Nonrelativistic noncommutative field theory and UV-IR mixing

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We study a non-commutative non-relativistic scalar field theory in $2+1$ dimensions. The theory shows the UV-IR mixing typical of QFT on non-commutative spaces. The one-loop correction to the two-point function turns out to be given by a δ function in momentum space. The one-loop correction to the four-point function is of logarithmic type. We also evaluate the thermodynamic potential at two-loop order. To avoid an IR singularity we have to introduce a chemical potential. The suppression of the non-planar contribution with respect to the planar one turns out to depend crucially on the value of the chemical potential.

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I. INTRODUCTION

Non-commutative field theories have an unconventional perturbative behavior [1–21]. New infrared singularities in the correlation functions appear even for massive theories [9–21]. This phenomenon is due to an interplay between the UV and IR induced by the Moyal phase appearing in the vertices. Recently some of the amplitudes of these non-commutative theories have been derived from string theory [22–28].

In this paper we analyze at the perturbative level the non-commutative version of a non-relativistic scalar field theory in $2+1$ dimensions in order to gather more information about the UV-IR mixing and the structure of degrees of freedom of non-commutative theories. Our motivation for investigating non-relativistic non-commutative field theories is twofold. In non-relativistic quantum theory the non-commutativity of space arises often in the effective description of charged particles carrying dipole momentum moving in strong magnetic fields [29]. It seems natural then to look for the by-now well-known UV-IR mixing in the context of non-relativistic quantum field theory. Another more theoretical motivation is that it might be easier to understand the effects of the non-commutativity of space in simpler setups than relativistic quantum field theory. We also would like to emphasize that due to an ordering ambiguity in the interaction vertices the particular model we are considering cannot be obtained as the non-relativistic limit of a relativistic field model.

As we will see, also in this non-relativistic example there is an interplay between the IR and UV behavior due to the Moyal phases. For the two-point function a singularity of delta-function type appears. This should be contrasted with the pole-like singularities found for relativistic theories [9]. For the four-point function we find a singularity of a more familiar, logarithmic type.

We study the system at finite temperature and non-zero chemical potential μ. We compute the thermodynamical potential up to two loops. The presence of the chemical potential provides another scale besides the non-commutativity scale and temperature. If $-\mu \gg T$, the non-planar contribution is strongly suppressed with respect to the planar one for thermal wavelengths smaller than the non-commutativity scale. This suggests a reduction of degrees of freedom running in the non-planar graphs at high temperature [20,21]. The limit of $-\mu \ll T$ is more involved. The non-planar graph does not appear to be strongly suppressed as a function of the temperature. It depends crucially on the ratio between the chemical potential and the non-commutativity scale.

The paper is organized as follows. In Sec. II, we study the two- and four-point functions up to one-loop. In Sec. III we study the free energy up to two loops. We give some conclusions in Sec. IV.

II. TWO- AND FOUR-POINT FUNCTIONS AT ONE LOOP

We will start by introducing the model. We will work in $2+1$ dimensions, where the non-commutativity affects only the spatial directions. Non-commutative R^2 is defined by the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2.1}$$

with $\theta^{\mu\nu} = \theta\epsilon^{\mu\nu}$. The algebra of functions on non-commutative R^2 is defined through the star product

$$(f*g)(x) := \lim_{y \to x} e^{(i/2)\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y} f(x)g(y). \tag{2.2}$$

Here x^μ are taken to be ordinary c numbers. We will study a self-interacting non-relativistic scalar field model, defined by the Lagrangian

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$$\mathcal{L}=\phi^{\dagger}\left(i\partial_t+\frac{\vec{\nabla}^2}{2}\right)\phi-\frac{g}{4}\phi^{\dagger}*\phi^{\dagger}*\phi*\phi. \qquad (2.3)$$

The star product has been dropped in the term bilinear in the fields. This is consistent since we can always delete one star in monomials of fields in the action. This is equivalent to neglecting total derivative terms. In ordinary space-time this model arises as the low energy limit of a real relativistic scalar field with ϕ^4 self-interaction. It has been studied in [30] as a model for applying renormalization to quantum mechanics with δ-function potential [31]. The model is scale invariant in ordinary space-time since scale transformations in a non-relativistic theory take the form $t\rightarrow\lambda^2 t,\ \vec{x}\rightarrow\lambda\vec{x}$. The scaling of t is due to the fact that in Eq. (2.3) the mass has been scaled out by redefining $t\rightarrow mt$. It has been shown that the theory acquires a scale anomaly upon quantization quite analogous to what happens in relativistic quantum field theory. Of course, in the case considered here scale invariance is already broken at the tree level by the non-commutativity scale $\sqrt{\theta}$.

In going from ordinary space-time to the non-commutative one an ordering ambiguity for the interaction term arises. We fix that ambiguity in Eq. (2.3) by putting the $\phi^{\dagger}$ fields to the left. The other possible ordering would have been to choose $\mathcal{L}_{int}=(-g'/4)\phi^{\dagger}*\phi*\phi^{\dagger}*\phi$. A relativistic complex scalar field model with both interaction vertices has been considered in [15]. There the authors showed that the theory was renormalizable at one-loop level only when $g=g'$ or $g=0$. We will later on show that no such restriction arises in the non-relativistic model (2.3).

Notice that the free theory is unchanged with respect to the ordinary case, since the star product has no effect on the quadratic terms in the action. Therefore all the effects of non-commutative space appear through the interaction vertices. The solutions to the free field equations can be written as Fourier transforms

$$\phi(\vec{x},t)=\int\frac{d^2k}{(2\pi)^2}a(\vec{k})e^{-i(\omega_k t-\vec{k}\cdot\vec{x})},$$
$$\phi^{\dagger}(\vec{x},t)=\int\frac{d^2k}{(2\pi)^2}a^{\dagger}(\vec{k})e^{i(\omega_k t-\vec{k}\cdot\vec{x})}, \qquad (2.4)$$

where $\omega_k=\vec{k}^2/2$. The propagator of the theory is given by

$$\langle T\phi(x)\phi^{\dagger}(y)\rangle=\int\frac{d^2k\,d\omega}{(2\pi)^3}\frac{ie^{-i(\omega t-\vec{k}\cdot\vec{x})}}{\omega-\dfrac{\vec{k}^2}{2}+i\epsilon}. \qquad (2.5)$$

We want to compute now the one-loop correction to the two-point function. This is given by the tadpole diagram of Fig. 1. In ordinary space-time we can employ a normal ordering prescription setting the tadpole to zero. In the non-commutative theory we expect a dependence of the tadpole on the external momentum due to the contribution of the non-planar diagrams. The planar and non-planar contributions are given by
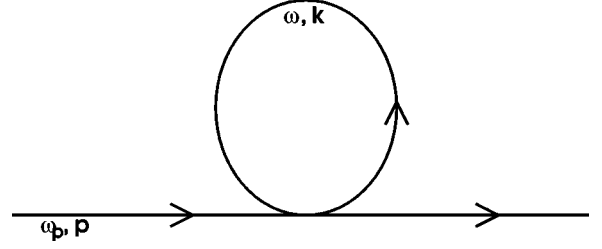

FIG. 1. The tadpole contribution to the self-energy.

$$-i\Sigma(E,\vec{p})=I_{planar}+I_{non\text{-}planar} \qquad (2.6)$$

and

$$I_{planar}=\frac{g}{2}\int\frac{d\omega\,d^2k}{(2\pi)^3}\frac{1}{\omega-\dfrac{\vec{k}^2}{2}+i\epsilon}, \qquad (2.7)$$

$$I_{non\text{-}planar}=\frac{g}{2}\int\frac{d\omega\,d^2k}{(2\pi)^3}\frac{\exp(i\vec{\tilde{p}}\cdot\vec{k})}{\omega-\dfrac{\vec{k}^2}{2}+i\epsilon},$$

where $\tilde{p}^{\mu}=\theta^{\mu\nu}p_{\nu}$. In order to do the ω integration we recall that $(x+i\epsilon)^{-1}=\mathcal{P}(1/x)-i\pi\delta(x)$. This leaves us with the $\vec{k}$ integrations

$$I_{planar}=\frac{-ig}{8\pi}\int_0^{\Lambda}k\,dk=\frac{-ig}{16\pi}\Lambda^2, \qquad (2.8)$$

$$I_{non\text{-}planar}=\frac{-ig}{16\pi^2}\int d\varphi\int_0^{\Lambda}e^{i\tilde{p}k\cos(\varphi)}k\,dk$$
$$=\frac{-ig}{8\pi\tilde{p}}\Lambda J_1(\tilde{p}\Lambda),$$

where we introduced a UV cutoff Λ and $J_1(x)$ denotes a Bessel function. The quadratic divergence from the planar part can be removed by adding a corresponding counterterm to the action, $\mathcal{L}_c=\delta\mu\phi^{\dagger}\phi$. The non-planar part reproduces the quadratic divergence for $\tilde{p}\rightarrow0$ since $\lim_{x\rightarrow0}[J_1(x)/x]=\frac{1}{2}$. In the limit $\Lambda\rightarrow\infty$, using $\int_0^{\infty}J_1(x)dx=1$, it is straightforward to show that the result from the non-planar diagram represents a delta function in polar coordinates in $\tilde{p}$ space. We find, then,

$$\Sigma(E,\vec{p})=\frac{g}{4\theta^2}\delta^2(\vec{p}). \qquad (2.9)$$

Thus the situation is rather analogous to what happened in relativistic field theories. The limits of $\Lambda\rightarrow\infty$ and $p\rightarrow0$ do not commute.

It is interesting to see that we can recover the delta-type singularity of the non-planar diagram as a limit of the relativistic case. The relativistic theory is given by the Lagrange density

$$\mathcal{L}_{rel} = \frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r*\phi_r*\phi_r*\phi_r.$$
(2.10)

The non-relativistic limit can be obtained as a $1/m$ expansion. We take off the fast oscillation due to the large mass and introduce dimensionless non-relativistic fields by defining

$$\phi_r = \frac{1}{\sqrt{2m}}(e^{-imt}\phi + e^{imt}\phi^\dagger).$$
(2.11)

To extract the non-relativistic limit we have to expand the vertex in Eq. (2.10) and compare with the vertex in Eq. (2.3). From the relativistic vertex we obtain

$$\mathcal{L}_{rel} = -\frac{\lambda}{4!m^2}\left(\phi^\dagger*\phi^\dagger*\phi*\phi + \frac{1}{2}\phi^\dagger*\phi*\phi^\dagger*\phi\right).$$
(2.12)

Note that the non-relativistic limit produces both possible orderings in the interaction. Therefore in the non-commutative case our model (2.3) is not the non-relativistic limit of a real relativistic scalar field. It turns out however that only the first vertex in Eq. (2.12) contributes to the non-planar tadpole diagram. We should be able then to obtain the result (2.9) from the relativistic case. Comparing Eq. (2.3) with Eq. (2.12) we see that $\lambda = 6mg$.[1] The non-planar contribution to the tadpole diagram in the relativistic theory is given by

$$I_{rel} = \frac{\lambda}{6(2\pi)^3}\int d^3k\,\frac{e^{i\tilde{p}_\mu k^\mu}}{k^2 - m^2}.$$
(2.13)

In order to evaluate the integral we switch to Euclidean momentum and use Schwinger parametrization. We obtain

$$I_{rel} = \frac{-i\lambda\pi^{3/2}}{6(2\pi)^3}\int_0^\infty \frac{d\alpha}{\alpha^{3/2}}e^{-\tilde{p}^2/4\alpha - \alpha m^2} = \frac{-i\lambda e^{-\tilde{p}m}}{24\pi\tilde{p}}.$$
(2.14)

There arises also a factor of $\frac{1}{2}$ which can be seen by noting that the integral I_{rel} defines the relativistic self-energy Σ_{rel}. The relativistic dispersion relation is $(p_0-m)(p_0+m)-\vec{p}^2 - \Sigma_{rel} = 0$. Setting $p_0 = m + E$ where E is the non-relativistic energy we can go to the non-relativistic limit by scaling $m \to \infty$ and $E \to 0$, keeping the product $Em = \omega$ fixed. This is the non-relativistic energy of dimension 2. In this way the relativistic dispersion relation becomes twice the non-relativistic one if we identify $\lim_{m\to\infty}\Sigma_{rel} = 2\Sigma_{non\text{-}rel}$. Substituting for λ it is then easy to show that

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[1] Recall that in Eq. (2.3) we have rescaled $t \to mt$ and $\mathcal{L} \to m\mathcal{L}$ in order to factor out the mass dependence.
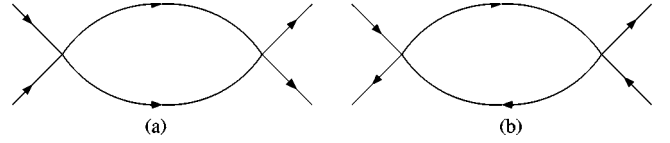

FIG. 2. The one-loop contribution to the four-point function. The diagrams with momentum flow as indicated in (b) vanish identically in non-relativistic field theory.

$$\lim_{m\to\infty} I_{rel} = -i\frac{g}{2}\delta^2(\tilde{\vec{p}}).$$
(2.15)

Thus we reproduce precisely the non-relativistic result (2.9).

If we formally sum all the tadpole diagrams contributing to the two-point function, we obtain the following modified dispersion relation:

$$\omega = \frac{\vec{p}^2}{2} + \frac{g}{4\theta^2}\delta^2(\vec{p}).$$
(2.16)

In the resummation one encounters arbitrary high powers of the δ functions. We just showed that the dispersion relation (2.16) arises also as the limit of the relativistic one. Therefore we expect it to be correct on physical grounds. Alternatively we could keep the cutoff and arrive at Eq. (2.16) with a suitable smeared δ function, for example keeping the mass at a large but finite value and using Eq. (2.14). The meaning of Eq. (2.16) is that the energy of the zero momentum states is shifted by an infinite amount. However, it is important to note that the delta function in the dispersion relation is integrable. Thus wave packets containing zero momentum components still will have finite energy.

We would like now to evaluate the four-point function at one loop (see Fig. 2). In a non-relativistic theory only the s channel contributes, since the t and u channels contain internal lines flowing both forward and backwards in time and this evaluates to zero in a non-relativistic theory. As shown in [15] the contributions from u and t channels make the relativistic complex scalar field non-renormalizable if one does not also include the second possible ordering for the vertex. It is thus the vanishing of u and t channels that allows us to ignore the second possible ordering in the vertex. The non-relativistic one-loop four-point function is [30]

$$\Gamma_4(\omega_i,\vec{p}_i,\Lambda) = -\frac{\lambda^2}{8\pi}\left(\log\frac{\Lambda^2}{E - \frac{\vec{P}^2}{4}} + i\pi\right),$$
(2.17)

where $E = \omega_1 + \omega_2 = \omega_1' + \omega_2'$ and $\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{q}_1 + \vec{q}_2$ are the center of mass energy and momentum, and $\omega_i, \vec{p}_i$ and $\omega_i', \vec{p}_i'$ the energy and momentum of the incoming and outcoming particles respectively; Λ is an UV cutoff.

The one-loop four-point function for the non-commutative case is given by

$$\Gamma_4 = \frac{i\lambda^2}{2}\cos\frac{\tilde{\vec{p}}_1 \cdot \vec{p}_2}{2}\cos\frac{\tilde{\vec{q}}_1 \cdot \vec{q}_2}{2}$$

$$\times \int \frac{d^2k \, d\omega}{(2\pi)^3} \frac{\cos^2\frac{\tilde{\vec{P}} \cdot \vec{k}}{2}}{\left(\omega - \frac{\vec{k}^2}{2} + i\epsilon\right)\left(E - \omega - \frac{(\vec{k} - \vec{P})^2}{2} + i\epsilon\right)}. \tag{2.18}$$

Using $\cos^2(x/2) = (1+\cos x)/2$ and writing $\cos x$ in terms of exponentials, we can separate the planar and non-planar contributions. After doing the ω integration and shifting $k \to k + P/2$ we get, for the non-planar part,

$$\Gamma_4^{non\text{-}planar} = -\frac{\lambda^2}{4}\cos\frac{\tilde{\vec{p}}_1 \cdot \vec{p}_2}{2}\cos\frac{\tilde{\vec{q}}_1 \cdot \vec{q}_2}{2}$$

$$\times \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\tilde{\vec{P}} \cdot \vec{k}}}{\vec{k}^2 - E + \frac{\vec{P}^2}{4} - 2i\epsilon}. \tag{2.19}$$

This integral can be analyzed by changing to polar coordinates in momentum space. For angles such that $\tilde{\vec{P}} \cdot \vec{k} > 0$, we can evaluate Eq. (2.19) by using a contour encircling the first quadrant of the $|k|$ complex plane. For angles such that $\tilde{\vec{P}} \cdot \vec{k} < 0$, it is convenient to use a contour in the $|k|$ plane encircling the fourth quadrant. Adding both contributions, we obtain the following result:

$$\Gamma_4^{non\text{-}planar} = -\frac{\lambda^2}{16}\cos\frac{\tilde{\vec{p}}_1 \cdot \vec{p}_2}{2}\cos\frac{\tilde{\vec{q}}_1 \cdot \vec{q}_2}{2}$$

$$\times \left[-Y_0\left(\tilde{P}\sqrt{E - \frac{\vec{P}^2}{4}}\right) + iJ_0\left(\tilde{P}\sqrt{E - \frac{\vec{P}^2}{4}}\right) \right], \tag{2.20}$$

where J_0 and Y_0 denote Bessel functions of first and second kinds, respectively. In order to better understand this expression, it is convenient to expand the Bessel functions for small $\tilde{P}$. Up to a real constant and $O(\tilde{P})$ terms, the result is

$$\Gamma_4^{non\text{-}planar} = -\frac{\lambda^2}{16\pi}\cos\frac{\tilde{\vec{p}}_1 \cdot \vec{p}_2}{2}\cos\frac{\tilde{\vec{q}}_1 \cdot \vec{q}_2}{2}\left(\ln\frac{1/\tilde{\vec{P}}^2}{E - \frac{\vec{P}^2}{4}} + i\pi \right). \tag{2.21}$$

The non-commutative phases regulate the otherwise divergent contribution coming from high momentum. The resulting dependence of the non-planar diagram on the external momentum is smoother than for the two-point function. The external momentum $\tilde{P}$ acts as an UV cutoff very much in the same way as in previously analyzed examples of relativistic theories [9,12,16,19].

III. FINITE TEMPERATURE BEHAVIOR

In this section we analyze the thermodynamics of our model. The physical reason to consider this system in a heat bath is to check if there is a reduction of degrees of freedom for the non-planar sector of the theory. In the case of relativistic theories this was shown to happen for thermal wavelengths smaller than the non-commutative length scale [20,21].

Before we embark on doing the calculation we remind the reader of the following formula:

$$\sum_n \frac{1}{i\omega_n - x} = -\frac{\beta}{2} - \frac{\beta}{e^{\beta x} - 1}, \tag{3.1}$$

where $\omega_n = 2\pi n/\beta$. The first term on the right-hand side (RHS) represents the zero temperature contributions. The resulting zero temperature divergences can be canceled by the introduction of appropriate couterterms.

We will compute the thermodynamic potential up to two loop. In order to cure infrared divergences we will introduce a chemical potential term $\mu\phi^\dagger\phi$ in our Lagrangian (2.3). The introduction of a chemical potential seems natural taking into account the renormalization properties of the theory at zero temperature. Notice that now three scales are present. Correspondingly we have two regimes of high temperature. By high temperature we mean a thermal wavelength[2] much smaller than the scale of non-commutativity or, equivalently, $T\theta \gg 1$. The physics is then still dependent on the chemical potential.[3] In the regime $-\mu \gg T$ we expect a classical particle picture to be valid. We will also investigate the regime $T \gg -\mu$, where classical field theory is a good approximation to quantum statistical mechanics.

The one-loop contribution to the thermodynamic potential $F = -T\log Z$ is given by

$$-T\int \frac{d^2k}{(2\pi)^2}\log(1 - e^{-\beta(\vec{k}^2/2 - \mu)}) = T^2\text{Li}_2(1 - e^{\mu/T}), \tag{3.2}$$

where Li_2 denotes the dilogarithm. The the two loop contribution is given by

$$I = \frac{g}{2}T^2\sum_{l,n}\int \frac{d^2p}{(2\pi)^3}\frac{d^2k}{(2\pi)^3}$$

$$\times \frac{\cos^2\frac{\tilde{\vec{p}} \cdot \vec{k}}{2}}{\left(i\omega_l - \frac{\vec{p}^2}{2} + \mu\right)\left(i\omega_n - \frac{\vec{k}^2}{2} + \mu\right)}. \tag{3.3}$$

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[2]The thermal wavelength of a non-relativistic system is given by $\lambda_T = 2\pi/\sqrt{T}$ ($m = 1$).

[3]In non-relativistic theory the chemical potential takes values in $(-\infty, 0)$.

As in the zero temperature case we substitute $\cos^2(x/2) = (1 + \cos x)/2$ separating the planar and non-planar parts.

Using formula (3.1) we obtain three contributions to the planar part. The $(T=0, T=0)$ is a temperature independent divergence. The $(T=0, T)$ contributions are divergent. They can be canceled by adding counterterm of the form of the chemical potential $\delta\mu\phi^\dagger\phi$. The (T, T) contribution can be easily integrated:

$$I_{planar} = \frac{g}{8\pi^2} T^2 [\ln(1 - e^{\mu/T})]^2. \qquad (3.4)$$

The non-planar contribution to the free energy contains again three pieces. The first one is temperature independent and finite. The $(T=0, T)$ contribution is

$$I_{non\text{-}planar}^{T=0} = \frac{g}{2} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{p}\cdot\vec{k}}}{e^{\beta(\vec{k}^2/2 - \mu)} - 1}$$

$$= \frac{g}{8\pi^2\theta^2} \frac{1}{e^{-\mu/T} - 1}. \qquad (3.5)$$

This can be interpreted as a one-loop contribution due to the shift in the dispersion relation (2.16). The (T, T) contribution is

$$I_{non\text{-}planar}^{T} = \frac{g}{8\pi^2} \int p\,dp\,k\,dk \frac{J_0(\tilde{p}k)}{(e^{\beta(k^2/2 - \mu)} - 1)(e^{\beta(k^2/2 - \mu)} - 1)}. \qquad (3.6)$$

Since $J_0 \leqslant 1$, we see that the non-planar contribution is suppressed with respect to the planar one. The strength of the suppression will depend on the value of the two dimensionless quantities θT and $-\mu/T$.

We will analyze first the regime $-\mu/T \gg 1$. In this limit we can substitute the Bose-Einstein distribution by the Maxwell-Boltzmann distribution. This corresponds to considering low densities for the thermal gas. This is the particle approximation to the quantum field theory. In this limit we can evaluate the integral explicitly:

$$I_{non\text{-}planar}^{T} = \frac{g}{8\pi^2} \frac{T^2}{1 + (\theta T)^2} e^{2\mu/T}. \qquad (3.7)$$

For $\theta T \ll 1$ planar and non-planar graphs give the same contribution. For $\theta T \gg 1$ there is a very strong suppression of the non-planar sector. The T^2 dependence of Eq. (3.4) is substituted by $1/\theta^2$. When T is larger than $1/\theta$ the thermal wavelength $\lambda_T \sim 1/\sqrt{T}$ becomes smaller than the radius of a Moyal cell. Equation (3.7) seems to indicate that the effective wavelength of the modes that circulate in the non-planar loop can not be smaller than the radius of the Moyal cell.

We analyze now the regime of small $-\mu/T \ll 1$. The classical thermal field theory approximation consists in dimensionally reducing the system along the Euclidean time direction or, equivalently, considering only the zero mode in the sum over Matsubara frequencies. In the limit of small

$-\mu/T$ this approximation is valid up to modes of momentum $k^2 < T$. On the other hand the non-commutative phases suppress modes of momentum $k^2 > 1/\theta$ as can be explicitly seen from the Bessel function appearing in (3.6). Therefore when $\theta T \gg 1$ and $-\mu/T \ll 1$ we expect that the classical field approximation will describe the leading behavior of the non-planar sector [21].[4] The integral (3.6) can be evaluated in this limit with the result

$$I_{non\text{-}planar}^{T} = \frac{g}{8\pi^2} T^2 G((-\mu\theta)^2), \qquad (3.8)$$

where $G(z) = G_{13}^{31}(z|_{000}^{0}) = (1/2\pi i)\int \Gamma(1+s)^3\Gamma(-s)z^s ds$ denotes a Meijer G function. The suppression of the non-planar sector with respect to the planar one appears in this case to be only logarithmic with the temperature. However, contrary to the previous case, the ratio between planar and non-planar contributions depends also on $-\mu\theta$. For $-\mu\theta$ large the function G tends to zero, implying an additional suppression for the non-planar sector. For $-\mu\theta$ small G diverges. This divergence is associated with the infrared problems of the theory at small chemical potential.

IV. DISCUSSION AND CONCLUSIONS

We have seen that the phenomena of UV-IR mixing is not only a characteristic of relativistic theories but also occurs in non-relativistic theories. The model we have considered is a non-commutative version of a $(2+1)$-dimensional model that describes many particle quantum mechanics with a delta function interaction. For the two-point function we have seen the appearance of an IR singularity of delta function type which changes the dispersion relation. For the four-point function we found a logarithmic singularity. Thus the non-relativistic model has an UV-IR mixing similar to the relativistic field theories studied so far. Since our model cannot be embedded in a natural way in a string theory, one might interpret this as evidence that the IR singularities are not connected to closed string states that do not decouple from the field theory in agreement with the results of Refs. [22,25].

The renormalizability of non-commutative field theories to all loop order is still an open problem [6,8]. The non-relativistic scalar field model might prove to be a simple and interesting toy model for such a study. The fact that some diagrams vanish identically (such as t- and u-channel contributions to the four-point function) could simplify a systematic study of renormalizability. That in resumming the self-energy insertions in the propagator one has to deal with powers of delta functions should not *a priori* be considered as an insurmountable obstacle. As we argued such a formal resummation is physically well motivated. Indeed the delta

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[4]Notice that in our case the two spatial directions are non-commutative. Therefore we expect the suppression of high momenta by θ to be more effective than in the cases studied in [21], where the classical approximation was applied to a system with odd spatial dimensions.

function appears also in the non-relativistic limit of the re-summed propagator of relativistic ϕ^4 theory.

We have also studied the two loop correction to the free energy and we have seen that the non-planar part of the theory is very sensitive to the value of the chemical potential. At large negative values it turns out that the non-planar part is strongly suppressed compared to the planar part. In this regime the behavior is similar to what has been found in relativistic theories in [21]. The thermal wavelength of the degrees of freedom in non-planar diagrams cannot become smaller that the non-commutativity scale. Therefore these degrees of freedom are suppressed at high temperature.

This interpretation is less clear at high temperature and small chemical potential. It turned out that the non-planar part is at most logarithmically suppressed. Given that these two regimes behave so differently it should be an interesting direction of further research to study the effects of a chemical potential also in relativistic, non-commutative field theories.

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