

Singularity-free space-time

F. J. Chinea and L. Fernández-Jambrina

Departamento de Física Teórica II, Universidad Complutense de Madrid, Ciudad Universitaria, 28040 Madrid, Spain

J. M. M. Senovilla*

Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

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We show that the solution published in the paper by Senovilla [Phys. Rev. Lett. **64**, 2219 (1990)] is geodesically complete and singularity-free. We also prove that the solution satisfies the stronger energy and causality conditions, such as global hyperbolicity, the strong energy condition, causal symmetry, and causal stability. A detailed discussion about which assumptions in the singularity theorems are not satisfied is performed, and we show explicitly that the solution is in accordance with those theorems. A brief discussion of the results is given.

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I. INTRODUCTION

Recently, one of us presented a solution to Einstein's field equations for a perfect-fluid energy-momentum tensor without any curvature singularity [1]. The solution has positive pressure and energy density everywhere, and it satisfies a realistic equation of state for hot radiation-dominated epochs. As we shall see, it also verifies the stronger causality requirements. Therefore, the question remains of whether or not the solution is geodesically complete and, if it is, how that fits in with the general conclusions of the very powerful singularity theorems [2,3].

The main result in this paper is that the mentioned solution is geodesically complete and, in fact, singularity-free (see Ref. [2] for definitions). We shall prove it in Sec. II by studying the equations of the non-spacelike geodesics and showing that they can always be extended to arbitrary values of the affine parameter. Once this important property has been established, in Sec. III we shall study which of the assumptions of the different singularity theorems are not verified in the solution. Since the energy and causality conditions will be proven to be satisfied, it will turn out that the so-called initial or boundary condition is the one that fails to hold. This condition usually refers to the existence of a causally trapped set [2,3] or to some fixed bound for the initial expansion of the geodesics. Thus, we shall see that the initial condition is *essential* for the development of singularities. Along with this analysis, we shall prove a number of nice and desirable properties of the solution, among them, global hyperbolicity and stable causality. Finally, we devote Sec. IV to give a brief discussion of the results herein presented.

Before getting into the main subject, let us give a summary of the solution of [1]. The line element is given by

$$ds^2 = C^4(at)C^2(3ar)(-dt^2 + dr^2) + (9a^2)^{-1}C^4(at)S^2(3ar)C^{-2/3}(3ar)d\phi^2 + C^{-2}(at)C^{-2/3}(3ar)dz^2, \quad (1)$$

where $a \neq 0$ is an arbitrary constant (which can be taken as positive) and we use the notation

$$S(u) \equiv \sinh(u), \quad C(u) \equiv \cosh(u).$$

The coordinates used are of cylindrical type, and their range is

$$-\infty < t, z < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \phi \leq 2\pi$$

such that ϕ and $\phi + 2\pi$ are identified. The metric (1) is a solution of Einstein's equations for a perfect fluid. The coordinates are comoving (or adapted to the fluid congruence) in the sense that the velocity vector has the form

$$\mathbf{u} = C^{-2}(at)C^{-1}(3ar)\partial/\partial t. \quad (2)$$

The fluid congruence is orthogonal to the spacelike hypersurfaces $t = \text{const}$ and therefore its rotation tensor vanishes. The expansion of the fluid has the expression

$$\theta = 3aS(at)C^{-3}(at)C^{-1}(3ar), \quad (3)$$

which is regular everywhere and positive for $t > 0$ (expanding phase) or negative for $t < 0$ (contracting phase). We also see that both for $t \rightarrow \pm\infty$ or $r \rightarrow \infty$ the expansion approaches zero.

However, the congruence of the fluid is not geodesic, the acceleration vector field being

$$\mathbf{a} = 3aS(3ar)C^{-3}(3ar)C^{-4}(at)\partial/\partial r. \quad (4)$$

From a physical point of view, this fact is of extreme importance for the absence of singularities in the solution. Because, as is well known, a nonvanishing acceleration means that there exists a spacelike *gradient of pressure* or, equivalently, a force which opposes the gravitational attraction. Thus, although for $t < 0$ there is a contraction,

*Also at Laboratori de Física Matemàtica, Societat Catalana de Física, I.E.C., Barcelona, Spain.

the gradient of pressure does not allow for the formation of a singularity and causes a bounce from which the matter expands. The fluid congruence is also shearing. The shear tensor can be found in [1].

The density and pressure of the fluid take the form

$$\chi\rho = 15a^2 C^{-4}(at)C^{-4}(3ar), \quad (5)$$

$$p = \rho/3, \quad (6)$$

where χ is the gravitational constant. From (5) we see that the maximum value of ρ at each time t is achieved at $r=0$. Moreover, the absolute maximum of the density (and pressure) occurs when $r=0$ and $t=0$, and its value is $\rho_M = 15a^2/\chi$. Therefore, we can interpret the constant a as the maximum energy density. Note that, since a is arbitrary, the maximum density can be chosen as large or small as we like. On the other hand, from (6) we learn that the equation of state is realistic for a space-time filled with incoherent radiation. Equations (5) and (6) also tell us that ρ and p are positive everywhere, which implies that the strong energy condition (or the timelike convergence condition; see [2]) is satisfied. From the same equations we can check that ρ and p are finite and well behaved everywhere, which means that there is no matter singularity (singularity in the Ricci tensor). It can further be shown, by computing the Weyl tensor (see Ref. [1]), that *all* the curvature invariants are regular on the whole space-time so that there is no curvature singularity in the metric at all.

The line element (1) admits an Abelian group of symmetries G_2 , and the Killing vectors $\partial/\partial\phi$ and $\partial/\partial z$ are globally defined. Both of them are spacelike, orthogonal to each other and orthogonally transitive, and therefore

the metric is cylindrically symmetric. The regularity condition [4] on the axis $r=0$ is satisfied so that the 2π periodicity of the coordinate ϕ is well defined and the so-called elementary flatness on the vicinity of the axis is assured. In this sense, the coordinate singularity which appears in (1) at $r=0$ is just that of cylindrical coordinates; other coordinates (Cartesian-like) do exist such that they provide a global chart on the manifold.

The coordinate t appearing in the metric is a *time function* (or cosmic time) in the sense that it (decreases) increases along every (past-) future-directed nonspacelike curve. This can be seen by checking that the gradient of t is timelike everywhere. As is well known, this is the necessary and sufficient condition for the stable causality condition to hold [2]. Thus, the space-time is causally stable which is the stronger causality requirement and it implies the weaker chronology and causality conditions.

Finally, let us remark that the solution is inextendible, which avoids the possibility of the existence of singularities hiding somewhere. This is apparent from the form of the metric (1); however, the best way to prove it is to show that all geodesics are complete and then the metric can be extended nowhere. In the next section we shall prove geodesic completeness for the solution.

II. GEODESIC COMPLETENESS

Our aim now is to show that every nonspacelike geodesic can be extended to arbitrary values of its affine parameter. To that end, we must analyze the equations of the geodesics in the metric (1), which after standard and straightforward calculations can be written as

$$\ddot{i} + 2aT(at)(\dot{i}^2 + \dot{r}^2) + 6aT(3ar)\dot{i}\dot{r} + (2/9a)T(at)S^2(3ar)C^{-8/3}(3ar)\dot{\phi}^2 - aC^{-7}(at)S(at)C^{-8/3}(3ar)\dot{z}^2 = 0, \quad (7)$$

$$\ddot{r} + 3aT(3ar)(\dot{i}^2 + \dot{r}^2) + 4aT(at)\dot{i}\dot{r} - (1/9a)S(3ar)C^{-5/3}(3ar)[3 - T^2(3ar)]\dot{\phi}^2 + aC^{-6}(at)S(3ar)C^{-11/3}(3ar)\dot{z}^2 = 0, \quad (8)$$

$$(9a^2)^{-1}C^4(at)S^2(3ar)C^{-2/3}(3ar)\dot{\phi} = K, \quad (9)$$

$$C^{-2}(at)C^{-2/3}(3ar)\dot{z} = L, \quad (10)$$

$$C^4(at)C^2(3ar)(-\dot{i}^2 + \dot{r}^2) + (9a^2)^{-1}C^4(at)S^2(3ar)C^{-2/3}(3ar)\dot{\phi}^2 + C^{-2}(at)C^{-2/3}(3ar)\dot{z}^2 = -\delta. \quad (11)$$

Here a dot means derivative with respect to the affine parameter and we use the notation

$$T(u) \equiv \tanh(u).$$

K and L are constants of motion along the geodesics due to the existence of two Killing vectors in the spacetime. Another constant of motion is denoted by δ , which must be taken equal to one or zero for timelike or null geodesics respectively. The fact that all functions involved in Eqs. (7)–(11) are nonsingular will grant us the existence and uniqueness of solutions.

The method of the proof is to give finite bounds for the first derivatives of the coordinates, which implies (see Ref. [5]) that the field is nonsingular and the geodesics are complete. We will also use the second derivatives of the coordinates to show that these cannot become singu-

lar. We shall only deal with geodesics propagating towards the future. Propagation towards the past can be treated in a similar way.

In Eq. (8), when (9) and (10) are taken into account, \ddot{r} is negative for positive t and increasing values of r . This follows from the fact that the positive term in \dot{z}^2 dominates over the negative term in $\dot{\phi}^2$ for large enough values of r . Negative values of t need not be considered, as t will end up becoming positive. It is obvious that r cannot diverge to infinity in a finite proper time with negative \ddot{r} .

Most of the geodesics approach $r=0$ without any problem. As r decreases, the term in \dot{r} dominates over the terms in \dot{r}^2 and \dot{i}^2 , while the term in \dot{z}^2 vanishes when approaching $r=0$. Hence, \ddot{r} is positive for decreasing r in the vicinity of $r=0$ for positive t . This means the geo-

desics cannot come near the axis quickly enough to become singular.

Also \dot{t} is negative for large values of t (notice that the term in \dot{t}^2 would not be relevant if r decreases), so that the geodesic does not grow faster than its tangent. The quantity \dot{z} does not become singular if r and t do not; $\dot{\phi}$ could have problems if r were zero, but we shall see that this is not possible. In order to proceed, we divide our study in steps, starting with the simpler geodesics and then going to the general case. In that way we shall be able to use some results several times as we progress on.

(1) *Geodesics in the fluid congruence.* This is the simplest case. The congruence of the fluid is defined by $\dot{r}=\dot{\phi}=\dot{z}=0$. From Eq. (4) it follows that the only possible geodesics lie in $r=0$. Then the system reduces to the single equation

$$\dot{t} = C^{-2}(at) \leq 1.$$

Obviously, these geodesics are complete.

(2) *Geodesics along the axis.* They are defined by $\dot{r}=\dot{\phi}=0, r=0$. The remaining equations are now

$$\begin{aligned} \dot{z} &= LC^2(at), \\ \dot{t} &= C^{-2}(at)[L^2C^2(at)+\delta]^{1/2} \leq (L^2+\delta)^{1/2}. \end{aligned}$$

Therefore, the derivatives are bounded everywhere and these geodesics are also complete. The coordinate t reaches infinity only when so does the affine parameter. This is a general property of all geodesics and we shall not refer to it in what follows.

(3) *Radial null geodesics.* Now $\dot{\phi}=\dot{z}=0, \delta=0$. After one integration the system reduces to

$$\dot{t} = |\dot{r}|, \quad \dot{r} = hC^{-4}(at)C^{-2}(3ar),$$

where h is a constant of integration. We see that $0 < |\dot{r}| = |\dot{t}| \leq |h|$, from where it follows that these geodesics are complete too. When one of the curves arrives at $r=0$ along a direction ϕ then it continues along the direction $\phi+\pi$ after crossing the axis without problems.

(4) *Radial timelike geodesics.* Again $\dot{\phi}=\dot{z}=0$, but now $\delta=1$. By introducing a new function v which, so to speak, parametrizes \dot{t} and \dot{r} , the system reduces in this case to

$$\dot{t} = C(v)A(r,t), \quad \dot{r} = S(v)A(r,t), \quad \dot{v} = B(r,t,v)A(r,t), \quad (12)$$

where

$$A(r,t) \equiv C^{-2}(at)C^{-1}(3ar), \quad (13)$$

$$B(r,t,v) \equiv -a[3T(3ar)C(v)+2T(at)S(v)]. \quad (14)$$

The previous reasoning about second derivatives allows us to conclude that these geodesics are complete.

(5) *Null geodesics with no angular velocity.* Now, $\dot{\phi}=0$ and $\delta=0$. Using the function v again, we can write the system of equations

$$\dot{t} = C(v)E(r,t), \quad \dot{r} = S(v)E(r,t), \quad \dot{v} = E(r,t)F(r,t,v),$$

where

$$E(r,t) \equiv |L|C^{-1}(at)C^{-2/3}(3ar), \quad (15)$$

$$F(r,t,v) \equiv -a[4T(3ar)C(v)+3T(at)S(v)]. \quad (16)$$

A reasoning similar to that given in the previous case leads us again to completeness for these null geodesics.

(6) *Null geodesics on the hypersurfaces $z=\text{const}$.* These are defined by $\dot{z}=0, \delta=0$. The system of equations can be written as

$$\dot{t} = C(v)M(r,t), \quad \dot{r} = S(v)M(r,t), \quad \dot{v} = M(r,t)D(r,v)$$

with

$$M(r,t) \equiv 3|aK|C^{-4}(at)C^{-2/3}(3ar)S^{-1}(3ar), \quad (17)$$

$$D(r,v) \equiv aC(v)[3T^{-1}(3ar)-4T(3ar)]. \quad (18)$$

In this case we can obtain one of the orbit equations by dividing \dot{r} by \dot{v} and then integrating. Doing so we get

$$C(v) = \alpha^{-1}S(3ar)C^{-4/3}(3ar),$$

where α is a positive constant. We see that, since $C(v) \geq 1$, the coordinate r can only take values between r_+ and r_- , with r_{\pm} defined by $S(3ar_{\pm})C^{-4/3}(3ar_{\pm}) = \alpha$. Thus $|v|$ is bounded and the geodesics are complete.

Among these geodesics there is one which is circular and goes round the axis of symmetry at $r=r_0 \equiv \text{arccosh}(2)/3a$. This can be identified as the special case in which $r_+ = r_- = r_0, v=0$.

(7) *General nonspacelike geodesics.* Using the v parametrization, we can rewrite the general system of Eqs. (7)–(11) in the form

$$\begin{aligned} \dot{t} &= (3|aK|)^{-1}M(r,t)G(r,t)C(v), \\ \dot{r} &= (3|aK|)^{-1}M(r,t)G(r,t)S(v), \\ \dot{v} &= (3|aK|)^{-1}M(r,t)H(r,t)G^{-1}(r,t), \end{aligned} \quad (19)$$

where we have defined

$$G(r,t) \equiv [9a^2K^2 + L^2C^6(at)S^2(3ar) + C^4(at)S^2(3ar)C^{-2/3}(3ar)(1-\delta)]^{1/2}, \quad (20)$$

$$\begin{aligned} H(r,t,v) &\equiv -a\{S(v)T(at)S^2(3ar)[3L^2C^6(at) + 2C^4(at)C^{-2/3}(3ar)(1-\delta)] \\ &\quad + C(v)T(3ar)[4L^2C^6(at)S^2(3ar) + 3C^4(at)S^2(3ar)C^{-2/3}(3ar)(1-\delta)] \\ &\quad + 9a^2K^2C(v)[4T(3ar) - 3T^{-1}(3ar)]\}. \end{aligned} \quad (21)$$

The only problem we could have appears when r approaches zero, but in this case the dominant terms are those of paragraph (6), so that the geodesics with $K \neq 0$ cannot reach the axis. Then they are complete. We can say that the term in $\dot{\phi}^2$ in (8) is centrifugal in the vicinity of the axis.

III. PROPERTIES OF THE SOLUTION AND ITS RELATION WITH SINGULARITY THEOREMS

From the results of the previous section it is obvious that every maximally extended null geodesic meets any of the hypersurfaces $t = \text{const}$. This is a sufficient condition such that every nonspacelike curve intersects the mentioned hypersurfaces exactly once [6]; in other words, the hypersurfaces $t = \text{const}$ are global Cauchy surfaces and the solution is globally hyperbolic. For this reason the solution is also causally simple [2], that is, for every compact set K , $J^+(K)$ is closed and its boundary coincides with $E^+(K)$.

Because of the fact that every spacelike hypersurface $t = \text{const}$ is a global Cauchy surface, and since t is a time function in the solution as explained in Sec. I, it follows that every nonspacelike curve (geodesic or not) can be extended to arbitrary values of its generalized affine parameter (because it has to meet all the Cauchy surfaces). This means that the solution is nonspacelike b complete (see Ref. [2]), that is to say, *singularity free*.

Furthermore, from expressions (5) and (6) we see that the energy-momentum tensor does not vanish anywhere. Therefore, given any nonspacelike vector v we have $R_{ab}v^av^b > 0$ so that, in addition to the strong energy condition, the generic condition is satisfied as well [3]. Thus, it remains to see which other conditions in the singularity theorems are not satisfied by the spacetime.

The simplest such theorem (Ref. [2], p. 274) requires the existence of a Cauchy surface such that the trace of its second fundamental form is bounded away from zero. In our case, the mentioned trace for any Cauchy surface $t = \text{const}$ is given by the expansion of the fluid congruence shown in Eq. (3). From this expression is obvious that θ is not bounded away from zero, because, for any t , θ goes to zero when $r \rightarrow \infty$. The significance of this result can be better understood as follows: given any Cauchy surface $t = t_1 < 0$, we have $\theta_{t_1} < 0$ and then there are points conjugate to the surface along every future-directed timelike geodesic orthogonal to the surface within a distance $-3/\theta_{t_1}$. However, as θ_{t_1} is not bounded away from zero, those distances do not have an upper bound. This allows for the existence of a maximal geodesic orthogonal to the surface up to any point q of its future, as is necessary, $t = t_1$ being a Cauchy surface.

Another property of the solution is that the Cauchy surface $\Sigma: t=0$ is a maximal spacelike hypersurface, i.e., one in which the trace of the second fundamental form vanishes. It follows that the spacetime is causally (or time) symmetric [7] so that $J^+(\Sigma)$, $I^+(\Sigma)$, and $D^+(\Sigma)$ are isometric to $J^-(\Sigma)$, $I^-(\Sigma)$, and $D^-(\Sigma)$, respectively. In Ref. [7], Tipler has shown that all geodesics are both future and past incomplete in a spacetime which contains a noncompact maximal Cauchy surface if, in addition to

the strong energy condition, the following is satisfied: there exist fixed positive constants b and c such that

$$\left| \int_0^b R_{ad}v^av^d d\tau \right| \geq c$$

for every timelike geodesic (with tangent vector v) intersecting Σ orthogonally at $\tau=0$, where τ is the affine parameter. However, this is not verified in solution (1), because for any given pair of constants b and c the above integral is always positive but not bounded below above zero. In fact, we can always choose geodesics (for r big enough initially) such that the integral takes a value as small as we like and less than any previously fixed c .

Let us consider then the classical singularity theorems. The first of them, chronologically speaking, is that of Penrose [8,2], which assumes the existence of both a noncompact Cauchy surface and a closed trapped surface, i.e., a compact (without boundary) spacelike two-surface in which the traces of the two null second fundamental forms have the same sign. Our space-time does not have a closed trapped surface. In order to prove it, suppose there was one. Since the surface is compact, it must have a point p where r reaches its maximum, so that the normal at p is a superposition of $\partial/\partial r$ and $\partial/\partial t$ at p . But if we compute the traces of both null second fundamental forms at p , we get

$$\begin{aligned} \chi_{\mp d}^d &= -g^{zz}n_{z,z}(p) - g^{\phi\phi}n_{\phi,\phi}(p) \\ &\quad + 2^{-1/2}aC^{-2}(at)S^{-1}(3ar) \\ &\quad \times [-T(at)T(3ar) \mp 3 \pm 2T^2(3ar)] \end{aligned}$$

from which it is clear that

$$\chi_{-d}^d \leq -2^{-1/2}aC^{-2}(at)S^{-1}(3ar)[1 + T(at)T(3ar)] < 0,$$

$$\chi_{+d}^d \geq 2^{-1/2}aC^{-2}(at)S^{-1}(3ar)[1 - T(at)T(3ar)] > 0,$$

since $n_{z,z}(p)$ and $n_{\phi,\phi}(p)$ are positive for outgoing normals and negative for ingoing normals at p . We see that the traces have opposite signs so that there are no closed trapped surfaces. This reasoning can be visualized in the following way. Take any closed compact two-surface in the manifold. This surface must be orthogonal to the in- and outgoing radial null geodesics somewhere. But the outgoing and ingoing radial null geodesics are expanding and contracting, respectively, everywhere.

The most famous and powerful singularity theorem was proven by Hawking and Penrose [9,2]. Our spacetime satisfies all conditions in the Hawking-Penrose theorem except for condition 4 in Ref. [2], which allows for three different possibilities. One of them is the existence of a closed trapped surface, which we have just shown is not satisfied in the solution. The second possibility is the existence of a point q such that on every past (future) null geodesic from q the expansion becomes negative. There is no such point in solution (1), since through any point in the manifold there are radial null geodesics which diverge if they are outgoing and future directed or ingoing and past directed. Another way to see that the point q does not exist is to remember that through any q there are null geodesics with $z = \text{const}$ which are bounded above and below in r . Thus, these geodesics can never

converge with the radial ones. The third possibility in the Hawking-Penrose theorem is the existence of a compact achronal set without edge. It is pretty obvious that there is not a set with those properties in the space-time under consideration. To prove it rigorously, take any achronal set in the manifold and a point q in the set. By using the radial geodesics we can always choose points $q_- \in I^-(q)$ and $q_+ \in I^+(q)$ such that $r(q_-) = r(q_+) > r(q)$, where for any point s we denote by $r(s)$ the value of the coordinate r at s . Since $q_+ \in I^+(q_-)$ and $r(q_+) = r(q_-)$, we can join q_- and q_+ with a future-directed world line of the fluid congruence. If the achronal set has no edge, this world line *must* intersect the set, and it will do it at a point \tilde{q} with $r(\tilde{q}) = r(q_-) = r(q_+) > r(q)$. We have thus proven that for any achronal set without an edge, the coordinate r cannot be bounded. It follows that any achronal set in the manifold cannot be both compact and without an edge.

Finally, let us remark that in Ref. [2] there appear two other singularity theorems due to Hawking (marked with numbers 3 and 4, pages 271 and 272). However, the conditions in these theorems are stronger than those of the Hawking-Penrose theorem which we have already proven not to hold in the solution. There are some other singularity theorems now available, but their hypotheses are mere variations of those here studied. In this sense, we believe that our study is somehow exhaustive. In the next section we shall give a brief discussion of the meaning of our results.

IV. DISCUSSION

We have proven that solution (1) is singularity free and is in accordance with the main singularity theorems *as well*. In fact, once the properties of the solution are known, it becomes rather obvious that it is free of singularities. Because the solution is globally hyperbolic, there cannot be any Cauchy horizon. However, the congruence of the fluid is trivially complete and through each point in the manifold there is a world line of the fluid congruence.

It is also clear that, because of the properties of the solution, any possible singularities would have to have some extension and therefore should manifest themselves in the curvature invariants; but they do not. However, to be on the safe side, we have preferred to perform a detailed analysis of which particular condition in the different singularity theorems failed to be verified in the space-time for the following reasons: first, to illustrate how the solution can avoid the development of singularities;

second, to get a deeper insight into the significance and application of the singularity theorems and, in this sense, to show explicitly that neither the energy or causality conditions are determinant by themselves for the appearance of singularities, not even for solutions filled with reasonable matter everywhere and such that all the matter is expanding at a given instant of time.

As is well known, reasonable singularity-free solutions are a rarity at the moment. This is especially true for solutions with cosmological properties. By this we do not mean solutions which can describe adequately the real observed Universe (which is not the case here), but rather solutions which *theoretically* must be considered as cosmological since they have matter everywhere and cannot be considered as interiors of some vacuum exteriors (for this it would be necessary a timelike surface of vanishing pressure). Therefore, the question arises of how many singularity-free solutions there possibly are and which particular properties they must obey.

It can be thought, for example, that the special equation of state (6) would have some importance in order to avoid the singularities. This is not true, for in Ref. [10] it has been shown that the solution here studied belongs to a larger family of singularity-free metrics for perfect fluids with no equation of state whatsoever. However, all the members in the family have in common that they are cylindrically symmetric and, in fact, every other solution found in [10] without this symmetry contains singularities. Thus, cylindrical symmetry could be somehow important for the avoidance of singularities. Of course, we must keep this conclusion as a mere hypothesis. On the other hand, the converse of this hypothesis is obviously false, and, in fact, if we replace $\cosh(at)$ for $\sinh(at)$ in (1) we obtain another solution of Einstein's equations with similar properties and a well-defined cylindrical symmetry but having a big-bang singularity in the finite past [1,11].

In any case, it is not clear to us why there should not be general solutions (with no symmetry) without singularities if they satisfy the main properties shown in this paper, that is to say, if they do not have any kind of causal trapped set, which reveals itself as an essential assumption in the singularity theorems.

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[1] J. M. M. Senovilla, Phys. Rev. Lett. **64**, 2219 (1990).

[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).

[3] J. Beem and P. Ehrlich, *Global Lorentzian Geometry*

(Dekker, New York, 1981).

[4] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).

[5] V. I. Arnold, *Ordinary Differential Equations* (MIT Press,

Cambridge, MA, 1973).

[6] R. Geroch, *J. Math. Phys.* **11**, 437 (1970).

[7] F. J. Tipler, *J. Math. Phys.* **18**, 1568 (1977).

[8] R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965).

[9] S. W. Hawking and R. Penrose, *Proc. R. Soc. London*

A314, 529 (1970).

[10] E. Ruiz and J. M. M. Senovilla, *Phys. Rev. D* (to be published).

[11] A. Feinstein and J. M. M. Senovilla, *Class. Quantum Grav.* **6**, L89 (1989).