Temperature-induced interaction: $\lambda\phi^4$ theory

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(Received 18 July 1985; revised manuscript received 28 October 1985)

We study the possibility that a field theory is interacting and bounded only above a certain temperature, while not existing below it. Within the Gaussian variational approximation used we find a phase for $\lambda\phi^4$ in four dimensions in which this phenomenon occurs.

I. INTRODUCTION

Quantum field theory is a never-ending story of potential surprises. In 3 + 1 dimensions our simplest theory is $\lambda\phi^4$, and yet very little is actually known about it. Does it exist as an interacting theory? This is not very likely if the bare coupling constant is positive: $\lambda_B > 0$.[1] However, the answer may be yes if it is negative and small (Ref. 2), $\lambda_B \lesssim 0$, which is known, since Symanzik's work,[3] not to lead necessarily to unboundedness (although his asymptotic-freedom argument is probably not enough to avoid it). In this work we would like to study another road to interaction: one induced by temperature. Could it be that a field theory only exists as an interacting theory above a given temperature with a coupling constant determined by the temperature-to-mass ratio? We have studied this question for $\lambda\phi^4$ theory and we find that within the approximations used, this possibility is realized in this model. Let us show how this happens.

Let us first recall Stevenson's main results[2] obtained at $T=0$ within a variational approach[4] ideally suited for investigating the question of existence. For $\lambda_B > 0$ the truly renormalizable theory, i.e., the one which remains finite when the UV cutoff is removed, $\Lambda\to\infty$ is trivial (noninteracting) or nonexistent (unbound from below). This reproduces the well-known triviality results.[1] Of course, the conclusion does not follow if one considers $\lambda\phi^4$ theory an effective theory where Λ has a physical meaning coming from the reduction of the original larger field theory to $\lambda\phi^4$ theory; one can then obtain a spontaneous-symmetry-breaking phase.[5] We will not follow this road but will consider $\lambda\phi^4$ theory alone as a renormalizable field theory. Then Stevenson finds that if λ_B goes towards zero from below as the cutoff is removed through

$$\lambda_B = -\frac{1}{6I_{-1}(m^2)}\left[1+\frac{1}{2\lambda I_1(m^2)}+O\left[\frac{1}{I_{-1}{}^2(m^2)}\right]\right], \tag{1.1}$$

where $I_{-1}(m^2)\sim\ln(\Lambda^2/m^2)$ and m and λ are the renormalized mass and coupling constant, the theory is interacting with

$$-8\pi^2<\lambda<0\ ,\quad m^2>0\ , \tag{1.2}$$

and the effective potential increases from the origin up to a certain value of the classical field and is flat from there on showing that the theory is stable and that no spontaneous symmetry breaking occurs.

This result and the technique by which it is obtained is our starting point. We will now perform a similar study but at finite temperature. Temperature is introduced in the usual way,[6] and we find it convenient to use the real-time formalism. Notice that although the variational approach is nonperturbative one can use the free-propagator temperature modifications[6] because our quantum trial field will be a free field.

II. THE ANALYSIS

The Hamiltonian density of $\lambda\phi^4$ theory is

$$\mathscr{H}=\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B{}^2\phi^2+\lambda_B\phi^4\ , \tag{2.1}$$

where the subscript B indicates that all parameters are bare. In order to use the variational method we will introduce the ansatz

$$\phi(x)=\phi_0+\int_{\mathbb{R}^3}\frac{d^3k}{(2\pi)^3 2\omega_k(\Omega^2)}[a_k(\Omega)e^{-ikx}+a_k^\dagger(\Omega)e^{ikx}]\ , \tag{2.2}$$

where both ϕ_0 and Ω^2 will be the variational parameters, and

$$\omega_k(\Omega^2)\equiv(\Omega^2+\mathbf{k}^2)^{1/2} \tag{2.3}$$

and the creation and annihilation operators satisfy the commutation relations

$$[a_{\mathbf{k}}(\Omega),a^\dagger_{\mathbf{k}'}(\Omega)]=(2\pi)^3 2\omega_k(\Omega^2)\delta^3(\mathbf{k}-\mathbf{k}')\ . \tag{2.4}$$

Then, the upper bound of the ground-state energy is given by the minimum of

$$V(\phi_0,\Omega(\phi_0))\equiv\langle 0_\Omega|\mathscr{H}|0_\Omega\rangle\,|_{\Omega=\Omega(\phi_0)}\ , \tag{2.5}$$

where $|0_\Omega\rangle$ is the normalized free-field vacuum.

One can easily compute the right-hand side of (2.5) and obtain

$$\langle 0_\Omega|\mathscr{H}|0_\Omega\rangle=I_1(\Omega^2)+\tfrac{1}{2}(m_B{}^2-\Omega^2)I_0(\Omega^2)+\tfrac{1}{2}m_B{}^2\alpha+\lambda_B\alpha^2+6\lambda_B\alpha I_0(\Omega^2)+3\lambda_B I_0{}^2(\Omega^2)\ , \tag{2.6}$$

where

$$\alpha\equiv\phi_0{}^2 \tag{2.7}$$

and

$$I_n(\Omega^2)\equiv\int_{\mathbf{R}^3}\frac{d^3k}{(2\pi)^3 2\omega_k(\Omega^2)}[\omega_k{}^2(\Omega^2)]^n \,. \tag{2.8}$$

Up until now, we have not included finite-temperature terms. In order to do that we have to realize that $I_0(\Omega^2)$ is the propagator of the theory and add to it the temperature corrections[6] obtaining

$$I_0^T(\Omega^2)=I_0(\Omega^2)+H_0(\Omega)\,, \tag{2.9}$$

where $\beta=1/(k_B T)$ and

$$H_0(\Omega)\equiv\frac{1}{(2\pi)^3}\int_{\mathbf{R}^4}\frac{d^4k}{\exp[\beta\omega_k(\Omega^2)]-1}\delta(k^2-\Omega^2)\,. \tag{2.10}$$

Equally,

$$I_1^T(\Omega^2)=I_1(\Omega^2)+H_1(\Omega) \tag{2.11}$$

with

$$H_1(\Omega)\equiv\frac{1}{(2\pi)^3}\int_{\mathbf{R}^4}\frac{\mathbf{k}^2+\Omega^2}{\exp[\beta\omega_k(\Omega^2)]-1}\delta(k^2-\Omega^2)d^4k\,. \tag{2.12}$$

Using Eqs. (2.9) and (2.11) the finite-temperature upper bound of the energy density is given by

$$\langle 0_\Omega|\mathcal{H}|0_\Omega\rangle=I_1(\Omega^2)+H_1(\Omega)+\tfrac{1}{2}(m_B{}^2-\Omega^2)[I_0(\Omega^2)+H_0(\Omega)]+\tfrac{1}{2}m_B{}^2\alpha+\lambda_B\alpha^2+6\lambda_B\alpha[I_0(\Omega^2)+H_0(\Omega)]$$
$$+3\lambda_B[I_0(\Omega^2)+H_0(\Omega)]^2\,. \tag{2.13}$$

Before continuing it will be useful to remove dimensions by multiplying all quantities with an adequate power of β. We will work from now on with these rescaled variables without changing the notation. All the relevant properties of $H_0(\Omega)$ and $H_1(\Omega)$ are given in the Appendix.

Formula (2.13) displays many divergences. Some of them will be removed by subtracting the zero-point energy. To eliminate the others one has to renormalize the mass and coupling constant. So, we start by regularizing the theory with a symmetric cutoff Λ in the divergent integral $I_n(\Omega^2)$, (2.8).

We choose the origin ($\alpha=0$) as the subtraction point and so

$$m^2\equiv 2\left.\frac{dV(\alpha,\Omega(\alpha))}{d\alpha}\right|_{\alpha=0}=m_B{}^2+12\lambda_B[I_0(\Omega_0{}^2)+H_0(\Omega_0)]+2\Omega_0'\left.\frac{\partial V(\alpha,\Omega(\alpha))}{\partial\Omega}\right|_{\alpha=0}\,, \tag{2.14}$$

$$\lambda\equiv\frac{1}{2}\frac{d^2V(\alpha,\Omega(\alpha))}{d\alpha^2}=\lambda_B+6\lambda_B\Omega_0'[\dot H_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]$$
$$+\tfrac{1}{2}\Omega_0'^2(\Omega_0{}^2 I_{-1}(\Omega_0{}^2)+\ddot H_1(\Omega_0)-H_0(\Omega_0)-2\Omega_0\dot H_0(\Omega_0)$$
$$+[\ddot H_0(\Omega_0)-I_{-1}(\Omega_0{}^2)+3I_{-2}(\Omega_0{}^2)\Omega_0{}^2]\{\tfrac{1}{2}(m_B{}^2-\Omega_0{}^2)+6\lambda_B[I_0(\Omega_0{}^2)+H_0(\Omega_0)]\}$$
$$+6\lambda_B[\dot H_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]^2)+\tfrac{1}{2}\Omega_0''\left.\frac{\partial V(\alpha,\Omega(\alpha))}{\partial\Omega}\right|_{\alpha=0}\,, \tag{2.15}$$

where $\Omega_0'\equiv d\Omega/d\alpha\,|_{\alpha=0}$ and $\Omega_0''\equiv d^2\Omega/d\alpha^2\,|_{\alpha=0}$. Also the formula

$$\frac{dI_n(\Omega^2)}{d\Omega}=(2n-1)\Omega I_{n-1}(\Omega^2) \tag{2.16}$$

has been repeatedly used.

On the other hand, Ω_0 is $\Omega(\alpha=0)$. To obtain it we will look for a solution $\Omega_0\geq 0$ of

$$\frac{\partial V(0,\Omega_0)}{\partial\Omega_0}=\dot H_1(\Omega_0)+[\dot H_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]\{\tfrac{1}{2}(m_B{}^2-\Omega_0{}^2)+6\lambda_B[I_0(\Omega_0{}^2)+H_0(\Omega_0)]\}-\Omega_0 H_0(\Omega_0)=0\,. \tag{2.17}$$

This equation could or could not have a solution. If not and if $\partial V(0,\Omega_0)/\partial\Omega_0<0$, the theory is unbound from below. If not but $\partial V(0,\Omega_0)/\partial\Omega_0>0$, then $\Omega_0=0$ gives the minimum of $V(0,\Omega_0)$. We leave this case for later and start assuming that it has a solution $\Omega_0\geq 0$. Consider first the subcase $\Omega_0\geq 0$. Then (2.14) becomes

$$m^2=m_B{}^2+12\lambda_B[I_0(\Omega_0{}^2)+H_0(\Omega_0)]\,, \tag{2.18}$$

and using (2.17) we see that

$$m^2=\Omega_0{}^2-\frac{2\Omega_0 H_0(\Omega_0)-2\dot{H}_1(\Omega_0)}{\Omega_0 I_{-1}(\Omega_0{}^2)}+O\left[\frac{1}{I_{-1}{}^2}\right] \tag{2.19}$$

so that $m^2>0$ too.

As Ω_0 has to be a minimum of $V(0,\Omega_0)$,

$$\frac{\partial^2 V(0,\Omega_0)}{\partial\Omega_0{}^2}>0\ , \tag{2.20}$$

which leads to the inequality

$$\ddot{H}_1(\Omega_0)-\frac{\dot{H}_1(\Omega_0)}{\Omega_0}-2\Omega_0\dot{H}_0(\Omega_0)+\Omega_0{}^2 I_{-1}(\Omega_0{}^2)+6\lambda_B[\dot{H}_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]^2+O\left[\frac{1}{I_{-1}}\right]>0\ . \tag{2.21}$$

Notice that this implies that λ as given by (2.15) is a convex function of Ω_0'.

We now subtract the zero-point energy. The result is

$$\begin{aligned}\epsilon(\alpha,\Omega)\equiv V(\alpha,\Omega)-V(0,\Omega_0)=&-\tfrac{1}{8}(\Omega^2-\Omega_0{}^2)^2 I_{-1}(\Omega_0{}^2)+\Sigma(\Omega^2,\Omega_0{}^2)\\&-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)[\Gamma(\Omega^2,\Omega_0{}^2)-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_{-1}(\Omega_0{}^2)]+\lambda_B\alpha^2+\tfrac{1}{2}\Omega_0{}^2\alpha\\&+3\lambda_B[\Gamma(\Omega^2,\Omega_0{}^2)-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_{-1}(\Omega_0{}^2)]^2\\&+6\lambda_B\alpha[\Gamma(\Omega^2,\Omega_0{}^2)-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_{-1}(\Omega_0{}^2)]+H_1(\Omega)-H_1(\Omega_0)\\&+6\lambda_B[H_0(\Omega)-H_0(\Omega_0)][\alpha+\Gamma(\Omega^2,\Omega_0{}^2)-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_{-1}(\Omega_0{}^2)]\\&+[3\lambda_B H_0(\Omega)-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)][H_0(\Omega)-H_0(\Omega_0)]-\frac{1}{2\Omega_0}(\Omega^2-\Omega_0{}^2)\dot{H}_1(\Omega_0)\\&+O\left[\frac{1}{I_{-1}}\right]\ ,\end{aligned} \tag{2.22}$$

where the following relations have been used:

$$\begin{aligned}&I_1(\Omega^2)-I_1(\Omega_0{}^2)=\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_0(\Omega_0{}^2)-\tfrac{1}{8}(\Omega^2-\Omega_0{}^2)^2 I_{-1}(\Omega_0{}^2)+\Sigma(\Omega^2-\Omega_0{}^2)\ ,\\&\Sigma(\Omega^2-\Omega_0{}^2)\equiv\frac{1}{128\pi^2}\left[2\Omega^4\ln\frac{\Omega^2}{\Omega_0{}^2}-(\Omega^2-\Omega_0{}^2)(3\Omega^2-\Omega_0{}^2)\right]+O\left[\frac{1}{\Lambda^2}\right]\ ,\\&I_0(\Omega^2)-I_0(\Omega_0{}^2)=-\tfrac{1}{2}(\Omega^2-\Omega_0{}^2)I_{-1}(\Omega_0{}^2)+\Gamma(\Omega^2,\Omega_0{}^2)\ ,\\&\Gamma(\Omega^2,\Omega_0{}^2)\equiv\frac{1}{16\pi^2}\left[\Omega^2\ln\frac{\Omega^2}{\Omega_0{}^2}-\Omega^2+\Omega_0{}^2\right]+O\left[\frac{1}{\Lambda^2}\right]=2\frac{d\Sigma(\Omega^2,\Omega_0{}^2)}{d\Omega^2}\geq 0\ ,\\&I_{-1}(\Omega^2)-I_{-1}(\Omega_0{}^2)=-\frac{1}{8\pi^2}\ln\frac{\Omega^2}{\Omega_0{}^2}+O\left[\frac{1}{\Lambda^2}\right]=-2\frac{d\Gamma(\Omega^2,\Omega_0{}^2)}{d\Omega^2}\ ,\\&I_{-1}(\Omega^2)=\frac{1}{8\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-2\right]+O\left[\frac{1}{\Lambda^2}\right]\ .\end{aligned} \tag{2.23}$$

It is easy to calculate Ω_0' by obtaining $\Omega(\alpha)$ which minimizes $\epsilon(\alpha,\Omega)$ in Eq. (2.22) and then performing its derivative at $\alpha=0$. This leads substituting into Eq. (2.15) to

$$\lambda=\lambda_B\frac{\Omega_0{}^2 I_{-1}(\Omega_0{}^2)+\ddot{H}_1(\Omega_0)-\dfrac{\dot{H}_1(\Omega_0)}{\Omega_0}-2\Omega_0\dot{H}_0(\Omega_0)-12\lambda_B[\dot{H}_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]^2}{\Omega_0{}^2 I_{-1}(\Omega_0{}^2)+\ddot{H}_1(\Omega_0)-\dfrac{\dot{H}_1(\Omega_0)}{\Omega_0}-2\Omega_0\dot{H}_0(\Omega_0)+6\lambda_B[\dot{H}_0(\Omega_0)-\Omega_0 I_{-1}(\Omega_0{}^2)]^2}\ . \tag{2.24}$$

The same result is obtained by direct minimization of λ as given in Eq. (2.15) with respect to Ω_0'.

For large Ω, (2.22) becomes

$$\epsilon(\alpha,\Omega)\underset{\Omega\gg 1}{\sim}\frac{\Omega^4}{8}I_{-1}(\Omega_0^2)[1+6\lambda_B I_{-1}(\Omega_0^2)]-\frac{\Omega^4}{64\pi^2}\ln\frac{\Omega^2}{\Omega_0^2}[1+12\lambda_B I_{-1}(\Omega_0^2)]\ , \tag{2.25}$$

in such a way that the condition

$$1+6\lambda_B I_{-1}(\Omega_0^2)\geq 0 \tag{2.26}$$

guarantees the stability of ϵ for large Ω at any fixed α. Then, the values of Ω which minimize (2.22) will be either $\Omega=0$ or solution of $\partial\epsilon/\partial\Omega=0$

$$\begin{aligned}-\tfrac{1}{2}I_{-1}(\overline{\Omega}^2)\overline{\Omega}\Bigg[&12\lambda_B\left[\alpha-\frac{\overline{\Omega}^2-\Omega_0^2}{2}I_{-1}(\Omega_0^2)+\Gamma(\overline{\Omega}^2-\Omega_0^2)\right]-\overline{\Omega}^2+\Omega_0^2-\frac{2\Omega_0H_0(\Omega_0)-2\dot{H}_1(\Omega_0)}{\Omega_0 I_{-1}(\Omega_0^2)}\\&+12\lambda_B[H_0(\overline{\Omega})-H_0(\Omega_0)]\Bigg]+6\lambda_B\dot{H}_0(\overline{\Omega})[\alpha-\tfrac{1}{2}(\overline{\Omega}^2-\Omega_0^2)I_{-1}(\Omega_0^2)+\Gamma(\overline{\Omega}^2-\Omega_0^2)]\\&+H_0(\overline{\Omega})[3\lambda_B\dot{H}_0(\overline{\Omega})-\overline{\Omega}]+\dot{H}_1(\overline{\Omega})+\dot{H}_0(\overline{\Omega})\{3\lambda_B[H_0(\overline{\Omega})-H_0(\Omega_0)]-\tfrac{1}{2}(\overline{\Omega}^2-\Omega_0^2)\}=0\ .\end{aligned} \tag{2.27}$$

A careful study along the lines of Ref. 2 leads to the conclusion that the only λ_B which eventually corresponds to a finite nontrivial theory is

$$\lambda_B=-\frac{1}{6I_{-1}(\Omega_0^2)}+O\left[\frac{1}{I_{-1}^2}\right]\ , \tag{2.28}$$

where the subdominant terms have to be positive or zero. The most remarkable fact is that it includes certain λ_B's which were not acceptable in the zero-temperature case where they led to an unphysical theory ($\lambda=\infty$). In particular,

$$\lambda_B=-\frac{1}{6I_{-1}(\Omega_0^2)}+O\left[\frac{1}{I_{-1}^3}\right]\ , \tag{2.29}$$

where analogously the subdominant terms cannot be negative is one of the λ_B's forbidden at zero temperature. The shape of $\epsilon(\alpha,\Omega(\alpha))$ depends on the form of the subdominant terms in (2.28) and also on Ω_0 in the following way. If (2.29) is the case then if Ω_0 is less than the value of the function

$$xe^{2\ddot{H}_0(x)} \tag{2.30}$$

at its minimum, the minimum of ϵ is in $\alpha=0$. As α increases, so does ϵ until it joins smoothly with the flat region $\epsilon(\alpha,\Omega=0)$, which is the minimum of $\epsilon(\alpha,\Omega)$ beyond a certain α. If Ω_0 is greater than the minimum of (2.30) but smaller than the root of

$$\frac{x^4}{128\pi^2}+H_1(x)-\frac{\pi^2}{30}-\frac{x}{2}\dot{H}_1(x)=0\ , \tag{2.31}$$

then the structure of ϵ is slightly more complicated but always has the minimum in $\alpha=0$ and the curve is continuous.

For Ω_0 larger than the root of Eq. (2.31), ϵ is entirely flat. If (2.29) is not the case then it will be

$$\lambda_B=-\frac{1}{6I_{-1}(\Omega_0^2)}+\frac{c}{6I_{-1}^2(\Omega_0^2)}+O\left[\frac{1}{I_{-1}^3}\right] \tag{2.32}$$

with $c>0$. If $c<1/16\pi^2$, then what we said above still applies although (2.30) becomes

$$xe^{2\ddot{H}_0(x)+4\pi^2c} \tag{2.33}$$

and (2.31) becomes

$$H_1(x)-\frac{\pi^2}{30}-\frac{x}{2}\dot{H}_1(x)+x^4\left[\frac{1}{128\pi^2}-\frac{c}{8}\right]=0\ . \tag{2.34}$$

If $c\geq 1/16\pi^2$, then as we see from (2.34) and (A9), ϵ is never entirely flat for any Ω_0.

The renormalized coupling constants are

$$\lambda=-\frac{1}{2\ddot{H}_0(\Omega_0)}<0 \tag{2.35}$$

in the first case and

$$\lambda=-\frac{1}{2[\ddot{H}_0(\Omega_0)+c]}<0 \tag{2.36}$$

in the second one.

Consider that $\Omega_0=0$ is not a solution of Eq. (2.17) so that $\partial V(0,\Omega_0)/\partial\Omega_0>0$. For reasons of continuity the minimum will be at $\Omega=0$ until some value of α, in such a way that we can write

$$\Omega_0'=\Omega_0''=0\ . \tag{2.37}$$

Also we have that

$$\frac{\partial V(0,\Omega_0)}{\partial\Omega_0}=\dot{H}_0(0)\{m_B{}^2+12\lambda_B[I_0(0)+H_0(0)]\}>0\ , \tag{2.38}$$

so that using (2.14) and (A6),

$$m^2<0\ . \tag{2.39}$$

Analogously we have that

$$\lambda=\lambda_B\ . \tag{2.40}$$

The energy density reads

$$\begin{aligned}\epsilon(\alpha,\Omega)=&-\frac{\Omega^4}{64\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-\frac{1}{2}\right]-\tfrac{1}{2}(\Omega^2-m^2)\left[H_0(\Omega)-\frac{\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-1\right]\right]\\&-6\lambda H_0(0)\left[H_0(\Omega)-\frac{\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-1\right]\right]+H_1(\Omega)-H_1(0)-\frac{m^2}{2}H_0(0)+6\lambda H_0{}^2(0)\\&+\tfrac{1}{2}m^2\alpha-6\lambda\alpha H_0(0)+\lambda\alpha^2+6\lambda\alpha\left[H_0(\Omega)-\frac{\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-1\right]\right]\\&+3\lambda\left\{H_0{}^2(\Omega)-H_0{}^2(0)+\left[H_0(0)+\frac{\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-1\right]\right]\left[\frac{\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\Omega^2}-1\right]\right]\right\}\ .\end{aligned} \tag{2.41}$$

Note that all IR divergences have canceled. For large Ω the behavior of ϵ is

$$\epsilon\underset{\Omega\gg 1}{\sim}\left[1+\frac{3\lambda}{4\pi^2}\ln\frac{4\Lambda^2}{\Omega^2}\right]\frac{\Omega^4}{64\pi^2}\ln\frac{4\Lambda^2}{\Omega^2}\ , \tag{2.42}$$

In order to get a nonzero λ it is necessary that λ_B is independent of the cutoff and positive.

As we have said, in a neighborhood of $\alpha=0$ the solution is $\Omega=0$ and ϵ is

$$\epsilon=-\frac{|m^2|}{2}\alpha+\lambda\alpha^2\ , \tag{2.43}$$

but beyond

$$\alpha>\frac{H_0(0)}{2\lambda}+\frac{4|m^2|\pi^2}{3\lambda} \tag{2.44}$$

the Ω which gives the minimum for ϵ is the solution of

$$\begin{aligned}\frac{\partial\epsilon}{\partial\Omega}=&\left\{6\lambda\left[\alpha-\frac{\bar\Omega^2}{16\pi^2}\left[\ln\frac{4\Lambda^2}{\bar\Omega^2}-1\right]\right]-\tfrac{1}{2}(\bar\Omega^2-m^2)\right\}\left[-\frac{\bar\Omega}{8\pi^2}\left[\ln\frac{4\Lambda^2}{\bar\Omega^2}-2\right]\right]-\bar\Omega H_0(\bar\Omega)+\bar{H}_1(\bar\Omega)\\&+6\lambda\dot{H}_0(\bar\Omega)[H_0(\bar\Omega)-H_0(0)]+3\lambda H_0(0)\frac{\bar\Omega}{8\pi^2}\left[\ln\frac{4\Lambda^2}{\bar\Omega^2}-2\right]=0\ ,\end{aligned} \tag{2.45}$$

which for large α implies

$$\epsilon\underset{\alpha\gg 1}{\sim}-2\lambda\alpha^2<0\ , \tag{2.46}$$

and so ϵ is unbounded from below.

If λ is zero then it is obvious from (2.43) that the theory is again unbounded below.

Finally, if $\Omega_0=0$ is a root of Eq. (2.17) we are in a limiting case of one of the previously studied cases. ϵ is at best flat. This ends the analysis.

III. PERTURBATION THEORY: COMPARISON

Stevenson showed[2] that the zero-temperature Gaussian effective potential does not reproduce perturbation theory; e.g., its $O(\hbar)$ contribution does not coincide with Coleman and Weinberg's one-loop result.[7] One can, however, reproduce the one-loop result if one expands the zero-temperature equation equivalent to (2.24) in λ_B, thereby working with perturbation theory, and keeping everywhere only $O(\hbar)$ terms. Let us quickly recall the steps at zero temperature (which just means dropping H_0 and H_1 everywhere).

The reintroduction of $\hbar$ is performed by multiplying the functions I_n, Σ, and Γ by $\hbar$. Equation (2.27) then leads, with the help of (2.19) to

$$\bar\Omega^2=m^2+12\lambda_B\alpha+O(\hbar) \tag{3.1}$$

and (2.24) to

$$\lambda=\lambda_B[1-18\lambda\hbar I_{-1}(m^2)]+O(\hbar^2)\ . \tag{3.2}$$

Using these expressions in (2.22) gives

$$V_0(\alpha)\equiv\epsilon_0(\alpha,\overline{\Omega})$$
$$=\tfrac{1}{2}m^2\alpha+\lambda\alpha^2+\hbar\Sigma(m^2+12\lambda\alpha,m^2)+O(\hbar^2)\ , \tag{3.3}$$

where the subscript 0 means zero temperature. This expression is the well-known one-loop result.[7]

Let us now perform the same steps at finite temperature. We would expect them to lead to the one-loop finite-temperature effective potential as given for instance by Dolan and Jackiw.[6] Surprisingly they do not.

From Eq. (2.27) one obtains

$$\overline{\Omega}^2=\Omega_0^2+12\lambda_B\alpha-\frac{2}{\Omega I_{-1}(\Omega^2)}\left[\dot{H}_1(\Omega)-\frac{\Omega}{\Omega_0}\dot{H}_1(\Omega_0)-\Omega[H_0(\Omega)-H_0(\Omega_0)]\right]\Bigg|_{\Omega=(\Omega_0^2+12\lambda_B\alpha)^{1/2}}$$
$$+O\left[\frac{1}{I_{-1}{}^2(\Omega^2)},\hbar\right]\ , \tag{3.4}$$

and from (2.19),

$$\Omega_0^2=m^2+O\left[\frac{1}{I_{-1}(m^2)}\right]\ , \tag{3.5}$$

whereas (2.24) gives

$$\lambda=\lambda_B\left[1-18\lambda\hbar I_{-1}(\Omega_0^2)+18\lambda\hbar\frac{1}{\Omega_0^2}\left[\ddot{H}_1(\Omega_0)-\frac{\dot{H}_1(\Omega_0)}{\Omega_0}\right]+O\left[\frac{1}{I_{-1}(m^2)},\hbar\right]\right]\ . \tag{3.6}$$

Using these expressions in (2.22) gives

$$V_T(\alpha)=\tfrac{1}{2}m^2\alpha+\lambda_B\alpha^2[1-18\lambda\hbar I_{-1}(m^2)]+\hbar\Sigma(m^2+12\lambda\alpha,m^2)$$
$$+\hbar\left[H_1((m^2+12\lambda\alpha)^{1/2})-H_1(m)-\frac{6\lambda\alpha}{m}\dot{H}_1(m)\right]+O(\hbar^2)\ . \tag{3.7}$$

If one defines the renormalized coupling constant and the renormalized mass (2.18) for $T=0$ one can write

$$V_T(\alpha)\equiv V_0(\alpha)+\hbar V_{\Delta T}(\alpha) \tag{3.8}$$

with

$$V_{\Delta T}(\alpha)=H_1((m^2+12\lambda\alpha)^{1/2})-H_1(m)-6\lambda\alpha\left[\frac{\dot{H}_1(m)}{m}-H_0(m)\right]\ . \tag{3.9}$$

The term proportional to $H_0(m)$ comes from (2.18).

This result, however, does not reproduce the temperature one-loop contribution[6] which normalized to zero at $\alpha=0$ is

$$V_{\Delta T}^{\rm pert}(\alpha)=\frac{1}{2\pi^2}\int_0^\infty dx\,x^2\ln\left[\frac{1-\exp[-(x^2+m^2+12\lambda\alpha)^{1/2}]}{1-\exp[-(x^2+m^2)^{1/2}]}\right]$$
$$=-\tfrac{1}{3}[H_1((m^2+12\lambda\alpha)^{1/2})-(m^2+12\lambda\alpha)H_0((m^2+12\lambda\alpha)^{1/2})-H_1(m)+m^2H_0(m)]\ . \tag{3.10}$$

Their respective expansions in α are

$$V_{\Delta T}(\alpha)=6\lambda H_0(m)\alpha+18\ddot{H}_0(m)\lambda^2\alpha^2+\frac{36\lambda^3}{m}\dddot{H}_0(m)\alpha^3+O(\alpha^4)\ ,$$
$$V_{\Delta T}^{\rm pert}(\alpha)=6\lambda H_0(m)\alpha+\frac{18\lambda^2}{m}\dot{H}_0(m)\alpha^2+\frac{36\lambda^3}{m^3}[m\ddot{H}_0(m)-\dot{H}_0(m)]\alpha^3+O(\alpha^4)\ . \tag{3.11}$$

What has happened? The steps which at zero temperature reproduce perturbation theory do not do so at finite temperature. It is easy to see that the root of the divergence lies in the finite-temperature vacuum expectation value of the kinetic energy of the Hamiltonian, i.e., in Eqs. (2.11) and (2.12). Let us quickly recall where our expressions (2.10) and (2.12) come from, and compare with Bardeen and Moshe's results.[8] Consider $\langle 0_\Omega | \phi^2(x) | 0_\Omega \rangle$. At zero temperature,

$$\langle 0_\Omega | \phi^2(x) | 0_\Omega \rangle^0 = I_0(\Omega^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\Omega^2+i\epsilon} , \tag{3.12}$$

and by using the real-time finite-temperature rule

$$\frac{i}{k^2-\Omega^2+i\epsilon} \to \frac{i}{k^2-\Omega^2+i\epsilon} + \frac{2\pi}{e^{\beta E}-1}\delta(k^2-\Omega^2) , \tag{3.13}$$

one obtains immediately (2.9) and (2.10). Alternatively, and following Bardeen and Moshe,[8] one can write in the imaginary time formalism

$$\langle 0_\Omega | \phi^2(x) | 0_\Omega \rangle^T = -\frac{1}{\beta} \sum_{n=0,\pm 1,\pm 2,\ldots} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2-\Omega^2} \tag{3.14}$$

with $k_0 \equiv \omega_n = i2\pi n/\beta$. It is well known that this expression leads to the previous one (Dolan and Jackiw[6]).

Consider now $\langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle$. At zero temperature

$$\langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^0 = I_1(\Omega^2) . \tag{3.15}$$

Furthermore, recall (2.16):

$$\frac{dI_1(\Omega^2)}{d\Omega} = \Omega I_0(\Omega^2) . \tag{3.16}$$

Now, Bardeen and Moshe assume that the relation (3.16) between (3.12) and (3.15) also holds at finite temperature, or in other words

$$\frac{dH_1(\Omega)}{d\Omega} = \Omega H_0(\Omega) , \tag{3.17}$$

where

$$\langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^T \equiv I_1(\Omega^2) + H_1(\Omega) . \tag{3.18}$$

It is easy to check from (3.17), (3.10), and (A1) that then $H_1(\Omega)$ as given by Eq. (3.17), which we will call $H_1^{\text{pert}}(\Omega)$, is precisely $V_{\Delta T}^{\text{pert}}(\alpha)$ (for a suitable constant of integration), i.e.,

$$H_1^{\text{pert}}((m^2+12\lambda\alpha)^{1/2}) - H_1^{\text{pert}}(m) = V_{\Delta T}^{\text{pert}}(\alpha) . \tag{3.19}$$

Obviously with this definition we would have retrieved perturbation theory as now (3.9) coincides with (3.19), recalling that H_1^{pert} satisfies (3.17).

Our point of view has been different. $H_1(\Omega)$ is defined by Eq. (3.18) and therefore should not, in principle, need another defining relation as (3.17). One can compute $H_1(\Omega)$ from (3.18) following steps as Bardeen and Moshe did for $H_0(\Omega)$ from (3.14). We first write

$$\langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^T = \lim_{x\to y} \langle 0_\Omega | T[\dot\phi(x)\dot\phi(y)] | 0_\Omega \rangle^T . \tag{3.20}$$

One can write (3.20) as

$$\langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^T = \lim_{x\to y} \partial_{x^0}\partial_{y^0} \langle 0_\Omega | T[\phi(x)\phi(y)] | 0_\Omega \rangle^T . \tag{3.21}$$

This quickly leads to

$$\begin{aligned} \langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^T &= -\frac{1}{\beta}\sum_n \omega_n^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2-\Omega^2} \\ &= -\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \\ &\quad -\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{k}^2+\Omega^2}{k^2-\Omega^2} , \end{aligned} \tag{3.22}$$

which performing the standard steps to go from the imaginary to real time formalism gives, neglecting the irrelevant Ω independent first term in the last expression of (3.22),

$$\begin{aligned} \langle 0_\Omega | \dot\phi^2(x) | 0_\Omega \rangle^T &= \int \frac{d^4k}{(2\pi)^4} (\mathbf{k}^2+\Omega^2) \left[\frac{i}{k^2-\Omega^2+i\epsilon} \right. \\ &\quad \left. + \frac{2\pi}{e^{\beta E}-1}\delta(k^2-\Omega^2) \right] . \end{aligned} \tag{3.23}$$

This is our expression for $H_1(\Omega)$, (2.12).

Our understanding of this interesting and puzzling situation is as follows. At $T=0$ the nonperturbative method we follow does not reproduce perturbation theory; this is not surprising. However, when the steps of taking $\Lambda\to\infty$ and $\lambda\to 0$ are reversed, i.e., first $\lambda\to 0$ and then $\Lambda\to\infty$, perturbation theory is recovered. This is also not surprising. At finite temperature, however, the reversal which leads to perturbation theory for $T=0$ is not anymore enough to recover it, as the nonperturbative character of the approach is also contained in the evaluation of the finite-temperature kinetic energy vacuum expectation value, which is independent of λ_B. Thus one only recovers perturbation theory if the finite-temperature kinetic energy vacuum expectation value is *computed as in perturbation theory;* if not, perturbation theory cannot be retrieved from our nonperturbative approach.

IV. CONCLUSIONS

We have shown that a $\lambda\phi^4$ theory with a bare coupling constant depending on Λ as

$$\lambda_B = -\frac{1}{6I_{-1}(m^2)}\left[1+\frac{c}{I_{-1}(m^2)}+O\left[\frac{1}{I_{-1}{}^2(m^2)}\right]\right] \tag{4.1}$$

with $0 \le c \le \frac{1}{16}\pi^2$, which, at zero temperature according to

Eqs. (1.1) and (1.2), does not lead to an interacting renormalized theory, but does so above a certain finite temperature. This phase has

$$m^2 > 0 \,, \tag{4.2}$$

and its coupling constant is given by

$$\lambda = -\frac{1}{2[\ddot{H}_0(m)+c]} < 0 \,. \tag{4.3}$$

If the approximations used are not misleading, temperature might play a far more important role in the construction of interacting field theories than what is generally thought. We have seen that it might even be the very source of stability and interaction.

These results are qualitatively different from the ones of perturbation theory, as we saw in Sec. III. At finite temperature our approach is just essentially different from a perturbative one. Our results should therefore be taken with care, but they indicate that very likely also the perturbative results should not be taken for granted. We do not have the means in this work to settle this question.

ACKNOWLEDGMENT

This work has been financially supported by Comisión Asesora de Investigación Cientifica y Tecnica (Spain).

APPENDIX

The functions $H_0(\Omega)$ and $H_1(\Omega)$ given by Eqs. (2.10) and (2.12) can be written as

$$H_0(\Omega) = \frac{1}{2\pi^2}\int_0^\infty dx \frac{[x(x+2\Omega)]^{1/2}}{e^{x+\Omega}-1} \tag{A1}$$

and

$$H_1(\Omega) = \frac{1}{2\pi^2}\int_0^\infty dx \frac{[x(x+2\Omega)]^{1/2}}{e^{x+\Omega}-1}(x+\Omega)^2 \,, \tag{A2}$$

where Ω is $|\Omega|$.

They have the following properties:

$$H_0(0) = \frac{1}{12} \,, \quad H_1(0) = \frac{\pi^2}{30} \,, \tag{A3}$$

$$H_0(\infty) = H_1(\infty) = 0 \,, \tag{A4}$$

$$\dot{H}_1(\Omega) \equiv \frac{dH_1(\Omega)}{d\Omega} = \Omega^2 \dot{H}_0(\Omega) - \Omega H_0(\Omega) \,, \tag{A5}$$

$$\dot{H}_0(\Omega) < 0 \,, \tag{A6}$$

$$\dot{H}_1(0) = 0 \,, \tag{A7}$$

$$\ddot{H}_0(\Omega) > 0 \,. \tag{A8}$$

Finally, using (A7) and (A8) one sees that

$$H_1(\Omega) - H_1(0) - \frac{\Omega}{2}\dot{H}_1(\Omega) < 0 \,. \tag{A9}$$

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