Bright and dark solitons in a Bose-Einstein condensate

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Abstract: We study dark and bright solitons in confined Bose-Einstein condensates of ultracold gases with repulsive and attractive atom-atom interaction. First, we compute analytic properties of the solitons in absence of the trap. Then, we consider the problem of the collision of a bright soliton with a barrier. This is studied by solving numerically the 1D Gross-Pitaevskii equation in the presence of a harmonic oscillator potential and a Rosen-Morse barrier. We discuss in detail the conditions for having a transmission coefficient of $T = 0.5$. Then, we study the evolution of the energies during this process and the nature of the part transmitted.

I. INTRODUCTION

Recently solitons have been a very active field of research in many areas of science such as astrophysics, molecular biology and non-linear optics [1]. The term soliton is associated to localized solitary wave packets that maintain their shape and have particle-like properties that allow them to strongly interact with other solitons and retain their identity [1] .

The existence of these type of solutions is a common feature of nonlinear equations. One ideal physical system to study nonlinear phenomena in quantum many-body physics are Bose-Eisntein condensates (BECs) in ultracold gases. As in BECs most of the atoms populate the same quantum state, the condensate can be described by a mean-field theory such as the Gross-Pitaevskii (GP) theory [2]. The nonlineality is a consequence of the inter-atomic interactions. In particular, depending on whether the interactions are attractive or repulsive the solitons that will appear will correspond to bright or dark solitons, respectively.

Both types of solitons correspond to a modulation of the density profile. Bright solitons [3, 4] are associated to a local maxima of the density presenting an increase of the density profile whereas dark solitons [5] correspond to a localized suppression of the background density. Dark solitons that present a total suppression of the background density are called completely dark solitons and appear when the propagating velocity is zero.

This work is organized as follows. First in Section II we will explain the mean-field theory used, then in Section III we will characterize solitons in homogeneous systems that are solutions of the 1D Gross-Pitaveskii equation. Then, in Section IV we will perform numerical simulations of bright solitons moving in the space inside an harmonic trap with a Rosen-Morse barrier at its center [6]. The use of a Rosen-Morse barrier will avoid the shape effects during the collision. In the last part of the study we will scrutinize under which conditions the initial soliton splits in two after colliding with the barrier. In Section V we will summarize the main conclusions.

II. THEORETICAL BACKGROUND

The evolution of BECs can be described using the time dependent mean-field Gross-Pitaveskii equation [2]. In absence of trapping potential, it reads,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} + g|\psi|^2\psi \tag{1}$$

where g measures the inter-atomic interaction strength. The GP equation reduces to the Schrödinger equation in absence of interactions. To study solitonic solutions it is useful to introduce a new set of units,

$$\tilde{z} = \frac{z}{\xi}, \qquad \tilde{v} = \frac{v}{c}, \qquad \tilde{t} = \frac{t}{\hbar/(|g|n_i)}, \tag{2}$$

where ξ is the healing length of the soliton, c is the sound speed and n_i will be specified in the following.

In solitonic units the Gross-Pitaveskii equation reads,

$$i\frac{\partial\psi(\tilde{z},\tilde{t})}{\partial\tilde{t}} = -\frac{\partial^2\psi(\tilde{z},\tilde{t})}{\partial\tilde{z}^2} + \frac{V_{ext}(\tilde{z})}{|g|n_i}\psi(\tilde{z},\tilde{t}) + \frac{g}{|g|}\frac{n}{n_i}\psi(\tilde{z},\tilde{t}), \tag{3}$$

where $n = |\psi(\tilde{z},\tilde{t})|^2$. The coefficient $\frac{g}{|g|}$ depends on the nature of the interaction, attractive for bright solitons and repulsive for dark solitons, and n_i are defined as the background density, n_∞, for dark solitons and as the central density, n_0, for bright solitons. If we consider the homogeneous case, $V_{ext} = 0$.

In Section III we will introduce an external potential for the numerical simulation, we will consider an harmonic trap with a Rosen-Morse barrier [7] at its center of the form,

$$\frac{V_{ext}(\tilde{z})}{n_0|g|} = \tilde{V}_b\frac{1}{\cosh^2\frac{\tilde{z}}{\tilde{\sigma}}} + \frac{1}{4}\tilde{\omega}^2\tilde{z}^2, \tag{4}$$

where $\tilde{\omega}$ is the frequency of the confinement, and $\tilde{V}_b$ and $\tilde{\sigma}$ are the parameters of the potential barrier expressed in the solitonic units.

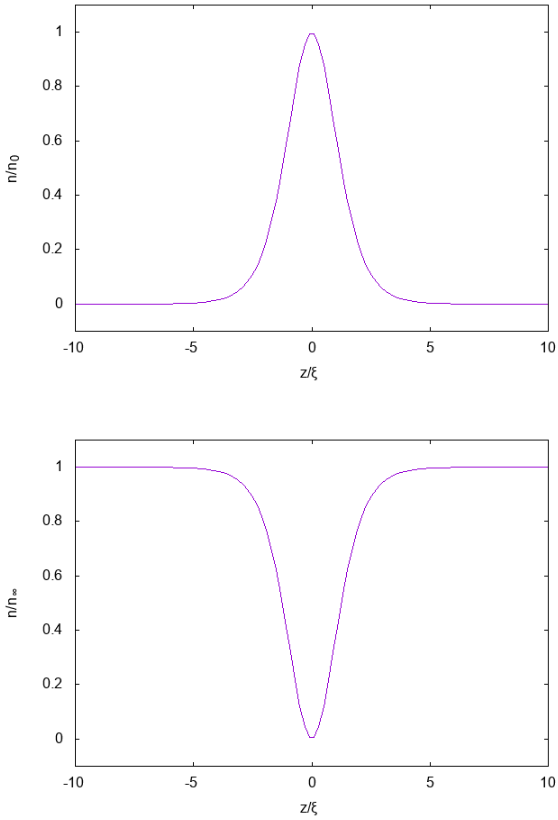

FIG. 1: Density profile of a stationary bright (top panel) and a dark (lower panel) soliton solution. The healing length, ξ, gives an idea of the soliton width.

III. DARK AND BRIGHT SOLITONS: HOMOGENEOUS SYSTEM

We focus the study on solitons whose wave functions are solution of the homogeneous case. As we mentioned previously, we consider separately dark and bright soltions. The dark soliton solution of the time dependent GP equation reads [2],

$$\psi(\tilde{z},\tilde{t}) = \sqrt{n_\infty}(i\tilde{v} + \sqrt{1-\tilde{v}^2}\tanh[(\frac{\tilde{z}}{\sqrt{2}} - \tilde{v}\tilde{t})\sqrt{1-\tilde{v}^2}]), \quad (5)$$

where $\tilde{v}$ is the propagation speed.

The bright soliton solution propagating at a velocity $\tilde{v}$ reads [8],

$$\psi(\tilde{z},\tilde{t}) = \sqrt{n_0}\frac{1}{\cosh\frac{\tilde{z}}{\sqrt{2}}}e^{i\tilde{z}\tilde{v}/\sqrt{2}}e^{i(\frac{1}{2}-\frac{\tilde{v}^2}{4})\tilde{t}}. \quad (6)$$

A. Chemical potential

If we consider the time-independent Gross-Pitaeveskii equation we can compute the chemical potential associated to solitons without velocity, $\tilde{v}=0$, knowing that the wave function introduced before are solution of the time independent GP,

$$-\frac{\partial^2\psi(\tilde{z})}{\partial\tilde{z}^2} - \frac{\mu}{|g|n_i} + \frac{g}{|g|}\frac{n}{n_i}\psi(\tilde{z}) = 0, \quad (7)$$

considering that for dark solitons we will have $g = |g|$ and $n_i = n_\infty$ and for bright solitons $g = -|g|$ and $n_i = n_0$.

$$\mu_{\text{dark}} = n_\infty|g|\,, \quad (8)$$

$$\mu_{\text{bright}} = -\frac{1}{2}|g|n_0\,, \quad (9)$$

these expressions agree with the ones in [9] and [2], respectively.

B. Number of atoms

We have considered that the thermodynamic variable is the chemical potential rather that the number of atoms. So, the number of atoms will be obtained,

$$N = -\frac{\partial E}{\partial\mu} = \int_{-\infty}^{\infty}|\psi(r)|^2 dr\,. \quad (10)$$

If we consider the wave-function introduced for bright solitons, the number of atoms is

$$N_{\text{bright}} = 2\sqrt{2}\xi n_0\,. \quad (11)$$

And following an analog procedure for dark solitons [8],

$$N_{\text{dark}} = -2\sqrt{2}\xi n_\infty\,. \quad (12)$$

We can notice the negative sign obtained for dark solitons. This sign can be interpreted due to the fact that dark solitons are a region of decreased density whereas bright solitons are a gain of density.

C. Energy

The energy of the soliton can be computed as,

$$E = \int_{-\infty}^{\infty}(\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V_{ext(r)}|\Psi|^2 + \frac{g}{2}|\psi|^4)dr. \quad (13)$$

We can separate three terms that contribute to the energy,

$$E = E_{\text{kin}} + E_{\text{pot}} + E_{\text{int}}, \quad (14)$$

where in the kinetic energy for bright solitons we consider two different terms $E_{kin} = E^v_{kin} + E^p_{kin}$. The first one is the kinetic energy associated to a velocity of the soliton and the second term will be the quantum pressure. As we are considering the homogeneous situation, the total energy will only depend on the kinetic and interaction term.

First we consider dark solitons

$$e_{\mathrm{kin}} = \frac{1}{|N|}\int_{\infty}^{\infty} \frac{\hbar^2}{2m}|\frac{d\Psi}{dz}|^2 dz = \frac{1}{3}(1-\tilde{v})^{3/2}|g|n_{\infty}\,. \quad (15)$$

And, for the interaction term we have to substract the background density to compute only the soliton part of the energy,

$$e_{\mathrm{int}} = \frac{1}{|N|}\int_{\infty}^{\infty} \frac{g}{2}(|\psi|^2 - n_{\infty})^2 dz = \frac{1}{3}(1-\tilde{v})^{3/2}|g|n_{\infty}\,. \quad (16)$$

The total energy becomes,

$$e_{\mathrm{dark}} = \frac{2}{3}(1-\tilde{v})^{3/2}|g|n_{\infty}. \quad (17)$$

For bright solitons, first we consider solitons at rest, so the only term that will contribute to the kinetic energy will be the associated to the quantum pressure,

$$e^p_{\mathrm{kin}} = \frac{E^p_{\mathrm{kin}}}{N} = \frac{1}{6}|g|n_0\,, \quad (18)$$

if we add velocity to bright solitons, there will be an extra term due to the velocity,

$$e^v_{\mathrm{kin}} = \frac{E^v_{\mathrm{kin}}}{N} = \frac{|g|n_0\tilde{v}^2}{2}\,. \quad (19)$$

The interaction term obtained following the same procedure used for dark solitons but without subtracting the background density for obvious reasons reads,

$$e_{\mathrm{int}} = \frac{E_{\mathrm{int}}}{N} = -\frac{1}{3}|g|n_0\,. \quad (20)$$

Finally, the total energy for bright solitons reads,

$$e_{\mathrm{bright}} = -\frac{1}{6}|g|n_0 + \frac{|g|n_0\tilde{v}^2}{2}\,. \quad (21)$$

D. Virial theorem

The virial theorem can be derived in 1D and can be used to check if the energies found previously for bright solitons satisfy this relation. To begin with we have to consider a transformation given a parameter λ,

$$\varphi(r) \to \varphi_\lambda(r) = \lambda^{\frac{1}{2}}\varphi(\lambda r). \quad (22)$$

The total energy considering this parameter is given by

$$e(\lambda) = \lambda e_{\mathrm{int}}(\varphi) + \lambda^2 e_{\mathrm{kin}}(\varphi), \quad (23)$$

as in the ground state is a minimum and corresponds to the energy of $\lambda = 1$

$$\frac{de(\lambda)}{d\lambda} = 0\,. \quad (24)$$

The relation that energies must satisfy is

$$e_{\mathrm{int}}(\varphi) + 2e_{\mathrm{kin}}(\varphi) = 0\,. \quad (25)$$

The results found in Eqs. (18) and (20) fulfill this relation.

E. Current density

We can also compute the current density associated to solitons. It will measure the propagation of matter in the system. The general expression reads [2],

$$j(r,t) = \frac{-i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*). \quad (26)$$

For dark solitons we obtain

$$j(r,t) = -\tilde{v}cn_{\infty}(1-\tilde{v}^2)\frac{1}{\cosh^2\left[(\frac{\tilde{z}}{\sqrt{2}} - \tilde{v}\tilde{t})\sqrt{1-\tilde{v}^2}\right]}, \quad (27)$$

if we subtract the density corresponding to a dark soliton with the wave function we have introduced before of the background density n_∞ we find

$$n_{\infty} - \rho = n_{\infty}(1-\tilde{v}^2)\frac{1}{\cosh^2\left[(\frac{\tilde{z}}{\sqrt{2}} - \tilde{v}\tilde{t})\sqrt{1-\tilde{v}^2}\right]}\,. \quad (28)$$

That allows us to express the current density in a compact form as,

$$j(r,t) = -v(n_{\infty} - \rho). \quad (29)$$

The movement of the dark soliton produces an associated current density with opposite sign as the velocity of propagation of the dark soliton. This is because dark solitons are a lack of particles in a background density.

For bright solitons we get,

$$j(z,t) = \frac{n_0 c\tilde{v}}{\cosh^2\frac{\tilde{z}}{\sqrt{2}}}. \quad (30)$$

The density associated with the wave function we are considering is,

$$\rho = \frac{n_0}{\cosh^2\frac{\tilde{z}}{\sqrt{2}}}, \quad (31)$$

and therefore we can write the current density in the familiar form,

$$j = v\rho, \quad (32)$$

which can be interpreted contrasting with the case of dark solitons as the particles contained in the soliton are the responsible to create the current density due to their movement.

IV. SIMULATION AND RESULTS

In this section we study numerically the temporal evolution of bright solitons using the Crank-Nicolson method, which is described in detail in [10-11]. We also introduce a harmonic oscillator potential and a Rosen-Morse barrier, it is important to point out that now we

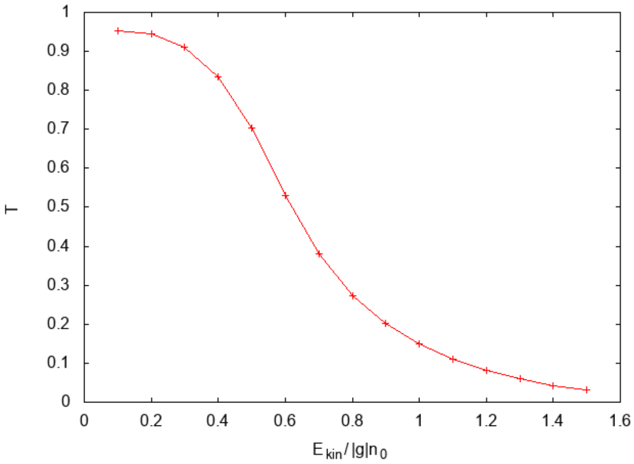

FIG. 2: Behaviour of the transmission coefficient T for different kinetic energy of the initial bright soliton. The initial soliton is at $\tilde{z} = 15$ and the parameters of the trap and the barrier are $\tilde{\omega} = 30\pi 10^{-3}$, $\tilde{V_b} = 0.8$ and $\tilde{\sigma} = 0.67$. The initial velocity of the bright soliton is changed to obtain a rang of kinetic energies. This figure agrees with Fig.2 of [6].

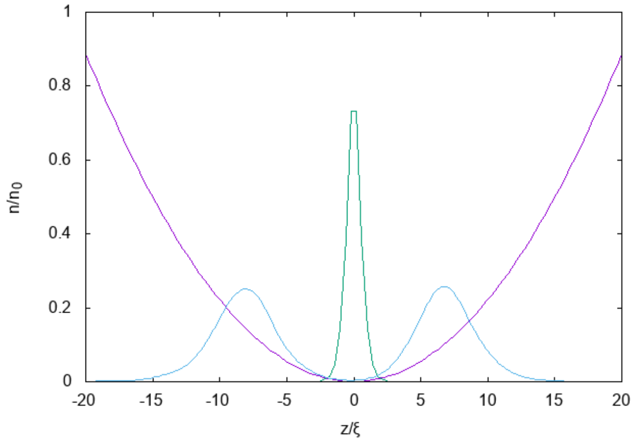

FIG. 3: Simulation of a solition crossing a Rosen-Morse barrier with an harmonic trap that fulfills the condition T=0.5, the solitons after the collision with the barrier have the same number of particles. This example corresponds to $\tilde{V_b} = 0.8$, $\tilde{\sigma} = 0.67$ and $\tilde{v} = 0.62624$.

are considering a non homogeneous system in contrast to the previous sections.

When a bright soliton collides with the Rosen-Morse barrier, the soliton splits into two different solitons where one of them crosses the barrier and the other is reflected. In Fig. 2 we show the fraction of the soliton that is transmitted as a function of the initial kinetic energy of the soliton. As seen in the figure, depending on the initial kinetic energy we can get almost all values for the transmission coefficient. Now we will focus on splitting the initial bright soliton in two equal parts, $T = 0.5$. With the data obtained for Figure 2 the velocity that will satisfy this condition corresponds to,

$$\tilde{v} = 0.62624\,. \tag{33}$$

In Fig. 3 we show the density profile after the collision for this initial velocity. The full evolution is shown in Fig. 4. The soliton is seen to split in two parts that seem to behave like soltions. In the particular case we are considering, imposing the condition $T = 0.5$, we have checked that the two parts can be adjusted to a soliton solution with a different n_0 which implies a different healing lenght. In our case the central density of the new solitons is found to be $n_0 = 0.26$. Qualitatively we have observed that the propagation speed for the two bright solitons is not the same, the transmitted soliton has a higher velocity than the reflected. After the splitting, the two solitons recombine after colliding with the barrier in such a way that in the end the soliton crosses the barrier.

Let us now study how the energies evolve during the collision. The interaction energy remains constant until the soliton reaches the barrier, see Fig. 5. When the soliton collides with the barrier there is an increment of the energy because during the first instant of the collision the soliton reduces its width. Then, after the collision, the interaction energy is smaller since the two remaining solitons are smaller. Regarding the kinetic energy, we can see how it increases when the soliton is closer to the

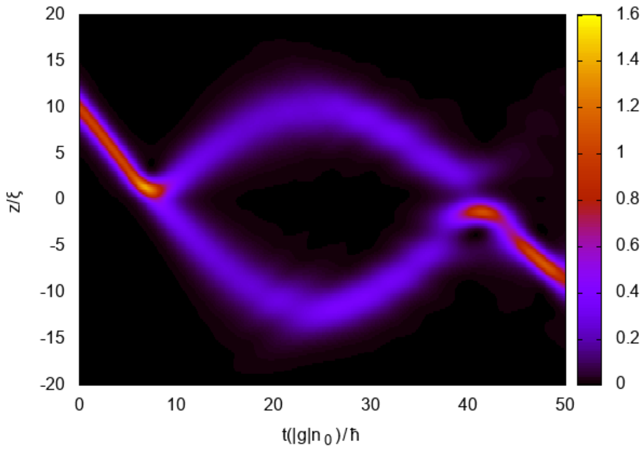

FIG. 4: Process of splitting and recombination of a bright soliton. At the time the soliton crosses the barrier, $\tilde{t} = 7$ it splits into two which move in opposite directions. Due to the harmonic oscillator where they are trapped, the two solitons collide again at $\tilde{t} = 40$ with the barrier and after the collision they recombine in a soliton that completely crosses the barrier. The color represents the values $|\psi(\tilde{z}, \tilde{t})|^2$ and its shown the evolution during $\tilde{t}$ of the position $\tilde{z}$.The initial position is $\tilde{z} = 15$, the velocity is $\tilde{v} = 0.62624$ and the parameters of the barrier and the trap $\tilde{V_b} = 0.8$, $\tilde{\sigma} = 0.67$ and $\tilde{\omega} = 30\pi 10^{-3}$.

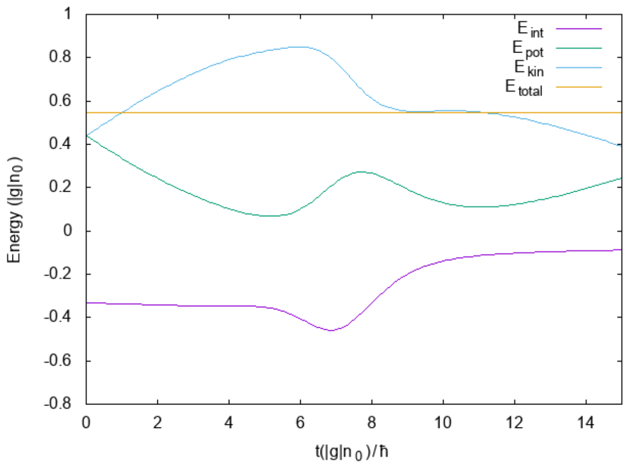

FIG. 5: Evolution of the energies of the soliton during the collision. In this figure is shown the interaction, kinetic and potential energy during the process of a bright soliton colliding with a Rosen-Morse barrier and the conservation of the total energy. The collision occurs at $\tilde{t}$=7. The energy is given in solitonic units, $|g|n_0$. This figure corresponds to the simulation shown in Fig. 4.

center of the harmonic oscillator. However, when the collision with the Rosen-Morse barrier is produced, it starts to decrease. The behavior of the potential energy can be understood in the following way. First it decreases, corresponding to the soliton approaching the center of the trap. At the collision it increases due to the barrier.

V. CONCLUSIONS

We have studied solitons in BECs using the Gross-Pitaveskii equation. We have been able to compute magnitudes such as the chemical potential, the number of atoms, the current density and the total energy for dark and bright solitons. We have also checked the virial theorem for the energies computed for bright solitons.

We have analyzed the propagation of bright solitons for a non homogeneous Gross-Pitaveskii equation, introducing a Rosen-Morse barrier and an harmonic oscillator. We have studied the transmission coefficient as a function of the initial kinetic energy. This has allowed us to find a relation between this coefficient and the velocity of the soliton and we have been able to numerically adjust this relation to obtain the condition for $T = 0.5$.

In addition, we have focused in this situation and simulated the process of collision and recombination of a bright soliton. And we have studied the energies during the collision process.

On the other hand, we have observed that the remaining parts of the soliton when it collides with the barrier can be described using the Eq. (6) with a different parameter n_0 and therefore with a different ξ which lead us to conclude that they are also bright solitons. Finally, as a possible continuation of this work we could study in more detail the velocity of both, the transmitted and reflected solitons.

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