each point in X has an open neighbourhood U for which $p^{-1}(U)$ decomposes as a disjoint union of open subsets U_i of Y such that the restriction of p to each U_i induces a homeomorphism of U_i with U. This definition and the one used above are equivalent, as it is shown in [7, Proposition 2.1.3]. Note that if $p : Y \to X$ is a cover, then p is surjective.

Definition 4.5. *A morphism between two covers $p_i : Y_i \to X$ ($i = 1, 2$) over X is given by a continuous map $f : Y_1 \to Y_2$ making the diagram*



commute. We say that f is an isomorphism of covers if f is a homeomorphism and its inverse is compatible with the above diagram.

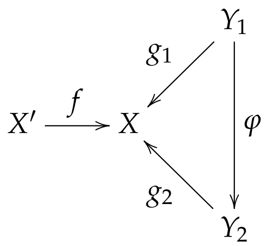
Proposition 4.6. *Consider three covers $p_i : Y_i \to X$, $i = 1, 2, 3$, and two morphisms between them $f_j : Y_j \to Y_{j+1}$, $j = 1, 2$. Then $f_2 \circ f_1$ is a morphism of covers.*



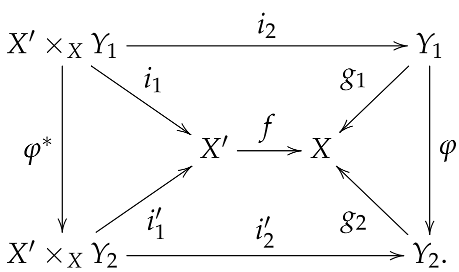
Proof. As $f_2 \circ f_1$ is a composition of two continuous maps, it is a continuous map. Now we have to see $p_1 = p_3 \circ (f_2 \circ f_1)$. By hypotesis $p_1 = p_2 \circ f_1$ and $p_2 = p_3 \circ f_2$, therefore $p_1 = p_2 \circ f_1 = (p_3 \circ f_1) \circ f_2 = p_3 \circ f_2 \circ f_1 = p_3 \circ (f_2 \circ f_1)$. □

Definition 4.7. *We say that Y is a trivial cover of X if it is isomorphic to a cover of the form $X \times F$, with F a discrete space. An isomorphism $\phi : Y \to X \times F$ is called a trivialisation of Y.*

Lemma 4.8. *Consider the following commutative diagram of continuous maps:*



then we get the commutative diagram:



(4.1)

If φ is a homeomorphism, then so is φ^* as well.

Proof. Consider the pullbacks $X' \times_X Y_1$ and $X' \times_X Y_2$:

$$\begin{array}{ccc} X' \times_X Y_1 & \xrightarrow{i_2} & Y_1 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ X' & \xrightarrow[f]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X' \times_X Y_2 & \xrightarrow{i'_2} & Y_2 \\ {\scriptstyle i'_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X' & \xrightarrow[f]{} & X. \end{array}$$

By hypotesis $g_1 = g_2 \circ \varphi$, so the following diagram

$$\begin{array}{ccc} X' \times_X Y_1 & \xrightarrow{\varphi \circ i_2} & Y_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X' & \xrightarrow[f]{} & X \end{array}$$

commutes. By definition of pullback (in **Top**) we have a (unique) continuous morphism $\varphi^* : X' \times_X Y_1 \to X' \times_X Y_2$, $\varphi^*(x', y_1) = (x', \varphi(y_1))$ which gives the commutative diagram (4.1).

Now if φ is a homeomorphism then $g_2 = g_1 \circ \varphi^{-1}$ and we can build the inverse of φ^* as above. It is trivial to check that φ^* is both injective and surjective. □

Lemma 4.9. *Let X and X' be two topological spaces, $f : X' \to X$ a continuous map and $p : Y \to X$ a cover. Then $q : Y' = X' \times_X Y \to X'$, $q(x', y) = x'$ is a cover of X'.*

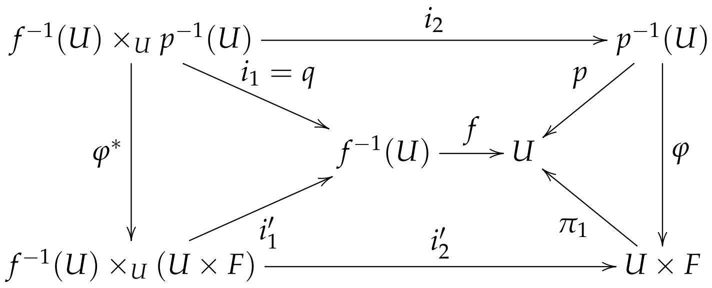
Proof. Consider for every $f(x') \in X$ with $x' \in X'$ the commutative diagram of the definition of covering space

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \pi_1} \\ & & U \end{array}$$

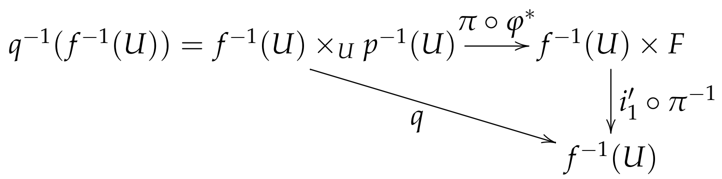
We have the commutative diagram

$$\begin{array}{ccccc} & & & & p^{-1}(U) \\ & & & {\scriptstyle p}\swarrow & \\ f^{-1}(U) & \xrightarrow{f} & U & & \downarrow{\scriptstyle \varphi} \\ & & & {\scriptstyle \pi_1}\nwarrow & \\ & & & & U \times F \end{array}$$

with $f^{-1}(U)$ a neighbourhood of x' (f is continuous). Apply last lemma to get the commutative diagram



with φ^* a homeomorphism. Consider the map $\pi : f^{-1}(U) \times_U (U \times F) \to f^{-1}(U) \times F$, $\pi(v, (f(v), d)) = (v, d)$. Clearly π is a homeomorphism, and so we get the commutative diagram



with $\pi \circ \varphi^*$ a homeomorphism. □

Definition 4.10. *Under the conditions of the above lemma, we say that* (X', f) *trivialises* Y *if the cover* Y' *is trivial.*

4.2 The category of covering spaces

With Proposition 4.6 we can define the category of covers of X, denoted by $\mathbf{Cov}(X)$, whose objects are covers of X and its morphisms are morphisms of covers.

4.3 Universal covering spaces

Definition 4.11. *Let* $f : Y \to X$ *be a map. The fiber of an element* $x_0 \in X$ *by* f*, denoted by* $f^{-1}(x_0)$*, is defined as*

$$f^{-1}(x_0) = \{y \in Y : f(y) = x_0\}.$$

Given a cover $p : Y \to X$ and a point $x_0 \in X$ we will denote the fiber of x_0 by the cover Y as $Y(x_0)$.

Definition 4.12. *A pointed cover of a pointed topological space (X, x_0) is a cover Y of X and a point $y_0 \in Y(x_0)$.*

Definition 4.13. *Let (X, x_0) be a pointed space. If the functor $Y \to Y(x_0)$ from **Cov(X)** to **Set** is representable by a pointed cover $(\widetilde{X}, \widetilde{x}_0)$, then we say that $(\widetilde{X}, \widetilde{x}_0)$ is a universal pointed cover of (X, x_0).*

Remark 4.14. Fixed a point $x_0 \in X$ we may define the category of covers of the pointed space (X, x_0), denoted by $\mathbf{Cov}(X, x_0)$, whose objects are pointed covers (Y, y_0) and its morphisms are morphisms of pointed covers (which are defined in the obvious way).
We can then define a universal pointed cover as the initial object of $\mathbf{Cov}(Y, y_0)$ for any object $Y \in \mathbf{Cov}(X)$ and any point $y_0 \in Y(x_0)$, i.e. as a pointed cover $(\widetilde{X}, \widetilde{x}_0)$ such that for every cover Y and for any $y_0 \in Y(x_0)$ there exists a unique morphism of pointed covers $f : \widetilde{X} \to Y$, $f(\widetilde{x}_0) = y_0$.

Definition 4.15. *We say that X is locally connected if each point has a basis of neighbourhoods consisting of connected open subsets.*

We are going to see a statement which will help us notice universal pointed covers if the space is locally connected:

Proposition 4.16. *Let X be locally connected. In order for a pointed cover $(\widetilde{X}, \widetilde{x}_0)$ of (X, x_0) to be universal, it is necessary and sufficient for $\widetilde{X}$ to be connected and to trivialise every cover of X.*

We need to introduce a few concepts before we prove last proposition:

Definition 4.17. *Consider two topological spaces X and Y and a continuous map $f : Y \to X$. We say that Y is Hausdorff over X if for every point $x_0 \in X$ and any two different points $y_1, y_2 \in Y(x_0)$ there exists two neighbourhoods V_1 of y_1 and V_2 of y_2 in Y such that $V_1 \cap V_2 = \emptyset$.*

Lemma 4.18. *Any cover $p : Y \to X$ is Hausdorff over X.*

Proof. Fix $x_0 \in X$ and consider the commutative diagram of the definition of cover. We claim that $U \times F$ is Hausdorff over U: if we have two different points $(u_1, f_1), (u_2, f_2) \in (U \times F)(x_0)$, then $u_1 = u_2 = x_0$ and thus $f_1 \neq f_2$. The sets of the form $U \times \{f\}$ with $f \in F$ are open in $U \times F$, so $U \times \{f_1\}$ is a neighbourhood of (u_1, f_1) and $U \times \{f_2\}$ is a neighbourhood of (u_2, f_2) such that $U \times \{f_1\} \cap U \times \{f_2\} = \emptyset$.

Now for any two different points $y_1, y_2 \in Y(x_0)$ consider two disjoint neighbourhoods V_1 of $\varphi(y_1)$ and V_2 of $\varphi(y_2)$. As φ is a homeomorphism $\varphi^{-1}(V_1 \cap V_2) = \varphi^{-1}(V_1) \cap \varphi^{-1}(V_2)$, and so we get two disjoint neighbourhoods $\varphi^{-1}(V_1)$ of y_1 and $\varphi^{-1}(V_2)$ of y_2. □

Proposition 4.19. *Consider two covers $p : Y \to X$, $q : Y' \to X$ and two morphisms of covers $f, g : Y \to Y'$. The set $A = \{y \in Y : f(y) = g(y)\}$ is open and closed in Y. In particular, if Y is connected and if there exists $y \in Y$ such that $f(y) = g(y)$, then $f = g$.*

Proof.

1) *A is closed.* We are going to see that $Y \backslash A$ is open. Consider any point $y \in Y \backslash A$. Since Y' is Hausdorff over X by last lemma, there exists two disjoint neighbourhoods V_f of $f(y)$, and V_g of $g(y)$. We have $W = f^{-1}(V_f) \cap g^{-1}(V_g)$ is a neighbourhood of y and $W \subset Y \backslash A$: if $w \in W$ then $f(w) \in V_f$ and $g(w) \in V_g$, and as $V_f \cap V_g = \emptyset$ we get $f(w) \neq g(w)$, thus $w \in Y \backslash A$.

2) *A is open.* Let $y \in A$. By definition we have $q \circ f(y) = p(y) = q \circ g(y)$. The map q is locally injective (easy to check using Remark 4.4), so there exists a neighbourhood V of $f(y) = g(y)$ such that $q|_V$ is injective. We have $f^{-1}(V)$ is a neighbourhood of y, and $f^{-1}(V) \subset A$: if $\tilde{y} \in f^{-1}(V)$ then $q \circ f(\tilde{y}) = p(\tilde{y}) = q \circ g(\tilde{y})$, but as $\tilde{y} \in f^{-1}(V)$ we get $f(\tilde{y}) = g(\tilde{y})$, thus $\tilde{y} \in A$.

□

Corollary 4.20. *With the same notations, if X is connected and if there exists $x \in X$ such that $f|_{Y(x)} = g|_{Y(x)}$, then $f = g$.*

Proof. Each connected component of Y has a point of $Y(x)$. □

Lemma 4.21. *Let X be connected, Y, Y' be two trivial covers of X and $f : Y \to Y'$ a morphism. Then (Y, f) is a cover of Y', trivial over each connected component of Y'.*

Proof. We can suppose $Y' = X \times F$ and $Y = X \times G$, where F and G are discrete. Then f is of the form $(x, u) \mapsto (x, \alpha(x, u))$, where $\alpha : X \times G \to F$ is a continuous map. Since X is connected and F is discrete, $\alpha(x, u)$ does not depend of x, so we can write $\alpha(x, u) = \varphi(u)$, where $\varphi(u)$ is an application from G to F. For $t \in F$ we have $f^{-1}(X \times \{t\}) = X \times \varphi^{-1}(t)$, and this space is a trivial cover of $X \times \{t\}$. □

Corollary 4.22. *Let Y, Y' be two covers of a locally connected space X and $f : Y \to Y'$ a morphism of covers. Then (Y, f) is a cover of Y'.*

Proof. Cover X with connected open sets trivialising both Y and Y'. □

Definition 4.23. *Given a cover $p : Y \to X$, a continuous section (or just a section) of Y is a continuous map $\sigma : X \to Y$ such that $p \circ \sigma = 1_X$. We say that σ goes through a point $y \in Y$ if $\sigma(p(y)) = y$.*

Definition 4.24. *Given a cover $p : Y \to X$ and $x \in X$, the degree of Y in x, denoted by $deg_x(Y)$, is the cardinality of the fibre $Y(x)$. If $X \neq \emptyset$ and $deg_x(Y)$ does not depend on x (for example, if X is connected) then we will just write $deg(Y)$ or $deg_X(Y)$.*

Proposition 4.25. *Let X be connected and locally connected, and let Y be a cover of X. If it exists $x_0 \in X$ such that for all $y \in Y(x_0)$ there exists a continuous section, then Y is trivial.*

Proof. Let Γ be the set of the continuous sections of Y with the discrete topology. The map $\varepsilon : X \times \Gamma \to Y$ defined by $\varepsilon(x,s) = s(x)$ is a morphism of covers and $(X \times \Gamma, \varepsilon)$ is a cover of Y by 4.22. We are going to see that ε is an isomorphism. We just need to show that for every connected component V of Y we have $\deg_V(X \times \Gamma, \varepsilon) = 1$. Let V be a connected component of Y, which is a cover of X, its projection over X is open and closed so it is equal to X. In particular $V \cap Y(x)$ is non empty: let $v \in V \cap Y(x)$, we have $\deg_v(X \times \Gamma, \varepsilon) = \deg_V(X \times \Gamma, \varepsilon) > 0$. If s and s' are two continuous sections of Y going through v, they coincide by Corollary 4.20, so $\deg_V(X \times \Gamma, \varepsilon) = 1$. □

Proof (of Proposition 4.16).

1) *If $(\widetilde{X}, \widetilde{x_0})$ is universal, then $\widetilde{X}$ is connected.* Let F be a closed open set of $\widetilde{X}$ containing $\widetilde{x_0}$. Define $f,g : \widetilde{X} \to X \times \{0,1\}$ by $f(\widetilde{x}) = (p(\widetilde{x}),0)$ for all $\widetilde{x} \in \widetilde{X}$ and $g(\widetilde{x}) = (p(\widetilde{x}),1)$ for all $\widetilde{x} \notin F$ and $g(\widetilde{x}) = (p(\widetilde{x}),0)$ for all $\widetilde{x} \in F$. We have $f(\widetilde{x_0}) = g(\widetilde{x_0})$ from which we conclude that $f = g$ by uniqueness of the universal property (see Remark 4.14). So $F = \widetilde{X}$ and thus $\widetilde{X}$ is connected.

2) *If $(\widetilde{X}, \widetilde{x_0})$ is universal, then $\widetilde{X}$ trivializes every cover of X.* Let Y be a cover of X. We are going to show that $\widetilde{X} \times_X Y$ is trivial. By Proposition 4.25 it is enough to see that for every $y \in Y(x_0)$ there exists a continuous section $\widetilde{X} \to \widetilde{X} \times_X Y$ going through $(\widetilde{x_0}, y)$. From the universal property of $\widetilde{X}$ there exists a morphism of covers $f : \widetilde{X} \to Y$ such that $f(\widetilde{x_0}) = y$, so $\widetilde{x} \mapsto (\widetilde{x}, f(\widetilde{x}))$ is the section we were looking for.

3) *Converse.* Let Y be a cover of X and $y \in Y(x_0)$. Since $\widetilde{X}$ trivialise Y, for $(\widetilde{x_0}, y)$ there exists a continuous section $\widetilde{X} \to \widetilde{X} \times_X Y$ going through $(\widetilde{x_0}, y)$. This section is unique as $\widetilde{X}$ is connected. This section corresponds to a unique morphism of covers $f : \widetilde{X} \to Y$ such that $f(\widetilde{x_0}) = y$.

□

Remark 4.26. The hypothesis 'X be locally connected' was only used at the necessity part of the proof. The sufficiency part is true without this hypothesis.

Corollary 4.27. *Let $(\widetilde{X}, \widetilde{x_0})$ be a universal pointed cover of a locally connected pointed space (X, x_0). Let $x \in X$ and $\widetilde{x} \in \widetilde{X}(x)$. Then $(\widetilde{X}, \widetilde{x})$ is a universal pointed cover of (X, x).*

Proof. The characterisation given in last proposition does not use base points. □

Definition 4.28. *Let $\widetilde{X}$ be a cover of X. We say that $\widetilde{X}$ is a universal cover of X if there exists a point $x_0 \in X$ and a point $\widetilde{x_0} \in \widetilde{X}(x_0)$ such that $(\widetilde{X}, \widetilde{x_0})$ is a universal pointed cover of (X, x_0).*

Using last corollary we see that a universal cover is a universal pointed cover for any pair $(x, \widetilde{x})$ such that $\widetilde{x} \in \widetilde{X}(x)$. We are going to see some conditions on X for which we can guarantee the existence of a universal cover.

Definition 4.29. *We say that X is semi-locally simply connected if each point $x \in X$ has a neighbourhood U such that the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.*

In other words, X is semi-locally simply connected if every loop in such neighbourhood U is homotopic in X to a constant path.

The spaces one generally meets are semi-locally simply connected: any open set in the plane or in $\mathbb{R}^n$, or any manifold, or any finite graph, is semi-locally simply connected. An example of a space that is not semi-locally simply connected is the shrinking wedge of circles or Hawaiian earring, the subspace $X \subset \mathbb{R}^2$ consisting of the circles of radius $\frac{1}{n}$ centred at the point $(\frac{1}{n}, 0)$ for $n = 1, 2, \ldots$

Theorem 4.30. *If X is connected, locally path connected and semi-locally simply connected then there exists a universal cover of X.*

Before proving the above theorem we need to prove some preliminary concepts:

Proposition 4.31. *If X is connected, locally path connected and semi-locally simply connected, then there exists a cover of X which is simply connected.*

Proof. See [1, pages 63-65] for its construction (note that if X is connected and locally path connected, then it is path connected). □

Lemma 4.32. *Let X be a locally path connected space. Then any cover $p : Y \to X$ is locally path connected.*

Proof. Let $y \in Y$ and consider an open neighbourhood V of y in Y. Recall by Remark 4.4 that there exists an open neighbourhood U of $p(y)$ and an open set $U_y \subset Y$ such that $y \in U_y$ and $p_y : U_y \to U$ is a homeomorphism. Then $V \cap U_y$ is homeomorphic to $p_y(V \cap U_y)$. This is an open set of $p(y)$, so it has a path connected neighbourhood W. Thus $p_y^{-1}(W) \subset V$ is a path connected neighbourhood of y. □

Lemma 4.33. *A cover of a simply connected and locally path-connected space is trivial.*

Proof. See [7, Lemma 2.4.4]. □

Remark 4.34. If X is locally path connected, then it is locally connected.

Proof (of Theorem 4.30). We have seen in Proposition 4.31 that there exists a simply connected cover $\widetilde{X}$ of X. In particular, $\widetilde{X}$ is connected. From Lemma 4.32 we see that $\widetilde{X}$ is locally path connected. Then by Lemma 4.33 every cover of $\widetilde{X}$ is trivial, so $\widetilde{X}$ trivialises any cover of X. Thus by Proposition 4.16 we get $\widetilde{X}$ is universal. □

4.4 Galois covering spaces

Given a cover $p : Y \to X$ we can define the set $\mathrm{Aut}(Y|X)$ given by all the morphisms of covers $f : Y \to Y$ such that f is a homeomorphism. It is trivial to check that $\mathrm{Aut}(Y|X)$ is a group with respect to composition. This group is called the group of automorphisms of Y over X.

Definition 4.35. *We say that a cover $p : Y \to X$ is Galois if $Aut(Y|X)$ acts transitively over each fibre of Y.*

Theorem 4.36. *Let X be connected and Y be a connected cover of X. The following conditions are equivalent:*

1) Y is Galois.

2) Y trivialises itself (i.e., $Y \times_X Y$ with $\pi_1 : Y \times_X Y \to Y$ the first projection is a trivial cover of Y).

We need a lemma before proving the above theorem:

Lemma 4.37. *Let X be connected and $p : Y \to X$ a cover. Then Y is a trivial cover of X if and only if for all $y \in Y$ there exists a continuous section $s : X \to Y$ such that $s(p(y)) = y$.*

Proof. The 'only if' part is clear. We are going to see the 'if' one. Let Γ be the set of the sections $X \to Y$ with the discrete topology. Consider the application $\varepsilon : X \times \Gamma \to Y$ defined by $\varepsilon(x, s) = s(x)$. The application ε is obviously continuous. It is a local homeomorphism as $X \times \Gamma$ and Y are locally homeomorphic to X, and it is surjective by hypothesis. The map is also injective: if $s(x) = s'(x')$ we have $x = x' = p(s(x))$, and the set of points where s and s' coincide is closed as Y is Hausdorff over X, open because p is locally injective and non-empty as it has x, so it is equal to X since X is connected. Therefore, ε is bijective and a local homeomorphism, so it is a homeomorphism. □

Proof (of Theorem 4.36). By Lemma 4.37 condition 2) is equivalent to:

2′) For every point of $Y \times_X Y$ there exists a continuous section going through it.

This condition is equivalent to:

2″) For all $(y, y') \in Y \times_X Y$ there exists a morphism of covers $f : Y \to Y$ such that $f(y) = y'$.

It is trivial to see that 1) $\Rightarrow$ 2′). We are going to see the converse. If 2″) is true, then for y and y' of the same fibre there exist two morphisms of covers $f, f' : Y \to Y$ such that $f(y) = y'$ and $f'(y') = y$. Thus $f \circ f'$ and $f' \circ f$ have a fixed point, so $f \circ f' = f' \circ f = 1_Y$ by Corollary 4.20. □

Corollary 4.38. *Let X be connected, locally path connected and semi-locally simply connected. The universal cover constructed in Theorem 4.30 is Galois.*

Proof. Since such universal cover trivialises every cover of X, it trivialises itself. □

Let X be connected and locally connected. Suppose that there exists a Galois cover $\widetilde{Y}$ of X and consider the category $\mathrm{Aut}(\widetilde{Y}|X)$-**Sets**. Denote by $\mathfrak{C}$ the category of covers of X trivialised by $\widetilde{Y}$. For every object Y in $\mathfrak{C}$ denote $S(Y)$ the set $\mathrm{Hom}_X(\widetilde{Y}, Y)$ of homomorphisms $f : \widetilde{Y} \to Y$ on which $\mathrm{Aut}(\widetilde{Y}|X)$ acts by $(g, f) \mapsto f \circ g^{-1}$. This assignment defines a functor $S : \mathfrak{C} \to \mathrm{Aut}(\widetilde{Y}|X)$-**Sets**.

Theorem 4.39. *The above functor S is an equivalence of categories.*

Proof. See [8, Theorem 4.5.3]. □

4.5 A relation between covering spaces and the fundamental group

Given a cover $p : Y \to X$, the fibre $p^{-1}(x_0)$ over a point $x_0 \in X$ carries a natural action by the group $\pi_1(X, x_0)$. This will be a consequence of the following lemma on 'lifting paths and homotopies':

Lemma 4.40. *Let $p : Y \to X$ be a cover, y_0 a point of Y and $x_0 = p(y_0)$.*

1) Given a path $f : [0,1] \to X$ with $f(0) = x_0$, there is a unique path $\widetilde{f} : [0,1] \to Y$ with $\widetilde{f}(0) = y_0$ and $p \circ \widetilde{f} = f$.

2) Assume moreover given a second path $g : [0,1] \to X$ homotopic to f. Then the unique $\widetilde{g} : [0,1] \to Y$ with $\widetilde{g}(0) = y_0$ and $p \circ \widetilde{g} = g$ has the same endpoint as $\widetilde{f}$, i.e., we have that $\widetilde{f}(1) = \widetilde{g}(1)$.

Proof. See [7, Lemma 2.3.2]. □

We can now construct the left action of $\pi_1(X, x_0)$ on the fibre $p^{-1}(x_0)$. Given $y \in p^{-1}(x_0)$ and $\alpha \in \pi_1(X, x_0)$ represented by a path $f : [0,1] \to X$ with $f(0) = f(1) = x_0$, we define $\alpha y := \widetilde{f}(1)$, where $\widetilde{f}$ is the unique lifting $\widetilde{f}$ to Y with $\widetilde{f}(0) = y$ given by part 1) of the lemma above. By part 2) of the lemma αy does not depend on the coice of f, and it lies in $p^{-1}(x_0)$ by construction. This is indeed a left action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$: $(\alpha * \beta)y = \alpha(\beta y)$ for $\alpha, \beta \in \pi_1(X, x_0)$. It is called the *monodromy action* on the fibre $p^{-1}(x_0)$.

Theorem 4.41. *Let X be connected, locally path connected and semi-locally simply connected. For any $x_0 \in X$ the functor $\Phi : Y \to Y(x_0)$ from* ***Cov***(X) *to $\pi_1(X, x_0)$-**Sets** is an equivalence of categories.*

We need a proposition before proving the above theorem:

Proposition 4.42. *If $p : Y \to X$ is a cover with Y simply connected and X locally path connected then $\pi_1(X, x_0) \cong Aut(Y|X)$*

Proof. See [6, Corollary 13.15]. □

Proof (of Theorem 4.41). Consider $\widetilde{X}$ the universal cover constructed in 4.30. Since $\widetilde{X}$ is a Galois cover of X (see Corollary 4.38) and it trivialises every cover of X, the functor $S : Y \mapsto \mathrm{Hom}(\widetilde{X}, Y)$ from **Cov**(X) to $\mathrm{Aut}(\widetilde{X}|X)$-**Sets** is an equivalence of categories by Theorem 4.39. Identify $\pi_1(X, x_0)$ and $\mathrm{Aut}(\widetilde{X}|X)$ by the isomorphism ε of the last proposition. By Proposition 2.7 we get an equivalence of categories $\varepsilon^* : \mathrm{Aut}(\widetilde{X}|X)$-**Sets** $\to \pi_1(X, x_0)$-**Sets**. By Proposition 1.13 $\varepsilon^* \circ S$ is an equivalence of categories. We are going to show that the functors $\varepsilon^* \circ S$ and Φ are naturally isomorphic so we can apply Proposition 1.10. From the universal property of $(\widetilde{X}, \widetilde{x_0})$, for any cover Y of X the map $\delta_{\widetilde{x_0}} : f \mapsto f(\widetilde{x_0})$ from $\mathrm{Hom}(\widetilde{X}, Y)$ to $Y(x_0)$ is bijective. It is then an isomorphism of $\pi_1(X, x_0)$-sets. The functoriality is obvious, so $\delta_{\widetilde{x_0}}$ is a natural isomorphism. □

Chapter 5

The Seifert–van Kampen theorem

Finally, we state the main theorem of this paper:

Theorem 5.1 (Seifert–van Kampen). *Let U_1 and U_2 be two subsets of X, both open or both closed such that $X = U_1 \cup U_2$, $V = U_1 \cap U_2$, U_1 and U_2 are connected, locally path connected and semi-locally simply connected. If $x_0 \in V$, then $\pi_1(X, x_0)$ is the pushout of $(f : \pi_1(V, v_0) \to \pi_1(U_1, x_0), g : \pi_1(V, v_0) \to \pi_1(U_2, x_0))$, i.e., $\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(V,x_0)} \pi_1(U_2, x_0)$.*

The proof of the above theorem needs some lemmas:

Lemma 5.2. *Let X and Y be two topological spaces, $f : X \to Y$ an application and (C_k) a finite family of closed sets of X such that $\bigcup_k C_k = X$. If $f\big|_{C_k}$ is continuous for all k, then f is continuous.*

Proof. Let D be a closed subset of Y. For all k, $f^{-1}(D) \cap C_k = (f\big|_{C_k})^{-1}(D)$ is closed in C_k, thus it is in X, and $f^{-1}(D) = \bigcup_k((f)^{-1}(D) \cap C_k)$ is closed. □

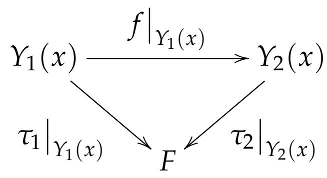
Lemma 5.3. *Let X be locally connected, U_1 and U_2 be two subspaces of X both open or both closed. We can suppose $X = U_1 \cup U_2$ and we denote $V = U_1 \cap U_2$ (if the U_i are open, they are locally connected and so is V; if the U_i are closed, then it is necessary to suppose U_1, U_2 and V are locally connected).*

The functor $\alpha : \boldsymbol{Cov}(X) \to \boldsymbol{Cov}(U_1) \times_{\boldsymbol{Cov}(V)} \boldsymbol{Cov}(U_2)$ defined by $\alpha(Y) = (Y\big|_{U_1}, Y\big|_{U_2}, 1_Y\big|_V)$ is an equivalence of categories.

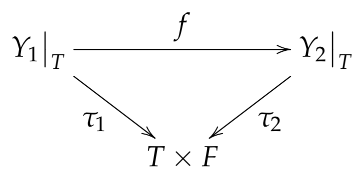
Proof.

1) *The functor α is full and faithfull.* If Y and Z are two covers of X and if $f_1 : Y\big|_{U_1} \to Z\big|_{U_1}$ and $f_2 : Y\big|_{U_2} \to Z\big|_{U_2}$ are two morphisms that coincide over V, then f_1 and f_2 are glued together in a morphism $f : Y \to Z$ (the case where U_1 and U_2 are both closed the continuity of f is given by Lemma 5.2).

2) *The functor α is essentially surjective.* Let $(Y_1, Y_2, f) \in \mathbf{Cov}(U_1) \times_{\mathbf{Cov}(V)} \mathbf{Cov}(U_2)$. Denote Y the space over X quotient of the disjoint union $Y_1 \sqcup Y_2$ by the equivalence relation identifying y with $f(y)$ for $y \in Y_1\big|_V$. The canonical injection $Y_1 \to Y_1 \sqcup Y_2$ is open and closed, and the canonical application $Y_1 \sqcup Y_2 \to Y$ is open if the U_i are open and closed if the U_i are closed. Therefore, the canonical injection $i_1 : Y_1 \to Y$ is an homeomorphism from Y_1 to $Y\big|_{U_1}$. Same holds for $i_2 : Y_2 \to Y$.
We are going to show that Y is a cover of X. It is obvious if the U_i are open. Suppose the U_i are closed and let $x \in X$. If $x \in X \backslash V$, then the point x is interior of one of the U_i, and Y is a cover over a neighbourhood of x. Suppose $x \in V$. Since V is locally connected, we can find a neighbourhood S of x in X such that $T = S \cap V$ is connected and such that $Y_1\big|_{S_1}$ and $Y_2\big|_{S_2}$ are trivial, with $S_i = S \cap U_i$. Let $\tau_i : Y_i\big|_{S_i} \to S_i \times F_i$ be its trivialisations. Since we can identify F_1 and F_2 by $\tau_2\big|_{Y_2(x)} \circ f\big|_{Y_1(x)} \circ (\tau_1\big|_{Y_1(x)})^{-1}$, we can suppose $F_1 = F_2 = F$ and that the diagram



is commutative. We have then that the diagram



is commutative since $\tau_2 \circ f$ and τ_1 are two morphisms from $Y_1\big|_T$ to $T \times F$ that coincide over x, so in at least one point of each connected component (recall Corollary 4.20). Consequently, τ_1 and τ_2 gather in a trivialisation $\tau : Y_S \to S \times F$; the continuity of τ and its inverse comes from Lemma 5.2. We have $Y \in \mathbf{Cov}(X)$ and $\alpha(Y) \cong (Y_1, Y_2, f)$.

□

From now on, we will denote $G = \pi_1(X, x_0)$, $G_i = \pi_1(U_i, x_0)$ and $H = \pi_1(V, x_0)$.

Lemma 5.4. *Consider the commutative diagram:*

$$\begin{array}{ccc} G_1 *_H G_2 & \xleftarrow{i_2} & G_2 \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle g} \\ G_1 & \xleftarrow[f]{} & H \end{array}$$

*Note that if $F \subset H$, then $f^* i_1^* F = g^* i_2^* F$. The functor $\gamma : (G_1 *_H G_2)$-**Sets** $\to$ G_1-**Sets** $\times_{H\text{-}\mathbf{Sets}}$ G_2-**Sets** defined by $\gamma(F) = (i_1^* F, i_2^* F, 1_{f^* i_1^* F})$ is an equivalence of categories.*

Proof. Consider the functor $\lambda : G_1$-**Sets** $\times_{H\text{-}\mathbf{Sets}}$ G_2-**Sets** $\to (G_1 *_H G_2)$-**Sets** defined by $\lambda(A, B, \varepsilon) = B$. It is obvious that γ and λ are well defined. It is also trivial to check that there exists a natural isomorphism $\eta : 1_{(G_1 *_H G_2)\text{-}\mathbf{Sets}} \cong \lambda\gamma$. We are going to see that there exists a natural isomorphism $\mu : 1_{G_1\text{-}\mathbf{Sets} \times_{H\text{-}\mathbf{Sets}} G_2\text{-}\mathbf{Sets}} \cong \gamma\lambda$. If $(A, B, \varepsilon) \in G_1$-**Sets** $\times_{H\text{-}\mathbf{Sets}}$ G_2-**Sets**, then the isomorphism ε yields an isomorphisms of sets $\varepsilon_* : A \to B$. So $(\varepsilon_*, 1_B) : (A, B, \varepsilon) \to (B, B, 1_B)$ is an isomorphism of G_1-**Sets** $\times_{H\text{-}\mathbf{Sets}}$ G_2-**Sets**. For every morphism $(q_1, q_2) : (A, B, \varepsilon) \to (A', B', \varepsilon')$ we have the commutative diagram of sets

$$\begin{array}{ccc} f^*A & \xrightarrow{\varepsilon} & g^*B \\ {\scriptstyle f^*q_1}\downarrow & & \downarrow{\scriptstyle g^*q_2} \\ f^*A' & \xrightarrow[\varepsilon']{} & g^*B'. \end{array}$$

Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon_*} & B \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ A' & \xrightarrow[\varepsilon'_*]{} & B' \end{array}$$

commutes, thus the diagram

$$\begin{array}{ccc} (A, B, \varepsilon) & \xrightarrow{(\varepsilon_*, 1_B)} & \gamma\lambda(A, B, \varepsilon) = (B, B, 1_B) \\ {\scriptstyle (q_1, q_2)}\downarrow & & \downarrow{\scriptstyle \gamma\lambda(q_1, q_2) = (q_2, q_2)} \\ (A', B', \varepsilon') & \xrightarrow[(\varepsilon'_*, 1_{B'})]{} & \gamma\lambda(A', B', \varepsilon') = (B', B', 1_{B'}) \end{array}$$

commutes too. □

Proof (of Theorem 5.1). We have the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Cov}(X) & \xrightarrow{\theta} & G\text{-}\mathbf{Sets} & \xrightarrow{v^*} & (G_1 *_H G_2)\text{-}\mathbf{Sets} \\
\downarrow \alpha & & \downarrow \beta & \swarrow \gamma & \\
\mathbf{Cov}(U_1) \times_{\mathbf{Cov}(V)} \mathbf{Cov}(U_2) & \xrightarrow[\Phi]{} & G_1\text{-}\mathbf{Sets} \times_{H\text{-}\mathbf{Sets}} G_2\text{-}\mathbf{Sets}, & &
\end{array}$$

where $\theta(Y) = Y(x_0)$, $\Phi(Y_1, Y_2, \varepsilon) = (Y_1(x_0), Y_2(x_0), \varepsilon_{x_0})$, $\alpha(Y) = (Y\big|_{U_1}, Y\big|_{U_2}, 1)$, $\beta(F) = (i_1^*F, i_2^*F, 1_{f^*i_1^*F})$ and $\gamma(F) = (i_1^*F, i_2^*F, 1_{f^*i_1^*F})$. Since the diagram

$$\begin{array}{ccccc}
\mathbf{Cov}(U_1) & & \xrightarrow{\Phi_1} & & G_1\text{-}\mathbf{Sets} \\
& \searrow^{\widetilde{i_1}} & & & \searrow^{f^*} \\
& & \mathbf{Cov}(V) & \xrightarrow{\varphi} & H\text{-}\mathbf{Sets} \\
& \nearrow^{\widetilde{i_2}} & & & \nearrow_{g^*} \\
\mathbf{Cov}(U_2) & & \xrightarrow[\Phi_2]{} & & G_2\text{-}\mathbf{Sets}
\end{array}$$

commutes with φ, Φ_1 and Φ_2 equivalences of categories by Theorem 4.41, by Theorem 1.26 we have Φ is an equivalence of categories. Now α and θ are equivalences of categories by Lemma 5.3 and Theorem 4.41 respectively, then so is β by Corollary 1.14. Since γ is an equivalence by Lemma 5.4, then by Proposition 1.13 so is v^*. Therefore, by Proposition 2.7 $v : G_1 *_H G_2 \to G$ is an isomorphism. □

Bibliography

[1] HATCHER, Allen. *Algebraic topology*. Cambridge [etc.]: Cambridge University Press, 2002. 544 p. ISBN 9780521795401.

[2] BORCEUX, Francis. *Handbook of categorical algebra I*. Cambridge: Cambridge University Press, 1994. 522 p. ISBN 0521441781.

[3] RIEHL, Emily. *Category theory in context*. Dover Publications, 2016. 272 p. ISBN 048680903X.

[4] SAUNDERS, Mac Lane. *Categories for the working mathematician*. New York [etc.]: Springer, cop. 1998. 314 p. ISBN 0387984038.

[5] MASSEY, William S. *A Basic course in algebraic topology*. New York [etc.]: Springer, 1991. 428 p. ISBN 038797430X.

[6] FULTON, William. *Algebraic topology: a first course*. New York [etc.] : Springer, cop. 1995. 430 p. ISBN 0387943269.

[7] SZAMUELY, Tamás. *Galois groups and fundamental groups*. Cambridge : Cambridge University Press, 2009. 270 p. ISBN 9780521888509.

[8] DOUADY, Régine. *Algèbre et théories galoisiennes*. Paris : Cassini, cop. 2005. 500 p. ISBN 9782842250058.

[9] BREDON, Glen E. *Topology and geometry*. Springer, 1993. 557 p. ISBN 0387979263.

[10] DIECK, Tammo Tom. *Algebraic topology*. Zürich : European Mathematical Society, 2008. 567 p. ISBN 9783037190487.

[11] JAMES, Ioan Mackenzie. *General topology and homotopy theory.* Springer Science & Business Media, 2012. 248 p. ISBN 9781461382836.

[12] EIE, Minking & CHANG, Shou-Te. *A course on abstract algebra*. London: World Scientific, cop. 2010. 359 p. ISBN 9789814271882.

[13] BOUC, Serge. *Biset functors for finite groups*. Heidelberg: Springer, cop. 2010. 299 p. ISBN 9783642112966.