Hamiltonian formalism for space-time noncommutative theories

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Space-time non-commutative theories are nonlocal in time. We develop the Hamiltonian formalism for non-local field theories in d space-time dimensions by considering auxiliary $(d+1)$-dimensional field theories which are local with respect to the evolution time. The Hamiltonian path integral quantization is considered and the Feynman rules in the Lagrangian formalism are derived. The case of non-commutative ϕ^3 theory is considered as an example.

I. INTRODUCTION

Space-time non-commutative field theories have peculiar properties due to their acausal behavior [1,2] and lack of unitarity [3]. In Ref. [4] it has been shown that there is a relation between the lack of unitarity and the obstruction to finding a decoupling limit of string theory in an electromagnetic background [5–10]. These theories have an infinite number of temporal and spatial derivatives, and therefore are non-local in time and space [11,12]. The initial value problem of a non-local theory requires one to give a trajectory or a finite piece of it [13]. The Euler-Lagrange (EL) equation is a constraint in the space of trajectories.

The Hamiltonian formalism for non-local theories was presented in [15]. In this paper we improve the formalism by clarifying the relation among the Lagrangian and Hamiltonian structures. We first consider an equivalent theory in a space-time of one dimension higher than that of the original theory. This space has ''two times'' and the dynamics is described in such a way that the evolution is local with respect to one of the times. For this equivalent theory one can construct the Hamiltonian. A characteristic feature of the Hamiltonian formalism for non-local theories is that it contains the EL equations as Hamiltonian constraints.

The Hamiltonian path integral for the $(d+1)$-dimensional field theory is constructed. The Lagrangian path integral formalism for the d dimensional theory is obtained by integrating out the momenta.

We apply the Hamiltonian formalism to time-like and light-like non-commutative theories [4]. As an example we consider the case of non-commutative ϕ^3 theory in d dimensions with space-time non-commutativity. The action contains the free Klein-Gordon Lagrangian and the interaction Lagrangian $L_i = (-g/3!) \int d\vec{x} \, \phi * \phi * \phi$, where $*$ refers to the Moyal product. We construct the Hamiltonian in $d+1$ dimensions. In the path integral quantization we get the Feynman rules that coincide with those used in Refs. [1–3]. The theory is unitary at the classical level (tree level) but is not unitary at one loop [3]. This analysis should shed new light on the structure of these theories. Knowledge of the Hamiltonian of time-like and light-like non-commutative field theories could also be useful to study the energy of their solitons.

II. EULER-LAGRANGE EQUATIONS FOR NON-LOCAL THEORIES

Unlike standard Lagrangians, which depend on the values of a finite number of derivatives at a given time, $q(t)$, $\dot{q}(t), \ldots, q^{(n)}(t)$, a non-local Lagrangian depends on a whole piece of the trajectory $q(t+\lambda)$, for all values of λ, that is, $L^{\text{non}}(t) = L([q(t+\lambda)])$. At best it can be written as a function of all time derivatives $q^{(j)}(t)$, $j = 0,1,2, \ldots$, at the same t. This means that the analogue of the tangent bundle for Lagrangians depending on positions and velocities is infinite dimensional. The action is

$$S[q] = \int dt \, L^{\text{non}}(t). \qquad (1)$$

The EL equation is obtained as the variation of functional (1) and is given by

$$\int dt \, E(t,t';[q]) = 0, \qquad (2)$$

where $E(t,t';[q]) = \delta L^{\text{non}}(t)/\delta q(t')$.

The EL equation must be understood as a functional relation to be satisfied by physical trajectories. It is not a differential system, as one is used to finding for local Lagrangians. In the latter case, the theorems of the existence and uniqueness of solutions enable one to interpret the EL equation as ruling the time evolution of the system, whose state at every instant of time is represented by a point in the space of initial data, e.g., $J_L = \{q, \dot{q}, \ldots, q^{(2n-1)}\}$, for a local Lagrangian of order n.

In the non-local case, if we denote the space of all possible trajectories as $J = \{q(\lambda), \lambda \in R\}$, Eq. (2) is a Lagrangian constraint defining the subspace $J_R \subset J$ of physical trajectories.

III. (1+1)-DIMENSIONAL FIELD THEORY DESCRIPTION OF NON-LOCAL THEORIES

Nevertheless, if we insist in defining a ''time evolution'' T_t for a given initial trajectory $q(\lambda)$, a natural choice is

$$q(\lambda) \xrightarrow{T_t} q(\lambda + t). \qquad (3)$$

We shall hence introduce new dynamical variables $Q(t,\lambda)$ such that

$$Q(t,\lambda) = q(\lambda + t). \qquad (4)$$

Thus, t is the ''evolution'' parameter and λ is a continuous parameter indexing the degrees of freedom. These new variables follow the evolution (4) above, and $Q(0,\lambda)$ can be seen as initial data in the local $(1+1)$-dimensional field theory.

In differential form, condition (4) reads

$$\dot{Q}(t,\lambda) = Q'(t,\lambda), \qquad (5)$$

where the overdot and prime respectively stand for ∂_t and ∂_λ.

We then consider the Hamiltonian system for the $(1+1)$-dimensional field Q with the Hamiltonian

$$H(t,[Q,P]) = \int d\lambda P(t,\lambda) Q'(t,\lambda) - \tilde{L}(t,[Q]), \qquad (6)$$

where P is the canonical momentum of Q. The phase space is thus T^*J with the fundamental Poisson brackets

$$\{Q(t,\lambda), P(t,\lambda')\} = \delta(\lambda - \lambda'). \qquad (7)$$

In the Hamiltonian (6), $\tilde{L}(t,[Q])$ is a functional defined by

$$\tilde{L}(t,[Q]) := \int d\lambda\, \delta(\lambda) \mathcal{L}(t,\lambda). \qquad (8)$$

The ''density'' $\mathcal{L}(t,\lambda)$ is constructed from $L^{non}(t)$ by replacing $q(t)$ by $Q(t,\lambda)$, the t derivatives of $q(t)$ by λ derivatives of $Q(t,\lambda)$ and $q(t+\rho)$ by $Q(t,\lambda+\rho)$. In this construction of the Hamiltonian λ inherits the signature of the original time t and is a time-like coordinate. Furthermore, the symmetry of the original Lagrangian is realized canonically in the enlarged space [14]. Note that $\mathcal{L}(t,\lambda)$ is local in t and is non-local in λ. H depends linearly on $P(t,\lambda)$ but does not depend on $\dot{Q}(t,\lambda)$.

The relation (5) naturally arises as the first Hamilton equation for Eq. (6). However, there is no *a priori* relationship between $P(t,\lambda)$ and $Q(t,\lambda)$, unlike what happens in the local case, and the second Hamilton equation

$$\dot{P}(t,\lambda) = P'(t,\lambda) + \frac{\delta \tilde{L}(t,[Q])}{\delta Q(t,\lambda)} \qquad (9)$$

does not imply any further restriction on $Q(t,\lambda)$. Thus, the Hamiltonian system (6) on T^*J is not so far equivalent to the non-local Lagrangian system of $L^{non}(t)$.

Now, instead of taking the whole phase space T^*J, we shall restrict ourselves to the subspace defined by the 1-parameter set of primary constraints [15]:

$$\varphi(t,\lambda,[Q,P]) \equiv P(t,\lambda) - F(t,\lambda,[Q]) \approx 0 \qquad (10)$$

with

$$F(t,\lambda,[Q]) := \int d\sigma\, \chi(\lambda, -\sigma) \mathcal{E}(t;\sigma,\lambda), \qquad (11)$$

where $\mathcal{E}(t;\sigma,\lambda)$ and $\chi(\lambda, -\sigma)$ are defined by

$$\mathcal{E}(t;\sigma,\lambda) = \frac{\delta \mathcal{L}(t,\sigma)}{\delta Q(t,\lambda)}, \quad \chi(\lambda, -\sigma) = \frac{\epsilon(\lambda) - \epsilon(\sigma)}{2}. \qquad (12)$$

Here $\epsilon(\lambda)$ is the sign distribution. The symbols ''$\equiv$'' and ''$\approx$'' respectively stand for ''strong'' and ''weak'' equality.

Further constraints are generated by requiring the stability of the primary ones. In the first step, we obtain

$$\dot{\varphi}(t,\lambda,[Q,P]) \equiv \varphi'(t,\lambda,[Q,P]) + \delta(\lambda) \psi_0(t,[Q]) \approx 0$$

where

$$\psi_0(t,[Q]) := \int d\sigma\, \mathcal{E}(t;\sigma,0) \approx 0 \qquad (13)$$

is the secondary constraint. Further constraints then follow by successive time differentiations of ψ_0. They can be written all together in a condensed form as

$$\psi(t,\lambda,[Q]) \equiv \int d\sigma\, \mathcal{E}(t;\sigma,\lambda) \approx 0. \qquad (14)$$

Therefore, the constrained Hamiltonian system defined by the Hamiltonian (6) and the primary constraints (10) resides in a reduced phase space $\Gamma \subset T^*J$ defined by Eqs. (10) and (14). Taking into account Eq. (4), the constraint (14) reduces to the EL equation (2) obtained from $L^{non}(t)$.

The constraints (10) and (14) belong to the second class in non-singular systems. In the next section we will show explicitly, for (non-singular) higher derivative Lagrangian system of order n, that they are used to reduce the phase space to $2n$ dimensions, reproducing the canonical Ostrogradski formalism [16]. Our formalism developed here turns out to be a generalization of the Ostrogradski formalism to the case of infinite order derivative theories. The infinite chain of second class constraints has also appeared in the description of boundary conditions as constraints [17]. Summarizing, the equivalence has been built in the $(1+1)$-dimensional Hamiltonian formalism of *local* field theories through the constraints (10) and (14). This type of equivalence between the Hamiltonian and Lagrangian formalism is different from the one in local theories [18].

IV. NON-SINGULAR HIGHER ORDER DERIVATIVE THEORIES

Here we would like to derive both the Lagrangian and Hamiltonian formalisms for non-singular higher order derivative theories from the Hamiltonian formalism of non-local theories developed in the last section [19].

Let us consider a regular higher derivative theory described by the Lagrangian $L(q, \dot{q}, \ddot{q}, \ldots, q^{(n)})$ and write the expressions obtained in the previous section for the non-local Lagrangian. As we embed the higher order theory in the non-local setting we start with the infinite dimensional phase space $T^*J(t) = \{Q(t, \lambda), P(t, \lambda)\}$. They are assumed to be expanded in the Taylor basis [20] as

$$Q(t, \lambda) \equiv \sum_{m=0}^{\infty} e_m(\lambda) q^m(t),$$

$$P(t, \lambda) \equiv \sum_{m=0}^{\infty} e^m(\lambda) p_m(t), \quad (15)$$

where $e^l(\lambda)$ and $e_l(\lambda)$ are orthonormal bases:

$$e^l(\lambda) = (-\partial_\lambda)^l \delta(\lambda), \quad e_l(\lambda) = \frac{\lambda^l}{l!}. \quad (16)$$

The coefficients in Eqs. (15) are new canonical variables,

$$\{q^m(t), p_n(t)\} = \delta^m{}_n, \quad (17)$$

and the Hamiltonian (6) is

$$H(t) = \sum_{m=0}^{\infty} p_m(t) q^{m+1}(t) - L(q^0, q^1, \ldots, q^n). \quad (18)$$

The momentum constraint (10) becomes an infinite set of constraints:

$$\varphi_m(t) = p_m(t) - \sum_{l=0}^{n-m-1} (-D_t)^l \frac{\partial L(t)}{\partial q^{l+m+1}(t)} \approx 0, \quad (19)$$

where

$$D_t = \sum_{r=0} q^{r+1} \frac{\partial}{\partial q^r}. \quad (20)$$

On the other hand, the constraint ψ in Eq. (14) in terms of the Taylor basis becomes

$$\psi^m(t) \equiv (D_t)^m \left[\sum_{l=0}^{n} (-D_t)^l \frac{\partial L(t)}{\partial q^l(t)} \right] \approx 0. \quad (21)$$

These constraints (19) and (21) are second class and are used to reduce the infinite dimensional phase space to a finite one, leading to the ordinary Ostrogradski Hamiltonian formalism. The operator D_t defined in Eq. (20) becomes a time evolution operator for q's using the first set of the Hamilton equation

$$\dot{q}^r = q^{r+1}. \quad (22)$$

Using this in Eq. (19) the constraints φ_m $(0 \leq m \leq n-1)$ coincide with the definition of the Ostragradsky momenta:

$$p_m \sim \sum_{l=0}^{n-m-1} (-\partial_t)^l \frac{\partial L(t)}{\partial(\partial_t^{l+m+1} q(t))} \quad (0 \leq m \leq n-1). \quad (23)$$

Now they can be solved for q^l $(n \leq l \leq 2n-1)$ as functions of the canonical pairs $\{q^j, p_j\}$ $(0 \leq j \leq n-1)$

$$q^l \approx q^l(q^0, q^1, \ldots, q^{n-1}, p_0, p_1, \ldots, p_{n-1})$$

$$(n \leq l \leq 2n-1). \quad (24)$$

They are combined with the constraints φ_l $(n \leq l \leq 2n-1)$,

$$\varphi_l = p_l \approx 0 \quad (n \leq l \leq 2n-1), \quad (25)$$

to form a second class set and can be used to eliminate the canonical pairs $\{q^l, p_l\}(n \leq l \leq 2n-1)$.

If we take into account Eq. (22) the constraint (21) for $m=0$ is the Euler-Lagrange equation for the original higher derivative Lagrangian:

$$\psi^0 \sim \sum_{l=0}^{n} (-\partial_t)^l \frac{\partial L(t)}{\partial(\partial_t^l q(t))} = 0. \quad (26)$$

The constraints (21) for $m>0$ are the time derivatives of the Euler-Lagrange equation (26) expressed in terms of q's. For a non-singular theory, all the constraints (21) can be rewritten as

$$q^l - q^l(q^0, q^1, \ldots, q^{n-1}, p_0, p_1, \ldots, p_{n-1}) \approx 0 \quad (l \geq 2n) \quad (27)$$

and can be paired with the constraints φ_l $(l \geq 2n)$,

$$\varphi_l = p_l \approx 0 \quad (l \geq 2n), \quad (28)$$

forming second class constraints. They are used to eliminate the canonical pairs $\{q^l, p_l\}$ $(l \geq 2n)$.

In this way the infinite dimensional phase space is reduced to a finite dimensional one. The reduced phase space is coordinated by $T^*J^n = \{q^l, p_l\}$ with $l = 0, 1, \ldots, n-1$. All the constraints are second class and we use the iterative property of Dirac brackets. The Dirac brackets for these variables have the standard form

$$\{q^m, p_n\}^* = \delta^m{}_n, \quad \{q^m, q^n\}^* = \{p_m, p_n\}^* = 0. \quad (29)$$

The Hamiltonian (6) in the reduced space is given by

$$H(t) = \sum_{m=0}^{n-1} p_m(t) q^{m+1}(t) - L(q^0, q^1, \ldots, q^n) \quad (30)$$

where q^n is expressed using Eq. (24) as a function of the reduced variables in T^*J^n. Note that if we consider the limit n going to infinity, the constraints (19) and (21) do not allow

one, in general, to reduce the dimensionality of the infinite dimensional phase space of the non-local system via Dirac brackets.

V. SYMPLECTIC FORMULATION OF THE EULER-LAGRANGE EQUATION

The Hamiltonian formalism presented in the last sections can be cast into a symplectic form as follows. The Poisson brackets (7) correspond to the symplectic two-form $\Omega \in \Lambda^2(T^*J)$:

$$\Omega = \int d\lambda \, \delta P(t,\lambda) \wedge \delta Q(t,\lambda), \qquad (31)$$

where δ stands for the functional exterior derivative.

In the constrained phase space $\Gamma_1 \subset T^*J$ defined by Eq. (10) only, the induced (pre)symplectic form is

$$\Omega_1 = \frac{1}{2} \int d\lambda \, d\lambda' \, \omega(t;\lambda,\lambda',[Q]) \delta Q(t,\lambda) \wedge \delta Q(t,\lambda') \qquad (32)$$

where

$$\omega(t;\lambda,\lambda',[Q]) = \chi(\lambda',-\lambda) \int d\sigma \frac{\delta \mathcal{E}(t;\sigma,\lambda)}{\delta Q(t,\lambda')}. \qquad (33)$$

The induced Hamiltonian is

$$H_1(t,[Q]) = \int d\lambda \, F(t,\lambda,[Q]) Q'(t,\lambda) - \tilde{L}(t,[Q]). \qquad (34)$$

The generator of the time evolution (4) is the vector field

$$X([Q]) = \int d\lambda \, \dot{Q}(t,\lambda) \frac{\delta}{\delta Q(t,\lambda)}. \qquad (35)$$

Now, $i(X)\Omega_1 + \delta H_1 = 0$ gives a first order formulation of the EL equation. Indeed a short calculation yields

$$i(X)\Omega_1 + \delta H_1 = -\int d\sigma \mathcal{E}(t;\sigma,0) \delta Q(t,0)$$

$$+ \int d\lambda d\lambda' \, \delta Q(t,\lambda')$$

$$\times [\dot{Q}(t,\lambda) - Q'(t,\lambda)] \omega(t;\lambda,\lambda'), \qquad (36)$$

whence the evolution (5) and the EL equation (2) follow from it.

VI. PATH INTEGRAL QUANTIZATION

Let us consider the Hamiltonian path integral quantization of the $(1+1)$-dimensional field theory associated with the Hamiltonian (6) for $L^{\mathrm{non}}(t)$. The path integral is given by

$$\int [dP(t,\lambda)][dQ(t,\lambda)]\mu$$

$$\times \exp\left(i \int dt d\lambda \{P(t,\lambda)[\dot{Q}(t,\lambda) - Q'(t,\lambda)] \right.$$

$$\left. + \tilde{L}(t)\delta(\lambda)\} \right). \qquad (37)$$

The integration is performed over the reduced phase space Γ and the measure [21,22] μ is

$$\mu = \det\begin{pmatrix} \{\varphi,\varphi\} & \{\varphi,\psi\} \\ \{\psi,\varphi\} & \{\psi,\psi\} \end{pmatrix} \delta(\varphi)\delta(\psi). \qquad (38)$$

First we consider the non-singular higher derivative Lagrangian system of order n. From the discussions of Sec. IV the constraints are arranged in a set in which the canonical variables (q^j,p_j) for $j \geq n$ are expressed in terms of the ones for $0 \leq j \leq n-1$. The measure becomes

$$\mu = \prod_{j=n}^{2n-1} \{\delta(p_j)\delta(q^j - \cdots)\} \prod_{k=2n}^{\infty} \{\delta(p_k)\delta(q^k - \cdots)\} \qquad (39)$$

where $\cdots$ terms are given as functions of $(q^i,p_i)(i=0,\ldots,n-1)$. Integrating over $(q^i,p_i)(i \geq n)$ Eq. (37) becomes

$$\int \prod_{i=0}^{n-1} dq^i dp_i \exp\left(i \int dt \sum_{i=0}^{n-1} p_i(\dot{q}^i - q^{i+1}) \right.$$

$$\left. + L(q^0,\ldots,q^n) \right) \qquad (40)$$

where q^n is given as a function of $(q^i,p_i)(i=0,\ldots,n-1)$. This is the Hamiltonian path integral of the Ostrogradski formalism. If we assume that non-local systems can be regarded as the infinite limit of n, the higher derivative system (40) becomes, by taking $n \to \infty$,

$$\int [dP(t,\lambda)][dQ(t,\lambda)]$$

$$\times \exp\left(i \int dt d\lambda \{P(t,\lambda)[\dot{Q}(t,\lambda) - Q'(t,\lambda)] \right.$$

$$\left. + \tilde{L}(t)\delta(\lambda)\} \right), \qquad (41)$$

where Q and P, which are $n \to \infty$ of (q^i,p_i) $(i=0,\ldots,n-1)$, are not restricted by the constraints in contrast to Eq. (37).

If we integrate out the momenta and use $\delta(\dot{Q}(t,\lambda) - Q'(t,\lambda))$, we get

$$\int [dq(t)]\exp\left(i \int dt \, L^{\mathrm{non}}(t) \right), \qquad (42)$$

which is the Lagrangian path integral formulation for the non-local theory.

VII. APPLICATION TO SPACE-TIME NON-COMMUTATIVE ϕ^3 THEORY

Space-time non-commutative theories have peculiar properties due to their acausal behavior and lack of unitarity. Here we would like to use the previous formalism to study the question of unitarity in these theories.

To fix the ideas we consider a non-commutative ϕ^3 theory with arbitrary non-commutativity in d dimensions. The Lagrangian density is given by

$$\mathcal{L}^{non}(x^\mu) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2$$

$$- \frac{g}{3!}\phi(x)*\phi(x)*\phi(x) \qquad (43)$$

where * is the star product defined by using a general antisymmetric background $\theta^{\mu\nu}$:

$$f(x)*g(x) = [e^{i(\theta^{\mu\nu}/2)\partial_\mu^\alpha\partial_\nu^\beta}f(x+\alpha)g(x+\beta)]_{\alpha=\beta=0}. \qquad (44)$$

The EL equation is

$$(\Box - m^2)\phi(x) - \frac{g}{2!}\phi(x)*\phi(x) = 0. \qquad (45)$$

x^0 in Eqs. (43)–(45) will be denoted as t hereafter. We introduce a ''new coordinate x^0'' which plays the role of λ in the previous discussion and introduce the field $Q(t,x^\mu)$ in $d+1$ dimensions. Now t is regard as ''evolution time'' and $x^\mu := (x^0,\vec{x})$ is a continuous Lorentzian index. Our metric conventions are $\eta_{tt} = \eta_{00} = -1, \eta_{ii} = +1$. The relation (4) in this case is

$$Q(t,x^0,\vec{x}) = \phi(t+x^0,\vec{x}). \qquad (46)$$

The Lagrangian density in $d+1$ dimensions for $Q(t,x^\mu)$ [see Eq. (8)] is

$$\mathcal{L}(t,x^\mu) = -\frac{1}{2}\partial_\mu Q(t,x)\partial^\mu Q(t,x) - \frac{m^2}{2}Q(t,x)^2$$

$$- \frac{g}{3!}Q(t,x)*Q(t,x)*Q(t,x), \qquad (47)$$

where now the derivatives in * are with respect to $x^\mu = (x^0,\vec{x})$. Note that this Lagrangian density is local in the evolution time t.

The momentum constraint (10) is given by

$$\varphi(t,x^\mu) = P(t,x^\mu) - \delta(x^0)Q'(t,x)$$

$$+ \frac{g}{2!}\int dx'\chi(x^0,-x'^0)\int dy_1 dy_2$$

$$\times K(y_1-x',y_2-x',x-x')Q(t,y_1)Q(t,y_2), \qquad (48)$$

where $Q'(t,x)$ denotes $\partial_{x^0}Q(t,x^\mu)$. Here K is the symmetric kernel of three star products:

$$f(x)*g(x)*h(x) = \int dy_1 dy_2 dy_3$$

$$\times K(y_1-x,y_2-x,y_3-x)$$

$$\times f(y_1)g(y_2)h(y_3). \qquad (49)$$

The Hamiltonian (6) is

$$H(t) = \int dx[P(t,x)Q'(t,x) - \mathcal{L}(t,x)\delta(x^0)]$$

$$= \int dx\left[P(t,x)Q'(t,x) + \delta(x^0)\left\{ -\frac{1}{2}Q'(t,x)^2 \right.\right.$$

$$+ \frac{1}{2}[\nabla Q(t,x)]^2 + \frac{m^2}{2}Q(t,x)^2$$

$$\left.\left. + \frac{g}{3!}Q(t,x)*Q(t,x)*Q(t,x)\right\} \right]. \qquad (50)$$

The Hamilton equations are

$$\dot{Q}(t,x) = Q'(t,x), \qquad (51)$$

$$\dot{P}(t,x) = P'(t,x) - \delta'(x^0)[Q'(t,x)]_{x^0=0} + \delta(x^0)$$

$$\times \{\nabla^2 Q(t,x) - m^2 Q(t,x)\}$$

$$- \frac{g}{2!}\int dx' dy_1 dy_2 \delta(x'^0)$$

$$\times K(y_1-x',y_2-x',x-x')$$

$$\times Q(t,y_1)Q(t,y_2). \qquad (52)$$

The stability of the constraint implies the new constraints (13):

$$\psi(t,x) \equiv (\nabla^2 - \partial_{x^0}^2 - m^2)Q(t,x)$$

$$- \frac{g}{2!}Q(t,x)*Q(t,x) = 0, \quad \text{at } x^0 = 0. \qquad (53)$$

By requiring further consistency we have an infinite number of constraints which can be written as, Eq. (14),

$$\psi(t,x) = 0 \quad \text{for } -\infty < x^0 < \infty. \qquad (54)$$

Using the Hamilton equation (51) for Q, Eq. (54) becomes the EL equation

$$(\nabla^2 - \partial_t^2 - m^2)Q(t,x) - \frac{g}{2!}Q(t,x)*Q(t,x) = 0, \quad (55)$$

where ∂_{x^0} on Q is replaced by ∂_t both in the first term and in the * product. It is the original non-local EL equation (45).

If we write the symplectic form and the Hamiltonian in terms of $Q(t,x)$, Eqs. (34) and (32), we have

$$\Omega = \int dx\, \delta(x^0)\, \delta Q'(t,x) \wedge \delta Q(t,x)$$

$$- \frac{g}{4}\int dx\, \delta(Q(t,x)*Q(t,x))\epsilon(x^0) \wedge \delta Q(t,x) \quad (56)$$

and

$$H = \int dx\, \frac{\delta(x^0)}{2}\{Q'(t,x)^2 + [\nabla Q(t,x)]^2 + m^2 Q(t,x)^2\}$$

$$- \frac{g}{4}\int dx[Q(t,x)*Q(t,x)]\epsilon(x^0)Q'(t,x). \quad (57)$$

These expressions can be rewritten in terms of $\phi(x)$ using Eq. (51), i.e., Eq. (46). In particular the interaction Hamiltonian becomes

$$H_i = -\frac{g}{4}\int dx[\phi(x)*\phi(x)]\epsilon(x^0)\dot{\phi}(x). \quad (58)$$

Note that the occurrence of time derivatives of any order in the interaction Hamiltonian is not forbidden in non-local theories. This property is clearly not satisfied by local theories.

Now we can perform the path integral quantization using Eq. (42) to obtain

$$\int [d\phi(x)]\exp\left[\int dx\left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2\right.\right.$$

$$\left.\left. - \frac{g}{3!}\phi(x)*\phi(x)*\phi(x)\right)\right], \quad (59)$$

from which we read the Lagrangian Feynman rules [23]. They coincide with the ones used in [3]. Therefore, it follows from [3] and [4] that non-commutative ϕ^3 theory with time-like non-commutativity is not unitary while non-commutative ϕ^3 theory with light-like non-commutativity is unitary.

Note added. Recently, Ref. [24] has also considered the Hamiltonian formalism for non-local theories.

ACKNOWLEDGMENTS

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