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Spin-2 gravitational trace anomaly

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The part proportional to the Euler-Poincaré characteristic of the contribution of spin-2 fields to the gravitational trace anomaly is computed. It is seen to be of the same sign as all the lower-spin contributions, making anomaly cancellation impossible. Subtleties related to Weyl invariance, gauge independence, ghosts, and counting of degrees of freedom are pointed out.

I. INTRODUCTION

Cancellation of anomalies has played an important role in theoretical particle physics. In this paper we finish a study concerning the cancellation of the gravitational trace anomaly in four dimensions, by computing the contribution which the anomaly gets from spin-2 particles. In fact, only one of the two pieces of the anomaly is computed, as it suffices to prove that no cancellation from the contributions from spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 is possible, as all contributions have the same sign. We indicate how the other piece can be computed, not an easy task, but exhaustion and the fact that the results obtained killed our original motivation have prevented us from actually doing it. The result obtained might seem to contradict results known from supergravity. This is however not the case. In supergravity the Lagrangian for spin-$\frac{3}{2}$ and spin-2 particles is not Weyl (local dilation) invariant. Thus the vacuum expectation value of the trace of the renormalized energy-momentum tensor receives not only the anomalous contribution but also one due to the explicit breaking of Weyl invariance at the classical level. In supergravity it is this total trace which plays the important role, it is the one which appears in a supermultiplet together with the divergences of the axial-vector current and supercurrent. Here our point of view will be different. We consider the anomaly cancellation issue in a setting unconstrained by supergravity. Weyl invariance will be our primary symmetry and in order to be sure that the trace computed is the genuine anomaly we will start from a Weyl-invariant Lagrangian.

We therefore consider a massless spin-2 field $\phi^{\mu\nu}$ of the same dimension as lower-spin fields coupled to an arbitrary classical gravitational field $g^{\mu\nu}$ and otherwise free. Nothing is assumed about the physical content of this field, in particular about its relation to quantum gravity. Thus $\phi^{\mu\nu}$ has mass dimension while $g^{\mu\nu}$ is dimensionless. Its Lagrangian will be Weyl invariant. Being quadratic in $\phi^{\mu\nu}$ the one-loop result coincides with the exact effective action, which is obtained by integrating out the spin-2 field. It is a functional of the classical gravitational field and its derivatives. This effective action contains UV divergences which require regularization. We will follow Brown and Cassidy[1] and Duff[2] in using dimensional regularization, which has several advantages as we will recall later. The regularized effective action has then a pole in $n-4$, n being the dimension of space-time. This pole has to be subtracted out with a counterterm. If the Lagrangian of the theory is Weyl invariant, then the residue of the counterterm is Weyl invariant too. Defining the counterterm $\Delta W(n)$ according to

$$W_{\text{ren}}=\lim_{n\to 4}[W(n)+\Delta W(n)] \tag{1.1}$$

means that it is of the form

$$\Delta W(n)=\frac{1}{(4\pi)^2}\frac{1}{n-4}\int d^nx\,(-g)^{1/2}[\alpha F(x)+\beta G(x)]\ , \tag{1.2}$$

where $g\equiv\det g_{\mu\nu}$ and

$$\begin{aligned} F&\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{3}R^2\ ,\\ G&\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2\ , \end{aligned} \tag{1.3}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}\equiv R^{\rho}{}_{\mu\rho\nu}$ the Ricci tensor, and $R\equiv R^{\mu}{}_{\mu}$ the scalar curvature. F is in four dimensions the Weyl tensor squared, $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$, and G leads, by the Gauss-Bonnet theorem, to the Euler-Poincaré characteristic. By using the well-known formulas

$$\begin{aligned} \frac{2}{[-g]^{1/2}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx\,[-g]^{1/2}F &=-(n-4)(F-\tfrac{2}{3}\Box R)\ ,\\ \frac{2}{[-g]^{1/2}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^nx\,[-g]^{1/2}G &=-(n-4)G\ , \end{aligned} \tag{1.4}$$

one obtains the trace anomaly from

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=\frac{2}{[-g]^{1/2}}g^{\mu\nu}\frac{\delta(\Delta W(n))}{\delta g^{\mu\nu}}\bigg|_{n=4}\ , \tag{1.5}$$

which gives

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=-\frac{1}{16\pi^2}[\alpha(F-\tfrac{2}{3}\Box R)+\beta G]\ . \tag{1.6}$$

Notice the strong restriction to the *a priori* general form

$$\langle T^{\mu}_{\text{ren}\ \mu}\rangle=-\frac{1}{16\pi^2}[\alpha(F-\tfrac{2}{3}\Box R)+\beta G+\gamma\Box R+\delta R^2] \tag{1.7}$$

as (1.6) implies $\gamma=\delta=0$.

The values of α and β for spin 0, $\frac{1}{2}$, and 1 are known and have been checked several times. Birrell and Davies[3] give most references concerning these spins. The values for spin $\frac{3}{2}$ have been computed recently (they were known for supergravity, which of course does not lead to $\gamma=\delta=0$ and for which α and β depend on the gauge parameter[4,5]). Putting all the results together one has, for physical massless fields (Majorana for fermions),

s	α	β
0	$\frac{1}{120}$	$-\frac{1}{360}$
$\frac{1}{2}$	$\frac{1}{40}$	$-\frac{11}{720}$
1	$\frac{1}{10}$	$-\frac{31}{180}$
$\frac{3}{2}$	$-\frac{9}{40}$	$-\frac{71}{720}$

For spin 1 and $\frac{3}{2}$ there is a further local invariance: gauge invariance. It allows one to reduce their 4 and 8 degrees of freedom, respectively, to the 2 physical ones which correspond to massless particles. This is done with one and two Faddeev-Popov ghosts, respectively, plus a Nielsen-Kallosh ghost for the spin-$\frac{3}{2}$ field. In Ref. 1 it was proven that using covariant gauges and dimensional regularization the effective action is gauge independent. The calculation in Ref. 6 is performed within the same assumptions. Thus the results for α and β shown in the table are gauge independent.

The aim of this work is to compute α and β for spin 2: our motivation is the possibility of anomaly cancellation in the context of Weyl invariance and the study of possible particle contents which follow from it. The spin-$\frac{3}{2}$ value for α has opposite sign than the values for the other spins. Thus anomaly cancellation is possible. This is not so for β, which has the same sign for all the spin values. We therefore concentrate first on β; if it comes to be negative again no anomaly cancellation is possible. This is, unfortunately, what we find.

Let us recall shortly results from previous computations: 't Hooft and Veltman[7] were the first ones to compute some of the coefficients of (1.7) for the Einstein-Hilbert Lagrangian. The others were computed by Critchley[8] and Fradkin and Tseytlin.[9] The results read

$$(\alpha+\beta)_{\rm TV}=\tfrac{53}{45}\ ,\quad \beta_{\rm TV}=\tfrac{149}{180}\ ,\quad \gamma_{\rm TV}=-\tfrac{29}{27}\ ,\quad \delta_{\rm TV}=\tfrac{1}{4}\ . \tag{1.8}$$

Christensen and Duff[5] calculated the same coefficients using index theorems. Their result is

$$(\alpha+\beta)_{\rm CD}=\tfrac{53}{45}\ ,\quad \beta_{\rm CD}=0\ ,\quad \gamma_{\rm CD}=\tfrac{106}{135}\ ,\quad \delta_{\rm CD}=\tfrac{239}{240}\ . \tag{1.9}$$

Neither result satisfies the constraints implied by (1.6). This is because all these authors start from Lagrangians which are, either straightforwardly or in a more subtle way, not Weyl invariant. Then the singular part of the effective action is not of the form (1.1), as Weyl invariance is explicitly broken already at the classical level. It is also known that then β, γ, and δ are gauge dependent, though not $\alpha+\beta$ [compare (1.8) with (1.9)]. These results thus do not correspond to the genuine gravitational trace anomaly of a massless spin-2 field, the one corresponding to a Weyl-invariant theory, which is what we want to compute here.

II. THE SPIN-2 WEYL-INVARIANT LAGRANGIAN

Consider a symmetric tensor field $\phi^{\mu\nu}$ of dimension M (we are not considering a higher-derivative theory, as would correspond to $\dim\phi^{\mu\nu}=M^0$). Weyl invariance is the local symmetry

$$g_{\mu\nu}(x)\to\Omega^2(x)g_{\mu\nu}(x)\ ,\quad \phi_{\mu\nu}(x)\to\Omega(x)\phi_{\mu\nu}(x)\ ,\quad \phi^{\mu\nu}(x)\to\Omega^{-3}(x)\phi^{\mu\nu}(x)\ . \tag{2.1}$$

The most general Weyl-invariant quadratic Lagrangian in presence of classical gravity is

$$L=\tfrac{1}{2}[\phi_{\alpha\beta;\mu}\phi^{\alpha\beta;\mu}-\tfrac{4}{3}\phi_{\mu\alpha}{}^{;\mu}\phi_{\beta}{}^{\alpha;\beta}+\tfrac{1}{6}R\,\phi_{\alpha\beta}\phi^{\alpha\beta}-2R^{\mu\nu}\phi_{\mu\alpha}\phi_{\nu}{}^{\alpha}+\tfrac{2}{3}\phi^{\sigma}{}_{\sigma;\mu}\phi^{\alpha\mu}{}_{;\alpha}-\tfrac{2}{9}R\,(\phi^{\alpha}{}_{\alpha})^2+R_{\mu\nu}\phi^{\mu\nu}\phi^{\alpha}{}_{\alpha}]$$
$$+c_1[\phi^{\sigma}{}_{\sigma;\mu}\phi^{\rho}{}_{\rho}{}^{;\mu}-\tfrac{1}{6}R\,(\phi^{\sigma}{}_{\sigma})^2]+c_2C^{\mu\nu\alpha\beta}\phi_{\mu\alpha}\phi_{\nu\beta}\ , \tag{2.2}$$

where c_1 and c_2 are arbitrary constants and the Weyl tensor is given by

$$C^{\mu\nu\alpha\beta}=R^{\mu\nu\alpha\beta}+\tfrac{1}{2}(g^{\mu\beta}R^{\alpha\nu}+g^{\nu\alpha}R^{\mu\beta}-g^{\mu\alpha}R^{\beta\nu}-g^{\nu\beta}R^{\mu\alpha})$$
$$+\tfrac{1}{6}R\,(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\ . \tag{2.3}$$

The term proportional to c_1 is nothing but the Weyl-invariant Lagrangian for the scalar field $\phi^{\sigma}{}_{\sigma}$, as expected from the fact that $\phi^{\mu\nu}$ has a spin-0 component, its trace. In order to get rid of this spin-0 component, and yet work with unconstrained fields, let us write in (2.2)

$$\phi^{\mu\nu}\equiv(g^{\mu}{}_{\alpha}g^{\nu}{}_{\beta}-\tfrac{1}{4}g^{\mu\nu}g_{\alpha\beta})\psi^{\alpha\beta}\ . \tag{2.4}$$

This ensures $\phi^{\mu}{}_{\mu}=0$ by working at the same time with the unconstrained field $\psi^{\mu\nu}$. One obtains

$$L=\tfrac{1}{2}[\psi_{\alpha\beta;\mu}\psi^{\alpha\beta;\mu}-\tfrac{4}{3}\psi_{\mu\alpha}{}^{;\mu}\psi_{\beta}{}^{\alpha;\beta}+\tfrac{1}{6}R\,\psi_{\alpha\beta}\psi^{\alpha\beta}$$
$$-2R^{\mu\nu}\psi_{\mu\alpha}\psi_{\nu}{}^{\alpha}+\tfrac{2}{3}\psi^{\sigma}{}_{\sigma;\mu}\psi^{\alpha\mu}{}_{;\alpha}-\tfrac{1}{3}\psi^{\sigma}{}_{\sigma;\mu}\psi^{\rho}{}_{\rho}{}^{;\mu}$$
$$-\tfrac{1}{6}R\,(\psi^{\alpha}{}_{\alpha})^2+R_{\mu\nu}\psi^{\mu\nu}\psi^{\alpha}{}_{\alpha}]$$
$$+c_2C^{\mu\nu\alpha\beta}\psi_{\mu\alpha}\psi_{\nu\beta}\ . \tag{2.5}$$

It is obvious from (2.4) that (2.5) has a further local invariance under the transformation

$$\psi^{\alpha\beta}\to\psi^{\alpha\beta}+sg^{\alpha\beta}\ , \tag{2.6}$$

where $s(x)$ is an arbitrary (local) scale. But there are more local invariances. Indeed, in flat space ($R_{\mu\nu\alpha\beta}=0$) (2.5) is invariant under the gauge transformation

$$\psi^{\alpha\beta}\to\psi^{\alpha\beta}+\Lambda^{,\alpha\beta}\ , \tag{2.7}$$

$\Lambda(x)$ being an arbitrary scalar function. We will perform later an expansion around flat space and thus (2.7) will imply that a gauge fixing is necessary: no flat-space propagator exists otherwise. But if we want our results for α and β not to depend on the gauge fixing the gauge symmetry (2.7) has to be exact, also in curved space-time. There is no problem to have it hold up to order R and $R_{;\mu}$. Indeed, with $c_2=1$ one finds that under

$$\psi^{\alpha\beta}\to\psi^{\alpha\beta}+\Lambda^{;\alpha\beta}-\tfrac{1}{2}R^{\alpha\beta}\Lambda \tag{2.8}$$

the variation of L as given by (2.5) is

$$\delta L\equiv\psi^{\mu\nu}S_{\mu\nu}\Lambda \tag{2.9}$$

with

$$S_{\mu\nu}=-R_{\mu\alpha\nu\beta}R^{\alpha\beta}+\tfrac{1}{2}R_{\mu\nu}{}^{;\alpha}{}_{\alpha}-\tfrac{1}{6}R_{;\mu\nu}+\tfrac{1}{3}RR_{\mu\nu}$$
$$+\tfrac{1}{4}g_{\mu\nu}(-\tfrac{1}{3}\Box R-\tfrac{1}{3}R^2+R_{\alpha\beta}R^{\alpha\beta})\ . \tag{2.10}$$

One can now expand (2.8) further nonlocally in order to cancel (2.9) but this will not be relevant to the anomaly computation, as $S_{\mu\nu}$ is such that $S^{\mu}{}_{\mu}=0$, so that being already quadratic in R it cannot contribute to the anomaly.

Thus (2.5) with $c_2=1$ is the Weyl-invariant Lagrangian with the largest possible local symmetry. For reasons which will become clear later on, we will choose a gauge fixing suggested by the symmetries (2.6) and (2.8):

$$L_G\equiv\tfrac{1}{2}a\,\psi^{\alpha\beta}N_{\alpha\beta\mu\nu}\psi^{\mu\nu} \tag{2.11}$$

with

$$aN^{\alpha\beta\mu\nu}=a[g_{\alpha\beta}(\nabla_\mu\nabla_\nu+\nabla_\nu\nabla_\mu-R_{\mu\nu})$$
$$+(\nabla_\alpha\nabla_\beta+\nabla_\beta\nabla_\alpha-R_{\alpha\beta})g_{\mu\nu}]$$
$$+g_{\alpha\beta}g_{\mu\nu}(a_1\Box+a_2R+a_3m^2)\ , \tag{2.12}$$

where a and a_i, $i=1,2,3$, are arbitrary gauge parameters. Notice that a mass has been introduced. The ghost fields corresponding to (2.12) will be considered later.

We will also need later an IR regulator, which we will choose to be a further mass term which we add to (2.5) and (2.11). It is

$$L_m=-\frac{m^2}{2}\psi_{\alpha\beta}\psi^{\alpha\beta}\equiv-\frac{m^2}{2}\psi^{\alpha\beta}I_{\alpha\beta\mu\nu}\psi^{\mu\nu}\ . \tag{2.13}$$

Of course the anomaly will be m^2 independent.

Putting everything together our Weyl-invariant gauge-fixed IR-regularized Lagrangian for a spin-2 field is

$$L_2^W\equiv-\tfrac{1}{2}\psi^{\alpha\beta}H^a_{\alpha\beta\mu\nu}(m)\psi^{\mu\nu} \tag{2.14}$$

with

$$H^a_{\alpha\beta\mu\nu}(m)\equiv H^a_{\alpha\beta\mu\nu}+m^2M_{\alpha\beta\mu\nu}\ , \tag{2.15}$$

where

$$M_{\alpha\beta\mu\nu}=I_{\alpha\beta\mu\nu}-a_3g_{\alpha\beta}g_{\mu\nu} \tag{2.16}$$

and

$$H^a_{\alpha\beta\mu\nu}=(\Box+\tfrac{1}{6}R)I_{\alpha\beta\mu\nu}-\tfrac{1}{3}(g_{\alpha\mu}\nabla_\beta\nabla_\nu+g_{\alpha\nu}\nabla_\beta\nabla_\mu+g_{\beta\mu}\nabla_\alpha\nabla_\nu+g_{\beta\nu}\nabla_\alpha\nabla_\mu)+(\tfrac{1}{6}-a)[g_{\alpha\beta}(\nabla_\mu\nabla_\nu+\nabla_\nu\nabla_\mu)+g_{\mu\nu}(\nabla_\alpha\nabla_\beta+\nabla_\beta\nabla_\alpha)]$$
$$-[(\tfrac{1}{3}+a_1)\Box+(\tfrac{1}{6}+a_2)R]g_{\alpha\beta}g_{\mu\nu}+(\tfrac{1}{2}+a)(R_{\alpha\beta}g_{\mu\nu}+R_{\mu\nu}g_{\alpha\beta})-(R_{\alpha\mu\beta\nu}+R_{\beta\mu\alpha\nu})\ . \tag{2.17}$$

The second-order differential operator is not of the Gilkey[10] type; i.e., it has derivatives apart from the Laplacian term.

In flat space (2.15) reads

$$H_a^{\alpha\beta}{}_{\mu\nu}(m,g_{\mu\nu}=\eta_{\mu\nu})=(\Box+m^2)\tfrac{1}{2}(\eta^\alpha{}_\mu\eta^\beta{}_\nu+\eta^\alpha{}_\nu\eta^\beta{}_\mu)-\tfrac{1}{3}(\eta^\alpha{}_\mu\partial^\beta\partial_\nu+\eta^\alpha{}_\nu\partial^\beta\partial_\mu+\eta^\beta{}_\mu\partial^\alpha\partial_\beta+\eta^\beta{}_\nu\partial^\alpha\partial_\mu)$$
$$+(\tfrac{1}{3}-2a)(\eta^{\alpha\beta}\partial_\mu\partial_\nu+\eta_{\mu\nu}\partial^\alpha\partial^\beta)-[(\tfrac{1}{3}+a_1)\Box+a_3m^2]\eta^{\alpha\beta}\eta_{\mu\nu}\ . \tag{2.18}$$

In momentum space the corresponding Green's function $\mathcal{G}^{(0)}$ satisfies

$$\{(-k^2+m^2)\tfrac{1}{2}(\eta^\alpha{}_\mu\eta^\beta{}_\nu+\eta^\alpha{}_\nu\eta^\beta{}_\mu)+\tfrac{1}{3}(\eta^\alpha{}_\mu k^\beta k_\nu+\eta^\alpha{}_\nu k^\beta k_\mu+\eta^\beta{}_\mu k^\alpha k_\nu+\eta^\beta{}_\nu k^\alpha k_\mu)$$
$$-(\tfrac{1}{3}-2a)(\eta^{\alpha\beta}k_\mu k_\nu+\eta_{\mu\nu}k^\alpha k^\beta)+[(\tfrac{1}{3}+a_1)k^2-a_3m^2]\eta^{\alpha\beta}\eta_{\mu\nu}\}\,\mathcal{G}^{\mu\nu}{}_{\rho\tau}{}^{(0)}(k)=-\tfrac{1}{2}(\eta^\alpha{}_\rho\eta^\beta{}_\tau+\eta^\beta{}_\rho\eta^\alpha{}_\tau)\ . \tag{2.19}$$

It is easy to obtain $\mathcal{G}^{(0)}$ from (2.19). We will however not give the general formulas but instead limit ourselves to two particular gauges, which are characterized by first not having poles at negative values of k^2 (tachyons) and second by having the minimal number of poles (two). They are (1) $a=-\frac{1}{6}$, $a_1=\frac{1}{2}$, $a_3=\frac{1}{2}$ ($a_2=-\frac{1}{6}$) for which

$$\mathcal{G}^{\mu\nu}{}_{\rho\tau}{}^{(0)}(k,1)=\tfrac{1}{2}(\eta^\mu{}_\rho\eta^\nu{}_\tau+\eta^\mu{}_\tau\eta^\nu{}_\rho-\eta^{\mu\nu}\eta_{\rho\tau})P$$
$$+(k^\mu k_\rho\eta^\nu{}_\tau+k^\mu k_\tau\eta^\nu{}_\rho+k^\nu k_\rho\eta^\mu{}_\tau$$
$$+k^\nu k_\tau\eta^\mu{}_\rho)PQ\ , \tag{2.20}$$

where

$$P\equiv\frac{1}{k^2-m^2},\quad Q\equiv\frac{1}{k^2-3m^2}\ , \tag{2.21}$$

and (2) $a=\frac{1}{12}$, $a_1=0$, $a_3=\frac{5}{16}$ ($a_2=0$) for which

$$\mathcal{G}^{\mu\nu}{}_{\rho\tau}{}^{(0)}(k,2)$$
$$=\tfrac{1}{2}(\eta^\mu{}_\rho\eta^\nu{}_\tau+\eta^\mu{}_\tau\eta^\nu{}_\rho)P+\tfrac{1}{8}(27Q-19P)\eta^{\mu\nu}\eta_{\rho\tau}$$
$$+[k^\mu k_\rho\eta^\nu{}_\tau+k^\mu k_\tau\eta^\nu{}_\rho+k^\nu k_\rho\eta^\mu{}_\tau+k^\nu k_\tau\eta^\mu{}_\rho$$
$$-3(k^\mu k^\nu\eta_{\rho\tau}+\eta^{\mu\nu}k_\rho k_\tau)]PQ\ . \tag{2.22}$$

We will perform the anomaly computation in the two gauges.

III. THE BASIC FORMULAS

We will present here the main formulas on which the calculation is based (see Refs. 3 and 6). The effective action W (neglecting ghosts and masses for the time being) is given by

$$\exp(iW) \equiv \int D\psi^{\alpha\beta} \exp\left[i \int d^4x\, L_2^W(x)\right] \propto (\det H_a)^{-1/2} . \quad (3.1)$$

We are using here operator notation. The Green's function

$$G_a^{\mu\nu}(x,y) = -i\langle T[\psi^{\mu\nu}(x)\psi_{\alpha\beta}(y)]\rangle \quad (3.2)$$

satisfies the differential equation

$$H_a^{\rho\tau}{}_{\mu\nu} G_a^{\mu\nu}{}_{\alpha\beta}(x,y) = -[-g(x)]^{-1/2}\delta(x-y) I^{\rho\tau}{}_{\alpha\beta}(x) . \quad (3.3)$$

Let us now introduce a Hilbert space with norm

$$\langle x,\mu\nu|y,\alpha\beta\rangle = [-g(x)]^{-1/2}\delta(x-y) I^{\mu\nu}{}_{\alpha\beta}(x) . \quad (3.4)$$

Then

$$G_a^{\mu\nu}(x,y) \equiv \langle x,\mu\nu|G_a|y,\alpha\beta\rangle \quad (3.5)$$

and (3.3) reads

$$H_a G_a = -1 . \quad (3.6)$$

From (3.1) one readily obtains

$$W = -\frac{i}{2}\ln\det(-G_a) = -\frac{i}{2}\operatorname{tr}\ln(-G_a) = -\frac{i}{2}\int d^4x [-g(x)]^{1/2}\langle x,\mu\nu|\ln(-G_a)|x,\mu\nu\rangle . \quad (3.7)$$

Recall furthermore

$$-G_a = H_a^{-1} = i\int_0^\infty ds\, e^{-isH_a} , \quad (3.8)$$

which implies, up to a constant,

$$\ln(-G_a) = -\ln H_a = \int_0^\infty \frac{ds}{s} e^{-isH_a} . \quad (3.9)$$

Thus the effective action is given by

$$W = -\frac{i}{2}\operatorname{tr}\int_0^\infty \frac{ds}{s} e^{-isH_a} . \quad (3.10)$$

Let us now prove that W does not depend on the gauge parameter a (nor on a_i), following steps similar to the ones performed in Refs. 1 and 6. Notice, using the invariance under (2.6) and (2.8) that

$$NH_a = aNH_1 = aN^2 \quad (3.11)$$

so that

$$NH_a^n = a^n NH_1^n = a^n N^{n+1} . \quad (3.12)$$

We are assuming here that N contains all the higher-order terms in the adiabatic expansion so that the gauge symmetry is exact. Thus under an infinitesimal change of a,

$$\delta W = -\frac{\delta a}{2}\operatorname{tr}\int_0^\infty ds\, N e^{-isH_a} = -\frac{\delta a}{2}\operatorname{tr}\int_0^\infty ds\, N e^{-isaN} = -i\frac{\delta a}{2a}\operatorname{tr}\int_0^\infty ds \frac{\partial}{\partial s} e^{-isaN} , \quad (3.13)$$

which is a constant and can thus be neglected.

Of course everything done up to now is formal and requires UV and IR regulators. The mass terms introduced in (2.12 and (2.13) will be our IR regulator: its standard imaginary part makes all the previous integrals convergent for large s.

The small-s UV divergences are regularized dimensionally. Recall the DeWitt-Schwinger[11] proper-time representation of the Green's function, which for coinciding arguments reads

$$G(x,x) = -i[-g(x)]^{-1/2}\frac{1}{(4\pi)^{n/2}} \times \int_0^\infty \frac{i\,ds}{(is)^{n/2}} e^{-im^2 s} F(x;is) , \quad (3.14)$$

which is based on (3.8). Equation (3.14) is valid for Lagrangians for which the mass term goes with the unit matrix, i.e., $a_3=0$. None of our gauges is of this type, but instead $H_a(m) = H_a + m^2 M$, $M \neq I$. If a_3 is such that M is invertible (our gauges are of this type), then working with $H_a M^{-1} + m^2 I$ one can use (3.14) for MG. Of course as $\det M$ does not depend on the Riemann tensor the anomaly computed from $H_a + m^2 M$ or $H_a M^{-1} + m^2 I$ will be the same. Similarly we will have, from (3.9),

$$\langle x,\mu\nu|\ln(-MG)|x,\alpha\beta\rangle = i[-g(x)]^{-1/2}\frac{1}{(4\pi)^{n/2}} \times \int_0^\infty \frac{i\,ds}{(is)^{n/2+1}} e^{-im^2 s} F^{\mu\nu}{}_{\alpha\beta}(x;is) . \quad (3.15)$$

Expanding $F(x;is)$ adiabatically in (is),

$$F^{\mu\nu}{}_{\alpha\beta}(x;is) = \sum_{j=0}^\infty a_j^{\mu\nu}{}_{\alpha\beta}(x)(is)^j \quad (3.16)$$

and recalling from (3.10) and (3.15) that

$$W = \tfrac{1}{2}\int \frac{d^4x}{(4\pi)^{n/2}} \int_0^\infty \frac{i\,ds}{(is)^{n/2+1}} e^{-im^2 s}\operatorname{Tr}F(x;is) \quad (3.17)$$

one obtains with the help of

$$\int_0^\infty \frac{ds}{s^{n/2-1}} e^{-im^2 s} = (im^2)^{(n/2)-2}\Gamma\left[2-\frac{n}{2}\right] \quad (3.18)$$

the following expression for the m^2-independent piece of the effective action:

$$W_{\rm div}(n) = \frac{1}{(4\pi)^2}\frac{1}{4-n}\int d^4x\, \operatorname{Tr} a_2(x) . \quad (3.19)$$

It only remains to compute $a_2(x)$. This will do as follows.

Introduce

$$\tilde{G}(x,y) \equiv [-g(x)]^{1/4} G(x,y) [-g(y)]^{1/4} \tag{3.20}$$

and its Fourier transform

$$\tilde{G}(x,x') \equiv \frac{1}{(2\pi)^n} \int d^n k \, e^{-ik\cdot y} \mathcal{G}(k) , \tag{3.21}$$

where y^μ are normal coordinates of x, the origin being at x', and $k\cdot y = k_\alpha \eta^{\alpha\beta} y_\beta$, so that one works in a localized momentum space. Notice that (3.15) can be written as

$$\langle x | \ln(-MG) | x \rangle = -\int_{m^2}^{\infty} MG(x,x) dm^2 , \tag{3.22}$$

where the m^2 integration brings down the extra power of (is) that appears in (3.15) as compared to (3.14). From (3.10),

$$W = \frac{i}{2} \int d^4x [-g(x)]^{1/2} \int_{m^2}^{\infty} dm^2 \, \mathrm{Tr} MG(x,x) , \tag{3.23}$$

which from (3.19) leads to

$$\mathrm{Tr} a_2 = \frac{i}{2} 16\pi^2 \lim_{n\to 4} (4-n) \int \frac{d^n k}{(2\pi)^n} \int_{m^2}^{\infty} dm^2 \, \mathrm{Tr} M \mathcal{G}^{(2)}(k) , \tag{3.24}$$

where $\mathcal{G}^{(2)}(k)$ is the second-order term in the adiabatic expansion (proportional to R^2, etc.) of $\mathcal{G}(k)$. All what remains is to compute $\mathcal{G}^{(2)}(k)$.

Before doing so, let us study the ghosts which go with our gauge fixing [(3.11) and (3.12)]. It corresponds to the generalized gauge conditions

$$g_{\mu\nu}\psi^{\mu\nu} = \phi_1 \tag{3.25}$$

and

$$(\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha - R_{\alpha\beta})\psi^{\alpha\beta} = \phi_2 . \tag{3.26}$$

Recall 't Hooft's device[12] for adding the gauge-fixing condition to the Lagrangian. It is based on the fact that

$$A \equiv \int D\phi_1 D\phi_2 \exp\left[i \int d^4x \, \phi_1 [a\phi_2 + \tfrac{1}{2}(a_1 \Box + a_2 R + a_3 m^2)\phi_1] \right] \tag{3.27}$$

is constant. Notice that no determinant appears in (3.27): we do not have Nielsen-Kallosh[13] ghosts. Then using the Faddeev-Popov procedure for quantizing a theory with local symmetries, one has, for the generating functional,

$$Z \propto \int D\psi^{\mu\nu} A \, \delta(g_{\mu\nu}\psi^{\mu\nu} - \phi_1) \delta((2\nabla_\alpha \nabla_\beta - R_{\alpha\beta})\psi^{\alpha\beta} - \phi_2) \times \det \begin{vmatrix} 4 & 2\Box - R \\ 2\Box - R & (2\nabla_\alpha\nabla_\beta - R_{\alpha\beta})(\nabla^\alpha\nabla^\beta + \nabla^\beta\nabla^\alpha - R^{\alpha\beta}) \end{vmatrix} \exp(iS_{\mathrm{cl}}[\psi]) , \tag{3.28}$$

where $S_{\mathrm{cl}}[\psi]$ is the classical action. The determinant in (3.28) is the Jacobian of (3.25) and (3.26) with respect to the transformations [(2.6) and (2.8)]. It corresponds to the ghosts. Notice it does not depend on the gauge parameters a and a_i. Equation (3.28) shows that the ghosts will be higher-derivative ghosts. There are two ways of avoiding these and thus all the complications of higher-derivative operators. First, work in a maximally symmetric space, for which

$$R_{\mu\nu\lambda\sigma} = K(g_{\nu\lambda}g_{\mu\sigma} - g_{\nu\sigma}g_{\mu\lambda}) , \quad R_{\mu\nu} = -3Kg_{\mu\nu}, \quad R = -12K . \tag{3.29}$$

Then the determinant of (3.28) is

$$\det \begin{vmatrix} 4 & 2\Box - R \\ 2\Box - R & 4\Box^2 - 2R\Box + \frac{R^2}{4} \end{vmatrix} = 12 \det\Box \det \left[\Box - \frac{R}{3} \right] . \tag{3.30}$$

There are therefore two different complex conformal scalar ghosts whose contribution has to be subtracted from the computation which will be performed in the next section.

The other option is to notice that $\alpha+\beta$, being the coefficient of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, does not depend on the Ricci tensor. Thus the determinant in (3.28) reduces to

$$\det \begin{vmatrix} 4 & 2\Box \\ 2\Box & 4\Box^2 \end{vmatrix} = 12 \det{}^2\Box . \tag{3.31}$$

We will not perform this second computation, as the value of β_2 obtained from the first one makes anomaly cancellation impossible. The computation of β_2 is already a major undertaking by itself.

Let us, before finishing this section, comment on the number of degrees of freedom our field $\psi^{\mu\nu}$, whose dynamics is governed by (2.5) with $c_2=1$, describes. It is a symmetric rank-2 tensor whose trace is projected out by (2.4): this gives 9 degrees of freedom. There are two complex ghosts, so that 5 degrees of freedom are left. This is too much for a spin-2 massless particle. This is a consequence of Weyl invariance. By having a further local invariance gauge invariance is reduced. Instead of being of the type of first derivatives of a vector function it is [recall (2.8)] of the type of second derivatives of a scalar function. Thus, instead of reducing $4\times 2=8$ degrees of freedom (the trace is not projected out) so that $10-8=2$ degrees of freedom are left, it only reduces $2\times 2=4$ degrees of freedom. In flat space these consequences of conformal invariance are well known:[14] the field describes also lower-spin particles. Taking a symmetric rank-2 tensor field and imposing Weyl invariance leaves no other choice: our spin-2 field describes more than just a massless spin-2 particle, as there is no Weyl-invariant description of a single massless spin-2 particle in terms of a symmetric rank-2 tensor field of dimension M with dynamics given by a local Lagrangian.

IV. THE COMPUTATION AND CONCLUSION

We will here compute a_2 for the gravitational background corresponding to (3.29), for which very important simplifications occur. This is a long computation performed with an algebraic program, so that only some intermediate steps will be given. As a check the computation has been performed in both gauges given in Sec. II. The final results coincide, as they should.

The following normal coordinates have been used

$$g_{\mu\nu}(y)=\eta_{\mu\nu}+Ky_\mu y_\nu(1+Ky^2)+O(K^3) \tag{4.1}$$

for which, up to K^2 included,

$$\Gamma^\mu_{\rho\sigma}=Ky^\mu(\eta_{\rho\sigma}+Ky_\rho y_\sigma)\ ,$$
$$-g=1+Ky^2(1+Ky^2)\ . \tag{4.2}$$

Recall that the covariant derivative acts on $\psi_{\mu\nu}$ according to

$$\nabla_\rho\psi_{\mu\nu}\equiv\psi_{\mu\nu;\rho}=\partial_\rho\psi_{\mu\nu}-\Gamma^\lambda_{\mu\rho}\psi_{\lambda\nu}-\Gamma^\lambda_{\nu\rho}\psi_{\mu\lambda}\ . \tag{4.3}$$

In this space

$$\Box R=F=0 \tag{4.4}$$

so that

$$\beta=\frac{a_2}{G} \tag{4.5}$$

with

$$G=24K^2\ . \tag{4.6}$$

Our starting equation is (3.3). In momentum space, and expanding in K,

$$\mathcal{G}^{\mu\nu}{}_{\alpha\beta}(k)=\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(0)}+K\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(1)}+K^2\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(2)}+\cdots\ ,$$
$$\mathcal{H}_a^{\mu\nu}{}_{\alpha\beta}\left[\frac{\partial}{\partial k}\right]=\mathcal{H}_a^{\mu\nu}{}_{\alpha\beta}{}^{(0)}+K\mathcal{H}_a^{\mu\nu}{}_{\alpha\beta}{}^{(1)}+K^2\mathcal{H}_a^{\mu\nu}{}_{\alpha\beta}{}^{(2)}+\cdots\ . \tag{4.7}$$

Obviously

$$\mathcal{H}_a^{\rho\tau}{}_{\mu\nu}{}^{(0)}\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(0)}=-\tfrac{1}{2}(\eta^\rho{}_\alpha\eta^\tau{}_\beta+\eta^\rho{}_\beta\eta^\tau{}_\alpha)\ , \tag{4.8}$$

which is precisely (2.19). In the two gauges we work $\mathcal{G}^{\mu\nu}{}_{\alpha\beta}$ is thus given by Eqs. (2.20) and (2.22). The K and K^2 equations are, omitting indices,

$$\mathcal{G}^{(1)}=\mathcal{G}^{(0)}\mathcal{H}_a^{(1)}\mathcal{G}^{(0)} \tag{4.9}$$

and

$$\mathcal{G}^{(2)}=\mathcal{G}^{(0)}\mathcal{H}_a{}^{(2)}\mathcal{G}^{(0)}+\mathcal{G}^{(0)}\mathcal{H}_a{}^{(1)}\mathcal{G}^{(1)}\ . \tag{4.10}$$

Some intermediate results are

$$\begin{aligned}\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(1)}(1)=&\,16PQ(P^2+PQ+Q^2)k^\mu k^\nu k_\alpha k_\beta-4P^3k^\mu k^\nu\eta_{\alpha\beta}\\&+2(2k^2P^4-6k^2Q^4-3P^3-4P^2Q-6PQ^2+12Q^3)(\eta^\mu{}_\alpha k^\nu k_\beta+\eta^\mu{}_\beta k^\nu k_\alpha+\eta^\nu{}_\alpha k^\mu k_\beta+\eta^\nu{}_\beta k^\mu k_\alpha)\\&+2P^2(2k^4P^2-3k^2P+1)\eta^{\mu\nu}\eta_{\alpha\beta}-(4k^4P^4-6k^2P^3+2P^2-7PQ+6Q^2)(\eta^\mu{}_\alpha\eta^\nu{}_\beta+\eta^\mu{}_\beta\eta^\nu{}_\alpha)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\mathcal{G}^{\mu\nu}{}_{\alpha\beta}{}^{(1)}(2)=&-8PQ(P^2+3PQ+Q^2)k^\mu k^\nu k_\alpha k_\beta-2(6k^2P^4-18k^2Q^4-9P^3+7P^2Q-15PQ^2+27Q^3)\eta^{\mu\nu}k_\alpha k_\beta\\&+(-12k^2P^4+36k^2Q^4+11P^3-2P^2Q+42PQ^2-63Q^3)k^\mu k^\nu\eta_{\alpha\beta}\\&-2(-2k^2P^4+6k^2Q^4+3P^3+P^2Q+3PQ^2-12Q^3)(\eta^\mu{}_\alpha k^\nu k_\beta+\eta^\mu{}_\beta k^\nu k_\alpha+\eta^\nu{}_\alpha k^\nu k_\beta+\eta^\nu{}_\beta k^\mu k_\alpha)\\&+\tfrac{1}{4}(76k^4P^4-108k^4Q^4-114k^2P^3+162k^2Q^3-53P^2+132PQ-99Q^2)\eta^{\mu\nu}\eta_{\alpha\beta}\\&+2(-2k^4P^4+3k^2P^3-P^2+2PQ-2Q^2)(\eta^\mu{}_\alpha\eta^\nu{}_\beta+\eta^\mu{}_\beta\eta^\nu{}_\alpha)\ .\end{aligned} \tag{4.12}$$

Notice that $\mathcal{G}^{(1)}$ is not symmetric, nor is there a reason for it to be so in the coordinate system used. To second order our results are, for the first gauge,

$$\begin{aligned}\mathrm{Tr}\mathcal{G}^{(0)}(1)\mathcal{H}_a^{(1)}(1)\mathcal{G}^{(1)}(1)=&\,192k^8P^7-640k^6P^6+720k^4P^5+384k^2P^4+160P^3-828P^2Q-444PQ^2+960k^8Q^7\\&-4224k^6Q^6+7968k^4Q^5-7008k^2Q^4+3120Q^3\ ,\\\mathrm{Tr}[M(1)-I]\mathcal{G}^{(0)}(1)\mathcal{H}_a^{(1)}(1)\mathcal{G}^{(1)}(1)=&\,192k^8P^7-480k^6P^6+480k^4P^5-768k^2P^4+96P^3-420P^2Q\\&+1452PQ^2-192k^8Q^7+960k^6Q^6-1392k^4Q^5+336k^2Q^4-264Q^3\ ,\\\mathrm{Tr}\mathcal{G}^{(0)}(1)\mathcal{H}_a^{(2)}(1)\mathcal{G}^{(0)}(1)=&\,64k^6P^6+128k^4P^5+438k^2P^4\\&+604P^3+202P^2Q-1626PQ^2+1344k^6Q^6+2208k^4Q^5+4842k^2Q^4-6720Q^3\ ,\\\mathrm{Tr}[M(1)-I]\mathcal{G}^{(0)}(1)\mathcal{H}_a^{2}(1)\mathcal{G}^{(0)}(1)=&-96k^6P^6-192k^4P^5-390k^2P^4+612P^3-66P^2Q-2718PQ^2-384k^6Q^6\\&-2112k^4Q^5+522k^2Q^4+4332Q^3\ ,\end{aligned} \tag{4.13}$$

and, for the second gauge,

$$
\begin{aligned}
\mathrm{Tr}\,\mathcal{G}^{(0)}(2)\mathcal{H}_a^{(1)}(2)\mathcal{G}^{(1)}(2) &= -96k^8P^7+\tfrac{2387}{4}k^6P^6+\tfrac{1521}{4}k^4P^5+1455k^2P^4-\tfrac{4661}{16}P^3-\tfrac{19\,461}{16}P^2Q+\tfrac{93\,237}{16}PQ^2+1248k^8Q^7 \\
&\quad -\tfrac{20\,199}{4}k^6Q^6+\tfrac{41\,691}{4}k^4Q^5-10\,005k^2Q^4-\tfrac{6555}{16}Q^3\ , \\
\mathrm{Tr}[M(2)-I]\mathcal{G}^{(0)}(2)\mathcal{H}_a^{(1)}(2)\mathcal{G}^{(1)}(2) &= 480k^8P^7-\tfrac{7185}{4}k^6P^6+\tfrac{3045}{4}k^4P^5+\tfrac{3135}{2}k^2P^4+\tfrac{11\,265}{16}P^3+\tfrac{5025}{16}P^2Q \\
&\quad -\tfrac{109\,545}{16}PQ^2-480k^8Q^7+\tfrac{7845}{4}k^6Q^6-\tfrac{18\,825}{4}k^4Q^5+\tfrac{10\,005}{2}k^2Q^4+\tfrac{48\,615}{16}Q^3\ , \\
\mathrm{Tr}\,\mathcal{G}^{(0)}(2)\mathcal{H}_a^{(2)}(2)\mathcal{G}^{(0)}(2) &= -\tfrac{1235}{4}k^6P^6-\tfrac{13}{4}k^4P^5+3744k^2P^4+\tfrac{12\,613}{16}P^3+\tfrac{2437}{16}P^2Q-\tfrac{249\,909}{16}PQ^2+\tfrac{5223}{4}k^6Q^6 \\
&\quad +\tfrac{1497}{4}k^4Q^5+8010k^2Q^4+\tfrac{83\,019}{16}Q^3\ , \\
\mathrm{Tr}[M(2)-I]\mathcal{G}^{(0)}(2)\mathcal{H}_a^{(2)}(2)\mathcal{G}^{(0)}(2) &= \tfrac{1425}{4}k^6P^6+\tfrac{15}{4}k^4P^5-\tfrac{8025}{2}k^2P^4-\tfrac{16\,065}{16}P^3+\tfrac{8655}{16}P^2Q+\tfrac{265\,185}{16}PQ^2-\tfrac{2085}{4}k^6Q^6 \\
&\quad +\tfrac{1965}{4}k^4Q^5-\tfrac{9495}{2}k^2Q^4-\tfrac{143\,655}{16}Q^3\ .
\end{aligned} \tag{4.14}
$$

It should be noted that in the above expressions total derivatives have been omitted. Using (4.10) and (3.24) one obtains, after performing the integrations,

$$\mathrm{Tr}\,a_2=\tfrac{22}{3}K^2 \tag{4.15}$$

for both gauges. The ghost contribution to $\mathrm{Tr}\,a_2$ is[3]

$$a_2|_{\mathrm{gh}}=2(\tfrac{29}{15}+\tfrac{269}{15})K^2\ , \tag{4.16}$$

which subtracted from (4.15) and using (4.5) leads to the final result

$$\beta_2=-\tfrac{27}{20}\ . \tag{4.17}$$

As the lower-spin values of β, this is also negative. No anomaly cancellation is possible.

For particles of spin up to 2 described by canonical dimension fields of scalar, spinor, vector, vector spinor, and symmetric rank-2 tensor type the gravitational trace anomaly cannot be vanishing.

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