Perturbations on domain walls and strings: A covariant theory

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A covariant formalism is developed for describing perturbations on vacuum domain walls and strings. The treatment applies to arbitrary domain walls in $(N+1)$-dimensional flat spacetime, including the case of bubbles of a true vacuum nucleating in a false vacuum. Straight strings and planar walls in de Sitter space, as well as closed strings and walls nucleating during inflation, are also considered. Perturbations are represented by a scalar field defined on the unperturbed wall or string world sheet. In a number of interesting cases, this field has a tachyonic mass and a nonminimal coupling to the world-sheet curvature.

I. INTRODUCTION

Vacuum domain walls and strings could be formed as topological defects at a phase transition in the early Universe [1]. Walls could also appear at the boundaries of true-vacuum bubbles nucleating in a false vacuum at first-order phase transitions [2], and strings could appear at the boundaries of circular holes nucleating in metastable domain walls. Moreover, it has recently been shown [3] that spherical domain walls and circular loops of string can spontaneously nucleate during the inflationary epoch in the early Universe. The physical properties of strings and walls have been extensively studied in recent years and still remain a fascinating topic of research.

The purpose of this paper is to study the dynamics of small perturbations on strings and walls. It will be assumed that the thickness of the defects is small compared to all the other relevant dimensions, so that they can be treated as infinitely thin lines and sheets. We shall see that in this case the perturbations can be described by a scalar field "living" in the $(1+1)$- or $(2+1)$-dimensional world sheet of the defect. Somewhat surprisingly, this field generally has a tachyonic mass (in some special cases the mass is equal to zero) and a nonminimal coupling to the world-sheet curvature. For vacuum bubbles and for strings and walls nucleating during inflation, the unperturbed world-sheet geometry is that of de Sitter space, and the field equation for perturbations can be solved analytically.

The evolution of perturbations on walls and strings can be important for cosmological applications. For instance, closed loops of string formed during inflation would all eventually collapse to black holes if they remained exactly circular. However, quantum fluctuations cause some deviations from circular shape, and by studying the dynamics of these fluctuations one should be able to determine the probability of black-hole formation.

Turok [4] has argued that quantum fluctuations on long strings during inflation can lead to a rapid growth of string energy and suggested the possibility of a self-consistent solution in which inflation is driven by fluctuating strings. To make his approach more rigorous one has to develop a covariant theory of perturbations on strings and then to devise a covariant regularization scheme for the string energy-momentum tensor. Here, we shall implement the first part of this program.

Another interesting problem is the evolution of perturbations on expanding vacuum bubbles. The question is whether or not perturbations on the bubble wall grow, so that the wall becomes more and more nonspherical as it expands. We shall see that the answer depends on whom you ask: different answers will be given by an external observer and by an observer "living" inside the wall. Evolution of perturbations on vacuum bubbles has been previously studied by Adams, Freese, and Widrow [5]. We agree with their conclusions and will comment on the relation of our work to theirs in Sec. V.

Our emphasis in this paper will be on developing a covariant classical theory of perturbations on strings and walls. Quantization and possible cosmological applications will be left as topics for future research.

The rest of this paper is organized as follows. In Sec. II we derive the action and the equation of motion for a domain wall separating regions of space with different vacuum energy densities. We do this for a wall moving in $(N+1)$-dimensional Minkowski space, so that for $N=2$ our results are applicable to a string bounding a hole in a planar domain wall. The equation for linearized perturbations on an arbitrary domain wall is derived in Sec. III. Solutions of this equation are obtained and discussed in Sec. IV for a planar domain wall and in Sec. V for an expanding vacuum bubble. Effects of gravity are ignored in Secs. II–V. In Sec. VI we study perturbations on walls and strings in an inflationary universe. This treatment applies to walls and strings nucleating during inflation, as well as to perturbations on straight strings and planar walls.

Our conclusions are summarized in Sec. VII. The validity of the linearized theory of perturbations is discussed in the Appendix.

II. THE ACTION AND THE EQUATIONS OF MOTION

In this section we shall derive the equations of motion for a domain wall of arbitrary shape separating two regions of space which may have different values of the vacuum energy density. As was mentioned in the Introduction, we shall work in the approximation in which the domain wall is infinitely thin, carrying an energy density per unit surface equal to its surface tension. We shall also ignore the effects of gravity.

For the sake of generality and also for its possible relevance in lower-dimensional problems we shall consider domain walls evolving in $(N+1)$-dimensional Minkowski space. As the wall evolves in time it will span an N-dimensional timelike hypersurface (world sheet) Σ which can be parametrized by coordinates $\xi^a, (a=0,1,\ldots,N-1)$. Denoting by $g_{ab}(a,b=0,1,\ldots,N-1)$ the induced metric on Σ, one can think of the history of the domain wall as a Riemannian manifold (Σ, g_{ab}). It is this manifold structure that we wish to emphasize throughout the present paper.

The action for a thin domain wall is [1]

$$S_w = -\sigma \int_\Sigma \sqrt{-g}\, d^N\xi \;, \tag{1}$$

where $g=\det g_{ab}$ and σ is the surface tension. To this we should add the vacuum action. Since the vacuum Lagrangian is just a constant potential energy density ρ_v, its contribution to the action is given by

$$S_v = -\epsilon \int dt \int_{\text{vol}} d^N x \;. \tag{2}$$

Here, $\epsilon=\rho_v^{(2)}-\rho_v^{(1)}$ is the difference of vacuum energy densities between the two sides of the wall, and the N-dimensional integral extends over the region occupied by the vacuum with $\rho_v=\rho_v^{(2)}$. Clearly, the boundary of that region is the domain wall itself. We can use this fact to write (2) as an integral over the boundary Σ, by introducing in (2) a factor of $1=\partial_\mu x^\mu/(N+1)$ and then applying Gauss's theorem. Upon so doing we find

$$S=S_w+S_v=-\int_\Sigma \sqrt{-g}\left[\sigma-\frac{\epsilon}{N+1}x^\mu n_\mu\right] d^N\xi \;. \tag{3}$$

Here, n^μ is the (spacelike) unit vector normal to Σ (our sign conventions are such that n^μ points in the direction of the region with $\rho_v=\rho_v^{(2)}$).

The quantities in (3) should be expressed as functions of ξ^a. There, x^μ is the position vector on the surface of the wall

$$x^\mu = x^\mu(\xi^a) \;. \tag{4}$$

The normal vector is characterized by the equations

$$n_\mu n^\mu = 1 \;, \quad n_\mu \partial_a x^\mu = 0 \;, \tag{5}$$

and the induced metric is given in terms of $x^\mu(\xi^a)$ as

$$g_{ab} = \partial_a x^\mu \partial_b x_\mu \;. \tag{6}$$

The action (3) is invariant under reparametrizations of Σ (x^μ and n_μ are scalars under reparametrizations).

Let us now consider the variation of the action with respect to small changes in the world sheet, $x^\mu \to x^\mu + \delta x^\mu$. Since only motion transverse to the wall is physically observable, we can write

$$\delta x^\mu = \phi n^\mu \;. \tag{7}$$

Here ϕ is assumed to be small, but is otherwise an arbitrary function of the coordinates ξ^a.

With the help of Eqs. (5) and (6) one can find the corresponding variations in the quantities appearing in the action. For instance, the new metric will be $g_{ab}=\partial_a(x^\mu+\phi n^\mu)\partial_b(x_\mu+\phi n_\mu)$ and to linear order in ϕ, we have

$$\delta g_{ab} = -2\phi K_{ab} \;, \tag{8}$$

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\, g^{ab}\delta g_{ab} = -\phi\sqrt{-g}\, g^{ab}K_{ab} \;,$$

where $K_{ab}=n_\mu\partial_a\partial_b x^\mu = -\partial_a n_\mu \partial_b x^\mu$ is the extrinsic curvature of Σ and g^{ab} is the inverse of g_{ab}. Similarly one finds

$$\delta n^\mu = -g^{ab}\phi_{,b}\partial_a x^\mu \;. \tag{9}$$

Using Eqs. (7)–(9) the variation of the action is

$$\delta S = \int_\Sigma \sqrt{-g}\left[\frac{\epsilon}{N+1}(\phi - x_\mu g^{ab}\phi_{,b}\partial_a x^\mu) + \left[\sigma - \frac{\epsilon}{N+1}x^\mu n_\mu\right]\phi g^{ab}K_{ab}\right] d^N\xi \;. \tag{10}$$

After an integration by parts we obtain the apparently complicated expression

$$\delta S = \int_\Sigma \sqrt{-g}\,(\sigma g^{ab}K_{ab}+\epsilon)\phi\, d^N\xi + \int_\Sigma \sqrt{-g}\,\frac{\epsilon}{N+1}x^\mu g^{ab}(\nabla_b\partial_a x_\mu - K_{ab}n_\mu)\phi\, d^N\xi \;, \tag{11}$$

where ∇_b stands for the covariant differentiation operator corresponding to the metric g_{ab}. However, the second integral in (11) vanishes identically because

$$\nabla_b \partial_a x^\mu \equiv K_{ab} n^\mu \;, \tag{12}$$

as can be shown by explicit evaluation of the left-hand side (LHS) and using the relation $\partial_c x_\mu \partial^c x_\nu = \eta_{\mu\nu} - n_\mu n_\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric.

So we are left with just one term in (11) and, from the variational principle $\delta S=0$, the equation of motion for the wall is

$$g^{ab}K_{ab} = -\frac{\epsilon}{\sigma} \;. \tag{13}$$

For the particular case that $\epsilon=0$, Eq. (13) reduces to the well-known result that minimal surfaces have vanishing mean curvature [6].

III. LINEARIZED PERTURBATIONS

In this section we shall prove the following result. Let x^μ be a known solution of the equations of motion (13). Then, the theory of linearized perturbations to x^μ is equivalent to the theory of a scalar field in the curved spacetime (Σ, g_{ab}) corresponding to the unperturbed world sheet. We shall determine the mass of this field and its coupling to the world-sheet curvature.

Let us denote by $\tilde{x}^{\mu}$ the perturbed solution. Again, only perturbations perpendicular to Σ need be considered:

$$\tilde{x}^{\mu}=x^{\mu}+\phi n^{\mu} . \tag{14}$$

Equation (14) is formally the same as Eq. (7), but now we are dealing with physical perturbations rather than mathematical variations. Here ϕ represents the proper magnitude of the perturbation, i.e., the deviation from the unperturbed solution as measured by an observer moving with the wall. (This can be easily seen by using a local inertial reference frame in which the unperturbed wall is at rest. In this frame the temporal component of n^{μ} vanishes.)

The equation for the linear perturbations can be obtained by applying Eq. (13) to the perturbed solution. Writing $\tilde{g}_{ab}=g_{ab}+\delta g_{ab}$ and $\tilde{K}_{ab}=K_{ab}+\delta K_{ab}$ for the perturbed quantities we have

$$g^{ab}\delta K_{ab}+K_{ab}\delta g^{ab}=0 . \tag{15}$$

From (8) one has

$$\delta g^{ab}=-g^{ac}g^{bd}\delta g_{cd}=2\phi K^{ab} . \tag{16}$$

Similarly, using (9),

$$\delta K_{ab}=-\partial_a(g^{cd}\phi_{,c}\partial_d x^{\mu})\partial_b x^{\mu}-\phi\partial_a n^{\mu}\partial_b n_{\mu} . \tag{17}$$

With the help of (12) we can write

$$\partial_a(g^{cd}\phi_{,c}\partial_d x^{\mu})\partial_b x_{\mu}=\nabla_a\partial_b\phi , \tag{18}$$

and it is also easy to check that

$$\partial_a n^{\mu}\partial_b n_{\mu}=K_a^c K_{cb} . \tag{19}$$

Using Eqs. (16)–(19) in (15), the equation for the perturbations reduces to

$$\Box\phi+K^{ab}K_{ab}\phi=0 , \tag{20}$$

where $\Box$ stands for the covariant d'Alembertian in the N-dimensional curved spacetime $\Box=g^{ab}\nabla_a\nabla_b$. Equation (20) can be put in terms of intrinsic quantities on the world sheet by using the contracted Gauss-Codazzi relation (see Ref. [7] for instance)

$$\mathcal{R}=(g^{ab}K_{ab})^2-K^{ab}K_{ab} , \tag{21}$$

where $\mathcal{R}$ is the world-sheet Ricci scalar. Using (21) and (13) we can finally write

$$-\Box\phi+\left[\mathcal{R}-\frac{\epsilon^2}{\sigma^2}\right]\phi=0 . \tag{22}$$

Equation (22) is the equation for a scalar field in a curved N-dimensional spacetime. The direct (nonminimal) coupling to the scalar curvature $\mathcal{R}$ is of the standard form $\xi\mathcal{R}\phi$ with $\xi=1$ [conformal coupling corresponds to [8] $\xi=(N-2)/4(N-1)$]. If $\epsilon\neq 0$ then the field has a negative mass squared which, in principle, may lead to instabilities. As we shall see in the following sections, this may sometimes be a matter of interpretation.

IV. LINEAR PERTURBATIONS ON PLANAR DOMAIN WALLS

The simplest case where one can apply the above formalism is the case where we have a planar wall perpendicular to, say, the z axis (we use Cartesian coordinates (t,z,x^i) in Minkowski space, $i=1,\ldots,N-1$). Its trajectory will be given by some function

$$z=z(t) , \tag{23}$$

which can be determined from Eq. (13). For this we need the induced metric

$$ds_{\Sigma}^2=-(1-\dot{z}^2)dt+\sum(dx^i)^2 , \tag{24}$$

which is obtained by substituting (23) into the usual $(N+1)$-dimensional Minkowski line element. We are using (t,x^i) as our coordinates on Σ and overdots stand for derivatives with respect to t. Now Eq. (5) can be used to find the normal vector

$$n_0=\frac{-\dot{z}}{\sqrt{1-\dot{z}^2}} , \quad n_z=\frac{1}{\sqrt{1-\dot{z}^2}} , \quad n_i=0 . \tag{25}$$

The only nonvanishing component of K_{ab} is [see after Eq. (8)]

$$K_{00}=n_{\mu}\partial_0^2 x^{\mu}=\frac{\ddot{z}}{\sqrt{1-\dot{z}^2}} . \tag{26}$$

Therefore, from (13), one finds the equation of motion

$$\frac{\ddot{z}}{(1-\dot{z}^2)^{3/2}}=\frac{\epsilon}{\sigma} , \tag{27}$$

which has the first integral

$$\frac{\sigma}{\sqrt{1-\dot{z}^2}}-\epsilon z=\text{const} . \tag{28}$$

Note that Eq. (28) is just the energy-conservation equation, which could as well have been used as the starting point: the vacuum energy swept out by the domain wall (which is ϵz per unit area) is entirely converted into kinetic energy of the domain wall [$\sigma(1-\dot{z}^2)^{-1/2}$ per unit area]. The solution to Eq. (28) has different qualitative behavior depending on whether or not ϵ is vanishing.

For $\epsilon=0$ the solution is $\dot{z}=$const and we can simply write $z=0$ by going to a suitable Lorentz frame. This is just the planar static domain wall. It is easy to study linearized perturbations to this solution. For this we note that (24) is flat space, so $\mathcal{R}=0$ and Eq. (22) reads

$$\Box\phi=0 , \tag{29}$$

where the box stands for the ordinary d'Alembertian in N-dimensional flat space. The mode solutions to (29) are plane waves of small amplitude propagating on the wall at the speed of light.

As is well known [1], *nonlinear* plane waves of arbitrary profile propagating on the wall at the speed of light, such as

$$z=f(t-x^1) , \tag{30}$$

are also solutions to the equations of motion (13) for

$\epsilon=0$. However, a superposition principle for such nonlinear waves does not apply. If the wave (30) is made to collide with some other nonlinear wave described by $z=g(t+x^1)$ they do not simply pass through each other but some complicated interaction between them takes place. In other words, $z=f(t-x^1)+g(t+x^1)$ is *not* a solution of (13). This raises the question of the stability of the nonlinear wave (30), that is, whether or not it will be seriously affected when it collides with a small disturbance traveling, say, in the opposite direction. This question can be answered by using our Eq. (22) in the spacetime corresponding to the wave solution (30). It is easy to check that this spacetime is, again, flat spacetime. So linearized perturbations to (30) will obey the wave equation (29), for which there are no growing mode solutions. Therefore the nonlinear traveling wave is stable against small perturbations.

Let us now examine the case $\epsilon\neq 0$. In this case the solution to Eq. (28) is the hyperbola

$$z^2-t^2=\frac{\sigma^2}{\epsilon^2}\ , \qquad (31)$$

where we have eliminated the integration constants by shifting the coordinates z and t. The induced metric (24) for this solution is also the flat-space metric, since it can be written as $ds_\Sigma^2=-d\tau^2+\sum(dx^i)^2$, where

$$\tau=\frac{\sigma}{\epsilon}\operatorname{arcsinh}\left[\frac{\epsilon t}{\sigma}\right] \qquad (32)$$

is the proper time measured by an observer moving with the wall. The perturbations as measured by this observer will obey the equation [see (22)]

$$-\Box\phi-\frac{\epsilon^2}{\sigma^2}\phi=0\ , \qquad (33)$$

whose mode solutions are

$$\phi_k\propto e^{-i(\omega_\pm\tau-\mathbf{k}\cdot\mathbf{x})}\ , \qquad (34)$$

with $\omega_\pm=\pm\sqrt{k^2-\epsilon^2/\sigma^2}$.

Due to the presence of a negative mass squared in (33), we have exponentially growing modes for $k^2<\epsilon^2/\sigma^2$:

$$\phi_k=Ae^{\lambda\tau}e^{i\mathbf{k}\cdot\mathbf{x}}\ , \qquad (35)$$

where $\lambda=[(\epsilon^2/\sigma^2)-k^2]^{1/2}$ and A is the initial amplitude. Equation (35) suggests that the solution (31) is unstable, since the amplitude of the proper perturbations increases with time while the wavelength remains fixed. Thus, the wall becomes more and more wiggly as it evolves.

It is amusing to realize that the magnitude of the perturbation as seen by an "outside" inertial observer at rest (at $z=0$, for instance), which to linear order is given by a Lorentz contraction factor times the proper perturbation,

$$\sqrt{1-\dot z^2}\phi=\frac{\phi}{\cosh(\epsilon\tau/\sigma)}\ , \qquad (36)$$

actually decreases with time. To linear order the perturbed solution as seen by this observer becomes closer and closer to the zero-order solution.

Note, however, that the linear approximation will break down at some point and the question of whether we can expect significant departures from the zero-order solution can only be settled by considering nonlinear terms in ϕ. In deriving Eq. (22) we have neglected (among others) terms such as $(\epsilon/\sigma)\phi_{,c}\phi^{c}_{,}$, which become as important as $(\epsilon^2/\sigma^2)\phi$ when the amplitude of the perturbation reaches a value of order $\phi\sim\sigma/\epsilon$. Beyond that the linear approximation is not expected to hold. After a time $t\sim\sigma^2/A\epsilon^2$ higher-order corrections will be important and, since for the comoving observer the domain wall appears to become wiggly, it is likely that nonlinear structures (such as cusps, for instance) may develop.

V. PERTURBATIONS ON SPHERICAL VACUUM BUBBLES

As mentioned in Sec. I, false vacua in field theories whose effective potentials have nondegenerate minima may decay through first-order phase transitions. These proceed via quantum nucleation of spherical bubbles of true vacuum in the "sea" of false vacuum [2]. The true and false vacua are separated by a spherical domain wall.

After nucleation, the bubble evolves classically and the domain-wall trajectory is given by [2]

$$R^2-t^2=H^{-2}\ , \qquad (37)$$

where R is the radius of the bubble and $H=\epsilon/N\sigma$. The bubble nucleates at $t=0$ with a finite radius $R_0=H^{-1}$ and zero velocity $\dot R=0$. Then it starts expanding with constant proper acceleration, asymptotically approaching the speed of light. The thin-wall approximation applies provided that the thickness of the wall is small compared with R_0.

One can readily check that (37) is a solution of the classical equation of motion. Imposing spherical symmetry in (13) we obtain, after one integration, the energy-conservation equation [analogous to Eq. (28)]

$$\frac{R^{N-1}S^{(N-1)}\sigma}{\sqrt{1-\dot R^2}}-R^N V^{(N-1)}\epsilon=\text{const}\ , \qquad (38)$$

where $S^{(N-1)}$ and $V^{(N-1)}$ stand for the surface and the volume of the unit $(N-1)$-sphere, respectively. The first term in (38) is the energy of the domain wall, while the second term is the energy removed from the sea of false vacuum by cutting out a spherical piece of radius R. In fact, the constant in RHS must be equal to zero since energy is conserved by the nucleation process. Using $S^{(N-1)}/V^{(N-1)}=N$ we can rewrite (38) as

$$R\sqrt{1-\dot R^2}=H^{-1}\ , \qquad (39)$$

which is readily satisfied by (37).

Let us proceed to study linear perturbations to the zero-order solution. This problem has already been addressed by Adams, Freese, and Widrow in Ref. [5] (for $N=3$). These authors use a noncovariant formalism in which the trajectory of the perturbed solution is represented as

$$r(t,\theta,\varphi)=R(t)+\Delta(t,\theta,\varphi)\ , \qquad (40)$$

where r is the distance to the origin of coordinates, θ and φ are the usual polar and azimuthal angles, $R(t)$ is the unperturbed solution (37), and Δ is the perturbation. They find a differential equation for Δ and solve it numerically for a number of different modes.

In our covariant formalism the perturbation is taken not at a fixed moment of time, but in the direction normal to the world sheet. The covariant perturbation ϕ is related to Δ by a Lorentz factor (to linear order in Δ)

$$\phi = \frac{\Delta}{\sqrt{1-\dot{R}^2}} \ . \tag{41}$$

To derive an equation for ϕ, we first note that the unperturbed world sheet (37) is a hyperboloid embedded in $(N+1)$-dimensional Minkowski space, and therefore the induced metric on the world sheet is that of N-dimensional de Sitter space. This metric is obtained by writing the Minkowski line element in spherical coordinates and then substituting $r=R(t)$ as given by (37). The result is

$$dS_{\Sigma}^2 = -d\tau^2 + H^{-2}\cosh^2(H\tau)d\Omega^{(N-1)} \ , \tag{42}$$

where $\tau = H^{-1}\operatorname{arcsinh}(Ht)$ and $d\Omega^{(N-1)}$ is the line element on the $(N-1)$-sphere.

The Ricci scalar in de Sitter space is $\mathcal{R} = N(N-1)H^2$ and Eq. (22) for the perturbation ϕ takes the form

$$-\Box\phi - NH^2\phi = 0 \ . \tag{43}$$

The theory of perturbations on vacuum bubbles is thus reduced to the theory of a scalar field in de Sitter space with a negative mass squared $m^2 = -NH^2$.

The analytic solution of Eq. (43) in the metric (42) has been found by Chernikov and Tagirov [9]. The field ϕ can be represented as

$$\phi = \sum_{LM} \phi_L(\tau) Y_{LM} \ ,$$

where Y_{LM} are the usual harmonics on the $(N-1)$-sphere. They satisfy $\tilde{\Delta} Y_{LM} = -J Y_{LM}$, where $\tilde{\Delta}$ is the Laplacian on the unit $(N-1)$-sphere and the eigenvalues are given by $J = L(L+N-2)$ with $L = 0, 1, \ldots, \infty$. The index M runs from $-L$ to L. The equation for ϕ_L is then

$$\frac{d^2\phi_L}{d\tau^2} + (N-1)H\tanh(H\tau)\frac{d\phi_L}{d\tau} + \left[\frac{J}{\cosh^2(H\tau)} - N\right] H^2\phi_L = 0 \ . \tag{44}$$

In terms of the new variables

$$Z = \tanh(H\tau) \ , \qquad \chi_L = (1-Z^2)^{(1-N)/4}\phi_L \ , \tag{45}$$

Eq. (44) is the associated Legendre equation

$$(1-Z^2)\chi_L'' - 2Z\chi_L' + \left[\nu(\nu+1) - \frac{\mu^2}{1-Z^2}\right]\chi_L = 0 \ . \tag{46}$$

Here $\nu = L + (N-3)/2$, $\mu = (N+1)/2$, and a prime denotes derivative with respect to Z. Its solutions are linear combinations of the Legendre functions $P_\nu^\mu(Z)$ and $Q_\nu^\mu(Z)$, and we can write

$$\phi_L = (1-Z^2)^{(N-1)/4}[C_1 P_\nu^\mu(Z) + C_2 Q_\nu^\mu(Z)] \ , \tag{47}$$

where C_1 and C_2 are arbitrary constants. The argument Z, defined in (45), runs from 0 to 1. It is perhaps easier to think of it as $Z = t/R$.

For the lowest modes $L=0$ and $L=1$ the above expression is not the general solution to Eq. (46) because for these modes $Q_\nu^\mu = 0$ if $N=2$ and $P_\nu^\mu = 0$ if $N=3$. Nevertheless, for arbitrary N, we have the solutions $\phi_{L=0} = (1-Z^2)^{-1/2}Z$ and $\phi_{L=1} = (1-Z^2)^{-1/2}$. These correspond to a time translation and a space translation of the unperturbed solution, respectively. Given that we know one solution for each L, a second one can be found by quadratures. For $N=2$ we have

$$\phi_{L=0} = (1-Z^2)^{-1/2}[C_1 Z + C_2(Z\arcsin Z + \sqrt{1-Z^2})] \ ,$$

$$\phi_{L=1} = (1-Z^2)^{-1/2}[C_1 + C_2(\arcsin Z + Z\sqrt{1-Z^2})] \ ,$$

and for $N=3$ we have

$$\phi_{L=0} = (1-Z^2)^{-1/2}[C_1 Z + C_2(1+Z^2)] \ ,$$

$$\phi_{L=1} = (1-Z^2)^{-1/2}[C_1 + C_2(3Z - Z^3)] \ .$$

As in the case of a planar wall, the negative mass squared of the field ϕ gives rise to solutions growing with time. In the limit of larger τ, Eq. (44) takes the form

$$\frac{d^2\phi_L}{d\tau^2} + (N-1)H\frac{d\phi_L}{d\tau} - NH^2\phi_L = 0 \ . \tag{48}$$

The mode solutions in this limit are $\phi_L \sim \exp(H\tau)$ and $\phi_L \sim \exp(-NH\tau)$. The corresponding solutions for the variable Δ defined in Eq. (40) can be found using the relation (41) between Δ and ϕ. For the growing modes this gives $\Delta_L \sim \text{const}$. Since Δ represents the perturbation as seen by an external observer, we conclude that at large times the amplitude of the perturbation "freezes" and the relative perturbation Δ/R rapidly goes to zero. Hence, to an external observer, the wall will appear more and more spherical as it expands [10].

The variable ϕ describes how the perturbed wall would appear to an observer sitting on the unperturbed wall. To find out how the perturbation is seen by an internal observer "living" in the world sheet, we have to examine the world-sheet metric perturbation that corresponds to the scalar perturbation ϕ. For the hyperboloid (37) we have $n^\mu = Hx^\mu$. (This is more easily seen by going to Euclidean space, where de Sitter space is a sphere of radius H^{-1}.) Therefore, the perturbed trajectory will be $\tilde{x}^\mu = (1+H\phi)x^\mu$, and the perturbed metric will be given by the simple expression

$$\tilde{g}_{ab} = \partial_a \tilde{x}^\mu \partial_b \tilde{x}_\mu = (1+H\phi)^2 g_{ab} + \phi_{,a}\phi_{,b} \ . \tag{49}$$

We shall restrict our analysis to linear order. Then we have

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \ ,$$

with $\Omega=(1+H\phi)$, and we can use the relation [8]

$$\tilde{R}^{\,a}_{\,b}=\Omega^{-2}R^a_b+(N-2)\Omega^{-1}\nabla_b\partial_c(\Omega^{-1})g^{ca}-(N-2)^{-1}\Omega^{-N}\Box(\Omega^{N-2})\delta^a_b$$

to find the perturbed Ricci tensor

$$\tilde{R}_{ab}=H^2(N-1)\tilde{g}_{ab}-H^3\phi(N-2)\tilde{g}_{ab}-H(N-2)\nabla_a\partial_b\phi\ , \tag{50}$$

where we have used (43). Note that for $N\leq 3$ all the information about the curvature is contained in the Ricci tensor. In fact, for $N>3$ this is also true, since the metric is conformally flat and therefore the Weyl tensor vanishes.

An interesting feature of (50) is that the perturbed Ricci scalar is equal to the unperturbed one, $\tilde{\mathcal{R}}=H^2N(N-1)$. For $N=2$ this readily implies that the "perturbed" space is also de Sitter space of the same curvature. Therefore we conclude that an internal observer on the perturbed wall does not "see" quite as much as one can see from the outside. In particular, for $N=2$ the internal observer does not "feel" any linear metric perturbations at all.

Finally, we discuss the validity of the linear approximation in ϕ. We saw that the tachyonic mass of the field ϕ brings about exponentially growing modes, and it is not surprising that eventually the linear approximation breaks down. We verified that this is indeed the case by comparing the terms in Eq. (43) to the nonlinear terms neglected in this equation. What *is* surprising is that one can define a new variable

$$\Phi=\frac{\phi}{2}\frac{2+H\phi}{1+H\phi}$$

for which the linear approximation is valid all the way to $t\to\infty$. For small ϕ, $|\phi|\ll H^{-1}$, $\Phi\approx\phi$, but when ϕ gets large this relation is replaced by $\Phi\approx\phi/2$. The linearized equation for Φ is the same as for ϕ, Eq. (43), but, as it is shown in the Appendix, nonlinear corrections in this case remain small if they were small initially. The variable Φ is related to Δ defined in Eq. (40) by a simple formula

$$\Phi=\frac{\Delta}{\sqrt{1-\dot{R}^2}}\ ,$$

which agrees with (41) when ϕ is small, but remains valid even when ϕ gets large.

VI. PERTURBATIONS ON STRINGS AND WALLS IN DE SITTER SPACE

So far we discussed topological defects evolving in flat spacetime. In this section we shall consider perturbations on strings and walls in an inflationary universe. We begin with the case of a string loop nucleating during inflation.

Mathematically, an inflating universe is described by a de Sitter space, which can be represented as a hyperboloid embedded in a five-dimensional Minkowski space

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2=H^{-2}\ . \tag{51}$$

Here, H is the expansion rate of the inflationary universe and X^α ($\alpha=0,\ldots,4$) is the position vector in Minkowski space. Then, the world sheet of the string after nucleation is given by [3]

$$-(X^0)^2+(X^3)^2+(X^4)^2=H^{-2}\ ,\quad X^1=X^2=0\ , \tag{52}$$

which is a $(1+1)$-dimensional hyperboloid of maximal "radius" H^{-1} embedded in the hyperboloid (51). Its internal geometry is that of $(1+1)$-dimensional de Sitter space.

We shall use the following coordinatization of the four-dimensional spacetime:

$$t=H^{-1}\ln[H(X^4+X^0)]\ ,\quad x^i=H^{-1}X^i(X^4+X^0)^{-1} \tag{53}$$

($i=1,2,3.$) The line element then takes the form

$$ds^2=-dt^2+a^2(t)(d\mathbf{x})^2\ , \tag{54}$$

where $a(t)=\exp(Ht)$. In these coordinates, a $t=$const slice of the world sheet (52) looks like an infinite straight string along the z axis (we use the notation $x^3=z$). The coordinization (53) differs from that in Ref. [3] by an interchange of the variables X^4 and X^1. There, the constant time slices of the world sheet (52) are circular loops of string, while the four-dimensional line element still takes the form (54).

From the preceding remarks it is clear that the straight string considered by Turok [4] and the circular loops considered in Ref. [3] are simply different slicings of the same hyperboloid (52), and for our present purposes they can be given unified treatment.

Using the conformal time $\tau=-H^{-1}\exp(-Ht)$ and z as the coordinates on the unperturbed world sheet, we can write the perturbed solution as

$$\tilde{x}^{\,\mu}=(\tau,a^{-1}\phi_1,a^{-1}\phi_2,z)\ . \tag{55}$$

Here $a(\tau)=-(H\tau)^{-1}$ and $\phi_A(A=1,2)$ are (small) arbitrary functions of τ and z which represent the proper displacements in the two directions normal to the string.

Let us now derive the effective action for such perturbations. The perturbed metric will be given by

$$\tilde{g}_{ab}=a^2(\tau)\eta_{\mu\nu}\tilde{x}^{\,\mu}{}_{,a}\tilde{x}^{\,\nu}{}_{,b}\ , \tag{56}$$

with $\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1)$. Denoting by $\tilde{g}$ the determinant of $\tilde{g}_{ab}$ and using the relation $\delta\sqrt{-g}=(1/2)\sqrt{-g}\,g^{ab}\delta g_{ab}$ it is straightforward to show that, to second order in ϕ_A,

$$\sqrt{-\tilde{g}}=\sqrt{-g}+\tfrac{1}{2}\sqrt{-g}\left[g^{ab}\phi^A{}_{,a}\phi^A{}_{,b}-H^2\phi^A\phi^A-H^2\tau\frac{d}{d\tau}(\phi^A\phi^A)\right]\ , \tag{57}$$

where g_{ab} is the unperturbed [$(1+1)$-dimensional de Sitter] metric and summation over the indices A is understood.

Inserting the last expression in the Nambu action

$S=-\mu\int\sqrt{-\tilde{g}}\,d^2\xi$ we obtain, after an integration by parts,

$$S=S_0-\frac{\mu}{2}\int\sqrt{-g}\,(\phi^A{}_{,a}\phi^{A}{}_{,}{}^{a}-NH^2\phi^A\phi^A)d^N\xi\ , \qquad (58)$$

where S_0 is the action of the unperturbed solution and $N=2$ for strings (μ is the string tension). For domain walls nucleating in de Sitter space we can follow a similar argument and we find the same expression (58) with $N=3$ (and with μ replaced by the surface tension σ). In this case one has only one normal direction and the index A can be suppressed.

From Eq. (58) we see that each one of the ϕ^A behaves like a free scalar field with a tachyonic mass $m^2=-NH^2$ in N-dimensional de Sitter space. This is the same mass that we found for the perturbations to vacuum bubbles discussed in the previous section.

VII. CONCLUSIONS

We have developed a covariant formalism for describing perturbations on topological defects, such as vacuum domain walls and strings. Perturbations are represented by a scalar field ϕ which "lives" on the unperturbed world sheet of the defect and has the meaning of a normal displacement of the world sheet.

For domain walls in Minkowski space the field ϕ satisfies the Klein-Gordon equation (22) with a nonminimal coupling to the world-sheet curvature and with a tachyonic mass proportional to the difference of vacuum energy densities on the two sides of the wall. (The field ϕ is massless if the two vacua are equivalent). The same equation applies to strings bounding the holes in a planar domain wall.

In a number of cosmological applications the unperturbed world sheet has the geometry of a de Sitter space. The most interesting examples are vacuum bubbles nucleating during a first-order phase transition, closed strings and walls nucleating during inflation and straight strings and walls in inflationary universe. We have shown that in all these cases the effective mass of the field ϕ (the combined contribution of the mass and curvature terms) is

$$m^2=-NH^2\ .$$

Here, $N=3$ for domain walls, $N=2$ for strings, and H is the expansion rate of the de Sitter space. For vacuum bubbles H is determined by the wall tension and false-vacuum energy, while for strings and walls in inflationary universe it is equal to the expansion rate of the universe.

Quantization of perturbations on strings and walls will require a careful treatment of zero modes [11]. These modes correspond to space-time translations of the world sheet in the case of nucleating bubbles and to de Sitter transformations in the case of defects nucleating during inflation. We shall discuss the quantum theory of perturbations in a separate paper.

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APPENDIX

In this appendix we discuss the validity of the linear approximation for perturbations on vacuum bubbles. In this approximation, the perturbations for large τ behave as $\phi\sim\exp(H\tau)$ (τ is the proper time measured by an observer moving with the unperturbed world sheet, see Sec. V). In deriving Eq. (22) we have neglected, among others, terms such as $H\phi^2$ compared to $H^2\phi$, so we expect that the linear theory will break down as soon as $\phi\sim H^{-1}$. We shall see, however, that it is possible to define a new field Φ for which the linear approximation holds all the way to $\tau\rightarrow\infty$.

The unperturbed world sheet of the vacuum bubble wall is given in spherical coordinates by

$$x^\mu=(t,r,\theta,\varphi)=(H^{-1}\sinh(H\tau),H^{-1}\cosh(H\tau),\theta,\varphi)\ . \qquad (A1)$$

The normal vector is $n^\mu=(\sinh(H\tau),\cosh(H\tau),0,0)$ and the perturbed world sheet can be expressed as

$$\tilde{x}^\mu=x^\mu+\phi n^\mu=((H^{-1}+\phi)\sinh(H\tau),(H^{-1}+\phi)\times\cosh(H\tau),\theta,\varphi)\ , \qquad (A2)$$

with $\phi=\phi(\tau,\theta,\varphi)$.

Before proceeding further, we should mention a limitation of the covariant formalism in the present case. Since, from Eq. (A2),

$$r^2-t^2=(H^{-1}+\phi)^2\geq 0\ , \qquad (A3)$$

the covariant description is not suitable for regions of the perturbed world sheet that lie inside the light cone from the origin. Note also from (A2) that the perturbation may reach the light cone $r=t\neq 0$ only for $\tau\rightarrow\infty$ (if at all) though this may correspond to finite r and t.

We shall use the coordinates

$$\tau=H^{-1}\text{arctanh}(t/r)\ ,\qquad \xi=H^{-1}\ln(H\sqrt{r^2-t^2})\ . \qquad (A4)$$

Notice that both on the unperturbed and the perturbed world sheet, this definition of τ reduces to the one used before for parametrizing these surfaces [see Eq. (A2)]. In terms of the coordinates (A4) the Minkowski line element reads

$$ds^2=e^{2H\xi}[-d\tau^2+d\xi^2+H^{-2}\cosh(h\tau)d\Omega^2]\ , \qquad (A5)$$

with $d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2$. The unperturbed world sheet is located at $\xi=0$ (constant ξ surfaces are three-dimensional hyperboloids), and its induced metric is

$$ds_\Sigma^2 = g_{ab}dx^a dx^b$$
$$= d\tau^2 + H^{-2}\cosh^2(H\tau)d\Omega^2 \ . \qquad (A6)$$

The position of the perturbed world sheet is described as a function $\xi=\xi(\tau,\theta,\varphi)$, and its induced metric can be written as

$$\tilde{g}_{ab} = e^{2H\xi}(g_{ab}+\xi_{,a}\xi_{,b}) \ . \qquad (A7)$$

Let us now derive the *exact* effective Lagrangian for the field ξ. The action can be written as (see Sec. II)

$$S = -\sigma\int_\Sigma \sqrt{-\tilde{g}}\,d\tau\, d\theta\, d\varphi + \epsilon\int_{\rm vol} d^4x \ ,$$

where the second integral extends over the interior of the perturbed bubble and d^4x stands for the four-dimensional volume element. It is straightforward to show that

$$\sqrt{-\tilde{g}} = \sqrt{-g}\,e^{3H\xi}(1+g^{ab}\xi_{,a}\xi_{,b})^{1/2} \ .$$

From (A5) the four-dimensional volume element is

$$e^{4H\xi}H^{-2}\cosh^2(H\tau)\sin\theta\, d\tau\, d\xi\, d\theta\, d\varphi \ ,$$

so we can write

$$S = -\sigma\int e^{3H\xi}\sqrt{-g}\,(1+g^{ab}\xi_{,a}\xi_{,b})^{1/2}d\tau\, d\theta\, d\varphi$$
$$+\epsilon\int e^{4H\xi'}\sqrt{-g}\,d\xi' d\tau\, d\theta\, d\varphi \ .$$

In the first integral of the preceding expression $\xi=\xi(\tau,\theta,\varphi)$, while in the second integral ξ' is an integration variable that runs from $\xi'=-\infty$ (the light cone) to $\xi'=\xi(\tau,\theta,\varphi)$ (the position of the wall). It should be noted that by using the coordinate ξ we are neglecting the volume of the interior light cone from the origin. This is just an infinite constant which does not contribute to the equations of motion. Since g does not depend on ξ' we can integrate over $d\xi'$ to obtain

$$S = -\sigma\int\sqrt{-g}\,(e^{3H\xi}\sqrt{1+g^{ab}\xi_{,a}\xi_{,b}} - \tfrac{3}{4}e^{4H\xi})d\tau\, d\theta\, d\varphi \ , \qquad (A8)$$

where we have used $H=\epsilon/3\sigma$ [see after Eq. (37)].

Let us introduce a new variable

$$\Phi = H^{-1}\sinh(H\xi) \ . \qquad (A9)$$

From Eqs. (A4) and (A3) we see that the new field Φ is related to the "old" ϕ through the equation

$$\Phi = \frac{\phi}{2}\,\frac{2+H\phi}{1+H\phi} \ . \qquad (A10)$$

In terms of Φ the action (A8) reads

$$S = -\sigma\int\sqrt{-g}\,[e^{3F}\sqrt{1+g^{ab}\Phi_{,a}\Phi_{,b}(F')^2}$$
$$-\tfrac{3}{4}e^{4F}]d\tau\, d\theta\, d\varphi \ , \qquad (A11)$$

with $F=\text{arcsinh}(H\Phi)$ and $F'=(1+H^2\Phi^2)^{-1/2}$. From the variational principle we find the nonlinear equation of motion

$$\Box\Phi + 3H[H\Phi G + (G-1)\sqrt{1+H^2\Phi^2}]$$
$$-g^{ab}\Phi_{,a}\partial_b \ln G = 0 \ , \qquad (A12)$$

with $G=(1+H^2\Phi^2+g^{ab}\Phi_{,a}\Phi_{,b})^{1/2}$ and $\Box = g^{ab}\nabla_a\nabla_b$ as usual. The linearized form of this equation is

$$-\Box\Phi - 3H^2\Phi = 0 \ , \qquad (A13)$$

as expected, since Φ and ϕ agree to linear order [see (A10)]. What is surprising is that the nonlinear terms that we have neglected in going from (A12) to (A13) do not become dominant as $\tau\to\infty$, but they remain small compared to the linear terms if the perturbations are small initially. To realize that this is the case it suffices to note that for $\tau\to\infty$, $\Phi\approx A(\theta,\varphi)\exp(H\tau)$, so $G\approx[1+H^2(A_{,\theta})^2+H^2(A_{,\varphi})^2(\sin\theta)^{-2}]^{1/2}$ and $\partial_\tau G\to 0$.

Finally, we should comment on the relation between Φ and the variable Δ used in Ref. [5]. From (40) and (37), $r-\Delta=(\tau^2+H^{-2})^{1/2}$. Squaring this equation and assuming $(\Delta/r)\ll 1$ (this turns out to be a self-consistent assumption) we find

$$\Phi = \cosh(H\tau)\Delta = \frac{\Delta}{\sqrt{1-\dot{R}^2}} \ , \qquad (A14)$$

where we have used (A2) and (A10), and $\dot{R}=dR/dt$ is the unperturbed velocity. Therefore, the comments in Sec. V about the behavior of Δ at late times can be made more rigorous by using Φ instead of ϕ and Eq. (A14) instead of (41).

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