

Neutrino oscillations

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We present an overview of neutrino oscillation theory. We develop a mathematical model of neutrino flavor change and discuss its implications. We compare predictions of a simplified version of the model to experimental evidence in order to check its validity.

I. INTRODUCTION

Neutrino existence was postulated by W. Pauli in 1930 in order to guarantee the energy, momentum and spin conservation in beta decays [1]. It was first detected by C. Cowan and F. Reines in 1956 [2]. Neutrinos are particles hard to detect since they are electrically neutral, with a very small rest mass and do not participate in the strong interaction. They are abundantly produced by nuclear reactors and accelerators, Earth's atmosphere and solar core.

Neutrino oscillation was predicted by Bruno Pontecorvo in 1957, in analogy with kaons oscillations [3]. A quantitative theory of neutrino flavor oscillation was first developed by Maki, Nakagawa, and Sakata in 1962 [4]. In this model, the flavor eigenstates differ from the mass eigenstates, and a neutrino produced in a certain flavor can be detected with a different flavor. The explanation for such transitions involves non-massless neutrinos. In 1968, R. Davis observed a deficit in the flux of solar neutrino [5], which constitutes the first experimental evidence of oscillations. Large detectors, such as Super-Kamiokande [6] and Sudbury Neutrino Observatory [7], have provided clear evidence of neutrino flavor change in the last two decades.

II. NEUTRINO MASSES AND FLAVORS

Experiments show that neutrinos which take part in the standard charged current (CC) and neutral current (NC) are divided in three types or flavors [8]: electron, ν_e , muon, ν_μ , and tauon, ν_τ . This distinction appears for dynamical reasons: ν_e is produced with e^+ or produces an e^- in CC weak interaction processes, and similar arguments apply to ν_μ and ν_τ . Henceforth, we will use Greek letters to refer to neutrino flavor eigenstates, $|\nu_\alpha\rangle$, and Latin letters to mass eigenstates, $|\nu_i\rangle$.

To explain the phenomenon of oscillations, we have to decompose a flavor eigenstate as a superposition of mass eigenstates. The flavor and mass eigenstates basis are related by a unitary matrix U :

$$|\nu_\alpha\rangle = \sum_{i=1}^3 U_{\alpha i}^* |\nu_i\rangle. \quad (1)$$

This unitary matrix is the Pontecorvo-Maki-Nakagawa-

Sakata (PMNS) matrix, similar to the quark mixing matrix (CKM). It can be parametrized in terms of three mixing angles and one CP violation phase.

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{12} & s_{12}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{12}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} = \cos(\theta_{ij})$, $s_{ij} = \sin(\theta_{ij})$, and δ is the CP violation phase. This parametrization is given in Particle Data Group [9].

As we will see, the 3-neutrino oscillation probabilities depend on the previous mixing angles and the neutrino squared mass differences, $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$. In a 3-neutrino mixing, only two of these parameters are independent. It is convenient to identify $|\Delta m_{21}^2|$ with the smaller of the two neutrino mass squared differences. Besides, we will number these masses in such a way that $m_1 < m_2$, so that $\Delta m_{21}^2 > 0$. Experiments indicate that Δm_{21}^2 and $|\Delta m_{31}^2|$ differ by approximately a factor of 30 [10]. Regarding the third neutrino mass, there are two possibilities:

1. Normal hierarchy (NH), if $m_1 < m_2 < m_3$. In this case, there are two light and one heavy neutrinos.
2. Inverted hierarchy (IH), if $m_3 < m_1 < m_2$. In this case, there are two heavy and one light neutrinos.

The existing data do not allow us to determine the actual mass hierarchy.

III. MATHEMATICAL DEVELOPMENT

A. Neutrinos in vacuum

Let $|\nu(t, \vec{x})\rangle$ be the neutrino state. At $(t, \vec{x}) = (0, 0)$ a neutrino is produced as the flavor eigenstate $|\nu_\alpha\rangle$. The initial state can be written as a superposition of mass eigenstates,

$$|\nu(0, 0)\rangle = |\nu_\alpha\rangle = \sum_j U_{\alpha j}^* |\nu_j\rangle. \quad (2)$$

The mass of this neutrino state is not well defined, and the time evolution of mass eigenstates produces a time

evolution of flavor eigenstates. For simplicity, let consider natural units, $c = \hbar = 1$, and let approximate the mass eigenstate to a plane wave with a well defined momentum and energy,

$$|\nu(\vec{x}, t)\rangle = \sum_j U_{\alpha j}^* e^{i(\vec{p}_j \vec{x} - E_j t)} |\nu_j\rangle. \quad (3)$$

In an ultrarelativistic approximation, one can take $x \sim t$ and expand the energy as

$$\vec{p}_j \vec{x} - E_j t \simeq t(p_j - E_j) \simeq t \left(E_j - \frac{m_j^2}{2E_j} - E_j \right) \simeq -\frac{m_j^2}{2E_j} t.$$

Since neutrino masses are negligible compared to its momentum, we can take $E_j \simeq |p|$, and $E_i \approx E_j \simeq E$. In this case, the neutrino state becomes

$$|\nu(\vec{x}, t)\rangle = \sum_j U_{\alpha j}^* e^{-\frac{m_j^2 t}{2E}} |\nu_j\rangle. \quad (4)$$

The probability amplitude of finding the neutrino in a flavor state $|\nu_\beta\rangle$ at the time t is

$$\begin{aligned} A_{\alpha\beta}(t) &= \langle \nu_\beta | \nu_\alpha(\vec{x}, t) \rangle = \sum_k \langle \nu_k | U_{\beta k} \sum_j U_{\alpha j}^* e^{-\frac{m_j^2 t}{2E}} |\nu_j\rangle \\ &= \sum_j U_{\beta j} U_{\alpha j}^* e^{-\frac{m_j^2 t}{2E}}, \end{aligned}$$

where $\langle \nu_i | \nu_j \rangle = \delta_{ij}$. Then, the probability of this flavor change is

$$\begin{aligned} P_{\alpha\beta}(t) &= |A_{\alpha\beta}(t)|^2 = \\ &= \sum_j U_{\beta j} U_{\alpha j}^* e^{-\frac{m_j^2 t}{2E}} \sum_k U_{\beta k}^* U_{\alpha k} e^{\frac{m_k^2 t}{2E}} \\ &= \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-\frac{\Delta m_{jk}^2 t}{2E}}. \end{aligned}$$

Finally, if we define $\varphi_{ij} = \frac{\Delta m_{ij}^2}{2E} L$ and take the ultrarelativistic approximation, $t \sim L$, then the probability becomes

$$P_{\alpha\beta}(L) = \sum_{j,k} U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-\varphi_{jk} i}. \quad (5)$$

B. Three neutrinos in vacuum

Equation (5) can be developed with three neutrino flavors as

$$P_{\alpha\beta} = \sum_{j=1}^3 U_{\beta j} U_{\alpha j}^* U_{\beta j}^* U_{\alpha j} + \sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} e^{-\varphi_{jk} i}.$$

With the Euler's identity and some trigonometric properties,

$$\begin{aligned} P_{\alpha\beta}(L) &= \sum_{j=1}^3 U_{\beta j} U_{\alpha j}^* U_{\beta j}^* U_{\alpha j} \\ &+ \sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \left(1 - 2 \sin^2 \left(\frac{\varphi_{jk}}{2} \right) \right) \\ &+ \sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \sin(\varphi_{jk}) i \end{aligned}$$

First, note that

$$\begin{aligned} &\sum_{j=1}^3 U_{\beta j} U_{\alpha j}^* U_{\beta j}^* U_{\alpha j} + \sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \\ &= \sum_{j,k=1}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} = \sum_{j=1}^3 U_{\beta j} U_{\alpha j}^* \sum_{k=1}^3 U_{\beta k}^* U_{\alpha k} = \delta_{\alpha\beta}, \end{aligned}$$

where $\sum U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta}$. By the symmetry of the terms of summation, we deduce that

$$\begin{aligned} &\sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \sin^2 \left(\frac{\varphi_{jk}}{2} \right) \\ &= \sum_{j>k}^3 (U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} + U_{\beta k} U_{\alpha k}^* U_{\beta j}^* U_{\alpha j}) \sin^2 \left(\frac{\varphi_{jk}}{2} \right) \\ &= 2 \sum_{j>k}^3 \text{Re} (U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin^2 \left(\frac{\varphi_{jk}}{2} \right), \end{aligned}$$

since $\sin^2(x)$ is an even function and the two terms in the summation are complex conjugates. A similar argument gives

$$\begin{aligned} &\sum_{\substack{j,k=1 \\ j \neq k}}^3 U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k} \sin(\varphi_{jk}) i \\ &= 2 \sum_{j>k}^3 \text{Im} (U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin(\varphi_{jk}), \end{aligned}$$

In conclusion,

$$\begin{aligned} P_{\alpha\beta}(L) &= \delta_{\alpha\beta} \\ &- 4 \sum_{j>k} \text{Re} (U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin^2 \left(\frac{\varphi_{jk}}{2} \right) \\ &+ 2 \sum_{j>k} \text{Im} (U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin(\varphi_{jk}) \end{aligned}$$

The probability is an oscillating function of L/E . Observe that the expression of the probability is an even function of Δm_{ij}^2 , and it cannot be used to determine the sign of squared mass differences. Experiments in vacuum cannot distinguish between hierarchies, NH or IH.

C. Neutrinos in matter

Neutrinos propagating through matter acquire an effective potential due to the coherent effect of forward scattering. When interaction proceeds from a Z^0 exchange, the process is called *neutral-current* (NC), whereas a W exchange is called *charged-current* (CC). The associated effective potentials are [11]

$$V_{NC}(x) = -\frac{1}{\sqrt{2}}G_F N_n(x),$$

$$V_{CC}(x) = \sqrt{2}G_F N_e(x),$$

where N_n and N_e are the neutron and electron number densities, and G_F is the Fermi constant.

Since scattering produced by CC only affects electron neutrinos, there are different contributions according to neutrino flavors. This phenomenon is called MSW effect, and makes it possible to distinguish between normal or inverted hierarchy [12].

D. Two neutrinos in vacuum

As we will see, in the standard framework with three neutrino flavors there are two subsystems decoupling from each other. Then, it is instructive to consider a model with only two neutrino flavors (α and β). In this case, there is one mixing angle, θ , and no CP violation phase. The mixing matrix becomes

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \quad (6)$$

From equation (5), the oscillation probability becomes

$$P_{\alpha\beta}(L) = \sin^2(2\theta) \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right). \quad (7)$$

We show $P_{\alpha\beta}$ as a function of Δm_{ij}^2 in Fig. 1. If the neutrino energy and the source-detector distance satisfy $\Delta m_{ij}^2 \ll 4E/L$, the oscillations do not have enough time to develop on the way to the detector. Then, the flavor change could not be observable when $\sin(\varphi_{ij}/2) \ll 1$. Moreover, note that the first maximum of $P_{\alpha\beta}$ occurs at $\Delta m_{ij}^2 = 2\pi E/L$, and therefore the condition $\varphi_{ij} \gtrsim 1$ provides the sensitivity to Δm_{ij}^2 .

On the other hand, after a sufficiently large distance $\Delta m_{ij}^2 \gg 4E/L$, it starts the average regime and the probability becomes

$$P_{\alpha\beta} = \frac{1}{2} \sin^2(2\theta). \quad (8)$$

In that case, we cannot estimate the value of Δm_{ij}^2 for a given value of probability.

Introducing regular units back in equation (7), it gives

$$P_{\alpha\beta}(L) = \sin^2(2\theta) \sin^2\left(1.27 \frac{\Delta m_{ij}^2}{\text{eV}^2} \frac{L}{\text{m}} \frac{\text{MeV}}{E}\right). \quad (9)$$

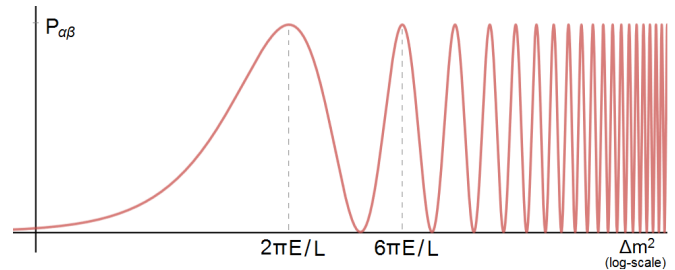


FIG. 1: Probability $P_{\alpha\beta}(\Delta m^2)$ of finding a flavor change as function of the squared mass difference, for a neutrino with energy E and traveled distance L , in natural units [equation (7)]. The oscillation amplitude is $\sin^2(2\theta)$.

IV. EXPERIMENTAL DATA

Neutrinos can be created in several ways, and each neutrino source has very different properties (flavor type, energy spectra, etc.). There are mainly four sources: the Sun, the Earth's upper atmosphere, nuclear reactors and accelerators. It is required to study each source separately. We will concentrate on the first two sources, and comment briefly on the third, since these are the experiments that have provided the main data for neutrino oscillations up to now.

A. Solar neutrinos

Solar neutrinos are produced by several nuclear reactions, such as the pp-cycle or through the CNO-cycle. Although only electron neutrinos are created by the sun, they have different properties: neutrinos coming from a pp-reaction have low energy and are very abundant, whereas ^8B reaction produces energetic and scarce neutrinos.

Several experiments have shown a deficit in neutrino fluxes, according to solar-model predictions, such as Homestake [13], GALLEX and GNO [14], and SNO [7] experiments. Now we can interpret the flux deficit as a neutrino flavor change, and use the experimental evidence to test our two neutrino oscillation model.

Experiment	Prediction	Data	Data/Prediction
^8B -ES [SNO]	$5.05_{-0.8}^{+1.0}$	2.39 ± 0.24	0.47 ± 0.12
^8B -CC [SNO]	$5.05_{-0.8}^{+1.0}$	1.76 ± 0.06	0.35 ± 0.08
^8B -NC [SNO]	$5.05_{-0.8}^{+1.0}$	5.09 ± 0.44	1.0 ± 0.20

TABLE I: Flux of ^8B neutrinos in $10^6 \text{ cm}^{-2}\text{s}^{-1}$, measured with elastic scattering (ES), CC ν_e -deuterium scattering (CC) and NC reactions (NC). Data taken from SNO [7].

If we take a ^8B neutrino energy of $E \simeq 9 \text{ MeV}$ [15] and an Earth-Sun distance of $L = 149.6 \cdot 10^6 \text{ km}$, SNO provides a sensitivity to Δm_{21}^2 of 10^{-10} eV^2 , according to equation (9). Since $\Delta m_{21}^2 \sim 10^{-5} \text{ eV}^2$, as the Particle Data Group states, we conclude that SNO is not an ac-

curate experiment to measure Δm_{21}^2 with our simplified model.

However, we can assume that the detected-predicted neutrino ratio is similar to the probability of no flavor change.

$$P_{ee} = \frac{\phi_{\text{data}}}{\phi_{\text{theo}}}. \quad (10)$$

It is instructive to depict the oscillation probability in terms of Δm_{21}^2 and θ_{12} , as shown in Fig. 2. In that case, the neutrino flux measured with CC reaction, given in Table I, allows to estimate the value of the oscillation parameters. The allowed region is inside the isoline

$$P_{e\mu} = 1 - 0.35 - 0.08 = 0.57,$$

considering the associated error of SNO measurements. The inequity appears because we cannot reject the oscillation to other flavors. From this contourplot, we deduce the minimal value of Δm_{21}^2 and estimate $\tan^2(\theta_{21})$:

$$\Delta m_{21}^2 > 4 \cdot 10^{-11} \text{ eV}^2, \quad \tan^2(\theta_{12}) = 1.0_{-0.8}^{+3.0} \quad (11)$$

This last result is comparable to the present world average, $\tan^2(\theta_{12}) = 0.4 \pm 0.1$, value deduced from $\sin^2(\theta_{12}) = 0.297 \pm 0.05$ [9].

Finally, note that different reactions have associated different matter interactions which invalidate our simplified model. The CC reaction is sensitive exclusively to ν_e , while the ES reaction also has a small sensitivity to ν_μ and ν_τ .

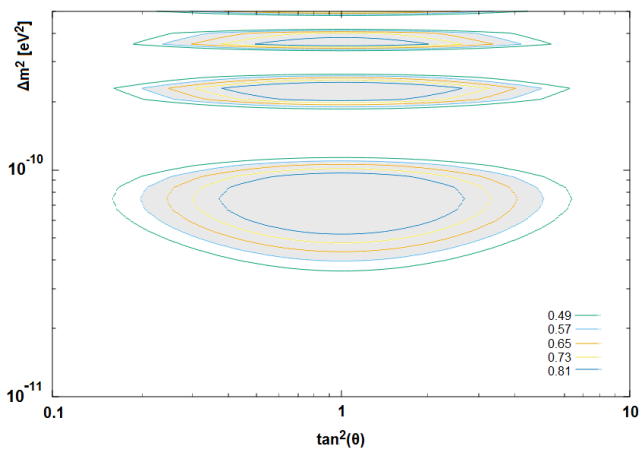


FIG. 2: Contourplot of our two neutrinos in vacuum model (9). Isolines represent the oscillation probability $P_{e\mu}$ on the $\tan^2(\theta_{21})/\Delta m_{21}^2$ plane for solar ^8B neutrinos measured in SNO experiment ($L \simeq 149.6 \cdot 10^9$ m, $E \simeq 9$ MeV). The allowed regions are inside the $P_{e\mu} = 0.57$ contours.

B. Atmospheric neutrinos

Atmospheric neutrinos are produced by cosmic rays on the upper atmosphere. Cosmic rays have an energy spectrum that extends up to extremely high energies. When

a cosmic particle collides it produces secondary particles (pions π^\pm , kaons K^\pm and muons μ^\pm) that decay and create atmospheric neutrinos.

For energies above 1 GeV, pions will decay before reaching the ground due to Lorentz space contraction. Therefore, neutrinos with energy $E \simeq 1$ GeV do not cross through the Earth and can be considered as in vacuum. Besides, we take an average distance of $L \simeq 9900$ km [9]. The double ratio

$$R = \left(\frac{\phi_\mu}{\phi_e} \right)_{\text{data}} / \left(\frac{\phi_\mu}{\phi_e} \right)_{\text{predicted}}, \quad (12)$$

where ϕ_α comes from ν_α and $\bar{\nu}_\alpha$ fluxes, should be 1 without an oscillation theory. According to Super Kamiokande [16], the measured ratio is $R = 0.66 \pm 0.06$. The deficit of μ -neutrinos may be interpreted as the presence of $\nu_\mu \leftrightarrow \nu_\tau$ oscillations, since electronic fluxes are the expected. Contourplot of $P_{\mu\tau}$ is shown in Fig. 3, and indicates the allowed values of $\sin^2(\theta_{23})$ and Δm_{23}^2 . This parameters must be inside the isoline

$$P_{\mu\tau} = 1 - 0.66 - 0.06 = 0.28,$$

considering the associated error of Super-Kamiokande measurements. From this plot we deduce the minimal value of Δm_{23}^2 and estimate $\sin^2(\theta_{23})$:

$$\Delta m_{23}^2 > 4.5 \cdot 10^{-5} \text{ eV}^2, \quad \sin^2(\theta_{23}) = 0.5 \pm 0.4 \quad (13)$$

This result is comparable to the present world average, $\sin^2(\theta_{23}) = 0.425_{-0.044}^{+0.190}$ (NH) or $\sin^2(\theta_{23}) = 0.589_{-0.205}^{+0.047}$ (IH) [9], depending on the hierarchy.

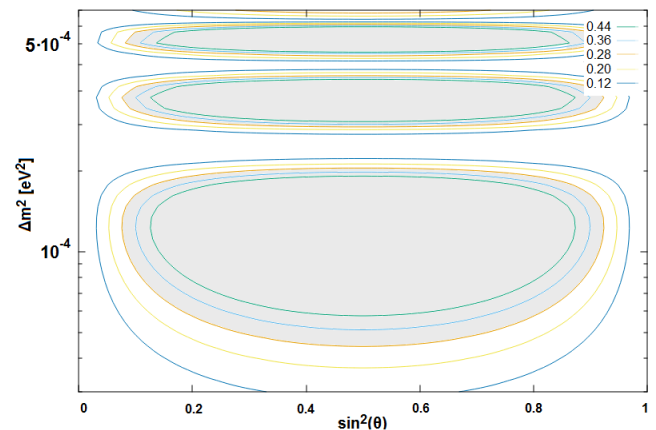


FIG. 3: Contourplot of our two neutrinos in vacuum model (9). Isolines represent the oscillation probability $P_{\mu\tau}$ on the $\sin^2(\theta_{23})/\Delta m_{23}^2$ plane for atmospheric neutrinos ($L \simeq 9.9 \cdot 10^6$ m, $E \simeq 10^3$ MeV). Allowed regions are inside the $P_{\mu\tau} = 0.28$ contours.

C. Reactor neutrinos

Nuclear reactors are isotropic sources of electron antineutrinos $\bar{\nu}_e$, which are abundantly produced by the

β -decay products in fission reactions [17]. Antineutrinos can be detected with the reaction

$$\bar{\nu}_e + p \rightarrow e^+ + n \quad \text{at } E_{th} = 1.8 \text{ MeV}. \quad (14)$$

Chooz [18] and Palo Verde [19] experiments have measured the energy spectrum of produced positrons in reaction (14). Ratios between non-oscillating theoretical prediction and experimental data were

$$R_{\text{Chooz}} = 1.01 \pm 2.8\%(\text{stat}) \pm 2.7\%(\text{syst}), \quad (15)$$

$$R_{\text{Palo Verde}} = 1.01 \pm 2.4\%(\text{stat}) \pm 5.3\%(\text{syst}). \quad (16)$$

The expected flux could be interpreted as a non-oscillating regime in our two neutrino model. Since both detectors have a baseline of $L \simeq 1$ km, it is possible to delimit squared mass difference Δm_{12}^2 ,

$$\Delta m_{12}^2 \ll \frac{1.8 \text{ MeV}}{1.27 \frac{\text{MeV}}{\text{m eV}^2} 1000 \text{ m}} \simeq 1.5 \cdot 10^{-3} \text{ eV}^2.$$

Other detectors with a longer baseline, such as KamLAND experiment [20], have measured reactor neutrino fluxes from $L \sim 180$ km. Its results offer accurate values for solar neutrino oscillation parameters.

V. CONCLUSIONS

We have summarized the current neutrino oscillation knowledge and made a review of the main experiments

that support this model. According to a general neutrino oscillation theory, we have developed a mathematical description of neutrino flavor change in vacuum. We have discussed the implications and limits of our model with respect to the neutrino masses, and commented how matter effects could determine the actual mass hierarchy. We have simplified our model to a two neutrino oscillation theory, and we have used it to explain deficits on experimental neutrino fluxes from different sources. We have estimated some oscillation parameters based on experimental values of solar and atmospheric detectors. For solar neutrino oscillations we find the mass solution to be $4 \cdot 10^{-11} \text{ eV}^2 < \Delta m_{21}^2 < 1.5 \cdot 10^{-3} \text{ eV}^2$, and $\sin^2(\theta_{12}) \sim 0.5$. The current world average for these parameters is [9] $\Delta m_{21}^2 = 7.37 \cdot 10^{-5} \text{ eV}^2$ and $\sin^2(\theta_{12}) = 0.297$. For atmospheric neutrino we find the lowest mass solution to be $\Delta m_{23}^2 > 4.5 \cdot 10^{-5} \text{ eV}^2$, and $\sin^2(\theta_{23}) \sim 0.5$. The current world average for these parameters is [9] $\Delta m_{23}^2 = 2.56 \cdot 10^{-3} \text{ eV}^2$ and $\sin^2(\theta_{23}) = 0.425$ or 0.589 (depending on the hierarchy).

We conclude that, although this simple analysis allows to deduce the existence of neutrino masses, and allows to put a lower limit on the neutrino mass differences, it is not sufficient to perform an estimation of the masses and angles, because to the existence of higher mass solutions due to the periodic nature of the probability expressions.

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