Mode decomposition and renormalization in semiclassical gravity

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We compute the influence action for a system perturbatively coupled to a linear scalar field acting as the environment. Subtleties related to divergences that appear when summing over all the modes are made explicit and clarified. Being closely connected with models used in the literature, we show how to completely reconcile the results obtained in the context of stochastic semiclassical gravity when using mode decomposition with those obtained by other standard functional techniques. [S0556-2821(99)08020-0]

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I. INTRODUCTION

The closed time path (CTP) functional formalism has been very useful to study the back-reaction effect in the context of semiclassical gravity (as well as other aspects of field theory) [1–4]. When considering the back-reaction problem in semiclassical gravity, one is usually interested only in the gravitational field dynamics whereas the quantum matter fields are treated as an environment [6,7]. The results obtained when integrating out the environmental degrees of freedom are closely connected with the influence functional [5], a statistical field theory method which has proved very fruitful to reveal the stochastic nature of open quantum systems (for applications to quantum Brownian motion models see Ref. [8]). In fact, it has been pointed out that semiclassical gravity [9,6] and effective theories in general should exhibit dissipation and noise [4]. To describe the stochastic character of the system dynamics due to the noise induced by the environment, Langevin-type equations are required. Thus, Einstein-Langevin equations have been used to address the back-reaction problem in the framework of semiclassical gravity [6,10,7,11].

When dealing with fields, mode decomposition can be a useful calculational tool since it makes the problem closer to quantum mechanical systems (free fields are treated as an infinite set of decoupled harmonic oscillators). The main advantage of this method is that the noise and dissipation kernels can be obtained in a rather direct way [12] and, in the context of semiclassical gravity, it provides a simple connection with the Bogoliubov coefficients (closely related to particle creation effects) [6,10]. For each mode no renormalization is required; the need for renormalization arises when considering an infinite number of degrees of freedom: it is precisely when summing over all the modes that one gets ultraviolet divergences. However, the appearance of distributional functions makes this sum rather subtle; the presence of such divergences is not always manifestly evident and misleading results may be obtained. In the semiclassical gravity context this is particularly important as one may overlook the need for counterterms to renormalize the divergences, which will imply the appearance of finite extra terms when addressing the back-reaction problem. These drawbacks do not arise in other treatments based on functional methods typical of quantum field theory (QFT) which make no use of mode decomposition, where renormalization seems to be more easily handled [1,7,13].

The aim of this Brief Report is to show how to reconcile the results obtained by means of a mode-decomposition approach with the results based on standard field theory techniques for renormalization in curved space-times [13]. In Sec. II we introduce the notation and the model that we are going to work with and evaluate the influence action perturbatively. A concrete example is considered in Sec. III, where sum over modes is performed revealing the appearance of divergences, and it is shown how they can be handled. In Sec. IV, the previous results are used to consider models treated in the literature which use mode decomposition in the context of stochastic semiclassical gravity and show how to reconcile these results with those obtained by usual functional methods.

II. MODE-DECOMPOSED EXPRESSION FOR THE INFLUENCE ACTION

To make the description as simple as possible we follow Ref. [6] and consider the whole action for a system described by the variable $x(t)$ with action $S[x(t)]$ and the environment, described by a free field $\phi(t,\vec{x})$ in flat space which has been decomposed in a complete set of modes $\{u_k(\vec{x})\}$: $\phi(t,\vec{x})=\Sigma_k q_k(t) u_k(\vec{x})$. The free action $S[\phi(t,\vec{x})]$ is local and at most quadratic in $\phi(t,\vec{x})$, since the field is linear. If that is also the case for the term $S^{int}[x(t),\phi(t,\vec{x})]$ describing the interaction with the system, the action terms for the environment may be written, after performing the spatial integrals and using the completeness relation for the modes, as $\Sigma_k S[q_k(t)]$ and $\Sigma_k S_k^{int}[x(t),q_k(t)]$, respectively, where the action for each mode $S[q_k(t)]$ corresponds to that of a harmonic oscillator. The dynamics is therefore equivalent to that of a set of decoupled harmonic oscillators interacting separately with the system:

$$S[x(t),\phi(t,\vec{x})]=S[x(t)]+\sum_k S[q_k(t)] + \sum_k S_k^{int}[x(t),q_k(t)], \tag{1}$$

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where $S_k^{int}[x(t),q_k(t)]=\int dt Q_k(q_k(t))h(x(t))$ with Q_k and h being some specific functions. The expression for the Feynman-Vernon influence functional [5] in the interaction picture is

$$F[x_+,x_-]=e^{iS_{IF}[x_+,x_-]}={}_I\langle 0\ \ \mathrm{in}|\prod_k T^-(e^{-iS_k^{int}[x_-(t)]})\times T^+(e^{iS_k^{int}[x_-(t)]})|0\ \ \mathrm{in}\rangle_I, \tag{2}$$

where T^+ and T^- correspond to the time ordering and anti-time ordering prescriptions, respectively. To obtain the influence action S_{IF}, we will treat the interaction term $S_k^{int}[x(t)]$ perturbatively. Taking the logarithm of Eq. (2) and expanding up to second order in S_k^{int}, we get

$$\begin{aligned}S_{IF}[x_+,x_-]\simeq\int dt[G_+(t)x_+(t)+G_-(t)x_-(t)]&+\frac{1}{2}\int dtdt'[G_{++}(t,t')x_+(t)x_+(t')\\&+G_{+-}(t,t')x_+(t)x_-(t')\\&+G_{-+}(t,t')x_-(t)x_+(t')\\&+G_{--}(t,t')x_-(t)x_-(t')],\end{aligned} \tag{3}$$

where

$$G_+(t)=-G_-(t)=\Sigma_k\langle Q_k(q_k(t))\rangle,$$

$$G_{\pm\pm}(t,t')=\Sigma_k i[\langle T^{\pm}Q_k(q_k(t))Q_k(q_k(t'))\rangle-\langle Q_k(q_k(t))\rangle\times\langle Q_k(q_k(t'))\rangle],$$

and

$$G_{+-}(t,t')=G_{-+}(t',t)=\Sigma_k-i[\langle Q_k(q_k(t))Q_k(q_k(t'))\rangle-\langle Q_k(q_k(t))\rangle\langle Q_k(q_k(t'))\rangle].$$

All the expectation values are considered with respect to the asymptotic *in* vacuum $|0\ \ \mathrm{in}\rangle_I$ in the interaction picture. Note that we have integrated out the environment degrees of freedom and S_{IF} depends only on the system variables.

It is important to separate the real and imaginary parts of the influence action because, as is well known [5,6], the imaginary part is related to the noise that the environment induces on the system, whereas the real part gives the averaged dynamics of the system. These are

$$\Re S_{IF}[x_+,x_-]=\sum_k\left[\int dt\langle Q_k(q_k(t))\rangle\Delta(t)+\frac{1}{2}\int dtdt'\Sigma(t)H_k(t,t')\Delta(t')\right], \tag{4}$$

$$\Im S_{IF}[x_+,x_-]=\sum_k\left[\frac{1}{2}\int dtdt'\Delta(t)N_k(t,t')\Delta(t')\right], \tag{5}$$

where we have defined $\Sigma(t)\equiv h(x_+(t))+h(x_-(t))$ and $\Delta(t)\equiv h(x_+(t))-h(x_-(t))$ and we have introduced

$$H_k(t,t')=A_k(t,t')-D_k(t,t')=-2D_k(t,t')\theta(t-t'), \tag{6}$$

which has been expressed in two alternative and equivalent ways for further use. Here the kernels A_k, D_k, and N_k are defined as follows:

$$D_k(t,t')=(-i/2)\langle[Q_k(q_k(t)),Q_k(q_k(t'))]\rangle$$

is the dissipation kernel and

$$N_k(t,t')=\tfrac{1}{2}\langle\{Q_k(q_k(t)),Q_k(q_k(t'))\}\rangle-\langle Q_k(q_k(t))\rangle\times\langle Q_k(q_k(t'))\rangle$$

is the noise kernel. The dissipation and noise kernels, which are related by the fluctuation-dissipation theorem, are antisymmetric and symmetric, respectively, under interchange of t and t'. On the other hand, the kernel $A_k(t,t')=(i/2)\mathrm{sgn}(t-t')\langle[Q_k(q_k(t)),Q_k(q_k(t'))]\rangle$ is symmetric and, as we will see, it is the part that gives rise to divergences.

III. SUM OVER ALL THE MODES AND NEED FOR RENORMALIZATION

For concreteness, let us now consider the case $Q_k(q_k(t))=(g/2)q_k(t)^2$ (g is a perturbative coupling constant) where $\phi(t,\vec{x})$ is a massless real scalar field satisfying the Klein-Gordon equation in Minkowski space-time. We can use the following conventions (note that the label for each mode, *k*, corresponds in fact to a three-dimensional vector):

$$\hat{q}_k(t)=\hat{a}_k f_k(t)+\hat{a}^\dagger_{-k}f^*_{-k}(t),$$

$$G_k^+(t,t')\equiv\langle\hat{q}_k(t)\hat{q}_{-k}(t')\rangle=f_k(t)f^*_{-k}(t'),$$

and

$$G_k^F(t,t')\equiv\langle T\hat{q}_k(t)\hat{q}_{-k}(t')\rangle=\theta(t-t')f_k(t)f^*_{-k}(t')+\theta(t'-t)f_{-k}(t')f^*_k(t),$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the creation and annihilation operators for each of the modes $u_k(\vec{x})$ in which the field $\phi(t,\vec{x})$ has been decomposed. When properly normalized, $f_k(t)=(2\pi)^{-3/2}(2\omega_k)^{-1/2}\exp(-i\omega_k t)$ with $\omega_k=(\vec{k}^2)^{1/2}$. Taking all this into account, we will have

$$D_k(t,t')=-\frac{i}{4}[G_k^+(t,t')^2-G_k^+(t',t)^2], \tag{7}$$

$$A_k(t,t')=\frac{i}{4}\mathrm{sgn}(t-t')[G_k^+(t,t')^2-G_k^+(t',t)^2], \tag{8}$$

$$N_k(t,t')=\frac{1}{4}[G_k^+(t,t')^2+G_k^+(t',t)^2]. \tag{9}$$

To perform the sum over all the modes, we note that we may write $A_k+iN_k=(i/2)G_k^{F2}$ and $D_k+iN_k=(i/2)G_k^{+2}$. Using the integral representations for G_k^F and G_k^+,

$$G_k^F(t,t')=-(2\pi i)^{-1}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}(\omega^2-\vec{k}^2+i\varepsilon)^{-1},$$

$$G_k^+(t,t')=\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}\delta(\omega^2-\vec{k}^2)\theta(\omega),$$

we obtain

$$G_k^F(t,t')^2=-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\times\int_{-\infty}^{\infty}\frac{d\omega}{[\omega^2-\vec{k}^2+i\varepsilon][(\omega-k^0)^2-\vec{k}^2+i\varepsilon]}, \tag{10}$$

$$G_k^+(t,t')^2=\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\int_{-\infty}^{\infty}d\omega\,\delta(\omega^2-\vec{k}^2)\times\theta(\omega)\,\delta((\omega-k^0)^2-\vec{k}^2)\,\theta(k^0-\omega). \tag{11}$$

When carrying out the sum over modes $[\Sigma_k\equiv V/(2\pi)^{n-1}\int d^{n-1}k]$ we note that the G_F^2 term will diverge. Thus, we use dimensional regularization to perform the integrations and then expand in powers of $(n-4)$, where n is the space-time dimension. The usual procedure gives

$$A(t-t')+iN(t-t')=-\frac{V}{32\pi^2}\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\times\left[\frac{1}{n-4}+\frac{1}{2}\ln\left(\frac{(k^0)^2+i\varepsilon}{\mu^2}\right)\right], \tag{12}$$

$$D(t-t')+iN(t-t')=\frac{iV}{32\pi^2}\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\theta(k^0), \tag{13}$$

where $A=\Sigma_kA_k$, $D=\Sigma_kD_k$, and $N=\Sigma_kN_k$. The second integral is finite, and thus we finally have the finite parts: $A_{ren}(t-t')=-(V/32\pi^2)\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\ln(k^0/\mu)^2$, $D(t-t')=i(V/32\pi^2)\int_{-\infty}^{\infty}dk^0e^{-ik^0(t-t')}\mathrm{sgn}(k^0)$, and $N(t-t')=(V/32\pi)\delta(t-t')$. The divergent part $A_{div}(t-t')=-[V/16\pi(n-4)]\delta(t-t')$ has been separated in such a way that the divergences may be absorbed by counterterms in $S[x]$. In other QFT contexts (e.g., two interacting scalar fields) [2,4] the finite contribution from the counterterms can be reabsorbed in the renormalized parameters. However, as we will see, in semiclassical gravity some logarithmic finite terms which cannot be reabsorbed arise in the counterterms.

IV. STOCHASTIC SEMICLASSICAL GRAVITY

As an example we consider the back reaction due to the effect of a small mass or a nonconformal coupling of the scalar field $\phi(x)$ on a flat Robertson-Walker model [6,10,13]. We have to make the following substitutions:

$$x(t)\to a(\eta),$$

$$Q_k(q_k(t))h(x(t))\to\frac{1}{2}\phi_k(\eta)^2\Delta\omega^2(a(\eta))=\frac{1}{2}\phi_k(\eta)^2[m^2+(\xi-\xi_c)\times R(a(\eta))]a(\eta)^2, \tag{14}$$

where η is the conformal time, $a(\eta)$ the scale factor, $R(a(\eta))$ the scalar curvature, m the scalar field mass, and ξ a dimensionless constant. In those previous works where Einstein-Langevin equations were derived using mode decomposition, divergences were not dealt with [6,10].

Let us now see how special care is needed with the sum of modes. Take for instance the second definition for H_k in Eq. (6) and note that, using the real part of Eq. (13), $D(\eta-\eta')=\Sigma_kD_k(\eta-\eta')$ may be written as $(V/16\pi^2)\mathrm{PV}(1/(\eta-\eta'))$. In this case, one would be inclined to write $H=\Sigma_kH_k=D(\eta,\eta')\theta(\eta-\eta')=(V/16\pi^2)\mathrm{PV}(1/(\eta-\eta'))\theta(\eta-\eta')$, but this is an ill-defined product of distributions which may give rise to divergences. A possible way to deal with this is by using, instead, the first definition in Eq. (6) and consider A and D separately:

$$H=\sum_k H_k=\sum_k A_k-\sum_k D_k, \tag{15}$$

where the first term in the last member will be ultraviolet divergent whereas the last term is finite. Now the divergence can be clearly identified and one may use the proper counterterm in dimensional regularization to cancel it:

$$S_g^{div}[a(\eta)]=\frac{(\xi-\xi_c)^2\mu^{n-4}}{32\pi^2(n-4)}\int d^nx\sqrt{-g}R^2=\frac{(\xi-\xi_c)^2}{32\pi^2(n-4)}V\int d^{n-3}x\left\{\frac{36}{n-4}\left(\frac{\ddot{a}}{a}\right)^2+36\left(\frac{\ddot{a}}{a}\right)\times\left[\ln(a\mu)\left(\frac{\ddot{a}}{a}\right)+\frac{2}{3}\left(\frac{\dot{a}}{a}\right)+\left(\frac{\dot{a}}{a}\right)^2\right]\right\}. \tag{16}$$

The second term in this integral, which is finite, will cause the appearance of extra terms when deriving the Einstein-

Langevin equation. Using now the results of the previous section, we get total agreement with those results reached by functional methods which do not use mode separation [13].

A very interesting connection between dissipation and fluctuations in the metric and particle creation has been revealed by Calzetta and Hu [6]. They computed the energy dissipated by the gravitational field per unit volume as $\rho_d = \int d\eta d\eta' [\partial H(\eta,\eta')/\partial\eta] \Delta\omega^2(\eta)\Delta\omega^2(\eta')$ (for simplicity we have considered that the asymptotic values of the scale factor are $a_{in} = a_{out} = 1$), and showed that it was equal to the energy density of the created particles, $\rho^{created}_{particles} = (2\pi)^{-3}\int 4\pi\nu^2 V\nu|\beta_\nu|^2 d\nu$, where β_ν is the Bogoliubov coefficient for the modes with frequency ν. However, formal use of divergent expressions was made in such a derivation. Our treatment shows clearly that the divergent part $A(\eta,\eta')$ of the kernel $H(\eta,\eta')$ decomposed according to Eq. (15) gives no contribution since it is symmetric under interchange of η and η' and hence the derivative will be antisymmetric:

$$\rho_d = \int d\eta d\eta' \frac{\partial H(\eta,\eta')}{\partial\eta} \Delta\omega^2(\eta)\Delta\omega^2(\eta')$$

$$= -\int d\eta d\eta' \frac{\partial D(\eta,\eta')}{\partial\eta} \Delta\omega^2(\eta)\Delta\omega^2(\eta'). \quad (17)$$

This integral is, therefore, manifestly finite and can be computed using the dissipation kernel obtained from Eq. (13), thus leading to the same result of Ref. [6] without the need to deal with divergent expressions.

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[1] R. D. Jordan, Phys. Rev. D **33**, 444 (1986); E. Calzetta and B. L. Hu, *ibid.* **35**, 495 (1987); A. Campos and E. Verdaguer, *ibid.* **49**, 1861 (1994).

[2] B. L. Hu, J. P. Paz, and Y. Zhang, in *Quantum Origin of Noise and Fluctuation in Cosmology*, The Origin of Structure in the Universe Conference at Chateau du Pont d'Oye, Belgium, 1992, edited by E. Gunzig and P. Nardone (NATO Advanced Study Institute Series Vol. 393) (Plenum, New York, 1993), p. 227.

[3] E. Calzetta and B. L. Hu, Phys. Rev. D **52**, 6770 (1995).

[4] E. Calzetta and B. L. Hu, Phys. Rev. D **55**, 3536 (1997).

[5] R. P. Feynman and F. L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963); R. P. Feynman and R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[6] E. Calzetta and B. L. Hu, Phys. Rev. D **49**, 6636 (1994).

[7] A. Campos and E. Verdaguer, Phys. Rev. D **53**, 1927 (1996).

[8] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983); B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992); **47**, 1576 (1993); M. Gell-Mann and J. B. Hartle, *ibid.* **47**, 3345 (1993).

[9] B. L. Hu, Physica A **158**, 399 (1989).

[10] B. L. Hu and A. Matacz, Phys. Rev. D **51**, 1577 (1995).

[11] R. Martín and E. Verdaguer, gr-qc/9811070; Phys. Rev. D **60**, 084008 (1999).

[12] B. L. Hu and A. Matacz, Phys. Rev. D **49**, 6612 (1994).

[13] E. Calzetta, A. Campos, and E. Verdaguer, Phys. Rev. D **56**, 2163 (1997).