

## Noise induced transitions in semiclassical cosmology

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A semiclassical cosmological model is considered which consists of a closed Friedmann-Robertson-Walker spacetime in the presence of a cosmological constant, which mimics the effect of an inflaton field, and a massless, non-conformally coupled quantum scalar field. We show that the back-reaction of the quantum field, which consists basically of a nonlocal term due to gravitational particle creation and a noise term induced by the quantum fluctuations of the field, are able to drive the cosmological scale factor over the barrier of the classical potential so that if the universe starts near a zero scale factor (initial singularity), it can make the transition to an exponentially expanding de Sitter phase. We compute the probability of this transition and it turns out to be comparable with the probability that the universe tunnels from “nothing” into an inflationary stage in quantum cosmology. This suggests that in the presence of matter fields the back-reaction on the spacetime should not be neglected in quantum cosmology. [S0556-2821(99)04108-9]

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### I. INTRODUCTION

A possible scenario for the creation of an inflationary universe is provided by cosmological models in which the universe is created by quantum tunneling from “nothing” into a de Sitter space. This creation is either based on an instanton solution or in a wave function solution which describes the tunneling in a simple minisuperspace model of quantum cosmology [1,2].

In the inflationary context one of the simplest cosmological models one may construct is a closed Friedmann-Robertson-Walker (FRW) model with a cosmological constant. The cosmological constant is introduced to reproduce the effect of the inflation field at a stationary point of the inflaton potential [1]. The dynamics of this universe is described by a potential with a barrier which separates the region where the scale factor of the universe is zero, where the potential has a local minimum, from the region where the universe scale factor grows exponentially, the de Sitter or inflationary phase. The classical dynamics of this homogeneous and isotropic model is thus very simple: the universe either stays in the minimum of the potential or it inflates.

The classical dynamics of the preinflationary era in such cosmological models may be quite complicated, however, if one introduces anisotropies, inhomogeneities or other fields. Thus, for instance, all anisotropic Bianchi models, except Bianchi type IX, are bound to inflate in the presence of a cosmological constant [3]. Also in the previous model but with an inhomogeneous scalar radiation field the universe may get around the barrier [4] and emerge into the inflationary stage even if initially it was not.

The emergence of an inflationary stage of the universe also seems to be aided by semiclassical effects such as particle creation which enhances the radiation energy density of

the preinflationary era and thus enlarges the set of inflating initial conditions [5,6].

In this paper we consider a semiclassical model consisting of a closed FRW cosmology with a cosmological constant in the presence of a quantum massless scalar field. This quantum field may be seen as linear perturbations of the inflaton field at its stationary point or as some other independent linear field. Because the field is free, the semiclassical theory is one loop exact. The expectation value in a quantum state of the stress-energy tensor of this scalar field influences by back-reaction the dynamics of the cosmological scale factor. There are here two main effects at play: on the one hand, since the field is not conformally coupled, particle creation will occur and, on the other hand, the quantum fluctuations of this stress-energy tensor induce stochastic classical fluctuations in the scale factor [7,8]. Thus the cosmological scale factor is subject to a history dependent term due to gravitational particle creation and also to noise due to these quantum fluctuations. We examine the possibility that a universe starting near the local minimum may cross the barrier and emerge into the inflationary region by the back-reaction of the quantum field on the scale factor. This is, in some sense, the semiclassical version of tunneling from nothing in quantum cosmology.

It is important to stress the difference between this calculation and the usual approach to quantum tunneling. The usual approach [9,10,11,12] begins with the calculation of an instanton or tunneling solution, which is a solution to the Euclidean classical (or sometimes semiclassical; see [13]) equations of motion. Because of symmetry, the scalar field is set to zero from the start. Its effect, if at all, is considered as a contribution to the prefactor of the tunneling amplitude [11], which is usually computed to one loop accuracy in the test field approximation. The effect of dissipation [14] or

even of particle creation [15] on quantum tunneling has been considered in some quantum mechanical systems but it ought to be noticed that to this date the effects of stress-energy fluctuations on the tunneling amplitude have not been considered in the literature, to the best of our knowledge. Even when the instanton is sought as a solution to semiclassical equations [13] this is done under approximations that effectively downplay the role of particle creation, and back-reaction fluctuations are not considered at all.

To underlie that the mechanism for barrier penetration to be investigated here is a different physical process than that computed from instantons in the test field approximation, we have chosen to ignore the quantum aspects of the gravitational field, so that in the absence of back reaction fluctuations the tunneling rate would be zero. From the point of view of the usual approximation, it could be said that our calculation amounts to a nonperturbative calculation of the tunneling amplitude, since the key element is that we go beyond the test field approximation, and consider the full effect of back reaction on the universe.

At least, in principle, it ought to be possible to combine both the usual and our approach. The whole scheme would resemble the derivation of the Hu-Paz-Zhang equations [16,17], once the subtleties of quantum cosmological path integrals are factored in [18].

In this paper, we follow the methodology of Langer's classic paper [19]; namely, we shall consider an ensemble of universes whose evolution is rendered stationary by the device that, every time a member of the ensemble escapes the barrier, it is captured and reemitted within the barrier. This fictitious stationary solution has a nonzero flux across the barrier, and the activation probability is derived from this flux.

Since semiclassical cosmology distinguishes a particular time (that when the quantum to classical transition takes place), it is meaningful to ask whether the stationary solution is relevant to the behavior of a solution with arbitrary initial data at the "absolute zero of time." The answer is that the stationary solution is indeed relevant, because the relaxation time which brings an arbitrary solution to the steady one is exponentially shorter than the time it takes to escape the barrier. We discuss this issue in detail in the Appendix, subsection 6.

The fact itself of assuming a semiclassical theory, i.e., where no gravitational fluctuations are included, indicates that our model must be invalid very close to the cosmological singularity. Therefore, we are forced to assume that some mechanism forces the universe to avoid this region, while being too weak to affect significantly the behavior of larger universes. For example, if we take the cosmological constant, in natural units, to be about  $10^{-12}$  [which corresponds to grand unified theory (GUT) scale inflation], then the presence of classical radiation with an energy density of order 1 (while the amount necessary to avoid recollapse in the classical theory is  $10^{12}$ ) would be sufficient. A more sophisticated possibility would be to appeal to some quantum gravitational effect, which could be as simple as Heisenberg's uncertainty principle, to make it impossible for the universe to linger for long times too close to the singularity.

Even with this simple setting, it is impossible to make progress without further simplifications, and we would like to give here a summary of the most significative ones. The most basic simplifying assumption is that the deviation from conformal coupling, measured by the parameter  $\nu$  to be introduced below [see Eq. (2)], is small. This will allow us to set up the problem as a perturbative expansion in  $\nu$ , whereby we shall stick to the lowest nontrivial order, namely  $O(\nu^2)$ . Of course, the quantity of highest interest, the escape probability itself, will turn out to be nonperturbative in  $\nu$ ; however, our procedure ought to capture its leading behavior.

Even to second order in  $\nu$ , the closed time path (CTP) effective action, whose variation yields the semiclassical equations for the universe scale factor, involves the calculation of several kernels. We have formal exact expressions for these kernels, but the results are too involved for further manipulation. This suggests a second simplification, namely, to substitute the exact kernels for their analogues as computed in a spatially flat universe with the same scale factor. Technically, this amounts to making a continuous approximation in the mode decomposition of the field. This is clearly justified when the separation between the frequencies for different modes is small, for example, as compared with the characteristic rate of the universe expansion. This condition holds for most orbits within the barrier, excepting maybe those where the universe never grows much larger than Planck's scales, a case which we shall not discuss, for the reasons given above.

The semiclassical evolution equations emerging from the CTP effective action differ from the usual Einstein equations in three main respects: (1) the polarization of the scalar field vacuum induces an effective potential, beyond the usual terms associated to spatial curvature and the cosmological constant; also the gravitational constants are renormalized by quantum fluctuations; (2) there appears a memory dependent term, associated to the stress-energy of particles created along the evolution; and (3) there appears a stochastic term associated to the quantum fluctuations of the scalar field. We shall focus our attention in the last two aspects, neglecting the one loop effective gravitational potential. It ought to be noted that, lacking a theory of what the bare potential is exactly like, the semiclassical theory does not uniquely determine the renormalized potential either. Moreover, the presence of stochasticity and memory are aspects where the semiclassical physics is qualitatively different from the classical one, not so for the modified effective potential. In any case, these corrections are very small unless very close to the cosmological singularity (where in any case the one loop approximation is unreliable, as implied by the logarithmic divergence of the quantum corrections). So assuming again that some mechanism will make it impossible for the universe to stay very close to the singularity, the neglect of the renormalized potential is justified.

Even after the neglect of the renormalized potential, the equations deriving from the CTP effective action are higher than second order, and therefore do not admit a Cauchy problem in the usual terms and also lead to possibly unphysical solutions. In order to reduce them to second order equations, and to ensure that the solutions obtained are physical,

it is necessary to implement an order reduction procedure as discussed by several authors [20]. This order reduction means that higher derivatives are expressed in terms of lower ones as required by the classical equations of motion. In this spirit, in the memory term, we substitute the history of a given state of the universe by the classical trajectory leading up to the same endpoint. Because the classical trajectory is determined by this end point, in practice this reduces the equations of motion to a local form, although no longer Hamiltonian.

The equations of motion for the model, after all these simplifications have been carried out, have the property that they do not become singular when the universe scale factor vanishes. As a consequence, the universe goes across the cosmic singularity and emerges in a new “cosmic cycle.” Because the escape time is generally much larger than the recollapse time, we may expect that this will happen many times in the evolution of a single trajectory. For this reason, our model describes a cyclic universe, being created and destroyed many times (but keeping the memory of the total amount of radiation and extrinsic curvature at the end of the previous cycle), and eventually escaping from this fate to become an inflationary universe. It should be noted that this does not detract from the rigor of our derivations, since it is after all a feature of the mathematical model, it being a matter of opinion whether it affects the application of our studies to the physical universe. For comparison we have studied a different, also mathematically consistent, model in which the universe undergoes a single cosmic cycle and obtain similar results (see the Appendix, subsection 7).

After this enumeration of the main simplifying assumptions to be made below, let us briefly review what we actually do. Our first concern is to derive the semiclassical equations of motion for the cosmological scale factor, by means of the CTP effective action. The imaginary terms in this action can be shown to carry information about the stochastic noises which simulate the effect on the geometry of quantum fluctuations of the matter field [21–27]. After these noises have been identified, the semiclassical equation is upgraded to a Langevin equation.

We then transform this Langevin equation into a Fokker-Planck equation, and further simplify it by averaging along classical trajectories. In this way, we find an evolution equation for the probability density of the universe being placed within a given classical trajectory. The actual universe jumps between classical trajectories, as it is subject to the non-Hamiltonian nonlocal terms and forcing from the random noises. Finding the above equation of evolution requires a careful analysis of both effects.

Finally, we investigate the steady solutions of this equation, and derive the escape probability therein. Again we are forced to consider the problem of very small universes, as the nontrivial steady solutions are nonintegrable in this limit. However, the solutions to the Wheeler-DeWitt equation associated to our model, which in this limit is essentially the Schrödinger operator for an harmonic oscillator, shows no singular behavior for small universes. Thus we shall assume that this divergence will be cured in a more complete model, and accept the nontrivial solution as physical.

The main conclusion of this paper is that the probability that the universe will be carried over the barrier by the sheer effect of random forcing from matter stress-energy fluctuations is comparable to the tunneling probability computed from gravitational instantons. This effect demonstrates the relevance of quantum fluctuations in the early evolution of the universe.

Besides its relevance to the birth of the universe as a whole, this result also may be used to estimate the probability of the creation of inflationary bubbles within a larger universe. We shall report on this issue in a further communication.

The plan of the paper is the following. In Sec. II we compute the effective action for the cosmological scale factor and derive the stochastic semiclassical back-reaction equation for such scale factor. In Sec. III we construct the Fokker-Planck equation for the probability distribution function of the cosmological scale factor which corresponds to the stochastic equation. In Sec. IV we use the analogy with Kramers’ problem to compute the probability that the scale factor crosses the barrier and reaches the de Sitter phase. In the concluding Sec. V we compare our results with the quantum tunneling probability. Some computational details are included in the different sections of the Appendix.

A short summary of this long Appendix is the following: Subsection 1 gives some details of the renormalization of the CTP effective action; subsection 2 explains how to handle the diffusion terms when the Fokker-Planck equation is constructed; in subsection 3, we formulate and discuss Kramer’s problem in action-angle variables; the short subsection 4, gives the exact classical solutions for the cosmological scale factor; in subsection 5, the averaged diffusion and dissipation coefficients for the averaged Fokker-Planck equation are derived; in subsection 6, the relaxation time is computed in detail; and finally in subsection 7, the calculation of the escape probability for the scale factor is made for a model which undergoes a single cosmic cycle.

## II. SEMICLASSICAL EFFECTIVE ACTION

In this section we compute the effective action for the scale factor of a spatially closed FRW cosmological model, with a cosmological constant in the presence of a quantum massless field coupled non-conformally to the spacetime curvature. The semiclassical cosmological model we consider is described by the spacetime metric, the classical source, which in this case is a cosmological constant, and the quantum matter sources.

### A. Scalar fields in a closed universe

The metric for a closed FRW model is given by

$$ds^2 = a^2(t)(-dt^2 + \tilde{g}_{ij}(x^k)dx^i dx^j), \quad i, j, k = 1, \dots, n-1, \quad (1)$$

where  $a(t)$  is the cosmological scale factor,  $t$  is the conformal time, and  $\tilde{g}_{ij}(x^k)$  is the metric of an  $(n-1)$ -sphere of unit radius. Since we will use dimensional regularization we work, for the time being, in  $n$ -dimensions.

Let us assume that we have a quantum scalar field  $\Phi(x^\mu)$ , where the Greek indices run from 0 to  $n-1$ . The classical action for this scalar field in the spacetime background described by the above metric is

$$S_m = - \int dx^n \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \Phi^* \partial_\nu \Phi + \left( \frac{n-2}{4(n-1)} + \nu \right) R \Phi^* \Phi \right], \quad (2)$$

where  $g_{00}=a^2$ ,  $g_{0i}=0$ ,  $g_{ij}=a^2 \tilde{g}_{ij}$ ,  $g$  is the metric determinant,  $\nu$  is a dimensionless parameter coupling the field to the spacetime curvature ( $\nu=0$  corresponds to conformal coupling), and  $R$  is the curvature scalar which is given by

$$R = 2(n-1) \frac{\ddot{a}}{a^3} + (n-1)(n-4) \frac{\dot{a}^2}{a^4} + (n-1)(n-2) \frac{1}{a^2}, \quad (3)$$

where an overdot means derivative with respect to conformal time  $t$ . Let us now introduce a conformally related field  $\Psi$ ,

$$\Psi = \Phi a^{(n-2)/2}, \quad (4)$$

and the action  $S_m$  becomes

$$S_m = \int dt dx^1 \dots dx^{n-1} \sqrt{\tilde{g}} \left[ \dot{\Psi}^* \dot{\Psi} - \frac{(n-2)^2}{4} \Psi^* \Psi - \nu a^2 R \Psi^* \Psi + \Psi^* \Delta^{(n-1)} \Psi \right], \quad (5)$$

where  $\Delta^{(n-1)}$  is the  $(n-1)$ -Laplacian on the  $(n-1)$ -sphere,

$$\Delta^{(n-1)} \Psi \equiv \frac{1}{\sqrt{\tilde{g}}} \partial_i (\sqrt{\tilde{g}} g^{ij} \partial_j \Psi). \quad (6)$$

Let us introduce the time dependent function  $U(t)$ ,

$$U(t) = -\nu a^2(t) R(t), \quad (7)$$

and the d'Alambertian  $\square = -\partial_t^2 + \Delta^{(n-1)}$  of the static metric  $\tilde{d}s^2 = a^{-2} ds^2$ . The action (5) may be written then as

$$S_m = \int dt dx^1 \dots dx^{n-1} \sqrt{\tilde{g}} \times \left[ \Psi^* \square \Psi - \frac{(n-2)^2}{4} \Psi^* \Psi + U(t) \Psi^* \Psi \right]. \quad (8)$$

When  $\nu=0$  this is the action of a scalar field  $\Psi$  in a background of constant curvature. The quantization of this field in that background is trivial in the sense that a unique natural vacuum may be introduced, the ‘‘in’’ and ‘‘out’’ vacuum coincide and there is no particle creation [28]. This vacuum is, of course, conformally related to the physical vacuum; see Eq. (4). The time dependent function  $U(t)$  will be considered as an interaction term and will be treated per-

turbatively. Thus we will make perturbation theory with the parameter  $\nu$  which we will assume small.

To carry on the quantization we will proceed by mode separation expanding  $\Psi(x^\mu)$  in terms of the  $(n-1)$ -dimensional spherical harmonics  $Y_{\vec{k}}^l(x^i)$ , which satisfy [29]

$$\Delta^{(n-1)} Y_{\vec{k}}^l(x^i) = -l(l+n-2) Y_{\vec{k}}^l(x^i), \quad (9)$$

where  $l=0,1,2,\dots$ ;  $l \geq k_1 \geq k_2 \geq \dots \geq k_{n-2} \geq 0$ ;  $\vec{k} = (k_1, \dots, \pm k_{n-2})$ . These generalized spherical harmonics form an orthonormal basis of functions on the  $(n-1)$ -sphere,

$$\int \sqrt{\tilde{g}} dx^1 \dots dx^{n-1} Y_{\vec{k}}^{l*}(x^i) Y_{\vec{k}'}^l(x^i) = \delta^{ll'} \delta_{\vec{k}\vec{k}'}, \quad (10)$$

and we may write

$$\Psi(x^\mu) = \sum_{l=0} \sum_{\vec{k}} \Psi_{\vec{k}}^l(t) Y_{\vec{k}}^l(x^i). \quad (11)$$

When  $\Psi$  is a real field, the coefficients  $\Psi_{\vec{k}}^l$  are not all independent; for instance in three dimensions we simply have  $\Psi_{\vec{k}}^{l*} = \Psi_{-\vec{k}}^l$ . Now let us substitute Eq. (11) into Eq. (5), use Eq. (9) and note that  $(n-2)^2/4 + l(l+n-2) = [l+1+(n-4)/2]^2$ . If we also introduce a new index  $k$  instead of  $l$  by  $k=l+1$ , so that  $k=1,2,\dots$ , we obtain

$$S_m = \int dt \sum_{k=1}^{\infty} \sum_{\vec{k}} [\dot{\Psi}_{\vec{k}}^{l*} \dot{\Psi}_{\vec{k}}^l - M_k^2 \Psi_{\vec{k}}^{l*} \Psi_{\vec{k}}^l + U \Psi_{\vec{k}}^{l*} \Psi_{\vec{k}}^l] \quad (12)$$

where

$$M_k \equiv k + \frac{n-4}{2}. \quad (13)$$

Note that the coefficients of Eq. (11),  $\Psi_{\vec{k}}^l(t)$ , are just functions of  $t$  (1-dimensional fields), and for each set  $(l, \vec{k})$  we may introduce two real functions  $\phi_{\vec{k}}^l(t)$  and  $\tilde{\phi}_{\vec{k}}^l(t)$  defined by

$$\Psi_{\vec{k}}^l \equiv \frac{1}{\sqrt{2}} (\phi_{\vec{k}}^l + i \tilde{\phi}_{\vec{k}}^l); \quad (14)$$

then the action (12) becomes the sum of the actions of two independent sets formed by an infinite collection of decoupled time dependent harmonic oscillators:

$$S_m = \frac{1}{2} \int dt \sum_{k=1}^{\infty} \sum_{\vec{k}} [(\dot{\phi}_{\vec{k}}^l)^2 - M_k^2 (\phi_{\vec{k}}^l)^2 + U(t) (\phi_{\vec{k}}^l)^2] + \dots, \quad (15)$$

where the ellipsis stands for an identical action for the real 1-dimensional fields  $\tilde{\phi}_{\vec{k}}^l(t)$ .

We will consider, from now on, the action for the 1-dimensional fields  $\phi_{\vec{k}}^l$  only. If our starting field  $\Phi$  in Eq.

(2) is real, the results from this ‘‘half’’ action, i.e. the written term in Eq. (15), are enough, if  $\Phi$  is complex, we simply have to double the number of degrees of freedom. Since  $M_k$  depends on  $k$  but not on  $\vec{k}$ , there is no dependence in the action on the vector  $\vec{k}$  and we can substitute  $\Sigma_{\vec{k}}$  by  $\Sigma_{\vec{k}1}$  which gives the degeneracy of the mode  $k$ . This is given by [11]

$$\sum_{\vec{k}} 1 = \frac{(2k+n-4)(k+n-4)!}{(k-1)!(n-2)!}. \quad (16)$$

Note that for  $n=2$ , i.e. when the space section is a circle  $\Sigma_{\vec{k}1}=2$ ; when  $n=3$ , which corresponds to the case of the ordinary spherical harmonics  $\Sigma_{\vec{k}1}=2k-1$  (or  $2l+1$  in the usual notation); and for  $n=4$ , which is the case of interest here, the space section of the spacetime are 3-spheres and we have  $\Sigma_{\vec{k}1}=k^2$ .

The field equation for the 1-dimensional fields  $\phi_{\vec{k}}^l(t)$  are, from Eq. (15),

$$\ddot{\phi}_{\vec{k}}^l + M_k^2 \phi_{\vec{k}}^l = U(t) \phi_{\vec{k}}^l, \quad (17)$$

which in accordance with our previous remarks will be solved perturbatively,  $U(t)$  being the perturbative term. The solutions of the unperturbed equation can be written as linear combinations of the normalized positive and negative frequency modes,  $f_k$  and  $f_k^*$  respectively, where

$$f_k(t) = \frac{1}{\sqrt{2M_k}} \exp(-iM_k t). \quad (18)$$

### B. Closed time path effective action

Let us now derive the semiclassical CTP effective action  $\Gamma_{CTP}$  for the cosmological scalar factor due to the presence of the quantum scalar field  $\Phi$ . The computation of the CTP effective action is similar to the computation of the ordinary (in-out) effective action, except that now we have to introduce two fields, the plus and minus fields  $\phi^\pm$ , and use appropriate ‘‘in’’ boundary conditions. These two fields basically represent the field  $\phi$  propagating forward and backward in time. This action was introduced by Schwinger [30] to derive expectation values rather than matrix elements as in the ordinary effective action, and it has been used recently in connexion with the back-reaction problem in semiclassical gravity [31,21,32]. Here we follow the notations and conventions of Refs. [7,8]

Note that since we are considering the interaction of the scale factor  $a$  with the quantum field  $\phi$ , in the CTP effective action we have now two scalar fields  $\phi^\pm$  and also two scale factors  $a^\pm$ . The kinetic operators for our 1-dimensional fields  $\phi_{\vec{k}}^l$  are given by  $A_k = \text{diag}(-\partial_t^2 - M_k^2 + U^+(t), \partial_t^2 + M_k^2 - U^-(t))$ . The propagators per each mode  $k$ ,  $G_k(t, t')$ , are defined as usual by  $A_k G_k = \delta$ , and are  $2 \times 2$  matrices with components  $(G_k)_{\pm\pm}$ .

To one loop order in the quantum fields  $\phi^\pm$  and at three level in the classical fields  $a^\pm$  the CTP effective action for  $a^\pm$  may be written as

$$\Gamma_{CTP}[a^\pm] = S_g[a^+] - S_g[a^-] + S_m^{cl}[a^+] - S_m^{cl}[a^-] - \frac{i}{2} \sum_{k=1}^{\infty} \sum_{\vec{k}} \text{Tr}(\ln G_k), \quad (19)$$

where  $S_g$  is the pure gravitational action,  $S_m^{cl}$  is the action of classical matter which in our case will include the cosmological constant term only, and  $G_k$  is the propagator for the mode  $k$  which solves Eq. (17). In principle the  $\Gamma_{CTP}$  depends on the expectation value in the quantum state of interest, the ‘‘in’’ vacuum here, of both  $a^\pm$  (the classical field) and of  $\phi^\pm$ . To get the previous expression we have substituted the solution of the dynamical equation for the expectation value of the scalar field which is  $\langle 0, in | \phi | 0, in \rangle = 0$ , so that there is no dependence on the expectation values of  $\phi^\pm$  in the effective action.

Because of the interaction term  $U(t)$  in Eq. (17), the propagator  $G_k$  cannot be found exactly and we treat it perturbatively. Thus we can write  $G_k = G_k^0 (1 - U G_k^0 + U G_k^0 U G_k^0 + \dots)$  where the unperturbed propagator is  $(G_k^0)^{-1} = \text{diag}(-\partial_t^2 - M_k^2, \partial_t^2 + M_k^2)$ . This unperturbed propagator has four components  $(G_k^0)_{++} = \Delta_{kF}$ ,  $(G_k^0)_{--} = -\Delta_{kD}$ ,  $(G_k^0)_{+-} = -\Delta_k^+$  and  $(G_k^0)_{-+} = \Delta_k^-$ , where  $\Delta_{kF}$ ,  $\Delta_{kD}$  and  $\Delta_k^\pm$  are the Feynman, Dyson and Wightman propagators for the mode  $k$ . This is a consequence of the boundary conditions which guarantee that our quantum state is the ‘‘in’’ vacuum  $|0, in\rangle$ . These propagators are defined with the usual  $i\epsilon$  prescription by

$$\begin{aligned} \Delta_{kF}(t-t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp[-i\omega(t-t')]}{\omega^2 - (M_k^2 - i\epsilon)} d\omega \\ &= -i[f_k(t)f_k^*(t')\theta(t-t') + f_k^*(t)f_k(t') \\ &\quad \times \theta(t'-t)], \end{aligned} \quad (20)$$

$$\begin{aligned} \Delta_{kD}(t-t') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp[-i\omega(t-t')]}{\omega^2 - (M_k^2 + i\epsilon)} d\omega \\ &= i[f_k^*(t)f_k(t')\theta(t-t') + f_k(t)f_k^*(t')\theta(t'-t)], \end{aligned} \quad (21)$$

$$\Delta_k^+(t-t') = if_k^*(t)f_k(t'), \quad \Delta_k^-(t-t') = -if_k(t)f_k^*(t'). \quad (22)$$

The trace term in the effective action (19) will now be expanded up to order  $\nu^2$ . The linear terms in  $\nu$  are tadpoles which are zero in dimensional regularization. Thus we can write the effective action as

$$\begin{aligned} \Gamma_{CTP}[a^\pm] &\approx S_g[a^+] - S_g[a^-] + S_m^{cl}[a^+] - S_m^{cl}[a^-] + T^+ \\ &\quad + T^- + T, \end{aligned} \quad (23)$$

where

$$T^\pm = -\frac{i}{4} \sum_{k=1}^{\infty} \sum_k \text{Tr}[U_\pm(G_k^0)_{\pm\pm} U_\pm(G_k^0)_{\pm\pm}],$$

$$T = \frac{i}{2} \sum_{k=1}^{\infty} \sum_k \text{Tr}[U_+(G_k^0)_{+-} U_-(G_k^0)_{-+}]. \quad (24)$$

The pure gravitational part of the action,  $S_g$ , includes the Einstein-Hilbert action and a quadratic counterterm which is needed for regularization of the divergences of Eq. (24),

$$S_g = \frac{1}{l_P^2} \int d^n x \sqrt{-g} R + \frac{\nu^2 \mu_c^{n-4}}{32\pi^2(n-4)} \int d^n x \sqrt{-g} R^2, \quad (25)$$

where  $\mu_c$  is an arbitrary mass scale which gives the correct dimension to the counterterm, and  $l_P^2 = 16\pi G$ , the square of the Planck length. To regularize the divergencies in  $T^\pm$  we need to expand the action (25) in powers of  $n-4$ . Using our metric (1), we can perform the space integration in Eq. (25) which leads to the volume to the  $(n-1)$ -sphere. Expanding now in powers of  $n-4$ , and recalling that the volume of the three-sphere is  $2\pi^2$  we may write  $S_g = S_g^{EH} + S_g^{div}$ , where the first term stands for the Einstein-Hilbert action in four dimensions and the second term is the first order correction in this expansion:

$$S_g[a] = \frac{2\pi^2}{l_P^2} \int dt 6a^2 \left( \frac{\ddot{a}}{a} + 1 \right), \quad (26)$$

$$S_g^{div}[a, \mu_c] = \frac{1}{16} \left\{ \frac{1}{n-4} \int dt U_1^2(t) + \int dt [U_1^2(t) \ln(a\mu_c) + 2U_1(t)U_2(t)] \right\}. \quad (27)$$

Here  $U_1(t)$  and  $U_2(t)$  are defined by the expansion of  $U$  in powers of  $n-4$ . That is, from Eqs. (7) and (3) we can write  $U(t) = U_1(t) + (n-4)U_2(t)$ , where

$$U_1 = -6\nu \left( \frac{\ddot{a}}{a} + 1 \right), \quad U_2 = -\nu \left( 2\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 5 \right). \quad (28)$$

The classical matter term  $S_m^{cl}$  includes in our case the cosmological constant  $\Lambda^*$  only. It can be understood as the term which gives the effect of the inflaton field at the stationary point of the inflaton potential [1]:

$$S_m^{cl}[a] = -2\pi^2 \int dt a^4 \Lambda^*. \quad (29)$$

### C. Computation of $T$ and $T^\pm$

Let us first compute  $T$  in Eq. (24), which may be written as

$$T = -\frac{i}{2} \sum_{k=1}^{\infty} \sum_k \int dt dt' U^+(t) \Delta_k^+(t-t') U^-(t') \Delta_k^-(t'-t). \quad (30)$$

Since this term will not diverge, we can perform the computation directly in  $n=4$  dimensions. In this case  $\Sigma_k^2 = k^2$  and  $M_k = k$ ; thus using Eqs. (22) and (18) we have

$$T = -\frac{i}{2} \int dt dt' \sum_{k=1}^{\infty} U^+(t) k^2 f_k^{*2}(t) f_k^2(t') U^-(t')$$

$$= -\int dt dt' U^+(t) D(t-t') U^-(t')$$

$$-i \int dt dt' U^+(t) N(t-t') U^-(t'), \quad (31)$$

where we have introduced the kernels  $D$  and  $N$  as

$$D(t-t') \equiv -\frac{1}{8} \sum_{k=1}^{\infty} \sin 2k(t-t') = -\frac{1}{16} \text{PV} \left[ \frac{\cos(t-t')}{\sin(t-t')} \right] \quad (32)$$

$$N(t-t') \equiv \frac{1}{8} \sum_{k=1}^{\infty} \cos 2k(t-t')$$

$$= \frac{1}{16} \left\{ \pi \left[ \sum_{n=-\infty}^{\infty} \delta(t-t'-n\pi) \right] - 1 \right\}, \quad (33)$$

and we have computed the corresponding series. The kernels  $D$  and  $N$  are called dissipation and noise kernel, respectively, using the definitions of [8]. It is interesting to compare with Refs. [7,8] where a spatially flat universe was considered. Our results may be formally obtained from that reference if we change  $\text{vol} \int_0^\infty dk$  there, where ‘‘vol’’ is the volume of the space section (assume for instance a finite box), by  $2\pi^2 \sum_{k=1}^\infty$ . In the spatially flat case the noise is a simple delta function (white noise), whereas here we have a train of deltas. Note also that we have, in practice, considered a real scalar field only since we considered only half of the action, i.e. the written part of Eq. (15). Thus for the complex scalar field we need to multiply these kernels by 2, i.e., the dissipation kernel is  $2D$  and the noise kernel is  $2N$ . Note also that the definition of the dissipation kernel here and in Ref. [21] differ by a sign.

Let us now perform the more complicated calculation of  $T^\pm$ . Since these integrals diverge in  $n=4$ , we work here in arbitrary  $n$  (dimensional regularization). From Eq. (24) and the symmetries of  $\Delta_{kF}$  and  $\Delta_{kD}$  we have

$$T^\pm = -\frac{i}{4} \int dt dt' U^\pm(t) \Delta_{F/D}^2(t-t') U^\pm(t'), \quad (34)$$

where we have introduced

$$\begin{aligned}\Delta_{F/D}^2(t-t') &\equiv \sum_{k=1}^{\infty} \left( \sum_k 1 \right) \Delta_{kF/D}^2(t-t') \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} I(\omega),\end{aligned}\quad (35)$$

where  $I(\omega)$  is defined after having made an integral in  $\omega$  with appropriate contour; recall the definitions (20) and (21). After using Eq. (16) and the definition (13) of  $M_k$ ,  $I(\omega)$  is given by

$$\begin{aligned}I(\omega) &= \pm \frac{i}{2(n-2)!} \\ &\times \sum_{k=1}^{\infty} \frac{(k+n-4)!}{(k-1)! \{ [k+(n-4)/2]^2 - (\omega/2)^2 \pm i0^+ \}} \\ &\equiv \pm \frac{i}{2(n-2)!} \sum_{k=1}^{\infty} a_k(\omega),\end{aligned}\quad (36)$$

where we have introduced the coefficients  $a_k$  in the last series expression. In the Appendix, subsection 1, we prove that this series diverges like  $1/(n-4)$ , and thus we can regularize it using Eq. (27). Furthermore, its imaginary part is finite and leads to the noise kernel  $N$  defined above.

Thus according to Eqs. (A1), (A3) and (A8) from the Appendix we can write Eq. (35) as

$$\Delta_{F/D}^2(t-t') = \mp \left( \frac{i}{4} \right) \left[ \frac{\delta(t-t')}{n-4} - \frac{1}{2\pi^2} K^\pm(t-t') \right],\quad (37)$$

where we have defined

$$K^\pm(t-t') \equiv 16\pi^2 [A(t-t') \pm iN(t-t')].\quad (38)$$

Here  $A(t-t')$  is a finite kernel which will be discussed below. We can now substitute Eq. (37) into Eq. (34) and use the expansion of  $U(t)$  in powers of  $n-4$  given in Eq. (28) to get

$$\begin{aligned}T^\pm &= \mp \left[ \frac{1}{n-4} \int dt (U_1^\pm)^2 + 2 \int dt U_1^\pm U_2^\pm \right. \\ &\quad \left. - \frac{1}{2\pi^2} \int dt dt' U_1^\pm(t) K^\pm(t-t') U_1^\pm(t') \right].\end{aligned}\quad (39)$$

#### D. The regularized CTP effective action

We are now in the position to compute the regularized semiclassical CTP effective action. Let us substitute in Eq. (23) the actions (26), (27) and (29), and the results (31) for  $T$  and (39) for  $T^\pm$ . It is clear that the divergent term in Eq. (39), i.e. the term proportional to  $1/(n-4)$ , will be cancelled by the divergent counterterm in Eq. (27). Also the terms  $\int dt U_1 U_2$  in these equations will cancel. Thus, we finally get the regularized semiclassical action

$$\Gamma_{CTP}[a^\pm] = S_{g,m}^R[a^+] - S_{g,m}^R[a^-] + S_{IF}^R[a^\pm],\quad (40)$$

where the regularized gravitational and classical matter actions are

$$\begin{aligned}S_{g,m}^R[a] &= \frac{2\pi^2}{l_p^2} \int dt 6a^2 \left( \frac{\ddot{a}}{a} + 1 \right) - 2\pi^2 \int dt a^4 \Lambda^* \\ &\quad + \frac{1}{16} \int dt U_1^2(t) \ln(a\mu_c).\end{aligned}\quad (41)$$

To write the remaining part,  $S_{IF}^R$ , we note that the kernels  $A$  and  $N$  in (38), satisfy the symmetries  $A(t-t') = A(t'-t)$  and  $N(t-t') = N(t'-t)$ . Taking into account also that  $D(t-t') = -D(t'-t)$  we obtain

$$\begin{aligned}S_{IF}^R[a^\pm] &= \frac{1}{2} \int dt dt' \Delta U(t) H(t-t') \{U(t')\} \\ &\quad + \frac{i}{2} \int dt dt' \Delta U(t) N(t-t') \Delta U(t'),\end{aligned}\quad (42)$$

where we have defined

$$H(t-t') = A(t-t'; \mu_c) - D(t-t'),\quad (43)$$

$$\Delta U = U^+ - U^-, \quad \{U\} = U^+ + U^-. \quad (44)$$

In Eq. (43) we have explicitly written that the kernel  $A$  depends on the renormalization parameter  $\mu_c$ . We note that this effective action has an imaginary part which involves the noise kernel  $N$ . However, because of the quadratic dependence of this term in  $\Delta U$ , it will not contribute to the field equations if we derive such equations from  $\delta\Gamma_{CTP}/\delta a^\pm|_{a^\pm=a} = 0$ . This, in fact, gives the dynamical equations for expectation values of the field  $a(t)$ .

However, we recall that we are dealing with the interaction of a ‘‘system,’’ our classical (one dimensional) field  $a(t)$ , with an ‘‘environment’’ formed by the degrees of freedom of the quantum system and that we have integrated out the degrees of freedom of the environment (note that in the effective action we have substituted the solutions of the field equations for the expectation value of the quantum field). In this case the regularized action  $S_{IF}^R$  can be understood as the influence action of the system-environment interaction, which describes the effect of the environment on the system of interest [33,34]. The imaginary part of the influence action is known [21–27] to give the effect of a stochastic force on the system, and we can introduce an improved semiclassical effective action,

$$\begin{aligned}S_{eff}[a^\pm; \xi] &= S_{g,m}^R[a^+] - S_{g,m}^R[a^-] \\ &\quad + \frac{1}{2} \int dt dt' \Delta U(t) H(t-t') \{U(t')\} \\ &\quad + \int dt \xi(t) \Delta U(t),\end{aligned}\quad (45)$$

where  $\xi(t)$  is a Gaussian stochastic field defined by the following statistical averages:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = N(t-t'). \quad (46)$$

The kernel  $H$  in the effective action gives a nonlocal effect (due to particle creation), whereas the source  $\xi$  gives the reaction of the environment into the system in terms of a stochastic force.

The formal derivation of the last term of Eq. (45) can be seen as follows. The Feynman-Vernon influence functional [33] of the system-environment interaction is defined from the influence action  $S_{IF}$  by  $F_{IF} = \exp(iS_{IF})$ . Note now that by using a simple path integral Gaussian identity, the imaginary part of Eq. (42) can be formally recovered in  $F_{IF}$  with the following functional Fourier transform  $F_{IF} = \int D\xi P[\xi] \exp\{i[\text{Re}(S_{IF}) + \int dt \xi(t)\Delta U(t)]\}$ , where

$$P[\xi] = \frac{\exp\left[-\frac{1}{2} \int dt dt' \xi(t) N^{-1}(t-t') \xi(t')\right]}{\int D\xi \exp\left[-\frac{1}{2} \int dt dt' \xi(t) N^{-1}(t-t') \xi(t')\right]},$$

can be interpreted as a Gaussian probability distribution for the field  $\xi$ . That is, the influence functional may be seen as the statistical average of  $\xi$  dependent influence functionals constructed with the ‘‘effective’’ influence action  $\text{Re}(S_{IF}) + \int dt \xi(t)\Delta U(t)$ . The physical interpretation of this result, namely, that the semiclassical equations are now the stochastic equations derived from such effective action, may be seen, for instance, in Ref. [22].

### E. Stochastic semiclassical back-reaction equation

The dynamical equation for the scale factor  $a(t)$  can now be found from the effective action (45) in the usual way, that is by functional derivation with respect to  $a^+(t)$  and then equating  $a^+ = a^- = a$ . These equations include the back-reaction of the quantum field on the scale factor. It is convenient to use a rescaled scale factor  $b$  and cosmological constant  $\Lambda$  defined by

$$b(t) \equiv \frac{\sqrt{24}\pi}{l_P} a(t), \quad \Lambda \equiv \frac{l_P^4}{12\pi^2} \Lambda^*. \quad (47)$$

The regularized action  $S_{g,m}^R$  becomes, after one integration by parts,

$$S_{g,m}^R[b] = -\frac{1}{2} \int dt \left[ \dot{b}^2 - b^2 + \frac{1}{12} \Lambda b^4 - \frac{9}{2} \nu^2 \left( \frac{\dot{b}}{b} + 1 \right)^2 \ln(b\bar{\mu}) \right], \quad (48)$$

where we have also rescaled the renormalization parameter  $\bar{\mu}$ . The remaining term in Eq. (45) does not change with this rescaling except that now  $U(t)$  should be written in terms of  $b$ ; thus according to Eq. (28) we have

$$U(t) = -6\nu \left( \frac{\dot{b}}{b} + 1 \right). \quad (49)$$

The dynamical equation for  $b(t)$  is

$$\left. \frac{\delta S_{eff}[b^\pm; \xi]}{\delta b^\pm} \right|_{b^\pm=b} = 0. \quad (50)$$

This equation improves the semiclassical equation by taking into account the fluctuations of the stress-energy tensor of the quantum field [35–37]. When averaged over  $\xi$  the equation leads to the usual semiclassical equation for the expectation value of  $b(t)$ .

Now this equation leads to the typical nonphysical runaway solutions due to the higher order time derivatives involved in the quantum correction terms. To avoid such spurious solutions we use the method of order reduction [20] into Eq. (50). In this method one assumes that Eq. (50) is a perturbative equation in which the perturbations are the quantum corrections. To leading order the equation reduces to the classical equation

$$\ddot{b} + b \left( 1 - \frac{1}{6} \Lambda b^2 \right) = O(\nu). \quad (51)$$

The terms with  $\dot{b}$  or with higher time derivatives in the quantum corrections of Eq. (50) are then substituted using recurrently the classical equation (51). In this form the solutions to the semiclassical equations are also perturbations of the classical solutions. Thus by functional derivation of Eq. (45), using Eq. (48), we can write the stochastic semiclassical back-reaction equation (50) as

$$\dot{p} = -V'(b) - \delta V'(b) + F(b, p, t) + J(\xi, b, p), \quad (52)$$

where a prime means a derivative with respect to  $b$ , and we have introduced  $p \equiv \dot{b}$ . The classical potential  $V(b)$  is

$$V(b) = \frac{1}{2} b^2 - \frac{\Lambda}{24} b^4, \quad (53)$$

and its local quantum correction is

$$\delta V(b) = -\frac{3\nu^2\Lambda}{4} \left[ \frac{1}{2} b^2 - \frac{\Lambda}{48} b^4 - p^2 \ln(b\bar{\mu}) \right], \quad (54)$$

where we have implemented order reduction in this term. On the other hand the term  $F(b, p, t)$  involves nonlocal contributions and may be written as

$$F(b, p, t) = -\frac{\partial U}{\partial b} I - \frac{d^2}{dt^2} \left( \frac{\partial U}{\partial \dot{b}} I \right) = 6\nu \left( \frac{d^2}{dt^2} \frac{1}{b} - \frac{\dot{b}}{b^2} \right) I, \quad (55)$$

where  $I(b, p, t)$  is defined by

$$I(b, p, t) \equiv \int_{-\infty}^{\infty} dt' H(t-t') U(t'). \quad (56)$$



After order reduction,  $U(t')$  must be evaluated on the classical orbit with Cauchy data  $b(t)=b$ ,  $p(t)=p$ , whereby it reduces to  $U = -\Lambda \nu b^2$ . The function  $J$  is the noise given by

$$J(\xi, b) = 6\nu \left[ \frac{d^2}{dt^2} \left( \frac{\xi}{b} \right) - \frac{\bar{b} \bar{\xi}}{b^2} \right]$$

and, after order reduction, by

$$J(\xi, b, p) = 6\nu \left[ \frac{\ddot{\xi}}{b} - \frac{2\dot{\xi}p}{b^2} + \frac{2\xi V'(b)}{b^2} + \frac{2\xi p^2}{b^3} \right], \quad (57)$$

with  $\xi(t)$  defined in Eq. (46) in terms of the noise kernel.

### F. Approximate kernels $N$ and $H$

To simplify the nonlocal term  $F(b, p, t)$  and the noise  $J(\xi, b, p)$  we will approximate the kernel  $H$  and the noise kernel  $N$ , keeping only the first delta function, i.e.  $n=0$ , in the train of deltas which define the noise kernel  $N$ . This amounts to take the continuous limit in  $k$  in the definition (33) of  $N$ . In fact, we take the sum in  $k$  as an integral and we get

$$N(u) = \int_0^\infty dk \cos 2ku = \frac{\pi}{16} \delta(u). \quad (58)$$

This is equivalent to assuming that the spacetime spatial sections are flat and of volume  $2\pi^2$ ; see Ref. [7]. Similarly the dissipation kernel  $D$  defined in Eq. (32) becomes

$$D(u) = -\frac{1}{8} \int_0^\infty dk \sin 2ku = -\frac{1}{16} \text{PV} \left( \frac{1}{u} \right). \quad (59)$$

The same approximation may be used to compute the kernel  $A$  defined in Eqs. (36)–(38). The computation of this kernel can be read directly from Eq. (A7) (see also Ref. [7]):

$$A(u) = -\frac{1}{8} \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega u} \ln \frac{|\omega|}{|\mu_c|} = \frac{1}{16} \text{Pf} \left( \frac{1}{|u|} \right) + \frac{1}{8} (\gamma + \ln \mu_c) \delta(u), \quad (60)$$

where  $\gamma$  is Euler's number and Pf means the Hadamard principal function whose meaning will be recalled shortly. To perform this last Fourier transform we write  $\ln|\omega| = \lim_{\epsilon \rightarrow 0^+} [\exp(-\epsilon|\omega|) \ln|\omega|]$ , use the integrals  $\int_0^\infty d\omega \ln \omega \cos(\omega u) \exp(-\epsilon\omega)$  and  $\int_0^\infty d\omega \cos(\omega u) \exp(-\epsilon\omega)$  which can be found in [40], and take into account that

$$\begin{aligned} & [2x \tan^{-1}(u/\epsilon) + \epsilon \ln(u^2 + \epsilon^2)] / (u^2 + \epsilon^2) \\ &= d[\ln(u^2 + \epsilon^2) \tan^{-1}(u/\epsilon)] / du. \end{aligned}$$

When  $\epsilon \rightarrow 0^+$  the last expression gives a representation of  $\pi \text{Pf}(1/|u|)$ . Finally, using Eqs. (59) and (60) the kernel of interest  $H(u) = A(u) - D(u)$  can be written as

$$H(u) = \frac{1}{8} \text{Pf} \left[ \frac{\theta(u)}{u} \right] + \frac{\gamma + \ln \mu_c}{8} \delta(u). \quad (61)$$

The distribution  $\text{Pf}(\theta(u)/u)$  should be understood as follows. Let  $f(u)$  be an arbitrary tempered function; then,

$$\int_{-\infty}^\infty du \text{Pf} \left[ \frac{\theta(u)}{u} \right] f(u) = \lim_{\epsilon \rightarrow 0^+} \left( \int_\epsilon^\infty du \frac{f(u)}{u} + f(0) \ln \epsilon \right). \quad (62)$$

The approximation of substituting the exact kernels by their flat space counterparts is clearly justified when the radius of the universe is large, which is when the semiclassical approximation works best. Once the local approximation for the noise kernel follows, the corresponding expression for  $D$  can be obtained by demanding that their Fourier transforms be related by the same fluctuation-dissipation relation as in the exact formula.

### III. FOKKER-PLANCK EQUATION

Now we want to determine the probability that a universe starting at the potential well goes over the potential barrier into the inflationary stage. In statistical mechanics this problem is known as Kramers' problem. To describe such process we have the semiclassical back-reaction equation (52), which is a stochastic differential equation (a Langevin type of equation). As is well known [38] to study this problem it is better to construct a Fokker-Planck equation, which is an ordinary differential equation for a distribution function. Thus, the first step will be to derive the Fokker-Planck equation corresponding to the stochastic equation (52). The key features of this stochastic equation are a potential given by the local potentials (53) and (54), a nonlocal term given by the function  $F$  and a noise term  $J$ . The classical part of the potential has a local minimum at  $b=0$ , then reaches a maximum and decreases continuously after that. The inflationary stage corresponds to the classical values of  $b$  beyond this potential barrier. If we start near  $b=0$ , the noise term will take the scale factor eventually over the barrier, but if we want to compute the escape probability, we need to consider both noise and nonlocality.

It should do no harm if we disregard the local quantum correction to the potential,  $\delta V(b)$ ; the reason is the following. This term is a consequence of renormalization, but in semiclassical gravity there is a two parameter ambiguity in terms which are quadratic in the curvature in the gravitational part of the action. This ambiguity is seen here only in the parameter  $\bar{\mu}$  because we have simply ignored the other possible parameter which was not essential in the renormalization scheme. Furthermore, we should not trust the semiclassical results too close to  $b=0$ , since the semiclassical theory should break down here. Thus the possible divergence at  $b=0$  may be disregarded and we should think of this renormalized term as just a small correction to the classical potential, as it is indeed for all radii of the universe unless  $b \ll 1$ . Thus the classical potential  $V(b)$  should contain the main qualitative features of the local renormalized potential.

To construct the Fokker-Planck equation let us introduce the distribution function

$$f(b,p,t) = \langle \delta(b(t)-b) \delta(p(t)-p) \rangle, \quad (63)$$

where  $b(t)$  and  $p(t)$  are solutions of Eq. (52) for a given realization of  $\xi(t)$ ,  $b$  and  $p$  are points in the phase space, and the average is taken both with respect to the initial conditions and to the history of the noise as follows. One starts by considering the ensemble of systems in phase space obeying Eq. (52) for a given realization of  $\xi(t)$  and different initial conditions. This ensemble is described by the density  $\rho(b,p,t) = \langle \delta(b(t)-b) \delta(p(t)-p) \rangle$ , where the average is over initial conditions. Next one defines the probability density  $f(b,p,t)$  as the statistical average over the realizations of  $\xi(t)$ , that is  $f(b,p,t) = \langle \rho(b,p,t) \rangle_\xi$

The next manipulations are standard [39]; we take the time derivative of  $f$ ,

$$\begin{aligned} \partial_t f = & \langle \dot{b}(t) \partial_{b(t)} \delta(b(t)-b) \delta(p(t)-p) \\ & + \delta(b(t)-b) \dot{p}(t) \partial_{p(t)} \delta(p(t)-p) \rangle, \end{aligned}$$

and note that  $\partial_{b(t)} \delta(b(t)-b) = -\partial_b \delta(b(t)-b)$ , and that  $\langle p(t) \delta(b(t)-b) \delta(p(t)-p) \rangle = p f(b,p,t)$ .

Performing similar manipulations for the other terms and using the equations of motion (52) we find

$$\frac{\partial f}{\partial t} = \{H, f\} - \frac{\partial}{\partial p} [F(b,p,t) f] - \frac{\partial}{\partial p} \Phi, \quad (64)$$

where we have defined

$$H(b,p) = \frac{1}{2} p^2 + V(b); \quad (65)$$

thus disregarding the potential  $\delta V(b)$  in Eq. (52), the curly brackets are Poisson brackets, i.e.

$$\{H, f\} = -p(\partial f / \partial b) + V'(b)(\partial f / \partial p),$$

and

$$\Phi = \langle J(\xi, b, p) \delta(b(t)-b) \delta(p(t)-p) \rangle. \quad (66)$$

Equation (64) is not yet a Fokker-Planck equation; to make it one we need to write  $\Phi$  in terms of the distribution function  $f$ . This term will be called the diffusion term since it depends on the stochastic field  $\xi(t)$ .

From Eqs. (66) and (57) we may write

$$\Phi = 6\nu \left[ \frac{C_2}{b} - \frac{2C_1 p}{b^2} + \left( \frac{2V'}{b^2} + \frac{2p^2}{b^3} \right) C_0 \right] \quad (67)$$

where

$$C_n = \left\langle \left( \frac{d^n}{dt^n} \xi(t) \right) \delta(b(t)-b) \delta(p(t)-p) \right\rangle, \quad (68)$$

for  $n=0,1,2$ . To manipulate the diffusion term of Eq. (64) we will make use of the functional formula for Gaussian averages [41],

$$\begin{aligned} \langle \xi(t) R[b(t), p(t)] \rangle = & \int dt' N(t-t') \\ & \times \left\langle \frac{\delta}{\delta \xi(t')} R[b(t), p(t)] \right\rangle, \quad (69) \end{aligned}$$

where  $R$  is an arbitrary functional of  $\xi(t)$ . Under the approximation (58) for the noise kernel

$$\begin{aligned} C_0 = & \frac{\pi}{16} \left\langle \frac{\delta}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle \Bigg|_{t' \rightarrow t} \\ = & -\frac{\pi \nu \Lambda}{8} b \frac{\partial}{\partial p} f(b,p,t), \quad (70) \end{aligned}$$

where we have used Eq. (A11) in the last step. The expressions for  $C_1$  and  $C_2$  are similarly obtained; first one uses the time translation invariance of the noise kernel to perform integration by parts, and then the problem reduces to taking time derivatives of Eq. (70). The results are (see the Appendix, subsection 2, for details)

$$C_1 = (\pi \nu \Lambda / 8) (b \partial_b f - p \partial_p f), \quad (71)$$

$$C_2 = (\pi \nu \Lambda / 8) (2p \partial_b f + V' \partial_p f + b V'' \partial_p f). \quad (72)$$

Finally, after substitution in Eq. (67) and using the equation of motion to lowest order we have

$$\Phi = -\frac{\pi \nu^2 \Lambda^2}{4} b^2 \frac{\partial f}{\partial p}, \quad (73)$$

which by Eq. (64) leads to the final form of the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \{H, f\} - \frac{\partial}{\partial p} [F(b,p,t) f] + \frac{\pi \nu^2 \Lambda^2}{4} b^2 \frac{\partial^2 f}{\partial p^2}. \quad (74)$$

We also notice that in the absence of a cosmological constant, we get no diffusion. This makes sense, because in that case the classical trajectories describe a radiation filled universe. Such a universe would have no scalar curvature, and so it should be insensitive to the value of  $\nu$  as well.

### A. Averaging over angles

We want to compute the probability that a classical universe trapped in the potential well of  $V(b)$  goes over the potential barrier as a consequence of the noise and nonlocality produced by the interaction with the quantum field, and ends up in the de Sitter phase. A universe that crosses this potential barrier will reach the de Sitter phase with some energy which one would expect will correspond to the energy of the quantum particles created in the previous stage. Note that this differs from the quantum tunneling from nothing approach in which the universe gets to the de Sitter stage

tunneling from the potential minimum  $b=0$  with zero energy. In practice, this difference will not be so important because as the universe inflates any amount of energy density will be diluted away.

For this computation we will follow closely the solution of Kramers' problem [42] reviewed in the Appendix, subsection 3. The three key features of such computation are, first, the introduction of action-angle canonical variables  $(J, \theta)$ ; second, the assumption that  $f$  depends on  $J$  only, i.e.  $f(J)$ ; and, third, the use of the averaged Fokker-Planck equation over the angle variable  $\theta$ . Of course, the Fokker-Planck equation in Kramers' problem, Eq. (A16), is much simpler than our equation (74) due to the nonlocal character of the latter; thus we need to take care of this problem, and it is quite remarkable that a relatively simple solution can be found.

Thus, let us consider Eq. (74), introduce  $(J, \theta)$  and assume that  $f(J)$ , then in the Appendix we see that the dissipation term which involves  $\partial^2 f / \partial p^2$  can be written in terms of derivatives with respect to  $J$  [see Eq. (A19)]. Since we now have  $\{H, f\} = 0$  we can write

$$\frac{\partial f}{\partial t} = \frac{\pi \nu^2 \Lambda^2}{4} b^2 \left[ \frac{1}{\Omega} \frac{\partial f}{\partial J} + \frac{p^2}{\Omega} \frac{\partial}{\partial J} \left( \frac{1}{\Omega} \frac{\partial f}{\partial J} \right) \right] - \left[ \frac{\partial}{\partial p} F(b, p, t) \right] f - \frac{pF}{\Omega} \frac{\partial f}{\partial J}. \quad (75)$$

Next we take the average of Eq. (75) with respect to the angle  $\theta$ . The averaged equation involves the two pairs of integrals  $\int d\theta b^2 p^2$ ,  $\int d\theta b^2$  and  $\int d\theta pF$ ,  $\int d\theta \partial_p F$ . The components of each pair are related by a derivative with respect to  $J$ . In fact, let us introduce

$$\mathbf{D}(J) = \frac{1}{2\pi\Omega} \int_0^{2\pi} d\theta b^2 p^2, \quad (76)$$

changing the integration variable to  $b$  (see the Appendix, subsection 3) this integral may be written as  $\Omega \oint db b^2 p$ , and using that  $\partial_p p|_b = \Omega/p$  we have

$$\frac{d\mathbf{D}}{dJ} = \frac{1}{2\pi} \int d\theta b^2. \quad (77)$$

Similarly, let us introduce

$$S(J) = \frac{1}{2\pi\Omega} \int_0^{2\pi} d\theta pF(b, p, t); \quad (78)$$

again by a change of integration variable this integral may be written as  $\Omega \oint db bF$  and by derivation with respect to  $J$  we get

$$\frac{dS}{dJ} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\partial}{\partial p} F(b, p, t). \quad (79)$$

Finally, the average of the Fokker-Planck equation (75) becomes

$$\frac{\partial f}{\partial t} = \frac{\pi \nu^2 \Lambda^2}{4} \frac{\partial}{\partial J} \left[ \frac{\mathbf{D}(J)}{\Omega} \frac{\partial f}{\partial J} \right] - \frac{\partial}{\partial J} (Sf). \quad (80)$$

This equation may be written as a continuity equation  $\partial_t f + \partial_J K = 0$ , where the probability flux  $K$  may be identified directly from Eq. (80). We see that, as in Kramers' problem, stationary solutions with positive flux  $K_0$  should satisfy

$$\frac{\pi \nu^2 \Lambda^2}{4} \frac{\mathbf{D}(J)}{\Omega} \frac{\partial f}{\partial J} - S(J)f = -K_0. \quad (81)$$

## B. Nonlocal contribution $S(J)$

We need to handle now the term  $S(J)$ , defined in Eq. (78). The problem here lies in the nonlocal term  $F(b, p, t)$  defined in Eqs. (55), (56), with  $U(t)$  given by Eq. (49). Since this term gives a quantum correction to a classical equation, we will adopt the order reduction prescription. Thus let us assume that  $b(t')$  and  $p(t')$  in the integral which defines  $F$  are solutions to the classical equations of motion with Cauchy data  $b(t) = b$  and  $p(t) = p$ ; then the integrand in  $F(b, p, t)$  will depend explicitly on time only through  $b$  and  $p$ . This means that the time dependence of  $U(t')$  may be written as  $U(b, p, t' - t)$ . If we now write the Cauchy data in terms of the action-angle variables  $(J, \theta)$ , since the equation of motion for the angle variable is simply  $\dot{\theta} = \Omega$ , we may write  $b[B(\theta, J), P(\theta, J), t] = b(\theta + \Omega t, J)$  and similarly for  $p$ . This means that we may substitute the time derivative operator  $d/dt$  by  $\Omega \partial / \partial \theta$  in  $F(b, p, t)$ .

Thus substituting Eqs. (55) and (56) into Eq. (78), using  $\Omega \partial / \partial \theta$  instead of  $d/dt$ , integrating by parts and using the expression for  $U(t)$  given by Eq. (49) we get

$$S = \frac{6\nu}{2\pi\Omega} \int_0^{2\pi} d\theta \left[ \frac{d}{dt} \left( \frac{\dot{p}}{b} \right) \right] I(t). \quad (82)$$

This may be simplified using the equation of motion (51) to lowest order; then changing  $d\theta$  by  $\Omega dt$  we have

$$S = -\frac{\nu^2 \Lambda^2}{2\pi} \int_0^{2\pi/\Omega} dt \left( \frac{d}{dt} b^2(t) \right) \int_{-\infty}^{\infty} dt' H(t-t') b^2(t'). \quad (83)$$

Note that this term is of order  $\nu^2 \Lambda^2$  as the diffusion term (75). Thus it is convenient to introduce  $\mathbf{S}$  by

$$S(J) = \frac{\pi \nu^2 \Lambda^2}{4} \mathbf{S}(J). \quad (84)$$

Now we can make use of Eq. (61) for the kernel  $H$  (note that the local delta term does not contribute), and introduce a new variable  $u = t - t'$ , instead of  $t'$  to write  $\mathbf{S}$  as

$$\mathbf{S}(J) = \frac{-1}{4\pi^2} \int_0^{2\pi/\Omega} dt \left( \frac{d}{dt} b^2(t) \right) \text{Pf} \int_0^{\infty} \frac{du}{u} b^2(t-u). \quad (85)$$

The equation for the stationary flux, Eq. (81), becomes

$$\frac{\mathbf{D}(J)}{\Omega} \frac{\partial f}{\partial J} - \mathbf{S}(J)f = -\frac{4}{\pi\nu^2\Lambda^2} K_0. \quad (86)$$

All that remains now is to find appropriate expressions for  $\mathbf{D}$  and  $\mathbf{S}$  in this equation and follow Kramers' problem in the Appendix to compute  $K_0$ . From now on, however, it is more convenient to use the energy  $E$  as a variable instead of  $J$ , where  $E=H(J)$  and thus we will compute  $\mathbf{D}(E)$  and  $\mathbf{S}(E)$  in what follows.

### C. Evaluating $\mathbf{S}$ and $\mathbf{D}$

Let us begin by recalling the basic features of the classical orbits. The most important feature of the classical dynamics is the presence of two unstable fixed points at  $p=0$ ,  $b=\pm 2\sqrt{E_s}$ , where  $E_s=3/(2\Lambda)$  is also the corresponding value of the "energy"  $E=p^2/2+V(b)$ . These fixed points are joined by a heteroclinic orbit or separatrix. Motion for energies greater than  $E_s$  is unbounded. For  $E\leq E_s$ , we have outer unbound orbits and inner orbits confined within the potential well. These periodical orbits shall be our present concern.

As it happens, the orbits describing periodic motion may be described in terms of elliptic functions (see the Appendix, subsection 4). The exact expression for the orbits leads to corresponding expressions for  $\mathbf{D}$  and  $\mathbf{S}$  (see the Appendix, subsection 5). Introducing a variable  $k$ ,

$$k^2 = \frac{1 - \sqrt{1 - E/E_s}}{1 + \sqrt{1 - E/E_s}}, \quad (87)$$

so that  $k^2 \sim E/4E_s$  for low energy, while  $k^2 \rightarrow 1$  as we approach the separatrix, we find

$$\begin{aligned} \mathbf{D}(E) &= \left(\frac{8E^2}{15\pi}\right) \frac{(1+k^2)^{3/2}}{k^4} \\ &\times \{2(1-k^2+k^4)E[k] - (2-3k^2+k^4)K[k]\}, \end{aligned} \quad (88)$$

where  $K$  and  $E$  are the complete elliptic integrals of the first and second kinds (see [44,45]):

$$\begin{aligned} K[k] &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \\ E[k] &= \int_0^1 dx \sqrt{\frac{(1-k^2x^2)}{(1-x^2)}}. \end{aligned} \quad (89)$$

The corresponding expression for  $\mathbf{S}$  is

$$\mathbf{S}(E) = \left(\frac{8E^2}{\pi^2}\right) \frac{(1+k^2)^2}{k^4} \{\alpha[k]E[k] - \gamma[k]K[k]\}, \quad (90)$$

where

$$\begin{aligned} \alpha[k] &= \int_0^\infty \frac{du}{u^2 \operatorname{sn}^2 u} \left\{ 1 - \left(\frac{1+k^2}{3}\right) \operatorname{sn}^2 u \right. \\ &\quad \left. - \left(\frac{u}{\operatorname{sn} u}\right) [1 - (1+k^2)\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u]^{1/2} \right\}, \end{aligned} \quad (91)$$

$\operatorname{sn} u$  being the Jacobi elliptic function, and

$$\begin{aligned} \gamma[k] &= \int_0^\infty \frac{du}{u^2 \operatorname{sn}^2 u} \left\{ 1 - \left(\frac{1+2k^2}{3}\right) \operatorname{sn}^2 u - \left(\frac{E[u,k]}{\operatorname{sn} u}\right) \right. \\ &\quad \left. \times [1 - (1+k^2)\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u]^{1/2} \right\}, \end{aligned} \quad (92)$$

where  $E[u,k]$  is an incomplete elliptic integral of the second kind:

$$E[u,k] = \int_0^{\operatorname{sn} u} dx \sqrt{\frac{(1-k^2x^2)}{(1-x^2)}}. \quad (93)$$

The conclusion of all this is that, while  $\mathbf{D}$  and  $\mathbf{S}$  individually behave as  $E^2$  times a smooth function of  $E/E_s$ , their ratio is relatively slowly varying. At low energy, we find  $\mathbf{D} \sim E^2/2$  and  $\mathbf{S} \sim E^2/4$ . As we approach the separatrix,  $\mathbf{D} \rightarrow 0.96E_s^2$  and  $\mathbf{S} \rightarrow 1.18E_s^2$ . Meanwhile, the ratio of the two goes from 0.5 to 1.23.

This means that we can write the equation for stationary distributions as

$$\frac{\partial f}{\partial E} - \beta(E)f = -\frac{4}{\pi\nu^2\Lambda^2 g(E)} \left(\frac{K_0}{E^2}\right), \quad (94)$$

where  $\beta$  and  $g$  are smooth order-1 functions. There is a fundamental difference with respect to Kramers' problem, namely the sign of the second term on the left hand side. In the cosmological problem, the effect of nonlocality is to favor diffusion rather than hindering it. We may understand this as arising from a feedback effect associated with particle creation (see [46]).

## IV. TUNNELING AMPLITUDE

Having found the reduced Fokker-Planck equation (94), we must analyze its solutions in order to identify the range of the flux  $K_0$ . We shall first consider the behavior of the solutions for  $E \leq E_s$ , and then discuss the distribution function beyond the separatrix. Since our derivation is not valid there, for this latter part we will have to return to an analysis from the equations of motion. For concreteness, in what follows it is convenient to choose the order of magnitude of the cosmological constant. We shall assume a model geared to produce GUT scale inflation, thus  $\Lambda \sim 10^{-12}$ , and correspondingly  $E_s \sim 10^{12}$  is very large in natural units.

### A. Distribution function inside the potential well

As we have already discussed, the approximations used in building our model break down at the cosmological singular-

ity, and therefore Eq. (94) cannot be assumed to hold in a neighborhood of  $E=0$ . Thus it is best to express the solution for  $f$  in terms of its value at  $E=E_s$ ,

$$f(E) = \frac{4K_0}{\pi\nu^2\Lambda^2} \left[ \sigma \exp\left(\int^E dE' \beta(E')\right) + f_p(E) \right], \quad (95)$$

where  $\sigma$  is an arbitrary constant and the particular solution  $f_p(E)$  is chosen to vanish at  $E=E_s$ ,

$$f_p(E) = \exp\left(\int^E dE' \beta(E')\right) \int_E^{E_s} \frac{dE'}{g(E')E'^2} \times \exp\left(-\int^{E'} dE'' \beta(E'')\right), \quad (96)$$

so that

$$f(E_s) \sim \frac{4K_0}{\pi\nu^2\Lambda^2} \sigma e^{\beta(E_s)E_s}. \quad (97)$$

Because of the exponential suppression, the particular solution is dominated by the lower limit in the integral, leading to

$$f_p(E) \sim \frac{1}{g(E)E^2[\beta(E)+2/E]} - \frac{e^{-\beta(E_s)(E_s-E)}}{g(E_s)E_s^2[\beta(E_s)+2/E_s]}. \quad (98)$$

For  $E \ll 1$  we see that  $f_p \sim E^{-1}$ , but this behavior cannot be extrapolated all the way to zero as it would make  $f$  non-integrable. However, we must notice that neither our treatment (i.e., the neglect of logarithmic potential corrections) nor semiclassical theory generally is supposed to be valid arbitrarily close to the singularity. Thus we shall assume that the pathological behavior of Eq. (94) near the origin will be absent in a more complete theory, and apply it only from some lowest energy  $E_\delta \sim 1$  on. There are still 12 orders of magnitude between  $E_\delta$  and  $E_s$ .

Since we lack a theory to fix the value of the constant  $\sigma$ , we shall require it to be generic in the following sense. We already know that  $f_p$  vanishes at  $E_s$ , by design, and then from the transport equation (94) we derive  $df_p/dE = -[g(E_s)E_s^2]^{-1}$  there. So unless  $\sigma \ll [\beta(E_s)g(E_s)E_s^2]^{-1} \exp[-\beta(E_s)E_s] \sim 10^{-24} \exp(-10^{12})$ ,  $f$  has a positive slope as it approaches the separatrix from below. We shall assume a generic  $\sigma$  as one much above this borderline value, so that for  $E \geq 1$  the right hand side of the reduced Fokker-Planck equation may be neglected, and  $f$  grows exponentially:

$$f(E) \sim \frac{4K_0\sigma}{\pi\nu^2\Lambda^2} e^{\beta(E)E}. \quad (99)$$

## B. Outside the well

Beyond the separatrix, all motion is unbounded and there is no analogue of action-angle variables; so we must return

to the original variables  $b, p$ . Also note that we are only interested in the regime when  $E \geq E_s$ ; that is, we shall not consider unbound motion below the top of the potential.

Let us first consider the behavior of classical orbits in the  $(b, p)$  plane. Our first observation is that as the universe gets unboundedly large, the effects of spatial curvature become irrelevant. This means that we may approximate  $U \sim -6\nu\dot{b}/b$ , and accordingly the classical equation of motion as  $\ddot{b} \sim \Lambda b^3/6$ .

In this regime, classical orbits are quickly drawn to a de Sitter type expansion, whereby they can be parametrized as

$$b(t') = \frac{b(t)}{1 + \sqrt{(\Lambda/12)b(t)(t-t')}}. \quad (100)$$

After substituting  $U \propto b^2$ , it is easily seen that the nonlocal term  $I$  is proportional to  $b^2(t)$ , and that therefore the nonlocal force  $F$  vanishes [see Eq. (55)]. Therefore what we are dealing with are the local quantum fluctuations of the metric, which one would not expect to act in a definite direction, but rather to provide a sort of diffusive effect. To see this, let us observe that if we look at the Fokker-Planck equation as a continuity equation; then we may write it as

$$\frac{\partial f}{\partial t} = -\vec{\nabla} \cdot \vec{K},$$

and this allows us to identify the flux. For example, if the Fokker-Planck equation reads

$$\frac{\partial f}{\partial t} = \frac{\partial A}{\partial b} + \frac{\partial B}{\partial p},$$

then whatever  $A$  and  $B$  are,  $\vec{K} = -A\hat{b} - B\hat{p}$ , where a caret denotes a unit vector in the corresponding direction. Rather than  $\hat{b}$  and  $\hat{p}$ , however, it is convenient to use the components of  $\hat{K}$  along and orthogonal to a classical trajectory. Since the energy  $E$  is constant along trajectories,  $\vec{\nabla} E$  lies in the orthogonal direction; so the orthogonal component is simply  $K_E$  or, since  $E=H(J)$ ,  $K_J$ .

Our whole calculation so far amounts to computing the mean value of  $K_J$  [see Eq. (80)]; indeed the first term acts as diffusion, opposing the gradients of  $f$ . The big surprise is the second term being positive, forcing a positive flux towards larger energies. Observe that, in particular, the mean flux across the separatrix is positive. Since for a stationary solution the flux is conserved, the flux must be positive across any trajectory. Now beyond the separatrix the term  $S$  of Eq. (80) is absent because  $F$  vanishes and, as we shall see,  $\mathbf{D}$  remains positive. So to obtain a positive flux, it is necessary that  $\partial f/\partial E < 0$ , as we will now show.

To compute  $\mathbf{D}$  beyond the separatrix, we observe that although there are no longer action-angle variables, we may still introduce a new pair of canonical variables  $(E, \tau)$ , where  $E$  labels the different trajectories and  $\tau$  increases along clas-

sical trajectories, with  $\dot{\tau}=1$ . It works as follows. The relationship between  $p$  and  $b$ ,  $p = \sqrt{2E + (\Lambda/12)b^4}$ , becomes, for low energy,

$$p = \sqrt{\frac{\Lambda}{12}}b^2 + \sqrt{\frac{12E}{\Lambda}} \frac{E}{b^2}. \quad (101)$$

This same relationship corresponds to a canonical transformation with generating functional  $W$ ,

$$W(b, E) = \sqrt{\frac{\Lambda}{12}} \frac{b^3}{3} - \sqrt{\frac{12E}{\Lambda}} \frac{E}{b},$$

and the new canonical coordinate  $\tau$  follows from

$$\tau = \frac{\partial W}{\partial E} = -\sqrt{\frac{12E}{\Lambda}} \frac{1}{b}. \quad (102)$$

Comparing with Eq. (100), this is just

$$\tau = -\sqrt{\frac{12}{\Lambda}} \frac{1}{b(t_0)} \left[ 1 + \sqrt{\frac{\Lambda}{12}} b(t_0)(t_0 - t) \right],$$

for some constant of integration  $t_0$ . Indeed  $\dot{\tau}=1$ , as it must.

Writing the Fokker-Planck equation (74) in the new variables  $(E, \tau)$  is an exercise in Poisson brackets, simplified by the approximation  $\partial b / \partial E \sim 0$  [to see that this approximation is justified we may go to one more order in  $E$  in the expressions for  $p$ ,  $W$  and  $\tau$  and we find that for large  $b$ ,  $\partial b / \partial E \sim -12/(5\Lambda b^3)$ ]. Thus from Eq. (74) with  $F=0$ , we get

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial \tau} + \frac{\pi \nu^2 \Lambda^3}{48} b^6 \frac{\partial^2 f}{\partial E^2}, \quad (103)$$

so that  $K_\tau = f$  (that is, the universe moves along the classical trajectory with  $\dot{\tau}=1$ ), and

$$K_E = -\frac{\pi \nu^2 \Lambda^3}{48} b^6 \frac{\partial f}{\partial E}$$

with only the normal diffusive term present, as was expected. Since  $K_E$  must be positive (at least in the average),  $f$  must decrease beyond the separatrix, as we wanted to show.

This result, in fact, can be made more quantitative if we note that Eq. (103) for a stationary distribution function  $f$  is essentially a heat equation which can be solved in the usual way. For this it is convenient to change to a new variable  $s = -1/(5\tau^5)$  which is positive semidefinite since the conformal time  $\tau$  is negative in the de Sitter region. The equation then can be written as

$$\frac{\partial f}{\partial s} = d \frac{\partial^2 f}{\partial E^2}, \quad (104)$$

where  $d \equiv 36\pi\nu^2$ . Its solution can be written as

$$f(E) = \frac{1}{\sqrt{4\pi ds}} \int dE' e^{(E-E')^2/4sd} h(E'), \quad (105)$$

where  $h(E')$  is a function which determines the value of  $f$  at  $\tau = -\infty$ . It is easy to compute that

$$\int_{-\infty}^0 d\tau f(E, \tau) \propto \int_0^\infty dE' \frac{h(E')}{(E-E')^{7/5}}, \quad (106)$$

which shows that, for large  $E$ ,  $f$  in fact decreases as  $E^{-7/5}$ .

### C. Tunneling amplitude

After the two previous subsections, we gather that the stationary solutions to the Fokker-Planck equation display a marked peak at  $E = E_s$ . We may now estimate the flux by requesting, as we do for Kramers' problem in the Appendix, that the total area below the distribution function should not exceed unity. Unless the lower cutoff  $E_\delta$  is very small (it ought to be exponentially small on  $E_s$  to invalidate our argument) the integral is dominated by that peak, and we obtain

$$K_0 \leq (\text{prefactor}) \exp[-\beta(E_s)E_s]. \quad (107)$$

The prefactor depends on  $\Lambda$ ,  $\nu$ ,  $g(1)$ ,  $\beta(1)$ ,  $\sigma$  and the details of the peak shape. Using  $E_s = 3/(2\Lambda)$ ,  $\beta(E_s) = 1.23$ , we get

$$K_0 \leq (\text{prefactor}) \exp\left(-\frac{1.84}{\Lambda}\right). \quad (108)$$

In the last section of the Appendix we have computed the flux when one considers a cosmological model with a single cosmic cycle. The result (A82) is qualitatively similar to this one; it just gives a slightly lower probability. This semiclassical result must now be compared against the instanton calculations.

## V. CONCLUSIONS

In this paper we have studied the possibility that a closed isotropic universe trapped in the potential well produced by a cosmological constant may go over the potential barrier as a consequence of back-reaction to the quantum effects of a nonconformally coupled quantum scalar field. The quantum fluctuations of this field act on geometry through the stress-energy tensor, which has a deterministic part, associated with vacuum polarization and particle creation, and also a fluctuating part, related to the fluctuation of the stress-energy itself. The result is that the scale factor of the classical universe is subject to a force due to particle creation and also to a stochastic force due to these fluctuations. We compute the Fokker-Planck equation for the probability distribution of the cosmological scale factor and compute the probability that the scale factor crosses the barrier and ends up in the de Sitter stage where  $b \sim \sqrt{12/\Lambda} \cosh(\sqrt{\Lambda/12}t')$ , where  $t'$  is cosmological time  $bdt = dt'$ , if it was initially near  $b \sim 0$ . The result displayed in Eq. (108) is that such probability is

$$K_0 \sim \exp\left(-\frac{1.8}{\Lambda}\right), \quad (109)$$

or a similar result, displayed in Eq. (A82), if we consider a cosmological model undergoing a single cosmic cycle. This result is comparable with the probability that the universe tunnels quantum mechanically into the de Sitter phase from nothing [1]. In this case from the classical action (48)  $S_{g,m}^R[b]$ ; i.e., neglecting the terms of order  $\nu$ , one constructs the Euclidean action  $S_E$ , after changing the time  $t=i\tau$ ,

$$S_E[b] = \frac{1}{2} \int d\tau \left[ \dot{b}^2 + b^2 - \frac{1}{12} \Lambda b^4 \right]. \quad (110)$$

The Euclidean trajectory is  $b = \sqrt{12/\Lambda} \cos(\sqrt{\Lambda/12} \tau')$ , where  $\tau'$  is Euclidean cosmological time (this is the instanton solution). This trajectory gives an Euclidean action  $S_E = 4/\Lambda$ . The tunneling probability is then

$$p \sim \exp\left(-\frac{8}{\Lambda}\right). \quad (111)$$

This result, which in itself is a semiclassical result, is comparable to ours, Eq. (109), but it is of a very different nature. We have ignored the quantum effects of the cosmological scale factor but we have included the back-reaction of the quantum fields on this scale factor. Also our universe reaches the de Sitter stage with some energy due to the particles that have been created. In the instanton solution only the tunneling amplitude of the scale factor is considered and the universe reaches the de Sitter phase with zero energy.

Taken at face value, our results seem to imply that the nonlocality and randomness induced by particle creation are actually as important as the purely quantum effects. This conclusion may be premature since after all Eq. (109) is only an upper bound on the flux. Nevertheless, our results show that ignoring the back-reaction of matter fields in quantum cosmology may not be entirely justified. We expect to delve further into this subject in future contributions.

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#### APPENDIX

To facilitate the reading of this appendix we repeat here the summary of its contents given in the Introduction: Subsection 1 gives some details of the renormalization of the CTP effective action; subsection 2 explains how to handle the diffusion terms when the Fokker-Planck equation is constructed; in subsection 3, we formulate and discuss Kramers' problem in action-angle variables; the subsection 4 gives the exact classical solutions for the cosmological scale factor; in subsection 5, the averaged diffusion and dissipation coefficients for the averaged Fokker-Planck equation are derived;

in subsection 6, the relaxation time is computed in detail; and finally in subsection 7, the calculation of the escape probability for the scale factor is made for a model which undergoes a single cosmic cycle.

#### 1. Divergences of $T$

Here we compute the finite imaginary part of the series defined in Eq. (36) and prove that the real part diverges like  $1/(n-4)$ . The finite real part of the series will not be found explicitly; its exact form is not needed in the calculation of this paper. Let us now call  $\varepsilon \equiv n-4$ , and call  $F(\omega)$  the series (36) which we can write in terms of the gamma functions as

$$\begin{aligned} F(\omega) &\equiv \sum_{k=1}^{\infty} a_k(\omega) \\ &= \sum_{k=1}^{\infty} \frac{\Gamma(k+\varepsilon+1)}{\Gamma(k)} \frac{1}{(k+\varepsilon/2)^2 - (\omega/2)^2 + i0^+} \\ &= \sum_{k=1}^{\infty} \frac{\Gamma(k+\varepsilon+1)}{\Gamma(k)} \left\{ \text{PV} \frac{1}{(k+\varepsilon/2)^2 - (\omega/2)^2} \right. \\ &\quad \left. - i\pi \delta[(k+\varepsilon/2)^2 - (\omega/2)^2] \right\} \\ &\equiv F_R + iF_I, \end{aligned} \quad (A1)$$

where we have used that  $(x \pm i0^+)^{-1} = \text{PV}(1/x) \mp i\pi \delta(x)$ . Let us first concentrate on the imaginary part  $F_I$  and compute, according to Eq. (35), its Fourier transform

$$\begin{aligned} \tilde{F}_I &\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} F_I \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\Gamma(k+\varepsilon+1)}{\Gamma(k)} \frac{\cos(k+\varepsilon/2)(t-t')}{k+\varepsilon/2}, \end{aligned} \quad (A2)$$

where we have used that  $2(k+\varepsilon/2) \delta[(k+\varepsilon/2)^2 - (\omega/2)^2] = \delta(k+\varepsilon/2+\omega/2) + \delta(k+\varepsilon/2-\omega/2)$ . Now the last expression is clearly convergent when  $\varepsilon=0$ ; thus we get

$$\tilde{F}_I = \frac{1}{2} \sum_{k=1}^{\infty} \cos k(t-t'). \quad (A3)$$

this series can be summed up and we get the train of deltas of Eq. (33), thus recovering the noise kernel, from  $\tilde{F}_I = 8N(t-t')$ .

Let us now see that the real part of the series diverges like  $1/\varepsilon$ . Using that  $\Gamma(x+1) = x\Gamma(x)$  the principal part of  $a_k$  can also be written as

$$a_k(\omega) = \frac{\Gamma(k+\varepsilon)}{\Gamma(k)} \frac{k+\varepsilon}{(k+\varepsilon/2)^2 - (\omega/2)^2}. \quad (A4)$$

It is clear from this expression that the divergences when  $\varepsilon=0$  come from the ratio of gamma functions in Eq. (A4) when  $k$  is large. Let us now separate the sum  $\sum_{k=1}^{\infty} a_k$

$= \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} a_k$  where  $N \gg 1$ . We can use now that for large  $x$ ,  $\Gamma(x) = \sqrt{2\pi} x^{-1/2} e^{-x} [1 + O(1/x)]$  and the definition of  $e$ ,  $e = \lim_{n \rightarrow \infty} (1 + 1/n)^n$ , to prove that  $\Gamma(k + \varepsilon)/\Gamma(k) = k^\varepsilon [1 + O(1/k)]$ . Substituting  $a_k$  by  $\bar{a}_k$ , defined by

$$\bar{a}_k = k^\varepsilon \left[ 1 + O\left(\frac{1}{k}\right) \right] \frac{k + \varepsilon}{(k + \varepsilon/2)^2 - (\omega/2)^2}, \quad (\text{A5})$$

in the second sum of the previous separation, we can write  $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} \bar{a}_k$ . Now we can use the Euler-Maclaurin summation formula [40] to write  $\sum_{k=N}^{\infty} \bar{a}_k = \int_N^{\infty} dk \bar{a}_k + \dots$ , where the ellipsis stands for terms which are finite since they depend on successive derivatives of  $\bar{a}_k$  at the integration limits. Thus we may write  $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N-1} a_k - \int_0^N dk \bar{a}_k + \int_0^{\infty} dk \bar{a}_k$ . The first sum and first integral of this last equation are finite for all  $\varepsilon$ ; thus we can take  $\varepsilon = 0$ , in which case  $a_k = \bar{a}_k = k/[k^2 - (\omega/2)^2]$ . The sum and integral may then be performed [writing  $2a_k = 1/(k + \omega/2) + 1/(k - \omega/2)$ ] and the  $\ln N$  which appears in both expressions cancel; the next to leading order terms differ by order  $O(1/N)$ . Therefore the divergence is in the last integral

$$\int_0^{\infty} dk \bar{a}_k = \int_0^{\infty} dk \frac{k^{\varepsilon+1}}{(k + \varepsilon/2)^2 - (\omega/2)^2}, \quad (\text{A6})$$

where here  $\varepsilon$  is an arbitrary parameter. This integral is easily computed [40], and when it is expanded in powers of  $\varepsilon$  we get

$$\int_0^{\infty} dk \bar{a}_k = - \left[ \frac{1}{n-4} + \frac{1}{2} \ln(\omega/2)^2 \right]. \quad (\text{A7})$$

Thus, according to Eqs. (35), (36) and (A1) we compute the Fourier transform of  $F_R$ ,

$$\tilde{F}_R = - \frac{\delta(t-t')}{n-4} - 8A(t-t'), \quad (\text{A8})$$

where  $A(t-t')$  stands for a finite kernel [see Eq. (60)].

## 2. Diffusion terms

We want to compute Eq. (66) which can be written as Eq. (67) in terms of the functions  $C_n$  ( $n=0,1,2$ ) of Eq. (68). The simplest function  $C_0$  can be written after using Eq. (69) as

$$C_0 = \int dt' N(t-t') \left\langle \frac{\delta}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle,$$

whereas to write the other two functions we observe that the noise kernel is translation invariant; so integrating by parts (in a distribution sense),

$$C_n = \int dt' N(t-t') \frac{\partial^n}{\partial t'^n} \left\langle \frac{\delta}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle.$$

We now use the local approximation for the noise kernel to get

$$C_0 = \frac{\pi}{16} \left\langle \frac{\delta}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle \Bigg|_{t' \rightarrow t},$$

and similarly  $C_1$  and  $C_2$ . As we know, this reduces to

$$C_0 = \frac{-\pi}{16} \left\langle \frac{\partial}{\partial b} \left\langle \frac{\delta b(t)}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle + \frac{\partial}{\partial p} \left\langle \frac{\delta p(t)}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle \right\rangle.$$

A functional derivative of the equations of motion leads to

$$\frac{d}{dt} \frac{\delta b(t)}{\delta \xi(t')} = \frac{\delta p(t)}{\delta \xi(t')},$$

$$\frac{d}{dt} \frac{\delta p(t)}{\delta \xi(t')} = -V''[b(t)] \frac{\delta b(t)}{\delta \xi(t')} + 6\nu \left[ \frac{1}{b(t')} \frac{d^2}{dt^2} \delta(t-t') + \frac{V'[b(t')]}{b^2(t')} \delta(t-t') \right],$$

where actually we are computing the right hand side only to lowest order in  $\nu$ . This suggests writing

$$\frac{\delta p(t)}{\delta \xi(t')} = G(t-t') \theta(t-t') + \frac{6\nu}{b(t')} \frac{d}{dt} \delta(t-t'),$$

$$\frac{\delta b(t)}{\delta \xi(t')} = R(t-t') \theta(t-t') + \frac{6\nu}{b(t')} \delta(t-t'), \quad (\text{A9})$$

which works provided

$$\frac{dR}{dt} = G, \quad R(0) = 0,$$

$$\frac{dG}{dt} = -V''[b(t)]R,$$

$$G(0) = 6\nu \left[ \frac{V'[b(t')]}{b^2(t')} - \frac{V''[b(t')]}{b(t')} \right] = 2\nu \Lambda b(t').$$

In the coincidence limit

$$\frac{\delta b(t)}{\delta \xi(t')} \Bigg|_{t' \rightarrow t} = 0, \quad \frac{\delta p(t)}{\delta \xi(t')} \Bigg|_{t' \rightarrow t} = 2\nu \Lambda b, \quad (\text{A10})$$

which leads to



$$\left\langle \frac{\delta}{\delta \xi(t')} \delta(b(t)-b) \delta(p(t)-p) \right\rangle \Big|_{t' \rightarrow t} = -2\nu\Lambda b \frac{\partial}{\partial p} f(b,p,t). \quad (\text{A11})$$

The diffusive terms also involve the first and second derivatives of the propagators with respect to  $t'$ . To find them, we make the following reasoning. We have just seen that, for example,  $R(t,t) \equiv 0$ ; therefore,

$$\frac{\partial}{\partial t'} R(t,t') \Big|_{t' \rightarrow t} = -\frac{\partial}{\partial t} R(t,t') \Big|_{t' \rightarrow t} = -G(t,t) = -2\nu\Lambda b. \quad (\text{A12})$$

With a slight adaptation, we also get

$$\frac{\partial}{\partial t'} G(t,t') \Big|_{t' \rightarrow t} = \frac{\partial}{\partial t} [G(t,t') \Big|_{t' \rightarrow t}] - \frac{\partial}{\partial t} G(t,t') \Big|_{t' \rightarrow t},$$

so that we have

$$\frac{\partial}{\partial t'} G(t,t') \Big|_{t' \rightarrow t} = 2\nu\Lambda p. \quad (\text{A13})$$

Iterating this argument, we find

$$\frac{\partial^2}{\partial t'^2} R(t,t') \Big|_{t' \rightarrow t} = -\frac{\partial}{\partial t} G(t,t) - \frac{\partial}{\partial t'} G(t,t') \Big|_{t' \rightarrow t},$$

where we have permuted a  $t$  and a  $t'$  derivative and used the equations of motion. From this we thus get

$$\frac{\partial^2}{\partial t'^2} R(t,t') \Big|_{t' \rightarrow t} = -4\nu\Lambda p. \quad (\text{A14})$$

The last formula of this type that we need is

$$\frac{\partial^2}{\partial t'^2} G(t,t') \Big|_{t' \rightarrow t} = \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial t'} G(t,t') \Big|_{t' \rightarrow t} \right] - \frac{\partial}{\partial t'} \frac{\partial}{\partial t} G(t,t') \Big|_{t' \rightarrow t},$$

which from the equations of motion leads to

$$\begin{aligned} \frac{\partial^2}{\partial t'^2} G(t,t') \Big|_{t' \rightarrow t} &= -2\nu\Lambda V'(b) + \frac{\partial}{\partial t'} V''[b(t)] R(t,t') \\ &= -2\nu\Lambda V'(b) - V''[b] 2\nu\Lambda b \\ &= -2\nu\Lambda \frac{\partial}{\partial b} (bV'[b]). \end{aligned} \quad (\text{A15})$$

### 3. Kramer's problem

For our purposes in this paper we call Kramer's problem [42] the computation of the "tunneling amplitude" or, more properly, the escape probability of a particle confined in a potential  $V(b)$ , such as Eq. (53) for instance, which has a maximum and a separatrix with an energy  $E_s$ . The particle is subject to a damping force  $\gamma p$  ( $p = \dot{b}$ ) and white noise with amplitude  $\gamma kT$ , according to the fluctuation-dissipation relation, where  $\gamma$  is a friction coefficient,  $k$  Boltzmann constant and  $T$  the temperature. The Fokker-Planck equation in this case is [38]

$$\frac{\partial f}{\partial t} = \{H, f\} + \gamma \frac{\partial}{\partial p} \left[ pf + kT \frac{\partial f}{\partial p} \right], \quad (\text{A16})$$

where  $H$  is given by Eq. (65). Since the particle is trapped in the potential, it undergoes periodic motion; in this case it is convenient to introduce action-angle variables [43] ( $J, \theta$ ) as canonical variables instead of  $(b, p)$ , thus making a canonical transformation  $b = B(\theta, J)$ ,  $p = P(\theta, J)$ . The action variable  $J$  is defined by

$$J = \frac{1}{2\pi} \oint p db. \quad (\text{A17})$$

Since  $p$  can be written in terms of  $b$  and  $H$ , substitution in Eq. (A17) and inversion implies that  $H = H(J)$ , and

$$\frac{\partial H}{\partial J} = \Omega(J) \quad (\text{A18})$$

is the frequency of the motion. The other canonical variable, the angle variable  $\theta$ , satisfies a very simple equation of motion  $\dot{\theta} = \Omega$  and changes from 0 to  $2\pi$ . At high energies, that is, near the separatrix when  $J \rightarrow J_s$ , the motion ceases to be periodic and  $\Omega \rightarrow 0$ . At low energies, let us assume that  $b = 0$  is a stable minimum of the potential; near this minimum the potential approaches the potential of a harmonic oscillator with frequency  $\omega$ ,  $V(b) \sim \omega b^2/2$  (in our case we simply have  $\omega = 1$ ), and then  $J \rightarrow 0$ ,  $H \sim \omega J$  and  $\Omega \sim \omega$ .

If  $\gamma = 0$ , then the solution to the Fokker-Planck equation is an arbitrary function of  $J$  and  $\theta - \Omega t$ . Stationary solutions are therefore functions of  $J$  alone. We may seek a general solution as

$$f(J,t) + \gamma \sum_{n \neq 0} c_n(J,t) e^{in(\theta - \Omega t)};$$

in this case we have  $\partial_p f = \partial_p J|_b \partial_J f$ . From Eq. (65) we have that  $p = [2(H - V(b))]^{1/2}$  and, consequently,  $\partial_J p|_b = \Omega/p$  whose inverse is  $\partial_p J|_b = p/\Omega$ . This can be used to write  $\partial^2 f / \partial p^2$  in terms of derivatives with respect to  $J$ , and since now  $\{H, f(J)\} = 0$  we can write the Fokker-Planck equation (A16) in the new variables as, keeping only first order terms,

$$\begin{aligned} \frac{\partial f}{\partial t} + \gamma \sum_{n \neq 0} \frac{\partial c_n}{\partial t}(J, t) e^{in(\theta - \Omega t)} \\ = \gamma \left\{ f + \frac{p^2}{\Omega} \frac{\partial f}{\partial J} + kT \left[ \frac{1}{\Omega} \frac{\partial f}{\partial J} + \frac{p^2}{\Omega} \frac{\partial}{\partial J} \left( \frac{1}{\Omega} \frac{\partial f}{\partial J} \right) \right] \right\}. \end{aligned} \quad (\text{A19})$$

Fourier expanding the coefficients on the right hand side we obtain a set of equations for the  $c_n$  coefficients. The equation for  $f$  itself follows from the average of this equation over the angle variable  $\theta$ . Let us change the integration variable in the definition (A17) of  $J$ ,  $db = \partial_\theta b|_J d\theta$ , taking into account that over a classical trajectory  $J$  is constant, and that  $\dot{\theta} = \Omega$  we have  $\partial_\theta b|_J = p/\Omega(J)$ . Thus, we can write Eq. (A17) as

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta p^2 = J\Omega(J). \quad (\text{A20})$$

Using this result we can now take the average of Eq. (A19) over  $\theta$ . This average reads, simply,

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial J} \left( J \left[ f + \frac{kT}{\Omega} \frac{\partial f}{\partial J} \right] \right). \quad (\text{A21})$$

As one would expect  $\exp(-E/kT)$  is a solution of this equation. Let us now see whether this equation, which is a transport equation, admits stationary solutions with positive probability flux [19]. Note that we may write this equation as a continuity equation  $\partial_t f + \partial_J K = 0$ , where the flux  $K$  can be read directly from Eq. (A21). Therefore a stationary solution with positive flux  $K_0$  should satisfy

$$\frac{kT}{\Omega} \frac{\partial f}{\partial J} + f = -\frac{K_0}{\gamma J}, \quad (\text{A22})$$

which can be integrated to give

$$f = \frac{K_0}{\gamma kT} e^{-E/kT} \int_J^{J_0} \frac{d\xi}{\xi} \Omega(\xi) e^{E(\xi)/kT}. \quad (\text{A23})$$

For any  $K_0$ ,  $f$  diverges logarithmically when  $J \rightarrow 0$ ; however, this is an integrable singularity in  $J$  and this is not a problem as we will see shortly. In our problem the action variable  $J$  satisfies that  $J \leq J_s$  and Eq. (A23) proves that there is a real and positive solution for any  $J$  in such a range, which corresponds to choosing  $J_0 = J_s$ .

Given a solution we may determine the flux  $K_0$ , imposing the condition that the probability of finding the particle trapped in the potential well should not be greater than unity [19], i.e.  $\int_0^{J_s} f(J) dJ \leq 1$ . This is equivalent to

$$1 \geq \frac{K_0}{\gamma kT} \left( \int_0^{J_s} \frac{d\xi}{\xi} \Omega(\xi) e^{E(\xi)/kT} \int_0^\xi dJ e^{-E/kT} \right). \quad (\text{A24})$$

Since the integral is regular at zero, it is dominated by the contribution from the upper limit, and the integral may be evaluated approximately. One gets

$$K_0 \leq \gamma \frac{\omega J_s}{kT} \exp\left(-\frac{E_s}{kT}\right), \quad (\text{A25})$$

where we have used that near the separatrix  $H \sim \omega J_s$ . Typically the flux is very small so that the probability of finding the particle in the potential well is nearly 1; therefore the value of  $K_0$  approaches the right hand side of Eq. (A25).

We should remark here that in the order reduction scheme that we are following, to compute the noise and the nonlocal terms we use the classical equations of motion. In fact, these terms have a quantum origin in our case and its computation is one of the tasks we have to perform in order to define our particular Kramer's problem. Thus the use of the action-angle variables, which is convenient for the classical equations of motion, is also convenient (after order reduction) in our approach to the Kramer's problem.

#### 4. A look at the orbits

In what follows, we shall quote extensively from Abramowitz and Stegun [44] (AS) and Whittaker and Watson [45] (WW).

The motion is described by the Hamiltonian

$$H = \frac{1}{2}(p^2 + b^2) - \frac{\Lambda}{24} b^4. \quad (\text{A26})$$

The energy is conserved, and on an energy surface  $H = E$ , the momentum is  $p^2 = 2E - b^2 + \Lambda b^4/12$ . The classical turning points correspond to  $p = 0$ . Introducing the separatrix energy  $E_s = 3/(2\Lambda)$ , we can write the four turning points as

$$b_\pm^2 = 4E_s \left[ 1 \pm \sqrt{1 - \frac{E}{E_s}} \right]; \quad (\text{A27})$$

two of them  $\pm b_-$  are inside the barrier, and two  $\pm b_+$  are outside it. The momentum can now be written as

$$p^2 = 2E \left( 1 - k^2 \frac{b^2}{b_-^2} \right) \left( 1 - \frac{b^2}{b_+^2} \right), \quad (\text{A28})$$

where we have introduced  $k^2 = (b_-/b_+)^2$ ; see Eq. (87). The equation for the orbit is  $b = b_- x(t)$ , where

$$x = \text{sn} \left[ \frac{b_+ t}{\sqrt{8E_s}}, k \right], \quad (\text{A29})$$

where sn is the Jacobi Elliptic Function (we follow the notation from WW 22.11; to convert to AS, put  $m = k^2$ , and see AS 16.1.5).

The Jacobi elliptic function is periodic with period  $4K[k]$ , where  $K$  is the complete elliptic integral of the first kind (AS 16.1.1 and 17.3.1) [see Eq. (89)]. The period in physical time is  $T = \sqrt{8E_s} 4K b_+^{-1}$ , and the frequency

$$\Omega = \frac{\pi b_+}{2\sqrt{8E_s} K[k]}. \quad (\text{A30})$$

### 5. D and S functions

The function **D** is given by

$$\mathbf{D}(J) = \frac{1}{2\pi} \int_0^{2\pi/\Omega} dt b^2 p^2,$$

which by introducing  $b = b_- x$  can be written as

$$\begin{aligned} \mathbf{D}(J) &= \frac{1}{2\pi} 4 \int_0^1 (b_- dx) (b_-^2 x^2) \sqrt{2E(1-k^2 x^2)(1-x^2)} \\ &= \frac{2}{\pi} \sqrt{2Eb_-^3} \sigma[k], \end{aligned} \quad (\text{A31})$$

where

$$\sigma[k] = \int_0^1 dx x^2 \sqrt{(1-k^2 x^2)(1-x^2)}. \quad (\text{A32})$$

Following a suggestion in WW 22.72, this can be reduced to complete elliptic integrals of the first and second kinds (we will need the third kind for the *S* function), to get the result quoted in the main text.

The function **S** is given by

$$\mathbf{S} = \frac{-1}{4\pi^2} \int_0^T dt \left[ \frac{d}{dt} b^2(t) \right] \text{Pf} \int_0^\infty \frac{du}{u} b^2(t-u).$$

Let us consider an orbit beginning at  $b(0) = 0$ , and divide the time interval in four quarters: (I)  $0 \leq t \leq T/4$ ; (II)  $T/4 \leq t \leq T/2$ ; (III)  $T/2 \leq t \leq 3T/4$ ; (IV)  $3T/4 \leq t \leq T$ . We have the following relationships: (I) in the first quarter,  $b_I = b(t)$ ,  $p_I = p(t)$ , (II) in the second quarter,  $b_{II}(t) = b_I(T/2 - t)$ ,  $p_{II} = -p_I(T/2 - t)$ , (III) in the third quarter,  $b_{III}(t) = -b_I(t - T/2)$ ,  $p_{III} = -p_I(t - T/2)$ , (IV) in the fourth quarter,  $b_{IV}(t) = -b_I(T - t)$ ,  $p_{IV} = p_I(T - t)$ . This suggests parametrizing time in terms of a unique variable  $\tau$ ,  $0 \leq \tau \leq T/4$ , as follows: (I) In the first quarter,  $t = \tau$ ; (II) in the second quarter,  $t = T/2 - \tau$ ; (III) in the third quarter,  $t = T/2 + \tau$ ; (IV) in the fourth quarter,  $t = T - \tau$ .

We can then write

$$\begin{aligned} \mathbf{S} &= \frac{-1}{2\pi^2} \int_0^{T/4} d\tau b(\tau) p(\tau) \text{Pf} \int_0^\infty \frac{du}{u} \left[ b^2(\tau - u) \right. \\ &\quad \left. - b^2\left(\frac{T}{2} - \tau - u\right) + b^2\left(\frac{T}{2} + \tau - u\right) - b^2(T - \tau - u) \right]. \end{aligned}$$

Since  $b^2$  is an even function of  $t$  with period  $T/2$ , we have

$$\mathbf{S} = \frac{-1}{\pi^2} \int_0^{T/4} d\tau b(\tau) p(\tau) \text{Pf} \int_0^\infty \frac{du}{u} [b^2(\tau - u) - b^2(\tau + u)],$$

and since the second integrand is obviously even,

$$\mathbf{S} = \frac{1}{\pi^2} \int_0^{T/4} d\tau b(\tau) p(\tau) \text{Pf} \int_{-\infty}^\infty \frac{du}{u} b^2(\tau + u). \quad (\text{A33})$$

To proceed, we must appeal to the addition theorem for elliptic functions (AS 16.17.1). Next we use the differential equation for Jacobi elliptic functions (AS 16.16.1) and integrate by parts to get

$$\mathbf{S} = \frac{32EE_s}{\pi^2} k^2 \text{Pf} \int_0^\infty \frac{du}{u^2} \text{sn}^2(u) \rho[k^2 \text{sn}^2(u)], \quad (\text{A34})$$

where

$$\rho[n] = \int_0^1 dx \frac{x^2 \sqrt{(1-k^2 x^2)(1-x^2)}}{[1-nx^2]}, \quad (\text{A35})$$

which can be expressed in terms of complete elliptic integrals

$$\rho = (-k^2) \left\{ \left[ c' + \frac{a}{k^2} \right] K[k] - \frac{a}{k^2} E[k] - c \Pi[n, k] \right\}, \quad (\text{A36})$$

where the last term is the complete elliptic integral of the third kind (AS 17.7.2), with  $\sin \alpha = k$ ,

$$\begin{aligned} a &= \frac{1}{n} \left[ \frac{1}{n} - \frac{(1+k^2)}{3k^2} \right], \\ c' &= \frac{1}{n} \left[ \frac{2}{3k^2} + \frac{1}{n} \left( \frac{1}{n} - \frac{(1+k^2)}{k^2} \right) \right], \\ c &= \frac{1}{n} \left[ \frac{1}{k^2} + \frac{1}{n} \left( \frac{1}{n} - \frac{(1+k^2)}{k^2} \right) \right]. \end{aligned}$$

Since in our application we always have  $n \leq k^2$ , we may use formulas AS (17.7.6) and (17.4.28) to get the result in the text [recall that  $E/E_s = 4k^2/(1+k^2)^2$ ].

### 6. Relaxation time

The aim of this section is to estimate the time on which a solution to the transport equation with arbitrary initial conditions relaxes to a steady solution as discussed in the main body of the paper, in Sec. IV. The way this kind of problem is usually handled [47] is to write the Fokker-Planck equation (74) in a way resembling a (Euclidean) Schrödinger equation

$$\frac{\partial f}{\partial t} = Lf. \quad (\text{A37})$$

Then if a complete basis of eigenfunctions of the *L* operator can be found,

$$Lf_n(p, q) = E_n f_n(p, q), \quad (\text{A38})$$

a generic solution to Eq. (A37) reads

$$f(p, q, t) = \sum c_n f_n(p, q) e^{E_n t}. \quad (\text{A39})$$

Therefore, provided no eigenvalue has a positive real part, the relaxation time is the inverse of the real part of the largest nonzero eigenvalue. The  $L$  operator may have purely imaginary eigenvalues, in which case it does not relax towards any steady solution.

This problem differs from the ordinary quantum mechanical one in several aspects, the most important being that the  $L$  operator does not have to be either Hermitian or anti-Hermitian. That is why the eigenvalues will be generally complex, rather than just real or imaginary. Also, it is important to notice that the ‘‘right’’ eigenvalue problem, Eq. (A38), is different from the ‘‘left’’ eigenvalue problem:  $g_n \tilde{L} = E'_n g_n$ . For example, for any  $L$  of the form  $L = \partial_i K^i$ , where the  $K$ 's are themselves operators,  $g_0 \equiv 1$  is a solution to this (left) equation (with zero eigenvalue), while it may not be a solution to Eq. (A38) at all.

### a. Our problem

In our case, the  $L$  operator can be read from Eq. (74). Since we are taking  $\nu$  as a small parameter, it is natural to write  $L = L^0 + L^1$ , where

$$L^0 f = \{H, f\}, \quad (\text{A40})$$

$$L^1 f = -\frac{\partial}{\partial p} [Ff] + \frac{\pi \nu^2 \Lambda^2}{4} b^2 \frac{\partial^2 f}{\partial p^2}. \quad (\text{A41})$$

The spectral decomposition of  $L^0$  is very simple. In action-angle variables,

$$L^0 f = -\Omega(J) \frac{\partial f}{\partial \theta}. \quad (\text{A42})$$

Imposing periodicity in  $\theta$  we find the following eigenvalues: 0 and

$$E_{n,\chi}^0 = -in\Omega(\chi), \quad (\text{A43})$$

with  $n$  integer (note that  $L^0$  is anti-Hermitian). The eigenvalue 0 is infinitely degenerate: any function of  $J$  alone is an eigenvector with zero eigenvalue. The  $E_{n,\chi}^0$  have eigenfunctions

$$f_{n,\chi}^0(J, \theta) = \frac{e^{in\theta}}{\sqrt{2\pi}} \delta(J - \chi), \quad (\text{A44})$$

and, barring accidental degeneracy (the ratio of frequencies for two different actions being rational), are nondegenerate. These eigenfunctions are normalized with the Hilbert product  $(g|f) = \int_0^{J_s} \int_0^{2\pi} dJ d\theta g^* f$  as  $(0n\xi|0n\chi) = \delta(\xi - \chi)$ , where here and in the rest of this section we use Dirac's notation.

Having solved the eigenvalue problem for  $L^0$ , it is only natural to see that of  $L$  as an exercise in time independent perturbation theory. There are three differences with the ordinary textbook problem: (1)  $L^1$  is neither Hermitian nor anti-Hermitian; (2) one of the eigenvalues of  $L^0$  is degener-

ate; (3) the eigenfunctions of  $L^0$  are not normalizable. In spite of this, the basic routine from quantum mechanics textbooks still works.

### b. Perturbations to nonzero eigenvalues

Let us seek the first order correction to  $E_{n,\chi}^0$ . We write the exact eigenvalue as  $E_{n,\chi} = E_{n,\chi}^0 + E_{n,\chi}^1 + \dots$  corresponding to the exact eigenfunction  $f_{n,\chi} = f_{n,\chi}^0 + f_{n,\chi}^1 + \dots$ , and obtain

$$L^1 f_{n,\chi}^0 + L^0 f_{n,\chi}^1 = E_{n,\chi}^1 f_{n,\chi}^0 + E_{n,\chi}^0 f_{n,\chi}^1. \quad (\text{A45})$$

For  $m \neq n$  we multiply both sides of the equation by  $f_{m,\xi}^{0*}$ , use that  $L^0$  is anti-Hermitian and integrate over  $J$  and  $\theta$ , to get

$$(0m\xi|1n\chi) = \frac{(0m\xi|L^1|0n\chi)}{E_{n,\chi}^0 - E_{m,\xi}^0}. \quad (\text{A46})$$

In the  $m = n \neq 0$  case, the same operation yields

$$E_{n,\chi}^1 (0n\xi|0n\chi) = (0n\xi|L^1|0n\chi) - [E_{n,\chi}^0 - E_{n,\xi}^0] (0n\xi|1n\chi), \quad (\text{A47})$$

and we may write

$$L^1 f_{n,\chi}^0 = \frac{e^{in\theta}}{\sqrt{2\pi}} [R + iI], \quad (\text{A48})$$

where

$$R = L^1 \delta(J - \xi) - n^2 \frac{\pi \nu^2 \Lambda^2}{4} b^2 \left( \frac{\partial \theta}{\partial p} \Big|_b \right)^2 \delta(J - \xi). \quad (\text{A49})$$

Whatever the imaginary part  $I$  is, it is not relevant to the relaxation time; in a similar way, the average of the first term in Eq. (A49) yields no term proportional to  $(0n\xi|0n\chi)$ . Therefore, we conclude that

$$\text{Re}[E_{n,\chi}^1] = -n^2 \frac{\pi \nu^2 \Lambda^2}{8\pi} \int_0^{2\pi} d\theta b^2 \left( \frac{\partial \theta}{\partial p} \Big|_b \right)^2 \Big|_{J=\chi}. \quad (\text{A50})$$

We see on dimensional grounds alone that the relaxation time (the inverse of this equation) will be of order  $E_s^2$  [recall that  $E_s = 3/(2\Lambda)$ ], much shorter than the average tunneling time, which is proportional to the inverse of Eq. (108).

The expression (A50) may be slightly simplified by using the identity  $(\partial \theta / \partial p)|_b = -(\partial b / \partial J)|_\theta$ , which follows from the transformation from one set of variables to the other being canonical. We may write

$$\text{Re}[E_{n,\chi}^1] = -n^2 \frac{\pi \nu^2 \Lambda^2}{32\pi} \int_0^{2\pi} d\theta \left( \frac{\partial b^2}{\partial J} \Big|_\theta \right)^2 \Big|_{J=\chi}. \quad (\text{A51})$$

We may Fourier transform  $b^2$  as a function of  $\theta$ , derive term by term, and use Parseval's identity, to conclude that, in any case,

$$|\text{Re}[E_{n,\chi}^1]| \geq n^2 \frac{\pi \nu^2 \Lambda^2}{(8\pi)^2} \left[ \frac{d}{dJ} \int_0^{2\pi} d\theta b^2 \right] \Big|_{J=\chi}. \quad (\text{A52})$$

The integral in this expression can be performed; recall that  $b = b_{-x}(t)$  where  $x(t)$  is given in Eq. (A29). We recall also that  $\Omega = \theta t$  with  $\Omega$  given in Eq. (A30), and then use as integration variable  $2\theta K[k]/\pi$ , where  $K[k]$  is the elliptic integral defined in Eq. (89), to get finally

$$|\text{Re}[E_{n,\chi}^1]| \geq n^2 \frac{\pi \nu^2 \Lambda^2}{(4\pi)} \left\{ \frac{d}{dJ} \left[ \frac{b_-^2}{k^2} \left( 1 - \frac{E[k]}{K[k]} \right) \right] \right\}^2. \quad (\text{A53})$$

Rather than a general formula, let us investigate the limiting cases. For  $J \rightarrow 0$ , we have  $J \sim E$ ,  $k^2 \sim E/(4E_s)$ ,  $b_-^2 \sim 2E$ ,  $E[k] \sim (\pi/2)(1 - k^2/4)$ , and  $K[k] \sim (\pi/2)(1 + k^2/4)$ . In this limit we thus get

$$|\text{Re}[E_{n,\chi}^1]| \geq n^2 \frac{\nu^2 \Lambda^2}{4}. \quad (\text{A54})$$

For  $J \rightarrow J_s$  (near the separatrix) we can use the following approximations:  $b_-^2 \sim 4E_s[1 - \sqrt{1 - E/E_s}]$ ,  $k^2 \sim [1 - 2\sqrt{1 - E/E_s}]$ ,  $K[k] \sim (1/2)\ln[16/(1 - k^2)] \sim (1/4)\ln[64/(1 - E/E_s)]$ ,  $E[k] \sim 1 + (1/4)\sqrt{1 - E/E_s}\{\ln[64/(1 - E/E_s)] - 1\}$ , and  $dE/dJ = \Omega \sim \pi/(2K[k])$ . Thus the correction to the eigenvalue diverges. In both cases, we get that the relaxation time is much smaller than the tunneling time.

### c. Perturbation of the zero eigenvalue

We now confront the harder problem of finding the first order correction to the zero eigenvalue. The idea, as in quantum mechanics, is that the first order eigenvalues shall be the eigenvalues of the restriction of  $L^1$  to the proper subspace of the zero eigenvalue, namely, the infinite dimensional space of all  $\theta$  independent functions. If  $f_{0,\chi}$  corresponds to an eigenfunction with null eigenvalue, the first order secular equation becomes

$$L^1 f_{0,\chi}^0 + L^0 f_{0,\chi}^1 = E_{0,\chi}^1 f_{0,\chi}^0. \quad (\text{A55})$$

We eliminate the second term in the left hand side of this equation by projecting back on  $\theta$  independent functions, by averaging over  $\theta$ . Fortunately the average over  $\theta$  of  $L^1$  acting on a  $\theta$  independent function is precisely what we did in Sec. III; so using Eqs. (80) and (84) we can write down the eigenvalue problem

$$\frac{\pi \nu^2 \Lambda^2}{4} \frac{d}{dJ} \left[ \frac{\mathbf{D}}{\Omega} \frac{d}{dJ} - \mathbf{S} \right] f = \lambda f, \quad (\text{A56})$$

where we call  $\lambda$  the eigenvalue, to avoid confusion with the energy. The left hand side of this equation is a sum of two terms, the first one being Hermitian, and the second undefined. However, if we introduce a new function  $\Psi$  by  $f = \Psi \exp[\frac{1}{2} \int^E dE' \beta(E')]$  where  $\beta = \mathbf{S}/\mathbf{D}$ , we can write

$$\frac{\pi \nu^2 \Lambda^2}{4} \left\{ \frac{d}{dJ} \frac{\mathbf{D}}{\Omega} \frac{d}{dJ} - \frac{1}{2} \frac{d\mathbf{S}}{dJ} - \frac{\Omega \mathbf{S}^2}{4\mathbf{D}} \right\} \Psi = \lambda \Psi. \quad (\text{A57})$$

Recall that we have seen in Sec. III that  $\mathbf{S}$  is an increasing function of  $E$  (or  $J$ ). Therefore, multiplying by  $\Psi^*$  and integrating, we see that  $\lambda$  must be real and negative. This is an important result.

Let us introduce a new non-negative parameter  $\alpha$ ,

$$\lambda = - \frac{\pi \nu^2 \Lambda^2}{8} \alpha, \quad (\text{A58})$$

write Eq. (A57) using  $E$  as independent variable instead of  $J$  ( $dE/dJ = \Omega$ ), and then introduce a new function  $\psi$  by  $\Psi = \psi/\sqrt{\mathbf{D}}$ . Finally Eq. (A57) becomes

$$- \frac{1}{2} \psi'' + V_\alpha(E) \psi = 0 \quad (\text{A59})$$

where

$$V_\alpha(E) = \frac{1}{4\mathbf{D}} \left\{ \frac{d\mathbf{S}}{dE} + \frac{\mathbf{S}^2}{2\mathbf{D}} + \mathbf{D}'' - \frac{\mathbf{D}'^2}{2\mathbf{D}} - \frac{\alpha}{\Omega} \right\}, \quad (\text{A60})$$

which looks like a Schrödinger equation with an unusual potential. We have therefore transformed the problem of finding the eigenvalues of Eq. (A57) into the question of for which values of  $\alpha$  a particle of zero energy has a bound state in the potential  $V_\alpha(E)$ .

To get an idea of what is going on, let us make the approximation  $\mathbf{D} \sim cE^2$ ,  $\mathbf{S} \sim \beta\mathbf{D}$ , where  $c$  and  $\beta$  are constant; then,

$$V_\alpha(E) = \frac{\beta}{4E^2} \left[ 2E + \frac{\beta}{2} E^2 - \frac{\alpha}{c\beta\Omega} \right]. \quad (\text{A61})$$

When  $\alpha = 0$ , we should get back some results of Sec. IV. Indeed, in this case the solutions for large  $E$  go like  $\exp(\pm \beta E/2)$ , which, after the equation relating  $f$  with  $\Psi$ , means that the solutions either are exponentially growing or bounded. The first ones correspond to steady solutions with nonzero flux (those in Sec. IV), while the second ones are the stationary solutions with no flux. Note that the change from  $\Psi$  to  $\psi$ , which we made previously, enforces the pathological  $E^{-1}$  low energy behavior we found in Sec. IV.

For  $\alpha \neq 0$ , the effective potential  $V_\alpha$  has two classical turning points, i.e. points where  $V_\alpha(E) = 0$ . For small  $E$  we find  $E_1 \sim \alpha/(2c\beta)$  [we use that  $\Omega(E_1) \sim 1$ ], and for large  $E$  we find  $E_2$  given by  $\Omega^{-1}(E_2) \sim c\beta^2 E_s^2/(2\alpha)$ , which, under the asymptotic form  $\Omega^{-1}(E) \sim \ln[64/(1 - E/E_s)]/(\sqrt{2}\pi)$ , is  $E_2 \sim E_s \{1 - 64 \exp[-\pi c\beta^2 E_s^2/(\sqrt{2}\alpha)]\}$ . The first classically allowed region sits precisely where the theory is unreliable, and we ought to disregard it as an artifact. Therefore the low  $\alpha$  eigenstates must be related to the presence of the second allowed region, near the separatrix. This is consistent with the fact that the zeroth order eigenvalues are  $-in\Omega$  [see Eq. (A43)], and so they tend to accumulate around 0 as we approach the separatrix.

In the second classically allowed region (large  $E$ ) we may approximate

$$V_\alpha(E) \sim \frac{\alpha}{4cE_s^2} \left[ \frac{1}{\Omega(E_2)} - \frac{1}{\Omega(E)} \right]. \quad (\text{A62})$$

As an estimate, we may look for values of  $\alpha$  such that  $V_\alpha$  satisfy a Bohr-Sommerfeld condition

$$\int_{E_2}^{E_s} dE \sqrt{-2V_\alpha(E)} \sim n\pi \quad (\text{A63})$$

(this only makes sense if we treat the separatrix as a turning point). To perform the integral, we introduce a new variable  $x = \ln[(1 - E_2/E_s)/(1 - E/E_s)]$ . The integral turns out to be  $n\pi \sim \sqrt{\alpha(1 - E_2/E_s)} \int_0^\infty dx \sqrt{x} e^{-x} / \sqrt{2\sqrt{2}\pi c}$ , and so the eigenvalues are the roots of

$$\alpha_n \exp\left(-\frac{\sqrt{2}\pi c \beta^2 E_s^2}{\alpha_n}\right) = \frac{n^2 \pi^2 c}{128}. \quad (\text{A64})$$

The relevant value of  $c$  being 0.96 near the separatrix (see the end of Sec. III), thus  $\beta \sim 1.23$ . Taking the logarithm of Eq. (A64), we find the lowest eigenvalue

$$\alpha_1 = \frac{\sqrt{2}\pi c \beta^2 E_s^2}{\ln(128\sqrt{2}\beta^2 E_s^2/\pi)} \left[ 1 + O\left(\frac{\ln \ln E_s}{\ln E_s}\right) \right]. \quad (\text{A65})$$

This is the result we were looking for. Going back to the beginning, we translate this into eigenvalues of the Fokker-Planck operator [see Eqs. (A38)] and (A56),

$$\lambda \sim -\frac{9\pi\nu^2}{32} \frac{\sqrt{2}\pi c \beta^2}{\ln(128\sqrt{2}\beta^2 E_s^2/\pi)}, \quad (\text{A66})$$

where we have used Eq. (A58) and that  $E_s = 3/(2\Lambda)$ . Thus we conclude that the relaxation time grows logarithmically with  $E_s$ , while the tunneling time grows exponentially. In fact, the tunneling time is proportional to the inverse of Eq. (108), and so it goes like  $\sim \exp(1.23 E_s)$ . Therefore it is totally justified to analyze tunneling under the assumption that all transient solutions have died out, and we only have the steady solutions discussed in Sec. IV.

## 7. Single cosmic cycle

The purpose of this section is to discuss whether it is possible to generalize the discussion of the paper to models with a single cosmic cycle. The basic problem is that an universe emerging from the singularity with a finite expansion rate is bound to lead to infinite particle production [48]. Therefore, in order to make sense, it is unavoidable to modify the behavior of the model close to the singularity, and there is no unique way to do this. Of course, a possibility is to assume that the singularity behaves as a perfectly reflecting boundary, which is equivalent to what we have done so far. Another possibility, to be discussed here, is that the evolution is modified for very small universes, so that  $p$  van-

ishes as  $b \rightarrow 0$ . For example, if the initial stages of expansion (and the final stages of collapse) are replaced by an inflationary (deflationary) period, then  $p \sim b^2$ ,  $\dot{p} \sim b^3$ , etc. We shall assume such an evolution in what follows. In these models, the singularity is literally pushed to the edge of time.

### a. $\mathbf{D}$ and $\mathbf{S}$ functions

The  $\mathbf{D}$  function is given by Eq. (76), where now we average over a half period only. However, the periodicity of the integrand is precisely  $T/2$ ; so the average over a half period is the same as the full average. Therefore,  $\mathbf{D} \sim E^2/2$  at low energy, and  $0.96E^2$  close to the separatrix as we had in the many cycles model.

For the function  $\mathbf{S}$ , let us begin from Eq. (78), modified to represent average over a half period,

$$S(J) = \frac{1}{\pi} \int_0^{T/2} dt p F(b, p, t), \quad (\text{A67})$$

and then use Eq. (55) for  $F$  and integrate by parts twice to get

$$S(J) = \frac{6\nu}{\pi} \left[ p \frac{dI}{dt} \frac{1}{b} \right]_0^{T/2} - \frac{6\nu}{\pi} \left[ \frac{I}{b} \right]_0^{T/2} + \frac{6\nu}{\pi} \int_0^{T/2} dt \left( \frac{\ddot{p}}{b} - \frac{p\dot{b}}{b^2} \right) I(b, p, t). \quad (\text{A68})$$

The discussion above on the approach to the singularity means that the integrated terms vanish. In the remaining term we use the equations of motion  $\ddot{b} = \dot{p} = -V'(b)$  and get

$$S(J) = \frac{\nu\Lambda}{\pi} \int_0^{T/2} dt \frac{db^2}{dt} I(b, p, t). \quad (\text{A69})$$

Next use Eqs. (56), (61), (49), (51) and the redefinition (84) to write

$$\mathbf{S} = -\frac{1}{2\pi^2} \int_0^{T/2} dt \frac{db^2}{dt} \text{Pf} \int_0^t \frac{du}{u} b^2(t-u), \quad (\text{A70})$$

where we have truncated the  $u$  integral to restrict it to the range where the equations of motion hold.

Instead of looking for a general expression, we shall only consider the low energy limit and the behavior close to the separatrix.

### b. Low energy limit

For low energy,  $b = (\sqrt{2E}/\Omega) \sin \Omega t$ . Substituting this into Eq. (A70), changing the order of integration and performing some simple integrations we obtain

$$\begin{aligned} \mathbf{S} &= \frac{E^2}{2\pi^2\Omega^4} \text{Pf} \int_0^{T/2} \frac{du}{u} [1 - \cos 2\Omega u + \pi \sin 2\Omega u] \\ &= 6.89 \frac{E^2}{2\pi^2\Omega^4}, \end{aligned} \quad (\text{A71})$$

where the last integration has been performed numerically. Thus,  $\mathbf{S}$  retains the main features as in the previous case, the most important being the sign and energy dependence.

### c. Close to the separatrix

Close to the separatrix, we must make allowance for the fact that the orbit spends an increasing amount of time near the turning point  $b_-$ . It is thus convenient to isolate the central portion of the orbit. Let us rewrite Eq. (A70) as

$$\begin{aligned} \mathbf{S} &= -\frac{1}{2\pi^2} \left[ \int_0^{T/4} dt \frac{db^2}{dt} \int_0^t \frac{du}{u} b^2(t-u) \right. \\ &\quad \left. + \int_{T/4}^{T/2} dt \frac{db^2}{dt} \int_0^t \frac{du}{u} b^2(t-u) \right]. \end{aligned} \quad (\text{A72})$$

Divide the  $u$  integral by quarter orbits, write  $t=T/2-t'$  in some of these integrals, and use the periodicity and parity of  $b^2$  and  $db^2/dt$ . We can then rewrite  $\mathbf{S}$  as

$$\mathbf{S} = \mathbf{A} + \mathbf{B} \quad (\text{A73})$$

where

$$\begin{aligned} \mathbf{A} &= -\frac{1}{2\pi^2} \left[ \int_0^{T/4} dt \frac{db^2}{dt} \int_0^{T/4+t} \frac{du}{u} b^2(t-u) \right. \\ &\quad \left. - \int_0^{T/4} dt \frac{db^2}{dt} \int_0^{T/4-t} \frac{du}{u} b^2(t+u) \right], \\ \mathbf{B} &= \frac{1}{2\pi^2} \left[ \int_0^{T/4} dt \frac{db^2}{dt} \int_t^{T/4+t} \frac{du}{u} b^2(t-u) \right. \\ &\quad \left. + \int_0^{T/4} dt \frac{db^2}{dt} \int_{T/4-t}^{T/2-t} \frac{du}{u} b^2(t+u) \right]. \end{aligned}$$

Observe that the factor  $db^2/dt$  effectively cuts off the  $t$  integrals at times much shorter than  $T/4$ . So we can take the limit  $T \rightarrow \infty$ , whereby  $\mathbf{A}$  converges to the expression for  $\mathbf{S}$  of the previous case, i.e. Eq. (A33). Here, our problem is to estimate  $\mathbf{B}$ .

Let us write  $\mathbf{B} = \mathbf{C} + \mathbf{D}$ , where

$$\begin{aligned} \mathbf{C} &= \frac{1}{2\pi^2} \int_0^{T/4} dt \frac{db^2}{dt} \int_0^{T/4} \frac{dv}{t+v} b^2(v), \\ \mathbf{D} &= \frac{1}{2\pi^2} \int_0^{T/4} dt \frac{db^2}{dt} \int_0^{T/4} \frac{dv}{T/2-v-t} b^2(v). \end{aligned}$$

To evaluate  $\mathbf{C}$ , we integrate by parts and take the limit  $T \rightarrow \infty$ ,

$$\begin{aligned} \mathbf{C} &= \frac{b_-^4}{2\pi^2} \ln\left(\frac{T}{4}\right) - \frac{1}{2\pi^2} \int_0^\infty dt \frac{db^2}{dt} \int_0^\infty dv \ln(t+v) \frac{db^2}{dv} \\ &\quad + O\left(\frac{1}{T}\right). \end{aligned} \quad (\text{A74})$$

Let us use the same argument in  $\mathbf{D}$ , take the limit and add  $\mathbf{C}$  to get  $\mathbf{B}$ . The final result is

$$\begin{aligned} \mathbf{S} &= \frac{1}{2\pi^2} \left\{ \int_0^\infty dt \frac{db^2}{dt} \int_0^\infty du \left[ \frac{1}{u} \{b^2(t+u) - b^2(t-u)\} \right. \right. \\ &\quad \left. \left. - \ln(t+u) \frac{db^2}{du} \right] + b_-^4 \ln\left(\frac{T}{2}\right) \right\}. \end{aligned} \quad (\text{A75})$$

Using that at the separatrix  $b = \sqrt{4E_s} \tanh(t/\sqrt{2})$ , the double integral in the above expression gives  $13.89E_s^2/(2\pi^2)$ , and we finally have

$$\mathbf{S} = 0.70E_s^2 + \frac{8E_s^2}{\pi^2} \ln\left(\frac{T}{2}\right). \quad (\text{A76})$$

For  $T$  we have the result [cf. Eq. (A30)]  $T = 4\sqrt{2}K[k]/(1 + \sqrt{1-E/E_s})$ , and when  $k \rightarrow 1$ ,  $K[k] \sim (1/4)\ln[64/(1 - E/E_s)]$ , and  $\mathbf{S}$  can then be written as

$$\mathbf{S} = 0.42E_s^2 + \frac{8E_s^2}{\pi^2} \ln\left(\ln\frac{64}{1-E/E_s}\right). \quad (\text{A77})$$

### d. Flux

We shall now show that, in spite of the divergence in  $\mathbf{S}$ ,  $f$  itself remains finite as we approach the separatrix. Basically, the arguments in Sec. IV still hold; so the equation to solve is

$$\frac{df}{dE} - \left[ \beta + \alpha \ln\left(\ln\frac{64}{1-E/E_s}\right) \right] f = 0, \quad (\text{A78})$$

where  $\beta = 0.44(0.42/0.96)$  and  $\alpha = 0.84(8/0.96\pi^2)$ . Let us now call  $64e^{-x} = 1 - E/E_s$ ; then  $dE = 64E_s e^{-x} dx$  and the equation becomes

$$\frac{df}{dx} - 64E_s [\beta + \alpha \ln x] e^{-x} f = 0, \quad (\text{A79})$$

which is well behaved as  $x \rightarrow \infty$ .

In order to estimate the flux, we now need the integral of  $f$  in a neighborhood of the separatrix, namely  $K^{-1} \sim \int dE f$ . With the same change of variables as above, we get

$$K^{-1} \sim 64E_s \int^\infty dx \exp\left[64E_s \int^x dx' (\beta + \alpha \ln x') e^{-x'} - x\right]. \quad (\text{A80})$$

The integral peaks when  $64E_s(\beta + \alpha \ln x)e^{-x} = 1$ , which defines  $x_0 = \ln(64E_s) + \ln(\beta + \alpha \ln x_0)$ , and thus

$$K^{-1} \sim \frac{1}{\beta + \alpha \ln x_0} \exp \left[ 64 E_s \int^{x_0} dx' (\beta + \alpha \ln x') e^{-x'} \right]. \quad (\text{A81})$$

In order to get back the old result when  $\alpha=0$ , we must assume a lower limit for the integral at  $x \sim \ln 64 \sim 4.16$ , which corresponds to  $E \sim 0$ . This limit is high enough that the integral is dominated by the lower limit ( $e^{-x} \ln x$  peaks below  $e$ ); so we finally obtain

$$K \sim (\beta + \alpha \ln x_0) \exp[-(\beta + 1.62\alpha)E_s] \\ \sim (\text{prefactor}) \exp\left(-\frac{2.71}{\Lambda}\right). \quad (\text{A82})$$

This result should be compared to our previous result (108) or (109). In spite of everything, we are still above the quantum tunneling probability (111). Thus, considering a cosmological model which undergoes a single cosmic cycle does not qualitatively change our conclusions.

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