Semiclassical back reaction in the formation of a straight cosmic string

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We derive the back reaction on the gravitational field of a straight cosmic string during its formation due to the gravitational coupling of the string to quantum matter fields. A very simple model of string formation is considered. The gravitational field of the string is computed in the linear approximation. The vacuum expectation value of the stress tensor of a massless scalar quantum field coupled to the string gravitational field is computed to one loop order. Finally, the back-reaction effect is obtained by solving perturbatively the semiclassical Einstein's equations.

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I. INTRODUCTION

Cosmic strings are macroscopic topological defects that may have been produced at phase transitions in the early universe [1]. They are predicted by gauge theories with spontaneous breakdown of symmetry whenever the unbroken subgroup contains a discrete symmetry as is the case of many grand unified theories (GUT's), although not in the simplest unified group SU(5). The GUT strings would have been produced when the universe was 10^{-34} sec old and had a temperature of $T \sim 10^{14}$–10^{16} GeV. The gravitational field of such strings may seed structure and, in fact, a network of these strings is an alternative to inflation for the generation of the universe structure [2–4].

There are two types of gravitational quantum effects associated with cosmic strings (or to any other topological defect) which result from the interaction of the string's gravitational field with any quantum field (i.e., matter) present: namely, particle creation and vacuum polarization. When a string forms, a sudden change in the gravitational field takes place which may translate into copious quantum pair production of particles in a way similar as electron-positron pairs are created by external electromagnetic fields. This effect has been considered by several authors in different settings which go from various models of string formation [5,6] to oscillating string loops [7]. The main conclusion is that even though very energetic particles may be created the cosmological significance of these is very small compared to the background radiation at the time of formation. Typically, the ratio of the energy density of particles created by the formation of the strings and the energy density of the radiation is of order $N^2(G\mu)^4$, where N is the number of particle species, G Newton's gravitational constant, and μ is the energy per unit length of the string; for GUT strings $\mu \sim 10^{22}$ g/cm, the square of the GUT mass, and thus $G\mu \sim 10^{-6}$. For particles created by oscillating loops such a ratio is much higher, $N^{3/2}(G\mu)^2$, but still cosmologically insignificant unless, of course, the number of particles is absurdly large; at GUT time we expect $N \sim 10^2$.

Vacuum polarization effects due to quantum fluctuations of matter fields have been much less studied. This is due, in part, to the fact that one does not expect very significant changes in the classical string network picture as a consequence of such effects, but also in part because the computation of such effects is difficult [8,9]. The vacuum expectation value of the stress tensor of quantum fields around the string is generally different from zero even for a static string; consequently, it is the source of a gravitational field which, in turn, modifies the classical gravitational field of the string. This is the back-reaction effect of quantum matter on the classical gravitational field. For example, the gravitational field outside an unperturbed (i.e., not wiggly) straight static string is described by the metric given by flat spacetime with a deficit angle in the plane perpendicular to the string [10–13]. The quantum stress tensor for conformally coupled scalar fields has been computed exactly by Linet [14] and by Helliwell and Konkowski [15] who found that the energy density goes like $N\hbar G\mu\, r^{-4}$, where r is the radial distance from the string axis. Such energy density creates in the weak field approximation a Newtonian potential outside the string of the order of $\Phi \sim N\hbar G\mu\, r^{-2}$. In fact, Hiscock [8] solved the semiclassical Einstein's equations to linear order in this case and found the back reaction in the gravitational field of the cosmic string. The result is that the spacetime surrounding the string is no longer flat with a deficit angle: The two-surface perpendicular to the string is hyperboloid (rather than a cone) and the corrections to the flat metric are of the order just described. Two consequences of this are clear: One is that a static string will exert Newtonian forces on surrounding nonrelativistic particles; the other is that when two cosmic strings approach they should feel increasingly strong attractive forces. The relative significance of these

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effects will be discussed later on.

In this paper we compute the back reaction on the gravitational field of a cosmic string when it is formed. Now in addition to pure vacuum polarization effects, as in the case of a static string, we have an effect due to the creation of particles which also contributes to the quantum stress tensor. The computation is made within the linear approximation of the semiclassical correction to Einstein's equations; i.e., we assume that the spacetime metric departs from the Minkowski metric by linear terms $h_{\mu\nu}$ only. This approximation is appropriate whenever the energy per unit length of the string satisfies that $G\mu \ll 1$, which is certainly true for GUT strings. Also the vacuum expectation value of the stress tensor for matter fields is computed perturbatively to first order in $h_{\mu\nu}$ and to the one loop order using the results by Horowitz [16] and Jordan [17] when the background metric is flat. More general results for the stress tensor in conformally flat backgrounds are known [18,19] but here we ignore the effect due to the cosmological expansion. To this order the stress tensor does not include the energy of the particles created, which is second order in $h_{\mu\nu}$; however, particle creation effects such as those due to transition elements from vacuum to two particles states are included in this linear order, see the discussion in Ref. [19].

To implement string formation we use a very simple macroscopic model for the classical stress tensor of the string. The field dynamics of string formation is too complicated an issue to be described at microscopic level and one is restricted to consider rough macroscopic models for this process. The models are either implemented by giving the gravitational field of the string [5,20] or by giving a prescribed form of the string stress tensor [6,7,21]. The results for particle creation are rather independent of specific models and thus we take the simplest model based on the stress tensor given in Ref. [7]. In this model the string is always there, first as a dust straight line source, and it is the string tension which grows in time; in this way the stress tensor is automatically conserved.

The plan and a summary of the main results of the paper are the following. In Sec. II we describe the model of string formation and derive its gravitational field in the linear approximation. We allow for the possibility that the string be a straight wiggly string. Straight wiggly strings are segments of long strings with small scale structure whose effective stress tensor may be described by a straight string with effective mass per unit length larger than the unperturbed one (i.e., with no small scale structure) and an effective tension which is also less than the mass per unit length [22]. They appear as long strings in the numerical simulations of string evolution [23] and may be the seeds of large scale structure [3,24]. As a consequence of linearity we can split the stress tensor in two parts: one which is static and whose gravitational field is easily solved and a time-dependent part whose gravitational field is found by solving the corresponding initial value problem. For simplicity we also use the sudden approximation, this introduces divergences in the gravitational field along the future light cone which must be regularized. To get rid of gauge effects which appear in the metric tensor we also compute the Riemann tensor. The evolution at large times of this tensor differs, of course, if the string is wiggly or unperturbed. In the first case we have Newtonian-like potentials which help in building wakes [3] whereas in the second the curvature tends to zero corresponding to flat space with a deficit angle.

In Sec. III the vacuum expectation value of the stress tensor outside the string for matter conformally coupled to the string's time dependent gravitational field is derived. It is seen that after a short transient period the stress tensor settles down to the values of a static, wiggly or unperturbed, string. In both cases, the energy density goes like $N\hbar G\mu r^{-4}$ as one expects from dimensional arguments. Note that, for GUT strings, the string radius is $r_0 \sim 10^{-30}$ cm, corresponding to the Compton wavelength of the GUT mass, and this radius set bounds on the energy density.

In Sec. IV the semiclassical Einstein's equations for our problem are solved to find the back reaction of the quantum matter on the gravitational field of the string. We can also split the stress tensor into a static source, which includes the static part of the classical source and of the quantum stress tensor, and a time-dependent part. The Riemann tensor is computed using some approximations in the stress tensor and we discuss specially its behavior at large times. The results are quite different if the string is wiggly or unperturbed. For wiggly strings there is a Newtonian potential at the classical level already and the quantum correction to such potential is too small to quantitatively modify wake formation behind a moving string. For unperturbed strings there is no Newtonian potential at the classical level but a Newtonian potential appears as a quantum correction and, thus, the string will exert a force on nonrelativistic matter. However, this effect would be only significant when r is very small, say, of the order of r_0, i.e., only at a microscopic scale. Perhaps back reaction might be significant to modify the dynamics of string crossing; however, we should keep in mind that when two strings approach at a distance of order r_0 the dynamics is dominated by the microscopic field dynamics and the effective macroscopic picture that we use breaks down.

II. CLASSICAL ANALYSIS

In this section we compute the gravitational field created by the string as it is formed. We use the following classical stress tensor for the formation of a straight string which lies along the z axis [7]:

$$T_c{}^{\mu}{}_{\nu} = \mu\delta(x)\delta(y)\,\mathrm{diag}(1,0,0,\tau\theta(t)), \tag{2.1}$$

where μ is the mass per unit length of the string, $\theta(t)$ the step function, and τ a parameter ($0 < \tau \leq 1$) used to modulate the tension $\mu\tau$ of the string. If $\tau = 1$, the string is an unperturbed cosmic string, whereas if $\tau < 1$, this tensor gives a macroscopic description of a straight but wiggly cosmic string. In this last case the effective equation of state for the string is [22,25] $\tau\mu^2 = \mu_0^2$, where μ_0 is the unperturbed mass per unit length (i.e., for a

GUT string $G\mu_0 \sim 10^{-6}$). From numerical simulations one has in the matter era the typical value [3] $\mu \sim 1.4\mu_0$ (i.e., $\tau \sim 0.5$). Note that when the string forms each segment may be approximated by an unperturbed cosmic string; a long segment becomes wiggly only after evolution of the string network by string intersection and by chopping off small loops. Thus back reaction is more important when an unperturbed string forms, since it takes the smallest time; however, we keep τ arbitrary in what follows since vacuum polarization will also exist once the string settles into a wiggly string and, in any way, keeping τ arbitrary allows a simple identification ot the time dependent effects.

In Ref. [21] a refinement of this model is considered in which the compensating issue [26] is discussed in some detail. This issue arises in more realistic scenarios when both ordinary matter and cosmic strings are present; it concerns the restrictions on the matter and metric perturbations due to the conservation laws of the total stress tensor (i.e., including matter and defects). Note also that in the classical stress tensor (2.1) we are using two simultaneous approximations. The first is the thin line approximation which assumes that the string has zero thickness, but, as we have emphasized earlier, a physical string has a radius r_0 which gives a cutoff radius inside which the approximation cannot be trusted. The second is the step approximation which assumes that the string is suddenly formed at time $t = 0$; the use of $\theta(t)$ instead of a smooth function which grows from zero to one in a certain time T requires a cutoff in the momentum of the quantum modes of the order of $1/T$. The time of formation T is bounded by the age of the universe when the string forms (for GUT strings $T < 10^{-34}$ sec).

In the weak field approximation the metric in Cartesian coordinates can be written as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\mid h_{\mu\nu} \mid \ll 1$, and we have the gauge freedom, due to infinitesimal coordinate changes $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, that

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} \tag{2.3}$$

for arbitrary $\xi^{\mu}(x)$. Einstein's equations for the metric perturbation $h_{\mu\nu}$ can be written in the harmonic gauge $(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^{\alpha}{}_{\alpha})_{,\nu} = 0$ as

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}, \tag{2.4}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}$ and we still have the freedom within the harmonic gauge of choosing functions $\xi^{\mu}(x)$ such that $\Box\xi^{\mu}(x) = 0$.

Since we start with a static prescribed metric before the string forms at $t = 0$, we can find the solution of (2.4) for $t > 0$ as a Cauchy problem. As is well known, such a solution is given by Kirchhoff's formula

$$\begin{aligned} h_{\mu\nu}(t,\vec{x}) = &\int_{\mathbf{R}^3} d^3\vec{x}' \left[\partial_{t'}h_{\mu\nu}(0,\vec{x}')\, D(t,\vec{x}-\vec{x}') + h_{\mu\nu}(0,\vec{x}')\, \partial_t D(t,\vec{x}-\vec{x}')\right] \\ &-16\pi G \int_0^t dt' \int_{\mathbf{R}^3} d^3\vec{x}'\, D_R(t-t',\vec{x}-\vec{x}')\, S_{\mu\nu}(t',\vec{x}'), \end{aligned} \tag{2.5}$$

where D_R is the retarded Green function for a massless scalar field, i.e., $D_R(x-x') = \frac{1}{2\pi}\,\delta[(x-x')^2]\,\theta(t-t')$ or $\frac{1}{4\pi}\,\delta(t-t'- \mid \vec{x}-\vec{x}' \mid)/ \mid \vec{x}-\vec{x}' \mid$, and $D = D_R - D_A$ is the Schwinger Green function (D_A is the advanced Green function), which may be written as $D(x-x') = \frac{1}{2\pi}\,\delta[(x-x')^2]\,[\theta(t-t') - \theta(t'-t)]$. The first integral in (2.5) is over the hypersurface $t' = 0$ and it is the solution of the homogenous equation which satisfies the boundary conditions given by the metric $h_{\mu\nu}$ and its first time derivative $\partial_{t'}h_{\mu\nu}$, at $t' = 0$. The second integral is the solution of the inhomogeneous equation which vanishes at $t = 0$ (i.e., the boundary conditions are implemented with the solution of the homogeneous equation) and has support (see D_R) on the truncated past light cone starting at $(t,\vec{x})$ and ending at the hypersurface $t' = 0$.

The solution of (2.4) for the stress tensor (2.1) is somewhat simplified if we write (2.1) as the sum of two stress tensors, both conserved, one of them $T^{(1)}_{\mu\nu}$ static and the other $T^{(2)}_{\mu\nu}$ time dependent: i.e., $T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu}$, with

$$T^{(1)\,\mu}{}_{\nu} = \mu\delta(x)\delta(y)\,\mathrm{diag}(1,0,0,0), \quad T^{(2)\,\mu}{}_{\nu} = \mu\tau\delta(x)\delta(y)\,\theta(t)\,\mathrm{diag}(0,0,0,1). \tag{2.6}$$

This leads us to write $h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$, where $h^{(1)}_{\mu\nu}$ is the static metric corresponding to the source $T^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ is the time-dependent part corresponding to the source $T^{(2)}_{\mu\nu}$. The field equation (2.4) for the static part $h^{(1)}_{\mu\nu}$ is now simply

$$\nabla^2 h^{(1)}_{\mu\nu} = 8\pi G\mu\delta(x)\delta(y)\,\mathrm{diag}(1,1,1,1), \tag{2.7}$$

which, using the cylindrical symmetry of the problem, has the solution

$$h^{(1)}_{tt} = h^{(1)}_{xx} = h^{(1)}_{yy} = h^{(1)}_{zz} = 4G\mu\,\ln\left(\frac{r}{R}\right) \equiv \alpha(r), \tag{2.8}$$

where $r^2 = x^2 + y^2$ and R is an arbitrary constant with dimensions of length.

The field equation for the time-dependent metric perturbation $h^{(2)}_{\mu\nu}$ is now given by

$$\Box h^{(2)}_{\mu\nu} = -8\pi G\mu\tau\delta(x)\delta(y)\,\theta(t)\,\mathrm{diag}(-1,1,1,-1), \tag{2.9}$$

whose solution as a Cauchy problem is given by (2.5), but we note that since the boundary conditions for these components are obviously $h^{(2)}_{\mu\nu}(0,\vec{x}) = 0$, $\partial_t h^{(2)}_{\mu\nu}(0,\vec{x}) = 0$ we are left with the second integral of (2.5) only. Using the second representation for the retarded Green function given above we have after a simple integration that the (unique) solution for $h^{(2)}_{\mu\nu}$ is

$$h^{(2)}_{tt} = h^{(2)}_{zz} = -h^{(2)}_{xx} = -h^{(2)}_{yy} = 4G\mu\tau \,\mathrm{arccosh}\left(\frac{t}{r}\right)\,\theta(t-r) \equiv \beta(t,r). \tag{2.10}$$

It is easy to check that the final metric perturbation $h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$ satisfies the harmonic gauge condition. We can write the gravitational field of the string in cylindrical coordinates (t, r, θ, z) as

$$ds^2 = (1+\alpha+\beta)\,dt^2 - (1-\alpha-\beta)\,dz^2 - (1-\alpha+\beta)\,(dr^2 + r^2 d\theta^2), \tag{2.11}$$

where $\alpha(r)$ and $\beta(t,r)$ are given by (2.8) and (2.10), respectively.

A few comments on the metric (2.11) are now in order. First, we see that the metric is continuous but its first derivatives are discontinuous along the light cone $t = r$. This is a consequence of the use of the step approximation: Had we used a smooth time-dependent function instead of $\theta(t)$ the derivatives would be smooth too. This means that we can only trust our results outside the spacetime region bounded by $t = r - T/2$ and $t = r + T/2$. To this we should add that due to the thin line approximation the results are only reliable for $r > r_0$, where r_0 is the string radius.

Second, we also note that the metric perturbations diverge at $r \to \infty$ (t fixed) and at $t \to \infty$ (r fixed) but these divergences, as we will see from the Riemann tensor, are only gauge effects; they are a consequence of the use of the harmonic gauge.

Third, one expects that when $t \gg r$ (r fixed) the metric becomes that of a static cosmic string, unperturbed if $\tau = 1$ or wiggly otherwise. But this again cannot be seen directly in the harmonic gauge. The coordinate change which puts the above metric in a suitable form to see this is rather messy and, since it gives no further light on this issue, we shall not write it here.

An important physical observable is the deficit angle of the two-surfaces t = const, z = const. for the metric (2.11). Following Ref. [13], given a closed piecewise smooth curve on a two-surface which encloses a regular region, the deficit angle is defined as the angle turned by a vector after being parallel transported once around the curve. An application of the Gauss-Bonnet theorem shows that this angle is given by the surface integral of the Gaussian curvature K over S: $\triangle\theta = \int_S K\,dS$. If the two-surface has circular symmetry, the deficit angle is defined as the deficit angle associated with circles of radius r. It is easy to see that in a spacetime described by a cylindrical symmetric metric the deficit angle of the two-surfaces t = const, z = const reduces to

$$\triangle\theta(t,r) = 2\pi\left[1 - \frac{1}{\sqrt{-g_{rr}}}\frac{\partial}{\partial r}\sqrt{-g_{\theta\theta}}\right], \tag{2.12}$$

which in the linear approximation becomes $\triangle\theta(t,r) = \pi\left(h^r_r - h^\theta_\theta - r\,h^\theta_{\theta,r}\right)$. Note that this definition differs from that used in Ref. [8]. For the metric (2.11) we obtain

$$\triangle\theta(t,r) = 4\pi G\mu\left[1 + \tau\frac{t}{\sqrt{t^2-r^2}}\,\theta(t-r)\right]. \tag{2.13}$$

It is clear that when $t \gg r$, r fixed, $\triangle\theta \to 4\pi G\mu\,(1+\tau)$, which is the deficit angle for a static string.

The properties of the metric (2.11) are better deduced from the Riemann tensor which is gauge independent. Note also that some Riemann components give the tidal forces on surrounding test particles. In the linear approximation we have

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\beta\partial_\nu\partial_\alpha + \delta^\rho_\nu\delta^\sigma_\alpha\partial_\mu\partial_\beta - \delta^\rho_\nu\delta^\sigma_\beta\partial_\mu\partial_\alpha - \delta^\rho_\mu\delta^\sigma_\alpha\partial_\nu\partial_\beta\right) h_{\rho\sigma}. \tag{2.14}$$

Following the separation of the metric perturbation into a static part and a time-dependent part, $h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$, the Riemann components can also be separated as $R^{(1)}_{\mu\nu\alpha\beta}$ and $R^{(2)}_{\mu\nu\alpha\beta}$; the last is easily identified because it is proportional to τ. For $r \neq 0$ they are

$$
\begin{aligned}
R_{tztz} &= -R_{xyxy} = \frac{2G\mu\tau}{r^2}\,\mathcal{P}f\left[\frac{b}{(b^2-1)^{3/2}}\,\theta(b-1)\right],\\
R_{xzxz} &= \frac{2G\mu}{r^2}\left\{\cos 2\theta - \tau\,\mathcal{P}f\left[\frac{b}{\sqrt{b^2-1}}\left(\cos 2\theta - \frac{\cos^2\theta}{(b^2-1)}\right)\theta(b-1)\right]\right\},\\
R_{xzyz} &= R_{txty} = \frac{2G\mu}{r^2}\,\sin 2\theta\left\{1-\frac{\tau}{2}\,\mathcal{P}f\left[\frac{b\,(2b^2-3)}{(b^2-1)^{3/2}}\,\theta(b-1)\right]\right\},\\
R_{txtx} &= \frac{2G\mu}{r^2}\left\{\cos 2\theta - \tau\,\mathcal{P}f\left[\frac{b}{\sqrt{b^2-1}}\left(\cos 2\theta + \frac{\sin^2\theta}{(b^2-1)}\right)\theta(b-1)\right]\right\},\\
R_{tzxz} &= R_{tyyx} = -\frac{2G\mu\tau}{r^2}\,\cos\theta\,\mathcal{P}f\left[\frac{1}{(b^2-1)^{3/2}}\,\theta(b-1)\right],
\end{aligned}
\tag{2.15}
$$

where we have introduced the new variable $b \equiv t/r$, instead of t, and where $\mathcal{P}f$ denotes the Hadamard finite part, which gives well-defined expressions, in the sense of distributions, on the light cone $b = 1$ (see Appendix A). R_{yzyz}, R_{tyty}, R_{tzyz}, and R_{txxy} can be obtained from R_{xzxz}, R_{txtx}, R_{tzxz}, and R_{tyyx}, respectively, by interchanging $\cos\theta$ and $\sin\theta$.

It is now clear that when $\tau = 1$ (unperturbed string) the Riemann tensor vanishes when $t \to \infty$ (r fixed), i.e., $\lim_{t\to\infty} R_{\mu\nu\alpha\beta} = 0$, and the spacetime becomes flat. But if $\tau \neq 1$ there are tidal forces among test particles which correspond to a Newtonian-like potential $h_{tt} \propto \ln r$. Note that a Riemann component such as R_{tztz}, which gives relative accelerations among particles along the direction of the string, is a transient term, i.e., $R_{tztz} = 0$ when $t < r$ (in regions not yet affected by the formation of the string) and $R_{tztz} \simeq 2G\mu\tau\, r^{-2}\, b^{-2}[1+\frac{3}{2}\,b^{-2}+O(b^{-4})]$ when $t > r$, which approaches zero very quickly.

Our next task is to obtain the quantum correction to this curvature tensor due to the quantum fluctuations of matter fields.

III. STRESS TENSOR FOR MATTER FIELDS

Quantum fluctuations of matter fields interacting with the gravitational field of a cosmic string give a non-null vacuum expectation value for the stress tensor of these matter fields, even if they are conformally coupled. For a free massless conformally coupled scalar field in a flat spacetime background with arbitrary linear gravitational perturbations (2.2) this stress tensor has been computed to one loop order by several authors [16–19] and it is given, to first order in $h_{\mu\nu}(x)$, by

$$
\langle T^{\mu\nu}(x)\rangle = -\frac{\alpha\hbar}{6}\,B^{\mu\nu}(x) + 3\alpha\hbar\int d^4y\, H(x-y,\bar{\mu})\,A^{\mu\nu}(y),
\tag{3.1}
$$

where $\alpha \equiv (2880\pi^2)^{-1}$,

$$
\begin{aligned}
B^{\mu\nu}(x) &= 2\eta^{\mu\nu}\,\Box\, G_\alpha{}^\alpha - 2G_\alpha{}^{\alpha,\mu\nu},\\
A^{\mu\nu}(x) &= -2\,\Box\, G^{\mu\nu} - \frac{2}{3}G_\alpha{}^{\alpha,\mu\nu} + \frac{2}{3}\eta^{\mu\nu}\,\Box\, G_\alpha{}^\alpha,
\end{aligned}
\tag{3.2}
$$

$G^{\mu\nu}$ is the Einstein tensor for the metric to first order in $h_{\mu\nu}$, and $H(x-y,\bar{\mu})$ is a propagator defined by

$$
H(x-y,\bar{\mu}) \equiv -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\left[\ln\left(\frac{-(p^2+i\epsilon)}{\bar{\mu}^2}\right) + 2\pi i\,\theta(p^2)\theta(-p^0)\right],
\tag{3.3}
$$

where $\bar{\mu}$ is an arbitrary renormalization scale. Notice that the second term in (3.1) is traceless and that it is the term proportional to $B^{\mu\nu}(x)$ which gives the trace anomaly in this case:

$$
\langle T^\mu{}_\mu(x)\rangle = -\frac{\alpha\hbar}{6}\,B^\mu{}_\mu(x) = \alpha\hbar\,\Box\, R,
\tag{3.4}
$$

where R is the scalar curvature to first order in $h_{\mu\nu}$. Using $\nabla_\mu G^{\mu\nu} = 0$, it is easily seen from (3.2) that the two terms in (3.1) are independently conserved.

We can now proceed to the computation of $\langle T^{\mu\nu}(x)\rangle$ to first order in $h_{\mu\nu}$. In Sec. III A we will calculate it outside the string, that is, for $r \neq 0$. In Sec. III B we will see how a generalization to include $r = 0$ can be made. In Sec. III C we give the approximation to the stress tensor which will be finally used to compute the back reaction on the string metric.

A. Quantum stress tensor outside the string

From (3.1) and (3.2) we see that all the dependence on the metric perturbations is in the Einstein tensor $G_{\mu\nu}$.

Now since the source of such gravitational perturbations is the classical string source $T_c^{\mu\nu}$ given in (2.1) we can use Einstein's equations $G^{\mu\nu}=-8\pi G\,T_c^{\mu\nu}$ in (3.2). Thus it is worth remarking that we do not need to know the explicit gravitational field created by the forming string; it suffices to know the explicit form of the classical stress tensor which produces that field. After the above substitution, since $T_c^{\mu\nu}$ is proportional to $\delta(x)\delta(y)$, both $B^{\mu\nu}(x)$ and $A^{\mu\nu}(x)$ have support on the string core. If we are interested in computing $\langle T^{\mu\nu}(x)\rangle$ outside the string, i.e., for $r\neq 0$, it is clear from (3.1) that the only contribution comes from the second term in (3.1) and that terms in $H(x-y,\bar\mu)$ with support on $x=y$ will not contribute.

Let us give a more suitable representation of $H(x-y,\bar\mu)$ for $x\neq y$. For this it is useful to define the four-vector

$$F_\alpha(x-y)\equiv-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\,\frac{\partial}{\partial p^\alpha}\left[\ln\left(\frac{-(p^2+i\epsilon)}{\bar\mu^2}\right)+2\pi i\,\theta(p^2)\theta(-p^0)\right]. \tag{3.5}$$

This vector can, on the one hand, be simply computed as

$$\begin{aligned}F_\alpha(x-y)&=-i\,\frac{\partial}{\partial x^\alpha}\int\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\left[\frac{1}{p^2+i\epsilon}+2\pi i\,\delta(p^2)\theta(-p^0)\right]\\&=i\,\frac{\partial}{\partial x^\alpha}D_R(x-y)=\frac{i}{\pi}\,(x-y)_\alpha\,\delta'\left[(x-y)^2\right]\theta(x^0-y^0),\end{aligned} \tag{3.6}$$

where δ' means derivative with respect to the argument of δ and we have used a well-known integral representation of the retarded Green function D_R and the form of D_R given in Sec. II.

On the other hand, integrating (3.5) by parts, we can also write $F_\alpha(x-y)$ as

$$F_\alpha(x-y)=i\,(x-y)_\alpha\,H(x-y,\bar\mu). \tag{3.7}$$

Now comparing (3.6) and (3.7) we get the following useful representation for H, when $x\neq y$:

$$H(x-y,\bar\mu)=\frac{1}{\pi}\,\delta'\left[(x-y)^2\right]\theta(x^0-y^0). \tag{3.8}$$

Thus using (3.8) we have, outside the string,

$$\langle T^{\mu\nu}(x)\rangle=\frac{3\alpha\hbar}{\pi}\int d^4y\,\delta'\left[(x-y)^2\right]\theta(x^0-y^0)A^{\mu\nu}(y), \tag{3.9}$$

with

$$A^{\mu\nu}(x)=16\pi G\,\eta^{\alpha\beta\mu\nu}{}_{\rho\sigma}\,\partial_\alpha\partial_\beta T_c^{\rho\sigma}, \tag{3.10}$$

where we have defined

$$\eta^{\alpha\beta\mu\nu}{}_{\rho\sigma}\equiv\eta^{\alpha\beta}\delta^\mu_\rho\delta^\nu_\sigma+\frac{1}{3}\,\eta_{\rho\sigma}\,(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\nu}\eta^{\alpha\beta}). \tag{3.11}$$

If we now write (see Ref. [17]), $\delta'\left[(x-y)^2\right]=\lim_{\lambda\to0^-}\delta'\left[(x-y)^2+\lambda\right]=\lim_{\lambda\to0^-}\frac{d}{d\lambda}\delta\left[(x-y)^2+\lambda\right]$ in (3.9), change variables, and use (2.1), we have explicitly

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle=48\alpha G\mu\hbar\,\eta^{\alpha\beta\mu\nu}{}_{\rho\sigma}\,\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\lim_{\lambda\to0^-}\frac{d}{d\lambda}\int d^4x'\,\delta(x'^2+\lambda)\theta(t')\\\times\delta(x-x')\delta(y-y')\,(\delta^\rho_t\delta^\sigma_t-\tau\theta(t-t')\,\delta^\rho_z\delta^\sigma_z)\end{aligned} \tag{3.12}$$

and the computation is now straightforward. It is clear from (3.12) that we have two types of terms in the stress tensor, $\langle T_{\mu\nu}\rangle=\tilde T^{(1)}_{A\,\mu\nu}+\tilde T^{(2)}_{A\,\mu\nu}$: the static terms $\tilde T^{(1)}_{A\,\mu\nu}$, which depend on the static part of $T_c^{\mu\nu}$; and the time-dependent terms $\tilde T^{(2)}_{A\,\mu\nu}$, which depend on the time dependent part, that is, the terms which are proportional to $\tau\theta(t-t')$. The subindex A stands for the part of the stress tensor related to the tensor $A^{\mu\nu}$ in (3.1). The static terms are easily computed, the $\int dx'\,dy'$ integration is trivial, and we use $\delta(x'^2+\lambda)$ to perform the $\int dt'$ integration, following Ref. [19]. These terms are all proportional to $1/r^4$. The time-dependent terms, although slightly more complicated, can be computed without difficulty. The final result for $r\neq0$ in the Cartesian coordinate basis is

$$\begin{aligned}&\langle T^t_t\rangle=-\frac{2\sigma}{r^4}\left[4-\tau\,f_{18}(b)\right],\qquad \langle T^z_z\rangle=\frac{4\sigma}{r^4}\left[1-\tau\,f_{12}(b)\right],\\&\langle T^a_t\rangle=\frac{3\sigma\,x^a}{r^5}\,\tau\,g(b),\qquad\qquad \langle T^a_b\rangle=\frac{\sigma}{r^4}\left\{3\delta^{ab}\left[2+\tau\,f_{12}(b)\right]-4\frac{x^ax^b}{r^2}\left[2+\tau\,f_{15}(b)\right]\right\},\end{aligned} \tag{3.13}$$

where $\sigma = 8\alpha G\mu\hbar$, the index $a = 1,2$ refers in x^a to $x^1 = x$, $x^2 = y$ (the string transversal coordinates), and

$$g(b) \equiv \mathcal{P}f\left[\frac{1}{(b^2-1)^{5/2}}\,\theta(b-1)\right], \quad f_C(b) \equiv \mathcal{P}f\left[\frac{b}{(b^2-1)^{5/2}}\,(2b^4-5b^2+C/4)\,\theta(b-1)\right]. \tag{3.14}$$

As above $\mathcal{P}f$ denotes the Hadamard finite part, which gives well-defined distributions on the light cone $b = 1$ (see Appendix A). It is easy to identify directly from (3.13) the static and time-dependent parts of the stress tensor: i.e., $\tilde{T}^{(1)}_{A\,\mu\nu}$ and $\tilde{T}^{(2)}_{A\,\mu\nu}$, respectively. Note that $\tilde{T}^{(1)}_{A\,\mu\nu}$ corresponds to the vacuum expectation value of the stress tensor due to a dust rod along the z axis with mass per unit length μ. It can be checked, computing the derivatives of the distributions (3.14) using the methods of Appendix A, that the $\langle T_{\mu\nu}\rangle$ just derived is conserved.

The most salient feature of $\langle T_{\mu\nu}\rangle$ is that it quickly settles down to the final static values. Note that when t grows keeping r fixed, $b \equiv t/r$ grows as t, and we can expand $g(b)$ and $f_C(b)$ in terms of b^{-1}: $g(b) \simeq b^{-5}[1+\frac{5}{2}\,b^{-2}+O(b^{-4})]$, $f_C(b) \simeq 2[1+O(b^{-4})]$. This means that $f_C(b)$ differs from the static value, 2, by terms of order b^{-4} and that $g(b)$ goes like b^{-5}. Thus $\langle T^{\mu\nu}(x)\rangle$ is effectively time dependent only when $t \sim r$; it reaches the static values very quickly.

The final static values of $\langle T_{\mu\nu}\rangle$, i.e., when $t \to \infty$, are read off from (3.13). In the polar coordinate basis $(\partial_t, \partial_r, \partial_\theta, \partial_z)$ they are

$$\lim_{t\to\infty}\langle T^{\mu}{}_{\nu}\rangle = -\frac{16\alpha G\mu\hbar}{r^4}\,\mathrm{diag}(4-2\tau, 1+\tau, -3-3\tau, -2+4\tau). \tag{3.15}$$

In particular when $\tau = 1$, i.e., the string is not wiggly, we get the well known results of Refs. [14,15].

B. Quantum stress tensor including $r = 0$

Since the quantum stress tensor is a source in the semiclassical Einstein's equations, we need to know $\langle T_{\mu\nu}\rangle$ in all the space-time. One way to do this is to try to generalize the previous calculation to include $r = 0$. Such a generalization should have the form

$$\langle T^{\mu\nu}(x)\rangle = \tilde{T}^{\mu\nu}_{B}(x) + \tilde{T}^{(1)\mu\nu}_{A}(x) + \tilde{T}^{(2)\mu\nu}_{A}(x), \tag{3.16}$$

where $\tilde{T}^{\mu\nu}_{B} \equiv -(\alpha\hbar/6)\,B^{\mu\nu}$ and $\tilde{T}^{(1)\mu\nu}_{A}$ and $\tilde{T}^{(2)\mu\nu}_{A}$ are the traceless tensors corresponding to the static and time-dependent parts, respectively, in the second term of (3.1). These should be some well-defined distributions which reduce to the expressions (3.13) for $r \neq 0$. Using Einstein's equations $G^{\mu\nu} = -8\pi G\,T_c^{\mu\nu}$ in (3.2), $\tilde{T}^{\mu\nu}_{B}$ can be expressed in terms of the classical stress tensor, and so its computation is straightforward. One finds the following non-null components in the Cartesian coordinate basis:

$$\begin{aligned}
\tilde{T}_{B\,t}^{\;t} &= -\frac{\pi}{3}\,\sigma(1+\tau\,\theta(t))\nabla^2(\delta(x)\delta(y)), \qquad \tilde{T}_{B\,t}^{\;a} = \frac{\pi}{3}\,\sigma\tau\,\delta(t)\,\partial_a(\delta(x)\delta(y)),\\
\tilde{T}_{B\,z}^{\;z} &= -\frac{\pi}{3}\,\sigma\left[(1+\tau\,\theta(t))\nabla^2(\delta(x)\delta(y)) - \tau\,\delta'(t)\,\delta(x)\delta(y)\right],\\
\tilde{T}_{B\,b}^{\;a} &= -\frac{\pi}{3}\,\sigma\Big\{\delta_{ab}\left[(1+\tau\,\theta(t))\nabla^2(\delta(x)\delta(y)) - \tau\,\delta'(t)\,\delta(x)\delta(y)\right]\\
&\quad -(1+\tau\,\theta(t))\partial_a\partial_b(\delta(x)\delta(y))\Big\}.
\end{aligned} \tag{3.17}$$

It is easy to see that the tensor $\tilde{T}^{\mu\nu}_{B}$ just found is conserved. From the expressions (3.17) it is clear that we can write $\tilde{T}^{\mu\nu}_{B} = \tilde{T}^{(1)\mu\nu}_{B} + \tilde{T}^{(2)\mu\nu}_{B}$ where, as always, $\tilde{T}^{(1)\mu\nu}_{B}$ refers to the static terms and $\tilde{T}^{(2)\mu\nu}_{B}$ to the time-dependent terms.

To generalize the calculation of the second term in (3.1) to include $r = 0$, we need to extend the representation (3.8) of the propagator $H(x-y,\bar{\mu})$ to all values of $(x-y)^\alpha$. Such an extension, which is derived in Appendix A, is

$$H(x-y,\bar{\mu}) = \lim_{\epsilon\to 0^+}\left\{\frac{1}{\pi}\,\delta'\left[(x-y)^2\right]\theta(x^0-y^0)\,\theta(|\,\vec{x}-\vec{y}\,|-\epsilon) + [\ln\bar{\mu}\epsilon + \gamma - 1]\,\delta^4(x-y)\right\}, \tag{3.18}$$

where γ is Euler's constant. For the static terms $\tilde{T}_A^{(1)\mu\nu}$ we find

$$\tilde{T}_A^{(1)\,t}{}_t = -4\pi\,\sigma\nabla^2 I(x,y), \quad \tilde{T}_A^{(1)\,z}{}_z = 2\pi\,\sigma\nabla^2 I(x,y),$$
$$\tilde{T}_A^{(1)\,a}{}_b = 2\pi\,\sigma\left(\delta_{ab}\nabla^2 - \partial_a\partial_b\right) I(x,y), \tag{3.19}$$

where $I(x,y) \equiv \int d^4x'\,H(-x',\bar{\mu})\,\delta(x+x')\delta(y+y')$. This integral is computed in Appendix A with the result

$$I = \frac{1}{2\pi}\mathcal{P}f\left(\frac{1}{r^2}\right) + \left(\ln\frac{\bar{\mu}}{2} + \gamma\right)\delta(x)\delta(y)$$
$$= \lim_{\epsilon\to 0^+}\left\{\frac{1}{2\pi}\frac{1}{r^2}\,\theta(r-\epsilon) + \left(\ln\frac{\bar{\mu}\epsilon}{2} + \gamma\right)\delta(x)\delta(y)\right\}. \tag{3.20}$$

Using the result given in Ref. [27], $\nabla^2\mathcal{P}f\left(r^{-2}\right) = 4\,\mathcal{P}f\left(r^{-4}\right) - 2\pi\nabla^2(\delta(x)\delta(y))$, one can substitute, in these expressions

$$\nabla^2 I = \frac{2}{\pi}\mathcal{P}f\left(\frac{1}{r^4}\right) + \left(\ln\frac{\bar{\mu}}{2} + \gamma - 1\right)\nabla^2(\delta(x)\delta(y)). \tag{3.21}$$

These results for $\tilde{T}_A^{(1)\mu\nu}$ agree with the static components of (3.13) when $r \neq 0$.

For the time-dependent terms $\tilde{T}_A^{(2)\mu\nu}$ the calculation is considerably more complicated. Thus, instead of computing these exactly, we will introduce the following approximations.

C. Approximated quantum stress tensor $\tilde{T}_A^{(2)\mu\nu}$

We introduce here some approximations in $\tilde{T}_A^{(2)\mu\nu}$. First, instead of working out the exact expresions for these terms to include $r = 0$, we will introduce a cutoff radius r_0 (which can be viewed as the physical string radius) and assume that the tensor $\tilde{T}_A^{(2)\mu\nu}$ derived in (3.13) is only valid for $r \geq r_0$. We will make some assumptions for the values of this tensor at $r < r_0$. At the end of the calculations we will take the limit $r_0 \to 0$. In addition to this, in the terms of (3.13) which correspond to $\tilde{T}_A^{(2)\mu\nu}$ we will substitute the distributions $f_C(b)$ and $g(b)$ by $2\,\theta(t-r)$ and zero, respectively. That is, we make a sudden approximation assuming that the values of $\tilde{T}^{(2)}_{A\,\mu\nu}$ change suddenly at the light cone $t = r$. This seems justified in view of the fact that such terms, as we have seen before, settle quickly to the final static values. But now, to ensure the conservation of $\tilde{T}^{(2)}_{A\,\mu\nu}$ on the light cone $t = r$, we need to add terms proportional to $\delta(t-r)$, which have the same singular behavior on the light cone as the stress tensor (3.13). Imposing also that $\tilde{T}_A^{(2)\,\mu}{}_\mu = 0$ we obtain the following approximated stress tensor for $r \geq r_0$:

$$\tilde{T}_A^{(2)\,\mu}{}_\nu = \frac{2\sigma\tau}{r^4}\left[\theta(t-r)\begin{pmatrix}2&&&\\&-1&&\\&&3&\\&&&-4\end{pmatrix} + r\,\delta(t-r)\begin{pmatrix}1&-1&&\\1&-1&&\\&&3&\\&&&-3\end{pmatrix}\right], \tag{3.22}$$

in polar coordinates. Now we have to make an assumption for the values of $\tilde{T}^{(2)}_{A\,\mu\nu}$ at $r < r_0$. We take a stress tensor with terms proportional to $\theta(t-r)$ and $r\,\delta(t-r)$ as before, but which is constant for $t > r$. Imposing that it is conserved, $\nabla_\mu\tilde{T}_A^{(2)\mu\nu} = 0$, and traceless, $\tilde{T}_A^{(2)\,\mu}{}_\mu = 0$, we find, for $r < r_0$,

$$\tilde{T}_A^{(2)\,\mu}{}_\nu = -\frac{2\sigma\tau}{r_0^4}\left[\theta(t-r)\begin{pmatrix}2&&&\\&1&&\\&&1&\\&&&-4\end{pmatrix} + r\,\delta(t-r)\begin{pmatrix}-1&1&&\\-1&1&&\\&&1&\\&&&-1\end{pmatrix}\right]. \tag{3.23}$$

Note that when we take the divergence of the complete $\tilde{T}_A^{(2)\,\mu}{}_\nu$ we have the step functions $\theta(r_0 - r)$ and $\theta(r - r_0)$ multiplying the above expressions which have to be derived too.

It can be seen also that the values of $\tilde{T}_A^{(2)\,\mu}{}_\nu$ for $r < r_0$ and $t > r$ obtained in (3.23) correspond to the values at $r = 0$ of the quantum stress tensor for a model based on the classical stress tensor for the string given in Ref. [11]. In such a model the string has a finite radius r_0 inside which the classical stress tensor is assumed to be constant; this tensor is given by

$$T_c{}^\mu{}_\nu = \varepsilon\,\theta(r_0 - r)\,\mathrm{diag}(1,0,0,\tau), \tag{3.24}$$

where we have introduced the τ parameter in order to allow for the possibility of a straight wiggly string. The string energy density ε is related to its energy per unit length μ (see Ref. [11]) by $4G\mu = 1 - \cos\left(r_0\sqrt{8\pi G\varepsilon}\right)$. At first order in $G\mu$ this relation gives $\pi G\varepsilon r_0^2 = G\mu + O(G^2\mu^2)$. Using the previous expressions (3.1) and (3.2), we can calculate the stress tensor $\langle T_{\mu\nu}\rangle$ to first order in $G\mu$ for a free massless conformally coupled scalar field with the classical source (3.24). The values of such tensor at $r = 0$ agree with these of the $\theta(t-r)$ terms in (3.23) and this gives further justification to the approximation taken.

IV. BACK-REACTION METRIC

In this section we compute the correction to the gravitational field due to the vacuum polarization of matter fields at one-loop order given in the previous section. For this we solve the semiclassical correction to Einstein's equations,

$$G^{\mu\nu}(x) = -8\pi G\,[T_c^{\mu\nu}(x) + \langle T^{\mu\nu}(x)\rangle], \tag{4.1}$$

in the linear approximation. Within this approximation the metric tensor can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{4.2}$$

where $h_{\mu\nu}$ is the metric perturbation due to the classical string stress tensor $T_c^{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ is the quantum correction to the metric which is of order $\hbar$. The classical part $h_{\mu\nu}$ has already been computed in Sec. II, and we note here that $\langle T^{\mu\nu}(x)\rangle$ depends on that part only, since including $\tilde{h}_{\mu\nu}$ in $\langle T^{\mu\nu}(x)\rangle$ would lead to terms of order $\hbar^2$ which we neglect. Thus Eq. (4.1) leads in the harmonic gauge $(\tilde{h}^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}\tilde{h}^{\alpha}{}_{\alpha})_{,\nu} = 0$ to the equation

$$\Box\tilde{h}_{\mu\nu} = -16\pi G\tilde{S}_{\mu\nu}, \tag{4.3}$$

where $\tilde{S}_{\mu\nu} \equiv \langle T_{\mu\nu}\rangle - \frac{1}{2}\,\eta_{\mu\nu}\langle T^{\alpha}{}_{\alpha}\rangle$. A formal solution of this equation is given as an initial value problem by (2.5), where one substitutes $h_{\mu\nu}$ by $\tilde{h}_{\mu\nu}$ and $S_{\mu\nu}$ by $\tilde{S}_{\mu\nu}$. Now from the decomposition (3.16) for the stress tensor and the separation of $\tilde{T}_{B\,\mu\nu}$ into a static and a time-dependent parts $\tilde{T}^{(1)}_{B\,\mu\nu}$ and $\tilde{T}^{(2)}_{B\,\mu\nu}$, it is clear that a similar separation can be performed in $\tilde{S}_{\mu\nu}$: i.e.,

$$\tilde{S}_{\mu\nu} = \tilde{T}^{(1)}_{A\,\mu\nu} + \tilde{S}^{(1)}_{B\,\mu\nu} + \tilde{T}^{(2)}_{A\,\mu\nu} + \tilde{S}^{(2)}_{B\,\mu\nu}, \tag{4.4}$$

where we use that $\tilde{T}_{A\,\mu\nu}$ is traceless. Then (4.3) leads to likewise separate quantum perturbations $\tilde{h}_{\mu\nu} = \tilde{h}^{(1)}_{A\,\mu\nu} + \tilde{h}^{(1)}_{B\,\mu\nu} + \tilde{h}^{(2)}_{A\,\mu\nu} + \tilde{h}^{(2)}_{B\,\mu\nu} \equiv \tilde{h}^{(1)}_{\mu\nu} + \tilde{h}^{(2)}_{\mu\nu}$. The main advantage of this separation is that the initial conditions at the surface $t=0$ for $\tilde{h}^{(2)}_{\mu\nu}$ are simply

$$\tilde{h}^{(2)}_{\mu\nu}\Big|_{t=0} = \partial_t\tilde{h}^{(2)}_{\mu\nu}\Big|_{t=0} = 0; \tag{4.5}$$

consequently, the explicit form of $\tilde{h}^{(2)}_{A\,\mu\nu}(t,\vec{x})$ is given by

$$\tilde{h}^{(2)}_{A\,\mu\nu}(t,\vec{x}) = -16\pi G\int_0^t dt'\int_{\mathbb{R}^3} d^3x'\,D_R(x-x')\,\tilde{T}^{(2)}_{A\,\mu\nu}(t',\vec{x}'). \tag{4.6}$$

Similarly $\tilde{h}^{(2)}_{B\,\mu\nu}$ is given by this expression changing $\tilde{T}^{(2)}_{A\,\mu\nu}$ by $\tilde{S}^{(2)}_{B\,\mu\nu}$.

We are only interested in finding these semiclassical perturbations $\tilde{h}_{\mu\nu}$ outside the string, that is, for $r \neq 0$.

A. Static part

For the static correction $\tilde{h}^{(1)}_{\mu\nu}$, Eq. (4.3) reduces to

$$\nabla^2\tilde{h}^{(1)}_{\mu\nu} = 16\pi G\,\tilde{S}^{(1)}_{\mu\nu}, \tag{4.7}$$

where $\tilde{S}^{(1)}_{\mu\nu} \equiv \tilde{T}^{(1)}_{A\,\mu\nu} + \tilde{S}^{(1)}_{B\,\mu\nu}$. It is very simple to solve these equations for $r \neq 0$ making use of the cylindrical symmetry of the problem. Notice that $\tilde{S}^{(1)}_{B\,\mu\nu}$ has support on $r=0$, and so from (3.13), we have in the polar coordinate basis $\tilde{S}^{(1)\,\nu}_{\;\mu} = \tilde{T}^{(1)\,\nu}_{A\;\mu} = -2\sigma\,r^{-4}\,\mathrm{diag}(4,1,-3,-2)$ for $r \neq 0$. It is now easy to see that the functions (which are defined in terms of the metric perturbations) $\tilde{h}^{(1)}_1 \equiv \tilde{h}^{(1)t}{}_t$, $\tilde{h}^{(1)}_2 \equiv \tilde{h}^{(1)z}{}_z$, $\tilde{h}^{(1)}_3 \equiv \tilde{h}^{(1)r}{}_r + \tilde{h}^{(1)\theta}{}_\theta$, and $\tilde{h}^{(1)}_4 \equiv \tilde{h}^{(1)r}{}_r - \tilde{h}^{(1)\theta}{}_\theta$ depend on r only and satisfy the equations, for $r \neq 0$,

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 4\,\frac{\delta_{k4}}{r^2}\right)\tilde{h}^{(1)}_k = \frac{64\pi\sigma G}{r^4}\,\gamma_k, \tag{4.8}$$

where $\gamma_1 = \gamma_4 = -2$ and $\gamma_2 = \gamma_3 = 1$. The harmonic gauge condition in terms of these functions can be written as $(\tilde{h}^{(1)}_1 + \tilde{h}^{(1)}_2 - \tilde{h}^{(1)}_4)_{,r} = 2r^{-1}\tilde{h}^{(1)}_4$. The general solution of these differential equations can be expressed as a linear combination of terms such as r^2, $\ln r$, r^{-2} and $r^{-2}\ln r$. Finally, after a gauge transformation, within the harmonic gauge, the solution can be written as

$$\begin{aligned} \tilde{h}^{(1)}_1 &= -32\pi\sigma G\,\tfrac{1}{r^2} + \sigma G\,A\,\ln\left(\tfrac{r}{R_1}\right), & \tilde{h}^{(1)}_2 &= 16\pi\sigma G\,\tfrac{1}{r^2} - \sigma G\,A\,\ln\left(\tfrac{r}{R_2}\right),\\ \tilde{h}^{(1)}_3 &= 16\pi\sigma G\,\tfrac{1}{r^2}, & \tilde{h}^{(1)}_4 &= 32\pi\sigma G\,\tfrac{1}{r^2}\,\ln\left(\tfrac{r}{R}\right), \end{aligned} \tag{4.9}$$

where the constants of integration R_1, R_2, and R have dimensions of length and A has dimensions of $(\text{length})^{-2}$. Notice that there is a gauge freedom within the harmonic gauge to set the constants R_1, R_2, and R to a fixed value, i.e., a change in the values of these constants is equivalent to an harmonic gauge transformation. The values of these constants in the solution for $r \neq 0$ can be determined if we solve Eqs. (4.7) including $r=0$. However, without working out the explicit solution, we can use dimensional arguments to see that A must be zero. Note that the only dimensional constant parameter which can be used to make $\tilde{h}^{(1)}_{\mu\nu}/\sigma G$ is $\bar{\mu}$, but since the dependence in this parameter must be logarithmic we are forced to take $A=0$; on the other hand, R will be proportional to $1/\bar{\mu}$. In fact, we can set $R = 1/\bar{\mu}$ after an harmonic gauge transformation for $r \neq 0$.

By doing a little more work, we can also arrive at the above solutions solving explicitly Eqs. (4.7) with the inclusion of $r=0$. In fact, from (3.17), and (3.19) it is easy to see that the equations for $\tilde{h}_1^{(1)}$, $\tilde{h}_2^{(1)}$, and $\tilde{h}^{(1)x}_{\ \ x} + \tilde{h}^{(1)y}_{\ \ y}$ are

$$\nabla^2 \left[\tilde{h}_1^{(1)} + 16\pi^2\sigma G \left(4\,I - \frac{1}{6}\,\delta(x)\delta(y) \right) \right] = 0,$$
$$\nabla^2 \left[\tilde{h}_2^{(1)} - 16\pi^2\sigma G \left(2\,I + \frac{1}{6}\,\delta(x)\delta(y) \right) \right] = 0,$$
$$\nabla^2 \left[\tilde{h}^{(1)x}_{\ \ x} + \tilde{h}^{(1)y}_{\ \ y} - 32\pi^2\sigma G \left(I + \frac{1}{3}\,\delta(x)\delta(y) \right) \right] = 0, \tag{4.10}$$

where I is given by (3.20). These give the solutions $\tilde{h}_1^{(1)} = -16\pi^2\sigma G(4\,I - \frac{1}{6}\,\delta(x)\delta(y))$, $\tilde{h}_2^{(1)} = 16\pi^2\sigma G(2\,I + \frac{1}{6}\,\delta(x)\delta(y))$, and $\tilde{h}^{(1)x}_{\ \ x} + \tilde{h}^{(1)y}_{\ \ y} = 32\pi^2\sigma G(I + \frac{1}{3}\,\delta(x)\delta(y))$, which for $r \neq 0$ reduce to (4.9) with $A = 0$. If we write the remaining two equations for $\tilde{h}^{(1)x}_{\ \ x} - \tilde{h}^{(1)y}_{\ \ y}$ and $\tilde{h}^{(1)x}_{\ \ y}$, using that $\delta(x)\delta(y) = \frac{1}{2\pi}\,\nabla^2 \ln r$, we see that for $r \neq 0$ one has $\tilde{h}_4^{(1)} = 32\pi\sigma G\, r^{-2}\, \ln(\kappa\bar{\mu} r)$, where κ is some numerical factor. So we finally have, for $r \neq 0$,

$$\begin{aligned} \tilde{h}_1^{(1)} &= -32\pi\sigma G\,\tfrac{1}{r^2}, & \tilde{h}_2^{(1)} &= 16\pi\sigma G\,\tfrac{1}{r^2}, \\ \tilde{h}_3^{(1)} &= 16\pi\sigma G\,\tfrac{1}{r^2}, & \tilde{h}_4^{(1)} &= 32\pi\sigma G\,\tfrac{1}{r^2}\,\ln \bar{\mu} r. \end{aligned} \tag{4.11}$$

This solution can also be obtained by the procedure which we will use next to obtain the time-dependent metric perturbations. Namely, we introduce a cutoff radius r_0 and take the limit $r_0 \to 0$ at the end of the calculation. For this we use an approximated stress tensor $\tilde{T}^{(1)}_{A\,\mu\nu}$ for $r < r_0$ as in Sec. III C.

Let us now consider the contribution $\triangle\tilde{\theta}^{(1)}$ to the deficit angle of the two-surfaces $t = \text{const}$, $z = \text{const}$ due to these static corrections to the metric. Substituting in (2.12) the values of $\tilde{h}^{(1)}_{\mu\nu}$ given by (4.11) we find

$$\triangle\tilde{\theta}^{(1)} = \frac{32\pi^2\sigma G}{r^2}. \tag{4.12}$$

We can now calculate the static quantum correction to the Riemann tensor. Using the expressions derived in Appendix B we find the following nonvanishing components for $r \neq 0$

$$\begin{aligned} \tilde{R}^{(1)}_{xzxz} &= \frac{16\pi\sigma G}{r^4}\,(2\cos 2\theta + 1), & \tilde{R}^{(1)}_{xzyz} &= \frac{32\pi\sigma G}{r^4}\,\sin 2\theta, & \tilde{R}^{(1)}_{xyxy} &= \frac{32\pi\sigma G}{r^4}, \\ \tilde{R}^{(1)}_{txtx} &= \frac{32\pi\sigma G}{r^4}\,(2\cos 2\theta + 1), & \tilde{R}^{(1)}_{txty} &= \frac{64\pi\sigma G}{r^4}\,\sin 2\theta, \end{aligned} \tag{4.13}$$

and also $\tilde{R}^{(1)}_{yzyz}$ and $\tilde{R}^{(1)}_{tyty}$, which are obtained from $\tilde{R}^{(1)}_{xzxz}$ and $\tilde{R}^{(1)}_{txtx}$ interchanging $\cos\theta$ by $\sin\theta$.

B. Time-dependent part

Let us first consider the contribution to the quantum perturbations to the metric $\tilde{h}^{(2)}_{A\,\mu\nu}$ coming from $\tilde{T}^{(2)}_{A\,\mu\nu}$ given by (3.22) and (3.23). This solution is given by (4.6). Defining, as we have done before, $\tilde{h}^{(2)}_{A_1} \equiv \tilde{h}^{(2)t}_{A\ \ t}$, $\tilde{h}^{(2)}_{A_2} \equiv \tilde{h}^{(2)z}_{A\ \ z}$, $\tilde{h}^{(2)}_{A_3} \equiv \tilde{h}^{(2)r}_{A\ \ r} + \tilde{h}^{(2)\theta}_{A\ \ \theta}$, $\tilde{h}^{(2)}_{A_4} \equiv \tilde{h}^{(2)r}_{A\ \ r} - \tilde{h}^{(2)\theta}_{A\ \ \theta}$, and $\tilde{h}^{(2)}_{A_5} \equiv \tilde{h}^{(2)r}_{A\ \ t}$; this leads for $(t - r) > 2r_0$ and $r > r_0$ to the solution

$$\tilde{h}^{(2)}_{A_k} = -\frac{32\sigma\tau G}{r^2} \int_0^\pi d\theta\, g_k(\theta) \left\{ \frac{1}{x_0^4} \int_0^{x_0} dx\, x \left[C_k + \bar{C}_k\, x\, \frac{\partial}{\partial b} \right] + \int_{x_0}^{\frac{1}{2}\left(\frac{b^2-1}{b-\cos\theta}\right)} \frac{dx}{x^3} \left[D_k + \bar{D}_k\, x\, \frac{\partial}{\partial b} \right] \right\} f(b; x, \theta), \tag{4.14}$$

where $b \equiv t/r$, $x_0 \equiv r_0/r$, and

$$\begin{aligned} &C_k \equiv -2\,\delta_{k1} + 4\,\delta_{k2} - 2\,\delta_{k3}, \quad \bar{C}_k \equiv \delta_{k1} + \delta_{k2} - 2\,\delta_{k3} + \delta_{k5}, \\ &D_k \equiv 2\,\delta_{k1} - 4\,\delta_{k2} + 2\,\delta_{k3} - 4\,\delta_{k4}, \quad \bar{D}_k \equiv \delta_{k1} - 3\,\delta_{k2} + 2\,\delta_{k3} - 4\,\delta_{k4} + \delta_{k5}, \\ &g_k(\theta) \equiv \begin{cases} 1, & k = 1,2,3 \\ \cos 2\theta, & k = 4 \\ \cos\theta, & k = 5, \end{cases} \\ &f(b; x, \theta) \equiv \ln\left[b - x + \sqrt{2x(\cos\theta - b) + b^2 - 1} \right] - \frac{1}{2}\,\ln(x^2 - 2x\cos\theta + 1). \end{aligned} \tag{4.15}$$

We now expand the solution (4.14) in terms of $x_0 \equiv r_0/r$, neglecting terms of order x_0 which will vanish in the limit $r_0 \to 0$. We find

$$\tilde{h}^{(2)}_{A_k} = -\frac{32\pi\sigma\tau G}{r^2}\, \left[\ln x_0\, \Phi^{\rm II}_k(b) + \Phi^{\rm I}_k(b) + O(x_0) \right], \tag{4.16}$$

where

$$\Phi_1^{\rm II} = -\Phi_3^{\rm II} \equiv \frac{1}{2}\,\frac{b}{(b^2-1)^{3/2}}, \qquad \Phi_2^{\rm II} \equiv 0, \qquad \Phi_4^{\rm II} \equiv \frac{1}{2}\,\frac{b}{(b^2-1)^{3/2}}\,(2b^2-3),$$

$$\Phi_5^{\rm II} \equiv \frac{1}{2}\,\frac{1}{(b^2-1)^{3/2}},$$

$$\Phi_1^{\rm I} \equiv \frac{1}{2}\left\{-1-\frac{1}{(b^2-1)}+\frac{b}{(b^2-1)^{3/2}}\left[\frac{9}{4}+\operatorname{arccosh} b-\ln 4(b^2-1)\right]\right\},$$

$$\Phi_2^{\rm I} \equiv 1+\frac{1}{2}\,\frac{b}{(b^2-1)^{3/2}},$$

$$\Phi_3^{\rm I} \equiv \frac{1}{2}\left\{-1+\frac{1}{(b^2-1)}-\frac{b}{(b^2-1)^{3/2}}\left[\frac{13}{4}+\operatorname{arccosh} b-\ln 4(b^2-1)\right]\right\},$$

$$\Phi_4^{\rm I} \equiv 3-\frac{(2b^6-2b^2-3)}{(b^2-1)}+\frac{1}{4}\,\frac{b}{(b^2-1)^{3/2}}\,(2b^2-1)\,(4b^4-11)+\operatorname{arccosh} b$$
$$+\frac{1}{2}\,\frac{b}{(b^2-1)^{3/2}}\,(2b^2-3)\left[3\ln 2b-2\operatorname{arccosh} b-\ln 4(b^2-1)-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right],$$

$$\Phi_5^{\rm I} \equiv \frac{1}{2}\left\{-\frac{b}{(b^2-1)}+\frac{1}{(b^2-1)^{3/2}}\left[\frac{5}{4}+b^2+\operatorname{arccosh} b-\ln 4(b^2-1)\right]\right\}. \tag{4.17}$$

It is easy to see that we can change in (4.16) $\ln x_0$ by $\ln(R/r)$, where R is an arbitrary constant with dimensions of length. For this we make the harmonic gauge transformation

$$\xi_t = -8\pi\sigma\tau G\,\frac{1}{r\sqrt{b^2-1}}\,\ln\frac{r_0}{R}, \qquad \xi_z = 0,$$
$$\xi_x = 8\pi\sigma\tau G\,\frac{b\cos\theta}{r\sqrt{b^2-1}}\,\ln\frac{r_0}{R}, \qquad \xi_y = 8\pi\sigma\tau G\,\frac{b\sin\theta}{r\sqrt{b^2-1}}\,\ln\frac{r_0}{R}. \tag{4.18}$$

We can also set $R = 1/\bar{\mu}$ and, after this, we can take the limit $r_0 \to 0$ in $\tilde{h}^{(2)}_{A\,k}$.

Let us now consider the terms $\tilde{h}^{(2)}_{B\,\mu\nu}$. Substituting $\tilde{T}^{(2)}_{A\,\mu\nu}$ by $\tilde{S}^{(2)}_{B\,\mu\nu}$ in (4.6) and defining $\tilde{h}^{(2)}_{B\,k}$ as before we find, for $r \neq 0$,

$$\tilde{h}^{(2)}_{B\,k} = -\frac{32\pi\sigma\tau G}{r^2}\,\frac{1}{6}\,\mathcal{P}f\left[\Phi_k^{\rm II}(b)\,\theta(b-1)\right], \tag{4.19}$$

where the functions $\Phi_k^{\rm II}(b)$ are defined in (4.17). It is easy to see that these terms can be eliminated with a gauge transformation of type (4.18). Thus we finally have, for $r \neq 0$ and $b > 1$,

$$\tilde{h}^{(2)}_{k} = -\frac{32\pi\sigma\tau G}{r^2}\left[\Phi_k^{\rm I}(b)-\ln\bar{\mu}r\,\Phi_k^{\rm II}(b)\right]. \tag{4.20}$$

We can now compute the time-dependent quantum contribution $\triangle\tilde{\theta}^{(2)}$ to the deficit angle of the two-surfaces $t=$ const, $z=$ const. Substituting in (2.12) the previous values of $\tilde{h}^{(2)}_{\mu\nu}$ we get

$$\triangle\tilde{\theta}^{(2)} = \frac{8\pi^2\sigma\tau G}{r^2}\left\{2-\frac{1}{(b^2-1)^2}\,(4b^4+13b^2-2)+\frac{3\,b}{(b^2-1)^{5/2}}\left[\frac{1}{4}\,(8b^4+8b^2-15)\right.\right.$$
$$\left.\left.+3\,(\ln 2b-\operatorname{arccosh} b)-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right]\right\}. \tag{4.21}$$

Expanding this expression in t^{-1} with r fixed one finds

$$\triangle\tilde{\theta}^{(2)} = \frac{16\pi^2\sigma\tau G}{r^2}\left[2-\frac{3}{b^4}+O\left(\frac{1}{b^6}\right)\right]; \tag{4.22}$$

that is, $\triangle\tilde{\theta}^{(2)}$ reaches its static value $32\pi^2\sigma\tau G\,r^{-2}$ very quickly. Note that this contribution due to the time-dependent terms is exactly the same as the contribution $\triangle\tilde{\theta}^{(1)}$ of (4.12) when the string is unperturbed ($\tau = 1$).

Finally we can calculate the time-dependent part of the Riemann tensor $\tilde{R}^{(2)}_{\mu\nu\alpha\beta}$. The substitution of the terms h_k of Appendix B by the terms $\tilde{h}^{(2)}_k$ of (4.20) gives, for $r \neq 0$ and $b > 1$,

$$
\begin{aligned}
\tilde{R}^{(2)}_{tztz} &= -\frac{24\pi\sigma\tau G}{r^4}\,\frac{b}{(b^2-1)^{7/2}}\,(2b^2+3),\\
\tilde{R}^{(2)}_{xzxz} &= -\frac{4\pi\sigma\tau G}{r^4}\left[8+\frac{3\,b}{(b^2-1)^{7/2}}\,(2b^2+3)+\cos 2\theta\left(16+\frac{15\,b}{(b^2-1)^{7/2}}\right)\right],\\
\tilde{R}^{(2)}_{xzyz} &= -\frac{4\pi\sigma\tau G}{r^4}\,\sin 2\theta\left(16+\frac{15\,b}{(b^2-1)^{7/2}}\right),\\
\tilde{R}^{(2)}_{txtx} &= -\frac{4\pi\sigma\tau G}{r^4}\left(4-\frac{3\,b}{(b^2-1)^{7/2}}\,(2b^2+3)+\cos 2\theta\left\{8+\frac{15\,b^2}{(b^2-1)^3}\,(2b^2+3)\right.\right.\\
&\quad -\frac{15\,b}{(b^2-1)^{7/2}}\left[\frac{1}{4}\,(8b^4+8b^2-11)+3\,(\ln 2b-\operatorname{arccosh} b)\right.\\
&\quad \left.\left.\left.-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right]\right\}\right),\\
\tilde{R}^{(2)}_{txty} &= -\frac{4\pi\sigma\tau G}{r^4}\,\sin 2\theta\left\{8+\frac{15\,b^2}{(b^2-1)^3}\,(2b^2+3)-\frac{15\,b}{(b^2-1)^{7/2}}\left[\frac{1}{4}\,(8b^4+8b^2-11)\right.\right.\\
&\quad \left.\left.+3\,(\ln 2b-\operatorname{arccosh} b)-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right]\right\},\\
\tilde{R}^{(2)}_{tzxz} &= \frac{24\pi\sigma\tau G}{r^4}\,\cos\theta\,\frac{1}{(b^2-1)^{7/2}}\,(4b^2+1),\\
\tilde{R}^{(2)}_{txxy} &= \frac{12\pi\sigma\tau G}{r^4}\,\sin\theta\left\{\frac{b}{(b^2-1)^3}\,(8b^4+14b^2+3)-\frac{(4b^2+1)}{(b^2-1)^{7/2}}\left[\frac{1}{4}\,(8b^4+8b^2-15)\right.\right.\\
&\quad \left.\left.+3\,(\ln 2b-\operatorname{arccosh} b)-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right]\right\},\\
\tilde{R}^{(2)}_{xyxy} &= -\frac{4\pi\sigma\tau G}{r^4}\left\{-4-\frac{1}{(b^2-1)^3}\,(16b^6+24b^4+39b^2-4)\right.\\
&\quad +\frac{3\,b}{(b^2-1)^{7/2}}\,(2b^2+3)\left[\frac{1}{4}\,(8b^4+8b^2-15)+3\,(\ln 2b-\operatorname{arccosh} b)\right.\\
&\quad \left.\left.-\frac{5}{4}\,\frac{1}{b^2}\,{}_3F_2\left(1,1,\frac{7}{2};2,4;\frac{1}{b^2}\right)\right]\right\},
\end{aligned}
\tag{4.23}
$$

where ${}_3F_2$ denotes a generalized hypergeometric function. As before $\tilde{R}^{(2)}_{yzyz}$, $\tilde{R}^{(2)}_{tyty}$, $\tilde{R}^{(2)}_{tzyz}$, and $\tilde{R}^{(2)}_{tyyx}$ are obtained interchanging $\cos\theta$ by $\sin\theta$ in $\tilde{R}^{(2)}_{xzxz}$, $\tilde{R}^{(2)}_{txtx}$, $\tilde{R}^{(2)}_{tzxz}$, and $\tilde{R}^{(2)}_{txxy}$, respectively. Notice that $\tilde{R}^{(2)}_{\mu\nu\alpha\beta}$ does not depend on the arbitrary parameter $\bar{\mu}$. This is due to the fact that, as we have seen before, the value of $\bar{\mu}$ can be changed by a gauge transformation. The static limit of these components, i.e., $\lim_{t\to\infty}\tilde{R}^{(2)}_{\mu\nu\alpha\beta} \equiv R^L_{\mu\nu\alpha\beta}$, is

$$
\begin{aligned}
R^L_{xzxz} &= -\frac{32\pi\sigma\tau G}{r^4}\,(2\cos 2\theta+1), \quad R^L_{xzyz} = -\frac{64\pi\sigma\tau G}{r^4}\,\sin 2\theta, \quad R^L_{xyxy} = \frac{32\pi\sigma\tau G}{r^4},\\
R^L_{txtx} &= -\frac{16\pi\sigma\tau G}{r^4}\,(2\cos 2\theta+1), \quad R^L_{txty} = -\frac{32\pi\sigma\tau G}{r^4}\,\sin 2\theta,
\end{aligned}
\tag{4.24}
$$

and R^L_{yzyz} and R^L_{tyty} are found interchanging $\cos\theta$ by $\sin\theta$ in $\tilde{R}^L_{xzxz}$ and $\tilde{R}^L_{txtx}$, respectively. From (4.23) we see that these final static values are quickly reached since the corrections for large times go at least as $b^{-4} = r^4/t^4$. The final semiclassical Riemann components are obtained by adding to the classical values (2.15) the back-reaction corrections (4.13) and (4.23).

V. CONCLUSIONS

In this paper we have derived the back reaction, due to quantum fluctuations of matter fields, on the gravitational field of a cosmic string during and after its formation. As matter fields we have just considered a massless conformally coupled scalar field but the results are easily extrapolated to N of such fields. As a model of cosmic string formation we take an initial thin straight rod whose tension grows suddenly (step approximation) from zero to a maximum value which corresponds to the mass per unit length if the string is unperturbed, or to a smaller tension if the string is wiggly.

Within the linear approximation of Einstein's equations we have first computed the metric perturbations and the corresponding curvature tensor for both unperturbed and wiggly strings. If the final string is unperturbed, the Newtonian potential vanishes and the spacetime becomes flat with a deficit angle, but if the string is wiggly there is a Newtonian force per unit mass which goes like $G\mu(1-\tau)r^{-1}$. This force may play an important role in the formation of wakes behind long strings [3].

We have then computed the vacuum expectation value of the stress tensor of matter coupled linearly to the string gravitational field. We have seen that after formation the stress tensor settles quickly to that of the final unperturbed or wiggly static string. The stress tensor has an energy density which goes like $N\hbar G\mu r^{-4}$ (as typical of Casimir-type energies), where we have assumed N matter fields which correspond to massless fields or with masses less than m_H, where m_H is the Higgs boson mass responsible for symmetry breaking.

With such a stress tensor as a source we have computed the perturbation induced on the gravitational field. We have also computed the Riemann tensor components, which give essentially the tidal forces, corresponding to such quantum corrections. The static part of this tensor correspond to a Newtonian force per unit mass which goes like $N\hbar G^2\mu r^{-3}$. The time-dependent part of the semiclassical perturbations to the metric and the Riemann tensor, on the other hand, are quite complicated. But, for a given radius, after a short time $t > r$ the curvature tensor becomes static. Unlike for the classical part the quantum correction to the gravitational field does not differ substantially if the final string is unperturbed or wiggly.

Let us now discuss the importance of the quantum corrections in the case of a wiggly string and the unperturbed string after their formation. In the case of a wiggly string, i.e., $\tau < 1$ (typically $\tau \sim 0.5$) [3], the ratio between the quantum F_q and classical F_c forces on surrounding nonrelativistic particles can be estimated in the static limit as $F_q/F_c \sim 64\pi\alpha N\hbar G\,(2-\tau)/\left[(1-\tau)\,r^2\right]$; i.e., if l_P is the Planck length, $F_q/F_c \sim 10^{-2}N\,(l_P/r)^2$. This means that, unless N is unreasonably large, the quantum effects are always negligible if $r \gtrsim 10 l_P$. Note that for a cosmic string the smallest value that r can take is $r_0 \sim \hbar/m_H$ and, since $\mu \sim m_H^2/\hbar$, we have $F_q/F_c(r) < F_q/F_c\,(r_0) \sim 10^{-2}NG\mu$ which for GUT strings is of order 10^{-5}–10^{-6}.

For an unperturbed string $\tau = 1$ there is no classical force F_c. Thus the quantum correction will be responsible for a Newtonian force on the classical matter surrounding the string. However, that force decreases like r^{-3} which means that it becomes negligible very quickly at macroscopic distances from the string. In this case it is better to consider the deficit angle which is an important physical observable. The deficit angle can be written as a sum of a classical term plus a term of quantum origin $\triangle\theta + \triangle\tilde{\theta}$, using the notation of the last section. As we have seen, for the classical part $\triangle\theta \sim G\mu$ whereas for the quantum part $\triangle\tilde{\theta} \sim 10^{-2}N\hbar G^2\mu r^{-2}$, so that $\triangle\tilde{\theta}/\triangle\theta \sim 10^{-2}N\,(l_p/r)^2$ which is of the same order as the ratio between the quantum and classical forces given before. The ratio of the transverse velocities on the string wakes due to the quantum effect and the classical deficit angle are also of this order. These results hint that whereas one should not expect any quantum effect at macroscopic distances on surrounding matter, quantum effects might start to become important near the string at microscopic distances. In particular, when two strings cross, there might be a correction, perhaps in the form of a Casimir-like force, due to quantum effects. However, in this case our classical picture breaks down and one must consider the dynamics of the Higgs fields themselves at microscopical level.

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APPENDIX A

1. Hadamard finite part

We have introduced in this paper some singular distributions denoted by the symbol $\mathcal{P}f$. They are generated by the Hadamard finite part of a divergent integral. See Refs. [27,28] for more details on these distributions. The idea is the following: Suppose that we have a function $h(x)$ which is Lebesgue integrable on all the intervals $(a+\epsilon, b)$, with $\epsilon > 0$, but which is not integrable on (a,b) (a, b are some real constants). Let us consider the integral

$$J(\epsilon) \equiv \int_{a+\epsilon}^{b} dx\, h(x), \tag{A1}$$

which diverges in the limit $\epsilon \to 0^+$, and assume that it is possible to separate it in two parts:

$$J(\epsilon) = I(\epsilon) + F(\epsilon), \tag{A2}$$

where $I(\epsilon)$ is a finite linear combination of negative powers of ϵ and positive powers of $\ln\epsilon$ and $F(\epsilon)$ has a finite well-defined limit as $\epsilon \to 0^+$ (in the sense that is independent of the way by which is obtained). Then the Hadamard finite part of the divergent integral $\int_a^b dx\, h(x)$ is defined by

$$\mathcal{F}p \int_a^b dx\, h(x) \equiv \lim_{\epsilon\to 0^+} F(\epsilon) = \lim_{\epsilon\to 0^+} \left\{ \int_{a+\epsilon}^{b} dx\, h(x) - I(\epsilon) \right\}; \tag{A3}$$

that is, we throw away the divergent terms in (A2) and then take the limit $\epsilon \to 0^+$. The definition is readily extended to functions $h(x)$ defined on all the real line which have singular points. All we have to do is to decompose the integral over $\mathbb{R}$ as a sum of integrals of the kind considered above. Then one can define distributions denoted by $\mathcal{P}f\,[h(x)]$ as

$$\int_{-\infty}^{\infty} dx\, \mathcal{P}f\,[h(x)]\, \varphi(x) \equiv \mathcal{F}p \int_{-\infty}^{\infty} dx\, h(x)\, \varphi(x), \quad \text{(A4)}$$

where $\varphi(x)$ is an arbitrary tempered test function. The definition can be easily generalized to the case of several variables whenever the divergent integrals can be reduced to one-dimensional divergent integrals.

As an example we calculate the derivative of the distribution $(b^2-1)^{-1/2}\,\theta(b-1)$ in detail. Notice that it is a distribution in a one-dimensional space (and also in a four-dimensional space if we set $b \equiv t/r$) since the integral $\int_{-\infty}^{\infty} db\,(b^2-1)^{-1/2}\,\theta(b-1)\,\varphi(b) = \int_1^{\infty} db\,(b^2-1)^{-1/2}\,\varphi(b)$ is convergent. To start with, it is easy to show that

$$\frac{d}{db}[(b^2-1)^{-1/2}\,\theta(b-1)] = \mathcal{P}f[-b\,(b^2-1)^{-3/2} \times\theta(b-1)]. \quad \text{(A5)}$$

For this, let us consider the integral

$$\int_{1+\epsilon}^{\infty} db\left[-b\,(b^2-1)^{-3/2}\right]\varphi(b) = \int_{1+\epsilon}^{\infty} db\,\frac{d}{db}\left[(b^2-1)^{-1/2}\right]\varphi(b) = -(b^2-1)^{-1/2}\,\varphi(b)\Big|_{b=1+\epsilon} - \int_{1+\epsilon}^{\infty} db\,(b^2-1)^{-1/2}\,\varphi'(b), \quad \text{(A6)}$$

where we have integrated by parts in the last step. This shows that

$$\begin{aligned}\int_{-\infty}^{\infty} db\,\mathcal{P}f\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1)\right]\varphi(b) &\equiv \mathcal{F}p\int_1^{\infty} db\left[-b\,(b^2-1)^{-3/2}\right]\varphi(b)\\ &= -\int_1^{\infty} db\,(b^2-1)^{-1/2}\,\varphi'(b)\\ &= \int_{-\infty}^{\infty} db\,\frac{d}{db}\left[(b^2-1)^{-1/2}\,\theta(b-1)\right]\varphi(b),\end{aligned} \quad \text{(A7)}$$

where the definition of the derivative of a distribution has been used, and this proves (A5). An explicit expression for the distribution $\mathcal{P}f\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1)\right]$ can be derived as a distributional limit. In fact, from (A6) we have

$$\begin{aligned}&\int_{-\infty}^{\infty} db\,\mathcal{P}f\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1)\right]\varphi(b)\\ &= \lim_{\epsilon\to 0^+}\left\{\int_{1+\epsilon}^{\infty} db\left[-b\,(b^2-1)^{-3/2}\right]\varphi(b) + \varphi(b)\,(b^2-1)^{-1/2}\Big|_{b=1+\epsilon}\right\}\\ &= \lim_{\epsilon\to 0^+}\left\{\int_{1+\epsilon}^{\infty} db\left[-b\,(b^2-1)^{-3/2}\right]\varphi(b) + \frac{1}{\sqrt{2\epsilon}}\,\varphi(1)\right\}\\ &= \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} db\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1-\epsilon) + \frac{1}{\sqrt{2\epsilon}}\,\delta(b-1)\right]\varphi(b),\end{aligned} \quad \text{(A8)}$$

which gives the sought expression

$$\begin{aligned}\frac{d}{db}\left[(b^2-1)^{-1/2}\,\theta(b-1)\right] &= \mathcal{P}f\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1)\right]\\ &= \lim_{\epsilon\to 0^+}\left[-b\,(b^2-1)^{-3/2}\,\theta(b-1-\epsilon) + \frac{1}{\sqrt{2\epsilon}}\,\delta(b-1)\right],\end{aligned} \quad \text{(A9)}$$

where this distributional limit has to be understood in the sense of (A8).

In a similar way it is easy to show that

$$\begin{aligned}\frac{d^2}{db^2}\left[(b^2-1)^{-1/2}\,\theta(b-1)\right] &= \mathcal{P}f\left[(2b^2+1)\,(b^2-1)^{-5/2}\,\theta(b-1)\right]\\ &= \lim_{\epsilon\to 0^+}\Bigg[(2b^2+1)\,(b^2-1)^{-5/2}\,\theta(b-1-\epsilon)\\ &\quad - \left(\frac{1}{(2\epsilon)^{3/2}} + \frac{1}{8}\,\frac{1}{\sqrt{2\epsilon}}\right)\delta(b-1) + \frac{3}{2}\,\frac{1}{\sqrt{2\epsilon}}\,\delta'(b-1)\Bigg].\end{aligned} \quad \text{(A10)}$$

Other Hadamard finite part distributions appearing in this paper are given by the distributional limits

$$\begin{aligned}
\mathcal{P}f\left[(b^2-1)^{-3/2}\,\theta(b-1)\right] &= \lim_{\epsilon\to 0^+}\left[(b^2-1)^{-3/2}\,\theta(b-1-\epsilon)-\frac{1}{\sqrt{2\epsilon}}\,\delta(b-1)\right],\\
\mathcal{P}f\left[\kappa\, p(b)\,(b^2-1)^{-3/2}\,\theta(b-1)\right] &= \kappa\, p(b)\,\mathcal{P}f\left[(b^2-1)^{-3/2}\,\theta(b-1)\right]\\
&= \lim_{\epsilon\to 0^+}\left[\kappa\, p(b)\,(b^2-1)^{-3/2}\,\theta(b-1-\epsilon)-\kappa\,\frac{p(1)}{\sqrt{2\epsilon}}\,\delta(b-1)\right],\\
g(b)\equiv\mathcal{P}f\left[(b^2-1)^{-5/2}\,\theta(b-1)\right] &= \lim_{\epsilon\to 0^+}\left[(b^2-1)^{-5/2}\,\theta(b-1-\epsilon)\right.\\
&\quad\left.-\left(\frac{1}{3}\,\frac{1}{(2\epsilon)^{3/2}}-\frac{5}{8}\,\frac{1}{\sqrt{2\epsilon}}\right)\delta(b-1)+\frac{1}{2}\,\frac{1}{\sqrt{2\epsilon}}\,\delta'(b-1)\right],\\
f_C(b)\equiv\mathcal{P}f\left[b\,(b^2-1)^{-5/2}\,(2b^4-5b^2+C/4)\,\theta(b-1)\right]&\\
&= \lim_{\epsilon\to 0^+}\left\{b\,(b^2-1)^{-5/2}\,(2b^4-5b^2+C/4)\,\theta(b-1-\epsilon)\right.\\
&\quad\left.+\left[\left(1-\frac{C}{12}\right)\frac{1}{(2\epsilon)^{3/2}}+\frac{1}{8}\left(5+\frac{C}{4}\right)\frac{1}{\sqrt{2\epsilon}}\right]\delta(b-1)-\frac{3}{2}\left(1-\frac{C}{12}\right)\frac{1}{\sqrt{2\epsilon}}\,\delta'(b-1)\right\},
\end{aligned} \tag{A11}$$

where $\kappa\neq 0$ is a constant and $p(b)$ is a polynomial in b with $p(1)\neq 0$.

2. Propagator $H(x,\bar{\mu})$

Let us consider the expression

$$\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)=\frac{1}{\pi}\lim_{\lambda\to 0^-}\frac{d}{d\lambda}\left[\theta(x^0)\,\delta(x^2+\lambda)\right], \tag{A12}$$

which is not a distribution in a four-dimensional space. To see this we consider the effect of integrating an arbitrary tempered test function $\phi(x)$ with such an expression:

$$\int d^4x\,\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\,\phi(x)=\frac{1}{4\pi}\int d^2\Omega\int_0^\infty\frac{dr}{r}\,\phi|_{t=r}-\frac{1}{4\pi}\int d^2\Omega\int_0^\infty dr\,\left.\frac{\partial\phi}{\partial t}\right|_{t=r}, \tag{A13}$$

where $r\equiv|\,\vec{x}\,|$. The integral in the first term is, in general, divergent. The Hadamard finite part of the divergent integral $\int_0^\infty dr\,r^{-1}\,\varphi(r)$, where $\varphi(r)$ is a tempered test function, is

$$\mathcal{F}p\int_0^\infty\frac{dr}{r}\,\varphi(r)=\lim_{\epsilon\to 0^+}\left[\int_\epsilon^\infty\frac{dr}{r}\,\varphi(r)+\ln\epsilon\,\varphi(0)\right]. \tag{A14}$$

Taking $\varphi(r)\equiv\int d^2\Omega\phi|_{t=r}$, one can define the finite part of the integral (A13). In this way we can define a distribution, which we call $\mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right]$, as

$$\begin{aligned}
\int d^4x\,\mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right]\phi(x) &\equiv \mathcal{F}p\int d^4x\,\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\,\phi(x)\\
&= \frac{1}{4\pi}\,\mathcal{F}p\int_0^\infty\frac{dr}{r}\,\varphi(r)-\frac{1}{4\pi}\int d^2\Omega\int_0^\infty dr\,\left.\frac{\partial\phi}{\partial t}\right|_{t=r}\\
&= \lim_{\epsilon\to 0^+}\left[\int_{|\vec{x}|\geq\epsilon}d^4x\,\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\,\phi(x)+\ln\epsilon\,\phi(0)\right]\\
&= \lim_{\epsilon\to 0^+}\int d^4x\left[\frac{1}{\pi}\,\theta(x^0)\,\theta(|\,\vec{x}\,|-\epsilon)\,\frac{d}{d(x^2)}\,\delta(x^2)+\ln\epsilon\,\delta^4(x)\right]\phi(x),
\end{aligned} \tag{A15}$$

from which we see that $\mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right]=\lim_{\epsilon\to 0^+}\left[\frac{1}{\pi}\,\theta(x^0)\,\theta(|\,\vec{x}\,|-\epsilon)\,\frac{d}{d(x^2)}\,\delta(x^2)+\ln\epsilon\,\delta^4(x)\right]$. From Eqs.

(3.6) and (3.7) we see that this distribution satisfies

$$x_\alpha \mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right] = x_\alpha\,\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2) = x_\alpha\,H(x,\bar\mu). \tag{A16}$$

Notice that (A16) is an equality between distributions: In the second member, the factor x_α destroys the divergences. This equality shows that the propagator $H(x,\bar\mu)$ must be equal to the distribution $\mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right]$ plus a distribution $\tilde H(x,\bar\mu)$ which satisfies $x_\alpha\,\tilde H(x,\bar\mu)=0$. It is easy to prove that such a distribution can only be a constant times the δ distribution, and so we have

$$H(x,\bar\mu) = \mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right] + C(\bar\mu)\,\delta^4(x), \tag{A17}$$

where $C(\bar\mu)$ is a constant involving the arbitrary mass scale $\bar\mu$. Note that we could not have applied this argument to Eq. (3.7) because $\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)$ is not a distribution. The constant $C(\bar\mu)$ can be computed by evaluation of the integral $\int d^4x\; H(x,\bar\mu)\,\theta(R-\mid\vec x\mid)$, with R constant, using, on the one hand, expression (A17) and, on the other hand, the Fourier transform representation

$$H(x,\bar\mu) = -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,e^{-ipx}\left[\ln\left(\frac{\mid p^2\mid}{\bar\mu^2}\right) - i\pi\,\theta(p^2)\,\mathrm{sgn}(p^0)\right]. \tag{A18}$$

This can be easily shown to be equivalent to the definition of $H(x,\bar\mu)$ in Eq. (3.3) using $\lim_{\epsilon\to0^+}\ln\left[-(p^2+i\epsilon)\right] = \ln\mid p^2\mid - i\pi\,\theta(p^2)$ and $2\,\theta(-p^0)-1=-\mathrm{sgn}(p^0)$. We obtain $C(\bar\mu)=\ln\bar\mu+\gamma-1$, and so we finally have

$$\begin{aligned} H(x,\bar\mu) &= \mathcal{P}f\left[\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d(x^2)}\,\delta(x^2)\right] + [\ln\bar\mu+\gamma-1]\,\delta^4(x),\\ &= \lim_{\epsilon\to0^+}\left\{\frac{1}{\pi}\,\theta(x^0)\,\theta(\mid\vec x\mid-\epsilon)\,\frac{d}{d(x^2)}\,\delta(x^2) + [\ln\bar\mu\epsilon+\gamma-1]\,\delta^4(x)\right\}. \end{aligned} \tag{A19}$$

This expression can be shown to be equivalent to

$$H(x,\bar\mu) = \lim_{\lambda\to0^-}\left\{\frac{1}{\pi}\,\theta(x^0)\,\frac{d}{d\lambda}\delta(x^2+\lambda) + \left[\frac{1}{2}\,\ln\left(-\lambda\bar\mu^2\right)+\gamma-1\right]\delta^4(x)\right\}, \tag{A20}$$

which is, in fact, the definition for the propagator H used by Horowitz in Ref. [16].

3. Calculation of the integral $I(x,y)$

We show here the calculation of the integral

$$I(x,y) \equiv \int d^4x'\; H(-x',\bar\mu)\,\delta(x+x')\delta(y+y'), \tag{A21}$$

which appeared in Sec. III. One can calculate this integral in two ways: using the representation (A18) for the propagator H as a Fourier transform or using expression (A19). With the representation (A18) for H we have

$$I(x,y) = -\int\frac{d^2p}{(2\pi)^2}\,e^{i\vec p\cdot\vec x}\,\ln\frac{\mid\vec p\mid}{\bar\mu}, \tag{A22}$$

where $\vec p\equiv(p^x,p^y)$, $\vec x\equiv(x,y)$ are vectors in a two-dimensional space. That is, I is the Fourier transform of a logarithm in a two-dimensional space. The result, which can be found in Ref. [27], is

$$\begin{aligned} I(x,y) &= \frac{1}{2\pi}\,\mathcal{P}f\left(\frac{1}{r^2}\right) + \left(\ln\frac{\bar\mu}{2}+\gamma\right)\delta(x)\delta(y)\\ &= \lim_{\epsilon\to0^+}\left\{\frac{1}{2\pi}\,\frac{1}{r^2}\,\theta(r-\epsilon) + \left(\ln\frac{\bar\mu\epsilon}{2}+\gamma\right)\delta(x)\delta(y)\right\}, \end{aligned} \tag{A23}$$

where $r\equiv\sqrt{x^2+y^2}$.

For completeness let us check that we can get this result using the representation (A19) for H:

$$I(x,y) = \lim_{\epsilon\to 0^+} \left\{ (\ln\bar{\mu}\epsilon + \gamma - 1)\,\delta(x)\delta(y) + \frac{1}{2\pi}\int_0^\infty dz\,\frac{1}{(z^2+r^2)^{3/2}}\,\theta(r^2+z^2-\epsilon^2) \right\}. \tag{A24}$$

Introducing $1 = \theta(r-\epsilon) + \theta(\epsilon - r)$ in the second term one has

$$\begin{aligned} I(x,y) &= \lim_{\epsilon\to 0^+} \left\{ (\ln\bar{\mu}\epsilon + \gamma - 1)\,\delta(x)\delta(y) + \frac{1}{2\pi}\,\theta(r-\epsilon)\int_0^\infty dz\,\frac{1}{(z^2+r^2)^{3/2}} + (1-\ln 2)\,\mathcal{G}(x,y;\epsilon) \right\} \\ &= \lim_{\epsilon\to 0^+} \left\{ \frac{1}{2\pi}\,\frac{1}{r^2}\,\theta(r-\epsilon) + (\ln\bar{\mu}\epsilon + \gamma - 1)\,\delta(x)\delta(y) + (1-\ln 2)\,\mathcal{G}(x,y;\epsilon) \right\}, \end{aligned} \tag{A25}$$

where

$$\begin{aligned} \mathcal{G}(x,y;\epsilon) &\equiv \frac{1}{2\pi}\,\frac{1}{(1-\ln 2)}\,\theta(\epsilon - r)\int_0^\infty dz\,\frac{1}{(z^2+r^2)^{3/2}}\,\theta(r^2+z^2-\epsilon^2) \\ &= \frac{1}{2\pi}\,\frac{1}{(1-\ln 2)}\,\frac{1}{r^2}\,\theta(\epsilon - r)\left(1 - \frac{\sqrt{\epsilon^2 - r^2}}{\epsilon}\right). \end{aligned} \tag{A26}$$

Now to prove (A23) it remains to be seen that $\lim_{\epsilon\to 0^+}\mathcal{G}(x,y;\epsilon) = \delta(x)\delta(y)$. For this we use the following theorem (see, for example, Ref. [29]). Suppose we have a function $f(x;\epsilon)$, with $x \in \mathbb{R}^n$, that satisfies the following conditions: (1) $f(x;\epsilon) \geq 0$ for $\|x\| \leq \kappa$, with $\kappa > 0$ some fixed constant; (2) in all the sets $a \leq \|x\| \leq \frac{1}{a}$, being $a > 0$ an arbitrary finite constant, $f(x;\epsilon)$ converges uniformly to zero as $\epsilon \to 0^+$; and (3) $\lim_{\epsilon\to 0^+}\int_{\|x\|\leq a,\, a>\epsilon} d^n x\, f(x;\epsilon) = 1$. Then it can be shown that $\lim_{\epsilon\to 0^+} f(x;\epsilon) = \delta^n(x)$. On the one hand we see that $\mathcal{G}(x,y;\epsilon) \geq 0$ and that $\mathcal{G}(x,y;\epsilon) = 0$ for $r \geq a > \epsilon$, and so, in the limit $\epsilon \to 0^+$, $\mathcal{G}(x,y;\epsilon)$ converges uniformly to zero for $r \geq a > 0$. On the other hand it is easy to prove that $\int_{r\leq a,\, a>\epsilon} dx\, dy\, \mathcal{G}(x,y;\epsilon) = 1$; thus, we have that $\lim_{\epsilon\to 0^+}\mathcal{G}(x,y;\epsilon) = \delta(x)\delta(y)$.

APPENDIX B: RIEMANN TENSOR IN THE LINEAR APPROXIMATION FOR $h_{\mu\nu}$ WITH CYLINDRICAL SYMMETRY

It is easy to check that the Riemann components (2.14) for $h_{\mu\nu}$ with cylindrical symmetry can be written in terms of $h_1 \equiv h^t_t$, $h_2 \equiv h^z_z$, $h_3 \equiv h^r_r + h^\theta_\theta$, $h_4 \equiv h^r_r - h^\theta_\theta$, and $h_5 \equiv h^r_t = -h^t_r$ in the form

$$\begin{aligned} R_{tztz} &= \frac{1}{2}\,h_{2,tt}, \qquad R_{xzxz} = \frac{1}{2}\left(\cos^2\theta\, h_{2,rr} + \sin^2\theta\,\frac{h_{2,r}}{r}\right), \\ R_{xzyz} &= \frac{1}{4}\sin 2\theta\left(h_{2,rr} - \frac{h_{2,r}}{r}\right), \\ R_{txtx} &= \frac{1}{4}\left(h_{3,tt} + \cos 2\theta\, h_{4,tt} - 2\cos^2\theta\, h_{1,rr} - 2\sin^2\theta\,\frac{h_{1,r}}{r} - 4\cos^2\theta\, h_{5,tr} - 4\sin^2\theta\,\frac{h_{5,t}}{r}\right), \\ R_{txty} &= \frac{1}{4}\sin 2\theta\left(h_{4,tt} - h_{1,rr} + \frac{h_{1,r}}{r} - 2\,h_{5,tr} + 2\,\frac{h_{5,t}}{r}\right), \qquad R_{tzxz} = \frac{1}{2}\cos\theta\, h_{2,tr}, \\ R_{txxy} &= \frac{1}{4}\sin\theta\left(-h_{3,tr} + h_{4,tr} + 2\,\frac{h_{4,t}}{r}\right), \quad R_{xyxy} = \frac{1}{4}\left(h_{3,rr} + \frac{h_{3,r}}{r} - h_{4,rr} - 3\,\frac{h_{4,r}}{r}\right). \end{aligned}$$

R_{yzyz}, R_{tyty}, R_{tzyz}, and R_{tyyx} can be obtained from the previous expressions interchanging $\cos\theta$ by $\sin\theta$ in R_{xzxz}, R_{txtx}, R_{tzxz}, and R_{txxy}, respectively.

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