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Particle creation in a colliding plane wave spacetime: Wave packet quantization

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We use wave packet mode quantization to compute the creation of massless scalar quantum particles in a colliding plane wave spacetime. The background spacetime represents the collision of two gravitational shock waves followed by trailing gravitational radiation which focus into a Killing-Cauchy horizon. The use of wave packet modes simplifies the problem of mode propagation through the different spacetime regions which was previously studied with the use of monochromatic modes. It is found that the number of particles created in a given wave packet mode has a thermal spectrum with a temperature which is inversely proportional to the focusing time of the plane waves and which depends on the mode trajectory.

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I. INTRODUCTION

Exact solutions representing the head on collision of two gravitational plane waves are some of the simplest exact dynamical spacetimes. They provide clear examples of highly nonlinear behavior in general relativity: when two plane waves collide the focusing effects of each exact plane wave lead to mutual focusing. This is revealed by the formation of either a spacetime singularity or a nonsingular Killing-Cauchy horizon at the focusing points of the two waves [1–3]. They may also be useful to provide local models for processes that may be taking place in our Universe as a result of gravitational waves produced in black hole collisions [4,5], the decay of cosmological inhomogeneous singularities [6], or by traveling waves in strongly gravitating cosmic strings [7,8].

Quantum effects in such dynamical spacetimes must surely be important and one expects particle production and vacuum polarization when a quantum field is coupled to such a background. Yurtsever [6] was the first to study field quantization on a colliding wave background; he considered the Kahn-Penrose [1] solution which may be interpreted as the collision of two impulsive plane waves. The solution which has curvature singularities at the focusing points of the plane waves allows for the definition of two physically meaningful vacuum states: an "in" vacuum state associated with the flat space before the collision of the two plane waves and an "out" vacuum state related to the flat spacetime regions behind the shock fronts also before the collision. The Bogoliubov coefficients relating the "in" and "out" creation and annihilation operators could be found only approximately in the long-wavelength limit. In this approximation the spectrum of created particles is consistent with a thermal distribution.

In a recent paper [9] we considered the quantization of a massless scalar field in the background of the collision of two plane waves which form a nonsingular Killing-Cauchy horizon. This solution describes the collision of two gravitational shock waves followed by trailing gravitational radiation [10]. The interaction region of the two waves is locally isometric to a region inside the event horizon of a Schwarzschild black hole with the Killing-Cauchy horizon corresponding to the event horizon [2,11,12]. Two unambiguous and physically meaningful quantum vacuum states may be defined: an "in" vacuum associated to the positive frequency mode solutions in the flat region before the collision of the waves and an "out" vacuum related to the positive frequency modes defined through the two null vector fields in the Killing-Cauchy horizon. Such a state, which is invariant under the symmetries associated with the horizon, corresponds to the unique *preferred vacuum state* defined by Kay and Wald [13] in spacetimes with bifurcate Killing horizons. It was found that the "in" vacuum contains a number of "out" particles which is proportional to the inverse of the frequency. In the long-wavelength limit the spectrum is consistent with a thermal spectrum at a temperature which is inversely proportional to the focusing time of the gravitational plane waves, in agreement with Yurtsever's result [6].

Our result however is exact: although the Klein-Gordon equation in the interaction region cannot be solved exactly the "in" modes become blueshifted towards the trailing points of the waves and can be propagated through this region by using the geometrical optics approximation. This is somewhat similar to the situation in a Schwarzschild black hole [14].

In this paper we want to reconsider this problem in the light of wave packet mode quantization instead of the monochromatic modes we used in Ref. [9]. There are several reasons that, we believe, justify this. One of the reasons is well known: the use of monochromatic plane wave modes leads to infinite expressions for the total number of particles created of a given frequency, whereas the number of particles created in a given wave packet mode is finite [14]. The second, and more important for us here, is that wave packet modes propagate in a simple way in a spacetime, such as ours, formed by the matching of different regions. Since the singular-free coordinates of one of the regions differ from the next region, the propagation of modes which are extended through all space finds serious difficulties. In Ref. [9] this could be done because of the blueshift of the modes in certain regions and the use there of the geometrical optics approximation which amounts to ray propagation. Since wave packets are localized in space and the packet maximum follows a well defined path in the spacetime, a natural approximation may be taken by which the propagation of the packets through the different regions becomes a simpler problem.

Another reason is that the wave packet formalism localizes the phenomena of particle creation: the one particle states defined with the wave packet modes have two labels; one gives information on the "energy" of the particle and the other on its "trajectory" (all within a certain range of values). Finally, we know that our colliding wave spacetime can be maximally extended through the Killing-Cauchy horizon with the extended Schwarzschild spacetime [10,15]. This is possible if one of the transversal coordinates of the plane waves is made cyclic. The resulting spacetime represents the collision of two plane waves propagating in a cylindrical universe and the creation of a black hole of mass proportional to the strength (or focusing time) of the plane waves. In a forthcoming paper we want to consider particle creation in the extended spacetime. Using the results of the present paper the calculation will become somewhat similar to that of stimulated emission by black holes [16]; the use of wave packet modes has proved useful also in this case (See Ref. [17]).

The plan of the paper is the following. In Sec. II we introduce wave packet mode quantization. In Sec. III we briefly review the geometrical properties of our colliding wave spacetime with special emphasis on the coordinates which are appropriate in the different regions. In Sec. IV we quantize a massless scalar field on the colliding wave background and propagate the "in" wave packet modes through the different regions. The advantage of wave packet modes is clearly seen in Sec. IV D when the modes are propagated through the interaction region. The Bogoliubov coefficients relating packet creation and annihilation operators are derived in Sec. IV F, and the creation of particles is derived in Sec. IV G, where we also compare our results with those of Ref. [9].

II. WAVE PACKETS

Let us consider a complete orthonormal family $\{f_\omega(x)\}$ of complex solutions of the Klein-Gordon equation for a massless scalar field ϕ (i.e., $\Box\phi=0$), which contain only positive frequencies with respect to a given timelike Killing vector $\partial/\partial t$, i.e., such that $\mathcal{L}_{\partial/\partial t} f_\omega(x) = -i\omega f_\omega(x)$ with $\omega>0$. Then it is possible to define a positive definite inner product between these solutions. The label ω is continuous and stands for the energy (or frequency) and x stands for the spacetime coordinates. Since these solutions have a well defined value for the energy (or frequency) we can call them *monochromatic modes*. It is true that since the energy of the monochromatic modes is well defined their space localization is completely uncertain in concordance with Heisenberg's uncertainty principle.

Now we want to work with a complete and orthonormal set of modes that are localized in space in some sense. To achieve this we can make an adequate superposition, within a small energy range, of continuous ω-labeled monochromatic modes in order to introduce an uncertainty in the energy and gain a certain information on spatial localization. We can do this as follows [14,17,18]: define

$$f_{\tilde\omega,n}(x)=\frac{1}{\sqrt{\epsilon}}\int_{\tilde\omega}^{\tilde\omega+\epsilon} d\omega\, e^{-in\omega} f_\omega(x) , \tag{1}$$

where the new labels $\tilde\omega$ and n are restricted to verify that $\tilde\omega/\epsilon\equiv j$ and $n\epsilon/2\pi\equiv l$ are integers, ϵ being a small and positive parameter. We will call this superposition a *wave packet*. Here n is a kind of Fourier label [the phase term $\exp(-in\omega)$ has l periods in the interval $(\tilde\omega,\tilde\omega+\epsilon)$], and $\tilde\omega$ is the lower extremum of the integration interval and gives information on the energy of the wave packet. It can be shown easily that a set of discrete $(\tilde\omega,n)$-labeled wave packets $\{f_{\tilde\omega,n}(x)\}$ is complete and orthonormal if the set of continuous ω-labeled monochromatic modes $\{f_\omega(x)\}$ is complete and orthonormal.

It is worth noticing that a general set of wave packets, $\{f_{\tilde\omega,n}(x)\}$, given by (1), satisfies the property

$$\sum_{j,l} |f_{\tilde\omega,n}(x)|^2=\int_0^\infty d\omega |f_\omega(x)|^2 , \tag{2}$$

which follows from the equality

$$\sum_{l=-\infty}^{\infty} e^{\pm in(\omega-\omega')}=\epsilon\delta(\omega-\omega') . \tag{3}$$

Note that the sums in (2) and (3) are over the integer labels (j,l) which could also be used to label the modes, i.e., $f_{j,k}(x)$, instead of the labels $(\tilde\omega,n)$ which we will used throughout.

We can now see in what sense the wave packets (1) are localized. Let us consider the generic wave packet (1) in terms of the modulus and phase of $f_\omega(x)$, i.e., $f_\omega(x)=|f_\omega(x)|e^{i\theta_\omega(x)}$, as

$$f_{\tilde\omega,n}(x)=\frac{1}{\sqrt{\epsilon}}\int_{\tilde\omega}^{\tilde\omega+\epsilon} d\omega\, e^{-in\omega+i\theta_\omega(x)}|f_\omega(x)| . \tag{4}$$

If the interval of integration in (1) is small enough so that $|f_\omega(x)|$ can be taken as approximately constant [this will be true in general provided that $|f_\omega(x)|$ has no singularities in the interval of integration], the $|f_\omega(x)|$ can be factorized out and we have an integral over the phase only. Now if the integrand's phase, i.e., $\Theta_\omega(x)\equiv -n\omega+\theta_\omega(x)$, oscillates rapidly over the range of integration (at least when $n\omega$ is big enough, and this is true for $l\gg 1$), then the integral roughly vanishes except at the stationary phase points, that is, when

$$\left.\frac{\partial\Theta_\omega(x)}{\partial\omega}\right|_{\omega=\tilde{\omega}}=0\ , \tag{5}$$

and we can use the stationary phase method [19,20] to evaluate (4). Note that in the equation (5) we have set $\omega=\tilde{\omega}$, after derivation, which is accurate provided the range of energy superposition is small enough. Equation (5) establishes a relation between the labels $(\tilde{\omega},n)$ of the wave packet and the spacetime coordinates x and so it represents a three-dimensional hypersurface. It determines the geometric locus of the spacetime points which give the main contribution to the integral (1); this will be a spacetime region labeled by $\tilde{\omega}$ and n and we will identify it as the *wave packet trajectory*, because outside this region the integral (1) roughly vanishes. It is worth noticing that if the monochromatic modes are labeled with some other continuous parameters, in addition to the energy, we can construct double or triple wave packets by superposition of monochromatic modes in a small range of these parameters. In this case we obtain surfaces or curves as the trajectories of the double or triple wave packet, respectively.

Finally we can make the following remark about the uncertainty of the energy and position of a wave packet. When we construct a wave packet we take a superposition of Klein-Gordon solutions with a well defined energy ω (the monochromatic modes) over a small interval $(\omega,\omega+\epsilon)$, and so the uncertainty in the energy of the wave packet is $\Delta\omega\simeq\epsilon$. To know the uncertainty in the position of the wave packet let us assume the following simple form for the monochromatic modes $f_\omega(x)=e^{-i\omega t}F_\omega(x^j)$ (where $x^j, j=1,2,3$, are space coordinates) which verify $\mathcal{L}_{\partial/\partial t}f_\omega(x)=-i\omega f_\omega(x)$. Then with the use of the stationary phase method we easily see that the wave packets (4) are peaked around values of time given by $t=-2\pi l\epsilon^{-1}$, with width $2\pi\epsilon^{-1}$ (l is an integer), and so the time uncertainty is $\Delta t\simeq 2\pi\epsilon^{-1}$, which reflects Heisenberg's uncertainty principle $\Delta\omega\Delta t\simeq 2\pi$.

III. COLLIDING PLANE WAVES GEOMETRY

We will work in a spacetime that describes the head on collision of two linearly polarized gravitational plane waves propagating in the z direction. This spacetime has four regions (see Fig. 1): a flat region (or region IV) at the past, before the arrival of the waves, two plane wave regions (regions II and III), and an interaction region (region I) where the waves collide and interact nonlinearly. The geometry of these regions is given by the following four metrics, in coordinates which are adapted to the Killing vectors ∂_x and ∂_y of the spacetime by (see Ref. [2] and references therein):

$$ds_{\rm I}^2=4L_1L_2[1+\sin(u+v)]^2du\,dv-\frac{1-\sin(u+v)}{1+\sin(u+v)}dx^2-[1+\sin(u+v)]^2\cos^2(u-v)dy^2\ , \tag{6}$$

$$ds_{\rm II}^2=4L_1L_2[1+\sin(u)]^2du\,dv-\frac{1-\sin(u)}{1+\sin(u)}dx^2-[1+\sin(u)]^2\cos^2(u)dy^2\ , \tag{7}$$

$$ds_{\rm III}^2=4L_1L_2[1+\sin(v)]^2du\,dv-\frac{1-\sin(v)}{1+\sin(v)}dx^2-[1+\sin(v)]^2\cos^2(v)dy^2\ , \tag{8}$$

$$ds_{\rm IV}^2=4L_1L_2du\,dv-dx^2-dy^2\ , \tag{9}$$

where u and v are two dimensionless null coordinates ($v+u$ is a time coordinate and $v-u$ a space coordinate) and L_1L_2 are two arbitrary positive length parameters, which represent the inverse of the strength (focusing time) of the waves. The boundaries of these four regions are $\{u=0,v\le 0\}$ between regions IV and II, $\{v=0,u\le 0\}$ between regions IV and III, $\{v=0,0\le u<\pi/2\}$ between regions II and I, and $\{u=0, 0\le v<\pi/2\}$ between regions III and I.

At the surfaces $u=\pi/2$ and $v=\pi/2$ on regions II and III, respectively, the determinants of the respective metrics vanish; this marks the focusing points of the waves and a coordinate singularity. This singularity can be avoided with the use of appropriate coordinates (harmonic coordinates) in which the causal structure of these spacetime regions is well posed. In these coordinates the surfaces $u=\pi/2$ and $v=\pi/2$ become spacetime lines (see [9,10] for details).

Region I (in Fig. 1, this is the triangle bounded by the lines $\{v=0,0\le u<\pi/2\}$, $\{u=0,0\le v<\pi/2\}$, and $u+v=\pi/2$) is locally isometric to a region of the interior of the Schwarzschild metric. This is easily seen with the coordinate transformation

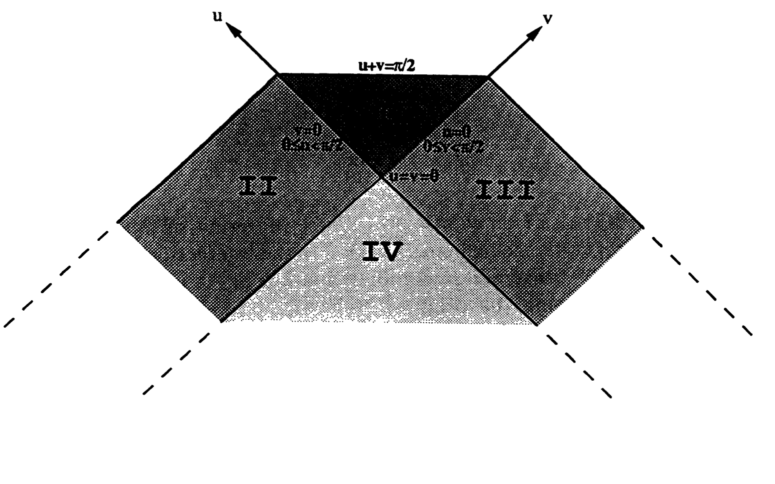

FIG. 1. Projection of the colliding plane wave spacetime in the (u,v) plane. One can see four regions: region IV is the flat region before the waves collide, regions II and III are the plane wave regions, and region I is the interaction region. The interaction starts at $u=v=0$, the lines $u=0$, $v<0$ and $v=0$, $u<0$ are the boundaries of region IV with regions II and III, respectively, and $v=0,0\le u<\pi/2$ and $u=0,0\le v<\pi/2$ are the boundaries of region I with regions II and III, respectively. At $u+v=\pi/2$ a Killing-Cauchy horizon is formed which one can see as a coordinate singularity of the metric in region I.

$$t=x,\quad r=M[1+\sin(u+v)]\ ,$$

$$\varphi=1+y/M,\quad \theta=\pi/2-(u-v)\ ,$$

where we have defined $M=\sqrt{L_1L_2}$. The metric (6) becomes

$$ds^2=\left[\frac{2M}{r}-1\right]^{-1}dr^2-\left[\frac{2M}{r}-1\right]dt^2$$
$$-r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ ,$$

which is the interior of the Schwarzschild metric. The surface $u+v=\pi/2$ corresponds to the black hole event horizon. The boundary $v=0$ corresponds to $r=M(1+\cos\theta)$ and $u=0$ corresponds to $r=M(1-\cos\theta)$. These are the boundaries of the plane waves; these boundaries join at $r=M$ (spacetime point of the collision) and also at the surface $u+v=\pi/2$ at $\theta=0$ and $\theta=\pi$. This region of the Schwarzschild interior does not include the singularity $r=0$ and thus the interaction region has no curvature singularities. The above local isometry is not global, however; the coordinates θ and ϕ are cyclic in the black hole case but, in the plane wave case, $-\infty<y<\infty$ and $-\infty<v-u<\infty$.

As in the Schwarzschild case, it is convenient to introduce a set of Kruskal-Szekeres-like coordinates to describe the interaction region, because the (u,v,x,y) coordinates become singular at the horizon. These coordinates will play an important role in the quantization of the field. First we introduce dimensionless time and space coordinates (ξ,η):

$$\xi=u+v,\quad \eta=v-u\ , \tag{10}$$

with the range $0\le\xi<\pi/2$, $-\pi/2\le\eta<\pi/2$ (we shall later see that in these coordinates the Klein-Gordon equation can be separated). Then we introduce a new time coordinate ξ^* related to the dimensionless time coordinate ξ by

$$\xi^*=2M\ln\left[\frac{1+\sin\xi}{2\cos^2\xi}\right]-M(\sin\xi-1)\ , \tag{11}$$

and a new set of null coordinates

$$\tilde U=\xi^*-x,\quad \tilde V=\xi^*+x\ . \tag{12}$$

Note that the transversal coordinate x appears in the coordinate transformation because it behaves badly at the horizon. Finally, we define

$$U'=-2M\exp\left[-\frac{\tilde U}{4M}\right]\le 0\ ,$$
$$V'=-2M\exp\left[-\frac{\tilde V}{4M}\right]\le 0\ , \tag{13}$$

and the metric in the interaction region (6) reads

$$ds_{\rm I}^2=\frac{2\exp[(1-\sin\xi)/2]}{1+\sin\xi}dU'dV'-M^2(1+\sin\xi)^2d\eta^2$$
$$-(1+\sin\xi)^2\cos^2\eta\, dy^2\ ,$$

with

$$U'V'=8M^2\frac{\cos^2\xi}{1+\sin\xi}\exp\left[\frac{\sin\xi-1}{2}\right]\ , \tag{14}$$

$$\frac{U'}{V'}=\exp\left[\frac{x}{2M}\right]\ . \tag{15}$$

The curves $\xi=$const and $x=$const are, respectively, hyperbolas and straight lines through the origin of coordinates ($U'=V'=0$), see Fig. 2. The Schwarzschild horizon (which is a Killing-Cauchy horizon for the spacetime) corresponds to the limit of the hyperbolas when $\xi\to\pi/2$, i.e., the "roofs" $V'=0$ or $U'=0$. Notice that the problem with the transversal coordinate x at the horizon is that all the lines $x=$const go through the origin of the (U',V') coordinates, so that all the range of x collapses into the point $V'=U'=0$, whereas the lines $U'=0$ and $V'=0$ represent $x=-\infty$ and $x=\infty$, respectively. One should recall that we have not represented the coordinate x in our picture of the collision (Fig. 1) in which only the (u,v) coordinates are shown; x is a transversal coordinate perpendicular to the propagation and adapted to the Killing vector ∂_x.

To understand the global geometry of the spacetime one needs a tridimensional picture where the boundary surfaces between the different regions have to be written in terms of appropriate nonsingular coordinates adapted to each region. See [9,10] for details.

It is worth noticing that most of the plane wave collisions produce true curvature singularities [2], but the spacetime described above is an example of a collision where the curvature singularity has been substituted by a Killing-Cauchy horizon (i.e., the surface $U'V'=0$ in the Kruskal-Szekeres-like coordinates).

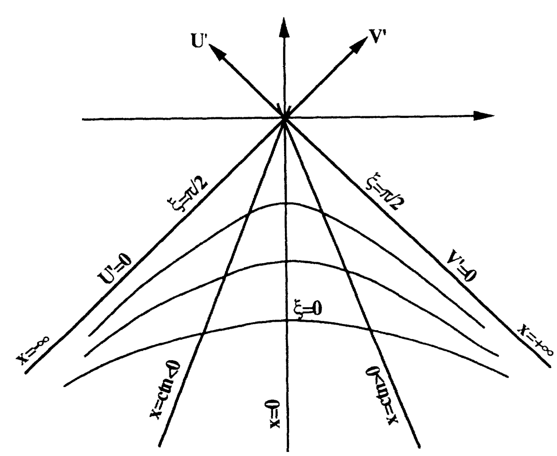

FIG. 2. The coordinates (ξ,x) in terms of the Kruskal-Szekeres-like coordinates (U',V') in the interaction region. The lines $\xi=$const are hyperbolas and the $x=$const are straight lines crossing the origin $U'=V'=0$. The Cauchy horizon is $\{U'=0,V'<0\}\cup\{V'=0,U'<0\}$ which corresponds to the limit of the hyperbolas as $\xi\to0$. The "roof" $U'=0$ corresponds to $x\to-\infty$ and the "roof" $V'=0$ to $x\to\infty$.

IV. WAVE PACKETS IN THE COLLISION OF TWO GRAVITATIONAL PLANE WAVES

A. Monochromatic mode quantization (summary)

In Ref. [9] the quantization of a massless scalar field ϕ was considered in the spacetime described above representing the head on collision of two gravitational plane waves. Let us now briefly summarize this quantization scheme. We introduce a massless scalar field ϕ on the colliding spacetime background, which satisfies the Klein-Gordon equation

$$\Box\phi=0 , \tag{16}$$

where

$$\Box\phi=(-g)^{-1/2}[(-g)^{1/2}g^{\mu\nu}\phi_{,\nu}]_{,\mu} ,$$

and g is the determinant of the metric. It has the generic plane wave solution

$$\phi=\frac{1}{\sqrt{FG}}f(u,v)e^{ik_x x+ik_y y} , \tag{17}$$

where the labels k_x and k_y are two separation constants which are interpreted physically as the momenta in the directions given by the Killing vectors ∂_x and ∂_y, which generate the plane symmetry of the whole spacetime. Introducing (17) in (16) one obtains

$$f_{,uv}-\left[\frac{(\sqrt{FG})_{,uv}}{\sqrt{FG}}-\frac{e^{-N}}{4}\left[\frac{k_x^2}{F^2}+\frac{k_y^2}{G^2}\right]\right]f=0 , \tag{18}$$

where the coefficients F, G, and e^{-N} come from the adoption of a generic metric for all colliding plane wave spacetime, adapted to ∂_x and ∂_y, i.e.,

$$ds^2=e^{-N(u,v)}du\,dv-F^2(u,v)dx^2-G^2(u,v)dy^2 . \tag{19}$$

One defines an initial vacuum state constructed with a complete orthonormal set of solutions of the Klein-Gordon equations (16) which are defined in the flat IV region, before the arrival of the plane waves, to be of positive frequency with respect to the timelike Killing vector ∂_{u+v}; these are the "in" monochromatic modes. Then one propagates these modes throughout the spacetime up to the horizon of region I (solving the appropriate boundary conditions imposed by the different classes of solutions for the Klein-Gordon equation (18) in the different regions of the spacetime). It is possible to define another "natural" vacuum state on the horizon of region I from a complete and orthonormal set of solutions of the Klein-Gordon equation (16), the "out" monochromatic modes, which are positive frequency solutions with respect to the vectors $\partial/\partial U'$ and $\partial/\partial V'$, which are two null Killing fields over the horizon [21].

Comparing the propagated "in" monochromatic modes and the "out" monochromatic modes on the horizon via a Bogoliubov transformation leads us to show that there is spontaneous creation of particles in this spacetime with a spectrum of "out" particles given by the formula [9]

$$\langle 0,\text{in}|N^{\text{out}}_{\omega-}|0,\text{in}\rangle=\frac{(2M)^3}{\pi}\frac{\delta_{m,-\bar{m}}}{8\pi M\omega_-}\int d\hat{k}_-\int dk\frac{|C_{l'}|^2}{e^k-1} . \tag{20}$$

Here $N^{\text{out}}_{\omega-}$ is the usual number operator for "out" particles, ω_-, l', and m' are labels of the "out" modes (ω_- is the energy label), $\hat{k}_-$, k, and m are dimensionless labels of the "in" modes, and $(2M)^3/\pi|C_{l'}|^2$ is a geometric factor whose coefficient $|C_{l'}|$ depends on $l', m', \hat{k}_-$, and k. This spectrum is inversely proportional to the inverse of the energy of the "out" particles which are produced and it is consistent, in the long-wavelength limit, i.e., $8\pi M\omega_- \ll 1$, with a thermal spectrum with temperature $T=(8\pi M)^{-1}$. The temperature is inversely proportional to the focusing time of the plane waves, given by the parameter $M=\sqrt{L_1L_2}$, and $(4M)^{-1}$ is also the surface gravity of the horizon. Note that the spectrum contains a logarithmic divergence; this divergence appears because the spectrum which has units of (length)^3, given by the factor $(2M)^3$, describes the total number of particles created in the whole spacetime volume. As is well known [22] this is characteristic of the use of continuous labeled modes and can be avoided using wave packets.

The propagation of the "in" monochromatic modes throughout the spacetime is a difficult task. This is due, essentially, to the fact that the monochromatic modes are defined in the whole spacetime and there is not a single singularity-free coordinate chart for all the four spacetime regions. Basically the problem is that the matching has to be done through all the spacetime points of the boundaries. Because of their spatial localization wave packet modes will be more easily matched. The matching will be done approximately on a single boundary point for each wave packet.

In what follows we introduce the wave packet formalism in the colliding spacetime background. As the first step we construct a complete set of "in" wave packets, from a superposition of positive frequency monochromatic "in" modes, in the four spacetime regions. Then we will propagate these "in" wave packets throughout the spacetime up to the horizon, with appropriate matching conditions. These matching conditions will ensure that the trajectory of the "in" wave packet will be smoothly connected through the boundaries between the different regions. Next we will construct a complete set of "out" wave packets on the surface of the horizon. Finally, we will relate the "in" and "out" wave packets via Bogoliubov transformation and compute the creation of particles in this formalism.

B. Flat region (region IV)

In the flat region the complete set of "in" modes is

$$u^{(\text{IV})}_{k_xk_yk_-}(u,v,x,y)=\frac{1}{\sqrt{2k_-(2\pi)^3}}e^{-ik_-v'-ik_+u'+ik_xx+ik_yy} , \tag{21}$$

where the labels k_x, k_y, and k_- are independent separation constants for the Klein-Gordon equation (16), and where u', v' are two-dimensional null coordinates related to the dimensionless null coordinates u, v by $v' = 2L_2 v, u' = 2L_1 u$. The label k_+ is determined by the relation

$$4k_+ k_- = k_x^2 + k_y^2 \ . \tag{22}$$

It was shown in Ref. [9] that these modes are well normalized on the hypersurface $\{u=0, v\le 0\} \bigcup \{v=0, u \le 0\}$.

The labels k_x and k_- are continuous but k_y is discrete if we take a cyclic spacetime in the y direction (this is not necessary but it is convenient if we want to maximally extend the colliding wave spacetime later on). We identify k_y with m/M where m is an integer. Our aim now is to discretize the continuous labels k_x and k_- by constructing a *double wave packet* as

$$u^{(\mathrm{IV})}_{\tilde{k}_x, n, k_y, \tilde{k}_-, q}(u,v,x,y) = \frac{1}{\sqrt{\epsilon\delta}} \int_{\tilde{k}_x}^{\tilde{k}_x+\epsilon} dk_x \int_{\tilde{k}_-}^{\tilde{k}_-+\delta} dk_- \frac{e^{-iqk_- - ink_x}}{\sqrt{2k_-(2\pi)^3}} e^{-ik_- v' - ik_+ u' + ik_x x + ik_y y} \ . \tag{23}$$

Note that, since we are only integrating over positive frequency monochromatic modes, the vacuum associated with the wave packets is the same as that defined by the monochromatic modes (21).

The integrand's phase is

$$\Theta^{(\mathrm{IV})} = -nk_x - qk_- - k_- v' - \left[\frac{k_x^2 + k_y^2}{4k_-} \right] u' + k_x x + k_y y \ , \tag{24}$$

where the relation (22) has been used. The trajectory of the double wave packet, in the sense given by the stationary phase method, i.e.,

$$\partial\Theta^{(\mathrm{IV})}/\partial k_x = \partial\Theta^{(\mathrm{IV})}/\partial k_- = 0 \ ,$$

is given by

$$x^{(\mathrm{IV})} = n + \frac{\tilde{k}_x}{\tilde{k}_-} L_1 u \ , \tag{25}$$

$$v'^{(\mathrm{IV})} = -q + \frac{\tilde{k}_x^2 + \tilde{k}_y^2}{2\tilde{k}_-^2} L_1 u \ , \tag{26}$$

which define a null geodesic in the flat region. That is, this double wave packet moves on a null trajectory.

C. Plane wave region (region II)

The complete set of solutions of the Klein-Gordon equation (16) in region II is easily found (see Ref. [9]) and is given by

$$u^{(\mathrm{II})}_{k_x k_y k_-}(u,v,x,y) = \frac{1}{\sqrt{2k_-(2\pi)^3}} \frac{1}{\cos u} \times \exp\left[-\frac{iL_1}{2k_-}[f(u)k_x^2 + g(u)k_y^2] - ik_- v' + ik_x x + ik_y y \right] \ , \tag{27}$$

where the two functions $f(u)$ and $g(u)$ are

$$f(u) = \frac{(1+\sin u)^2}{2\cos u}(9 - \sin u) + \tfrac{15}{2}\cos u - \tfrac{15}{2}u - 12 \ ,$$
$$g(u) = \tan u \ . \tag{28}$$

The labels k_x, k_y, and k_- have the same meaning as in the flat region IV because this expression for the solutions of the Klein-Gordon equation in region II matches smoothly (i.e., in a continuous and differentiable way) with the respective solutions (21) on the boundary between regions II and IV, i.e., $\{u=0, v\le 0\}$.

In analogy with the previous case, we now construct a double wave packet

$$u^{(\mathrm{II})}_{\tilde{k}_x, n, k_y, \tilde{k}_-, q}(u,v,x,y) = \frac{1}{\sqrt{\epsilon\delta}} \int_{\tilde{k}_x}^{\tilde{k}_x+\epsilon} dk_x \int_{\tilde{k}_-}^{\tilde{k}_-+\delta} dk_- \frac{e^{-iqk_- - ink_x}}{\sqrt{2k_-(2\pi)^3}} \times \frac{1}{\cos u} \exp\left[-i\frac{L_1}{2k_-}[f(u)k_x^2 + g(u)k_y^2] - ik_- v' + ik_x x + ik_y y \right] \ . \tag{29}$$

The integrand's phase is given by

$$\Theta^{(\mathrm{II})} = -nk_x - qk_- - k_- v' - [f(u)k_x^2 + g(u)k_y^2]\frac{L_1}{2k_-} + k_x x + k_y y \ , \tag{30}$$

and the trajectory of the double wave packet, i.e.,

$$\partial\Theta^{(\mathrm{II})}/\partial k_x = \partial\Theta^{(\mathrm{II})}/\partial k_- = 0 \ ,$$

is

$$x^{(\mathrm{II})}=n+\frac{\tilde{k}_x}{\tilde{k}_-}L_1 f(u)\ , \tag{31}$$

$$v'^{(\mathrm{II})}=-q+[f(u)\tilde{k}_x^2+g(u)\tilde{k}_y^2]\frac{L_1}{2\tilde{k}_-^2}\ , \tag{32}$$

which represents a null geodesic traveling throughout the plane wave region II. Note that this trajectory matches smoothly (i.e., in a continuous and differentiable way) with the trajectory of the wave packet in region IV, i.e., (25) and (26), on the boundary $\{u=0, v\le 0\}$. Since $f(0)=g(0)$ and $df/du(0)=dg/du(0)=0$ the following matching conditions are satisfied:

$$x^{(\mathrm{IV})}(u=0)=x^{(\mathrm{II})}(u=0)\ ,$$

$$v'^{(\mathrm{IV})}(u=0)=v'^{(\mathrm{II})}(u=0)\ ,$$

$$\left.\frac{dx^{(\mathrm{IV})}}{du}\right|_{u=0}=\left.\frac{dx^{(\mathrm{II})}}{du}\right|_{u=0}\ ,$$

$$\left.\frac{dv'^{(\mathrm{IV})}}{du}\right|_{u=0}=\left.\frac{dv'^{(\mathrm{II})}}{du}\right|_{u=0}\ .$$

This is not surprising because the monochromatic modes which we have used to define the double wave packets in the two regions match smoothly on the boundary $\{u=0, v\le 0\}$.

Before going further, we can extract some information on the meaning of the new discrete labels n and q. First, note that at the boundary between regions II and IV, $\{u=0, v\le 0\}$, we have from (25), (26) and (31), (32),

$$x_0\equiv x^{(\mathrm{IV})}(0)=x^{(\mathrm{II})}(0)=n\ ,$$

$$v_0'\equiv v'^{(\mathrm{IV})}(0)=v'^{(\mathrm{II})}(0)=-q\ ,$$

which means that a wave packet labeled by n and q has a peak on the null geodesic which crosses the flat region into the plane wave region at the spacetime coordinates $x_0=n$ and $v_0'=-q$. Note that for wave packets in region III we can repeat the same discussion.

D. Interaction region (region I)

In this region mode propagation is more difficult; instead of relying on the calculations given in Ref. [9], we will start the discussion from the beginning. First, let us consider that the Klein-Gordon equation (16) in this region can be separated by taking

$$\phi(\xi,\eta,x,y)=e^{ik_x x+ik_y y}\psi_{\alpha k_x}(\xi)\varphi_{\alpha k_y}(\eta)\ , \tag{33}$$

and reduced to equations for $\psi_{\alpha k_x}(\xi)$ and $\varphi_{\alpha k_y}(\eta)$. The coordinates ξ,η are related to the usual null coordinates u,v by (10), and the two new equations read

$$\psi_{,\xi\xi}-(\tan\xi)\psi_{,\xi}+\left[\alpha+\frac{\hat{k}_x^2}{4}\frac{(1+\sin\xi)^4}{\cos^2\xi}\right]\psi=0\ , \tag{34}$$

$$\varphi_{,\eta\eta}-(\tan\eta)\varphi_{,\eta}+\left[\alpha-\frac{\hat{k}_y^2}{4}\frac{1}{\cos^2\eta}\right]\varphi=0\ , \tag{35}$$

where α is a dimensionless separation constant and k_x,k_y are the same labels as in regions IV or II. We use the notation $\hat{a}=2Ma$; therefore $\hat{k}_x,\hat{k}_y$ are dimensionless parameters.

These differential equations have singular points at $\xi=\pi/2$, i.e., on the horizon of region I, and at $\eta=\pm\pi/2$ respectively. To avoid them we can perform the following change of variables:

$$d\xi^*=M(1+\sin\xi)^2\arccos\xi\, d\xi,\quad d\eta^*=M\arccos\eta\, d\eta\ .$$

With appropriate integration constants, ξ^* reduces to (11) and η^* to

$$\eta^*=M\ln\left[\frac{1+\sin\eta}{\cos\eta}\right]\ . \tag{36}$$

We can still introduce a new function $\gamma_{\alpha k_x}(\xi)$ instead of $\psi_{\alpha k_x}(\xi)$ by $\gamma=\frac{1}{2}(1+\sin\xi)\psi$, and we arrive at the following equations for $\gamma_{\alpha k_x}(\xi)$ and $\varphi_{\alpha k_y}(\eta)$:

$$\gamma_{,\xi^*\xi^*}+\left[k_x^2+\frac{\cos^2\xi}{M^2(1+\sin\xi)^4}\left[\alpha+\frac{2\sin\xi}{1+\sin\xi}\right]\right]\gamma=0\ , \tag{37}$$

$$\varphi_{,\eta^*\eta^*}+\left[-k_y^2+\frac{\cos^2\eta}{M^2}\alpha\right]\varphi=0\ , \tag{38}$$

which have no singular points in the regions of interest.

Recall that we are looking for a solution of the scalar field ϕ in the interaction region, restricted to satisfy certain boundary conditions imposed by the wave packets traveling from regions II (or III) into region I on the boundary $\{v=0, 0\le u<\pi/2\}$ (or $\{u=0, 0\le v<\pi/2\}$). Thus the general solution for ϕ in terms of the new functions $\gamma_{\alpha k_x}(\xi),\varphi_{\alpha k_y}(\eta)$ is given by

$$\phi(\xi,\eta,x,y)=e^{ik_x x+ik_y y}\sum_\alpha C_\alpha\frac{2\gamma_{\alpha k_x}(\xi)}{1+\sin\xi}\varphi_{\alpha k_y}(\eta)\ , \tag{39}$$

where the coefficients C_α depend on α and the separation constants used to label the monochromatic modes in region II, i.e., k_x,k_y, and k_-. Now we will try to obtain all possible information on the coefficients C_α in (39). We know that expression (39) gives the general solution (once k_x and k_y are fixed) for a massless scalar field in the interaction region, and the coefficients C_α have to be such that the appropriate boundary conditions are satisfied. If we want to match the monochromatic modes defined in region I with those of region II (27) (or III) we have to perform a sum over all possible values of α in (39); this was one of the main difficulties in Ref. [9]. In the following discussion we will show that when we match the spatially localized wave packets traveling from region II (or III) into region I (29) with wave packets defined using the general solution (39), there is only one coefficient C_α which carries the main contribution. That is, we will find a single value of α for which the infinite linear combination (39) can be approximated by a single term. Furthermore, we will be able to find the value of the coefficient C_α for that particular value of α.

We will proceed as follows. First, we will find an approximate solution of Eqs. (37) and (38) for $\gamma_{\alpha k_x}(\xi)$ and $\varphi_{\alpha k_y}(\eta)$, respectively, near the boundary of region I with region II, which will lead to an approximate solution of (39) near this boundary. With this solution we will construct a wave packet in analogy with (23) or (29) and then we will find the particular value of α and the phase of the coefficient C_α which allow the matching of this wave packet with the wave packet (29) traveling from region II. Next, we will find an approximate expression for the general solution (39) near the horizon of region I, i.e., the surface $\xi=\pi/2$, with the particular values of α and the phase of C_α calculated before. Such a solution becomes exact on the surface of the horizon and it will be used to construct a wave packet there.

Let us start with the approximate solution of (39) near the boundary between regions II and I. First note that the solution ϕ of the Klein-Gordon equation in the plane wave regions (regions II or III) takes an exact WKB form; i.e., it can be written as $\phi=C\exp(iS)$, where C and S are two real functions of the spacetime coordinates. This is directly related to the fact that the geometrical optics approximation is exact in the single plane wave regions; i.e., the rays of the Klein-Gordon solutions (the lines perpendicular to the constant phase surfaces) follow null geodesics. Thus we can expect that an approximate solution of the massless scalar field in the interaction region close to the boundaries with the plane wave regions can be obtained with the WKB method; this is physically related to the fact that near the boundaries the colliding plane waves superpose linearly. In fact, from Eqs. (37) and (38) we can see that they admit WKB solutions given by (see, for example, [20])

$$\gamma_{\rm WKB}(\xi^*)=\frac{C}{\sqrt{Q(\xi^*)}}\exp\left[\pm i\int_{\xi_0^*}^{\xi^*}Q(\xi^*)d\xi^*\right], \tag{40}$$

$$\varphi_{\rm WKB}(\eta^*)=\frac{C}{\sqrt{R(\eta^*)}}\exp\left[\pm i\int_{\eta_0^*}^{\eta^*}Q(\eta^*)d\eta^*\right], \tag{41}$$

where $Q(\xi^*)$ and $R(\eta^*)$ are

$$Q^2(\xi^*)=k_x^2+\frac{\cos^2\xi}{M^2(1+\sin\xi)^4}\left[\alpha+\frac{2\sin\xi}{1+\sin\xi}\right], \tag{42}$$

$$R^2(\eta^*)=-k_y^2+\frac{\cos^2\eta}{M^2}\alpha\,. \tag{43}$$

These WKB solutions can be used provided $Q(\xi^*)$ and $R(\eta^*)$ do not vanish and they become accurate solutions when $Q(\xi^*)$ and $R(\eta^*)$ change slowly, i.e., when $|dQ(\xi^*)/d\xi^*|\ll Q^2(\xi^*)$ and $|dR(\eta^*)/d\eta^*|\ll R^2(\eta^*)$. Note that Eq. (37) admits a WKB approximation of this kind throughout region I. Furthermore, this approximation becomes asymptotically exact on the surface of the horizon of region I, i.e., $\xi=\pi/2$, and it is a really good approximation near the boundaries between regions I and II (or I and III) provided the separation constant α is large enough: in fact, we will show that to ensure the correct matching between wave packets traveling from region II into I (or from region III into I) with a single term in the infinite sum (39), it is necessary that the terms containing α in (42) and (43), which also contain factors $\cos^2\xi$ or $\cos^2\eta$, be dominant. Recall that when we approach the horizon, i.e., the surface $\xi=\pi/2$, on the boundary between regions I and II (or I and III), then both $\cos^2\xi$ and $\cos^2\eta$ go rapidly to zero. This means that α must be big enough to compensate the decreasing behavior of $\cos^2\xi$ and $\cos^2\eta$. Equation (38), on the other hand, admits a WKB approximation only near the boundary between regions I and II (or I and III) under the assumption of large α [this is because of the minus term in (43)]. Fortunately, we know its exact solution when it is written in the form (35); this solution is given in terms of the *associated Legendre polynomials* and that will be the solution we will take in the regions where the WKB approximation is not valid.

Let us now evaluate the form of $\gamma_{\alpha k_x}(\xi)$ and $\varphi_{\alpha k_y}(\eta)$ near the boundary between regions I and II, i.e., $\{\xi=-\eta, 0<\xi<\pi/2\}$. Although $Q(\xi^*)$ and $R(\eta^*)$, as given by (42) and (43), are the complete terms that appear in the WKB formulas, if we assume that α is large only the first two terms in powers of α^{-1} are relevant. With this expansion, performing the integrals in (40) and (41) and choosing the minus sign in the exponent of (40) and the plus sign in the exponent of (41), we have

$$\phi(\xi,\eta,x,y)=e^{ik_xx+ik_yy}\sum_\alpha C_\alpha\frac{1}{\sqrt{\cos\xi\cos\eta}}\frac{M}{\sqrt{\alpha}}\exp\left[i\sqrt{\alpha}[(\eta-\xi)-(\eta_0-\xi_0)]\right.$$
$$\left.-i\{k_x^2[f(\xi)-f(\xi_0)]+k_y^2[g(\eta)-g(\eta_0)]\}\frac{M^2}{2\sqrt{\alpha}}+O(\alpha^{-3/2})\right], \tag{44}$$

where the functions f and g are given by (28).

Next we construct a *double wave packet* as

$$\phi^{(\mathrm{I})}_{\bar{k}_x,n,k_y,\bar{k}_-,q}(\xi,\eta,x,y)=\frac{1}{\sqrt{\epsilon\delta}}\int_{\bar{k}_x}^{\bar{k}_x+\epsilon}dk_x\int_{\bar{k}_-}^{\bar{k}_-+\delta}dk_-\,e^{-iqk_--ink_x}\times\phi^{(\mathrm{I})}_{k_xk_yk_-}(\xi,\eta,x,y)\,, \tag{45}$$

and write the coefficient C_α and phase by

$$C_\alpha\equiv|C_\alpha|e^{i\theta_\alpha}\,. \tag{46}$$

The integrand's phase in (45) is then

$$\tilde{\Theta}^{(\mathrm{I})}=-qk_{-}-nk_x+k_xx+k_yy+\theta_\alpha-2(u-u_0)\sqrt{\alpha}$$
$$-\{k_x^2[f(\xi)-f(\xi_0)]+k_y^2[g(\eta)-g(\eta_0)]\}\frac{M^2}{2\sqrt{\alpha}}$$
$$+O(\alpha^{-3/2})\ . \tag{47}$$

Now $\tilde{\Theta}^{(\mathrm{I})}$ provides us with a natural way of matching the wave packets in region II and the wave packets in region I. Let us assume that the matching is good enough if the trajectories of the wave packets in regions II and I are joined in a continuous and differentiable way on the point $(u=u_0,v=0)$ of the boundary between these regions. This means that $\tilde{\Theta}^{(\mathrm{I})}$ must satisfy the three constraints

$$\tilde{\Theta}^{(\mathrm{I})}|_{(u=u_0,v=0)}=\Theta^{(\mathrm{II})}|_{(u=u_0,v=0)}\ , \tag{48}$$

$$\left.\frac{\partial\tilde{\Theta}^{(\mathrm{I})}}{\partial u}\right|_{(u=u_0,v=0)}=\left.\frac{\partial\Theta^{(\mathrm{II})}}{\partial u}\right|_{(u=u_0,v=0)}\ , \tag{49}$$

$$\left.\frac{\partial\tilde{\Theta}^{(\mathrm{I})}}{\partial v}\right|_{(u=u_0,v=0)}=\left.\frac{\partial\Theta^{(\mathrm{II})}}{\partial v}\right|_{(u=u_0,v=0)}\ . \tag{50}$$

Using (47) and (30), these constraints give, respectively,

$$\theta_\alpha=-[k_x^2f(u_0)+k_y^2g(u_0)]\frac{L_1}{2k_{-}}\ , \tag{51}$$

$$2\sqrt{\alpha}+[k_x^2\dot{f}(u_0)-k_y^2\dot{g}(u_0)]\frac{M^2}{2\sqrt{\alpha}}$$
$$=[k_x^2\dot{f}(u_0)+k_y^2\dot{g}(u_0)]\frac{L_1}{2k_{-}}\ , \tag{52}$$

$$[k_x^2\dot{f}(u_0)+k_y^2\dot{g}(u_0)]\frac{M^2}{2\sqrt{\alpha}}=2L_2k_{-}\ , \tag{53}$$

where $\dot{f}=df/du$, $\dot{g}=dg/du$ are, using (28),

$$\dot{f}(x)=\frac{(1+\sin x)^4}{\cos^2 x},\quad \dot{g}(x)=\frac{1}{\cos^2 x}\ . \tag{54}$$

Note that the constraint (51) gives the value of the phase θ_α for the coefficient C_α in (39), and the constraints (52) and (53) define the value of α. These two constraints are compatible; this is not surprising because we have already noticed that the WKB approximation is the natural approximation near those boundaries. The value of α is

$$\sqrt{\alpha}\simeq[k_x^2\dot{f}(u_0)+k_y^2\dot{g}(u_0)]\frac{L_1}{4k_{-}}\ , \tag{55}$$

where we assume an expansion in even powers of $\cos u_0$ and we recall that $\dot{f}(u_0)$ and $\dot{g}(u_0)$ are of the order $(\cos u_0)^{-2}$. Then α goes like $(\cos u_0)^{-4}$; this justifies our assumption that $\alpha\cos^2\xi\gg 1$ or $\alpha\cos^2\eta\gg 1$ because both terms go like $(\cos u_0)^{-2}$ in the matching point $(u=u_0,v=0)$. Note that $(\cos u_0)^{-1}$ increases rapidly when u_0 differs from zero, but even when $u_0=0$ (55) with the use of (22) gives $\sqrt{\alpha}\simeq k_+L_1$ and the wave packet (45) still matches with (29).

It is important to note that this matching fixes the phase of the wave packet but does not ensure its normalization. In fact, the normalization condition fixes the modulus of the coefficient C_α, but it is convenient to postpone the calculation of this term until we have the form of the wave packet close to the horizon of region I, where the wave packet normalization condition is much simpler.

Our next step is to identify the integrand's phase defining the wave packet (45) near the horizon $\xi=\pi/2$. To do this we can still use the WKB approximation (40) of Eq. (37) near the horizon; this means that we can take, asymptotically,

$$\gamma_{\mathrm{WKB}}(\xi^*)=\frac{1}{\sqrt{|k_x|}e^{-i|k_x|\xi^*}}\ , \tag{56}$$

where we have chosen the minus sign in the phase in order to be consistent with (44). From (39) the scalar field ϕ is then given by

$$\phi^{(\mathrm{I})}(\xi,\eta,x,y)=\frac{e^{ik_yy}}{\sqrt{|k_x|}}\sum_\alpha C_\alpha\varphi_{\alpha k_y}(\eta)\times\begin{cases}e^{-i|k_x|(\xi^*-x)}, & k_x\ge 0,\\ e^{-i|k_x|(\xi^*+x)}, & k_x\le 0,\end{cases} \tag{57}$$

where $\xi^*-x=\tilde{U}$, $\xi^*+x=\tilde{V}$, $U'=-2M\exp(-\tilde{U}/2M)$, and $V'=-2M\exp(-\tilde{V}/2M)$, following Eqs. (12) and (13). Notice that, when $k_x\ge 0$, the scalar field reaches the "roof" $V'=0$ of the horizon (strictly speaking, the rays of ϕ, i.e., the lines normal to the constant phase surfaces of ϕ, reach the "roof" $V'=0$), and when $k_x\le 0$, they reach the "roof" $U'=0$ of the horizon. This asymptotic solution has been obtained because near to the horizon $\xi=\pi/2$ the dominant term in (42) is k_x^2. However, it is not possible to obtain such an asymptotic solution for Eq. (38), because of the minus term in (43). Fortunately, as we have said, we can go back to the untransformed equation (35) which is the equation for the *associated Legendre polynomials* $P_{\alpha k_y}(\sin\eta)$. Note that in the cyclic case $Mk_y=m$ is an integer and we can take $\alpha=l(l+1)$ for l integer; then $e^{ik_yy}P_{\alpha k_y}(\sin\eta)$ is proportional to the *spherical harmonic* $Y_l^m(y/M,\pi/2-\eta)$. The matching of the packets requires that α is given by (55); therefore we can taken $l=l_\alpha\simeq\sqrt{\alpha}$, where l_α is the closest integer to $\sqrt{\alpha}$ for large α. Now using the asymptotic form (57) of ϕ near the horizon we can construct a double packet as in (45). The integrand's phase defining the wave packet is given by

$$\Theta^{(\mathrm{I})}=-qk_{-}-nk_x+k_yy-k_x(-x\pm\xi^*)+\theta_\alpha\ , \tag{58}$$

where the upper sign in (58) stands for $k_x \geq 0$ and the lower sign stands for $k_x \leq 0$, and θ_α is the phase for the coefficient C_α given in (51). As we have said, the coefficients C_α are restricted to satisfy a normalization condition, to make sure that the scalar field ϕ is well normalized on the horizon. The appropriate inner product of any two solutions ϕ_1, ϕ_2 of the Klein-Gordon equation is given on the horizon by (see [9])

$$\langle \phi_1, \phi_2 \rangle = -i4M \int \cos\eta \, d\eta \, dy \left[\int_{-\infty}^{0} (\phi_1 \overleftrightarrow{\partial}_{V'} \phi_2^*) \Big|_{U'=0} dV' + \int_{-\infty}^{0} (\phi_1 \overleftrightarrow{\partial}_{U'} \phi_2^*) \Big|_{V'=0} dU' \right] . \quad (59)$$

Since, as we have just seen,

$$C_l(l, k_x, k_y, k_-) \equiv C_\alpha \propto e^{i\theta_\alpha} \delta(l - l(k_x, k_y, k_-))$$

where $l(k_x, k_y, k_-) = l_\alpha$ and θ_α is given by (51), the normalization condition with respect to (59) leads to

$$C_l(l, k_x, k_y, k_-) = \frac{e^{i\theta_\alpha}}{2M} \left[\frac{l_\alpha}{4\pi k_-} \right]^{1/2} \delta(l - l_\alpha) . \quad (60)$$

Finally with the asymptotic value (58) for the phase near the horizon ($\xi = \pi/2$) we can use $\partial\Theta^{(I)}/\partial k_x = \partial\Theta^{(I)}/\partial k_- = 0$ to obtain the trajectory of the wave packet near the horizon:

$$q = [\tilde{k}_x^2 f(u_0) + \tilde{k}_y^2 g(u_0)] \frac{L_1}{2\tilde{k}_-^2} , \quad (61)$$

$$x^{(I)} = n \pm \xi^* + \frac{\tilde{k}_x}{\tilde{k}_-} f(u_0) L_1 . \quad (62)$$

Notice that $n + (\tilde{k}_x/\tilde{k}_-) L_1 f(u_0) \equiv x^{(I)}(0)$ is the value of the coordinate x when the wave packet trajectory coming from region II crosses the boundary $v = 0$ into region I, as one can see from (31); then (62) reads

$$x^{(I)} = x^{(I)}(0) \pm \xi^* . \quad (63)$$

This is the equation for a null geodesic in a region close to the horizon as can be seen by writing it in the form $x^{(I)} \approx x^{(I)}(0) \mp \ln(\cos\xi)$, where $x^{(I)}(0)$ is a constant for a fixed wave packet [(11) has been used]. It is worth noticing that this is the equation for null geodesics close to the horizon as one can see in the Appendix of [9].

Equation (63) can be written, using (12) and (13), as

$$U' = -2M e^{x^{(I)}(0)/4M} \equiv U_0', \quad \tilde{k}_x \geq 0 , \quad (64)$$

$$V' = -2M e^{-x^{(I)}(0)/4M} \equiv V_0', \quad \tilde{k}_x \leq 0 . \quad (65)$$

That is, the trajectories of the "in" wave packets, in the region near the horizon, are straight lines of $U' = U_0' = \text{const}$ for $\tilde{k}_x \geq 0$ and straight lines of $V' = V_0' = \text{const}$ for $\tilde{k}_x \leq 0$.

E. "Out" wave packets

Since at the horizon the fields $\partial/\partial U'$ and $\partial/\partial V'$ become two null Killing vector fields we can define a new complete set of "out" modes on region I. We define solutions with positive frequency ω_+ with respect to the vector $\partial/\partial U'$ on the $V' = 0$ "roof" of the horizon and positive frequency ω_- with respect to the vector $\partial/\partial V'$ on the $U' = 0$ "roof" of the horizon [9,13,21]. These modes, close to the horizon, are given by

$$u^{\text{out}}_{\omega_- l' m'}(U', V', \eta, y) = \frac{1}{2M\sqrt{(2\pi)2\omega_-}} Y_{l'}^{m'}(y/M, \pi/2 - \eta) \times e^{-i\omega_+ U' - i\omega_- V'} , \quad (66)$$

where the labels ω_-, m', and l' are the three independent separation constants of the Klein-Gordon equation (16) using the asymptotic metric on the horizon of region I, and the label ω_+ is given by

$$16M^2 \omega_+ \omega_- = l'(l'+1) . \quad (67)$$

If we restrict ourselves, now, to the cyclic case the labels l' and m' are both discrete but ω_- is continuous. We can transform ω_- into a discrete label by constructing a *single wave packet* as

$$u^{\text{out}}_{\tilde{\omega}_-, n', l', m'}(U', V', \eta, y) = \frac{1}{\sqrt{\epsilon'}} \int_{\tilde{\omega}_-}^{\tilde{\omega}_- + \epsilon'} d\omega_- \frac{Y_{l'}^{m'}(y/M, \pi/2 - \eta)}{2M\sqrt{(2\pi)2\omega_-}} \times e^{-in'\omega_- - i\omega_+ U' - i\omega_- V'} . \quad (68)$$

The integrand's phase is given by

$$\Theta = -n'\omega_- - \omega_+ U' - \omega_- V' - m' \frac{y}{M} , \quad (69)$$

and the wave packet trajectory (i.e., $\partial\Theta/\partial\omega_- = 0$), using relation (67), is

$$\tilde{\omega}_+ U' = (n' + V') \tilde{\omega}_- . \quad (70)$$

Let us discuss these trajectories. On the "roof" $U' = 0$ the trajectory (70) is $V' = -n' = \text{const} \leq 0$, the coordinate V' is always negative [see (13)]; thus, n' is positive and, for a fixed wave packet, constant. This relation means that on the "roof" $U' = 0$ the wave packet trajectories are straight lines with $V' = \text{const} = -n'$. Similarly, on the "roof" $V' = 0$, the trajectories (70) are $(\tilde{\omega}_+/\tilde{\omega}_-) U' = n'$; this means that they are straight lines with coordinate $U' = \text{const} = (\tilde{\omega}_-/\tilde{\omega}_+) n'$; now the label n' is negative since the coordinate U' is negative [see (13)]. Therefore wave packets with n' positive are localized on the $U' = 0$ "roof" of the horizon and wave packets with n' negative are localized on the $V' = 0$ "roof" of the horizon. These "roofs" of the horizon are depicted in Fig. 3.

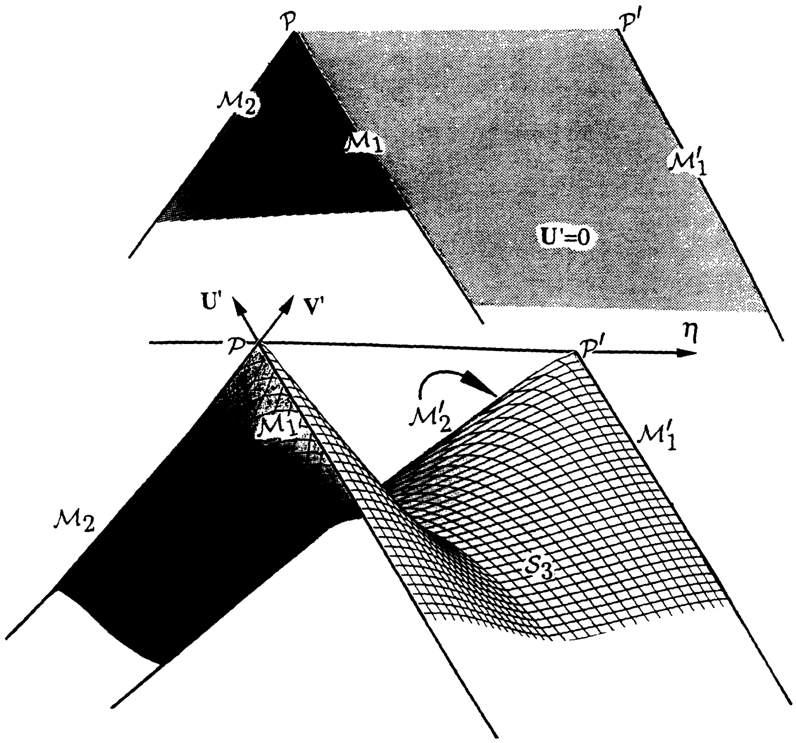

FIG. 3. Three-dimensional plot of the interaction region (region I) in which all its boundaries are shown, using nonsingular Kruskal-Szekeres-like coordinates. The surface $\mathcal{S}_3$ and the lines $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}'_1$, and $\mathcal{M}'_2$, are the boundaries of region I and the two plane wave regions II and III, the points $\mathcal{P}$ and $\mathcal{P}'$ correspond to $U'=0, V'=0, \eta=-\pi/2$, and to $U'=0, V'=0, \eta=\pi/2$, respectively, and they are *folding singularities* [9,10,24]. The Cauchy horizon is the "roof" $\{U'=0, V'<0\} \cup \{V'=0, U'<0\}$ and region I is enclosed between the surface $\mathcal{S}_3$ and the "roof."

F. Bogoliubov coefficients

Using the well defined inner product on the surface of the horizon (59), we can compute the Bogoliubov transformation coefficients relating the propagated "in" wave packet modes from the flat region (region IV) up to the horizon of region I and the "out" wave packet modes defined on the horizon. The two Bogoliubov coefficients, defined in the usual way (i.e.,

$$\alpha_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'} = \langle u^{\text{in}}_{\tilde k_x,\tilde k_-,n,q,m}, u^{\text{out}}_{\tilde\omega_-,n',l',m'} \rangle$$

and

$$\beta_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'} = -\langle u^{\text{in}}_{\tilde k_x,\tilde k_-,n,q,m}, u^{\text{out}*}_{\tilde\omega_-,n',l',m'} \rangle$$

are

$$\left.\begin{matrix}\alpha_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'}\\ \beta_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'}\end{matrix}\right\} = \mp \frac{1}{\sqrt{\epsilon\delta\epsilon'}}\int_{\tilde k_x}^{\tilde k_x+\epsilon} dk_x \int_{\tilde k_-}^{\tilde k_-+\delta} dk_- \int_{\tilde\omega_-}^{\tilde\omega_-+\epsilon'} d\omega_- \, e^{-ink_x-iqk_-\pm in'\omega_-} \times \frac{i(2M)(\pm 1)^m}{\sqrt{\pi|k_x|\omega_-}}\delta_{m,\pm m'}|C_{l'}|e^{i\theta_\alpha}\Gamma(1+i4M|k_x|) \times e^{\pm 2\pi M|k_x|}(2M\omega_\pm)^{-i4M|k_x|}, \quad (71)$$

where $l' \simeq \sqrt{\alpha}$ and θ_α are given by (55) and (51), respectively. The upper signs in (71), except for the label $\omega_\pm$, stand for the α coefficient and the lower signs for the β coefficient. The label $\omega_\pm$ is the same for both coefficients and its double sign means $k_x \geq 0$ for the upper sign and $k_x \leq 0$ for the lower sign.

In the approximation of small integration intervals ϵ, ϵ', and δ, and following the stationary phase technique, one can take the nonoscillating terms in the integrals as constants. For that reason we can perform a convenient separation in the Bogoliubov coefficients into modulus and phase as

$$\left.\begin{matrix}\alpha_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'}\\ \beta_{\tilde k_x,\tilde k_-,n,q,m;\tilde\omega_-,n',l',m'}\end{matrix}\right\} = \mp \frac{i(2M)(\pm 1)^m}{\sqrt{\pi|\tilde k_x|}} \times |\Gamma(1+i4M|\tilde k_x|)| \times e^{\pm 2\pi M|\tilde k_x|}\mathcal{R}_{\alpha/\beta}, \quad (72)$$

where we have made use of Stirling's formula for the Γ function [23], i.e.,

$$\Gamma(z)=z^{z-1/2}e^{-z}\sqrt{2\pi}\left[1+\frac{1}{12z}+\frac{1}{288z^2}+O(z^{-3})\right];$$

it can be written as

$$\Gamma(1+iy)=|\Gamma(1+iy)|\exp\left[i\left[\frac{y}{2}\ln(1+y^2)-y+\tfrac{1}{2}\arctan y\right]\right] \tag{73}$$

for values $y>0$, and can be written as

$$\Gamma(1+iy)=|\Gamma(1+iy)|\exp(y\ln y-y+\pi/4), \tag{74}$$

for $y>1$. We have also defined $\mathcal{R}_{\alpha/\beta}$ as

$$\mathcal{R}_{\alpha/\beta}=\frac{\delta_{m,\pm m'}}{\sqrt{\epsilon\delta\epsilon'}}\int_{\tilde{k}_x}^{\tilde{k}_x+\epsilon}dk_x e^{-ink_x+i\Omega}\int_{k_-}^{\tilde{k}_-+\delta}dk_-|C_{l'}|e^{-iqk_-+i\theta_\alpha}\int_{\tilde{\omega}_-}^{\tilde{\omega}_-+\epsilon'}\frac{d\omega_-}{\sqrt{\omega_-}}e^{\pm in'\omega_- -i4M|k_x|\ln(2M\omega_\pm)}, \tag{75}$$

where

$$\Omega=4M|k_x|\ln(4M|k_x|)-4M|k_x|+\pi/4;$$

it comes from Stirling's formula (74), where we have assumed $4M|k_x|>1$ for simplicity. This assumption does not affect our results because the purpose is to relate a single "in" wave packet with a fixed label $\tilde{k}_x$ to a single "out" wave packet with a fixed label $\tilde{\omega}_-$. Of course we can follow the same steps for the case $4M|k_x|>0$ using the more accurate version of Stirling's formula (73). We have kept the terms $|C_{l'}|$ and $(\omega_-)^{-1/2}$ in the integrands for convergence.

The terms $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ act like localizing terms in the sense that the three integrals of which they consist have a well localized peak. The integrand's phases of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ (Θ_α and Θ_β, respectively) are

$$\Theta_{\alpha/\beta}=-nk_x-qk_-\pm n'\omega_-+\Omega+\theta_\alpha-4M|k_x|\ln(2M\omega_\pm), \tag{76}$$

and the peaks of $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are in the spacetime points which satisfy

$$\partial\Theta_{\alpha/\beta}/\partial_x=\partial\Theta_{\alpha/\beta}/\partial k_-=\partial\Theta_{\alpha/\beta}/\partial\omega_-=0.$$

Solving these three equations for Θ_α we find, respectively,

$$n=\frac{\partial\theta_\alpha}{\partial k_x}\pm4M\ln\left[\frac{2|\tilde{k}_x|}{\tilde{\omega}_\pm}\right], \tag{77}$$

$$q=\frac{\partial\theta_\alpha}{\partial k_-}, \tag{78}$$

$$n'=-4M\frac{\tilde{k}_x}{\tilde{\omega}_-}, \tag{79}$$

where the upper sign in (77) stands for $\tilde{k}_x\geq0$ and the lower sign for $\tilde{k}_x\leq0$. Similarly, from the equations for Θ_β, we obtain

$$\bar{n}=\frac{\partial\theta_\alpha}{\partial k_x}\pm4M\ln\left[\frac{2|\tilde{\bar{k}}_x|}{\tilde{\omega}_\pm}\right], \tag{80}$$

$$\bar{q}=\frac{\partial\theta_\alpha}{\partial k_-}, \tag{81}$$

$$n'=4M\frac{\tilde{\bar{k}}_x}{\tilde{\omega}_-}, \tag{82}$$

where the two first equations are functionally the same as (77) and (78) but with a bar over the "in" labels, i.e., $\tilde{\bar{k}}_x,\bar{n},\bar{q}$, since these three equations come from the β coefficient which relates an "out" wave packet with an "in" anti wave packet (i.e., a wave packet constructed by the superposition of monochromatic modes of negative frequency). On the other hand, (77), (78), and (79) come from an α coefficient and so relate an "out" wave packet with an "in" wave packet. In fact, notice that Eqs. (77), (78), and (79) give a relation between the labels of the "in" wave packets (i.e., $\tilde{k}_x,\tilde{k}_y,\tilde{k}_-,n$, and q) and the labels of the "out" wave packets (i.e., ω_-,n',l', and m') and Eqs. (80), (81), and (82) give a relation between the labels of the "in" anti wave packets (i.e., $\tilde{\bar{k}}_x,\tilde{\bar{k}}_y,\tilde{\bar{k}}_-,\bar{n}$, and $\bar{q}$) and the labels of the "out" wave packets (i.e., ω_-,n',l', and m'). When the previous equations are satisfied the terms $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ have a peak, otherwise they roughly vanish.

Let us now extract some more information from Eqs. (77)–(82). For instance, Eq. (77) [or (80)] can be written as

$$|\tilde{k}_x|=\tfrac{1}{2}\omega_\pm e^{\pm x^{(\mathrm{I})}(0)/4M}=\begin{cases}-\omega_+U_0'/(4M), & \tilde{k}_x\geq0,\\ -\omega_-V_0'/(4M), & \tilde{k}_x\leq0,\end{cases} \tag{83}$$

where we have used (51), (64), (65), and the fact that

$$n+(\tilde{k}_x/\tilde{k}_-)L_1f(u_0)=x^{(\mathrm{I})}(0)$$

is the value of the coordinate x when the "in" wave packet coming from region II crosses the boundary $\{v=0,0\leq u<\pi/2\}$ into region I [see (31)]. Note that U_0' or V_0' are the coordinates on the horizon reached for the wave packets. From (83), Eqs. (77) and (80) can be seen as *redshift formulas*, because they relate the energy of an "out" wave packet (i.e., ω_-) with the energies of an "in" wave packet and an "in" anti wave packet, respectively, (the energy label for the "in" wave packets in the region near the horizon is $|\tilde{k}_x|$). Note also from (83) that the *redshift coefficient* is given by the value of the coordinate U or V on the horizon reached by the "in" wave or anti wave packets and by the surface gravity of the horizon, i.e., $(4M)^{-2}$. Equations (79) and (82) are *position formulas* because they give information on the trajectories of the "in" wave and anti wave packets which are related to a given "out" wave packet. In fact, the label n' appearing in these formulas can be written, in analogy to (83), as

$$|n'|=\begin{cases} -\dfrac{\tilde{\omega}_+}{\tilde{\omega}_-}U_0, & \tilde{k}_x\geq 0, \\ -V_0, & \tilde{k}_x\leq 0. \end{cases} \tag{84}$$

Recall that the label n' of the "out" wave packet is related to its position on the horizon [see (70)]; when $n'\geq 0$ the "out" wave packet is localized on the $U'=0$ "roof" at coordinate $V'\equiv V'_{\max}=-n'=\text{const}$, and when $n'\leq 0$ it is localized on the $V'=0$ "roof" at coordinate $U'\equiv U'_{\max}=(\tilde{\omega}_-/\tilde{\omega}_+)n'=\text{const}$. From (79) we see that when $n'\geq 0$ then $\tilde{k}_x\leq 0$, and vice versa, and from (82) that when $n'\geq 0$ then $\tilde{\tilde{k}}_x\geq 0$, and vice versa. Equation (78) [or (81)], with the use of (51), is the same as (61) and it does not give us any additional information.

Putting all this together we can give the following interpretation. An "out" wave packet $(\tilde{\omega}_-,n')$, which is localized at the coordinate $V'_{\max}$ on the "roof" $U'=0$ ($U'_{\max}$ on the "roof" $V'=0$) of the horizon, is related to an "in" wave packet with momentum along the x axis $\tilde{k}_x=\tilde{\omega}_-V'_{\max}/4M$ ($\tilde{k}_x=-\tilde{\omega}_+U'_{\max}/4M$) which reaches the horizon at coordinate $V'_0=V'_{\max}$ on the "roof" $U'=0$ ($U'_0=U'_{\max}$ on the "roof" $V'=0$), and to an "in" anti wave packet with the same momentum along the x axis, but with opposite sign, $\tilde{\tilde{k}}_x=-\tilde{\omega}_-V'_{\max}/4M$ ($\tilde{\tilde{k}}_x=\tilde{\omega}_+U'_{\max}/4M$), which reaches the horizon at coordinate $\bar{U}'_0=(\tilde{\omega}_-/\tilde{\omega}_+)V'_{\max}$ on the "roof" $V'=0$ ($\bar{V}'_0=(\tilde{\omega}_+/\tilde{\omega}_-)U'_{\max}$ on the "roof" $U'=0$).

G. Particle creation

Following the formalism of quantum field theory on curved spacetime, spontaneous particle creation is directly related to the β Bogoliubov coefficient. In fact, the number of "out" particles in a given wave packet mode with labels $(\tilde{\omega}_-,n',l',m')$ [i.e., the number of quanta in the wave packet mode $(\tilde{\omega}_-,n',l',m')$] in the "in" vacuum is given by the sum over the "in" labels of the squared modulus of the β coefficient [22], (72), that is,

$$\langle 0,\text{in}|N^{\text{out}}_{\tilde{\omega}_-,n',l',m'}|0,\text{in}\rangle = \sum_{\tilde{k}_x,\tilde{k}_-,m,n,q}|\beta_{\tilde{k}_x,\tilde{k}_-,m,n,q;\omega_-,n',l',m'}|^2 . \tag{85}$$

Here $N^{\text{out}}_{\tilde{\omega}_-,n',l',m'}$ is the number operator of "out" particles defined in the standard way as

$$N^{\text{out}}_{\tilde{\omega}_-,n',l',m'}=a^{\dagger\text{out}}_{\tilde{\omega}_-,n',l',m'}a^{\text{out}}_{\tilde{\omega}_-,n',l',m'} ,$$

where $a^{\dagger\text{out}}_{\tilde{\omega}_-,n',l',m'}$ and $a^{\text{out}}_{\tilde{\omega}_-,n',l',m'}$ are the "out" wave packet creation and annihilation operators, respectively. With these operators we can write the field operator ϕ as a combination of "out" wave and anti wave packets, i.e.,

$$\phi(x)=\sum_{\tilde{\omega}_-,n',l',m'} a^{\text{out}}_{\tilde{\omega}_-,n',l',m'}u^{\text{out}}_{\tilde{\omega}_-,n',l',m'}(x) + a^{\dagger\text{out}}_{\tilde{\omega}_-,n',l',m'}u^{*\text{out}}_{\tilde{\omega}_-,n',l',m'}(x) .$$

To evaluate the sum in (85) it is worth noticing that the $\mathcal{R}_\beta$ coefficient in (72), given by (75), satisfies the two equalities

$$\sum_{\tilde{\omega}_-,n',l',m'}|\mathcal{R}_\beta|^2=\frac{1}{4(2M)^3} , \tag{86}$$

$$\sum_{\tilde{k}_x,\tilde{k}_-,m,n,q}|\mathcal{R}_\beta|^2=\frac{1}{4(2M)^3} . \tag{87}$$

These equalities follow from the general property (2) of wave packets and from the form of $C_{l'}$ given by (60). Note also, in order to make sense of the square of $C_{l'}$ according to (60), we consider the product of a Dirac's δ with a Kronecker's δ.

These two equalities and the fact that the $\mathcal{R}_\beta$ term has a peak when the relations (80)–(82), between the "in" labels (i.e., k_x,k_y,k_-,n,q) and the "out" labels (i.e., ω_-,n',l',m') are satisfied, allow us to write approximately

$$\mathcal{R}_\beta\simeq\frac{1}{2(2M)^{3/2}}\delta_{\tilde{k}_x,n'\tilde{\omega}-(4M)^{-1}}\delta_{n,\bar{n}(\tilde{\omega}_-,n',l',m')} \times\delta_{k_-,\bar{k}_-(\tilde{\omega}_-,n',l',m')}\delta_{q,q(\tilde{\omega}_-,n',l',m')}\delta_{m,-m'} , \tag{88}$$

where $n=\bar{n}(\tilde{\omega}_-,n',l',m')$ is given by (80), $q=q(\tilde{\omega}_-,n',l',m')$ by (81), and $\delta_{k_-,\bar{k}_-(\tilde{\omega}_-,n',l',m')}$ is given by δ_{l',l_α} which appears from the squared modulus of (60). Then the number of "out" particles created in the "out" packet mode $(\tilde{\omega}_-,n',l',m')$, Eq. (85), is simply

$$\langle 0,\text{in}|N^{\text{out}}_{\tilde{\omega}_-,n',l',m'}|0,\text{in}\rangle\equiv N^{\text{out}}_{\tilde{\omega}_-,n'} =\frac{1}{e^{8\pi M\tilde{\omega}_-(|n'|/4M)}-1} , \tag{89}$$

which can be interpreted as a thermal spectrum for each fixed value of the label n', with a temperature

$$T=\frac{1}{8\pi M(|n'|/4M)} . \tag{90}$$

This spectrum depends on the dimensionless label $n'/4M$, i.e., on the trajectory of the wave packet, but it is thermal for all wave packet modes with the same trajectory. This is quite different from the black hole case [14] where the temperature is independent of the packet trajectory [17] and depends only on the surface gravity, $\kappa=(4M)^{-1}$. In Ref. [9] we discuss how the black hole case can be seen in some sense as the time reversal of the colliding wave case. The physical interpretation of this n'-dependent temperature follows from the fact that we are computing the particles produced on the wave packet mode $(\tilde{\omega}_-, n', l', m')$, which is localized on the horizon by n', given by (70), so that these particles may be "localized" in the same position on the horizon. Note that when $n' \simeq 0$, i.e., near the bifurcation point $U' = V' = 0$ of the horizon, the temperature is higher.

This spectrum is in agreement with the spectrum of particles created on the monochromatic modes (20), i.e., particles with a well defined momentum but not localized in space, because the one particle Fock space can be decomposed on a basis given in terms of the monochromatic labels $|\omega\rangle$, or, alternatively, on a basis given in terms of the wave packet labels $|\tilde{\omega}, n'\rangle$. The space is the same but, of course, the particle interpretation is different.

For the monochromatic modes, discussed in Ref. [9], the spectrum of particles created (20) is inversely proportional to the energy of the "out" modes (i.e., ω_-), with a proportionality factor $(8\pi M)^{-1}$. The relation between the number operators $N^{\text{out}}_{\omega_-}$ and $N^{\text{out}}_{\tilde{\omega}_-, n'}$ is given in terms of the Bogoliubov coefficients relating the monochromatic modes and the wave packet modes. Such a transformation has no β Bogoliubov coefficient because the positive frequency wave packets are constructed with positive frequency monochromatic modes only. The α Bogoliubov coefficient is

$$\alpha_{\omega_-;\omega_-,n'} = \frac{e^{-in'\omega}}{\sqrt{\epsilon'}}[\theta(\omega_- - \tilde{\omega}_-) - \theta(\omega_- - \{\tilde{\omega}_- + \epsilon'\})] ,$$

where, here, $\theta(x)$ is the usual Heaviside step function; thus the relation between the monochromatic and wave packet annihilation operators is given by

$$a^{\text{out}}_{\omega_-} = \sum_{\tilde{\omega}_-, n'} \alpha^*_{\omega_-;\omega_-,n'} a^{\text{out}}_{\tilde{\omega}_-,n'} = \frac{1}{\sqrt{\epsilon'}} \sum_{n'} e^{-in'\omega_-} a^{\text{out}}_{\omega_-,n'} .$$

Then the monochromatic number operator can be written as

$$N^{\text{out}}_{\omega_-} = a^{\dagger\text{out}}_{\omega_-} a^{\text{out}}_{\omega_-} = \frac{1}{\epsilon'} \sum_{n', \tilde{n}'} e^{-i(n'-\tilde{n}')\omega_-} a^{\dagger\text{out}}_{\omega_-,n'} a^{\text{out}}_{\omega_-,\tilde{n}'} ,$$

and we can approximate

$$\langle 0,\text{in}| N^{\text{out}}_{\omega_-} |0,\text{in}\rangle \simeq \frac{1}{\epsilon'} \sum_{n'} \langle 0,\text{in}| a^{\dagger\text{out}}_{\omega_-,n'} a^{\text{out}}_{\omega_-,n'} |0,\text{in}\rangle = \frac{1}{\epsilon'} \sum_{n'} N^{\text{out}}_{\omega_-,n'} ,$$

as the phase term $\exp[-i(n'-\tilde{n}')\omega_-]$ is fast oscillating (except when $n' = \tilde{n}'$), because $n' - \tilde{n}' = (l - \tilde{l})2\pi/\epsilon'$ with l and l' integers and ϵ' a small positive parameter. Then

$$\langle 0,\text{in}| N^{\text{out}}_{\omega_-} |0,\text{in}\rangle \simeq \frac{2M}{\pi} \frac{1}{8\pi M \omega_-} \int_0^\infty \frac{dk}{e^k - 1} , \tag{91}$$

where we have assumed $8\pi M \omega_- \ll 1$ in order to approximate $k \equiv (8\pi M \omega_-)n'/4M$ as a continuous dimensionless variable. The inverse proportionality in ω_- and the logarithmic divergence, in (20), have been recovered. The extra factor $(2M)^2$ in (20) is due to the fact that two monochromatic labels in Ref. [9] were, in fact, discrete and this factor guarantees the correct normalizations.

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[1] K. Kahn and R. Penrose, Nature (London) **229**, 185 (1971).
[2] J. B. Griffiths, *Colliding Waves in General Relativity* (Clarendon, Oxford, 1991).
[3] P. Szekeres, Nature (London) **228**, 1183 (1970).
[4] P. D. D'Eath, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
[5] V. Ferrari, P. Pendenza, and G. Veneziano, Gen. Relativ. Gravit. **20**, 1185 (1980).
[6] U. Yurtsever, Phys. Rev. D **40**, 360 (1989).
[7] D. Garfinkle, Phys. Rev. D **41**, 1112 (1989).
[8] D. Garfinkle and T. Vachaspati, Phys. Rev. D **42**, 1960 (1990).
[9] M. Dorca and E. Verdaguer, Nucl. Phys. **B403**, 770 (1993).
[10] S. A. Hayward, Class. Quantum Grav. **6**, 1021 (1989).
[11] S. Chandrasekhar and B. Xanthopoulos, Proc. R. Soc. London **A408**, 157 (1986).
[12] V. Ferrari and J. Ibañez, Gen. Relativ. Gravit. **19**, 405 (1987).
[13] B. S. Kay and R. M. Wald, Phys. Rep. **207**, 49 (1991).
[14] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[15] U. Yurtsever, Phys. Rev. D **38**, 1706 (1988).
[16] R. M. Wald, Phys. Rev. D **13**, 3176 (1976).
[17] J. A. Audretsch and R. Müller, Phys. Rev. D **45**, 513 (1992).
[18] J. A. Audretsch and R. Müller, University of Konstanz report, 1993 (unpublished).
[19] C. Eckart, Rev. Mod. Phys. **20**, 399 (1948).
[20] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (Addison-Wesley, Palo Alto, CA, 1964).
[21] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[22] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[24] R. A. Matzner and F. J. Tipler, Phys. Rev. D **29**, 1575 (1984).

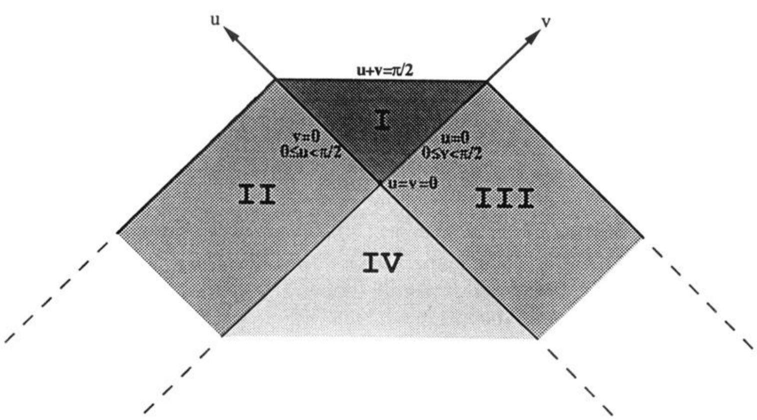

FIG. 1. Projection of the colliding plane wave spacetime in the (u,v) plane. One can see four regions: region IV is the flat region before the waves collide, regions II and III are the plane wave regions, and region I is the interaction region. The interaction starts at $u=v=0$, the lines $u=0$, $v<0$ and $v=0$, $u<0$ are the boundaries of region IV with regions II and III, respectively, and $v=0, 0\leq u<\pi/2$ and $u=0, 0\leq v<\pi/2$ are the boundaries of region I with regions II and III, respectively. At $u+v=\pi/2$ a Killing-Cauchy horizon is formed which one can see as a coordinate singularity of the metric in region I.

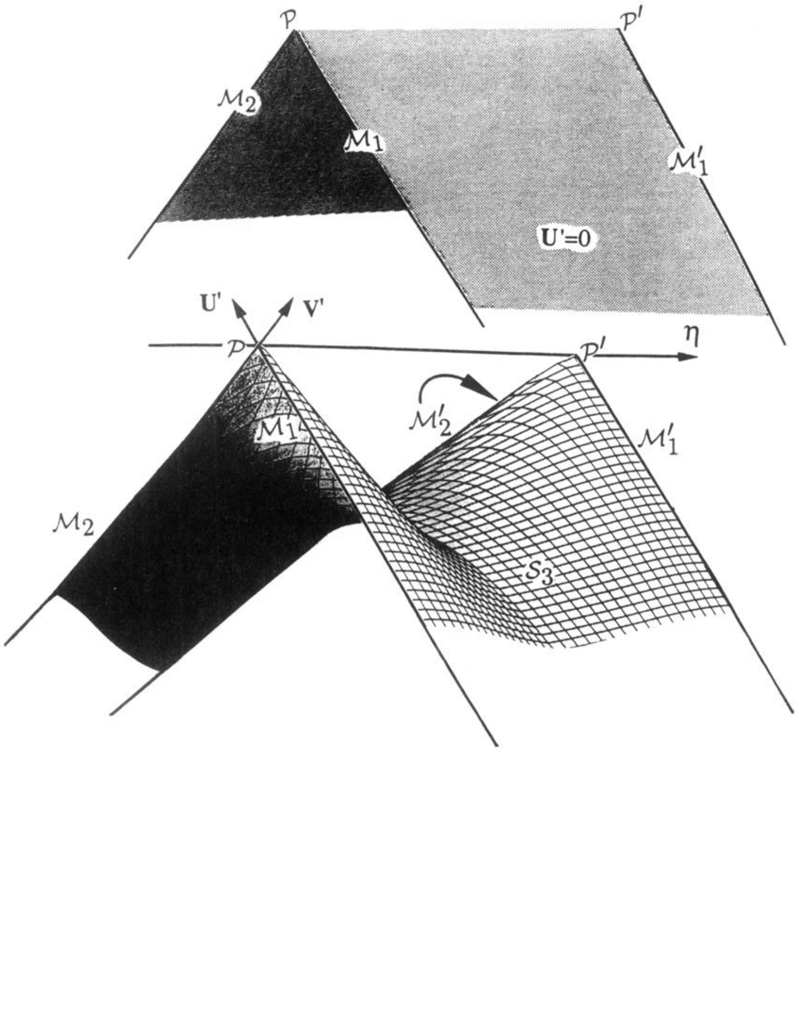

FIG. 3. Three-dimensional plot of the interaction region (region I) in which all its boundaries are shown, using nonsingular Kruskal-Szekeres-like coordinates. The surface $\mathcal{S}_3$ and the lines $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}'_1$, and $\mathcal{M}'_2$, are the boundaries of region I and the two plane wave regions II and III, the points $\mathcal{P}$ and $\mathcal{P}'$ correspond to $U'=0, V'=0, \eta=-\pi/2$, and to $U'=0, V'=0, \eta=\pi/2$, respectively, and they are *folding singularities* [9,10,24]. The Cauchy horizon is the "roof" $\{U'=0, V'<0\} \bigcup \{V'=0, U'<0\}$ and region I is enclosed between the surface $\mathcal{S}_3$ and the "roof."